Poles and Thresholds and Unstable Particles.

P. V. Landshoff (*)

St. John's College - Cambridge

(ricevuto il 5 Novembre 1962)

Summary. — Consequences are investigated of assuming that transition amplitudes display some analyticity in the coupling constants. This enables one to demonstrate the existence of singularities associated with unstable and virtual particles, similar to those required by unitarity for stable particles. It is simple to determine on which Riemann sheets lie the unstable and virtual particle singularities and one may also show that the stable particle singularities are absent from certain sheets.

1. – Introduction.

Several years ago Peierls (1) suggested that an unstable particle would manifest itself as a complex pole on an unphysical sheet of a transition amplitude. Although little progress has been made in formulating a field theory containing unstable particles, it is now generally accepted that this complex pole idea is a sufficent description within the framework of pure S-matrix theory.

According to the work of Polkinghorne (2) and of Stapp (3) simple unitarity and analyticity assumptions for the S-matrix lead to the result that the existence of poles corresponding to stable particles implies also the presence of all the singularities found in perturbation theory. These are represented by Landau-Cutkosky diagrams (which look just like Feynman diagrams) and include, for example, the normal thresholds. It has been pointed out that the

(*) NATO Fellow.

(1) R. E. Peierls: *Proc. of 1954 Glasgow Conference.*

(2) J. C. Polkinghorne: *Nuovo Cimento*, **23**, 360 (1962).

(3) H. P. Stapp: *Phys. Rev.*, **125**, 213 (1962).

unstable-particle poles should also generate similar singularities ([4]). For the case of the two-particle threshold this has been demonstrated by BLANKENBECLER *et al.* ([5]) and more recently by ZWANZIGER ([6]). The method of Zwanziger is similar to that of Polkinghorne and so will apply generally. The positions of the singularities are simple to determine, but it is not so easy to decide on which Riemann sheet they lie. This is a principal concern of this paper. The method to be used here also provides a very simple rederivation of the proof of the existence of the singularities.

It is sometimes hoped that in the eventual solution of the equations of the S-matrix formalism the coupling constants will be uniquely determined from some sort of self-consistency requirement. At the present level of sophistication, however, it seems allowable to vary a coupling constant and examine how the theory changes. It will be assumed that if, for some values of the various coupling constants, the theory contains an unstable particle, then this particle can be made stable by increasing one of the coupling constants, g. It will be further assumed that everything changes analytically in this process and it will be necessary to allow the coupling constant g to become complex, though remaining near to the real axis. (This is merely a mathematical device and it does not matter that the theory is not unitary for complex g.) It is supposed that there is encountered no fixed branch point on the real axis in the complex g-plane. This assumption appears to be valid in potential theory: it is difficult to find a potential which violates it ([*]).

Since we know about the singularities associated with the particle when it is stable, these assumptions allow us to infer the properties of the unstable particle singularities by analytic continuation. In particular, we have an immediate demonstration that they do indeed exist. The procedure also gives some information about singularities associated with stable bound states. Here, however, it must be remembered that a stable bound state cannot always be made unstable by decreasing the coupling constant. An example is provided by the deuteron, which becomes virtual rather than unstable when the strength of the binding potential is reduced. If we take account of this possibility and so do gain information about all stable bound states, it is relevant to remember the principle of Feynman that *all* stable particles can be regarded as bound states.

It will be found that singularities associated with unstable particles are similar in many respects to those for stable particles. In particular those

---

([4]) P. V. LANDSHOFF: quoted in ref. ([2]).

([5]) R. BLANKENBECLER, M. L. GOLDBERGER, S. W. MACDOWELL and S. B. TREIMAN: *Phys. Rev.*, **123**, 692 (1961).

([6]) D. ZWANZIGER: preprint (Berkeley).

([*]) E. SQUIRES: private communication.

other than the poles occur in all partial waves, a fact that may be relevant in the current controversy about cuts in the Regge l-plane, when the partial waves are continued to nonphysical values of l.

