Parameter Norm Penalties

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Regularization Strategies

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Topics in Parameter Norm Penalties

1. Overview (limiting model capacity)
2. L^2 parameter regularization
3. L^1 regularization

Limiting Model Capacity

- Regularization has been used for decades prior to advent of deep learning
- Linear- and logistic-regression allow simple, straightforward and effective regularization strategies
  - Adding a parameter norm penalty $\Omega(\theta)$ to the objective function J :

$$\boxed{\tilde{J}(\theta; X, y) = J(\theta; X, y) + \alpha\Omega(\theta)}$$

    - where $\alpha \varepsilon [0,\theta)$ is a hyperparameter that weight the relative contribution of the norm penalty term Ω
      - Setting α to 0 results in no regularization. Larger values correspond to more regularization

Norm Penalty

- When our training algorithm minimizes the regularized objective function

$$\boxed{\tilde{J}(\theta; X, y) = J(\theta; X, y) + \alpha\Omega(\theta)}$$

  - it will decrease both the the original objective J on the training data and some measure of the size of the parameters θ
- Different choices of the parameter norm Ω can result in different solutions preferred
  - We discuss effects of various norms

No penalty for biases

- Norm penalty Ω penalizes only weights at each layer and leaves biases unregularized
  - Biases require less data to fit than weights
  - Each weight specifies how variables interact
    - Fitting weights requires observing both variables in a variety of conditions
- Each bias controls only a single variable
  - We do not induce too much variance by leaving biases unregularized
- $\boldsymbol{w}$ indicates all weights affected by norm penalty
- $\boldsymbol{\theta}$ denotes both $\boldsymbol{w}$ and biases

Different or Same αs for layers?

- Sometimes it is desirable to use a separate penalty with a different α for each layer

$$\boxed{\tilde{J}(\theta; X, y) = J(\theta; X, y) + \alpha\Omega(\theta)}$$

- Because it can be expensive to search for the correct value of multiple hyperparameters, it is still reasonable to use same weight decay at all layers to reduce search space

L^2 parameter Regularization

- Simplest and most common kind
- Called *Weight decay*
- Drives weights closer to the origin
  - by adding a regularization term to the objectve function

$$\boxed{\Omega(\theta) = \frac{1}{2} || w ||_2^2}$$

- In other communities also known as *ridge regression* or *Tikhonov regularization*

Gradient of Regularized Objective

- Objective function (with no bias parameter)

$$\boxed{\tilde{J}(w;X,y) = \frac{\alpha}{2} w^T w + J(w;X,y)}$$

- Corresponding parameter gradient

$$\boxed{\nabla_w \tilde{J}(w;X,y) = \alpha w + \nabla_w J(w;X,y)}$$

- To perform single gradient step, perform update:

$$\boxed{\boldsymbol{w} \leftarrow \boldsymbol{w} - \varepsilon\left(\alpha \boldsymbol{w} + \nabla_{\boldsymbol{w}} J\left(\boldsymbol{w}; X, \boldsymbol{y}\right)\right)}$$

- Written another way, the update is

$$\boxed{\boldsymbol{w} \leftarrow (1 - \varepsilon\alpha)\boldsymbol{w} - \varepsilon\nabla_{\boldsymbol{w}} J\left(\boldsymbol{w}; X, \boldsymbol{y}\right)}$$

  – We have modified learning rule to shrink $\boldsymbol{w}$ by constant factor $1\text{-}\varepsilon\alpha$ at each step

