A discussion on physical scaling laws for astrophysical objects

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Abstract

This project presents plots showcasing relationships between mass, radius, and density of nearly all known astrophysical objects, from asteroids in our solar system to supermassive black holes at the centers of galaxies. These plots include nearly 200,000 data points as well as theoretical predictions for scaling based on idealized physics. Discussion includes the explanation of these visualizations themselves, explanations and derivations of assumed scaling laws, as well as limitations of current theories. All data and code used to produce these results are available at a publicly accessible GitHub repository, here.
Keywords: scaling laws, mass, radius, density, relations

Contents

1 Introduction—a large scale overview of plots

1.1 Mass-radius relations

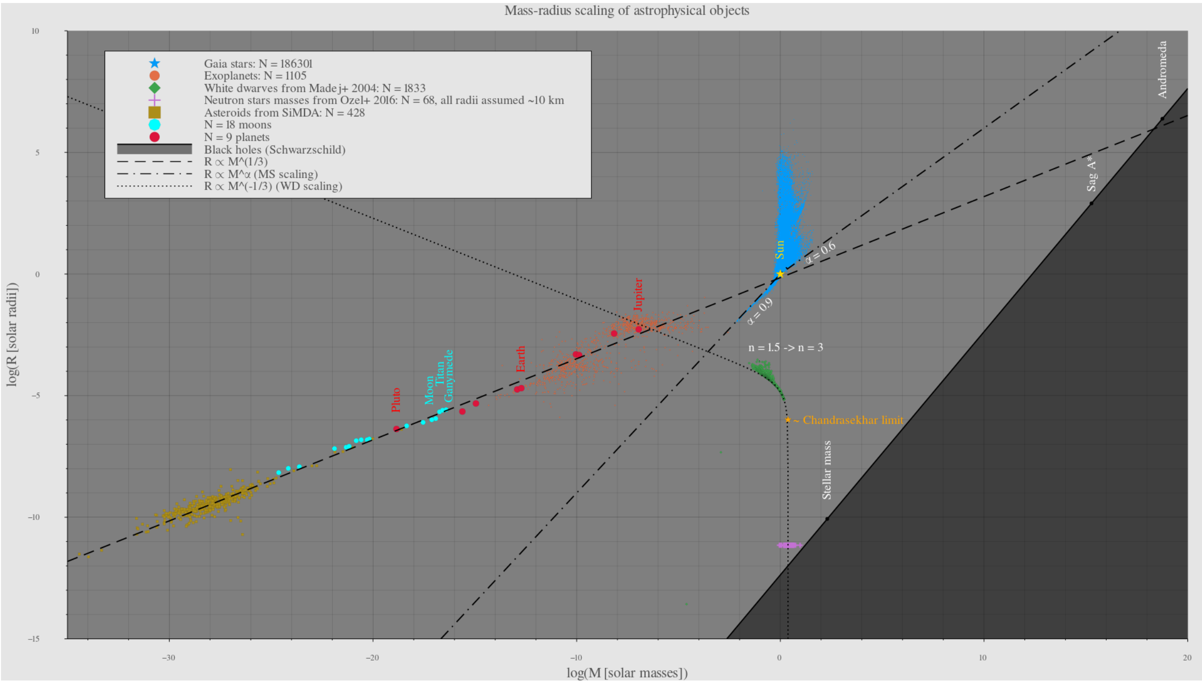

Figure 1: The mass-radius scaling of singular astrophysical objects at all scales. Both axes are in natural log space to allow everything to fit in the same plot and to showcase power law dependencies, which exist as straight lines in log space.

1.2 Mass-density relations

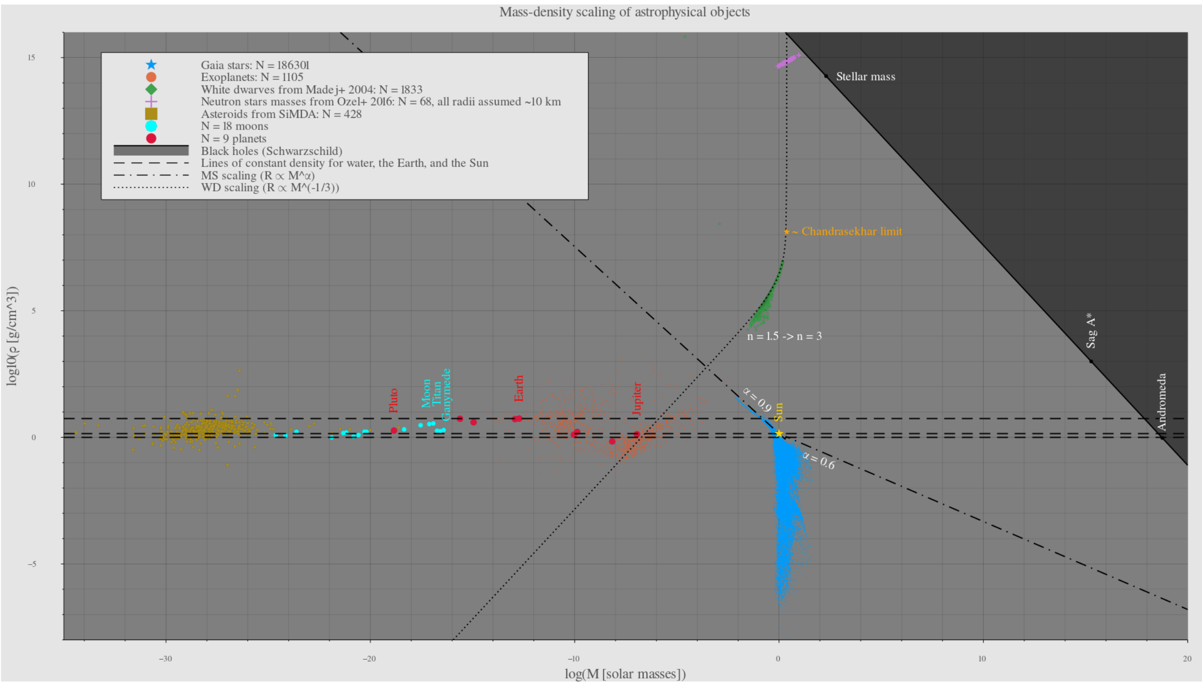

Figure 2: The mass-density scaling of singular astrophysical objects at all scales. The y-axis is in log10 space while the x-axis is still in natural log space.

