Contact interactions and P-wave dominated few-body systems

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We analyse predictions of regularized two- and three-body contact interactions in $A = 2 \dots 8$ body systems. The interactions are renormalized with observables which are associated with totally symmetric spatial wave functions, specifically, the deuteron and triton ground states. We apply these interactions to systems whose number of fermionic constituents exceeds the number of internal degrees of freedom, thereby precluding a totally symmetric spatial wave function.

For such A-body systems, we find a characteristic interaction range in form of a regulator parameter $\lambda_A \in \{0.01\,\mathrm{fm}^{-1}, \dots, 10\,\mathrm{fm}^{-1}\}$ such that for $\Lambda \lesssim \lambda_A$ a stable state exists, while for any $\Lambda \gtrsim \lambda_A$ the system is unstable with respect to its lowest breakup threshold. Specifically, we $\lambda_A \approx \beth \cdot A$, *i.e.* , a linear dependence of the critical effective range on the number of constituents. This relation and the proximity of the interactions, which were used to obtain it, to unitarity, support the first main conclusion of this work: Any momentum-independent, spherically symmetric interaction, which is constrained by shallow two- and three-particle states, *i.e.* , by totally spatially symmetric states, and a non-zero effective range

Furthermore, we find no evidence for a resonance pole close to the scattering threshold in the limit $\Lambda \to \infty$.

I. OVERTURE

Contact interactions stabilize the unitary A-boson system with respect to certain thresholds which are set by an infinite number of stable 3-body states, each of which is correlated with a pair (one shallow and one deep) of 4-body states. A pair of 5-body states is then found for each of the deep 4-body states. This pattern generalizes such that a stable pair of a deep, and a shallow $A+1$-body state is found for each A-body state which is deep relative to its $A-1$-body threshold. With each particle, the binding energy increases with the number of pairs in the system which interact via a unitary attraction. Nothing hinders all particles to reside in the state which maximises the attraction. These "bosonic" states are the ground states of fermions whose internal space is of some dimension $n \geq A^*$. The dynamics of fermionic systems with $n < A$ subject to pairwise contact interaction is the subject of this note.

For $\hat{V} = \sum\limits_{i<j} \lim\limits_{\lambda\to\infty} \delta_\lambda(\boldsymbol{r}_i - \boldsymbol{r}_j)$ we substantiate two conjecture with numerical evidence:

Fermion contact instability. *$\nexists$ stable A-body state ($A > N_{internal}$) for particles interacting solely at zero-range, and there momentum independently ("contact interactions").*

For the theoretical description of nuclei, this theorem implies that the description of 6-helium, for example, must take into account the effective range of the nuclear interaction.

Finite-range stability. *$\forall$ non-zero, attractive, momentum-dependent (i.e. , finite-range) interactions $\exists A_c : \forall N \geq A_C,$ partitions F of $N :$ $B(N) > \sum\limits_{n\in F} B(n)$.*

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[*] For nucleons $n = 4$, *i.e.* , two iso-spin and two spin degrees of freedom.

II. λ_c SPECULATION

If two systems A and B have similar critical binding ranges, $\lambda_c(A) \approx \lambda_c(B)$ we indicate this with $A \sim B$.

Observation 1 ($\begin{smallmatrix}\square\\\square\end{smallmatrix} < \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$)**.** *Although, the third particle is forced into a higher orbital excitation, it is stable for even shorter ranges compared with the two-fermion system. The higher orbital suggests a higher angular-momentum barrier which demands an even longer-ranged attraction. Expressing the wave function in cluster coordinates, which are in this case identical to Jacobi coordinates, the*

III. RESONATING-GROUP DERIVATION OF THE CORE-PARTICLE INTERACTION

$$\big\langle\phi_A\big|\big(\hat{T}_{\boldsymbol{R}} + \hat{V}_{\text{A,A+1}} - E\big)\hat{A}\left[\phi_A\psi(R)\right]\big\rangle = 0 \tag{1}$$

$$\Rightarrow\quad \left(\hat{T}_{\boldsymbol{R}} - E\right)\big\langle\phi_A\big|\hat{A}\left[\phi_A\psi\right]\big\rangle + \big\langle\phi_A\big|\hat{V}_{\text{A,A+1}}\big|\hat{A}\left[\phi_A\psi\right]\big\rangle = 0 \tag{2}$$

with an average over the internal coordinates of the fragment:

$$\big\langle\ldots\big\rangle = \prod_{i=1}^{A-1}\int d^3\bar{\boldsymbol{r}}_{\text{i}} \tag{3}$$

which we chose to express relative to its center or mass, *i.e.* , we use the following set of coordinates

$$\bar{\boldsymbol{r}}_{\text{i}} := \boldsymbol{r}_{\text{i}} - \frac{\sum_{i=1}^{A}\boldsymbol{r}_{\text{i}}}{A} \quad i \in \{1,\ldots,A\} \tag{4}$$

$$\boldsymbol{R} := \boldsymbol{r}_{\text{A+1}} - \frac{\sum_{i=1}^{A}\boldsymbol{r}_{\text{i}}}{A} \tag{5}$$

$$\boldsymbol{R}_{\text{cm}} := \frac{\sum_{i=1}^{A+1}\boldsymbol{r}_{\text{i}}}{A+1} \tag{6}$$

which comprises A independent vectors because $\bar{\boldsymbol{r}}_{\text{A}} = -(\bar{\boldsymbol{r}}_1 + \ldots + \bar{\boldsymbol{r}}_{\text{A-1}})$ for $A+1$ equal-mass particles located at single-particle coordinates $\boldsymbol{r}_{1,\ldots,\text{A+1}}$.

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}\bar{\boldsymbol{r}}_{\text{i}}^2} = e^{-a\sum^{A-1}\bar{\boldsymbol{r}}_{\text{i}}^2 - a\sum_{i<j}^{A-1}\bar{\boldsymbol{r}}_{\text{i}}\cdot\bar{\boldsymbol{r}}_{\text{j}}} \tag{7}$$

$$\psi \to \int d^3\boldsymbol{R}'\ \delta^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big)\psi(\boldsymbol{R}') \tag{8}$$

$$\hat{A} = \mathbb{1} - \hat{P}(\boldsymbol{r}_{\text{A}} \leftrightarrow \boldsymbol{r}_{\text{A+1}}) \tag{9}$$

$$\hat{P}(\boldsymbol{r}_{\text{A}} \leftrightarrow \boldsymbol{r}_{\text{A+1}})\left[\boldsymbol{R} = \boldsymbol{r}_{\text{A+1}} - \frac{\sum_i^A\boldsymbol{r}_{\text{i}}}{A}\right] = -A^{-1}\boldsymbol{R} - (1+A^{-1})\sum_i^{A-1}\bar{\boldsymbol{r}}_{\text{i}} \tag{10}$$

