11f. Consider

$$f(z)=\sum_{k=0}^{\infty}\frac{k}{k^2+4}z^k.$$

For $z=1$, this sum diverges; for $z=-1$, this sum converges. For $z=e^{i\theta}$, $\theta\neq 0,\pi$, do as much as you can.

14. Circle?

54. Very involved.

Continued from previous lecture?:

Consider $f:U\to\mathbb{R}$ with U connected, $a\in U$.

Lemma 1. *Suppose $f(a)=0$ and $f^{(n)}(a)\neq 0$ for some n. Then there is some $r>0$ such that $D(a,r)\subset U$ and $f(z)$ has no zeros in $D(a,r)\setminus\{a\}$.*

Define $D(a,r)^*=D(a,r)\setminus\{a\}$.

Proof. Take f holomorphic over $\overline{D(a,R)}\subset U$. Expanding at a,

$$f(z)=f(a)+\cdots+\frac{f^{(k)}(a)}{k!}(z-a)^k+\cdots$$

where $f^{(k)}(a)$ is the first nonzero derivative of f at a; this exists because there is at least one nonzero derivative, as assumed.

Then

$$g(z)=\sum_{\ell=k}^{\infty}\frac{f^{(\ell)}(a)}{\ell!}(z-a)^{\ell-k},$$

and

$$f(z)=(z-a)^k g(z).$$

Note that $g(a)=\frac{f^{(k)}(a)}{k!}\neq 0$.

Since $g(z)$ converges in a small disk $D(a,r)$ and is holomorphic, and since $g(a)\neq 0$, then there exists some small $r\leq R$ such that $g(z)$ has no zeros in $D(a,r)^*$. □

Corollary 1. *If $f^{(n)}(a)=0$ for all $n\geq 0$, then $f(z)=0$ in some small disk $D(a,r)$.*

Now, this raises the question: can a holomorphic function f be zero over some small disk, but nevertheless be nonzero somewhere else over the same connected region U? Answer: No!

Take

$$\Sigma_n=\{p\in U : f^{(n)}(p)=0\}.$$

Note that this is closed, because each $f^{(n)}$ is continuous.

Then

$$\Sigma_\infty=\bigcap_{n=0}^{\infty}\Sigma_n$$

is also closed.

Then, $p\in\Sigma_\infty$ implies that $f^{(n)}(p)=0\forall n$, which implies $f(z)=0$ near p. Thus Σ_∞ is closed.

Lemma 2. *$\Sigma_\infty\subset U$ is open.*

Proof. Let $p\in\Sigma_\infty\subset U$. To show Σ_∞ is open at p, we need only to find some $\epsilon>0$ such that $D(p,\epsilon)\subset\Sigma_\infty$.

Since $p\in\Sigma_\infty$, $f^{(n)}(p)=0\forall n$. Thus the power series expansion of f at p is identically zero, and thus for some small ϵ, $f(z)=0\forall z\in D(p,\epsilon)$, and thus $D(p,\epsilon)\subset\Sigma_\infty$. □

The final result follows from noting that Σ_∞ can only be both open and closed in U iff it is empty or equal to U.

Theorem 1. *Let $U \subset \mathbb{R}$ be connected, and let $f : U \to \mathbb{R}$ be holomorphic. Suppose $\{p_n\} \to p$ in U such that $p_n \neq p \forall n$, and suppose $f(p_n) = 0 \forall n$. Then $f = 0$ on all of U.*

Proof. We want to show that all the derivatives of f at p are zero. But f is continuous, so $f(p) = 0$. We claim that this means $f^{(n)}(p) = 0$ for all $n \geq 1$; otherwise, p is the only zero of f in some $D(p, \epsilon)$ with $\epsilon > 0$ by the previous result; this contradicts $p_n \to p$ with $p_n \neq p$ and $f(p_n) = 0$ (because some p_n must be in $D(p, \epsilon)^*$).

This proves that

$$\Sigma_\infty = \{\alpha \in U : f^{(n)}(\alpha) = 0 \forall n\}$$

is nonempty, since it contains p.

