Prop. (deformation inv.). Let $f : U \to \mathbb{R}$ be a holomorphic function, and let $\gamma_1, \gamma_2 \subset U$ be two arcs. Then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

if one of the following holds:

1. Both γ_1 and γ_2 are closed, and one is a (continuous) deformation of the other.
2. Both γ_1 and γ_2 are arcs from α to β, and one is a deformation of the other (fixing the endpoints).

Remark. When talking about U, γ_1 deformed to γ_2, etc., always consider a deformation contained entirely in U. For instance, one cannot deform across a hole in the domain U.

Proof. First, prove claim 1; 2 is very similar.

Key technique: Consider curve $\gamma \subset U$, and a slight perturbation $\gamma' \subset U$ near some point a. Then, we can define $\gamma' = \gamma + \Gamma$ where Γ is a small loop containing γ'. Since Γ is small, consider first the case where Γ lies entirely in some disc $D \subset U$. Then (because f is holomorphic) f has an antiderivative F on D, and $\int_\Gamma f(z)dz = F(\Gamma(b)) - F(\Gamma(a)) = 0$, because Γ is closed.

Accumulating such perturbations over all the points of the curve, any continuous deformation can be achieved.

Proof. We call γ_1, γ_2 (both closed) sufficiently close if there are closed arcs $\Gamma_1, \ldots, \Gamma_N$ such that

1. $\gamma_1 - \gamma_2 = \Gamma_1 + \cdots + \Gamma_N$,
2. each $\Gamma_i \subset D(a_i, r_i) \subset U$ for some $a_i \in U$, $r_i > 0$.

Over the discs $D(a_i, r_i)$ we find holomorphic antiderivatives $F_i : D(a_i, r_i) \to \mathbb{R}$ such that $F_i' = f\,|_{D(a_i,r_i)}$. Then

$$\int_{\Gamma_i} f(z)dz = \int_{\Gamma_i} F_i'(z)dz = 0$$

(i.e. int. of derivative of holomorphic function over closed arc is zero).

This means that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz + \sum_{i=1}^{n} \int_{\Gamma_i} f(z)dz = \int_{\gamma_2} f(z)dz.$$

In general, suppose γ_1, γ_2 are closed and γ_2 is a deformation of γ_1 within U. Then we can find closed arcs $\tilde\gamma_1, \tilde\gamma_2, \ldots, \tilde\gamma_n$ such that $\gamma_1 = \tilde\gamma_1$, $\gamma_2 = \tilde\gamma_n$, and $\tilde\gamma_{k+1}$ is suff. close to $\tilde\gamma_k$ for $k = 1, 2, \ldots, n-1$ (i.e. we can break up the continuous deformation into finitely many sufficiently close perturbations). Note that this would immediately imply the desired equality (progressing through each of the sufficiently close perturbations, i.e.

$$\int_{\gamma_1 = \tilde\gamma_1} f = \int_{\tilde\gamma_2} f = \cdots = \int_{\tilde\gamma_n = \gamma_2} f.$$

Note: consider two curves $\gamma_1, \gamma_2 : [0,1] \to U$ where $\gamma_1(0) = \gamma_1(1)$, $\gamma_2(0) = \gamma_2(1)$. Then γ_1 deforms to γ_2 iff there is some continuous function $\Gamma : [0,1]^2 \to U$ such that $\gamma_1(t) = \Gamma(0,t)$, $\gamma_2(t) = \Gamma(1,t)$, and $\Gamma(s,0) = \Gamma(s,1)$ for any s (i.e. for every s, $\Gamma(s,\cdot)$ is a closed curve).

Bonus: Note then that the image of Γ is another compact set, so there is some $\epsilon > 0$ such that for any $\Gamma(s,t)$ in the image of Γ, an ϵ-neighborhood of $\Gamma(s,t)$ lies entirely in U. Then, note that Γ is continuous on a compact set, and is thus uniformly continuous, so there is some $\delta > 0$ such that if $|s' - s| < \delta$ and $|t' - t| < \delta$, then $|Gamma(s',t') - \Gamma(s,t)| < \epsilon$. Take $\frac{1}{n} < \delta$, and $\Gamma(0,\cdot), \Gamma(1/n,\cdot), \ldots, \Gamma(n/n,\cdot)$ will be the desired family of sufficiently close curves. To extend to the case of non-closed curves with endpoints α, β, fix $\Gamma(s,0) = \alpha$ and $\Gamma(s,1) = \beta$ instead of $\Gamma(s,0) = \Gamma(s,1)$.