These plots are extensions with up-to-date data of the plots presented on page 4.9 of the lecture notes. The top case (1) shows how the radius of various astrophysical objects depends on mass, while the bottom (2) represents how the density of astrophysical objects depends on mass. The data for stars is from [2], for exoplanets from the Caltech/NASA exoplanet archive, for white dwarves from [12], for neutron stars from [13], for asteroids from [11], and for moons/planets from NASA/JPL pages. Both figures can be downloaded and viewed as full high-definition figures from the GitHub repository (putting big plots like this into LaTeXnever looks great...). Black lines in both figures denote theoretical scaling laws, and while the rest of this paper will delve deeper into the methodology here there are a few fun large scale tidbits that we can read off of the plot immediately I'd like to point out now:

- In the mass-density plot, it's easy to see that both Saturn and the black hole at the center of the Andromeda galaxy would "float" in a hypothetical, astrophysically scaled bathtub! In the mass-radius plot we see the line of constant density intersects the Schwarzschild radius theoretical line also right at about Andromeda, showcasing the same result.
- The degenerate matter white dwarf line creates an interesting triangle in both plots, and in both we see that the exoplanets take on an almost parabolic shape as they go between the degenerate core regime and the small main sequence mass regime.
- While there's a lot going on in the Gaia data, we can clearly see a match to what we expect for the main sequence. We also see what looks like another "branch" higher up that I believe is likely the "second main sequence" of He burning as Sun-like stars progress towards becoming red giants!

We'll now delve deeper into the different populations shown in the main figures presented in the previous section, attempting to explain why the data look the way they do as well as deriving the observed scaling laws for each, as well as elaborating on what we cannot currently accurately model from approximate analytic theory.

2 Trends in "small" bodies (from asteroids to $\sim$ Jupiter sized planets)

Let's zoom in now a bit on our original figures and look just in the regime where planets, moons, asteroids, and comets should reside.

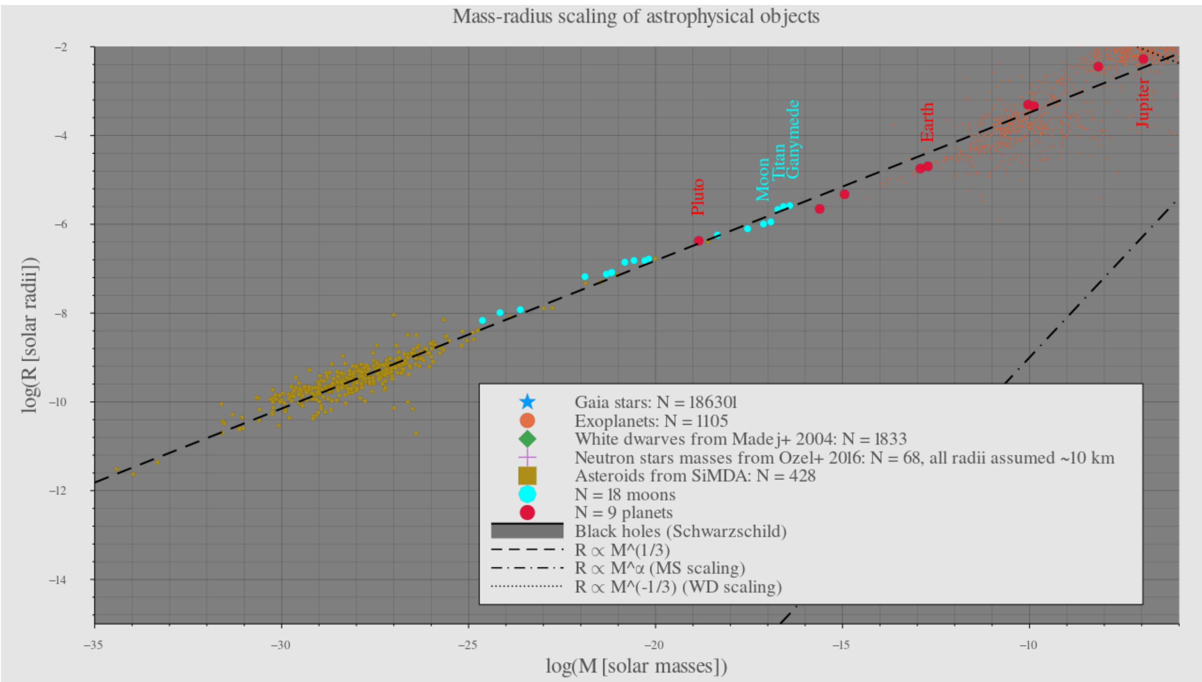

Figure 3: The mass-radius scaling of small bodies in the universe, a zoom in subsection of figure 1

We can clearly see that objects up to $\sim$ Jupiter-size follow a $\sim$ constant density model, i.e.

$$R = \left(\frac{M}{\frac{4}{3}\pi\rho}\right)^{\frac{1}{3}} \longrightarrow R \propto M^{\frac{1}{3}}$$

