I. DIMER-DIMER SCATTERING

A. Resonating-group description

The two-fragment resonating-group equation

$$\int \left\{ \phi_A^* \phi_B^* \left(-\frac{\hbar^2}{2\mu} \mathbf{\Delta}_R - E + \mathcal{V}_{AB} \right) \mathcal{A}_{AB} \left[\phi_A \phi_B \chi(\boldsymbol{R}) \right] \right\} d\boldsymbol{r}_{A,B}^{\text{internal}} = 0 \tag{1}$$

embodies the assumption of a rapid internal motion relative to the slow relative motion between clusters A and B. The product $[\phi_A \phi_B \chi(\boldsymbol{R})]$ represents the asymptotic state whose evolution is dictated by $G_0(z) = \left[\hat{T}_0 + \hat{V}_A + \hat{V}_B - z \right]^{-1}$ whenever the two fragments A and B are sufficiently far apart such that only particles within each of them interact with $\hat{V}_{A,B}$ while the inter-fragment potential $\hat{V}_{AB}$ is zero. We assume that we can attain an approximation of the fragment-internal wave functions $\phi_{A,B}$ for all "reasonable" interactions and desired accuracies with a Gaussian basis:

$$\phi_A := \sum_n^{N_f} c_n \cdot e^{-\alpha_n \sum_{i=1}^{N_A} (\boldsymbol{r}_i - \boldsymbol{R}_A)^2} \quad ; \begin{array}{ll} N_f & : \text{basis dimension} \\ N_A & : \text{number of particles in fragment } A \\ \boldsymbol{r}_i & : \text{single-particle coordinates} \\ \boldsymbol{R}_A & = \left(\sum_i^{N_A} m_i \right)^{-1} \sum_i^{N_A} m_i \, \boldsymbol{r}_i \end{array} \quad . \tag{2}$$

If the two fragments contain identical particle species, antisymmetrization is required between these, *i.e.* , the inter-cluster antisymmetrizer comprises elements $\mathfrak{P}$ of the symmetric group of the particle labels in cluster A and B ($\mathfrak{S}_{AB}$),

$$\mathcal{A}_{AB} := \mathbb{1} + \sum_{\mathfrak{p} \in \mathfrak{S}_{AB}} (-1)^{\mathfrak{p}} \, \mathfrak{P} \quad . \tag{3}$$

The interaction, in contrast, is effective only if it involves particles in different clusters, which are **not** of the same species. Here, the focus is on zero-range interactions between two and three particles. The latter are one way to avoid a collapse of the three-body system which would occur with two-body zero-range potentials, only.

$$\mathcal{V}_{AB} = C(\lambda) \sum_{\substack{i \in A \\ j \in B}} \delta_\lambda^{(3)}(\boldsymbol{r}_i - \boldsymbol{r}_j) + D(\lambda) \sum_{\substack{i,j,k \\ i \in A \Rightarrow j \vee k \in B}} \delta_\lambda^{(3)}(\boldsymbol{r}_i - \boldsymbol{r}_j) \, \delta_\lambda^{(3)}(\boldsymbol{r}_i - \boldsymbol{r}_k) \quad . \tag{4}$$

The subscripted contacts indicate a specific regularization, namely, for one Cartesian dimension

$$\delta^{(1)}(x) = \lim_{\lambda \to \infty} \sqrt{\frac{\lambda}{\pi}} \cdot e^{-\lambda x^2} \tag{5}$$

The nucleon-nucleon interaction assumes this form in the leading-order of the EFT($\not\pi$) , where the two coupling strengths are renormalized, *e.g.* , to a dimer and a trimer binding energy, respectively. Thereby, the parameter α in (**??**) is correlated with a physical observable and its λ-regulator dependence should, even if the fragment's wave function is not subjected directly to a renormalization constraint, vanish for $\lim_{\lambda \to \infty}$. We assume from here-on, that all considered fragments a within the range of the applicability of the microscopic theory. In case of EFT($\not\pi$) , *e.g.* , we shall assume the existence of a functional correlation between $\alpha_{^2H,^3He,^4He} = f_{^2H,^3H,^4He}(a_{\text{ff}}, B(^3H))$. Analytic forms of these functions f remain obscure, and we rely on reasonable estimates, *e.g.* , for $\alpha_{^2H}$ the dimer S-wave function can be derived in closed form

$$\phi(r) = \frac{e^{-\frac{r}{a_{\text{ff}}}}}{\sqrt{2\pi a_{\text{ff}}} r} \quad . \tag{6}$$

Foreseeing the importance of the spacial extend of the fragment's state in light of the non-local character of the effective interaction, demanding that the average radius ensuing from this form equals that of the ansatz (**??**), which yields:

$$\alpha_{^2H} = \frac{3}{2} \, a_{\text{ff}}^{-2} \tag{7}$$

where the fermion-fermion scattering length $a_{\rm ff}$ is a converging function of the regulator λ if one enforces a λ independent fermion-fermion binding energy to renormalize the interaction, *i.e.* , $C(\lambda)$. To refine this choice, we adjust this value once to obtain a known dimer-dimer results which was obtained in a microscopic four-body calculation, *e.g.* , to obtain for $\lambda \to \infty$ the result in (9) and subsequently scale the α's of other systems with the size of those relative to the dimer.