Theorem (Cauchy integral formula). Let $f : U \to \mathbb{R}$ be holomorphic, and let $D(a,r)$ be a disc such that its closure $\overline{D(a,r)} \subset U$. Then for any $z_0 \in D(a,r)$,

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z-a|=r} \frac{f(z)}{z - z_0} dz.$$

Proof: We want to evaluate the integral

$$\oint_{|z-a|=r} \frac{f(z)}{z - z_0} dz.$$

We look at $F(z) = \frac{f(z)}{z-z_0}$, where F is defined on $U \setminus \{z_0\}$ which is holomorphic over its domain. Note that the integral of F over a circle is invariant up to deformation in $U \setminus \{z_0\}$; in particular, we can deform the circle into an infinitesimal loop about z_0. Then, note that for any $\epsilon > 0$, because f is continuous, there is some $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| \leq \delta$. Then

$$\oint_{|z-a|=r} \frac{f(z)}{z-z_0} dz = \oint_{|z-z_0|=\delta} \frac{f(z)}{z-z_0} dz.$$

Parametrizing the curve by $z = z_0 + \delta e^{i\theta}$ for $\theta \in [0, 2\pi]$,

$$\oint_{|z-z_0|=\delta} \frac{f(z)}{z-z_0} dz = \int_0^{2\pi} \frac{f(z_0 + \delta e^{i\theta})}{\delta e^{i\theta}} \delta i e^{i\theta} d\theta = i \int_0^{2\pi} f(z_0 + \delta e^{i\theta}) d\theta.$$

Then, note that

$$\left| i \int_0^{2\pi} f(z_0 + \delta e^{i\theta}) d\theta - 2\pi i f(z_0) \right| = \left| \int_0^{2\pi} (f(z_0 + \delta e^{i\theta}) - f(z_0)) d\theta \right| \leq \int_0^{2\pi} \left| f(z_0 + \delta e^{i\theta}) - f(z_0) \right| d\theta \leq \int_0^{2\pi} \epsilon d\theta = 2\pi\epsilon,$$

so the integral approaches $2\pi i f(z)$ as $\delta \to 0$. Thus,

$$2\pi i f(z) = \lim_{\delta \to 0} \oint_{|z-a|=r} \frac{f(z)}{z-z_0} dz = \oint_{|z-a|=r} \frac{f(z)}{z-z_0} dz,$$

since the integral is independent of δ, and thus

$$f(z) = \frac{1}{2\pi i} \oint_{|z-a|=r} \frac{f(z)}{z-z_0} dz.$$

Application. Evaluate:

$$\oint_{|z|=2} \frac{z^2+3z}{(z-3)(z+1)} dz.$$

Note that $\frac{z^2+3z}{(z-3)(z+1)}$ is holomorphic on $\mathbb{R} \setminus \{-1, 3\}$. Thus, over the disc $|z| \leq 2$, the function is holmorphic except at $z = -1$, so we can deform the loop to

$$\oint_{|z|=2} \frac{z^2+3z}{(z-3)(z+1)} dz = \oint_{|z+1|=1} \frac{z^2+3z}{(z-3)(z+1)} dz = \oint_{|z-(-1)|=1} \frac{(z-3)^{-1}(z^2+3z)}{(z-(-1))} dz.$$

Using the integral formula,

$$\oint_{|z-(-1)|=1} \frac{(z-3)^{-1}(z^2+3z)}{(z-(-1))} dz = 2\pi i(-1-3)^{-1}((-1)^2 + 3(-1)) = 2\pi i \frac{1-3}{-4} = \pi i.$$

Now, consider

$$\oint_{|z|=2} \frac{z^2+3z}{(z-1)(z+1)} dz.$$

This can be done similarly, except by deforming the large circle into two smaller circles surrounding the two discontinuities $z = 1$ and $z = -1$.

In general,

$$f(z) = \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{w-z} dw.$$

Next time, will show that

$$f'(z) = \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-z)^2} dw,$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-z)^{n+1}} dw$$