1 Complex Line Integrals

Let $U \subset \mathbb{R}$ be an open subset, $f : U \to \mathbb{R}$ is a continuous function, and $\gamma : [a, b] \to U$ is a (continuous) arc (i.e.

$$\gamma : t \mapsto (x(t), y(t)) \equiv x(t) + iy(t) \equiv z(t).$$

The line integral

$$\int_\gamma f(z)dz \equiv \int_a^b \frac{d}{dt} f(z(t))\dot{z}(t)dt$$

(note the alternative notation $dz(t) \equiv \dot{z}(t)dt$). Alternatively, we can expand $f = u + iv$, to yield (in real line integral form)

$$\int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy).$$

Green's theorem for $\mathbb{R}$ line integrals is as follows:

Theorem. Let U be a simply connected region, and $f : U \to \mathbb{R}$ is holomorphic. Then

$$\int_\gamma f(z)dz$$

is path-independent (i.e. it depends only on the endpoints $\gamma(a), \gamma(b)$, not on the path in between).

Theorem. Let U be a simply connected region, and let $f : U \to \mathbb{R}$ be holomorphic. Then there is a holomorphic function $g : U \to \mathbb{R}$ such that

$$\frac{\partial}{\partial z} g(z) = f(z)$$

(i.e. antiderivatives over simply connected regions exist).

Proof. Let $\alpha \in U$ be a fixed point. For any $z \in U$, pick a continuous arc γ_z running from α to z, and along γ, define g by the definite/explicit path integral

$$g(z) = \int_{\gamma_z} f(w)dw,$$

where $\gamma_z(a) = \alpha$ and $\gamma_z(b) = z$.

Claim. This integral is well-defined. Justification: From previous theorem, this integral is path-independent (since U is simply connected, f is holomorphic), and thus attains only one value for all valid γ_z (which start at α and end at z).

Check:

$$\frac{\partial}{\partial x} g(z) = f(z), \frac{\partial}{\partial y} g(z) = if(z).$$

For the former equation, consider the particular choice of γ_z where, for $t \in [c, b]$, $\gamma_z(t) = x_0 + (x - x_0)\frac{t-c}{b-c} + iy$, where $z = x + iy$ (i.e. γ_z is the combination of a curve from α to $x_0 + iy$ and a straight line segment from $x_0 + iy$ to $x + iy$). Denote by γ_1 the restriction of γ_z to $[a, c]$ (i.e. the first part), and denote by γ_2 the function

$$\gamma_2(t) = x_0 + (x - x_0)\frac{t - c}{b - c} + iy,$$

i.e. γ_2 runs from $x_0 + iy$ to $x + iy$. Then

$$g(z) = \int_{\gamma_1} f(w)dw + \int_{\gamma_2} f(w)dw = \int_{\gamma_1} f(w)dw + \int_{x_0}^{x} f(t + iy)dt,$$

so

$$\frac{\partial}{\partial x} g(z) = f(x + iy) = f(z).$$

For the second equation, consider some $z = x + iy$, and fix some y_0. Then, consider the particular curves γ_1 from α to $x + iy_0$, and the curve γ_2 running from $x + iy_0$ to $x + iy = z$. Then by path independence, if γ_z is any curve from α to z,

$$\int_\gamma f(w)dw = \int_{\gamma_1} f(w)dw + \int_{\gamma_2} f(w)dw = \int_{\gamma_1} f(w)dw + \int_{y_0}^{y} f(x + it)idt,$$

so

$$\frac{\partial}{\partial y}\int_\gamma f(w)dw = if(x+iy) = if(z).$$

Then $g_x + ig_y = 0$, so g is holomorphic. Furthermore,

$$\frac{\partial}{\partial z}g(z) = \frac{1}{2}\left(g_x(z) - ig_y(z)\right) = f(z).$$

Weak extension of theorem: Let U be a disk, and $p \in U$. Suppose $f : U \to \mathbb{R}$ is continuous, and f is holomorphic on $U \setminus \{p\}$. Then there is a holomorphic $g : U \to \mathbb{R}$ such that

$$\frac{\partial}{\partial z}g(z) = f(z)$$

Proof: Fix $\alpha \in U$, and define

$$g(z) = \int_\gamma f(w)dw,$$

where γ runs from α to z. Step 1: Show that $g(z)$ is also path-independent.

Step 2: verify g is C^1, and verify that $\partial_x g(z) = f(z)$ and $\partial_y g(z) = if(z)$ (note that $\partial_x \equiv \frac{\partial}{\partial x}$ and similarly for y).

Step 1: Since U is an open disk, $\int_\gamma f(z)dz$ is independent of paths iff for any simple closed arc $\Gamma \subset U$, $\int \Gamma f(z)dz = 0$. This makes sense since for any two paths γ_1, γ_2 with common endpoints, γ_1 concatenated with the inverse of γ_2 forms a closed arc, with path integral

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz,$$

so the integrals are the same iff $\int_\gamma f(z)dz$.

Caveat: the two curves γ_1 and γ_2 may intersect in complex manner, but we can use continuity to perturb the two arcs γ_1 and γ_2 slightly, so that they are non-intersecting and their combination is simple. The difference between this integral and the original is negligible (i.e. it is the integral over the tiny line segments added to cover the perturbation), and so is arbitrarily small; thus the difference between the integrals can satisfy this only if it is zero.

Tactic: This result is trivial for any curves not surrounding the excised point P. For curves that do surround P, create two curves, γ_- and γ_+, which each cover one half of the area, excepting a very small ϵ-sized region around P. Thus, if Γ is the starting closed arc,

$$\int_{\Gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\partial B_\epsilon(p)} f(z)dz.$$

The first two terms are closed curves not surrounding P, and are zero by the previous result. For the $\partial B_\epsilon(p)$ term,

$$\left|\int_{\partial B_\epsilon(p)} f(z)dz\right| = \left|\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(p)} f(z)dz\right| \leq \mathrm{len}(\partial B_\epsilon(p)) \cdot \max_{\partial B_\epsilon(p)} |f(z)|,$$

and the maximum is bounded since f is continuous, so as $\epsilon \to 0$, the curve length around P goes to zero, and this term also vanishes.

A similar approach for curves running through P (shift slightly to one side and bound the error).

In general, define $g(z) = \int_\gamma f(z)dz$, with γ running from α to z. Is this integral path-independent? To proceed, must show that it is.

Application: Let $f : U \to \mathbb{R}$ where U is a disk and f is holomorphic. Define

$$h(z) = \begin{cases} \dfrac{f(z) - z_0}{z - z_0} & \text{where } z \neq z_0. \\ \dfrac{\partial}{\partial z}f(z) & \text{where } z = z_0. \end{cases}$$

Next time: define $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ when the limit exists (not always).