$$\hat{P}(\boldsymbol{r}_{\text{A}} \leftrightarrow \boldsymbol{r}_{\text{A+1}})\left[\delta^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big)\right] = (2\pi)^{-3}\int ds\ e^{is\left(A^{-1}\boldsymbol{R} + (1+A^{-1})\sum_i^{A-1}\bar{\boldsymbol{r}}_{\text{i}} + \boldsymbol{R}'\right)} \tag{11}$$

$$\hat{P}(\boldsymbol{r}_{\text{A}} \leftrightarrow \boldsymbol{r}_{\text{A+1}})\left[\phi_A\psi\right] = \frac{A}{A+1}(2\pi)^{-3}\prod_{m\in\{x,y,z\}}\int_{\mathbb{R}^2} d(s,\boldsymbol{R}_m')\ e^{-\frac{a\epsilon_1}{2}\boldsymbol{R}_m^2 - is\left(\frac{\boldsymbol{R}_m}{A+1} + \frac{A}{A+1}\boldsymbol{R}_m'\right)} \tag{12}$$

$$\cdot\, e^{-\frac{a\epsilon_1}{2}\sum\limits_{i<A}\bar{\boldsymbol{r}}_{\text{m,i}}^2 - \frac{a}{A}\sum\limits_{i\neq j<A}\bar{\boldsymbol{r}}_{\text{m,i}}\bar{\boldsymbol{r}}_{\text{m,j}} + \frac{a\boldsymbol{R}_m}{A}\sum\limits_{i<A}\bar{\boldsymbol{r}}_{\text{m,i}} - is\sum\limits_{i<A}\bar{\boldsymbol{r}}_{\text{m,i}}} \times\psi(\boldsymbol{R}') \tag{13}$$

$$\hat{V}_{\mathrm{A,A+1}} = C_0(\Lambda) \sum_{i\leq A} \delta^{(3)}_{\Lambda}\big(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}\big) \tag{14}$$

$$+ D_1(\Lambda) \sum_{\substack{i\neq j\\ \leq A}} \Big\{ \delta^{(3)}_{\Lambda}\big(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}\big)\delta^{(3)}_{\Lambda}\big(\boldsymbol{r}_{\mathrm{j}} - \boldsymbol{r}_{\mathrm{A+1}}\big) + \delta^{(3)}_{\Lambda}\big(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}\big)\delta^{(3)}_{\Lambda}\big(\boldsymbol{r}_{\mathrm{j}} - \boldsymbol{r}_{\mathrm{j}}\big)\Big\} \tag{15}$$

An effective potential which considers the interaction potential – which weighs a configuration of particles within a limited parameter space as circumscribed by the effective theory – and the identity of particles – reflected in the implied (anti)symmetry of the wave function – is defined by

$$\int_{\mathbb{R}} d^3\boldsymbol{R'}\ \hat{V}_{\mathrm{eff}}(\boldsymbol{R},\boldsymbol{R}')\psi(\boldsymbol{R}') := \big\langle \phi_A \big| \hat{V}_{\mathrm{A,A+1}} \big| \hat{A}\left[\phi_A\psi\right] \big\rangle - \Big(\hat{T}_{\boldsymbol{R}} - E\Big) \big\langle \phi_A \big| \hat{P}(\boldsymbol{r}_{\mathrm{A}} \leftrightarrow \boldsymbol{r}_{\mathrm{A+1}}) \left[\phi_A\psi\right] \big\rangle \ \ , \tag{16}$$

a function of

$$\big\{\boldsymbol{R},\boldsymbol{R}',a,\Lambda,C_0,D_1,A\big\} \ \ .$$

Unorthodoxly, we do thus not distinguish between overlap/exchange kernel and direct/exchange interaction.

A. Overlap Kernel

The so-called exchange kernel measures the overlap of the fragment wave function, on the left-hand side, and the total wave function on the right.

$$\mathscr{K}(R) := \big\langle \phi_A \big| \hat{A}\left[\phi_A\psi\right] \big\rangle \ \ .$$

The demand for a totally antisymmetric wave function preserves the dependence of this quantity on the quantum numbers which characterize the fragment state ϕ_A. This dependence dissappears only in the approximation $\hat{A} = \mathbb{1}$ for orthonormal ϕ_A's. Assuming that the particles with label A and $A+1$, respectively, are identical,

$$\mathscr{K}(R) = \int d(\bar{\boldsymbol{r}}_{1\ldots\mathrm{A}},\boldsymbol{R}') e^{-\frac{a}{2}\sum_i^{A+1}\bar{\boldsymbol{r}}_{\mathrm{i}}^2} \Big(\mathbb{1} - \hat{P}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{A+1}})\Big) \left[e^{-\frac{a}{2}\sum_i^{A+1}\bar{\boldsymbol{r}}_{\mathrm{i}}^2} \delta^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big)\right] \psi(\boldsymbol{R}') \ \ .$$

We find

$$\begin{aligned} &\hat{P}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{A+1}})\phi_A = \\ \hat{P}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{A+1}})\delta^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big) &= \frac{A}{A+1}\ \delta^{(3)}\big(\bar{\boldsymbol{r}}_{\mathrm{A}} - \frac{\boldsymbol{R}}{A+1} - \frac{A}{A+1}\boldsymbol{R}'\big) \\ &= (2\pi)^{-3} \prod_{m\in\{x,y,z\}} \int_{\mathbb{R}} ds\ e^{-is\left(\bar{\boldsymbol{r}}_{\mathrm{m,A}} - \frac{\boldsymbol{R}_m}{A+1} - \frac{A}{A+1}\boldsymbol{R}'_m\right)} \end{aligned}$$

We proceed with

$$\big\langle \phi_A \big| \hat{A}\left[\phi_A\psi\right] \big\rangle = \int d^3\boldsymbol{R}' \sum_{k\in\{\mathrm{D,EX}\}} A_k \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} d(\bar{r}_{1,\ldots,\mathrm{A-1}},s)\ e^{-\frac{1}{2}\bar{\boldsymbol{r}}^{\intercal}\widehat{M}_k\bar{\boldsymbol{r}} + \mathcal{S}_k^{\intercal}\cdot\bar{\boldsymbol{r}} - isB_k} \right)^{\vec{3}} \times\psi(\boldsymbol{R}') \tag{17}$$

$$= \left\{ \delta^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big) - \frac{A}{A+1}\left(\pi^{-1}(A-1)^{-1}\right)^{3/2} e^{-\alpha_k\boldsymbol{R}^2 - \beta_k\boldsymbol{R}'^2 - \gamma_k\boldsymbol{R}\cdot\boldsymbol{R}'} \right\} \tag{18}$$

$$\times \left(\left(\frac{\pi}{a}\right)^{A-1} A^{-1}\right)^{3/2} \psi(\boldsymbol{R}') \ \ , \tag{19}$$

with $\vec{3}$ indicating a product of prefactors×exponentials for the spatial coordinates, *i.e.* , in the exponents, components are to be replaced by vectors, and prefactors are n-cubed. Note that the combinatoric factors A_k are included only once and not for each dimension, separately.

