- Professor: Jun Li, 383Z, 723-4508, jli@math.stanford.edu.
- Course website: math.stanford.edu/ jli/.
- Every week: HW due Wed. 5pm to his office.
- Solutions will be posted on Coursework.
- Office hours: Mon, Tues?, Wed 2:05-2:40; Tues 12-1, 3-4.
- One evening? midterm (1.5 hr), one final (3 hr, Fri. Jun. 8 8:30-11:30am).
- Course grade: 20% homework, 30% midterm, 50% final.
- All information available on course website.

This course is primarily concerned with functions of one complex variable (effectively two real variables); inherently concerned with line integrals (i.e. Math 52).

1 Complex Numbers

The complex plane can be represented as an ordered pair $(x, y) \in \mathbb{R} \times \mathbb{R}$, corresponding to element $x + iy \in \mathbb{R}$. $i^2 = -1$. These numbers arise naturally from the problem of finding quadratic roots for equations $ax^2 + bx + c = 0$, i.e.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Real roots correspond to $b^2 - 4ac \geq 0$. If $b^2 - 4ac < 0$, then we create a pair of roots

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a},$$

i.e. $\sqrt{-5} = \sqrt{5}\sqrt{-1} = \sqrt{5}i$. Complex numbers are defined as a sum $z = x + iy$, where x and y are real numbers (z conventionally denotes such a complex number).

Really hoping this gets more interesting later on. . .

Alternatively, complex numbers $z = x + iy$ can also be represented in polar form $z = r\cos\theta + ir\sin\theta = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the counterclockwise angle from the positive x axis to z. This derives from the Euler equation, $e^{i\theta} = \cos\theta + i\sin\theta$. For a canonical form, take r nonnegative and take $\theta \in [0, 2\pi)$.

Example: $\theta = \frac{\pi}{6}$, $\frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{\frac{\pi}{6}i}$.

Operations:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- Complex conjugate: if $z = x + iy$ with x, y real, $\bar{z} = x - iy$. If $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$.
- $|x + iy| = \sqrt{x^2 + y^2}$, $\left|re^{i\theta}\right| = |r|$. Note: $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$, or $z\bar{z} = re^{i\theta}re^{-i\theta} = r^2e^0 = r^2$. This depends on the fact that $e^ae^b = e^{a+b}$ for complex a, b (not yet proved).
- $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. Thus, if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.

Note that there is a correspondence from (x, y) to $(z, \bar{z})$, since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$.

2 Complex-valued Functions

By comparison, $\mathbb{R}$-valued functions can be graphed with the relation $y = f(x)$, e.g. $f(x) = x^2 + 3x$. Two-argument functions (i.e. $\mathbb{R}$-valued functions on the real plane $\mathbb{R}^2$) can be represented by $z = f(x, y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ assigns to each pair $(x, y) \in \mathbb{R}^2$ a value $f(x, y) \in \mathbb{R}$, e.g. $f(x, y) = x^2 + y^2$. Can also be graphed in $\mathbb{R}^3$ in the natural way.

$\mathbb{R}$-valued functions on the complex plane $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ which assigns to every complex number $z = x + iy \in \mathbb{R}$ an associated complex number $f(z) \in \mathbb{R}$. Can equivalently view $\mathbb{R}$ as $\mathbb{R}^2$ with the natural bijection $x + iy \mapsto (x, y)$.

Constructive definition: A complex-valued function f is given by

$$f : x + iy \mapsto u(x, y) + iv(x, y)$$

where u, v are $\mathbb{R}$-valued functions on the real plane.

Examples:

- $$f(z) = f(x,y) = f(x+iy) = (x+iy)^2 = z^2 = (x^2 - y^2) + i(2xy) = u(x,y) + iv(x,y)$$
  where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. z^2 and $u(x,y) + iv(x,y)$ are two canonical forms for writing the function.
- $$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + i(e^x \sin y) = u(x,y) + iv(x,y)$$
  where $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Again, canonical forms e^z and $u(x,y) + iv(x,y)$.
- $$g(z) = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = u(x,y) + iv(x,y)$$
  where $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Again, canonical forms $z\bar{z}$ and $u(x,y) + iv(x,y)$.

3 Calculus

For the moment, define $f : \mathbb{R}^2 \to \mathbb{R}$. Then $f(x,y) = u(x,y) + iv(x,y)$ is C^1 iff both u and v are C^1 (i.e. have continuous partial derivatives).

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \end{aligned}$$

Consider $f(x,y) = (x+iy)^2 = (x^2 - y^2) + i(2xy)$. Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + i(2y) \\ \frac{\partial f}{\partial y} &= -2y + i(2x). \end{aligned}$$

Interesting question: How do we consider $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$? Intuitive response: $f(z) = z^2$, so $\frac{\partial f}{\partial z} = 2z$ and $\frac{\partial f}{\partial \bar{z}} = 0$ (view z and $\bar{z}$ as independent variables for this differentiation).

Next lecture: how to compute $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$. Method: Use the change of variables $z = x + iy$ and $\bar{z} = x - iy$, and apply chain rule formally (i.e. the correspondences $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$ from before). Holomorphic functions are those where $\frac{\partial f}{\partial \bar{z}} = 0$, based on this definition.