I. DIMER-DIMER SCATTERING

The two-fragment resonating-group equation

$$\int \left\{ \phi_A^* \phi_B^* \left(-\frac{\hbar^2}{2\mu} \boldsymbol{\Delta}_R - E + \mathcal{V}_{AB} \right) \mathcal{A}_{AB} \left[\phi_A \phi_B \psi(\boldsymbol{R}) \right] \right\} d\boldsymbol{r}_{A,B}^{\text{internal}} = 0 \tag{1}$$

embodies the assumption of a rapid internal motion relative to the slow relative motion between clusters A and B. Ockham's ansatz for spatially symmetric fragments employs a one-parameter Gaussian

$$\phi_A := e^{-\alpha \sum_{i=1}^{A} (\boldsymbol{r}_i - \boldsymbol{R}_A)^2} \quad ; \begin{array}{ll} \boldsymbol{r}_i & : \text{single-particle coordinates} \\ \boldsymbol{R}_A & : \text{core centre of mass} \end{array} \quad . \tag{2}$$

The parameter representation $\psi(\boldsymbol{R}) = \int d\boldsymbol{R}' \delta^{(3)}(\boldsymbol{R} - \boldsymbol{R}')\psi(\boldsymbol{R}')$ allows for a translation of the inter-cluster antisymmetrizer $\mathcal{A}_{AB}$ into a non-local integro-differential equation which, in general, assumes the form

$$(\hat{T} - E)\ \chi(\boldsymbol{r}) + \mathcal{V}^{(1)}(\boldsymbol{r})\ \chi(\boldsymbol{r}) + \int d^{(3)}\boldsymbol{r}'\ \mathcal{V}^{(2)}(\boldsymbol{r}, \boldsymbol{r}', E)\ \chi(\boldsymbol{r}') = 0 \tag{3}$$

with the radial coordinates denoting the spatial separation between the two fragments. If these fragments are two-body S-wave bound states comprised of equal-mass fermions, the effective potentials which derive from a zero-range fermion-fermion – *e.g.* , $\mathrm{EFT}(\not{\pi})$ LO – interaction are given for a two- and three-species system – *e.g.* , four neutrons assuming a bound di-neutron, and 4-hydrogen, respectively. We denote the former as (ab):(ab) (scale invariant), and the latter $(ab)\therefore(ca)$(discretely scale invariant, Thomas collapse of (abc)). The characteristic three-body scale in an $(ab)\therefore(ca)$ system flows into the effective dimer-dimer potentials, while in the absence of such a scale in the zero-range two-body limit, the effective potentials are parametrized by the dimer, *i.e.* , a two-body observable, only.

In detail,

(ab):(ab)**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 2\, C_0(\lambda) \cdot \left(\frac{2\alpha}{2\alpha + \lambda} \right)^{3/2} \cdot e^{-\frac{2\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} \quad , \tag{4}$$

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r}, \boldsymbol{r}', E) = 8\, \alpha^{3/2} \cdot e^{-\alpha \boldsymbol{r}'^2} \cdot \left[\frac{\hbar^2}{2\mu}\ (4\alpha^2 \boldsymbol{r}^2 - 2\alpha) \cdot e^{-\alpha \boldsymbol{r}^2} + E \cdot e^{-\alpha \boldsymbol{r}^2} - 2\ C_0(\lambda) \cdot \left(\frac{2\alpha}{2\alpha+\lambda} \right)^{3/2} \cdot e^{-\alpha \cdot \frac{2\alpha + 3\lambda}{2\alpha+\lambda} \boldsymbol{r}^2} \right] \quad . \tag{5}$$

$(ab)\boldsymbol{\therefore}(ca)$**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = 3 \cdot C_0(\lambda) \cdot \left(\frac{2\alpha}{2\alpha+\lambda} \right)^{3/2} \cdot e^{-\frac{2\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} \tag{6}$$