$$A_{\mathrm{D(irect)}} = 1 \;\;;\;\; \widehat{M}_{\mathrm{D}} = \mathtt{MAT}(\text{A-1}) = \begin{pmatrix} 4a & & & \\ & 4a & (2a)_{\triangledown} & \\ (2a)_{\triangle} & & \ddots & \\ & & & 4a \end{pmatrix} \;\;;\;\; \mathcal{S}_{\mathrm{D}} = \mathbf{0} \;\;;\;\; B_{\mathrm{D}} = (R - R') \tag{20}$$

$$\det \widehat{M}_{\mathrm{D}} = 2^{A-1} A\; a^{A-1} \tag{21}$$

$$A_{\mathrm{EX(change)}} = -\frac{A}{A+1}(2\pi)^{-3} \;\;;\;\; B_{\mathrm{EX}} = \left(\frac{R}{A+1} + \frac{A}{A+1}R'\right) \tag{22}$$

The A dependence in A_{EX} stems from a rescaling of $\delta(R, R')$, and $(2\pi)^{-3}$ from the Fourier representation of it.

$$\widehat{M}_{\mathrm{EX}} = \begin{pmatrix} a\epsilon_3 & & & \\ & a\epsilon_3 & (a\epsilon_1)_{\triangledown} & \\ (a\epsilon_1)_{\triangle} & & \ddots & \\ & & & a\epsilon_3 \end{pmatrix} \;\;;\;\; \mathcal{S}_{\mathrm{EX}} = \begin{pmatrix} \frac{aR}{A} - is \\ \vdots \\ \frac{aR}{A} - is \end{pmatrix} \;\;;\;\; \epsilon_{1(3)} := \frac{(3)A-1}{A} \tag{23}$$

$$\det \widehat{M}_{\mathrm{EX}} = 2^{A-2}(A + A^{-1})\; a^{A-1} \tag{24}$$

After the cluster-coordinate integration, the exponent reads

$$\frac{1}{2}\mathcal{S}_{\mathrm{EX}}^{\intercal}\widehat{M}_{\mathrm{EX}}^{-1}\mathcal{S}_{\mathrm{EX}} - isB_{\mathrm{EX}} = \frac{\epsilon_1}{2(A^2+1)}aR^2 - \frac{A-1}{A^2+1}iRs - \frac{\epsilon_1}{2(1+A^{-2})}s^2 \tag{25}$$

$$-is\left(\frac{R}{A+1} + \frac{A}{A+1}R'\right) \tag{26}$$

From the last formula, we identify the coefficients for the quadratic and linear s terms which parametrize the one-dimensional Gaussian integral over s. After this integration, the $\delta^{(3)}(\boldsymbol{R}-\boldsymbol{R}')$, as present for the direct interaction is transformed into a Gaussian which is non-zero even if the direction of the relative distance between the centre of mass of the A bodies and the odd man differs from its parameter value due to the interchange of $\boldsymbol{r}_{\mathrm{A}}$ and $\boldsymbol{r}_{\mathrm{A+1}}$:

$$\exp\left[-\underbrace{\frac{3A^2-1}{2A(A+1)(A^2-1)}}_{:=\alpha_{\mathrm{EX}}}a\,\boldsymbol{R}^2 - \underbrace{\frac{A(A^2+1)}{2(A+1)(A^2-1)}}_{:=\beta_{\mathrm{EX}}}a\,\boldsymbol{R}'^2 - \underbrace{\frac{2A^2}{(A+1)(A^2-1)}}_{:=\gamma_{\mathrm{EX}}}a\,\boldsymbol{R}\cdot\boldsymbol{R}'\right] \tag{27}$$

Note that all matrices are chosen to be symmetric in order to employ

$$\int d^3\boldsymbol{v}e^{-\frac{1}{2}\boldsymbol{v}^{\intercal}\widehat{M}\boldsymbol{v}+\mathcal{S}^{\intercal}\cdot\boldsymbol{v}} = (2\pi)^{(A-1)/2}\left(\det\widehat{M}\right)^{-1/2} e^{\frac{1}{2}\mathcal{S}^{\intercal}\widehat{M}^{-1}\mathcal{S}}$$

For the direct term (corresponding to the $\mathbb{1}$ of the antisymmetrizer), only one (A-1)-dimensional Gaussian integral over the $\bar{\boldsymbol{r}}_{\mathrm{i}}$'s has to be evaluated. The s integral recovers $\delta^{(3)}(\boldsymbol{R}-\boldsymbol{R}')$, yielding a constant which is naturally local (no $\boldsymbol{R}'$ dependence). The permutation encodes the probability to find the particles arranged in ϕ_A and $\phi_B(=1)$ with a single particle of the A system to be present at a distance $\boldsymbol{R}$ from its centre of mass while the B particle resides "within" A in a non-local structure. The form of this structure results from the Gaussian integration over the quadratic s terms.

B. Partial-wave projection

All complications related to the projection of Eq.(2) into partial waves arise even in the absence of an interaction and shall be described below.

$$\left(\hat{T}_{\boldsymbol{R}} - E\right)\langle\phi_A|\hat{A}\left[\phi_A\psi\right]\rangle = 0 \tag{28}$$

$$\left(\hat{T}_{\boldsymbol{R}} - E\right)\int d^3\boldsymbol{R}'\left\{\delta^{(3)}(\boldsymbol{R}-\boldsymbol{R}') + c_{\rm EX}e^{-\alpha_k\boldsymbol{R}^2-\beta_k\boldsymbol{R}'^2-\gamma_k\boldsymbol{R}\cdot\boldsymbol{R}'}\right\}\psi(\boldsymbol{R}') \tag{29}$$

which in virtue of

$$c_{\rm EX} := -\frac{A}{A+1}\left(\pi(A-1)\right)^{-3/2} \quad ; \quad \psi(\boldsymbol{R}) = R^{-1}\sum_{lm}\psi_{lm}(R)Y_{lm}(\hat{\boldsymbol{R}}) \tag{30}$$

$$\hat{T}^l_R = \frac{\hbar}{2}\underbrace{\frac{(A+1)}{mA}}_{=\mu^{-1}_{\rm red}}\left(-\frac{d^2}{dR^2}+\frac{l(l+1)}{R^2}\right) \quad ; \quad R\int d^2\hat{\boldsymbol{R}}\; Y^*_{l'm'}(\hat{\boldsymbol{R}}) \to (\ldots) \tag{31}$$

