Supersymmetric Quantum Mechanics

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Abstract

Supersymmetric quantum mechanics (SUSY QM) is a powerful tool for generating new potentials with known spectra departing from an initial solvable one. In these lecture notes we will present some general formulae concerning SUSY QM of first and second order for one-dimensional arbitrary systems, and we will illustrate the method through the trigonometric Pöschl-Teller potentials. Some intrinsically related subjects, as the algebraic structure inherited by the new Hamiltonians and the corresponding coherent states will be analyzed. The technique will be as well implemented for periodic potentials, for which the corresponding spectrum is composed of allowed bands separated by energy gaps.

1 Introduction

The techniques based on the factorization method, which try to identify the class of Hamiltonians whose spectral problem can be algebraically solved, have attracted for years people's attention. In these lecture notes we will elaborate a brief survey of this subject; since we shall show that the factorization method, supersymmetric quantum mechanics (SUSY QM), and intertwining technique are equivalent procedures for generating new solvable potentials departing from an initial one, these names will be indistinctly used to characterize the technique we are interested in. For a collection of books and review articles concerning factorization method and related subjects the reader can seek [1]- [24].

I would like to emphasize that these lecture notes on SUSY QM are a sequel of those delivered by myself in 2004 at the Latin-American School of Physics [23]. For people wanting to have a global view of the way our group addresses the subject, and to know more details of our achievements, I recommend to read both, this article and [23], because they complement to each other.

From a historical viewpoint, it is nowadays accepted that the factorization method was introduced for the first time by Dirac in 1935 in order to derive algebraically the spectrum of the harmonic oscillator [25]. In coordinates such that $\hbar = \omega = m = 1$ the oscillator

Hamiltonian admits the following factorizations:

$$H = aa^{+} - \frac{1}{2} = a^{+}a + \frac{1}{2},\tag{1}$$

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}, \quad a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right). \tag{2}$$

From these equations it is straightforward to derive the intertwining relationships,

$$Ha = a(H-1), \quad Ha^{+} = a^{+}(H+1),$$
 (3)

which lead to the commutators ruling the standard Heisenberg-Weyl algebra:

$$[H, a] = -a, \quad [H, a^+] = a^+, \quad [a, a^+] = 1.$$
 (4)

These expressions and the fact that H is positive definite allow to generate the eigenfunctions $\psi_n(x)$ and corresponding eigenvalues $E_n = n + 1/2, \ n = 0, 1, \ldots$ of H.

The second important advance in the subject was done by Schrödinger in 1940, who realized that the factorization method can be as well applied to the radial part of the Coulomb problem [26, 27]. In this case the explicit factorizations in appropriate units read:

$$H_{\ell} = a_{\ell}^{+} a_{\ell} - \frac{1}{2\ell^{2}} = a_{\ell+1} a_{\ell+1}^{+} - \frac{1}{2(\ell+1)^{2}}, \tag{5}$$

$$H_{\ell} = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2r^2} - \frac{1}{r},\tag{6}$$

$$a_{\ell} = \frac{1}{\sqrt{2}} \left(\frac{d}{dr} + \frac{\ell}{r} - \frac{1}{\ell} \right), \quad a_{\ell}^{+} = \frac{1}{\sqrt{2}} \left(-\frac{d}{dr} + \frac{\ell}{r} - \frac{1}{\ell} \right).$$
 (7)

As in the previous case, these factorizations imply some intertwining relationships:

$$H_{\ell+1}a_{\ell+1}^+ = a_{\ell+1}^+ H_{\ell}, \quad H_{\ell-1}a_{\ell} = a_{\ell}H_{\ell},$$
 (8)

from which it can be derived once again the radial eigenfunctions $R_{n\ell}(r)$ and eigenvalues $E_n = -1/(2n^2)$, $n = \ell + 1, \ell + 2, \ldots$ for the Hamiltonian H_{ℓ} .

In a set of works starting from 1941, Infeld and collaborators put forward the method [28,29], culminating their research with a seminal paper in which it was classified a wide set of Schrödinger-type Hamiltonians (indeed four families) solvable through the factorization method [1]. The main tool for performing this classification were the factorizations

$$H_m = A_m^+ A_m + \epsilon_m = A_{m+1} A_{m+1}^+ + \epsilon_{m+1}, \tag{9}$$

$$H_m = -\frac{1}{2}\frac{d^2}{dx^2} + V_m(x),\tag{10}$$

$$A_m = \frac{1}{\sqrt{2}} \left[\frac{d}{dx} + \beta_m(x) \right], \quad A_m^+ = \frac{1}{\sqrt{2}} \left[-\frac{d}{dx} + \beta_m(x) \right], \tag{11}$$

which lead straightforwardly to the intertwining relationships

$$H_{m+1}A_{m+1}^+ = A_{m+1}^+ H_m, \quad H_{m-1}A_m = A_m H_m.$$
 (12)

With these ingredients, Infeld and Hull were able to derive elegantly the eigenfunctions and eigenvalues for the hierarchy of Hamiltonians H_m [1]. Moreover, the very technique allowed them to determine the several forms of the potentials $V_m(x)$ admitting the factorization treatment. In this way, after [1] the idea that the factorization methods was exhausted started to dominate; indeed, at some time it was thought that for a potential V(x) to be solvable through factorization, it just has to appear in the Infeld-Hull classification.