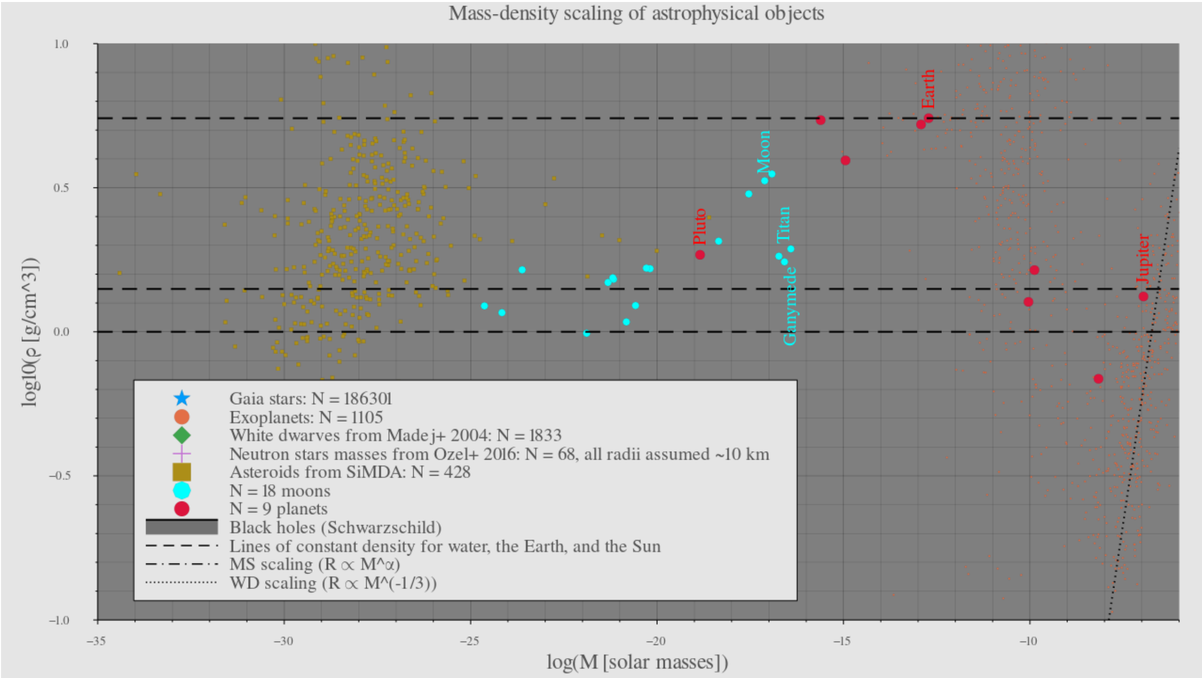

Figure 4: The mass-density scaling of small bodies in the universe, a zoom in subsection of figure 2

From the mass-density plot we can see that this constant density has some spread, but overall that scatter is limited to within roughly an order of magnitude. We note that the giant planets (Jupiter, Saturn, Uranus, and Neptune) are more gaseous and thus have a lower average density, while similarly some moons and asteroids are not as tightly packed as their planetary counterparts, giving them a lower average density as well. Getting data on the asteroids was tricky to come by, and most have significant relative errors here (up to 100%), so the large scatter in the asteroid population is at least partially due to this. There is significantly less error in most of the other measurements, particularly as you go up in size and closer to Earth.

This constant density model trend starts to break down past Jupiter sized planets as we approach significant fractions of a stellar mass, which we'll explore now!

3 Fusing stellar population trends

Let's now zoom in on the area occupied by the stellar population (that's actively fusing—we'll get to remnants later—in our original plots and explore what scaling laws exist for stars in this regime.

The points plotted here are retrieved from Berger+ (2020)[2], which used Gaia DR2 data to fit stellar properties, of which I've used their inferred mass, radius, and density values for the nearly 200,000 stars included. Since Gaia is sampling stars in our local universe in no preferential order or direction, we expect this to be a representative sample of current stellar populations (assuming conditions in the Milky Way are representative of average star-forming conditions everywhere). Thus, the density of points observed in different locations on this plot can be thought if in a sense as a probability distribution of where we expect to find most stars through time. As expected, we see the greatest density of points along the main sequence, which is modelled by the dash-dot line. This is a simplified but generally accepted model in stellar astrophysics, that there exist scalings along the main sequence between mass and various other physical parameters, including the radius of the star. These scalings were first noticed by Hertzsprung and Russel almost 100 years ago, as the famous HR diagrams in astronomy can also be thought of as mass-luminosity relations. In the mid-20th century this was extended into thinking about mass-radius relations as our ability to characterize stars improved, with McCrea 1950 being the earliest example I have noticed cited in discussion on mass-radius relationships[7].

The idea is simple—when plotting inferred physical properties of stars against each other in log space (every astronomer's favorite tool) it was noticed that, especially for smaller masses, there appeared to be a tight regime of radii that corresponded to a given mass. We then fit a line to this data, and the slope gives the inferred power-law dependence of the form:

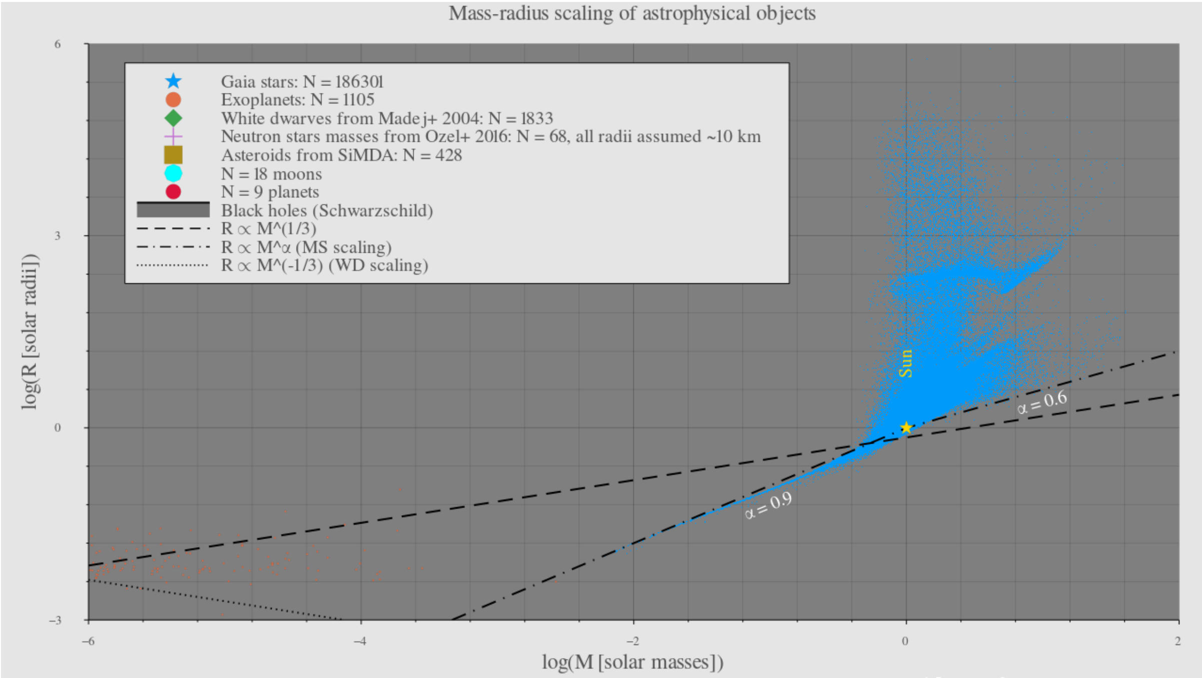

Figure 5: The mass-radius scaling of fusing stars in our universe, a zoom in subsection of figure 1

