The Convolution Operation

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Topics in Convolutional Networks

- Overview

1. The Convolution Operation
2. Motivation
3. Pooling
4. Convolution and Pooling as an Infinitely Strong Prior
5. Variants of the Basic Convolution Function
6. Structured Outputs
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9. Random or Unsupervised Features
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11. Convolutional Networks and the History of Deep Learning

Plan of discussion

1. What is convolution?
2. Convolution: continuous and discrete cases
3. Convolution in two dimensions
4. Discrete convolution viewed as matrix multiplication

What is convolution?

- Convolution is an operation on two functions of a real-valued argument
- Examples of the two functions
  - Tracking location of a spaceship by a laser sensor
    - A laser sensor provides a single output $x(t)$, the position of spaceship at time t
  - w a function of a real-valued argument
    - If laser sensor is noisy, we want a weighted average that gives more weight to recent observations
    - Weighting function is $w(a)$ where a is age of measurement
- Convolution is the weighted average or smoothed estimate of the position of the spaceship
  - A new function s

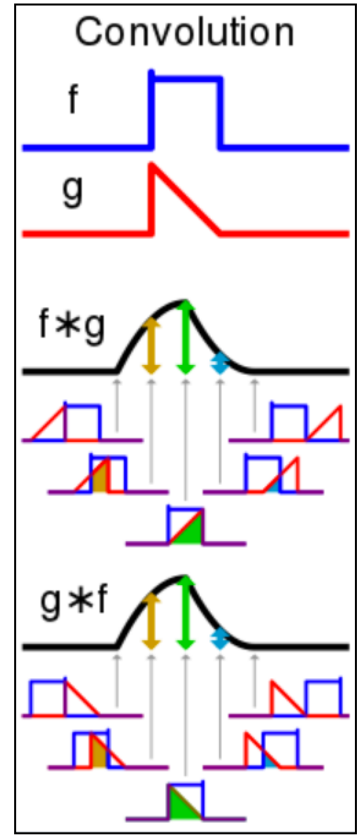
$$\boxed{s(t) = \int x(a)w(t-a)\,da}$$

What is Convolution?

- One-dimensional continuous case
  - Input $f(t)$ is convolved with a kernel $g(t)$