The parameter representation $\chi(\boldsymbol{R}) = \int d\boldsymbol{R}'\delta^{(3)}(\boldsymbol{R}-\boldsymbol{R}')\chi(\boldsymbol{R}')$ allows for a translation of the inter-cluster antisymmetrizer $\mathcal{A}_{AB}$ into a non-local integro-differential equation which, in general, assumes the form

$$(\hat{T}-E)\ \chi(\boldsymbol{r}) + \mathcal{V}^{(1)}(\boldsymbol{r})\ \chi(\boldsymbol{r}) + \int d^{(3)}\boldsymbol{r}'\ \mathcal{V}^{(2)}(\boldsymbol{r},\boldsymbol{r}',E)\ \chi(\boldsymbol{r}') = 0 \tag{8}$$

with the radial coordinates denoting the spatial separation between the two fragment's mass centres. If these fragments are bound states of the spectrum of the single-particle interaction – here taken as in (4) – and if the energy of the relative motion between these fragments E is small relative to their binding **and** excitation energies, it is in order to attach physical meaning to the equation of motion which follows after the averaging over internal degrees of freedom in (1). That being, an approximation of the scattering characteristics of the two cluster.

The effective potentials which feed into this equation are then parametrized by the underlying, microscopic interaction strengths, *e.g.* , for EFT($\not\pi$) LO, $C(\lambda)$ and $D(\lambda)$. Consequently, the extent to which such a single-particle theory is able to describe many-body features, *e.g.* , four neutrons assuming a bound di-neutron, 5-body resonances, 6-helium halos, or 16-oxygen, is then assessed by the dependence of the respective many-body amplitudes on the short-distance regulator scale λ.

We commence with the four-body system, dimer-dimer scattering therein. There are two scenarios which embody the transition between Bose and Fermi dynamics, namely a system of two dimers, each one composed of the same two species of a fermion. We denote this system as (ab):(ab) – a hypothetical dineutron, for example. Such a system is by all we know scale invariant, which means that its low-energy behaviour is independent of λ, and does not require the additional three-body renormalization constraint manifest in $D(\lambda)$. This arrangement has been studied very thoroughly and indeed, the remarkable universality of the ratio between dimer-dimer and fermion-fermion attraction – measured by scattering lengths $a > 0$ – has been numerically discovered [**?** **?**] and confirmed in numerous approaches [**? ? ? ?**]:

$$\frac{a_{\rm dd}}{a_{\rm ff}} \approx 0.6\ \ . \tag{9}$$

If a third species is present in one of the clusters, the dynamics change fundamentally and lead to a collapse of the three-body state comprised of these three different equal-mass, two-body-contact-interacting particles. Universal scale invariance reduces to a discrete scale invariance. Expressed less abstractly, parts of the three-body spectrum show cyclic behaviour instead of convergence. We denote this composition as $(ab)\therefore(ca)$. The dimer-dimer potential of such a four-body system inherits the characteristic three-body scale through its dependence on $D(\lambda)$. This four-body amplitude is then correlated to one two-body and one three-body datum, while (ab):(ab), in the absence of such a three-body scale in the zero-range two-body limit, depends on a two-body observable, only.

With the above formulas, it is arguably painful, but straight forward to arrive at the specific forms for the effective potentials as given below.

(ab):(ab)**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 2\,C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{2\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} \quad , \tag{10}$$

61

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r},\boldsymbol{r}',E) = \; 8\;\alpha^{\frac{3}{2}} \cdot e^{-\alpha\boldsymbol{r}'^2} \cdot \left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2} - 2\;C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\alpha\cdot\frac{2\alpha+3\lambda}{2\alpha+\lambda}\boldsymbol{r}^2}\right] \;. \tag{11}$$

(ab)**∴**(ca)**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = 3\cdot C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{2\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} \tag{12}$$

$$+D(\lambda)\left(\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3} \cdot e^{-\frac{4\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} + \left(\frac{4\alpha^2}{4\alpha^2+6\alpha\lambda+\lambda^2}\right)^{3/2} \cdot e^{-\frac{4\alpha\lambda(\alpha+\lambda)}{4\alpha^2+6\alpha\lambda+\lambda^2}\boldsymbol{r}^2}\right) \tag{13}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r},\boldsymbol{r}',E) = \tag{14}$$

$$8\;\alpha^{\frac{3}{2}} \cdot \Bigg\{\; e^{-\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)} \cdot \left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha) + E\right] \tag{15}$$

