Stability of fermionic states with contact theories

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(Dated: March 15, 2020)

We analyze the stability of systems composed of isomassive fermions in which the number of particles is larger than the number fermionic flavours. To this end, the leading order of a momentum- and flavour-independent contact effective field theory is renormalized to shallow dimer and trimer states. The regulator dependence of the stability is assessed as a function of particle number and of the proximity of the two-body interaction to unitarity. The systems become unstable with respect to decays into spatially symmetric fragments if the regulator-induced effective ranges are below a certain threshold. This critical range decreases when the number of particles is increased assuming a minimum range which is significantly larger than zero. The closer the system is to unitarity, the more particles are needed to attain the minimum. At unitarity, the critical range tends to zero parabolically with the particle number.

We elaborate on the consequences of our results for the systematic description of any system system close to unitarity. For the latter, the study indicates an inclusion of momentum-dependent interaction terms in order to describe P-wave stable systems such as ^{6}Li and ^{7}Li effectively.

PACS numbers:

Introduction If each particle of a set of A isomassive fermions can be distinguished by an internal degree of freedom, the dynamics change significantly with this number A exceeding the dimension d of the flavour space. For $A \leq d$, the system can realize bosonic behaviour in a totally spatially symmetric state while mixed symmetry is demanded if $A > d$. If the mutual interaction is flavour independent, this change is solely a consequence of the statistical properties of the particles. A minimal theory which covers non-trivial few-body phenomena in both bosonic and fermionic sectors is a resonant two-body interaction [1]. It pertains to a specific class of systems whose interaction range is significantly smaller than the resultant two-body correlation/scattering length rendering, *e.g.* , the three-body Efimov effect [2], the Tjon [3] and Phillips [4] correlations, and the spectrum of the multi-boson system [5] as its universal consequences.

Correlations between that same resonant two-body interaction and three- and four-body systems driven by fermionic substructures were also found in form of the absence of shallow resonant [6] and bound states [7] in isomassive two-flavour three- and four-body systems, respectively. It thus appears that the description of fermionic systems which exhibit such peculiar structure, *e.g.* , a hypothetical three-neutron resonance, needs to consider an additional scale which relaxes unitarity. If a specific realization of the discrete scale invariance in the three-boson [8] system with a S-wave three-body contact provides for such a scale is unknown. In addition, finite two-body scales beyond the scattering length, the difference between particle number and flavour-space dimension $(A-d)$, the total number A, or d could provide such a scale.

Here, we explore these possibilities with an effective field theory (EFT) applied to a variety of few-body problems. This entails renormalization as a systematic way to trace the effect of short-distance scales in a range of A-body observables for various flavour-space dimensions.

Theoretical framework The development of a minimal EFT for non-relativistic point particles exhibiting two- and three-body shallow states has been studied extensively (*e.g.* Refs.[8–13]). The theory is defined as a perturbative series and can be refined systematically to attain a desired accuracy. Its Hamiltonian formulation at leading order (LO) comprises zero-range two- and three-body vertices which depend on the renormalization parameter Λ

$$H = -\sum_{i<j} \frac{\hbar^2}{2m} \boldsymbol{\nabla}_{ij}^2 + C^\Lambda \sum_{i<j} \delta_\Lambda(\boldsymbol{r}_i - \boldsymbol{r}_j) + D^\Lambda \sum_{\substack{i<j<k \\ \text{cyc}}} \delta_\Lambda(\boldsymbol{r}_i - \boldsymbol{r}_j)\delta_\Lambda(\boldsymbol{r}_i - \boldsymbol{r}_k). \quad (1)$$

In the expansion of any ensuing amplitude, the LO is represented by all Born terms depending solely on the coupling constants C^Λ and D^Λ. Parameters representing the aforementioned refinements enter perturbatively at

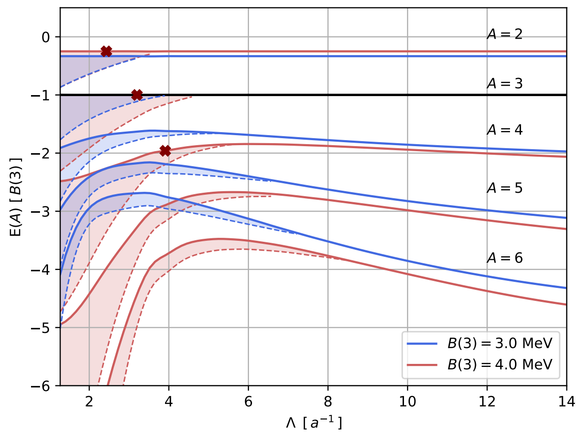

FIG. 1: (Color online) LO Cutoff dependence of ground-state energies of A bosons (solid) and $A \oplus 1$(dashed) systems obtained with $B(2) = 1$ MeV and $B(3) = 3$ and 4 MeV (blue and red) from (1). With scattering volume set to zero, the $A \oplus 1$ systems destabilize at smaller Λ_c (red crosses, $B(3) = 4$ MeV).