yields

$$\left(\hat{T}^l_R - E\right)\psi_{lm}(R) \tag{32}$$

$$+c_{\rm EX}\int dR'\;R'e^{-\beta_{\rm EX}R'^2}\left(\hat{T}^l_R - E\right)\left[Re^{-\alpha_{\rm EX}R^2}\int d^2\;(\hat{\boldsymbol{R}},\hat{\boldsymbol{R}}')Y^*_{l'm'}(\hat{\boldsymbol{R}})e^{-\gamma_{\rm EX}\boldsymbol{R}\cdot\boldsymbol{R}'}Y_{lm}(\hat{\boldsymbol{R}}')\right]\psi_{lm}(R') = 0$$

in which we substitute

$$e^{-\gamma_{\rm EX}\boldsymbol{R}\cdot\boldsymbol{R}'} = e^{ii\gamma_{\rm EX}\boldsymbol{R}\cdot\boldsymbol{R}'} = 4\pi\sum_{LM}i^L j_L(i\gamma_{\rm EX}RR')Y^*_{LM}(\hat{\boldsymbol{R}})Y_{LM}(\hat{\boldsymbol{R}}') \tag{33}$$

to arrive at

$$\left(\hat{T}^l_R - E\right)\psi_{lm}(R) + 4\pi\, i^l\, c_{\rm EX}\int dR'\;R'e^{-\beta_{\rm EX}R'^2}\left(\hat{T}^l_R - E\right)\left[Re^{-\alpha_{\rm EX}R^2}j_l(i\gamma_{\rm EX}RR')\right]\psi_{lm}(R') = 0\ . \tag{34}$$

Enter the interaction potential does not alter this basic structure but adds terms of the same type:

$$\left(\hat{T}^l_R - E\right)\psi_{lm}(R) + \sum_k c_k\int dR'\;R'e^{-\beta_k R'^2}\hat{O}_k(R)\left[Re^{-\alpha_k R^2}j_l(i\gamma_k RR')\right]\psi_{lm}(R') = 0 \tag{35}$$

with

$$c_{\rm EX} = -4\pi i^l\;\frac{A}{A+1}\left(\pi(A-1)\right)^{-3/2} \quad \text{and} \quad \hat{O}_{\rm EX} = \left(\hat{T}^l_R - E\right) \tag{36}$$

C. Direct- and exchange interaction

Besides the interaction derived above is a consequence of the identity of particles and the ensuing demand for an antisymmetric spatial wave function. This effective potential presents a deviation from a free motion between the A-body core and a particle even if the particle-particle interaction is zero. For specific non-zero two- and three-particle interactions we derive the contributions affecting the cluster-particle relative potential below. Specifically, we calculate the second term of Eq. (2) employing the interaction Eq.(14) and Eq.(15).

For a Gaussian representation of the contact interactions,

$$\delta_\Lambda^{(3)}(\boldsymbol{x}) = e^{-\frac{\Lambda}{4}\boldsymbol{x}^2} \quad ,$$

and $\hat{P} \in \left\{\mathbb{1}, \hat{P}_{A,A+1}\right\}$, one arrives at

$$\left\langle \phi_A \middle| \hat{V}_{\mathrm{A,A+1}} \middle| \hat{P}\left[\phi_A \psi\right] \right\rangle = \int d^3\boldsymbol{R}' \sum_{k=1}^{7} A_k \left(\int_{\mathbb{R}} \dots \int_{\mathbb{R}} d(\bar{r}_{1,\dots,\mathrm{A-1}}, s)\; e^{-\frac{1}{2}\bar{\boldsymbol{r}}^\intercal \widehat{M}_k \bar{\boldsymbol{r}} + \mathcal{S}_k^\intercal \cdot \bar{\boldsymbol{r}} - isB_k} \right)^{\vec{3}} \times \psi(\boldsymbol{R}')$$

$$= \int d^3\boldsymbol{R}' \sum_{k=1}^{7} A_k \left(\sqrt{\frac{(2\pi)^{A-1}}{\det \widehat{M}_k}} \int_{\mathbb{R}} ds\; e^{\frac{1}{2}\mathcal{S}_k^\intercal \widehat{M}_k^{-1} \mathcal{S}_k - isB_k} \right)^{\vec{3}} \times \psi(\boldsymbol{R}') \tag{37}$$

writing the exponent in the form $\frac{1}{2}\boldsymbol{s}^\intercal \widehat{M}_{s,k}\boldsymbol{s} + \boldsymbol{B}_{s,k}^\intercal \cdot \boldsymbol{s}$ translates the s integral into

$$= \int d^3\boldsymbol{R}' \sum_{k=1}^{7} A_k \left(\sqrt{\frac{(2\pi)^{A-1}}{\det \widehat{M}_k}} \cdot \sqrt{\frac{2\pi}{\det \widehat{M}_{s,k}}} \right)^3 \times e^{-\alpha_k \boldsymbol{R}^2 - \beta_k \boldsymbol{R}'^2 - \gamma_k \boldsymbol{R}\cdot\boldsymbol{R}'} \times \psi(\boldsymbol{R}')$$

$$= \sum_{k=1}^{7} A_k \left(\frac{(2\pi)^A}{\det \widehat{M}_k \;\det \widehat{M}_{s,k}} \right)^{\frac{3}{2}} \int d^3\boldsymbol{R}'\; e^{-\alpha_k \boldsymbol{R}^2 - \beta_k \boldsymbol{R}'^2 - \gamma_k \boldsymbol{R}\cdot\boldsymbol{R}'} \times \psi(\boldsymbol{R}') \tag{38}$$

i.e. , a structure which is analogous to Eq.(17). In order to obtain the coefficients, interacting pairs and triples are organized in seven terms, each of which corresponding to the same exponent. Specifically, we found the following arrangement convenient (details in Sec.III C 1).

$$C_0(\Lambda) \sum_{i \leq A} \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}) = \left\{ \sum_{i<A} \delta^{(3)}(\bar{\boldsymbol{r}}_{\mathrm{i}} - R) + \delta^{(3)}\big(-\sum_{i<A} \bar{r}_{\mathrm{i}} - R\big) \right\} \cdot C_0(\Lambda) \tag{39}$$

$$= \left\{ (A-1)\,\delta_\Lambda^{(3)}(\bar{\boldsymbol{r}}_1 - R) + 1 \cdot \delta_\Lambda^{(3)}\big(-\sum_{i<A} \bar{r}_{\mathrm{i}} - R\big) \right\} \cdot C_0(\Lambda) \tag{40}$$

$$D_1(\Lambda) \sum_{\substack{i<j \\ \mathrm{cyc}}}^{A} \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{j}} - \boldsymbol{r}_{\mathrm{A+1}}) = \left\{ \binom{A-1}{2} \delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_2 - \boldsymbol{r}_{\mathrm{A+1}}) \right. \tag{41}$$

$$\left. + \; (A-1)\,\delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_{\mathrm{A+1}}) \right\} \cdot D_1(\Lambda) \tag{42}$$