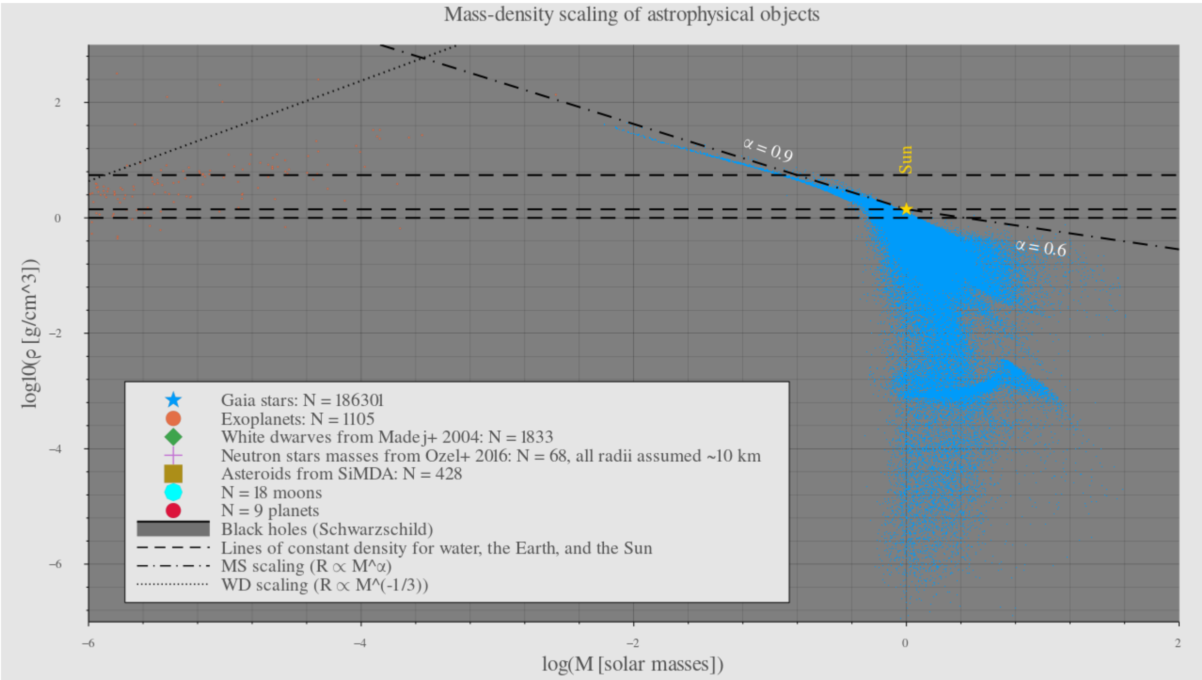

Figure 6: The mass-density scaling of fusing stars in the universe, a zoom in subsection of figure 2

$$R \propto M^{\alpha}$$

If we work in solar units this is really just $R = M^{\alpha}$ as the logarithm of 1 is zero. Intuitively we expect this relationship as more massive stars should be burning through their fuel faster, leading to a higher pressure force opposing gravity, leading to a larger sized star. This works pretty well for small stellar masses ($M \lesssim M_{\odot}$), as we notice from the Gaia data plotted here that there is indeed a pretty tight correlation between M and R, but it becomes harder to do for larger masses. For small mass stars it is generally assumed that α is $\approx$ 0.8-0.9 (here I've plotted 0.9 from James Imamura at UO[9]) and for larger mass stars it is assumed to be $\approx$ 0.6. This is still an area of active research, and while some people try to fit the distribution with higher order polynomials most use a break like this, breaking the distribution up into two or more segmented fits (Eker+ 2018[7]). What does this break signify? Lower mass stars have a convective envelope that high mass stars lack. Energy is able to escape from the interior of the star faster if convection is present, and this means stars with convective envelopes actually contract slightly relative to the non-convective model, giving a lower value of α for the high mass regime. While these scalings have been verified in numerical simulations of stars, they all have roots in observations and plots like this.

If the stars follow a mass-radius relation, they should also follow a mass-mean density relationship as well, i.e.

$$\langle \rho \rangle = \frac{M}{\frac{4}{3}\pi R^3} \longrightarrow \langle \rho \rangle \propto M^{1-3\alpha}$$

If we examine the density version of the plot we can see that for low-masses the $\alpha \approx 0.9$ curve does alright, but as we go to higher masses it performs worse. This is mostly just a plotting artifact, as if we look at the slope itself it looks to match the tail of the main sequence decently, just that the y-intercept is off, and our α here is not an actual fit to the data but rather just using a standard number as a sort of approximate sanity check, and any errors in the true value of α for this data are multiplied 3x when converting to the density version of the plot.

What about the regimes that clearly don't match to what we expect from our scaling laws? These stars are not on the main sequence! While they don't match our main sequence scaling results, there are still a couple interesting things to notice here:

1. There is a clear "second main sequence" type branch of stars, which is likely core burning of elements heavier than Hydrogen. This is likely as this "second main sequence" happens for both a significant period of time and for a significant fraction of stars (remember that any regions of significant density on this plot must be statistically probable for the stellar population in our region of the Milky Way). This also appears to have two main "branches" like in the normal main sequence case, likely related to the process in the core (i.e. higher mass stars have the CNO cycle, while lower mass stars can only fuse He).

2. Outside of this "second main sequence" we see a large smattering of stars above and below this regime. These are our "giant" type stars (mostly red giants based on mass distribution). As the core becomes degenerate and no longer capable of fusion stars fuse lighter elements in shells above their cores, and with these nuclear reactions now much closer to the surface of the star they puff up and expand to large radii and low average densities. Higher mass stars undergo this phase quicker, and stars also lose significant amounts of mass during the giant phases due to stronger stellar winds, so this explains why we see an increased density of data points in the more leftward regime of the Gaia sample. The largest stars captured in this dataset appear to have radii of $\sim 100 - 400 R_\odot$ and masses of $\sim 7 - 10 M_\odot$.

4 Trends in stellar endpoints

We will now move our focus to the most exotic regimes of these plots, turning our attention to the physics behind some of the universe's densest and mind-boggling objects. We will progress in order of progenitor mass—we'll first start with white dwarves, and how their behavior is possible to decently well explain with analytic approximations, then move on to neutron stars, discussing how their behavior and scaling is still very much an open area of hotly-debated research, before finally ending our journey in the depths of black holes.

