Residue theorem allows us to calculate definite integrals for closed loops. Remark:

$$\int_{-\infty}^{\infty} \equiv \lim_{R\to\infty} \int_{[-R,R]}.$$

Combining this with arc Γ_R^+ (the upper semicircle with radius R),

$$\int_{[-R,R]} + \int_{\Gamma_R^+} = 2\pi i \sum \text{Residues} \in \mathbb{R}.$$

Verify: $\lim_{R\to\infty} \int_{\Gamma_R^+} = 0$, which then implies $\int_{-\infty}^{\infty} = 2\pi i \sum \text{Residues}$.

Using the example $\frac{1}{1+z^4}$, note that

$$\operatorname{Res}(e^{i\pi/4}) = \lim_{z\to e^{i\pi/4}} \frac{z - e^{i\pi/4}}{1+z^4} = \frac{1}{4z^3} = \frac{1}{4}e^{-i3\pi/4}.$$

Similarly calculate residue for $e^{i3\pi/4}$, and plug in.

For quick check,

$$\int_{\Gamma_R^+} \frac{1}{1+z^4}dz \to 0$$

as $R \to \infty$ because the degree of $(1+z^4)^{-1}$ is -4.

Remark: $\int_{-\infty}^{\infty} \frac{q(x)}{p(x)}dx$ when $p(x)$ has no real roots and the degree of $q(x)$ is at most degree of $p(x)$ minus 2 can be calculated this way.

Example: $\int_0^{2\pi} \frac{d\theta}{a+\sin\theta}$ for $a > 1$. Strategy: note that θ ranging from 0 to 2π is like a parametrization of the unit circle. Take $z(\theta) = e^{i\theta}$. Then $dz(\theta) = ie^{i\theta}d\theta$, so $d\theta = \frac{1}{ie^{i\theta}}dz = \frac{dz}{iz}$. Thus

$$\int_{|z|=1} \frac{dz/(iz)}{a + \frac{z-z^{-1}}{2i}} = \int_{|z|=1} \frac{2dz}{z^2 + 2iaz - 1}.$$

Now, apply the residue theorem, and consider the poles; by calculation

$$z = -ia \pm \sqrt{1-a^2}.$$

Considering these roots, note that the product of the two roots is 1, so one of the two roots is inside, and one is outside the loop. The roots can be rewritten as

$$z = i(-a \pm \sqrt{a^2-1}),$$

where $i(-a+\sqrt{a^2-1})$ is inside the loop. Thus

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2\pi i \operatorname{Res}(-ia + i\sqrt{a^2-1}).$$

In general, this substitution $z = e^{i\theta}$ can be used to calculate closed loop (circular) integrals of rational functions on $\sin\theta$, $\cos\theta$.

Exercise 47. Calculate $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}dx$. First, try $\int_{[-R,R]} + \int_{\Gamma_R^+}$, as before.

Looking at

$$\int_{\Gamma_R^+} \frac{1}{2}\frac{e^{iz}+e^{-iz}}{1+z^4}dz,$$

note that the latter term e^{-iz} explodes as $z \to i\infty$. Thus, this approach will not work. Instead, use

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}dx = \int_{-\infty}^{\infty} \frac{\Re(e^{ix})}{1+x^4}dx = \Re\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4}dx\right).$$

Now, instead integrate over the boundary of the rectangle with vertices $R, R+iR, -R+iR, -R$. Then,

$$\int_{R\to R+iR} \frac{e^{iz}}{1+z^4}dz = \int_0^R \frac{e^{iR-t}}{1+(R+it)^4}dz,$$

where the top is bounded and the denominator is of order R^4, so the integral goes to zero. The other branches can be checked similarly.

Another note: in general, we can say

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2},$$

but the same trick does not work for

$$\int_0^\infty \frac{x^{1/4}}{1+x^2}dx.$$

Now, define $z^{1/4}$ holomorphic over $D(0,R) \setminus [0,R)$ (i.e. take a sheaf). Take our loop integral where

$$\begin{aligned}\gamma_1(t) &= t, t \in [0,R],\\ \gamma_2(t) &= Re^{i\theta}, \theta \in [0,2\pi],\\ \gamma_3(t) &= se^{2\pi i}, s \in [R,0].\end{aligned}$$

We write γ_3 as such to denote a slight gap between γ_1 and γ_3, over the break in the definition of $z^{1/4}$. Take

$$\int_{\gamma_1+\gamma_2+\gamma_3} \frac{z^{1/4}}{1+z^2}dz = \int_0^R \frac{t^{1/4}}{1+t^2}dt + \int_0^{2\pi} \frac{R^{1/4}e^{i\theta/4}}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta + \int_R^0 \frac{(se^{2\pi i})^{1/4}}{1+(se^{2\pi i})^2}ds.$$

Note that the γ_2 integral goes to zero, because R^2 dominates. Thus

$$\text{RHS} = \int_0^R \frac{t^{1/4}}{1+t^2} - \int_0^R \frac{e^{i\pi/2}s^{1/4}}{1+s^2}ds,$$

Thus

$$2\pi i \sum \text{Res} = (1-e^{2\pi i})I,$$

where I is the desired integral.