$$D_1(\Lambda) \sum_{\substack{i<j \\ \mathrm{cyc}}} \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{j}} - \boldsymbol{r}_{\mathrm{j}}) = \left\{ \binom{A-1}{2} \delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \right. \tag{43}$$

$$+ \; (A-1)\delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_1) \tag{44}$$

$$\left. + \; (A-1)\delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_{\mathrm{A+1}}) \delta_\Lambda^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_{\mathrm{A}}) \right\} \cdot D_1(\Lambda) \quad , \tag{45}$$

and written as a function of cluster, *i.e.* , integration variables:

$$\begin{aligned}
\hat{V}_{A,A+1} = C_0(\Lambda)\cdot \Big\{ & (A-1)\,\delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - R\big) + 1\cdot\delta_\Lambda^{(3)}\big(\sum_{i<A}\bar{\boldsymbol{r}}_{\rm i}+R\big)\Big\}_{k=1,2} \\
+\ D_1(\Lambda)\cdot\Big\{ & \binom{A-1}{2}\Big(\delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - R\big)\delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_2 - R\big) + \delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - R\big)\delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - \bar{\boldsymbol{r}}_2\big)\Big)_{k=3,5} \\
& +(A-1)\Big(\delta_\Lambda^{(3)}\big(\sum_{i<A}\bar{\boldsymbol{r}}_{\rm i}+R\big)\delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - R\big) + \delta_\Lambda^{(3)}\big(\sum_{i<A}\bar{\boldsymbol{r}}_{\rm i}+R\big)\delta_\Lambda^{(3)}\big(\sum_{i<A}\bar{\boldsymbol{r}}_{\rm i}+\bar{\boldsymbol{r}}_1\big) \\
& +\ \delta_\Lambda^{(3)}\big(\bar{\boldsymbol{r}}_1 - R\big)\delta_\Lambda^{(3)}\big(\sum_{i<A}\bar{\boldsymbol{r}}_{\rm i}+\bar{\boldsymbol{r}}_1\big)\Big)_{k=4,6,7}\Big\}\ . \qquad (46)
\end{aligned}$$

From these 7 terms, the matrices and vectors which parametrize the integral in Eq.(38) are read off (the blue subscripts refer to the parameter set which results from the respective term in the brackets):

$$
\widehat{M}_1 = \begin{pmatrix} a\epsilon_3 + \frac{\Lambda^2}{2} & & & \\ & a\epsilon_3 & (a\epsilon_1)_{\triangledown} & \\ (a\epsilon_1)_{\triangle} & & \ddots & \\ & & & a\epsilon_3 \end{pmatrix} \qquad \mathcal{S}_1 = \begin{pmatrix} \frac{aR}{A} - is + \frac{\Lambda^2}{2} R \\ \frac{aR}{A} - is \\ \vdots \\ \frac{aR}{A} - is \end{pmatrix} \qquad B_1' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{4}\right) R^2
$$

$$
\widehat{M}_2 = \begin{pmatrix} a\epsilon_3 + \frac{\Lambda^2}{2} & & & \\ & a\epsilon_3 + \frac{\Lambda^2}{2} & (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangledown} & \\ (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangle} & & \ddots & \\ & & & a\epsilon_3 + \frac{\Lambda^2}{2} \end{pmatrix} \qquad \mathcal{S}_2 = \begin{pmatrix} \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \\ \vdots \\ \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \end{pmatrix} \qquad B_2' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{4}\right) R^2
$$

$$
\widehat{M}_3 = \begin{pmatrix} a\epsilon_3 + \frac{\Lambda^2}{2} & & & & \\ & a\epsilon_3 + \frac{\Lambda^2}{2} & (a\epsilon_1)_{\triangledown} & & \\ & & a\epsilon_3 & & \\ & (a\epsilon_1)_{\triangle} & & \ddots & \\ & & & & a\epsilon_3 \end{pmatrix} \qquad \mathcal{S}_3 = \begin{pmatrix} \frac{aR}{A} - is + \frac{\Lambda^2}{2} R \\ \frac{aR}{A} - is + \frac{\Lambda^2}{2} R \\ \frac{aR}{A} - is \\ \vdots \\ \frac{aR}{A} - is \end{pmatrix} \qquad B_3' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{2}\right) R^2
$$

$$
\widehat{M}_4 = \begin{pmatrix} a\epsilon_3 + \Lambda^2 & & & \\ & a\epsilon_3 + \frac{\Lambda^2}{2} & (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangledown} & \\ (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangle} & & \ddots & \\ & & & a\epsilon_3 + \frac{\Lambda^2}{2} \end{pmatrix} \qquad \mathcal{S}_4 = \begin{pmatrix} \frac{aR}{A} - is \\ \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \\ \vdots \\ \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \end{pmatrix} \qquad B_4' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{2}\right) R^2
$$

$$
\widehat{M}_5 = \begin{pmatrix} a\epsilon_3 + \Lambda^2 & a\epsilon_1 - \frac{\Lambda^2}{2} & & & \\ a\epsilon_1 - \frac{\Lambda^2}{2} & a\epsilon_3 + \frac{\Lambda^2}{2} & (a\epsilon_1)_{\triangledown} & & \\ & & a\epsilon_3 & & \\ & (a\epsilon_1)_{\triangle} & & \ddots & \\ & & & & a\epsilon_3 \end{pmatrix} \qquad \mathcal{S}_5 = \begin{pmatrix} \frac{aR}{A} - is + \frac{\Lambda^2}{2} R \\ \frac{aR}{A} - is \\ \vdots \\ \frac{aR}{A} - is \end{pmatrix} \qquad B_4' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{4}\right) R^2
$$

$$
\widehat{M}_6 = \begin{pmatrix} a\epsilon_3 + \frac{5}{2}\Lambda^2 & a\epsilon_1 + \frac{3}{2}\Lambda^2 & \dots & a\epsilon_1 + \frac{3}{2}\Lambda^2 \\ a\epsilon_1 + \frac{3}{2}\Lambda^2 & a\epsilon_3 + \Lambda^2 & & \\ \vdots & & \ddots & (a\epsilon_1 + \Lambda^2)_{\triangledown} \\ & (a\epsilon_1 + \Lambda^2)_{\triangle} & & \\ a\epsilon_1 + \frac{3}{2}\Lambda^2 & & & a\epsilon_3 + \Lambda^2 \end{pmatrix} \qquad \mathcal{S}_6 = \begin{pmatrix} \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \\ \vdots \\ \frac{aR}{A} - is - \frac{\Lambda^2}{2} R \end{pmatrix} \qquad B_6' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{4}\right) R^2
$$