$$-\,C(\lambda)\cdot\left(e^{-(\alpha+\lambda)(\boldsymbol{r}^2+\boldsymbol{r}'^2)-2\lambda\boldsymbol{r}'\cdot\boldsymbol{r}} + 2\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\alpha\cdot\left(\frac{2\alpha+3\lambda}{2\alpha+\lambda}\boldsymbol{r}^2+\boldsymbol{r}'^2\right)}\right) \tag{16}$$

$$-\;D(\lambda)\left(\left(\frac{\alpha}{\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{2\alpha^2+4\alpha\lambda+\lambda^2}{2(\alpha+\lambda)}(\boldsymbol{r}^2+\boldsymbol{r}'^2)-\frac{\lambda^2}{\alpha+\lambda}\boldsymbol{r}\cdot\boldsymbol{r}'} + \left(\frac{2\alpha}{2\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{2\alpha^2+5\alpha\lambda+\lambda^2}{2\alpha+\lambda}\boldsymbol{r}^2-(\alpha+\lambda)\boldsymbol{r}'^2-2\lambda\boldsymbol{r}\cdot\boldsymbol{r}'}\right)\Bigg\} \tag{17}$$

(abc)**∴**(a)**:**

$$\mathcal{V}^{(1)}_{(abc)\therefore(a)}(\boldsymbol{r}) = 2\cdot C(\lambda)\left(\frac{3\alpha}{3\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{3\alpha\lambda}{3\alpha+\lambda}\boldsymbol{r}^2} \tag{18}$$

$$+8\cdot D(\lambda)\left(\left(\frac{3\alpha^2}{12\alpha^2+16\alpha\lambda+\lambda^2}\right)^{3/2} \cdot e^{-\frac{12\alpha\lambda(\alpha+\lambda)}{12\alpha^2+16\alpha\lambda+\lambda^2}\boldsymbol{r}^2}\right) \tag{19}$$

$$\mathcal{V}^{(2)}_{(abc)\therefore(a)}(\boldsymbol{r},\boldsymbol{r}',E) = \tag{20}$$

$$\alpha^{\frac{3}{2}}\;3^3\;\left(\frac{3}{2}\right)^{3/2} \cdot \Bigg\{\; e^{-\frac{3}{16}\alpha(5\boldsymbol{r}^2+6\boldsymbol{r}\cdot\boldsymbol{r}'+5\boldsymbol{r}'^2)} \cdot \frac{1}{8}\left[\frac{\hbar^2}{2\mu}\;\frac{1}{2^6}\left(3\cdot(5\alpha\boldsymbol{r}+3\alpha\boldsymbol{r}')^2 - 40\,\alpha\right) + E\right] \tag{21}$$

$$-\,2\;C(\lambda)\cdot\left(\frac{\alpha}{4\alpha+\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{3}{4}\,\frac{\alpha}{4\alpha+\lambda}\cdot\left((5\alpha+8\lambda)\boldsymbol{r}^2+6(\alpha+\lambda)\boldsymbol{r}\cdot\boldsymbol{r}'+(5\alpha+2\lambda)\boldsymbol{r}'^2\right)} \tag{22}$$

$$-\,D(\lambda)\cdot\left(\frac{\alpha}{4\alpha+5\lambda}\right)^{\frac{3}{2}} \cdot e^{-\frac{3}{4}\,\frac{1}{16\alpha+20\lambda}\cdot\left((20\alpha^2+52\alpha\lambda+27\lambda^2)\boldsymbol{r}^2+6(4\alpha^2+8\alpha\lambda+3\lambda^2)\boldsymbol{r}\cdot\boldsymbol{r}'+(20\alpha^2+28\alpha\lambda+3\lambda^2)\boldsymbol{r}'^2\right)}\Bigg\} \tag{23}$$

B. Solution of the resonating-group equation

Above, any low-energy two-fragment observable was found encoded in the solution of (8), which assumes the generic form

$$\sum_{n=1}^{N_{\text{loc}}} \hat{\eta}_n \; e^{-w_n \boldsymbol{R}^2} \chi(\boldsymbol{R}) - \sum_{n=1}^{N_{\text{n-loc}}} \int \left\{ \hat{\zeta}_n \, e^{-a_n \boldsymbol{R}^2 - b_n \boldsymbol{R}\cdot\boldsymbol{R}' - c_n \boldsymbol{R}'^2} \right\} \chi(\boldsymbol{R}') d\boldsymbol{R}' = 0 \tag{24}$$

with $\;\hat{\eta}_n, \hat{\zeta}_n, w_n, a_n, b_n, c_n\;$ dependent upon $\;C(\lambda), D(\lambda), \alpha, \lambda, E, A, B$.

