No-bound-state theorem for P-wave systems

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We show that momentum-independent contact interactions can stabilize composites of non-relativistic, identical fermions only in a totally symmetric, spatial state. If the number of constituents exceeds the number of accessible internal-symmetry states or another reason precludes the total symmetry of the coordinate-space part of the wave function, a zero-range S-wave interaction cannot bind the system with respect to breakup into substructures. This theorem holds for an arbitrary number of fermion species and spatial dimensions.

a. Overture The problem of a finite number of identical, point-like fermions interacting with another is generic for many physical systems. In the absence of other degrees of freedom, gauge fields for instance, the interaction between the fermions is purely due to contact interactions. These contact interactions, however, might exhibit an arbitrarily complicated momentum dependence. The problem we address in the following is whether an interaction which comprises only those contact terms without momentum dependence (zero-range S-wave interactions) can, *in principle*, sustain a fermionic, bound A-body state with a totally symmetric spatial wave function.

b. The no-bound-P-wave-state theorem We consider momentum-independent contact interactions pertinent for an A-body system of the general structure

$$\hat{V} = \sum_{\substack{i\neq j \\ \widetilde{\boldsymbol{S}}}} \hat{P}_2(\widetilde{\boldsymbol{S}})\delta_{ij}^{(d)} + \sum_{\substack{i\neq j\neq k \\ \widetilde{\boldsymbol{S}}}} \hat{P}_3(\widetilde{\boldsymbol{S}})\delta_{ij}^{(d)}\delta_{jk}^{(d)} + \dots \quad . \tag{1}$$

The spatial component is given by d-dimensional delta-functions, $\delta_{ij}^{(d)} = \delta^{(d)}(\boldsymbol{r}_i - \boldsymbol{r}_j)$, which combine with a dependence on the internal degrees of freedom (collectively denoted by $\widetilde{\boldsymbol{S}} \overset{e.g.}{=} (SU(2)$ spin, SU(N) isospin)). For the operators corresponding to the latter, $\hat{P}_{n\leq A}(\widetilde{\boldsymbol{S}})$, we assume rank-0 tensors *w.r.t.* to rotations in space, and their commutation with the antisymmetrizing operator $\mathcal{A}$ in order to have $\left[\hat{H}, \mathcal{A}\right] = 0$.

For the wave function, we assume the existence of a complete orthonormal, single-particle basis in which the state of the *interacting* A-body system, with (d) spatial and (N) internal degrees of freedom ($\widetilde{r} \equiv (\boldsymbol{r}, \boldsymbol{j})$), can be expanded:

$$\Psi\left(\widetilde{\boldsymbol{r}}_1, \dots, \widetilde{\boldsymbol{r}}_A\right) = \sum_{\boldsymbol{n},\boldsymbol{m}} \left(c_{mn} \prod_{i=1}^{A} \phi_{n_i}(\boldsymbol{r}_i)\ \xi_{m_i} \right) \equiv \sum \Psi_{\boldsymbol{nm}} \quad , \tag{2}$$

with $\left\langle \Psi_{\boldsymbol{nm}} \middle| \Psi_{\boldsymbol{n'm'}} \right\rangle = \delta_{\boldsymbol{mm'}}\delta_{\boldsymbol{nn'}}$. As we exclude totally symmetric states in space, $\exists\ (i,j) \mid i \neq j\ :\ n_i \neq n_j$. Thereby we exclude distributions of all A fermions into a single, spatial state k, $\boldsymbol{n} = \delta_{ki} n_i$.

Next, we select one of the expansion terms, $\Psi_{\boldsymbol{nm}}$, arbitrarily. It is helpful, to visualize its behaviour *w.r.t.* permutations with a Young tableau. For $A = 3$, *e.g.* , we have $\phi_{\boldsymbol{k}}^{123} \equiv \phi_a(1)\phi_a(2)\phi_b(3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ in mind, with row labels a and b and two occupied internal states. Particles 1 and 2 occupy the same spatial shell a, and hence $\phi_{\boldsymbol{k}}^{213(321)} = (-)\phi_{\boldsymbol{k}}^{123}$. With an arbitrary horizontal line, we define a partition of the A coordinates: an upper set I, and a lower set II. The three-body example has only one such partition, namely, $I = \{1,2\}$ and $II = \{3\}$. The Hamiltonian can always be written as a sum of partition-internal operators, a kinetic-energy term for the relative motion between the partition, and the interaction between particles in different partitions:

$$\hat{H} = \hat{H}_I + \hat{H}_{II} + \hat{T}_{\boldsymbol{R}} + \hat{V}_{I-II} \quad . \tag{3}$$

We assume knowledge of the spectrum of the partitions $\hat{H}_{I(II)}\Phi_{I(II)} = E_{I(II)}\Phi_{I(II)}$, which allows us to write the stationary A-body Schrödinger equation for a single term of the wave-function expansion

$$\left(\hat{T}_{\boldsymbol{R}} + \hat{V}_{I-II} - E_{\text{total}} + E_I + E_{II}\right) \mathcal{A}\left(\Phi_I \Phi_{II} \phi_{\boldsymbol{R}}\right) = 0 \quad . \tag{4}$$

It is crucial that conditions were neither made when selecting the term in the expansion, *i.e.* , the specific Young tableau, nor when partitioning that term. A "horizontal" partition is always possible by the definition of the system under investigation. Only those configuration with all particles in one shell cannot be partitioned. To those pure "S-wave" states, the theorem does not apply.

