(Late to class, missed :( )

One solution: restrict to a smaller open set U such that a continuous inverse (e.g. $\log z$, $e^{\alpha \log z}$) can be defined.

(Erased! :[ )

If f is continuously defined up to the boundary of U, we can speak of integration over the boundary (where the integral can then be calculated by the sum of the residues inside U).

Apply this to $f(z) = \frac{z^{1/4}}{1+z^2}$ where $U = \{|z| < R\} \setminus [0, R]$.

Note that f is holomorphic on U, U is bounded, and f is C^0 up to ∂U with the understanding that f has different values on the upper and lower sides of $[0, R]$. Note that the value of f on the outer circle Γ_R of radius R is unambiguous.

Calculating, if γ_+ is the line $[0, R]$ on the "upper side" of the boundary,

$$f|_{\gamma_+} = \frac{r^{1/4}}{1+r^2},$$

but for γ_- on the lower side,

$$f|_{\gamma_-} = \frac{(re^{i2\pi})^{1/4}}{1+(re^{i2\pi})^2} = \frac{r^{1/4}e^{i\pi/2}}{1+r^2}.$$

To make this rigorous, consider this taken as a limit of curves near, but not on $[0, R]$, and take limits.

For example: consider $\int_0^\infty \frac{dx}{x^2+6x+8}$. Note first that $x^2 + 6x + 8 = (x + 4)(x + 2)$. Note that the usual method of integrating over a large circle does not work, because we are integrating from 0 to ∞, not $-\infty$ to ∞.

Trick: create a multi-valued function for the purposes of this integral. Consider

$$\frac{1}{2\pi i}\oint_{\partial U} \frac{\log z}{z^2+6z+8} dz.$$

As before,

$$\log z|_{\gamma_+} = \log r$$

and

$$\log z|_{\gamma_-} = \log r + 2\pi i.$$

The $\log r$ cancels in integration, the outer circle integrates to zero, and the difference between the γ_+ and γ_- integrals (the $2\pi i$ term) creates the desired result.

Exercise 62. Prove

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}$$

where $\alpha \notin \mathbb{Z}$.

Solution. Via experience, look at

$$\frac{1}{(z+\alpha)^2}\frac{\cos \pi z}{\sin \pi z}.$$

This is based on needing every integer z to be a pole, and adding a factor $\frac{1}{(z+\alpha)^2}$ to make the value of the residue correct. This introduces another pole of order 2 at $z = -\alpha$.

Use an integral over a large box, sum the residues and solve (ideally, $\frac{\pi^2}{\sin^2 \pi\alpha}$ will arise from the residue for $z = -\alpha$, while the sum will arise from the residues at each $z \in \mathbb{Z}$.

In particular, take R to be a non-integer (i.e. integer plus $\frac{1}{2}$ so that $\frac{1}{(z+\alpha)^2}$ $\frac{1}{R^2}$ and $\left|\frac{\cos \pi z}{\sin \pi z}\right|$ is bounded).

Meromorphic function. Take $U \subset \mathbb{R}$ open.

Definition. A subset $S \subset U$ is discrete if for any $p \in S$ there is some $r > 0$ such that $D(p, r)^* \subset U \setminus S$ (equivalently, p is not a limit point of S).

Definition. f is meromorphic on U (an open subset of $\mathbb{R}$) iff there is a closed, discrete subset $S \subset U$ such that

1. $f : U \setminus S \to \mathbb{R}$ is holomorphic.
2. For every $p \in S$, p is either a removable singularity or a pole of f (i.e. not an essential singularity).

Example: $\frac{p(z)}{q(z)}$ where p, q polynomials with no common factors is meromorphic on $\mathbb{R}$.

Definition. The extended complex plane $\bar{C} = \mathbb{R} \cup \{\infty\}$ is defined such that a (punctured) neighborhood of ∞ is a set $\{z \in \mathbb{R} : |z| > R\}$ for some $R \in (0, \infty)$.

Intuitively, think of the complex plane as the stereographic projection of a sphere less a point; ∞ covers this extra point to form a full sphere.

Using this definition, consider $f : U \to \mathbb{R}$ holomorphic. We say ∞ is an isolated singularity of f if there is an R such that $\{|z| > R\} \subset U$.

Definition. We say $f : U \to \mathbb{R}$ has a removable singularity (resp. pole; resp. essential singularity) at ∞ if

1. ∞ is an isolated singularity of f.
2. $f(1/z)$ has a removable singularity (resp. pole; resp. essential singularity) at 0.

We say that meromorphic f on $\mathbb{R}$ is meromorphic at ∞ (or meromorphic on $\bar{\mathbb{R}}$) if ∞ is an isolated singularity and is not an essential singularity.

Example: A rational function $f(z) = \frac{p(z)}{q(z)}$ with p, q polynomials with no common factors is a meromorphic function on $\bar{\mathbb{R}}$.

Theorem 1. *A meromorphic function on $\bar{\mathbb{R}}$ is a rational function.*

This theorem is the last problem on the assignment.

Strategy: First argue that there are only finitely many poles (use the isolated point property on $\bar{C}$).

Now take $(z - \alpha_1) \cdots (z - \alpha_n) f(z)$, which no longer has poles at $\mathbb{R}$. At this point, either we can use Liouville or there must be a pole at ∞, which means there are only finitely many zeros inside $\mathbb{R}$. Divide out the zeros and repeat. If there is still a pole at ∞, take $1/f(z)$ to achieve a contradiction.