4.1 White dwarves

White dwarves are the most common stellar endpoint for stars (our own Sun will eventually become one)—they are the degenerate cores full of elements the star was not massive enough to fuse, with the outer layers blown off by the strong stellar winds driven by shell burning in the late red giant phase of the star's life. Stars with masses less than $\approx 7 M_\odot$ will end their lives as white dwarves, with the smallest stars ($M \lesssim 0.5 M_\odot$ ending as degenerate He and the rest as degenerate C/O white dwarves[6]. Star's live their lives in a constant battle against the never-ending inward pressure induced by gravity—while they are fusing nuclear fusion provides the counterbalancing pressure, but after they can no longer fuse

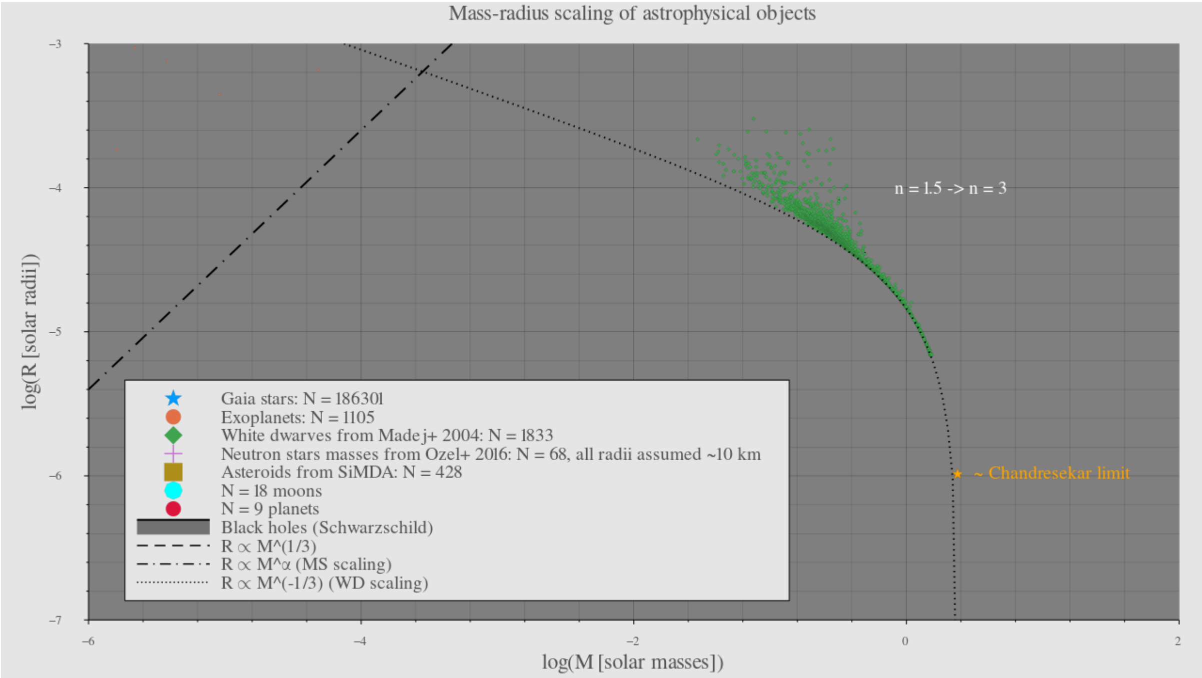

Figure 7: The mass-radius scaling of white dwarves in our universe, a zoom in subsection of figure 1

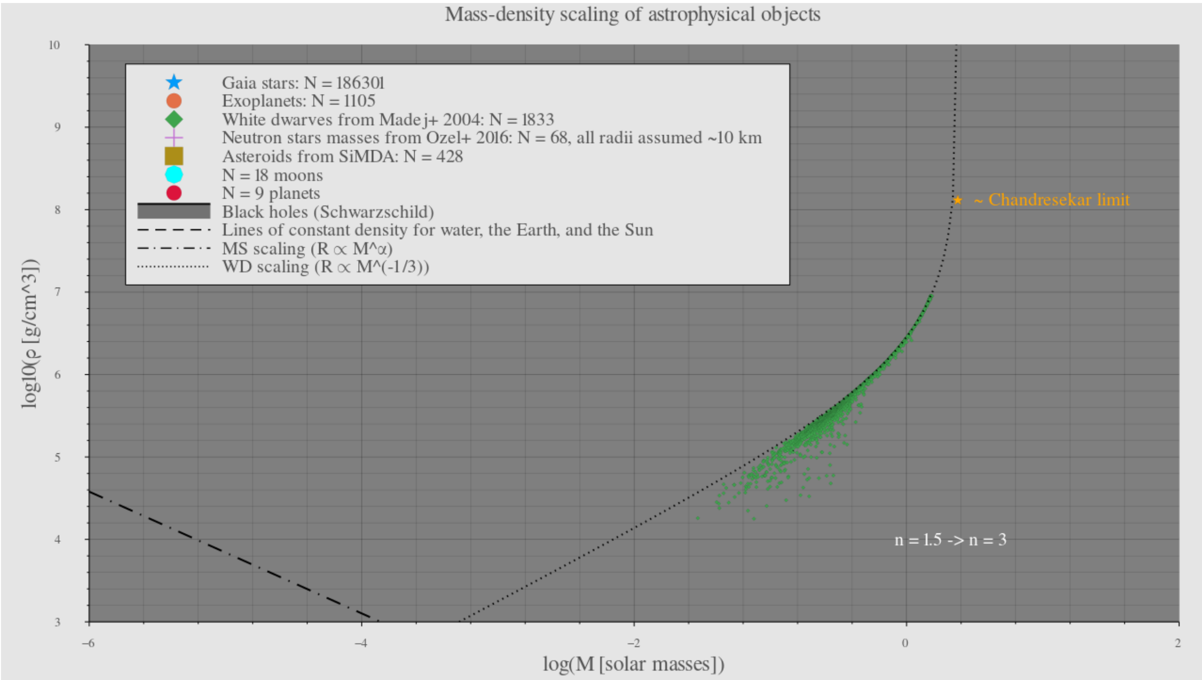

Figure 8: The mass-density scaling of white dwarves in the universe, a zoom in subsection of figure 2

degeneracy pressure takes over. White dwarves are supported by electron degeneracy pressure, a direct result of the Heisenberg uncertainty principle. Since no two electrons can occupy the same place in momentum / energy space, as gravity collapses the star free electrons are forced to occupy higher and higher energy states, providing a pushback pressure against the infall of gravity. This pressure balances the force of gravity at $\sim$ Earth-sized radii—a dramatic decrease in size from the extent of hundreds of solar radii the star may have been in its red giant phase prior to this! This also only works up to a certain point—the Chandrasekhar mass ($M_{Ch} \approx 1.456 M_{\odot}$)—where the equation of state becomes fully relativistic and this pressure is no longer able to stop gravity.

Polytropes are given by an equation of state that goes like:

$$P = K\rho^{1+\frac{1}{n}}$$

Where n is the polytropic index. For a non-relativistic star $n = 3/2$, and thus $P \propto \rho^{\frac{5}{3}}$, but in the relativistic case $n = 3$, giving $P \propto \rho^{\frac{4}{3}}$, and this lower pressure is unable to support the star in the fully relativistic case, leading to collapse after the Chandrasekhar mass.