The projection onto $\Phi_I \Phi_{II}$ leads to the equation

$$\Big\langle\ \Phi_I\Phi_{II}\ |\mathcal{A}\left[\Phi_I\Phi_{II}\left(\hat{T}_{\boldsymbol{R}}-E_{\rm total}+E_I+E_{II}\right)\phi_{\boldsymbol{R}}\right]\Big\rangle+\underbrace{\Big\langle\ \Phi_I\Phi_{II}\ \Big|\ \hat{V}_{I-II}\ \mathcal{A}\left[\Phi_I\Phi_{II}\phi_{\boldsymbol{R}}\right]\Big\rangle}_{\equiv V_{\rm EX}}=0\quad . \tag{5}$$

The subject of this article is to prove that

$$V_{\rm EX}=0\quad\begin{array}{c}\text{for momentum-independent}\\ \text{zero-range interactions.}\end{array} \tag{6}$$

For an arbitrary tuple of interacting particles and a single-particle expansion as in (**??**), only the identity **1** and certain permutations within the tuple yield non-zero matrix elements. Any permutation which affects a particle which does not partake in the interaction under consideration will be put into a state which is orthogonal to the one it occupies in $\Phi_I\Phi_{II}$. As the interaction cannot compensate this shift to another state, the resulting matrix element is zero.

The contribution of the first term of (**??**) to $V_{\rm EX}$ is then

$$\Big\langle\ \Phi_I\Phi_{II}\ \Big|\ \hat{P}_2(\widetilde{\boldsymbol{S}})\delta_{ij}^{(d)}\ (\mathbf{1}-(i\leftrightarrow j))\left[\Phi_I\Phi_{II}\phi_{\boldsymbol{R}}\right]\ \Big\rangle = \Big\langle\ \phi_{n_i}^{(i)}\phi_{n_j}^{(i)}\ \Big|\ \phi_{n_i}^{(i)}\phi_{n_j}^{(i)}\ \Big\rangle\cdot \Big[\Big\langle\ \xi_{m_i}^{(i)}\xi_{m_j}^{(j)}\ \Big|\ \hat{P}_2\ \Big|\ \xi_{m_i}^{(i)}\xi_{m_j}^{(j)}\ \Big\rangle-\Big\langle\ \xi_{m_i}^{(i)}\xi_{m_j}^{(j)}\ \Big|\ \hat{P}_2\ \Big|\ \xi_{m_i}^{(j)}\xi_{m_j}^{(i)}\ \Big\rangle\Big] \tag{7}$$

where only the integration over the coordinates of particle i does not yield unity. The minus sign stems from the asymmetry of the spatial part of the wave function, which follows from particles residing in different partitions. Hence, exchanging $(i\leftrightarrow j)$ in the spin part of the wave function has no effect. Thus, last term of (**??**) does not change its value and thus cancels the other spin matrix element for an arbitrary pair-interaction operator $\hat{P}_2$.

If we consider an arbitrary contribution to $V_{\rm EX}$, *i.e.* , a term from (**??**) for a specific operator structure and particle tuple, it is not obvious that the matrix element which involves not just one but an arbitrary number of pairs vanishes, too. Instead of a single permutation, any permutation $(\hat{p}\in S_{|D|})$ of the symmetric group of the interacting set of particles (D), could yield non-zero matrix elements. The generalization of (**??**) is

$$\sum_{p\in S_{|D|}}(-)^p\Big\langle\ \Phi_I\Phi_{II}\ \Big|\ \hat{P}_{|D|}\prod_{i\neq j\in D}\delta_{ij}^{(d)}\ \hat{p}\ \left[\Phi_I\Phi_{II}\phi_{\boldsymbol{R}}\right]\ \Big\rangle\propto \sum_{p\in S_{|D|}}(-)^p\Big\langle\ \prod_{i\in D}\xi_{m_i}^{(i)}\ \Big|\ \hat{P}_{|D|}\ \Big|\ \prod_{i\in D}\xi_{m_{\hat{p}(i)}}^{\hat{p}(i)}\ \Big\rangle\quad . \tag{8}$$

The $|D|!$ terms of this sum cancel as follows.