1 Summary

Properties of holomorphic functions:

- Invariance of line integrals: for $f : U \to \mathbb{R}$ holomorphic, with γ_1 γ_2 in U closed (i.e. γ_1 can be deformed continuously to γ_2 in U; also called homotopy equivalence) then $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$.
  Proof: Link $\Gamma = \gamma_1 - \gamma_2$ to form a closed loop. Then show that f has an antiderivative F for some region containing all of Γ, and
  $$\oint_\Gamma f(z)dz = \oint_\Gamma F'(z)dz = 0.$$
- Cauchy integral formula: for f holomorphic over $\overline{D(a,r)}$, then
  $$f(z) = \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{w-z} dw$$
  for all $z \in D(a,r)$.
  Proof: deform the curve $|w-a| = r$ to $|w-z| = \epsilon$, and show that
  $$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \oint_{|w-z|=\epsilon} \frac{f(w)}{w-z} dw = f(z).$$
- Corollary: for f holomorphic over U,
  $$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-z)^n} dw,$$
  which leads to the power series expansion:
  $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$
  when $z \in D(a,r)$ and $\overline{D(a,r)} \subset U$.
  Proof:
  $$\frac{1}{w-z} = \frac{1}{w-a} \frac{1}{1 - \frac{z-a}{w-a}}$$
  where
  $$\left| \frac{z-a}{w-a} \right| < 1 - 0^+.$$
  For functions f holomorphic on $\overline{D(a,r)}$ and $z \in D(a,r)$,
  $$\left| f^{(n)}(z) \right| \leq \frac{n! r}{(r - |z-a|)^n} M$$
  where $M = \sup_{|z-a|=r} |f(z)|$.
  Liouville Thm: No non-constant bounded entire function. Remark: "entire" refers to a function $f : \mathbb{R} \to \mathbb{R}$ holomorphic.
- Corollary: Power series expansions are unique. Restatement: if
  $$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n = \sum_{n=0}^{\infty} b_n (z-\alpha)^n$$
  on some $D(\alpha, r)$ with $r > 0$, then $a_n = b_n \forall n$.
  Proof: $f^{(n)}(\alpha) = n! a_n$ for all n.
  To show this, note that $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$. Then, differentiating termwise (due to uniform convergence),
  $$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z-\alpha)^{n-k}.$$
  Thus substituting $z = \alpha$,
  $$f^{(k)}(\alpha) = \frac{k!}{(k-k)!} a_k = k! a_k \iff a_k = \frac{f^{(k)}(\alpha)}{k!}$$

2 Uniform limits of holomorphic functions

Note that in general, sequences of differentiable (i.e. C^1) functions can have uniform limits that are only continuous, not differentiable.

Theorem 1. *Let $f_n : U \to \mathbb{R}$ be a sequence of holomorphic functions. Suppose that for any compact $E \subset U$, $f_n|_E$ converges uniformly to a function on E. Then there is a holomorphic function $f : U \to \mathbb{R}$ such that $f_n \to f$ uniformly on every compact subset $E \subset U$. Furthermore, the same convergence holds for $f_n^{(k)} \to f^{(k)}$ for any nonnegative integer k.*

Proof. By assumption, for any $z \in U$, we can view $\{z\} \subset U$ as a compact set. Then $f_n(z)$ converges. Define $f(z) = \lim_{n\to\infty} f_n(z)$ (i.e. take f to be the pointwise limit of the f_n).

Now choose some $r > 0$ such that $\overline{D(a,r)} \subset U$ (because U is open). Then for each n,

$$f_n(z) = \frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f_n(w)}{w-z}dw.$$

Note that the Cauchy integral determines not only the function at its values on a circle, it also determines its derivatives.

Note furthermore that $f_n(w) \to f(w)$ along the boundary $\partial D(a,r)$. Then

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f_n(w)}{w-z}dw.$$

Because $f_n \to f$ uniformly on $\partial D(a,r)$,

$$f(z) = \frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f(w)}{w-z}dw$$

(Note: be more rigorous about this in homework.) Then because any integral of this form is holomorphic, f is holomorphic. ☐

3 Zeros of a Holomorphic Function

Another thing we want to study is the local behavior of holomorphic functions.

Take $f : U \to \mathbb{R}$ holomorphic, $\alpha \in U$, and some $D(\alpha,r) \subset U$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n.$$

In particular,

$$f(z) - f(\alpha) = a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots + a_n(z-\alpha)^n + \cdots.$$

Case 1: $a_j = 0 \forall j = 1, 2, \ldots$. Then $f(z)$ is a constant, and is equal to $f(\alpha)$.

Case 2: At least one of the a_j nonzero for some j; WLOG assume $a_1 = \cdots = a_{n-1} = 0$, $a_n \neq 0$ (where n could be 1). Then

$$f(z) - f(\alpha) = a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots = (z-\alpha)^n(a_n + a_{n+1}(z-\alpha) + \cdots).$$

The inside is now a power series, and it can be shown that this series converges on $D(\alpha,r)$. Then, say

$$(z-\alpha)^n(a_n + a_{n+1}(z-\alpha) + \cdots) = (z-\alpha)^n g(z).$$

I.e. define

$$g(z) = a_n + a_{n+1}(z-\alpha) + \cdots$$

and $g(z)$ will be holomorphic on $D(\alpha,r)$, and $g(\alpha) = a_n \neq 0$.

This is the (local) factorization property. Either $f(z) = f(\alpha)$ near $z = \alpha$, or $f(z) - f(\alpha) = (z-\alpha)^n g(z)$, where $g(\alpha) \neq 0 \iff f^{(n)}(\alpha) \neq 0$.

Theorem 2. *Let $U \subset \mathbb{R}$ be open, connected. Take $z_n \to z_0 \in U$, $z_n \neq z_0$. Suppose $f : U \to \mathbb{R}$ is holomorphic and $f(z_n) = 0$ for all n. Then $f(z) = 0$ for $z \in U$. (i.e. the zeros of a holomorphic function cannot accumulate at a point.)*

Proof. First, we show that there is a $D(z_0, r) \subset U$ such that

$$f|_{D(z_0,r)} = 0$$

Suppose for contradiction this is not true. Note in particular that $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$, since f is continuous. Then

$$f(z) - f(z_0) = (z - \alpha)^n g(z)$$

for some holomorphic function $g(z)$ on $D(z_0, r)$, where $g(z_0) \neq 0$ But then

$$f(z_i) = (z_i - z_0)^n g(z_i) = 0$$

for each i. But $z_i - z_0 \neq 0$, so $g(z_i) = 0$ for all i. Then $g(z_0) = \lim_{i\to\infty} g(z_i) = 0$, which is a contradiction. Thus $f^{(k)}(z_0) = 0 \forall k$. Now we let

$$\Sigma = \{p \in U : f^{(k)}(p) = 0 \forall k\}.$$

If $\Sigma = U$ we are done.

Claims:

- Σ is closed in U.
- Σ is open in U.

Note that Σ is closed, because it is the intersection of sets $E_k = \{z \in U : f^{(k)}(z) = 0\}$, each of which is closed.

Σ is open because of the previous property? (Not fully fleshed out, will cover next class?) ☐