$$+ D_0(\lambda) \cdot \left(\left(\frac{2\alpha}{2\alpha+\lambda} \right)^3 \cdot e^{-\frac{4\alpha\lambda}{2\alpha+\lambda}\boldsymbol{r}^2} + \left(\frac{2\alpha}{\sqrt{(2\alpha+\lambda)^2 + 2\alpha\lambda}} \right)^3 \cdot e^{-\frac{4\alpha\lambda(\alpha+\lambda)}{4\alpha^2 + 6\alpha\lambda + \lambda^2}\boldsymbol{r}^2} \right) \tag{7}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r}, \boldsymbol{r}', E) = 8\ \alpha^{3/2} \cdot \Bigg(e^{-\alpha \boldsymbol{r}'^2} \cdot \left[\frac{\hbar^2}{2\mu}\ (4\alpha^2 \boldsymbol{r}^2 - 2\alpha) \cdot e^{-\alpha \boldsymbol{r}^2} + E \cdot e^{-\alpha \boldsymbol{r}^2} \right] \tag{8}$$

$$- C_0(\lambda) \cdot e^{-(\alpha+\lambda)(\boldsymbol{r}^2 + \boldsymbol{r}'^2) - 2\lambda \boldsymbol{r}' \cdot \boldsymbol{r}} - 2\ C_0(\lambda) \cdot \left(\frac{2\alpha}{2\alpha+\lambda} \right)^{3/2} \cdot e^{-\alpha \cdot \left(\boldsymbol{r}'^2 + \frac{2\alpha+3\lambda}{2\alpha+\lambda} \boldsymbol{r}^2 \right)} \tag{9}$$

$$-\ D_0(\lambda) \cdot \left(\frac{\alpha}{\alpha+\lambda} \right)^{3/2} \cdot e^{-\frac{2\alpha^2 + 4\alpha\lambda + \lambda^2}{2(\alpha+\lambda)}(\boldsymbol{r}^2 + \boldsymbol{r}'^2) - \frac{\lambda^2}{\alpha+\lambda} \boldsymbol{r} \cdot \boldsymbol{r}'} \tag{10}$$

$$- D_0(\lambda) \cdot \left(\frac{2\alpha(\alpha+\lambda)}{2\alpha^2 + 3\alpha\lambda + \lambda^2} \right)^{3/2} \cdot e^{-\frac{2\alpha^2 + 5\alpha\lambda + \lambda^2}{2(\alpha+\lambda)}\boldsymbol{r}^2 - (\alpha+\lambda)\boldsymbol{r}'^2 - 2\lambda \boldsymbol{r} \cdot \boldsymbol{r}'} \Bigg) \tag{11}$$

It is in order to consider the following limits:

zero-range or contact limit: $\lambda \gg \alpha$

local approximation: $\int d^{(3)}\boldsymbol{r'}\; \mathcal{V}^{(2)}(\boldsymbol{r},\boldsymbol{r'},E)\; \chi(\boldsymbol{r'}) \overset{E\to 0}{\approx} \chi(\boldsymbol{r}) \cdot v^{(2)}(\boldsymbol{r}) \cdot \int d^{(3)}\boldsymbol{r'}\; v^{(2)}(\boldsymbol{r'})$.

Assuming an unnaturally large dimer scale emergent from a relatively short-ranged fermion-fermion interaction, the zero-range approximation is justified and the ensuing dimer-dimer potentials read:

(zero-range) (ab):(ab)**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 2\,(2\alpha)^{3/2}\;\frac{C_0(\lambda)}{\lambda^{3/2}} \cdot e^{-2\alpha\boldsymbol{r}^2} \quad , \tag{12}$$

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r},\boldsymbol{r'},E) = \; 8\;\alpha^{3/2} \cdot e^{-\alpha\boldsymbol{r'}^2} \cdot \left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2} - 2\,(2\alpha)^{3/2}\;\frac{C_0(\lambda)}{\lambda^{3/2}}\cdot e^{-3\alpha\boldsymbol{r}^2}\right] \quad . \tag{13}$$

(zero-range) $(ab)\boldsymbol{\therefore}(ca)$**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = 3\,(2\alpha)^{3/2}\;\frac{C_0(\lambda)}{\lambda^{3/2}}\cdot e^{-2\alpha\boldsymbol{r}^2} + 2\,(2\alpha)^3\;\frac{D_0(\lambda)}{\lambda^3}\cdot e^{-4\alpha\boldsymbol{r}^2} \tag{14}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r},\boldsymbol{r'},E) = 8\;\alpha^{3/2}\cdot\Bigg(e^{-\alpha\boldsymbol{r'}^2}\cdot\left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2}\right] \tag{15}$$

$$-C_0(\lambda)\cdot e^{-\lambda(\boldsymbol{r}+\boldsymbol{r'})^2} - 2\;(2\alpha)^{3/2}\;\frac{C_0(\lambda)}{\lambda^{3/2}}\cdot e^{-\alpha\boldsymbol{r'}^2 - 3\alpha\boldsymbol{r}^2} \tag{16}$$

$$-\alpha^{3/2}(1+2^{3/2})\;\frac{D_0(\lambda)}{\lambda^{3/2}}\cdot e^{-\frac{\lambda}{2}(\boldsymbol{r}+\boldsymbol{r'})^2}\Bigg) \tag{17}$$