2. – Poles.

For ease of exposition we consider a model in which there are two spinless particles A having no intrinsic quantum numbers. The imposition of selection rules, such as baryon conservation, would change the analysis in an important but easily determined way.

Suppose that for a given value of the coupling constant g the elastic scattering amplitude $A+A \to A+A$ has a pole P corresponding to an unstable particle. There is a cut along the real axis in the complex s-plane, attached to the branch point at the $2A$ normal threshold, and the physical values of the amplitude are obtained by approaching the cut from above. The pole P lies in the lower half-plane, on the unphysical Riemann sheet reached by passing through the cut from above. Thus if P lies not far from the real axis we can see, by pushing the cut down over P and so exposing it, that it effectively lies close to the physical region and so may be expected to produce a resonance in the physical amplitude (*).

We suppose that we can now increase g through real values so that P changes from representing an unstable particle to represent a stable bound state. Then P is on the real axis below the $2A$ threshold and is on the physical sheet. Since P has changed sheet it has passed through the branch cut and in fact it has passed through the branch point itself. This we infer from the reflection principle of complex variable theory, which operates because the amplitude is real on the real axis below the cut [7]. Thus when P is complex there must be a complex conjugate pole P' on the unphysical sheet reached by going through the cut in the opposite direction. (This is actually the same sheet as that occupied by P, since there are only two sheets associated with the $2A$ branch point [8].) If P passed through the cut at any place other than at the threshold, P' would too and so both would be on the physical sheet. Thus unless we are prepared to have the bound states occurring in pairs, we must suppose that the poles actually pass through the threshold, as this is the only way by which one can have changed sheets and the other not, while con-

(*) Here we have implicitly assumed that the resonance lies below the $3A$ threshold. Otherwise, to reach the pole from the physical sheet it is presumably necessary to pass through both the $2A$ and the $3A$ cuts.

(7) We suppose there are no anomalous thresholds. The analysis could simply be altered to accommodate these.

(8) W. ZIMMERMANN: *Nuovo Cimento*, **21**, 249 (1961).

forming to the reflection principle. Notice that P and P' necessarily have the same spin, as the analysis may equally be applied to the partial amplitude containing P rather than to the whole amplitude.

We have said that when a pole actually passes through the branch point it may or may not change sheet. We must remove this ambiguity for later applications and this is done by giving g a small imaginary part so that the poles avoid the branch point. On the basis of what has been said in the last paragraph their paths are now as drawn in Fig. 1*a*. In this figure both paths pass above the branch point (*) and P passes through the cut, but not P'. If we give g an imaginary part of opposite sign both poles pass below and now it is P' that passes through the cut to become the bound state, while P remains on the unphysical sheet. The final results, when g is real again, are the same. This is in accord with our supposition of analyticity in g.

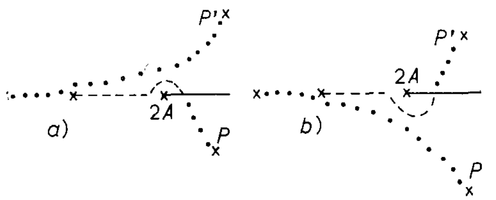

Fig. 1. – The paths of the poles P, P' in the complex s-plane when the transition is directly from an unstable particle to a stable bound state. Parts of a path on the physical sheet are drawn with a dashed line and on an unphysical sheet with a dotted line. The two cases (*a*) and (*b*) are for different signs of Im g.

If we start with a stable bound state P and find, on decreasing g, that it becomes virtual rather than unstable, the picture is as in Fig. 2*a* or Fig. 2*b*, according to which sign is given to Im g. P now ends up on the real axis on the second sheet. Notice that in this situation, unlike the previous one, we are not forced to conclude that the bound state pole is accompanied by a « shadow » pole on the second sheet. It seems to be an open question whether this is necessarily so for some other reason. We shall not need to assume it here.