$$\boxed{(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau}$$

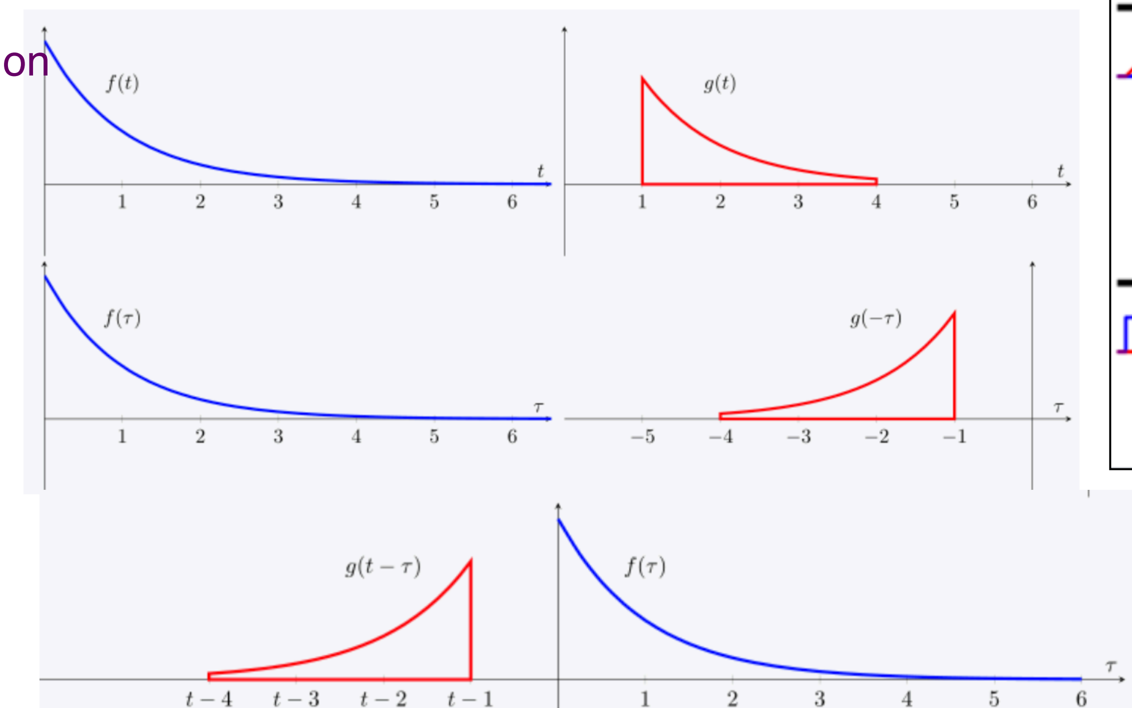
Note that $(f * g)(t) = (g * f)(t)$



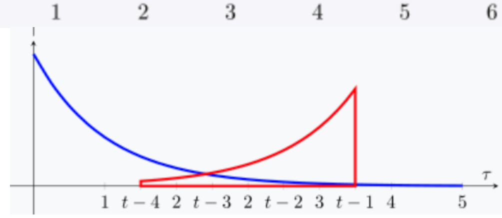
1.Express each function in terms of a dummy variable τ

2. Reflect one of the functions $g(\tau) \rightarrow g(-\tau)$

3. Add a time offset t, which allows $g(t-\tau)$ to slide along the τ axis

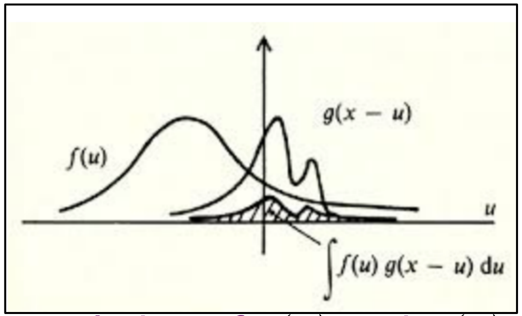


4. Start t at $-\infty$ and slide it all the way to $+\infty$
Wherever the two functions intersect find the integral of their product



https://en.wikipedia.org

Definition of convolution of input and kernel



Convolution of $f(u)$ and $g(u)$

- Convolution is a new function s, the weighted average of x

$$s(t) = \int x(a)w(t-a)\,da$$

  - This operation is typically denoted with an asterisk

$$s(t) = (x * w)(t)$$

  - w needs to be a valid pdf, or the output is not a weighted average
  - w needs to be 0 for negative arguments, or we will look into the future
- In convolution network terminology the first function x is referred to as the *input*, the second function w is referred to as the *kernel*
- The output s is referred to as the feature map

Convolution with Discrete Variables

- Laser sensor may only provide data at regular intervals
- Time index t can take on only integer values
  - x and w are defined only on integer t

$$\boxed{s(t) = (x * w)(t) = \sum_{a=-\infty}^{\infty} x(a)w(t-a)}$$

- In ML applications, input is a multidimensional array of data and the kernel is a multidimensional array of parameters that are adapted by the learning algorithm
  - These arrays are referred to as tensors
- Input and kernel are explicitly stored separately
  - The functions are zero everywhere except at these points

