# 1 Power series expansion

Consider $f : U \to \mathbb{R}$ holomorphic, and $D(a, r) \subset U$.

**Theorem 1.** *$f(z)|_{D(a,r)} = f(a)+f'(a)(z-a)+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\cdots$ where the power series converges over $D(a, r)$, and $=$ holds on $D(a, r)$ too.*

*Proof.* Take $z \in D(a, r)$, and take $r'$ such that $z \in D(a, r') \subset \overline{D(a, r')} \subset D(a, r)$. Then

$$f(z) = \frac{1}{2\pi i}\oint_{|w-a|=r'} \frac{f(w)}{w-z}dw.$$

Note that

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\frac{1}{1-\frac{z-a}{w-a}}.$$

Observation: in the integral, $|w - a| = r'$, $|z - a| \leq r' - \epsilon$ for some $\epsilon > 0$. Then

$$\left|\frac{z-a}{w-a}\right| \leq \frac{r'-\epsilon}{r'} = 1 - \frac{\epsilon}{r'} < 1.$$

Thus

$$\frac{1}{1-\frac{z-a}{w-a}} = \sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n.$$

Thus

$$\frac{1}{w-z} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

converges uniformly for $|w - a| = r'$, $|z - a| \leq 1 - \frac{\epsilon}{r'}$.

By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{|w-a|=r}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{|w-a|=r'}\left(f(w)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right)dw.$$

Due to the uniform convergence, we can extract the sum from the integral, so

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{|w-a|=r'}\left(f(w)\frac{(z-a)^n}{(w-a)^{n+1}}\right)dw = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{|w-a|=r'}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n,$$

where the interior integral is equal to $\frac{f^{(n)}(a)}{n!}$. Thus

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n.$$

□

The argument implies that this series converges and is equal to $f(z)$ for $|z - a| \leq r' - \epsilon$, with $r' < r$ and $\epsilon > 0$. But for any $z \in D(a, r)$, we may choose $r'$ and $\epsilon$ satisfactory. Thus the series converges over all of $D(a, r)$. General argument: pick a circle slightly smaller than $r$, use uniform convergence and the Cauchy integral over the circle to translate the sum outside the integral (also, take the Cauchy integral and expand it as a power series with ratio $\frac{z-a}{w-a}$).

Note that the power series converges over all of $D(a, r)$, but converges uniformly over any slightly smaller disc $D(a, r')$, and particularly over its closure as well.

Note also that uniform convergence allows for operations like differentiating under the sum? (not sure here, didn't hear that properly)

Example: $e^z = e^x(\cos y + i \sin y)$. $(e^z)' = e^z$, so $(e^z)^{(n)}|_{z=0} = 1$. By the above,

$$e^z = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots.$$

Remark: using the root test,

$$r = \frac{1}{\limsup \sqrt[n]{1/n!}} = \infty,$$

since $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Note that

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

are both holomorphic. Additionally,

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}), \cosh z = \frac{1}{2}(e^{z} + e^{-z}).$$

Note that almost all operations with the power series (sum, product, etc.) can be done with the power series, due to its strong convergence (i.e. geometric).

## 2 Cauchy Estimate

**Theorem 2.** *Take $f : U \to \mathbb{R}$ holomorphic, and $\overline{D(a,r)} \subset U$, and*

$$M = \sup_{w \in \partial D(a,r)} |f(w)|.$$

*Then*

$$\left|f^{(n)}(a)\right| \leq \frac{n!}{r^n} M.$$

*Proof.* This follows from the Cauchy integral. The formula yields

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}} dw.$$

Thus

$$\left|f^{(n)}(a)\right| \frac{n!}{2\pi} \left| \oint_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}} dw \right| \leq \frac{n!}{2\pi} \oint_{|w-a|=r} \left| \frac{f(w)}{(w-a)^{n+1}} \right| |dw|.$$

Recall that $dw$ is complex, so it also requires an absolute value. Thus

$$\frac{n!}{2\pi} \oint_{|w-a|=r} \left| \frac{f(w)}{(w-a)^{n+1}} \right| |dw| = \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f(a+re^{i\theta})}{(re^{i\theta})^{n+1}} \right| \left|ire^{i\theta}\right| d\theta.$$

Then, $\left|f(a+re^{i\theta})\right| \leq M$, so

$$\frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f(a+re^{i\theta})}{(re^{i\theta})^{n+1}} \right| \left|ire^{i\theta}\right| d\theta \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^{n+1}} r d\theta = \frac{n!}{r^n} M.$$

□

**Theorem 3.** *Under same conditions, take $z \in D(a,r)$. Then*

$$f^{(n)}(z) \leq \frac{rn!}{(r-|z-a|)^n} M.$$

*Proof.* Then the proof follows the same structure as the previous, except that the denominator becomes $\left|re^{i\theta} - (z-a)\right| \geq r - |z-a|$ instead of $\left|re^{i\theta}\right| = r$. Thus

$$\left|f^{(n)}(z)\right| \leq \frac{rn!}{(r-|z-a|)^{n+1}} M.$$

□

## 3 Liouville Theorem

**Theorem 4.** *A bounded holomorphic function $f : \mathbb{R} \to \mathbb{R}$ is a constant function.*

*Proof.* $f$ bounded iff $|f(z)| \leq M$ for some $M$. Then, take any large disc of radius $R$ centered at some point $a \in \mathbb{R}$. Applying the previous estimate,

$$|f'(a)| \leq \frac{M}{R}.$$

As $R$ can be arbitrarily large, $|f'(a)| = 0$. Thus, $f' = 0$ identically for all $a \in \mathbb{R}$, and $f$ is constant. □

## 4 Gauss' Fundamental Theorem of Algebra

**Theorem 5.** *Any positive degree polynomial with complex coefficients (defined on $\mathbb{R}$) has at least one root in $\mathbb{R}$.*

*Proof (one of many).* Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ (WLOG assume leading coeff. 1), with $n \geq 1$, and $p$ has no roots in $\mathbb{R}$. Take

$$f(z) = \frac{1}{p(z)}.$$

Then, show that $f : \mathbb{R} \to \mathbb{R}$ is holomorphic. Following that, show that $f(z)$ is bounded on $\mathbb{R}$ (i.e. when $z$ is of very large magnitude, $z^n$ term dominates, so $p(z)$ also very large (and for smaller $z$, bound by compactness argument over sufficiently large closed ball).

In particular,

$$\frac{1}{p(z)} = \frac{1}{z^n}\frac{1}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}.$$

Pick $R$ such that for $|z| \geq R$,

$$\left|a_{n-k}z^{-k}\right| \leq \frac{1}{n+2}.$$

Then $|z| \geq R \geq 10(n+1)^2$,

$$\left|\frac{1}{p(z)}\right| \leq \left|\frac{1}{z^n}\right| \frac{1}{1 - \frac{n}{n+2}} \leq 1.$$

Liouville Thm. implies $p$ constant, but this contradicts the positive degree of $p$. □