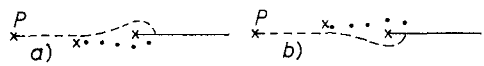

Fig. 2. – The transition from bound state to virtual state. (*a*) and (*b*) again represent different signs for Im g.

A third possibility (9), not represented by either Fig. 1 or Fig. 2, is that P represents all three types of state as g is changed. Thus initially it is complex, representing an unstable particle. As g is increased it moves to meet its shadow P' on the real axis below the threshold. Since it is still on the unphysical sheet it now represents a virtual state. As g is further increased it moves along the real axis up to the threshold and through the cut on to the physical sheet,

(*) While g is complex the requirements of the reflection principle are relaxed and the poles do not occupy complex conjugate positions.

(9) I am grateful to Dr. R. J. EDEN for pointing this out.

so that we now have a bound state. The paths of the poles are drawn in Fig. 3. There are four different possibilities according to which sign is given to Im g, first to avoid the poles coalescing and then to avoid the pole passing through the branch point.

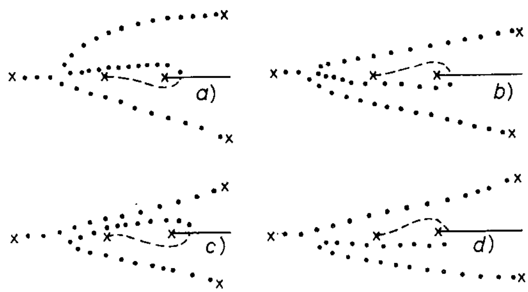

Fig. 3. – The transition from bound state to unstable particle via a virtual state. The four cases represent different signs given to Im g to avoid the poles coalescing and to avoid the pole passing through the branch point.

We may derive a result from the fact that when g is such that there is no stable bound state there is no pole P on the physical sheet. If we now increase g so that such a pole appears, we see by tracing the paths, for any of Fig. 1, 2, 3, that there is not now a pole in the same place on the unphysical sheet (*). This result is apparently not valid in perturbation theory, where stable particle poles seem to occur in the same place on every sheet, though perhaps it would emerge if the series could be summed in some way. It may, of course, be proved generally from unitary: the partial amplitude on the second sheet is obtained (5) by multiplying that on the first sheet by S^{-1}.

3. – Thresholds.

According to POLKINGHORNE (2) and STAPP (3) generalized unitarity demands that in a theory containing stable particles there be singularities corresponding to the Landau-Cutkosky diagrams and the discontinuities across the attached cuts are given (10) by the formula derived by CUTKOSKY (11) for perturbation theory. Thus when g is large enough for binding the diagram of Fig. 4 yields normal thresholds on the physical sheet, corresponding to $3A$ and $(A+P)$. As g is reduced P moves in a known way (Fig. 1, 2 or 3), hence also the $(A+P)$ threshold, and we have the result that

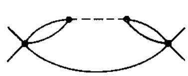

Fig. 4. – Landau-Cutkosky diagram discussed in Section 3. The solid lines represent A-particles, the broken lines the poles P or P'.

(*) Except, possibly, for discrete values of g; though the unitarity argument below would require that its residue then vanish.

(10) J. C. POLKINGHORNE: *Nuovo Cimento*, **25**, 901 (1962).

(11) R. CUTKOSKY: *Journ. Math. Phys.*, **1**, 429 (1960).

also for an unstable or a virtual particle P there is an $(A+P)$ branch point. Using the information in Fig. 1, 2 or 3 we see that this branch point has moved through the $3A$ cut on to the neighbouring unphysical sheet. In the unstable particle case the reflection principle tells us that there is also a complex conjugate branch point $(A+P')$.

By analytic continuation we find that the Cutkosky formula still gives the discontinuity across the $(A+P)$ cut, even when P is not stable. The form given by CUTKOSKY [11] is rather unsuitable for continuation, as it involves δ-functions, and it is convenient to transform these by some method such as that used by POLKINGHORNE [2]. According to ZWANZIGER [6] the $(A+P)$ branch point for unstable P produces a woolly cusp in the scattering amplitude.