To study effect on entire training

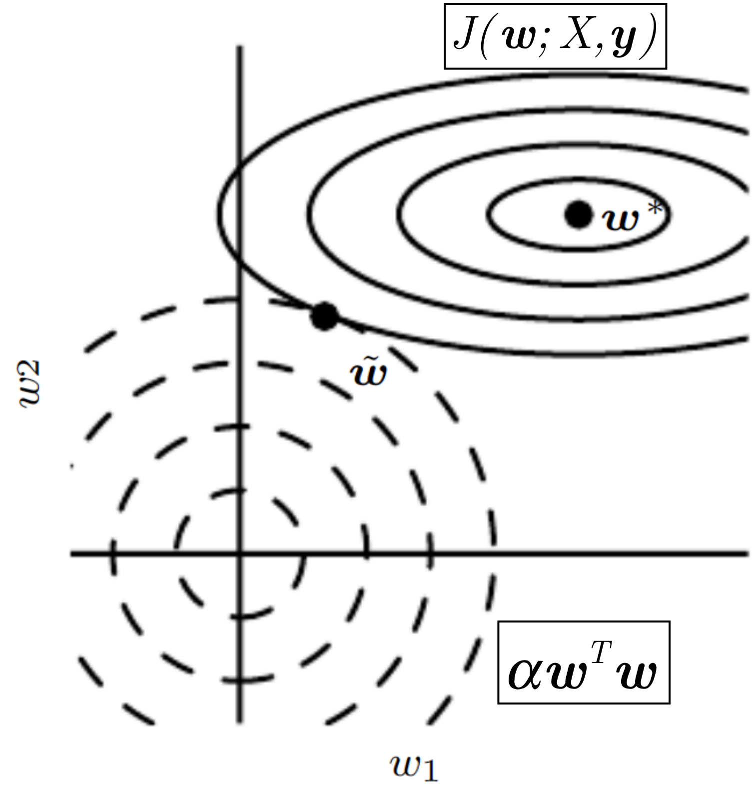
- Make quadratic approximation to the objective function in the neighborhood of minimal unregularized cost $\boldsymbol{w}^*=\arg\min_{\boldsymbol{w}} J(\boldsymbol{w})$
- The approximation is given by

$$J(\boldsymbol{w}^*)+½(\boldsymbol{w}\text{-}\boldsymbol{w}^*)^T H(\boldsymbol{w}\text{-}\boldsymbol{w}^*)$$

- Where H is the Hessian matrix of J wrt $\boldsymbol{w}$ evaluated at $\boldsymbol{w}^*$

Illustration of L^2 regularization

- Effect on value of optimal $\boldsymbol{w}$
- Solid ellipses:
  - contours of equal value of unregularized objective J
- Dotted circles:
  - contours of equal value of L^2 regularizer
- At point $\boldsymbol{w}$ competing objectives reach equilibrium



Along w_1, eigen value of Hessian of J is small. J does not increase much when moving horizontally away from $\boldsymbol{w}^*$. Because J does not have a strong preference along this direction, the regularizer has a strong effect on this axis. The regularizer pulls w_1 close to 0.

Along w_2, J is very sensitive to movements away from $\boldsymbol{w}^*$. The corresponding eigenvalue is large, indicating high curvature. As a result, weight decay affects the position of w_2 relatively little

L^1 Regularization

- While L^2 weight decay is the most common form of weight decay there are other ways to penalize the size of model parameters
- L^1 regularization is defined as

$$\boxed{\Omega(\boldsymbol{\theta}) = \left\|\boldsymbol{w}\right\|_1 = \sum_i \left|w_i\right|_1}$$

  – which is the sum of the absolute values of the individual parameters

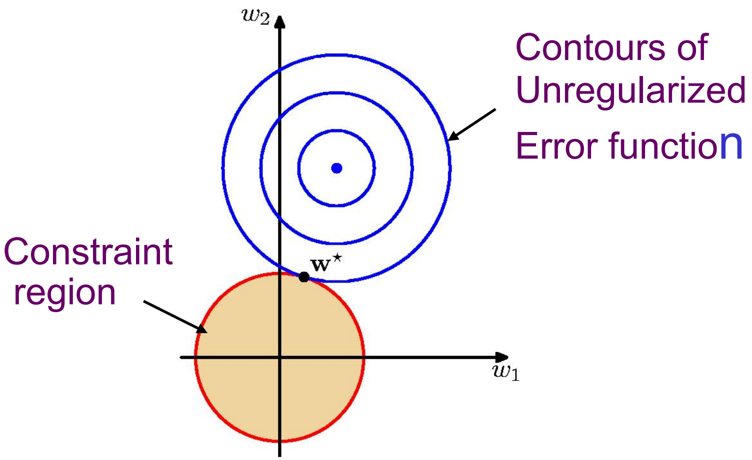
Sparsity and Feature Selection

- The sparsity property induced by L^1 regularization has been used extensively as a feature selection mechanism
  - Feature selection simplifies an ML problem by choosing subset of available features
- LASSO (Least Absolute Shrinkage and Selection Operator) integrates an L^1 penalty with a linear model and least squares cost function
  - The L^1 penalty causes a subset of the weights to become zero, suggesting that those features can be discarded

Sparsity with Lasso constraint

- With q=1 and λ is sufficiently large, some of the coefficients w_j are driven to zero
- Leads to a sparse model
  - where corresponding basis functions play no role
- Origin of sparsity is illustrated here:

Quadratic solution where $w_1{}^*$ and $w_0{}^*$ are nonzero



Minimization with Lasso Regularizer A sparse solution with $w_1{}^*$=0