We identify $n = 1$ with the quantities which result in the course of the antisymmetrization and integration from the kinetic part of (1), and therefore have

$$\hat{\eta}_1 = \left(-\frac{\hbar^2}{2\mu} \boldsymbol{\Delta}_R - E \right) \quad , \quad w_1 = 0 \quad , \quad \hat{\zeta}_1 = \left(-\frac{\hbar^2}{2\mu} \boldsymbol{\Delta}_R - E \right) \zeta_1 \quad . \tag{25}$$

Acting with the derivative on the integral kernel produces the structures as listed for the (ab):(ab) and (ab)∴(ca) systems in equations (10) to (23).

a. In partial waves

$$\text{Expanding} \;\; \chi(\boldsymbol{R}) = R^{-1} \sum_{lm} \phi_{lm}(R) Y_{lm}(\hat{\boldsymbol{R}}) \;\; \text{and projecting from the left with} \;\; R \int d^2\hat{\boldsymbol{R}} \; Y^*_{lm}(\hat{\boldsymbol{R}}) \tag{26}$$

before substituting

$$e^{-b\boldsymbol{R}\cdot\boldsymbol{R}'} = 4\pi \sum_{LM} i^L j_L(ibRR') Y^*_{LM}(\hat{\boldsymbol{R}}) Y_{LM}(\hat{\boldsymbol{R}}') \;\; ; \;\; \boldsymbol{R}\cdot\boldsymbol{R}' = -\sqrt{3} \left[\boldsymbol{R}_p \otimes \boldsymbol{R}'_{-p} \right]^{00} = \frac{4\pi}{3} RR' \sum_p (-)^p \, Y_{1p} Y'_{1-p}$$

$$\text{and} \quad \boldsymbol{r}_m = \sqrt{\frac{4\pi}{3}} r Y_{1,m}(\hat{\boldsymbol{r}}) \tag{27}$$

yields for

partial wave l:

$$0 \;=\; \left(\frac{\hbar^2}{2\mu} \left[-\partial_R^2 + \frac{l(l+1)}{R^2} \right] - E \right) \phi_{lm}(R) + \sum_{n=2}^{N_{\text{loc}}} \eta_n \; e^{-w_n R^2} \phi_{lm}(R) \tag{28a}$$

$$- \int dR' \; \phi_{lm}(R') \; (4\pi RR') \left[\zeta_1 \cdot e^{-a_1 R^2 - c_1 R'^2} \cdot \left\{ \left[-(4a_1^2 R^2 + b_1^2 R'^2 - 2a_1) + \frac{l(l+1)}{R^2} - \frac{2\mu}{\hbar^2} E \right] i^l j_l(ib_1 RR') \right.\right.$$

$$\left.\left. - \, (4a_1 b_1) \cdot RR' \cdot \sum_L i^L j_L(ib_1 RR') \; \hat{L} \; \hat{l} \; \begin{pmatrix} L & 1 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \; \triangle_{L1l} \right\} + \sum_{n=2}^{N_{\text{n-loc}}} \zeta_n \, i^l j_l(ib_n RR') \cdot e^{-a_n R^2 - c_n R'^2} \right] \tag{28b}$$

with a 3-j symbol $\neq 0 \Leftrightarrow L + 1 + l =$ even, and $\triangle_{j_1 j_2 j_3} \neq 0 \Leftrightarrow j_3 = |j_1 - j_2|, \ldots, j_1 + j_2$.

S-wave $l = 0$:

$$0 \;=\; \left(-\frac{\hbar^2}{2\mu} \partial_R^2 - E \right) \phi_S(R) + \sum_{n=2}^{N_{\text{loc}}} \eta_n \; e^{-w_n R^2} \phi_S(R) \tag{29a}$$

$$- \int dR' \phi_S(R') (4\pi RR') \left[\sum_{n=2}^{N_{\text{n-loc}}} \zeta_n \cdot j_0(ib_n RR') \cdot e^{-a_n R^2 - c_n R'^2} \right. \tag{29b}$$

$$+ \, \zeta_1 \cdot e^{-a_1 R^2 - c_1 R'^2} \cdot \left\{ \left[-(4a_1^2 R^2 + b_1^2 R'^2 - 2a_1) - \frac{2\mu}{\hbar^2} E \right] j_0(ib_1 RR') - 4a_1 b_1 \cdot RR' \cdot \frac{i}{\sqrt{3}} \; j_1(ib_1 RR') \right\}$$

P-wave $l = 1$:

$$0 = \left(\frac{\hbar^2}{2\mu}\left[-\partial_R^2 + \frac{2}{R^2}\right] - E\right)\phi_{1m}(R) + \sum_{n=2}^{N_{\rm loc}} \eta_n \; e^{-w_n R^2}\phi_{1m}(R) \tag{30a}$$

$$- \int dR' \phi_{1m}(R')(4\pi RR') \left[\; \sum_{n=2}^{N_{\text{n-loc}}} \zeta_n \cdot i \cdot j_1(ib_n RR') \cdot e^{-a_n R^2 - c_n R'^2} \right. \tag{30b}$$

$$+ \zeta_1 \cdot e^{-a_1 R^2 - c_1 R'^2} \cdot \left\{ \left[-(4a_1^2 R^2 + b_1^2 R'^2 - 2a_1) + \frac{2}{R^2} - \frac{2\mu}{\hbar^2} E \right] i \cdot j_1(ib_1 RR') \right.$$

$$\left. \left. - 4a_1 b_1 \cdot RR' \cdot \left(\frac{1}{\sqrt{3}} \; j_0(ib_1 RR') - \frac{2}{\sqrt{15}} \; j_2(ib_1 RR') \right) \right\} \right] \tag{30c}$$