$$
\widehat{M}_7 = \begin{pmatrix} a\epsilon_3 + \frac{5}{2}\Lambda^2 & a\epsilon_1 + \Lambda^2 & \dots & a\epsilon_1 + \Lambda^2 \\ a\epsilon_1 + \Lambda^2 & a\epsilon_3 + \frac{\Lambda^2}{2} & & \\ \vdots & & \ddots & (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangledown} \\ & (a\epsilon_1 + \frac{\Lambda^2}{2})_{\triangle} & & \\ a\epsilon_1 + \Lambda^2 & & & a\epsilon_3 + \frac{\Lambda^2}{2} \end{pmatrix} \qquad \mathcal{S}_7 = \begin{pmatrix} \frac{aR}{A} - is + \frac{\Lambda^2}{2} R \\ \frac{aR}{A} - is \\ \vdots \\ \frac{aR}{A} - is \end{pmatrix} \qquad B_7' = -\frac{i}{s}\left(\frac{a\epsilon_1}{2} + \frac{\Lambda^2}{4}\right) R^2
$$

$$
B_k = B_k' + \underbrace{\left(\frac{R}{A+1} + \frac{A}{A+1} R'\right)}_{\xrightarrow{\hat{P} = \mathbb{1}} R - R'}
$$

$$
\epsilon_1 = \begin{cases} 2 \\ \frac{A-1}{A} \end{cases} \qquad \epsilon_3 = \begin{cases} 4 & \text{if } \hat{P} = \mathbb{1} \\ \frac{3A-1}{A} & \text{if } \hat{P} = \hat{P}_{A,A+1} \end{cases} \qquad \text{red} = 0 \text{ if } \hat{P} = \mathbb{1}
$$

These matrices enable the derivation of

$$\begin{aligned}
\left\langle \phi_A \middle| \hat{V}_{\mathrm{A,A+1}} \middle| \hat{P}\left[\phi_A\psi\right] \right\rangle =& \sum_{k=1}^{7} A_k \left(\frac{(2\pi)^{A-1}}{\det \widehat{M}_k}\right)^{\frac{3}{2}} e^{-\alpha_k \boldsymbol{R}^2} \psi(\boldsymbol{R}) \\
&- \sum_{k=8}^{14} \frac{A}{A+1}(2\pi)^{-3} A_k \left(\frac{(2\pi)^A}{\det \widehat{M}_k \cdot \det \widehat{M}_{s,k}}\right)^{\frac{3}{2}} \int d^3\boldsymbol{R'} e^{-\alpha_k \boldsymbol{R}^2 - \beta_k \boldsymbol{R'}^2 - \gamma_k \boldsymbol{R}\cdot\boldsymbol{R'}} \psi(\boldsymbol{R'}) \\
=& (2\pi)^{\frac{3}{2}(A-1)} \sum_{k=1}^{7} A_k \int d^3\boldsymbol{R'} \left[\left(\det \widehat{M}_k^{\mathrm{dir}}\right)^{-\frac{3}{2}} \delta^{(3)}(\boldsymbol{R}-\boldsymbol{R'}) e^{-\alpha_k^{\mathrm{dir}} \boldsymbol{R'}^2} \right. \\
&- \left. \frac{A}{A+1}(2\pi)^{-\frac{3}{2}} \left(\det \widehat{M}_k^{\mathrm{ex}} \cdot \det \widehat{M}_{s,k}\right)^{-\frac{3}{2}} e^{-\alpha_k^{\mathrm{ex}} \boldsymbol{R}^2 - \beta_k^{\mathrm{ex}} \boldsymbol{R'}^2 - \gamma_k^{\mathrm{ex}} \boldsymbol{R}\cdot\boldsymbol{R'}} \right] \psi(\boldsymbol{R'})
\end{aligned} \tag{47}$$

1. *Interacting notes*

We represent the microscopic, *i.e.* , particle-particle interaction with properly renormalized two- and three-body contacts: Particles interact only if two or three of them occupy the same point in space. The intensity of the interaction depends on whether two or three particles collide:

$$\hat{V} = C_0(\Lambda) \sum_{i<j}^{X} \delta_\Lambda^{(3)}\left(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{j}}\right) + D_1(\Lambda) \sum_{\substack{i<j<k \\ \text{cyclic}}}^{X} \delta_\Lambda^{(3)}\left(\boldsymbol{r}_{\mathrm{i}} - \boldsymbol{r}_{\mathrm{j}}\right) \delta_\Lambda^{(3)}\left(\boldsymbol{r}_{\mathrm{j}} - \boldsymbol{r}_{\mathrm{k}}\right) \quad .$$

For $X \leq A$, the interaction yields stable, self-bound X-body ground states with totally symmetric spatial wave functions, because we investigate systems whose internal space is precisely A-dimensional. For $X = A+1$, we isolate the interaction of particle $A+1$ with the A-body fragment:

$$\begin{aligned}
\hat{V} = \hat{V}_A + \hat{V}_{A,A+1} =& \sum_{i<j}^{A} \delta_{i,j} + \sum_{i}^{A-1} \delta_{i,A+1} + \delta_{A,A+1} \\
&+ \sum_{\substack{i<j<k \\ \text{cyc}}}^{A} \delta_{i,j,k} + \sum_{\substack{i<j \\ \text{cyc}}}^{A-1} \delta_{i,j,A+1} + \sum_{\substack{i \\ \text{cyc}}}^{A-1} \delta_{i,A,A+1} \\
\hat{V}_{A,A+1} \overset{\langle\ldots\rangle}{\sim}& \underbrace{(A-1)\,\delta_{1,A+1} + 1 \cdot \delta_{A,A+1}}_{(39),(40)} + \underbrace{2 \cdot \binom{A-1}{2} \delta_{1,2,A+1}}_{\Rightarrow(41)+(43)} + \underbrace{3 \cdot (A-1)\,\delta_{1,A,A+1}}_{(42)+(44)+(45)} \quad .
\end{aligned}$$

Matrix elements in the $\phi_A\psi$ basis as defined in Eqs.(7) and (8) are identical ($\overset{\langle\ldots\rangle}{\sim}$) for left and right-hand-side operators.