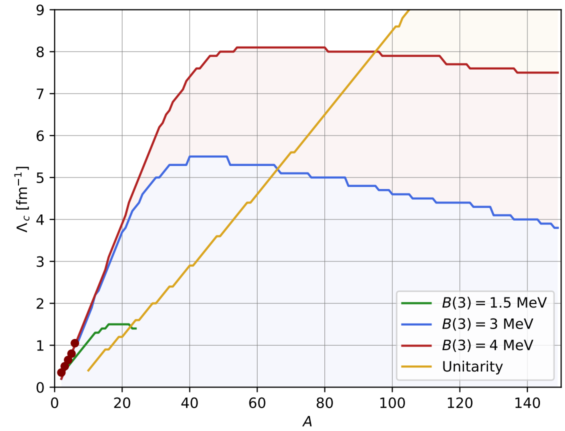

FIG. 2: (Color online) Dependence of the critical cutoff Λ_c on the number of core particles A. SVM few-body results are shown for $A < 8$ (dots, $B(3) = 4$ MeV) along with single-channel resonating-group approximations for $A \lesssim 100$ (lines). The unitarity limit (yellow) was realized with $B(2) \to 0^+$ and $B(3) = 3$ MeV and deviations from it with $B(2) = 1$ MeV and $B(3) \in \{1.5, 3, 4\}$ MeV (green, blue, red). In the shaded regions, the respective theories do sustain bound $A \oplus 1$ states, while systems above the lines are unstable.

the order given naïvely by their mass dimension. Specific to this work is the Gaussian regulator $\delta_\Lambda(\boldsymbol{x}) \propto \Lambda^3 e^{-\frac{\Lambda^2}{4}\boldsymbol{x}^2}$. It induces a Λ dependence in C^Λ and D^Λ which was calibrated to the energy of a single bound state in the two- ($B(2)$) and three-body ($B(3)$) system, respectively. Whether or not the Λ convergence of another amplitude depends on the specific choice for $B(2)$ and $B(3)$ classifies the corresponding observable as universal or emergent. The problem is specified through five parameters: the particle's mass (here, $m = 938$ MeV), the number of particles (A) and flavours (d), and the dimer and trimer binding energies.

Using this structure, we consider a class of few-body systems with $A \oplus 1$ statistics [27] as they approach the unitarity limit via increasing the numerator instead of the denominator of the ratio $\upsilon := B(3)/B(2)$. Specifically we use: $B(2) = 1$ MeV with $B(3) \in \{1.5, 3, 4\}$ MeV; and $B(2) = 0^+$ with $B(3) = 3$ MeV. The nuclear pionless EFT (EFT($\not\pi$)) is renormalized separately to yield the deuteron and triton binding energies of $B(2) = 2.22$ MeV and $B(3) = 8.48$ MeV, respectively. Technically, the necessary fits employ Stochastic-Variational (SVM, [14]) and Resonating-Group (RGM, [15]) variational diagonalizations for D^Λ. C^Λ was determined via a Numerov-type integration of the appropriate one-dimensional radial Schrödinger equation.

Results The bosonic ground states of a theory with Hamiltonian of type (1) have been analysed in detail numerically (see *e.g.* Refs.[16–20]). The SVM-predicted bosonic-like ground state energies for $A < 7$ are shown in fig. 1. We find convergent behaviour as $\Lambda \to \infty$ (renormalization-group invariance). At unitarity, we find the ratio $B(4)/B(3)$ consistent with Refs. [21, 22]. In addition, we find $B(A)/B(3)\Big|_{\upsilon=3} < B(A)/B(3)\Big|_{\upsilon=4}$ which implies that the universal ratios $B(A)/B(3)$ are approached from below when taking the unitarity limit.

Now, we extend the analysis to $A \oplus 1$ systems. In those, our SVM calculations with anti-symmetric wave function but total orbital angular momentum $L_{\text{total}} = 0$ yield no stable states which confirms the intuitive demand for mixed spatial symmetry. Even if expected due to Pauli repulsion, this result is non-trivial because of the numerous angular couplings between particles in many-fermion systems.