To proceed further we shall fix attention, for definiteness, on the paths of the poles shown in Fig. 2. The cases of Fig. 1 and 3 may be discussed similarly, with similar results. As P describes a path round the $2A$ branch point as shown in either Fig. 2*a* or 2*b*, the $(A+P)$ threshold describes a similar path round the $3A$ branch point. Thus using Fig. 2*a* and 2*b* in turn, we see that, since $(A+P)$ is on the physical sheet in the stable case, in the virtual case it is on both the unphysical sheets reached by going through the $3A$ cut in either direction. (These sheets are not the same, because unlike the $2A$ cut the $3A$ cut has more than two sheets.) We label these sheets ± 1 and call the physical sheet the zeroth; the sheet $+1$ is reached from sheet 0 by going *down* through the cut.

We have said that $(A+P)$ is on sheet 0 in the stable case. It is not, however, on sheets ± 1. This result is less familiar (*) than the corresponding one discussed at the end of the last section for the pole P. The proof is simple: if it were initially on sheet $+1$ and we took the path of Fig. 1*b* to make P virtual it would come up on to sheet 0. This is not allowed by the theorem [12] that the only singularities in the physical region are the stable particle normal thresholds.

If we start with P stable, take the path of Fig. 2*a* to make it unstable and return along the path of Fig. 2*b* we must arrive back in the initial configuration. Thus we conclude that in the stable case the sheets occupied by the $(A+P)$ branch point are $0, \pm 2, \pm 4, \ldots$, while in the virtual case it is on the sheets $\pm 1, \pm 3, \pm 5, \ldots$. This is provided we give thought to the $(A+P)$ cut, which trails after the branch point in the transitions. It is perhaps worth

(*) It suggests, for example, that since π may be regarded as an $\mathcal{N}\overline{\mathcal{N}}$ bound state the $\pi\mathcal{N}$ threshold is absent from the sheet reached through the $\mathcal{N}\overline{\mathcal{N}}\mathcal{N}$ cut. Note, however, that there is no reason to suppose it is absent from the sheet reached through the $\pi\pi\mathcal{N}$ cut, because neither π nor $\mathcal{N}$ is a bound state of a pair of the particles involved in this threshold.

(12) R. J. EDEN: *Phys. Rev.*, **121**, 1566 (1960). See also P. V. LANDSHOFF: *Phys. Lett.*, **3**, 116 (1962).

remarking that the same results apply to the complete amplitude and not just the contribution corresponding to Fig. 4. In this case there is also the $2A$ cut, but during the discussion this can be bent out of the way into the upper half-plane, say, and so does not cause complications.

To provide further illustration of the methods used in this paper we turn brief attention to the Landau-Cutkosky diagram of Fig. 5. When P is stable we know there are three thresholds on the physical sheet: $4A$, $(2A+P)$ and $2P$. In the usual way we decrease g from the stable situation, taking the path corresponding to Fig. 1*a*, say. The situation for the value of g such that the $(2A+P)$ and $2P$ thresholds lie directly above the $4A$ is drawn in Fig. 6*a*, where we have only drawn those branch points that are on the physical sheet. It can be seen that both $(2A+P)$ and $2P$ pass down through the $4A$ cut on to the corresponding unphysical sheet, giving the final configuration of Fig. 6*b*. The reflection principle tells us there are corresponding branch points $(2A+P')$ and $2P'$ on the unphysical sheet reached from the physical sheet by passing up through the $4A$ cut. We could also have shown this instead by varying g along the path of Fig. 1*b*. The branch points $(2A+P')$ and $2P'$ must also be present in the stable case, on either of the unphysical sheets reached through the $4A$ cut.

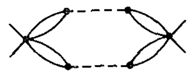

Fig. 5. – Landau-Cutkosky diagram discussed in Section **3**. The solid lines represent A-particles, the broken lines the poles P or P'.