The data points plotted here (from Madej+ 2004) are assuming that these white dwarves are all degenerate C cores. The data came with the choice of using this or using an He core, but I chose to use the carbon core as given the lifespan of low mass main sequence stars it seems more likely that these white dwarves came from progenitors of the higher mass variety, meaning they should have C/O cores.

The theoretical curve plotted here is derived from a polytropic star, where the polytropic index transitions from n = 1.5 to n = 3 as we approach the relativistic limit. Let's derive this, following along with lecture notes 4 [6] in conjunction with some notes by Robin Ciardullo at Penn State[5]. The mass enclosed in a polytropic star is given by:

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi r_n^3 \rho_c \int_0^{\xi'} \xi^2 \theta^n d\xi$$

Where $\xi = \frac{r}{r_n}$ is a dimensionless radius, with $r_n = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{\frac{(1-n)}{2n}}$

Where the right side is in the form of the Lane-Emden equation, which reads:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^3 \frac{d\theta}{d\xi}\right) = -\theta^n$$

This is a dimensionless form of Poisson's equation that describes the gravitational potential of a spherically symmetric, polytropic fluid under Newtonian gravity.

We can substitute this in to get:

$$M(\xi') = 4\pi r_n^3 \rho_c \int_0^{\xi'} -\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = 4\pi r_n^3 \rho_c \xi'^2 \left(\frac{d\theta}{d\xi}\right)_{\xi'}$$

Which, plugging in for r_n, gives:

$$M(\xi') = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c^{\frac{(3-n)}{2n}} \xi'^2 \left(-\frac{d\theta}{d\xi}\right)_{\xi'}$$

The radius is:

$$r(\xi') = r_n \xi' = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{\frac{(1-n)}{2n}} \xi'$$

And we can use this in conjunction with the mass equation to solve for ρ_c:

$$\rho_c = \left(\left[\frac{M}{4\pi}\right]\left[\frac{4\pi G}{(n+1)K}\right]^{\frac{3}{2}}\left[-\xi'^2\left(\frac{d\theta}{d\xi}\right)_{\xi'}\right]^{-1}\right)^{\frac{2n}{3-n}}$$

Finally, we can substitute this into the radius relation to obtain:

$$r = (4\pi)^{\frac{1}{n-3}}\left[\frac{(n+1)K}{G}\right]^{\frac{n}{3-n}}\left[-\xi'^2\left(\frac{d\theta}{d\xi}\right)_{\xi'}\right]^{\frac{n-1}{3-n}} M^{\frac{1-n}{3-n}}$$

Which tells us that, for a polytrope, $\boxed{R \propto M^{\frac{1-n}{3-n}}}$ and $\boxed{M \propto R^{\frac{3-n}{1-n}}}$.

There are a few interesting limits here:

1. if $n = 0$, this corresponds to an incompressible equation of state, and our scaling simply goes like $R \propto M^{\frac{1}{3}}$, which is what we saw previously for the asteroids and planets!

2. if $n = 1.5$, we recover the ideal gas equation of state in the non-relativistic limit, and this gives a scaling of $R \propto M^{-\frac{1}{3}}$. This is counter-intuitive as it means that adding mass *shrinks* the star, but this is precisely what must happen in order to increase the degeneracy pressure to counterbalance gravity! This scaling illustrates the behavior shown for white dwarves of smaller masses.

3. if $n = 3$, we now have the equation of state for a relativistic gas ($\gamma = 4/3$) and we notice that R is no longer proportional to M, and $M \propto R_0$, i.e. any value of R will give the same value of M. What is this magical M? The Chandrasekhar mass!

In the ultra-relativistic limit, we know that $K = 1.23 \times 10^{15} \mu_e^{-4/3}$. The value of $\xi'^2 \left(-\frac{d\theta}{d\xi}\right)_{\xi'} = \Theta_n$ must be computed numerically for $n = 3$—doing so gives a value of $\Theta_3 \approx 2.02$. Plugging these results into our expressing for the mass then gives us:

$$M_{Ch} \approx 5.8 \frac{M_\odot}{\mu_e^2}$$

Recalling that the mean molecular weight per free electron is given by $\mu_e \approx \frac{2}{1+x}$ gives that, for a white dwarf where $X \approx 0$, $\mu_e \approx 2$. This tells us that the Chandrasekhar mass should be roughly $\boxed{1.456 M_\odot}$.

But what about transitioning between the relativistic and non-relativistic regimes? To generate an equation describing the full behavior we must consider the contributions of both equations of state, which we can approximate like:

$$P \approx \left[\left(K_1 \rho^{5/3}\right)^{-2} + \left(K_2 \rho^{4/3}\right)^{-2}\right]^{-1/2}$$

Using the equations of stellar structure we can rearrange this to read:

$$R \approx \frac{K_1}{GM^{1/3}} \left(1 - \frac{G^2 M^{4/3}}{K_2^2}\right)^{1/2}$$

Which, after evaluating the limiting cases previously described, we can get to a nice equation that approximates what the mass dependence a white dwarf's radius has:

$$\boxed{R_{WD} \approx 0.0126 R_\odot \left(\frac{2}{\mu_e}\right)^{5/3} \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2}}$$

This is the equation used in showing what the white dwarf scaling should theoretically look like (dotted black line in plots, using $\mu_e = 2$ as justified above), and it does a pretty good job! We can see that it recovers the $R \propto M^{-1/3}$ scaling for small M as well as turning relativistic (going to $n = 3$ polytrope) with an asymptote at the Chandrasekhar mass. This last bit of the derivation follows some notes from Prof. Adam Burrows at Princeton[4], which skip fewer steps than I just did.