We do now interpret these potentials as vertices of interacting dimer fields – the physical nature of the fields is inessential for the following; quite generally, we applied a transformation on a renormalized contact interaction, and we are now interested in whether or not this transformation, *i.e.* , the RGM averaging over fragment-internal, "frozen" degrees of freedom, preserves the renormalized character of amplitudes of the image theory – whose regularization is inherited from the renormalized fermion-fermion interaction.

We commence the analysis of the renormalizability of the transformed dimer-dimer theory under the assumption that the transformation does not affect the power-counting rules. That means, solutions of a Schrödinger equation with and interaction as given by the non-local potentials shall be well-behaved for $\lambda \to \infty$. Renormalizing the fermion-fermion amplitude yields

$$C_0(\lambda) \propto \lambda \tag{18}$$

and arguably two scenarios for the three-body parameter:

$$\text{Three-body spectrum with one single shallow bound state } \neq f(\lambda):\;\; D_0(\lambda) \propto e^{\kappa\lambda} \tag{19}$$

$$\text{Three-body spectrum with a tower of Efimov-type states with the shallowest } \neq f(\lambda):\;\; \text{countably infinite poles} \tag{20}$$

Revisit the effective potentials, considering $\lambda \to \infty$:

(zero-range) (ab):(ab)**:**

$$\mathcal{V}^{(1)}_{(ab):(ab)}(\boldsymbol{r}) = 0 \tag{21}$$

$$\mathcal{V}^{(2)}_{(ab):(ab)}(\boldsymbol{r},\boldsymbol{r'},E) = \; 8\;\alpha^{3/2}\cdot e^{-\alpha\boldsymbol{r'}^2}\cdot\left[\frac{\hbar^2}{2\mu}\;(4\alpha^2\boldsymbol{r}^2 - 2\alpha)\cdot e^{-\alpha\boldsymbol{r}^2} + E\cdot e^{-\alpha\boldsymbol{r}^2}\right] \quad . \tag{22}$$

In words, the sub-threshold dimer-dimer amplitude depends on the microscopic interaction only through the character of a single dimer as parametrized with α. Although, no analytic form of the functional relation $\alpha = f(\aleph)$ is known, its existence implies a universal low-energy dimer-dimer system, thereby conforming with the "Petrov ratio" $a_{dd}/a_{ff} \approx 0.6$.

(zero-range) $(ab)\therefore(ca)$**:**

$$\mathcal{V}^{(1)}_{(ab)\therefore(ca)}(\boldsymbol{r}) = c_1 \mathtt{P}[\lambda] \cdot e^{-4\alpha \boldsymbol{r}^2} \tag{23}$$

$$\mathcal{V}^{(2)}_{(ab)\therefore(ca)}(\boldsymbol{r},\boldsymbol{r}',E) = 8\ \alpha^{3/2} \cdot \left(e^{-\alpha \boldsymbol{r}'^2} \cdot \left[\frac{\hbar^2}{2\mu}\ (4\alpha^2 \boldsymbol{r}^2 - 2\alpha) \cdot e^{-\alpha \boldsymbol{r}^2} + E \cdot e^{-\alpha \boldsymbol{r}^2} \right] - c_2 \mathtt{P}[\lambda] \cdot e^{-\frac{\lambda}{2}(\boldsymbol{r}+\boldsymbol{r}')^2} \right) \tag{24}$$

`ECCE` this structure, which I want to discuss/have an opinion on.

The specific form of the polynomials depends on the implemented three-body renormalization condition, and so do the values of the non-equal constants $c_{1,2}$. Yet, regardless of the specific shape, the induced λ dependence will translate into dimer-dimer observables which consequently do not have a well defined $\lim\limits_{\lambda\to\infty}$!