When projecting the spatial component of the variational basis onto $L_{\text{total}} = 1$, we find $A \oplus 1$ systems for A between 2 and 6 to sustain stable state ($B(A \oplus 1) > B(A)$) for $\Lambda \approx 0.1\,\text{fm}^{-1}$. In order to assess the universal character of these bound states, we vary the cutoff ($1.2\ a^{-1} < \Lambda < 60\ a^{-1}$) for all considered υ. Increasing the cutoff, *i.e.* , decreasing the interaction range while approaching the contact limit, unbinds the $A \oplus 1$ systems at some critical value Λ_c (fig. 1). The interaction range at which an $A \oplus 1$ system disintegrates decreases with the number of particles $A = d$ comprising its spatially symmetric core. For small systems, this relation is almost linear, $\Lambda_c \propto A < 7$. To assess the dependence for $A > 6$, we employ a single-channel effective two-fragments resonating-group local approximation (to be detailed in upcoming communication in extension of

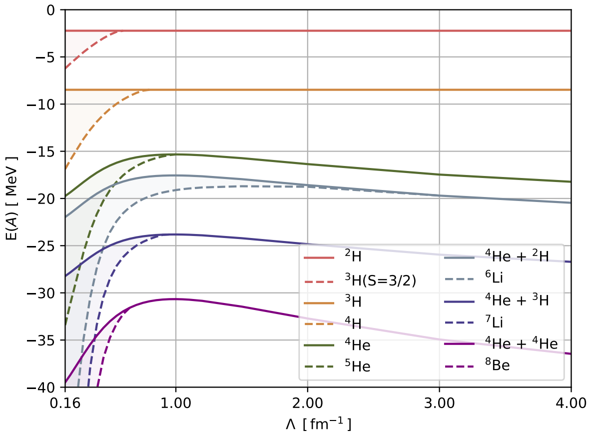

FIG. 3: (Color online) Cutoff dependence of nuclear ground-state energies obtained in LO EFT($\not\pi$). For $A < 5$, solid lines represent nuclei with spatially symmetric ground-state wave functions. For $A > 4$, solid lines mark the lowest decay threshold into two bosonic fragments.

Refs. [15, 23]). This approximation turns the few-body into a two-body problem between a "frozen" core and the one residual particle which is forced out of the Pauli shell.

The halo character of the problem motivates a one-parameter representation of the symmetric core as a product of harmonic oscillator ground states because of the increasingly large gap between $B(A)$ and $B(A-1)$ which does not allow for excitations of the bosonic core by the out-of-shell particle. For $A < 7$, the parameter is fitted to the SVM results for the rms radius of the core. For larger A, we match the core wave function with the drop model formula $r_{\mathrm{rms}} \propto A^{1/3}$ as successfully employed in Ref. [5].

Thereby, we find Λ_c to increase up to a maximum number of particles A^*. Both, A^* and the associated $\Lambda_c(A^*)$ increase with υ (compare maxima of the two curves in fig. 2) and we observe $\lim_{\upsilon\to\infty} A^* > 100$. Based on this, existence of A^* appears as an artifact of the deviation from unitarity. As a finite Λ_c indicates a drastic change of the system's behaviour, its location around or above a conceivable breakdown scale implies the breakdown of the theory. For systems with breakdown scales below Λ_c, our results expose the instability of $A \oplus 1$ system as universal.

To substantiate the conjecture of such a universal instability for the nuclear case and more particles outside the Pauli shell, we fit the experimental deuteron and triton binding energies, $B(2) = 2.22$ MeV and $B(3) = 8.48$ MeV, respectively. Furthermore, we invoke a $SU(4)$-symmetric version of EFT($\not\pi$) [24]. For $A \leq 4$, we observe an instability pattern qualitatively identical to the ones found earlier (compare fig. 1 with fig. 3). Hence, the three-parameter theory predicts correctly the experimentally established instability of nuclei in the ^{3}H$(\frac{3}{2}^-)$, ^3n, ^{4}H, ^{3,4}Li, and ^{5}He channels. In contrast, the $A = 6$ and 7 lithium isotopes are known to sustain bound states in the respective $J^\pi = 1^+$ and $\frac{3}{2}^-$ channels. In these channels, we find particle-stable systems ($4\oplus 2$ and $4\oplus 3$) only below critical cutoffs $\Lambda_c \approx 1.5\,\mathrm{fm}^{-1}$ and $1\,\mathrm{fm}^{-1}$. For larger Λ, the systems break into an α-particle and a deuteron or triton. Furthermore, we find ^{8}Be ($4^2 \oplus 0$, $J^\pi = 0^+$), which is considered to be stable in absence of Coulomb repulsion [25, 26], to α-decay for $\Lambda_c \approx 0.7\,\mathrm{fm}^{-1} \approx m_\pi^{-1}$. Thus, Λ_c is of the same order as the nuclear breakdown scale, which identifies the stability as a cutoff artifact. We explain the loss of stability with the increasing number of particles in different Pauli shells heuristically with the smaller spatial extent of the pertinent nulcear fragments, *i.e.* , the α particle, the triton, and the deuterium. The larger the rms radius of the fragment, the larger is its overlap with the symmetric α core which increase the attraction between the two.