4.2 Neutron stars

Let us now consider an even more exotic stellar end-point: the neutron star. Neutron stars are left behind from progenitors of $8M_\odot \lesssim M_i \lesssim 25M_\odot$. They are formed in the violent supernova explosions that rip apart high mass stars at the ends of their lives, or (less commonly) through a massive white dwarf star passing the Chandrasekhar mass that does not explode in a type Ia supernova. Neutron stars operate in much the same way white dwarves do—is is again degeneracy pressure that is halting the further collapse of the star, but this time it is neutrons being forced into unocuppied higher energy states that provides this pressure instead of electrons. The neutron is ≈ 1800 times the mass of the electron, and we can roughly intuit the scale size for the neutron star to thus be ≈ 1800 times smaller than the white dwarf was, down to scale sizes of 5-15 km. While the exact maximum mass of a neutron star is not known, it is thought to be approximately 2.2-2.9 solar masses (the so-called Tolman-Oppenheimer-Volkoff limit[10]). Because the neutrons are much more tightly packed than the electrons in a white dwarf, additional nuclear forces are at play in determining the pressure exerted, and most neutron stars are additionally thought to rotate very quickly and contain strong magnetic fields. These factors are only marginally important in the white dwarf case as the degeneracy pressure provided by the electrons themselves is by far the most important contribution to the pressure and thus our approximate equation of state can produce sensible results, but in the neutron star case all of these factors must be considered and the equation of state is considerably more complicated and, as of this writing, unknown. There are also likely even fancier quantum mechanical effects at play as neutron stars probe the low temperature and large density region of the QCD phase diagram, which is expected to have exotic matter and phase transitions[8].

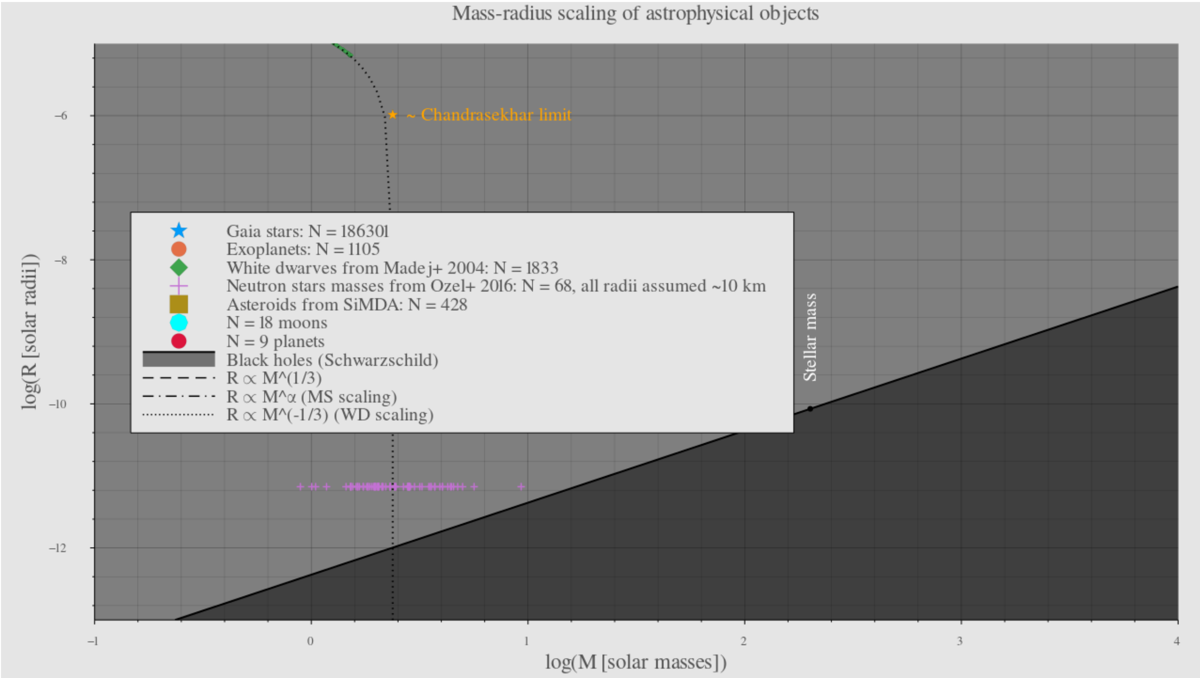

Figure 9: A zoom in subsection of figure 1 showing the neutron star regime. Note there is no current accepted scaling law between mass and radius for neutron stars.

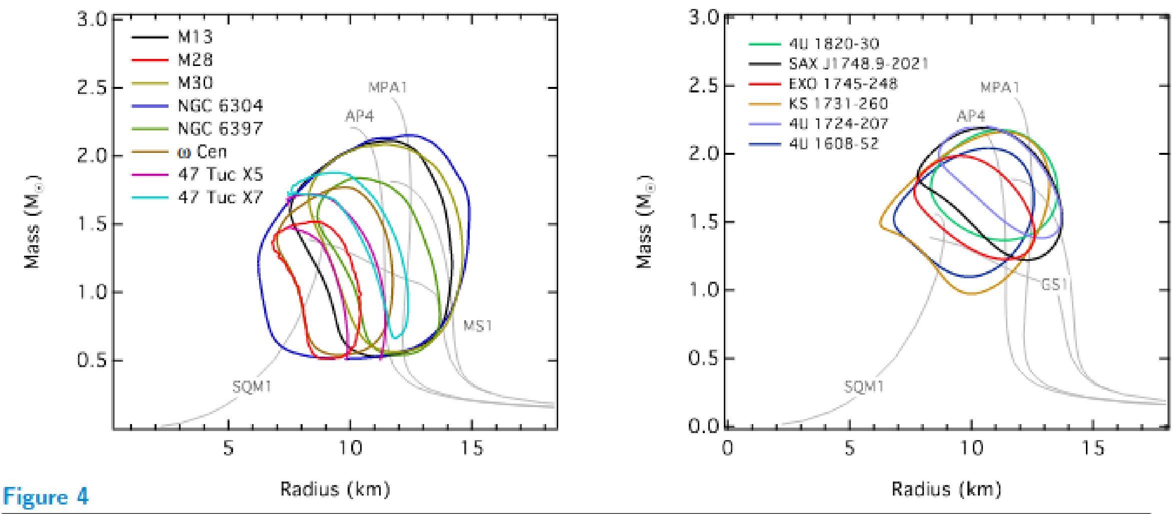

Figure 4

The combined constraints at the 68% confidence level over the neutron star mass and radius obtained from (Left) all neutron stars in low-mass X-ray binaries during quiescence (Right) all neutron stars with thermonuclear bursts. The light grey lines show mass-relations corresponding to a few representative equations of state (see Section 4.1 and Fig. 7 for detailed descriptions.)

Figure 10: Figure 4 from Ozel & Fraire (2016)[13], with their accompanying caption.

We can see in 9 that the regime where neutron stars can exist is quite small on our original plot—assuming a radius of order 10km they would become black holes at $\sim 3M_{\odot}$ as expected, and their radii (for a Chandrasekhar massed NS) can't go much smaller than $\sim$ 5 km as likewise they would now be dense enough to collapse into a black hole. Figure 11 shows some of the most up-to-date modelling results attempting to constrain the true mass-radius relationship for neutron stars. We see that their radii have 1 σ confidence intervals of between 5-15 km, and their masses between 0.5-2.5 $M_{\odot}$. The grey lines indicate different hypothetical equations of state that would give us an analytic result, but unfortunately none of them currently seem to significantly better fit the data, and they extrapolate to significantly different behavior outside of the measured regime. This is an area of hotly debated research that will continue for many years to come!