C. dimer-dimer

For the (ab):(ab) and $(ab)\therefore(ca)$ systems, we already derived all coefficients (see (10) to (23)) and simplify (29) with $b_1 = 0$ to obtain the S-wave component

$$0 = \left(-\frac{\hbar^2}{2\mu}\partial_R^2 - E\right)\phi_{lm}(R) + \sum_{n=2}^{N_{\text{loc}}} \eta_n \; e^{-w_n R^2}\phi_{lm}(R) \tag{31a}$$

$$+\int (4\pi R R') \left\{\zeta_1 \cdot e^{-a_1 R^2 - c_1 R'^2}\left[(4a_1^2 R^2 - 2a_1) + \frac{2\mu}{\hbar^2}E\right] - \sum_{n=2}^{N_{\text{n-loc}}} \zeta_n \, j_0(i b_n R R') \cdot e^{-a_n R^2 - c_n R'^2}\right\} \phi_{lm}(R')\, dR' \; . \tag{31b}$$

The coefficients to be used with this equation are listed in table I.

		$\boxed{(ab):(a)}$		
n	η	w		
2	$8\,C(\lambda)\left(\frac{\alpha}{4\alpha+\lambda}\right)^{3/2}$	$\frac{4\alpha\lambda}{4\alpha+\lambda}$		
	ζ	a	b	c
1	$\frac{64}{27}\;(2\,\alpha)^{3/2}\left(\frac{\hbar^2}{2\mu}\right)$	$\frac{10}{9}\alpha$	$\frac{16}{9}\alpha$	$\frac{10}{9}\alpha$
2	$-\frac{64}{27}\;(2\,\alpha)^{3/2}\;C(\lambda)$	$\frac{2}{9}(5\alpha+8\lambda)$	$\frac{16}{9}(\alpha+\lambda)$	$\frac{2}{9}(5\alpha+2\lambda)$
		$\boxed{(abc)\therefore(a)}$		
n	η	w		
2	$2\,C(\lambda)\left(\frac{3\alpha}{3\alpha+\lambda}\right)^{3/2}$	$\frac{3\alpha\lambda}{3\alpha+\lambda}$		
3	$8\,D(\lambda)\left(\frac{3\alpha^2}{12\alpha^2+16\alpha\lambda+\lambda^2}\right)^{3/2}$	$\frac{12\alpha\lambda(\alpha+\lambda)}{12\alpha^2+16\alpha\lambda+\lambda^2}$		
	ζ	a	b	c
1	$\frac{3^4}{2^4}\;\sqrt{\frac{3}{2}}\cdot\alpha^{3/2}\left(\frac{\hbar^2}{2\mu}\right)$	$\frac{15}{16}\alpha$	$\frac{18}{16}\alpha$	$\frac{15}{16}\alpha$
2	$-3^4\;\sqrt{\frac{3}{2}}\cdot\left(\frac{\alpha}{4\alpha+\lambda}\right)^{3/2}\cdot\alpha^{3/2}\;C(\lambda)$	$\frac{3\alpha(5\alpha+8\lambda)}{4(4\alpha+\lambda)}$	$\frac{9\alpha(\alpha+\lambda)}{2(4\alpha+\lambda)}$	$\frac{3\alpha(5\alpha+2\lambda)}{4(4\alpha+\lambda)}$
3	$-\frac{3^4}{2}\;\sqrt{\frac{3}{2}}\cdot\left(\frac{\alpha}{4\alpha+5\lambda}\right)^{3/2}\cdot\alpha^{3/2}\;D(\lambda)$	$\frac{3(20\alpha^2+52\alpha\lambda+27\lambda^2)}{4(16\alpha+20\lambda)}$	$\frac{9(4\alpha^2+8\alpha\lambda+3\lambda^2)}{2(16\alpha+20\lambda)}$	$\frac{3(20\alpha^2+28\alpha\lambda+3\lambda^2)}{4(16\alpha+20\lambda)}$
		$\boxed{(abcd)\therefore(a)}$		
n	η	w		
2	$24\,C(\lambda)\left(\frac{2\alpha}{8\alpha+3\lambda}\right)^{3/2}$	$\frac{8\alpha\lambda}{8\alpha+3\lambda}$		
3	$24\,D(\lambda)\left(\frac{2\alpha^2}{8\alpha^2+11\alpha\lambda+\lambda^2}\right)^{3/2}$	$\frac{8\alpha\lambda(\alpha+\lambda)}{8\alpha^2+11\alpha\lambda+\lambda^2}$		
	ζ	a	b	c
1	$\frac{2^{12}}{5^4}\;\sqrt{\frac{1}{3}}\cdot\alpha^{3/2}\left(\frac{\hbar^2}{2\mu}\right)$	$\frac{68}{75}\alpha$	$\frac{64}{75}\alpha$	$\frac{68}{75}\alpha$
2	$-3\cdot\frac{2^{12}}{5^3}\cdot\left(\frac{\alpha}{3\alpha+\lambda}\right)^{3/2}\cdot\alpha^{3/2}\;C(\lambda)$	$\frac{4\alpha(17\alpha+27\lambda)}{25(3\alpha+\lambda)}$	$\frac{64\alpha(\alpha+\lambda)}{25(3\alpha+\lambda)}$	$\frac{4\alpha(17\alpha+7\lambda)}{4(4\alpha+\lambda)}$
3	$-3\cdot\frac{2^{15}}{5^3}\cdot\left(\frac{\alpha^2}{12\alpha+16\alpha\lambda+\lambda^2}\right)^{3/2}\cdot\alpha^{3/2}\;D(\lambda)$	$\frac{4\alpha(68\alpha^2+176\alpha\lambda+91\lambda^2)}{25(12\alpha^2+16\alpha\lambda+\lambda^2)}$	$\frac{4\alpha(64\alpha^2+128\alpha\lambda+48\lambda^2)}{25(12\alpha^2+16\alpha\lambda+\lambda^2)}$	$\frac{4\alpha(68\alpha^2+96\alpha\lambda+11\lambda^2)}{25(12\alpha^2+16\alpha\lambda+\lambda^2)}$
		$\boxed{(ab):(ab)}$		
n	η	w		
2	$2\,C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3/2}$	$\frac{2\alpha\lambda}{2\alpha+\lambda}$		
	ζ	a	b	c
1	$8\;\alpha^{3/2}\left(\frac{\hbar^2}{2\mu}\right)$	α	0	α
2	$-16\;\alpha^{3/2}\,C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3/2}$	$\alpha\frac{2\alpha+3\lambda}{2\alpha+\lambda}$	0	α

