1 Harmonic Functions

Holomorphic Cauchy-Riemann equations:

$$u_x = v_y \text{ and } u_y = -v_x$$
$$f_x + if_y = 0 (= 2\frac{\partial}{\partial \bar{z}} f)$$
$$f_y = if_x$$

Definition: $u : \Omega \to \mathbb{R}$, C^2 function is harmonic iff

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0.$$

Proposition: Let $f(z) : \Omega \to \mathbb{R}$ be a C^2 holomorphic function. Then $\Re f$ and $\Im f$ are harmonic.

Proof. Write $f = u + iv$. Then

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

so u is harmonic. Similarly, v is harmonic. □

2 Antiderivative

Proposition: Let D be a disk and F be a holomorphic function on D. Then there is a holomorphic function $G(z)$ on D such that

$$\frac{\partial}{\partial z} G(z) = F(z).$$

Compare: $h(t) \in C^0$ means that if we take $g(t) = \int_0^t h(u)du$, then $g'(t) = h(t)$.

Proof. Suppose $G(z)$ exists. Then, consider $a + ib$ fixed, and some other complex variable $x + iy$. Then define h by

$$h(t) = G(a + t(x - a), b + t(y - b)).$$

Then

$$G(x, y) = h(1) = G(a, b) + \int_0^1 h'(t)dt.$$

where $h(0) = G(a, b)$. But

$$h'(t) = G_x(a + t(x - a), b + t(y - b))\frac{d}{dt}(a + t(x - a)) + G_y(a + t(x - a), b + t(y - b))\frac{d}{dt}(b + t(y - b))$$

so

$$h'(t) = (x - a)G_x(a + t(x - a), b + t(y - b)) + (y - b)G_y(a + t(x - a), b + t(y - b)).$$

But we defined

$$\frac{\partial}{\partial z} G = \frac{1}{2}(G_x - iG_y) = F.$$

In conjunction with $G_x + iG_y = 0$,

$$G_x = F, G_y = iF.$$

Then

$$h'(t) = (x-a)F(a+t(x-a), b+t(y-b))+(y-b)iF(a+t(x-a), b+t(y-b)) = [(x-a)+i(y-b)]F(a+t(x-a), b+t(y-b)).$$

Since F is holomorphic, it is certainly differentiable, so it is integrable, and thus

$$G(x, y) = G(a, b) + \int_0^1 [(x - a) + i(y - b)]F(a + t(x - a), b + t(y - b))dt.$$

Now, to prove this function is satisfactory, take this as the definition of G (and, WLOG, assume $G(a,b) = 0$ and $a = b = 0$ by translation substitution). We need to check the equations

$$\frac{\partial}{\partial \bar{z}} G(z) = 0, \frac{\partial}{\partial z} G(z) = F(z).$$

For the former, note that now

$$G(x,y) = \int_0^1 (x+iy)F(tx,ty)dt.$$

Differentiating,

$$G_x(x,y) = \int_0^1 [F(tx,ty) + (x+iy)tF_x(tx,ty)]dt$$

and similarly

$$G_y(x,y) = \int_0^1 [iF(tx,ty) + (x+iy)tF_y(tx,ty)]dt.$$

Then note that

$$iG_y(x,y) \int_0^1 [-F(tx,ty) + (x+iy)tiF_y(tx,ty)]dt.$$

But since F is holomorphic, $iF_y = -F_x$. Thus $iG_y = G_x$, which by the Cauchy-Riemann eq. forms above, implies that $\frac{\partial}{\partial \bar{z}} G = 0$.

Latter left as exercise. □

Alternately, we can consider

$$G(x,y) = \int_0^1 [xF(tx,ty) + iyF(tx,ty)]dt,$$

note that this is the path integral from $(0,0)$ to (x,y) so this can be rewritten as

$$G(x,y) = \int_0^1 F(tx,ty)d[(x+iy)t] = \int_0^1 F(tx,ty)(d(tx) + id(ty)) = \int_\gamma F(x,y)(dx+idy)$$

where $\gamma : [0,1] \to D$ is given by $\gamma(t) = (tx,ty)$. Thus this is a line integral over γ, which can be written as

$$G(z) = \int_\gamma F(w)dw = \int_0^z F(w)dw,$$

where w ranges from 0 to z on a straight line.

3 Line Integral

Considering $f(z) = u(x,y) + iv(x,y)$, define

$$\gamma : t \mapsto a(t) + ib(t), t \in [0,1].$$

Define

$$\int_\gamma f(z)dz = \int_\gamma (u(x,y)dx - v(x,y)dy) + i\int_\gamma (v(x,y)dx + u(x,y)dy) (= \int_\gamma (u(x,y) + iv(x,y))(dx+idy)).$$

Substitution yields

$$\int_\gamma f(z)dz = \int_0^1 [u(a(t),b(t))a'(t) - v(a(t),b(t))b'(t)]dt + i\int_0^1 [v(a(t),b(t))a'(t) + u(a(t),b(t))b'(t)]dt.$$

Note that by Stokes' theorem, path integrals $\int_\gamma a(x,y)dx + b(x,y)dy$ are path-independent iff $a_y = b_x$. Thus, the above integral is path-independent iff $u_y = v_x$ and $u_x = -v_y$, i.e. iff f is holomorphic.

Writing out explicitly,

$$\int_\gamma f(z)dx + if(z)dy$$

is path-independent iff

$$\frac{\partial}{\partial y}f(z) = \frac{\partial}{\partial x}if(z)$$

i.e. $f_y = if_x$, or f is holomorphic.

Note: path-independence is important, remember this.