Thinking on this has also spawned an interesting new hypothetical class of compact star—the pion star [3]. These hypothetical objects would be stars made up of a Bose-Einstein condensate of charged pions taking the place of neutrons, which represent the lightest excitations in QCD. This is advantageous for modelling because the equation of state is known for this kind of strange matter, and thus we can directly plot where they would exist (if they existed). Unfortunately it does not appear likely that they could last long enough to be observed in the real universe, but I've included a figure below from the paper proposing their existence showing their relationship to the black hole curve to get a sense for where they might lie on our overall plot should they really exist.

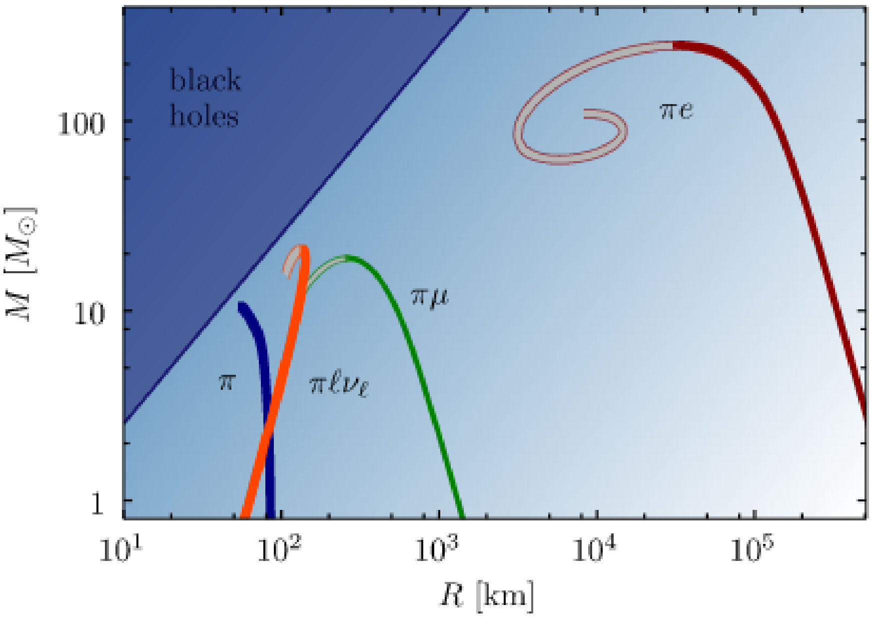

FIG. 4. Mass-radius relations of various scenarios for pion stars. Shown is a pure pion star (π), a pion-electron system (πe), and a pion-muon system ($\pi\mu$), together with a system containing both lepton families in chemical equilibrium ($\pi\ell\nu_\ell$). The filled (open) segments mark the gravitationally stable (unstable) solutions (for details, see the text). The dark blue area marks the region excluded by causality, and the background color represents the compactness $\beta \propto M/R$ of the objects (darker colors indicate more compact stars). The widths of the curves indicate statistical errors and the uncertainty in the lattice pion mass.

Figure 11: Figure 4 Brandt et al 2018[3], with their accompanying caption.

4.3 Black holes

Finally, the last regime on our plot we have not explored yet is that of black holes. There are obviously no data points to show for observed radii or densities of black holes (the closest we've gotten is the EHT, but this is still several times the actual scale of the event horizon, and regardless we should not be able to see any photons from the actual event horizon itself demarcating it), but we do have several masses which I've included in the plot, using the Schwarzschild scaling for the radius:

$$R_s = \frac{2GM}{c^2}$$

Where the mean density is calculated assuming a spherical volume described by R_s, which gives the interesting scaling of $\rho \propto M^{-2}$—as the mass increases the density goes *down*. This behavior is interesting as its the only type of object we've considered that behaves like this! There are some semantics here—the black hole is really infinitely dense at the singularity, with no density everywhere else, like a universe-scale Dirac delta function—but these semantics make no difference for *our* experience with the black hole, as for all intents and purposes the matter can be distributed anyway it likes inside of the event horizon and its effects on our universe will be the same. This means that, to us, the black hole at the center of the Andromeda galaxy ($M \approx 10^8 M_\odot$) is indistinguishable gravitationally from an equivalently sized giant sphere with radius of R_s of water—the densities are the same!

The largest stars ($M > 25M_\odot$) end their lives in supernovae, with their cores too massive to be held up even by neutron degeneracy pressure and thus collapsing into black holes. The "stellar mass" point in the plots represents a 10 solar mass black hole that could be created by such an event. While the details of this process are hard to model, the general picture is agreed upon by almost all astrophysicists. Where there is still significant intrigue/disagreement is how the universe creates *supermassive* black holes, like the one at the center of the Milky Way or Andromeda. There is a huge gulf between these black holes and the population of stellar mass black holes that is assumed to be created by supernovae. While the logical answer is to assume many of these stellar mass black holes have simply merged to form supermassive black holes, there are problems with this observationally and theoretically. If this is how they are formed, then we should probably see some black holes filling in the parameter space between stellar mass and supermassive black holes, but until very recently we had no evidence of this. LIGO has since provided a single example[1], but this is still an area of the parameter space we have not really been able to explore or characterize. The theoretical problems arise from the timescales dynamicists assume it takes for black holes to merge—currently we think it simply takes too long for black holes to collide that you could meaningfully accumulate enough stellar mass black holes into the supermassive black holes we see today.

5 Conclusions and references

Fitting almost every known type of astrophysical object into a single plot was a fun and exciting challenge, and it's very cool to see how we've explained a lot of the scaling in the universe, as well as seeing where we still have work to do. While we think we understand planets and white dwarves pretty well, we're still refining our models of stellar evolution, and we have significant knowledge gaps to fill in on neutron stars and black holes. It's also interesting to look into the cases at the edges of the regimes we think we understand—i.e. how do we characterize the largest exoplanets/brown dwarves? It's also remarkable to me that most of this plot is empty—we'd truly be having a much harder go at things if physics were even slightly more complicated, and it's not often that I appreciate the *simplicity* of the universe (I'm usually complaining the other way, but the big picture like this puts it into perspective).

I was also surprised at how easy it was to write these 10 pages about just two plots...I suppose this is good practice for the rest of my academic career!

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