The study of the trajectories of the bound-state poles through the respective $\alpha{-}d$, $\alpha{-}t$, and $\alpha{-}\alpha$ thresholds at $\Lambda > \Lambda_c$ is crucial for the usefulness of EFT($\not\pi$) for the description of these nuclear channels. This will tell whether a perturbative insertion of subleading order must move a shallow scattering pole to a stable one, or if a novel non-perturbative mechanism has to account for the creation of the pole anew. At unitarity, a resonant $A \oplus 1$ pole cannot be fixed to a specific energy as a consequence of scale invariance [28]. In this absence of scales, the resonance can only converge to threshold or diverge to infinity for $\Lambda \to \infty$. In the first case, three-body unitarity would be a universal consequence of the resonant two-body interaction. In the second, the pole is an unphysical artifact which disappears with the regulator. In contrast, for $A > 2$, scale invariance is broken, and the associated emergence of a scale could pin the resonance to a finite energy. As of now, such a study has not been done.

For the conception of an extension of the EFT($\not\pi$) which predicts also the particle-stable character of ^{6,7}Li and ^{8}Be in the zero-range limit, we analyze the mechanism behind the stability of these systems for $\Lambda < \Lambda_c$. Of all artifacts introduced by the finite range of the regulated contact interaction, the finite effective range in the two-body S-wave channel and a non-zero attractive two-body P-wave interaction are expected to dominate. Both contribute to the attraction in the $A \oplus 1$ system but their relative significance in this role is obscure. In other words, the finite-range interaction does not only describe a finite but large S-wave scattering length but also other finite parameters of the effective-range expansion of the S-wave amplitude like the effective range r_0. Furthermore, the scattering volume a_1 of the two-nucleon P-wave amplitude is non-zero as well. To shed light on their relative importance, we project the two-body interaction in an asymmetric internal state. This forces two interacting particles into an even spatial state ($L = 0, 2, ..$) removing any spatially asymmetric contributions. In effect, a reduction of Λ_c^A by about 50% is observed (crosses in fig. 1). Hence, the finite r_0 and a_1 seem to be of similar significance for the stability of the corresponding nuclear systems.

Conclusion We find that a non-relativistic system of $d+1$ particles with identical masses and a d-dimensional internal flavour space cannot sustain a stable state if its dynamics is constrained by representations of two- and three-body momentum-independent contact interactions which are renormalized to yield a resonant two-body state and a single bound three-body state and whose residual finite-range is below a critical value. If the range of the regulated contact interactions, however, surpasses the critical range, the $L_{\text{total}} = 1$ ground state of the $d+1$ particles is stable with respect to breakup into a spatially symmetric d-body ground state and a single free particle. This critical range decreases with the system's increasing particle number d. For a set of interactions close to unitarity, the critical range reaches a minimum at a certain number of particles. At unitarity, we observe that the critical range keeps decreasing up to $d \sim 100$ particles. Similarly, with the pionless EFT at leading order, we find the nuclear systems ^{6}Li, ^{7}Li, and ^{8}Be unstable, contrary to expectations.

We investigate the finite effective-range and scattering volume of two bodies as the dominant parameters affecting stability. Both were found of similar importance to bind the studied systems at small cut-off. With the vanishing of these parameters in the contact limit, the result questions the capability of such theories for the description of P-wave-stable states and asks for a study of the renormalization-group running of hypothetical shallow scattering poles. Only RG-stable poles might be stabilized with insertions of sub-leading operators, while in their absence they would have to be created anew. The latter result would have fundamental consequences for the effective-field-theory formulation.

Acknowledgments We thank N. Barnea, M. Birse, U. van Kolck, N. Walet for insightful discussions. M.S. was supported by the Czech Science Foundation GACR Grant No.19-19640S. L.C. and J.K. acknowledge support from Espace de Structure et de ractions Nuclaire Thorique (ESNT, http://esnt.cea.fr) at CEA-Saclay, where this work was partially carried out. L.C. was also supported by the Pazy Foundation and by the Israel Science Foundation Grant No. 1308/16.

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[27] We refer to a system of A d-flavour fermions with $A > d$ as $d^p \oplus b$ with p d-uplets of fermions which can share the same spatial quantum states, and $b < d$ are the residual particles which are not enough to fill the flavour degeneracy of the states. This notation exposes the number of particles not on-shell for the Pauli interaction. For example, in this notation the nuclear ^{6}Li is noted as $4 \oplus 2$; A hypothetical system of 33 atoms of Helium-3 (fermions with spin 1/2) are noted as $2^{16} \oplus 1$;

[28] We thank U. van Kolck for clarifying discussions on this issue.