If Σ_∞ is both closed and open in some connected domain U, and is also nonempty, then $\Sigma_\infty = U$. □

Corollary 2. *Suppose $f, g : U \to \mathbb{R}$ are both holomorphic, with U connected. Suppose $\gamma \subset U$ is some (piecewise C^1?) arc in U, and*

$$f|_\gamma = g|_\gamma.$$

Then $f = g$ on U.

Proof. By the previous result, $f - g$ has infinitely close zeros (along γ), so $f - g = 0$ on all of U. □

Example: $\sin 2x = 2 \sin x \cos x$, for $x \in \mathbb{R}$. This implies $\sin(2z) = 2 \sin z \cos z$ for all $z \in \mathbb{R}$, since we can take any arc $\gamma \subset \mathbb{R} \subset \mathbb{R}$, and use the previous corollary.

Expanding on connectedness:

U is connected iff for all $p, q \in U$, there is some arc γ connecting p and q. Take some subset $V \subset U$ that is both closed and open in U, then assume $p \in V$, and assume for contradiction $q \notin V$. Since γ is continuous, the preimage $\gamma^{-1}(U)$ is both closed and open in $[0, 1]$, but the only such set is all of $[0, 1]$; i.e. $q \in V$, which is a contradiction. Thus, either V contains both p and q, or neither; thus V is empty or $V = U$.

Theorem 2. *Riemann extension theorem (i.e. removable singularity): Suppose $f : D(a, r)^* \to \mathbb{R}$ is holomorphic. Suppose $\lim_{z \to a} f(z) = \alpha$ exists in $\mathbb{R}$. Then f extends to a holomorphic function on $D(a, r)$, i.e. there exists $\tilde{f} : D(a, r) \to \mathbb{R}$ holomorphic such that*

$$\tilde{f}|_{D(a,r)^*} = f.$$

Proof. Note that there is only one possible extension, i.e.

$$\tilde{f}(z) = \begin{cases} \alpha = \lim\limits_{w \to a} f(w) & \text{if } z = a \\ f(z) & \text{otherwise} \end{cases}$$

since $\tilde{f}$ must be C^0, i.e. continuous.

We need to show that $\tilde{f}$ is holomorphic. One way is to show that

$$\tilde{f}'(a) = \lim_{h \to 0} \frac{\tilde{f}(a + h) - \tilde{f}(a)}{h}$$

exists, but this may be difficult.

Instead, use a power series expansion, or rather prove the result for a modified function with similar power series. In particular, show $g(z) = (z - a)\tilde{f}(z)$ is holomorphic on $D(a, r)^*$, which has no difficulty. Then

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h} = \lim_{h \to 0} \frac{h\tilde{f}(a + h)}{h} = \tilde{f}(a) = \alpha,$$

so g is holomorphic.

Then, either $g(z) = 0$ for all $z \in D(a, r)$, in which case $\tilde{f} = 0$ is holomorphic; or g is not identically zero, and $g(z) = (z - a)^k \tilde{g}(z)$, with $\tilde{g}(a) \neq 0$, $k \geq 1$ (with $\tilde{g}$ holomorphic).

Then $\tilde{f}(z) = \frac{g(z)}{z-a} = (z - a)^{k-1}\tilde{g}(z)$ is holomorphic.

Another alternative: Morera's theorem: If $f \in C^0(D(a,r))$ and $\oint_\gamma f = 0$ for all closed arcs γ, then f is holomorphic on $D(a,r)$. The proof of this uses the existence of an antiderivative F of $\tilde{f}$ on a disc (using the taxicab-style path from a to z for integration; this is well-defined iff integrals on closed paths are zero), after which direct calculation implies that the Cauchy-Riemann equations are satisfied, and F is holomorphic. Then, $\tilde{f} = F'$ is holomorphic.

Left to verify: $\tilde{f}$ as defined has zero integral over every closed curve γ, and in particular every rectangular closed curve. Idea: over the integration path, cut the loop into two (if needed) pieces avoiding the point a, and use continuity at a (we know closed curves not surrounding a will have zero integral by holomorphicity of f). $\square$