Assumption of a tightly-bound, spatially-symmetric A-body core:

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}\bar{\boldsymbol{r}}_\mathrm{i}^2} = e^{-a\sum^{A-1}\bar{\boldsymbol{r}}_\mathrm{i}^2 - a\sum_{i<j}^{A-1}\bar{\boldsymbol{r}}_\mathrm{i}\cdot\bar{\boldsymbol{r}}_\mathrm{j}}$$

The contact theory:

$$\hat{V} = C_0(\Lambda)\sum_{i<j}^{X}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i}-\boldsymbol{r}_\mathrm{j}\big) + D_1(\Lambda)\sum_{\substack{i<j<k\\ \text{cyclic}}}^{X}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i}-\boldsymbol{r}_\mathrm{j}\big)\,\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{j}-\boldsymbol{r}_\mathrm{k}\big)$$

Enter indistinguishability:

$$\hat{A} = \mathbb{1} - \hat{P}(\boldsymbol{r}_\mathrm{A} \leftrightarrow \boldsymbol{r}_\mathrm{A+1})$$

"Freeze" the core and extremize the action *w.r.t.* the relative motion between the one particle outside of the core:

$$\Big(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + \mathbb{N}^{-1}\big\langle\phi_A\big|\hat{V}\big|\phi_A\big\rangle\Big)\,\chi(\boldsymbol{R})$$

$$-\mathbb{N}^{-1}\int d\boldsymbol{R}'\Big[\big\langle\phi_A\big|\left(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + \hat{V}\right)\hat{P}\{\big|\phi_A\big\rangle\delta_\Lambda^{(3)}\big(\boldsymbol{R}-\boldsymbol{R}'\big)\Big]\,\chi(\boldsymbol{R}') = 0$$

$\hat{V} = C_\Lambda\sum_{i<A}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{A+1}\big)$ and $\hat{P} = 0$ *i.e.* direct 2-body contribution only:

$$\left(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + C_\Lambda(A-1)\;8\left(4 + \left(\frac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{-3/2}\cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}\Lambda^2}\boldsymbol{R}^2}\right)\chi(\boldsymbol{R}) = 0$$

$$\downarrow$$

$$-\frac{\hbar}{2m}\frac{A+1}{A}\partial_R^2 + \left(\frac{\hbar}{2m}\frac{A+1}{A}\frac{l(l+1)}{R^2} + \frac{C_\Lambda(A-1)\;8}{\left(4+\left(\frac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{3/2}}\cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}\Lambda^2}\boldsymbol{R}^2}\right)\psi_{lm}(R) = 0$$

$$\Leftrightarrow -\frac{\hbar}{2m}\partial_R^2 + \left(\frac{\hbar}{2m}\frac{l(l+1)}{R^2} + \frac{8\;C_\Lambda A(A-1)(A+1)^{-1}}{\left(4+\left(\frac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{3/2}}\cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}\Lambda^2}\boldsymbol{R}^2}\right)\psi_{lm}(R) = 0$$

with

$$E_\mathrm{relative} = \epsilon_A - E_\mathrm{total} \quad \text{and} \quad \mathbb{N} = \big\langle\phi_A\big|\phi_A\big\rangle$$

Assumption of a tightly-bound, spatially-symmetric A-body core:

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}\bar{\boldsymbol{r}}_\mathrm{i}^2} = e^{-a\sum^{A-1}\bar{\boldsymbol{r}}_\mathrm{i}^2 - a\sum_{i<j}^{A-1}\bar{\boldsymbol{r}}_\mathrm{i}\cdot\bar{\boldsymbol{r}}_\mathrm{j}}$$

The contact theory:

$$\hat{V} = C_0(\Lambda)\sum_{i<j}^{X}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{j}\big) + D_1(\Lambda)\sum_{\substack{i<j<k\\ \text{cyclic}}}^{X}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{j}\big)\,\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{j} - \boldsymbol{r}_\mathrm{k}\big)$$

Enter indistinguishability:

$$\hat{A} = \mathbb{1} - \hat{P}\big(\boldsymbol{r}_\mathrm{A} \leftrightarrow \boldsymbol{r}_\mathrm{A+1}\big)$$

“Freeze” the core and extremize the action *w.r.t.* the relative motion between the one particle outside of the core:

$$\Big(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + \mathbb{N}^{-1}\big\langle\phi_A\big|\hat{V}\big|\phi_A\big\rangle\Big)\,\chi(\boldsymbol{R})$$
$$-\mathbb{N}^{-1}\int d\boldsymbol{R}'\left[\big\langle\phi_A\big|\left(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + \hat{V}\right)\hat{P}\{\big|\phi_A\big\rangle\delta_\Lambda^{(3)}\big(\boldsymbol{R} - \boldsymbol{R}'\big)\right]\chi(\boldsymbol{R}') = 0$$

$\hat{V} = C_\Lambda\sum\limits_{i<A}\delta_\Lambda^{(3)}\big(\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{A+1}\big)$ and $\hat{P} = 0$ *i.e.* direct 2-body contribution:

$$\left(\hat{T}_{\boldsymbol{R}} - E_\mathrm{rel} + C_\Lambda(A-1)\,8\left(4 + \left(\tfrac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{-3/2}\cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}\Lambda^2}\boldsymbol{R}^2}\right)\chi(\boldsymbol{R}) = 0$$

$$\downarrow$$

$$-\frac{\hbar}{2m}\frac{A+1}{A}\partial_R^2 + \left(\frac{\hbar}{2m}\frac{A+1}{A}\frac{l(l+1)}{R^2} + \frac{C_\Lambda(A-1)\ 8}{\left(4+\left(\frac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{3/2}} \cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}\Lambda^2}\boldsymbol{R}^2}\right)\psi_{lm}(R)$$

$$\Leftrightarrow -\frac{\hbar}{2m}\partial_R^2 + \left(\frac{\hbar}{2m}\frac{l(l+1)}{R^2} + \frac{8\ C_\Lambda A(A-1)(A+1)^{-1}}{\left(4+\left(\frac{A-1}{A}\right)a^{-1}\Lambda^2\right)^{3/2}} \cdot e^{-\frac{\Lambda^2}{4+A^{-1}(A-1)a^{-1}}}\right.$$

with

$$E_{\text{relative}} = \epsilon_A - E_{\text{total}} \quad \text{and} \quad \mathbb{N} = \left\langle \phi_A \middle| \phi_A \right\rangle$$