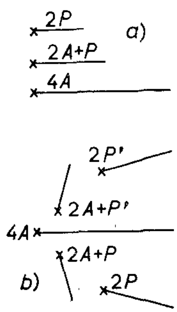

Fig. 6. – (*a*) Positions of those singularities that are on the physical sheet for Fig. 5, corresponding to some intermediate point on the path of Fig. 1*a*; (*b*) final positions of the singularities. Only $4A$ is on the physical sheet.

Although no proof is immediately apparent, one might expect to find a further branch point corresponding to $(P+P')$. This would lie on the real axis between the $2P$ and the $2P'$ branch points. It would not cross any cuts in the transition from stable P to unstable P and so would be on the same unphysical sheet all the time. Some information as to which sheet it could be on may be obtained again from the theorem that the only singularities on the real axis in the physical region are the stable-particle normal thresholds ([12]). Thus, from the final configuration we see that it cannot be on the sheet reached from the physical sheet through the $4A$ cut. From the initial configuration it is not on that sheet reached by passing through both the $2P$ and $(2A+P)$ cuts, though it could be on the sheet reached through the $2P$ cut alon If so, in the final configuration one would have to go first downwards through the $4A$ and then through the $2P$. The reflection principle then would tell us we could also go up through the $4A$ then through the $2P'$, in the final and so also in the initial configuration.

4. – Other singularities.

One expects to find numerous other singularities on unphysical sheets. The simplest of these are the « pseudo »-thresholds, whose presence is familiar in perturbation theory. For example, for a two-particle intermediate state the normal threshold is at $s=(m_1+m_2)^2$ and the pseudo-threshold is found at $s=(m_1-m_2)^2$ by passing through the normal threshold cut. This is known (8) independently of perturbation theory for this example. The analysis of this paper will deal equally with pseudo-thresholds for unstable particles.

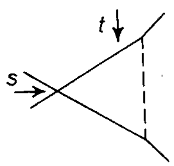

Fig. 7. – The Landau-Cutkosky diagram discussed in Section 4.

It will also apply to more general Landau-Cutkosky singularities, of both first and second types (13). CHALLIFOUR (14) has pointed out that an unstable or virtual pole in a crossed channel will also lead to singularities on unphysical sheets associated with the s-channel. He gives the simplest example, the « anomalous threshold » diagram of Fig. 7. When P is stable this gives a singularity on the real axis in the s-plane. As P becomes unstable, so that it is reached through a normal threshold cut in the t-channel, this singularity passes through the cut in the s-plane associated with the same diagram for which P is replaced by $2A$. It is clear from the previous analysis that for any diagram containing P-lines the corresponding singularities move, when P becomes unstable, through the cuts associated whith the diagrams for which P is replaced by $2A$.

* * *

I am much indebted to Mr J. CHALLIFOUR, Dr. G. F. CHEW, Dr. R. J. EDEN, Dr. J. GOLDSTONE, Mr. D. I. OLIVE and Dr J. C. POLKINGHORNE for vital discussions. My interest in this work arose out of a seminar by Mr. J. GUNSON.

(13) D. B. FAIRLIE, P. V. LANDSHOFF, J. NUTTALL and J. C. POLKINGHORNE: *Journ. Math. Phys.*, 3, 594 (1962).
(14) J. CHALLIFOUR: private communication.

RIASSUNTO (*)

Si esaminano le conseguenze dell'ipotesi che le ampiezze di transizione presentino qualche analiticità nelle costanti d'accoppiamento. Questo ci permette di dimostrare l'esistenza di singolarità associate alle particelle instabili e virtuali, simili a quelle richieste per le particelle stabili dall'unitarietà. È semplice determinare su quale dei foglietti di Riemann giacciono le singolarità delle particelle instabili e virtuali e si può anche dimostrare che le singolarità delle particelle stabili sono assenti da certi foglietti.

(*) *Traduzione a cura della Redazione.*

1677