		$\boxed{(ab)\therefore(ca)}$		
n	η	w		
2	$3\,C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3/2}$	$\frac{2\alpha\lambda}{2\alpha+\lambda}$		
3	$D(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3}$	$\frac{4\alpha\lambda}{2\alpha+\lambda}$		
4	$D(\lambda)\left(\frac{2\alpha}{\sqrt{(2\alpha+\lambda)^2+2\alpha\lambda}}\right)^{3}$	$\frac{4\alpha\lambda(\alpha+\lambda)}{4\alpha^2+6\alpha\lambda+\lambda^2}$		
	ζ	a	b	c
1	$8\ \alpha^{3/2}\left(\frac{\hbar^2}{2\mu}\right)$	α	0	α
2	$-8\ \alpha^{3/2}C(\lambda)$	$\alpha+\lambda$	2λ	$\alpha+\lambda$
3	$-16\ \alpha^{3/2}\,C(\lambda)\left(\frac{2\alpha}{2\alpha+\lambda}\right)^{3/2}$	$\alpha\frac{2\alpha+3\lambda}{2\alpha+\lambda}$	0	α
4	$-8\ \alpha^{3/2}D(\lambda)\left(\frac{\alpha}{\alpha+\lambda}\right)^{3/2}$	$\frac{2\alpha^2+4\alpha\lambda+\lambda^2}{2(\alpha+\lambda)}$	$\frac{\lambda^2}{\alpha+\lambda}$	$\frac{2\alpha^2+4\alpha\lambda+\lambda^2}{2(\alpha+\lambda)}$
5	$-8\ \alpha^{3/2}D(\lambda)\left(\frac{2\alpha(\alpha+\lambda)}{2\alpha^2+3\alpha\lambda+\lambda^2}\right)^{3/2}$	$\frac{2\alpha^2+5\alpha\lambda+\lambda^2}{2\alpha+\lambda}$	2λ	$\alpha+\lambda$

TABLE I: Defining coefficients for the local and non-local effective RGM interaction to be used in (29) to (31b) for the scale-invariant $(ab){:}(ab)$ and $(ab){:}(a)$, the discrete-scale-invariant $(ab)\therefore(ca)$ dimer-dimer, and the $(abc)\therefore(a)$ trimer-atom configurations. Gaussian widths w, a, b, c have units length^{-2}, the coupling strengths η (local) and ζ_1 (exchange kernel) scale as energies, and the non-local $\zeta_{n\geq 2}$ as an energy density, energy$\cdot$length^{-3}.

D. trimer-atom

The non-local potential of this system is derived to obtain characteristics of elastic scattering of a fermion off a trimer, where the latter contains a fermion identical to the projectile. With the 2- and 3-fermion coupling strengths renormalized to the deuteron and triton binding energies respectively, and all particle masses set to the average nucleon mass, results are parametrized by the regulator λ, only, as the core size α is a dependent quantity albeit its analytical form $\alpha(\lambda, B(2), B(3), m_N)$ is unknown.