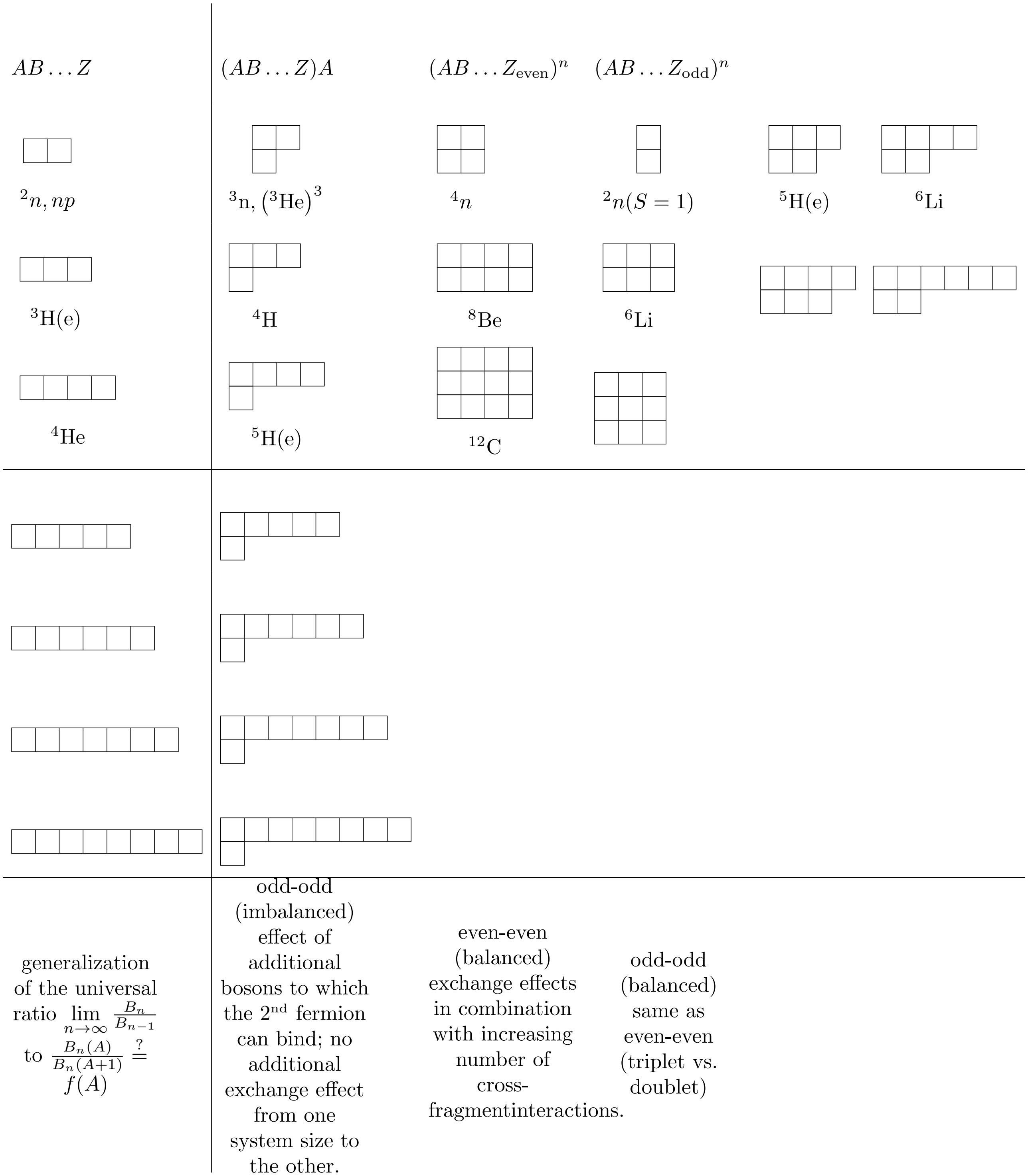
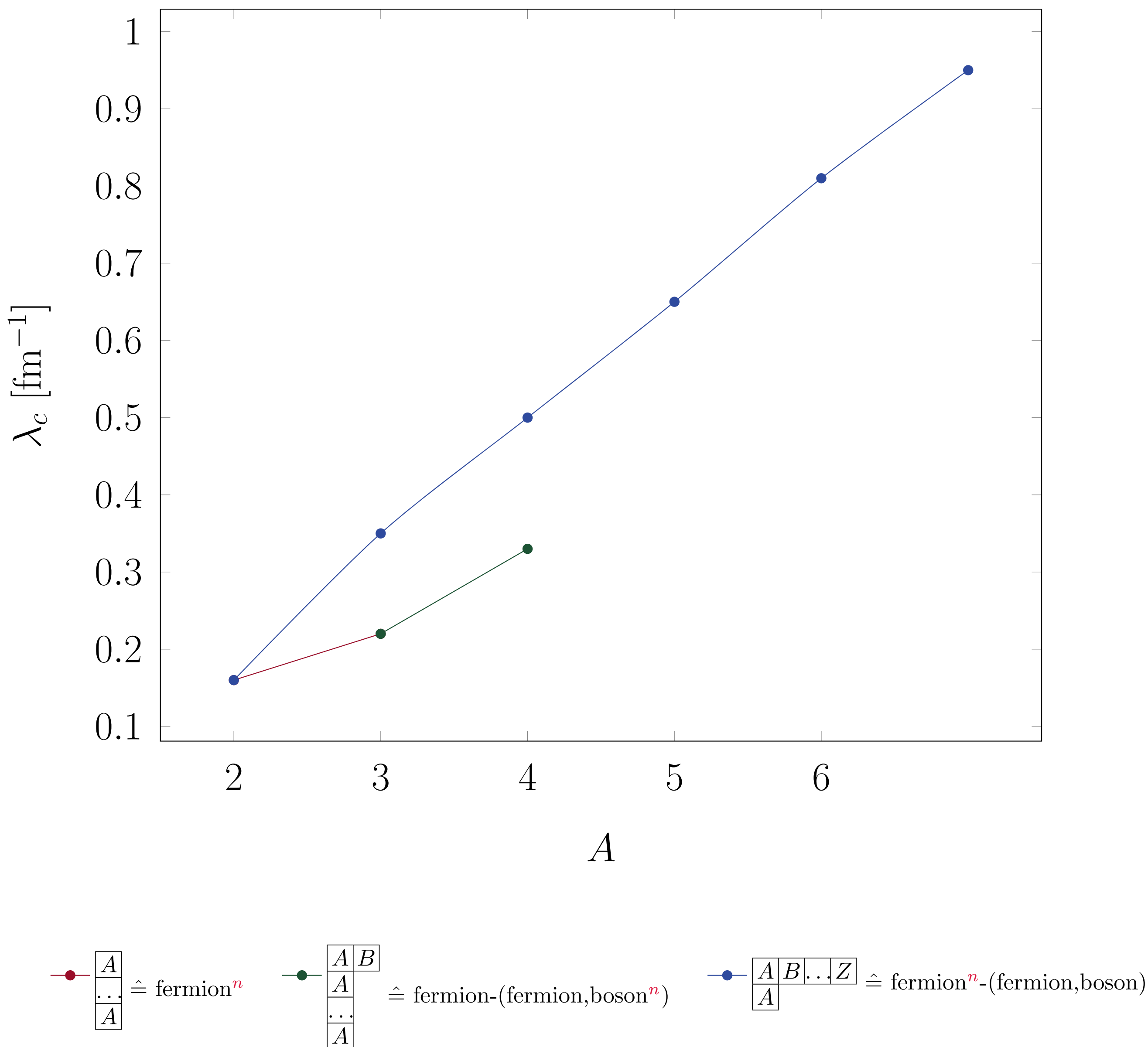


FIG. 1. Classification of A-body systems according to particle number and accessible internal states.



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
λ_c [fm^{-1}]
2
3
4
5
6
A
≙ fermion^n
≙ fermion-(fermion,boson^n)
≙ fermion^n-(fermion,boson)

FIG. 2. .