Convolution in discrete case

- Here we have discrete functions f and g

$$(f * g)[t] = \sum_{\tau=-\infty}^{\infty} f[\tau] \cdot g[t-\tau]$$

$f[t]$							
	0	1	2	-1	1	-3	0

$g[t-\tau]$			
	1	0	-1

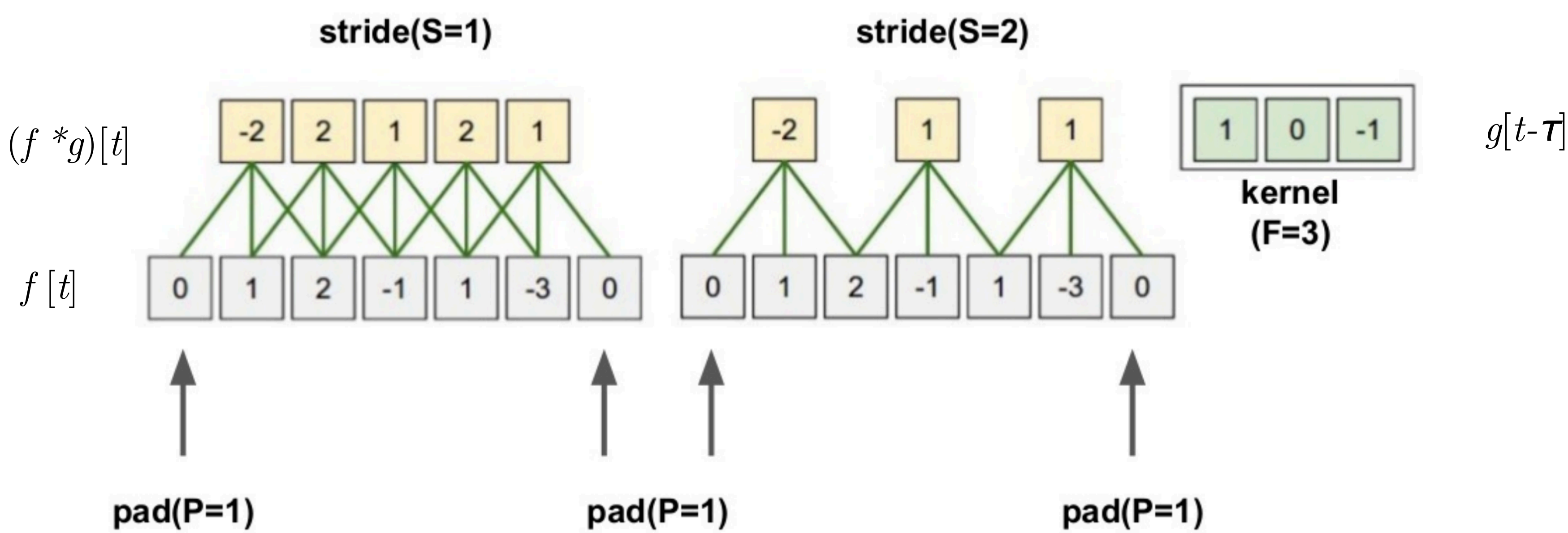
Computation of 1-D discrete convolution

Parameters of convolution:

Kernel size (F)

Padding (P)

Stride (S)



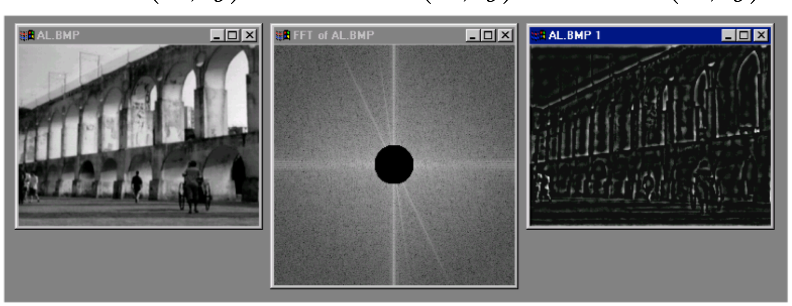
Two-dimensional convolution

- Convolutions over more than one axis
- If we use a 2D image I as input and use a 2D kernel K we have

$$\boxed{S(i,j) = (I * K)(i,j) = \sum_m \sum_n I(m,n)K(i-m, j-n)}$$

Sharply peaked kernel K for edge detection

$I(i\ ,\ j)$ $K(i\ ,\ j)$ $S(i\ ,\ j)$



Kernels K_1-K_4 for line detection

-1	-1	-1
2	2	2
-1	-1	-1

Horizontal lines

-1	2	-1
-1	2	-1
-1	2	-1

Vertical lines

-1	-1	2
-1	2	-1
2	-1	-1

45 degree lines

2	-1	-1
-1	2	-1
-1	-1	2

135 degree lines

Commutativity of Convolution

- Convolution is commutative. We can equivalently write:

$$\boxed{S(i,j) = (K * I)(i,j) = \sum_m \sum_n I(i-m, j-n) K(m,n)}$$

- This formula is easier to implement in an ML library since there is less variation in the range of valid values of m and n
- Commutativity arises because we have flipped the kernel relative to the input
  - As m increases, index to the input increases, but index to the kernel decreases