E. Limiting cases and renormalizability

It is in order to consider the following limits:

zero-range or contact limit: $\lambda \gg \alpha$

local approximation: $\int d^{(3)}\boldsymbol{r}'\ \mathcal{V}^{(2)}(\boldsymbol{r},\boldsymbol{r}',E)\ \chi(\boldsymbol{r}') \overset{E\to 0}{\approx} \chi(\boldsymbol{r})\cdot v^{(2)}(\boldsymbol{r})\cdot\int d^{(3)}\boldsymbol{r}'\ v^{(2)}(\boldsymbol{r}')$.

Assuming an unnaturally large dimer scale emergent from a relatively short-ranged fermion-fermion interaction, the zero-range approximation is justified and the ensuing dimer-dimer potentials read:

(zero-range) $(ab){:}(ab)$**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 2\,(2\alpha)^{\frac{3}{2}}\ \frac{C(\lambda)}{\lambda^{\frac{3}{2}}}\cdot e^{-2\alpha\boldsymbol{r}^2}\quad , \tag{32}$$

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r},\boldsymbol{r}',E) =\ 8\ \alpha^{\frac{3}{2}}\cdot e^{-\alpha\boldsymbol{r}'^2}\cdot\left[\frac{\hbar^2}{2\mu}\ (4\alpha^2\boldsymbol{r}^2-2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2}+E\cdot e^{-\alpha\boldsymbol{r}^2}-2\,(2\alpha)^{\frac{3}{2}}\ \frac{C(\lambda)}{\lambda^{\frac{3}{2}}}\cdot e^{-3\alpha\boldsymbol{r}^2}\right]\quad . \tag{33}$$

(zero-range) $(ab)\therefore(ca)$**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = 3\,(2\alpha)^{\frac{3}{2}}\ \frac{C(\lambda)}{\lambda^{\frac{3}{2}}}\cdot e^{-2\alpha\boldsymbol{r}^2} + 2\,(2\alpha)^3\ \frac{D(\lambda)}{\lambda^3}\cdot e^{-4\alpha\boldsymbol{r}^2} \tag{34}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r},\boldsymbol{r}',E) = 8\ \alpha^{\frac{3}{2}}\cdot\Bigg(e^{-\alpha\boldsymbol{r}'^2}\cdot\left[\frac{\hbar^2}{2\mu}\ (4\alpha^2\boldsymbol{r}^2-2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2}\right] \tag{35}$$

$$-C(\lambda)\cdot e^{-\lambda(\boldsymbol{r}+\boldsymbol{r}')^2} - 2\ (2\alpha)^{\frac{3}{2}}\ \frac{C(\lambda)}{\lambda^{\frac{3}{2}}}\cdot e^{-\alpha\boldsymbol{r}'^2-3\alpha\boldsymbol{r}^2} \tag{36}$$

$$-\alpha^{\frac{3}{2}}(1+2^{\frac{3}{2}})\ \frac{D(\lambda)}{\lambda^{\frac{3}{2}}}\cdot e^{-\frac{\lambda}{2}(\boldsymbol{r}+\boldsymbol{r}')^2}\Bigg) \tag{37}$$

(zero-range) $(abcd)\therefore(abcd)$**:**

The zero-range approximation yields also a form of the interaction between two fragments with each composed of four four-component fermions with equal mass. This pertains to the effective interaction between two α nuclei.

Compared with the dimer-dimer interaction, the local part:

$$\mathcal{V}^{(1)}_{(abcd)\therefore(abcd)}(\boldsymbol{r}) = \frac{C(\lambda)}{\lambda^{\frac{3}{2}}}\cdot\frac{32}{\sqrt{3}}\,\alpha^{\frac{3}{2}}\cdot e^{-\frac{4}{3}\alpha\boldsymbol{r}^2} + \frac{D(\lambda)}{\lambda^3}\cdot 48\sqrt{2}\,\alpha^3\cdot e^{-2\alpha\boldsymbol{r}^2}\quad, \tag{38}$$

comprises **stronger** but **shorter-ranged** interactions.

The non-local component assumes the following form:

$$\mathcal{V}^{(2)}_{(abcd)\therefore(abcd)}(\boldsymbol{r},\boldsymbol{r}',E) = \quad -48\sqrt{2}\ \alpha^{3/2}\cdot\Bigg[$$

$$E\cdot\left\{e^{-2\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)} - 3^{-3/2}\frac{10}{3}e^{-\frac{10}{3}\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)}\cosh[\frac{16}{3}\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}']\right\} \tag{39}$$

$$-\frac{\hbar^2}{2\mu}\cdot\Bigg\{e^{-2\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)}((4\alpha\boldsymbol{r})^2-4\alpha) \tag{40}$$

