Disk-wind math

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Following from the $\hat{n}$ given in Waters and the rate of strain tensor terms in spherical coordinates from Batchelor we verify and expand the result given in CM96:

$$\begin{aligned}\hat{n}\cdot\mathbf{\Lambda}\cdot\hat{n} = &\sin^2 i\left[\frac{\partial v_r}{\partial r}\sin^2\phi + \left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right)\sin\phi\cos\phi + \frac{v_r}{r}\cos^2\phi\right]\\ &-\sin i\cos i\left[\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right)\sin\phi + \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\cos\phi\right]\\ &+\cos^2 i\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)\end{aligned} \tag{1}$$

Where in arriving at the form above we have assumed all of the $\frac{\partial}{\partial\phi}$ operator terms are 0 (axisymmetric) and the disk is in the equatorial plane ($\theta = \frac{\pi}{2}$) which allows us to significantly simplify $\hat{n} = (\sin\theta\cos\phi\sin i + \cos\theta\cos i)\,\hat{r} + (\cos\theta\cos\phi\sin i - \sin\theta\cos i)\,\hat{\theta} - (\sin\phi\sin i)\,\hat{\phi}$. Waters uses a ϕ convention that differs from CM96 by $-\frac{\pi}{2}$, and applying this to equation one gives us:

$$\begin{aligned}\hat{n}\cdot\mathbf{\Lambda}\cdot\hat{n} = &\sin^2 i\left[\frac{\partial v_r}{\partial r}\cos^2\phi - \left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right)\sin\phi\cos\phi + \frac{v_r}{r}\sin^2\phi\right]\\ &-\sin i\cos i\left[\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right)\cos\phi - \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\sin\phi\right]\\ &+\cos^2 i\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)\end{aligned} \tag{2}$$

In CM96 they assume that $v_r \approx 0$ but that there is an acceleration related to the escape velocity, ie $\frac{\partial v_r}{\partial r} \approx 3\sqrt{2}\frac{v_\phi}{r}$, where $v_\phi = \sqrt{\frac{GM}{r}}$ is the Keplerian v_ϕ, which gives us $\frac{\partial v_\phi}{\partial r} = \frac{-v_\phi}{2r}$.

But what are the θ terms? Following in the footsteps of CM96 it makes sense to assume that on average $v_\theta \approx 0$ for the same reason $v_r \approx 0$, but similarly we will assume a particle may be lifted by the wind and accelerated to the local escape velocity (but now in the $\hat{\theta}$ direction) such that $\frac{\partial v_\theta}{\partial\theta} \approx \frac{v_{esc}}{(H/R)}$ and $\frac{\partial v_\theta}{\partial r} \approx \frac{\partial v_r}{\partial r}$. Since $v_\phi(r) \rightarrow \frac{\partial v_\phi}{\partial v_\theta} = 0$, and we also set $\frac{\partial v_r}{\partial\theta} = 0$. **But should it**

be? Should a tiny change in theta (lifting off the disk) then allow the thing to be radially accelerated away? Maybe this one should be also be like $\frac{v_{esc}}{(H/R)}$...

Plugging in these approximations reduces equation two to:

$$\begin{aligned}
\hat{n} \cdot \mathbf{\Lambda} \cdot \hat{n} =& 3\frac{v_\phi}{r}\sin^2 i \cos\phi \left[\sqrt{2}\cos\phi + \frac{\sin\phi}{2}\right] \\
& - \sin i \cos i \left[\frac{1}{r} 3\sqrt{2}\frac{v_\phi}{r}\cos\phi\right] \\
& + \cos^2 i \left(\frac{1}{r}\frac{v_{esc}}{(H/R)}\right)
\end{aligned} \tag{3}$$

Rescaling v_ϕ into units of r_s gives us $v_\phi = \sqrt{\frac{1}{2r'}}$ (where $r' = r/r_s$ and the extra factor of c is just absorbed into an overall normalizing constant since we normalize by the flux anyways). Similarly the H/R dependence can be absorbed into an overall constant, and we can convert all our $1/r$ terms to be in terms of r_s and absorb the extra scaling terms into our normalization, so that we get an equation that just shows us the r dependence of the line of sight velocity gradient:

$$\begin{aligned}
\hat{n} \cdot \mathbf{\Lambda} \cdot \hat{n} \approx \frac{\mathrm{d}v_l}{\mathrm{d}l} \approx & \frac{3\sqrt{\frac{1}{2r}}}{r}\cos\phi \left[\sin^2 i \left(\sqrt{2}\cos\phi + \frac{\sin\phi}{2}\right) - \frac{\sin i \cos i}{r}\right] \\
& + \cos^2 i \frac{1}{\sqrt{r^3}}
\end{aligned} \tag{4}$$

This is the form used currently in the fitting routine in the code. Since the $\sin i \cos i$ term is divided again by the radius and we are usually at $r_s > 1000$ this term is usually very small and does not contribute, but the inclination dependence makes the $\cos^2 i$ term very important at low inclinations and changes the overall shape of the line profile significantly.