Cross-Correlation

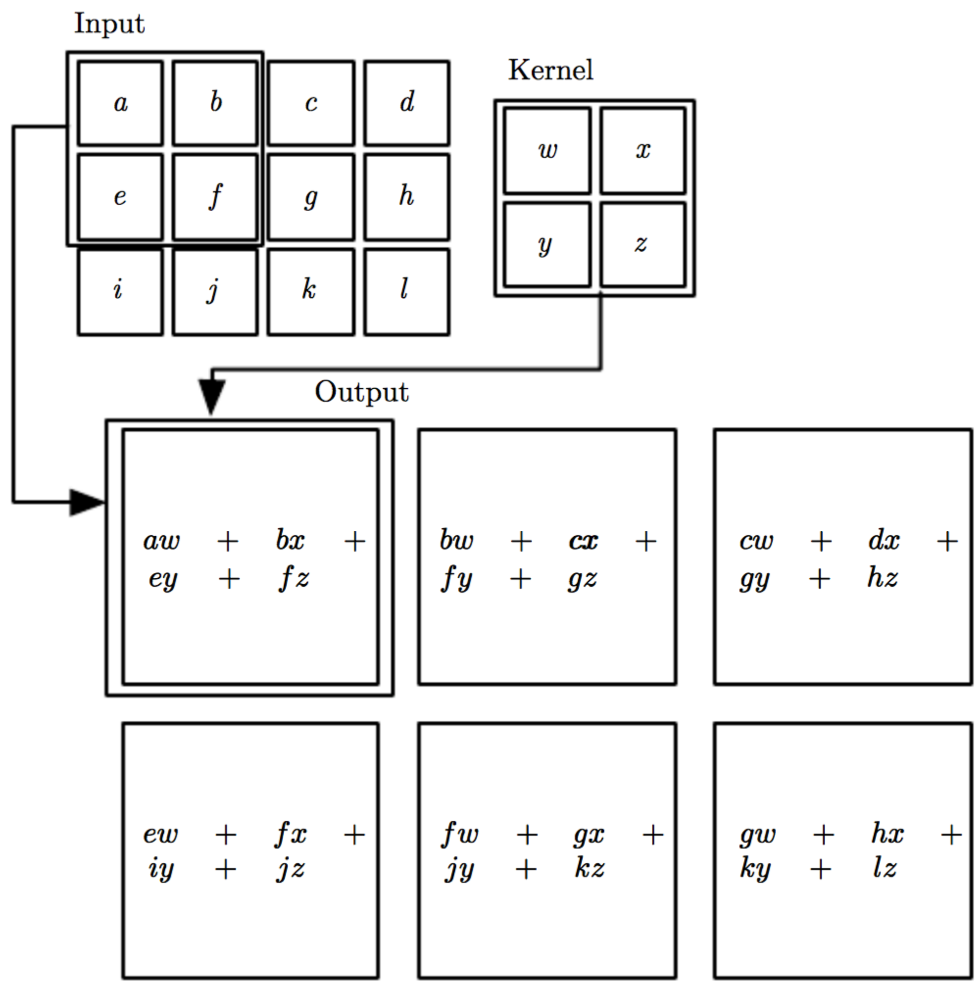
- Same as convolution, but without flipping the kernel

$$\boxed{S(i,j) = (K * I)(i,j) = \sum_m \sum_n I(i+m, j+n) K(m,n)}$$

- Both referred to as convolution, and whether kernel is flipped or not
- In ML, learning algorithm will learn appropriate values of the kernel in the appropriate place

Example of 2D convolution

- Convolution without kernel flipping applied to a 2D tensor
- Output is restricted to case where kernel is situated entirely within the image
- Arrows show how upper-left of input tensor is used to form upper-left of output tensor



Discrete Convolution Viewed as Matrix multiplication

- Convolution can be viewed as multiplication by a matrix
- However the matrix has several entries constrained to be zero
- Or constrained to be equal to other elements
  - *For univariate discrete convolution: Univariate Toeplitz* matrix:
    - Rows are shifted versions of previous row

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

  - 2D case: *doubly block circulant matrix*
    - It corresponds to convolution

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$