$$+\frac{4}{3^{7/2}}\frac{10}{3}\cdot e^{-\frac{10}{3}\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)}\cosh[\frac{16}{3}\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}']\left(\left((10\alpha\boldsymbol{r})^2+(8\alpha\boldsymbol{r}')^2-15\alpha\right)-160\ \boldsymbol{r}\cdot\boldsymbol{r}'\tanh[\frac{16}{3}\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}']\right)\Bigg\}$$

$$+\frac{C(\lambda)}{\lambda^{3/2}}\cdot\frac{16}{\sqrt{2}}\alpha^{3/2}\cdot\left\{e^{-4\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)}\cosh[4\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}'] + 2e^{-2\alpha(3\boldsymbol{r}^2+2\boldsymbol{r}'^2)}\cosh[8\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}'] - \frac{4}{3^{3/2}}e^{-\frac{3}{2}\alpha(5\boldsymbol{r}^2+3\boldsymbol{r}'^2)}\right\} \tag{41}$$

$$+\frac{D(\lambda)}{\lambda^3}\cdot 128\alpha^3\cdot\Bigg\{e^{-6\alpha(\boldsymbol{r}^2+\boldsymbol{r}'^2)}\cosh[8\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}'] + \frac{4}{5^{3/2}}\cosh[\frac{48}{5}\alpha\ \boldsymbol{r}\cdot\boldsymbol{r}']\left[e^{-\frac{2}{5}\alpha(19\boldsymbol{r}^2+11\boldsymbol{r}'^2)} - e^{-\frac{2}{5}\alpha(11\boldsymbol{r}^2+11\boldsymbol{r}'^2)}\right]$$

$$-2^{-5/2}e^{-2\alpha(2\boldsymbol{r}^2+\boldsymbol{r}'^2)}\Bigg\} \tag{42}$$

$$\Bigg]\ . \tag{43}$$

This structure is parity-even, and the signs of terms as they relate to one-body ($\propto E,\mu^{-1}$) and two/three-body ($C(\lambda)/D(\lambda)$) interactions are identical to the ones of the dimer-dimer potential, *i.e.* , the non-local form switches the character from attractive to repulsive and *vice versa*!

We do now interpret these potentials as vertices of interacting dimer fields – the physical nature of the fields is inessential for the following; quite generally, we applied a transformation on a renormalized contact interaction, and we are now interested in whether or not this transformation, *i.e.* , the RGM averaging over fragment-internal, "frozen" degrees of freedom, preserves the renormalized character of amplitudes of the image theory – whose regularization is inherited from the renormalized fermion-fermion interaction.

We commence the analysis of the renormalizability of the transformed dimer-dimer theory under the assumption that the transformation does not affect the power-counting rules. That means, solutions of a Schrödinger equation with and interaction as given by the non-local potentials shall be well-behaved for $\lambda \to \infty$. Renormalizing the fermion-fermion amplitude yields

$$C(\lambda) \propto \lambda \tag{44}$$

and arguably two scenarios for the three-body parameter:

$$\text{Three-body spectrum with one single shallow bound state } \neq f(\lambda): \; D(\lambda) \propto e^{\omega\lambda} \tag{45}$$

$$\text{Three-body spectrum with a tower of Efimov-type states with the shallowest } \neq f(\lambda): \; \text{countably infinite poles} \tag{46}$$

Revisit the effective potentials, considering $\lambda \to \infty$:

(zero-range) (ab):(ab)**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 0 \tag{47}$$

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r},\boldsymbol{r}',E) = \; 8\;\alpha^{\frac{3}{2}} \cdot e^{-\alpha\boldsymbol{r}'^2} \cdot \left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2}\right] \quad . \tag{48}$$

In words, the sub-threshold dimer-dimer amplitude depends on the microscopic interaction only through the character of a single dimer as parametrized with α. Although, no analytic form of the functional relation $\alpha = f(\aleph)$ is known, its existence implies a universal low-energy dimer-dimer system, thereby conforming with the "Petrov ratio" $a_{\rm dd}/a_{\rm ff} \approx 0.6$.

(zero-range) $(ab)\therefore(ca)$**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = c_1\mathtt{P}[\lambda]\cdot e^{-4\alpha\boldsymbol{r}^2} \tag{49}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r},\boldsymbol{r}',E) = 8\;\alpha^{\frac{3}{2}}\cdot\left(e^{-\alpha\boldsymbol{r}'^2}\cdot\left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2-2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2}\right] - c_2\mathtt{P}[\lambda]\cdot e^{-\frac{\lambda}{2}(\boldsymbol{r}+\boldsymbol{r}')^2}\right) \tag{50}$$

`ECCE` this structure, which I want to discuss/have an opinion on.

The specific form of the polynomials depends on the implemented three-body renormalization condition, and so do the values of the non-equal constants $c_{1,2}$. Yet, regardless of the specific shape, the induced λ dependence will translate into dimer-dimer observables which consequently do not have a well defined $\lim\limits_{\lambda\to\infty}$!

II. SUMMARY OF PHYSICAL INSIGHT WHICH CAN BE GAINED ACCESSED WITH THE ABOVE

universality of fragment-fragment scattering: