Markov Network Independencies

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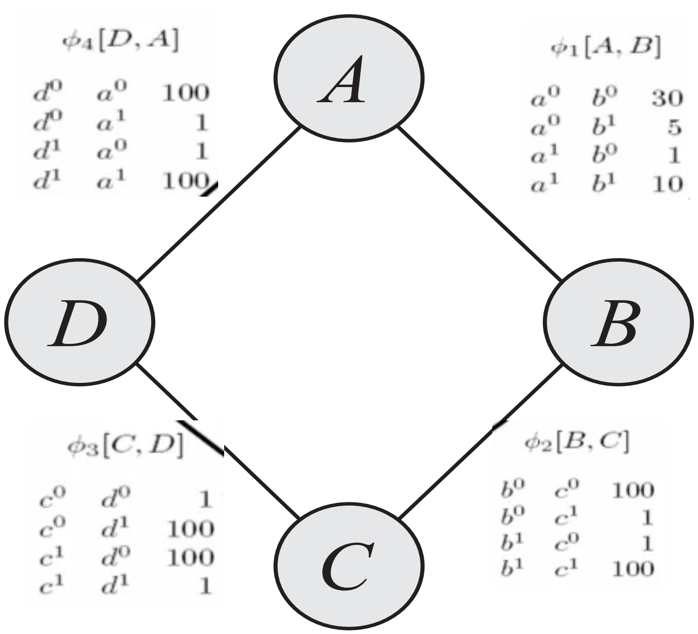
Topics

- Markov Network Independencies
  - Basic Independencies
  - Independencies Revisited
  - From Distributions to Graphs

Markov network captures independencies of interactions

Misconception MN with factors



$\phi_4[D, A]$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

$\phi_1[A, B]$

a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\phi_3[C, D]$

c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\phi_2[B, C]$

b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$$P(a,b,c,d) = \frac{1}{Z}\phi_1(a,b)\cdot\phi_2(b,c)\cdot\phi_3(c,d)\cdot\phi_4(d,a)$$

where

$$Z = \sum_{a,b,c,d}\phi_1(a,b)\cdot\phi_2(b,c)\cdot\phi_3(c,d)\cdot\phi_4(d,a)$$

As with BNs tight connection between factorization and independence properties:

P supports $(X \perp Y | Z)$ *iff* we can write distribution as

$$P(\chi) = \phi_1(X,Z)\,\phi_2(Y,Z)$$

Consider now: $\boxed{P(A,B,C,D) = \left[\frac{1}{Z}\phi_1(A,B)\cdot\phi_2(B,C)\cdot\right]\phi_3(C,D)\cdot\phi_4(D,A)}$

From this decomposition we can infer that $(B \perp D | A,C)$

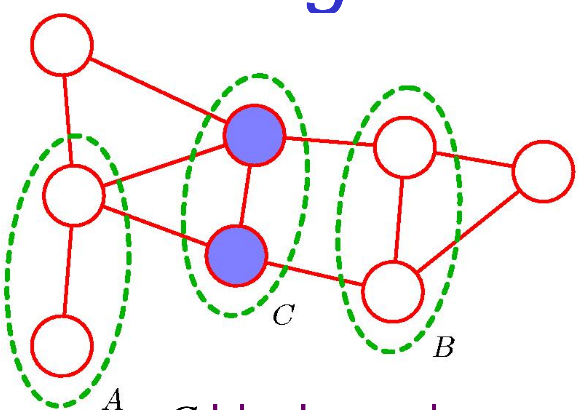
Proof: the first grouping yields a factor with $\{D,\{A,C\}\}$

The second grouping yields a factor with $\{B,\{A,C\}\}$

Similarly we can infer $(A \perp C | B,D)$

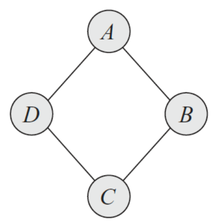
Basic Independencies

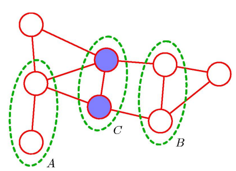
- As in Bayesian Networks, graph structure in a Markov network encodes a set of independence assumptions
- In a MN Probabilistic influence flows along the undirected paths in the graph and *"blocked" if we condition on intervening nodes*
  - i.e., we know their values
- We state this formally, next



C blocks nodes of A and B

Active Path Definition

- Let $\mathcal{H}$ be a Markov network with nodes $\chi=\{X_1,..X_n\}$,
- Let X_1-…-X_k be a path in $\mathcal{H}$
- Let $\boldsymbol{Z} \subseteq \chi$ be a set of observed variables
- A Path X_1-…-X_k is *active* given $\boldsymbol{Z}$ if none of X_i is in $\boldsymbol{Z}$
  - Ex 1: 

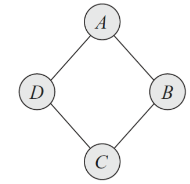
    If the observed set $Z=\{B\}$, path A-D-C is active
  - Ex 2: 

    If the observed set $Z=\{C\}$, paths between nodes of $\{A\}$ and $\{B\}$ are inactive

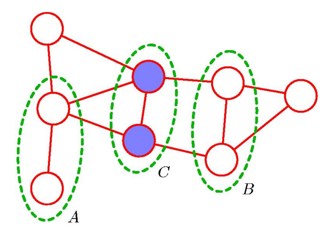
- We can define separation in the graph when there is no active path, next

Separation and Global Independencies

1. Set of nodes $\boldsymbol{Z}$ *separates* sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ denoted $\mathrm{sep}_{\mathcal{H}}(X;Y\,|\,Z)$ if there is no active path between any $X \in \boldsymbol{X}$ and $Y \in \boldsymbol{Y}$

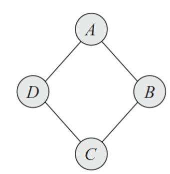


- Ex 1: $Z=\{B,D\}$ separates A and C
  - i.e., $\mathrm{sep}_{\mathcal{H}}(A;C\,|\,B,D)\}$ or there is no active path between A and C



- Ex 2: C separates A and B

2. *Global independencies* associated with $\mathcal{H}$ are
   - $\boxed{I(\mathcal{H})=\{(X\perp Y|Z):\ \mathrm{sep}_{\mathcal{H}}(X;Y|Z)\}}$
   - Independencies in $I(\mathcal{H})$ guaranteed to hold for every distribution P over $\mathcal{H}$



$I(\mathcal{H})=\{(A\perp C|B,D),$
$(B\perp D|A,C)\}$

Definition of Separation Leads to a disadvantage

- With a superset of Z, separation still holds
  - If $sep_{\mathcal{H}}(X;Y|Z)$ then $sep_{\mathcal{H}}(X;Y|Z')$ for any $Z' \supset Z$
- If separation is taken as definition of independencies, we restrict ability to encode non-monotonic independence relations
  - Non-monotonic reasoning is quite useful
    - E.g., intercausal reasoning with BNs
      - Two diseases are independent, but dependent given some common symptom
  - Such independence properties cannot be expressed as a Markov network

Factorization and Independencies

- Can show connection between independence properties implied by a Markov structure and factorizing a distribution over the graph
- Analogous to Bayesian Networks
  - Let $\mathcal{G}$ be a BN for a set of random variables χ and P be a distribution over χ.
  - If P factorizes according to $\mathcal{G}$, i.e.,product of CPDs, then $\mathcal{G}$ is an I-map of P
    - i.e., independencies $I(\mathcal{G}) \subsetneq I$

Formalizing independencies in MNs and distributions

- Gibbs Distribution
  - A distribution P_Φ is a Gibbs distribution parameterized by a set of factors $\Phi = \{\phi_1(D_1),..,\phi_K(D_K)\}$
  - If defined as follows

$$P_\Phi(\mathrm{X}_1,..\mathrm{X}_n) = \frac{1}{Z}\tilde{P}(\mathrm{X}_1,..\mathrm{X}_n)$$

where

$$\tilde{P}(\mathrm{X}_1,..\mathrm{X}_n) = \prod_{i=1}^{m}\phi_i(D_i)$$

is an unnomalized measure and

$$Z = \sum_{\mathrm{X}_1,..\mathrm{X}_n}\tilde{P}(\mathrm{X}_1,..\mathrm{X}_n) \text{ is a normalizing constant}$$

called the partition function

D_i are sets of random variables

Soundness of Separation Criterion

- *Theorem* 1 (from factorization to independencies):
  - Let P be distributed over $\chi=\{X_1,..X_n\}$ and $\mathcal{H}$ a Markov structure over χ
  - If P is a Gibbs distribution that factorizes over $\mathcal{H}$, (i.e., every D_i in $\mathcal{H}$ is a clique), then $\mathcal{H}$ is an I-map for P (i.e., every independency in $\mathcal{H}$ holds in P)
- *Theorem* 2: Hammersley-Clifford

  (other direction: from independencies to factorization)
  - if $\mathcal{H}$ is an I-map for P then P factorizes over $\mathcal{H}$
  - Holds only for positive distributions ($P > 0$ for every assignment)

Positive Distribution

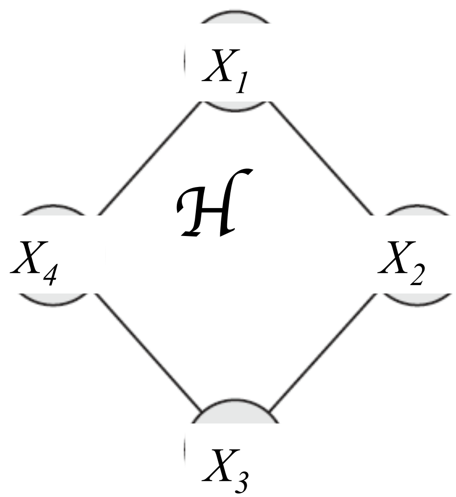
- A distribution P is said to be positive if for all events $\alpha \, \varepsilon \, S$ such that $\alpha \neq \emptyset$ we have that $P(\alpha) > 0$
- Ex: A non-positive distribution
  - 16 possible values
    - Distribution P: 8 have value $1/8$ rest are zero

X_1	X_2	X_3	X_4	$P(X)$
0	0	0	0	1/8
1	0	0	0	1/8
1	1	0	0	1/8
1	1	1	0	1/8
0	0	0	1	1/8
0	0	1	1	1/8
0	1	1	1	1/8
1	1	1	1	1/8

Rest 8 probs are 0

Non-positive distribution consistent with $\mathcal{H}$

- Four binary random variables
- Global Independencies in graph $\mathcal{H}$:
  - Consider $X_1 — X_2 — X_3 — X_4 — X_1$
    - implies $(X_1 \perp X_3 | X_2, X_4)$
- P also satisfies this
  - For the assignment $X_2=1,\ X_4=0$
  - $P(X_1=1|X_2=1,X_4=0)=1$
    - Rest are zero
    - Thus X_1 is independent of X_3



X_1	X_2	X_3	X_4	$P(X)$
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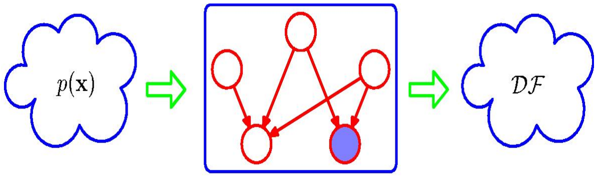
1	1	1	0	1/8
0	0	0	1	1/8
0	0	1	1	1/8
0	1	1	1	1/8
1	1	1	1	1/8

Rest 8 probs are 0

- Global independencies hold ⇒ $\mathcal{H}$ is an I-map for P
- But P does not factorize according to $\mathcal{H}$ (Proof by contradiction)

Graphical Model as Filter



p(x) is allowed to pass through only if
It satisfies independencies in graph
This set is denoted *DF* or *UF* (for BN or MN)

- UI is set of distributions that are consistent with set of conditional independence statements read from the undirected graph using graph separation
- UF are set of distributions that can be expressed as factorization of the form

$$p(\mathrm{x}) = \frac{1}{Z}\prod_C \psi_C(\mathrm{x}_C)$$

- *Hammersley-Clifford theorem* states that UI and UF are identical

Independencies in Bayesian Networks

- Bayesian networks have two types of independencies
  - Local independencies
    - Each node is independent of its non-descendants given it parents
  - Global independencies
    - Induced by d-separation
- These two sets of independencies are equivalent
  - One implies the other

Three Independencies of an MN

1. Pairwise Independencies (defined next slide)
   – Pairwise $I_p(\mathcal{H})$
2. Local independencies (defined shortly)
   – Markov Blanket $I_\ell(\mathcal{H})$
3. Global independency $I(\mathcal{H})$
   – Identify three sets of nodes A, B and C
   – To test conditional independence property

$$A \perp B \mid C$$

   – Consider all possible paths from nodes in set A to nodes in set B
      • If all such paths pass through one or more nodes in C, then path is blocked and independence holds

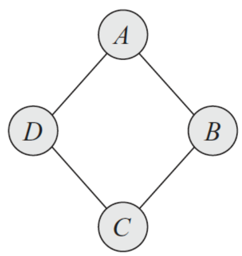
Pairwise Independency $I_p(\mathcal{H})$

- If $\mathcal{H}$ is a MN its *pairwise* independencies are

$$\boxed{I_p(\mathcal{H})=\{(X \perp Y|\chi\text{-}\{X,Y\}):X\text{—}Y \notin \mathcal{H}\}}$$

Meaning: When X, Y are **not** directly connected
i.e., $X\text{—}Y \notin \mathcal{H}$,
they are **independent** given all other variables

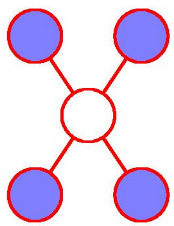
– Example:



$$I_p(\mathcal{H})=\{(A \perp C|B,D),(B \perp D|A,C)\}$$

Markov Blanket Independency $I_\ell(\mathcal{H})$

- Analogous to local independencies in Bayesian networks
  - We can block all influences by conditioning on its immediate neighbors
    - Node is conditionally independent of all nodes given its immediate neighbors
- For graph $\mathcal{H}$ the Markov blanket of X in $\mathcal{H}$ is the set of neighbors of X in $\mathcal{H}$



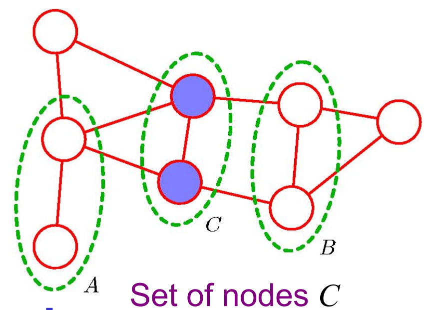
- Local independencies associated are
  - $\boxed{I_\ell(\mathcal{H})=\{(X \perp \chi-\{X\}-MB_{\mathcal{H}}(X)|MB_{\mathcal{H}}(X)):X \in \chi\}}$

Relationship between Markov properties

- Three independencies of network structure $\mathcal{H}$
- $I_p(\mathcal{H})$ is strictly weaker than $I_\ell(\mathcal{H})$ is strictly weaker than $I(\mathcal{H})$
- For positive distributions all three are equivalent

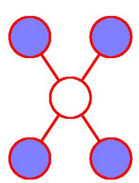
Separation in Markov Networks

- Markov network encodes a set of conditional independencies
- Probabilistic influence flows
  - in undirected paths



Set of nodes C Separates sets A and B

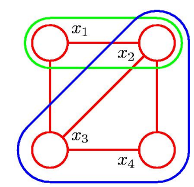
- Blocked if we condition on intervening nodes
  - Every path from any node in A to B passes through C
  - No explaining away
    - Testing for independence simpler than in directed graphs
  - Alternative view
    - Remove all nodes in set C together with all their connecting links
    - If no paths from A to B then conditional independence holds
- Markov blanket



  - A node is conditionally independent of all other nodes conditioned only on its neighbors

Factorization Properties

- Factorization rule corresponds to conditional independence test
- Notion of locality needed
- Consider two nodes x_i and x_j not connected by a link
  - They are conditionally independent given all other nodes in graph
    - Because there is no direct path between them and
    - All other paths pass through nodes that are observed and hence those paths are blocked
  - Expressed as



$$\boxed{p(x_i, x_j \mid \mathrm{x}_{\setminus\{i,j\}}) = p(x_i \mid \mathrm{x}_{\setminus\{i,j\}})\, p(x_j \mid \mathrm{x}_{\setminus\{i,j\}})}$$

  - Where $\mathrm{x}_{\setminus\{i,j\}}$ denotes set x of all variables with x_i and x_j removed
- For conditional independence to hold
  - factorization is such that x_i and x_j do not appear in the same factor
    - No path between them other than going through others
  - leads to graph concept of clique