Revealing himself against this idea, in 1984 Mielnik performed a simple generalization of the factorization method [30]. He proposed to look for the most general first-order differential operators b, b^+ factorizing the harmonic oscillator Hamiltonian in the way:

$$H = bb^{+} - \frac{1}{2}, \quad b = \frac{1}{\sqrt{2}} \left[\frac{d}{dx} + \beta(x) \right], \quad b^{+} = \frac{1}{\sqrt{2}} \left[-\frac{d}{dx} + \beta(x) \right].$$
 (13)

Mielnik was able to determine the most general form of $\beta(x)$. Moreover, it turns out that the product b^+b supplies a new Hamiltonian

$$\widetilde{H} = b^+ b + \frac{1}{2} = -\frac{1}{2} \frac{d^2}{dx^2} + \widetilde{V}(x),$$
(14)

$$\widetilde{V}(x) = \frac{x^2}{2} - \beta'(x) + 1.$$
 (15)

Hence, the following intertwining relationships are valid:

$$\widetilde{H}b^{+} = b^{+}(H+1), \quad Hb = b(\widetilde{H}-1),$$
 (16)

which interrelate the eigenfunctions of the harmonic oscillator Hamiltonian H with those of \widetilde{H} and vice versa.

Mielnik's work was a breakthrough in the development of the factorization method, since it opened new ways to explore exactly solvable potentials in quantum mechanics. In particular, his generalization was quickly applied to the radial part of the Coulomb problem [31]. The important resultant expressions for that system become:

$$H_{\ell} = A_{\ell}^{+} A_{\ell} - \frac{1}{2\ell^{2}}, \quad \widetilde{H}_{\ell-1} = A_{\ell} A_{\ell}^{+} - \frac{1}{2\ell^{2}},$$
 (17)

$$\widetilde{H}_{\ell-1}A_{\ell} = A_{\ell}H_{\ell}, \quad H_{\ell}A_{\ell}^{+} = A_{\ell}^{+}\widetilde{H}_{\ell-1}.$$
 (18)

They allow to generate the eigenfunctions and eigenvalues of $\widetilde{H}_{\ell-1}$ departing from those of H_{ℓ} . Notice that, in the same year the connections between the factorization method with the Darboux transformation and Witten's supersymmetric quantum mechanics were established by Andrianov et al [32, 33] and Nieto [34] respectively.

In 1985 Sukumar [35,36] pushed further Mielnik's factorization by applying it to general one-dimensional potentials V(x) and arbitrary factorization energies ϵ in the way

$$H = AA^{+} + \epsilon, \quad \widetilde{H} = A^{+}A + \epsilon,$$
 (19)

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \quad \widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x), \tag{20}$$

$$A = \frac{1}{\sqrt{2}} \left[\frac{d}{dx} + \beta(x) \right], \quad A^+ = \frac{1}{\sqrt{2}} \left[-\frac{d}{dx} + \beta(x) \right]. \tag{21}$$

These relationships imply that

$$\widetilde{H}A^{+} = A^{+}H, \quad HA = A\widetilde{H},$$
 (22)

which allow to generate the eigenfunctions and eigenvalues of \widetilde{H} from those of H.

Along the years, several particular cases and applications of the generalized factorization of [30, 31, 35] have been performed successfully [37]- [63]. Up to this point, however, the factorization operators, becoming the intertwiners in the equivalent formalism, were differential operators of first order. A natural generalization, posed in 1993 by Andrianov and collaborators [64,65], requires that Eq.(22) be valid but A,A^+ are substituted by differential operators of order greater than one. Through this higher-order case, it is possible to surpass the restriction arising from the first-order method that we can only modify the ground state energy of the initial Hamiltonian. In 1995 Bagrov and Samsonov proposed the same generalization, and they offered an alternative view to this issue [66]. On the other hand, our group arrived to the subject in 1997 [67]- [71], although there were several previous works related somehow to this generalization [72]- [76]. It is worth to mention explicitly some important contributions that members of our group have made to the factorization method and related subjects.

- Generation of SUSY partner of the harmonic oscillator [30, 68, 69]
- Construction of coherent states for SUSY partners of the harmonic oscillator [73–75,77] and for general one-dimensional Hamiltonians [78].
- Determination of SUSY partners of the radial oscillator potential [76]
- Characterization of systems ruled by polynomial Heisenberg algebras and connection with SUSY partners of the oscillator [77, 79–81]
- Analysis of SUSY partners of the radial Coulomb problem [31,82–84]
- Implementation of SUSY transformations involving complex factorization 'energies' [85]- [87]
- Study of the confluent algorithm for second-order SUSY transformations and applications [84, 88, 89]
- Generation of SUSY partners of the Pöschl-Teller potentials [90]- [92]
- Application of SUSY techniques to the Lamé and associated Lamé potentials [93]- [99]

It is important to mention that some other groups have elaborated the same subject from different viewpoints, e.g., N-fold supersymmetry by Tanaka and collaborators [100]- [104], hidden nonlinear supersymmetries by Plyushchay et al [105]- [109], etc.

2 Standard supersymmetric quantum mechanics

In this section we are going to show that the factorization method, intertwining technique and supersymmetric quantum mechanics are equivalent procedures for generating new solvable potentials departing from an initial one. The most straightforward way to see this equivalence is starting from the technique which involves just first-order differential intertwining operators.

2.1 Intertwining technique

Suppose we have two Schrödinger Hamiltonians

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \quad H_1 = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x), \tag{23}$$

which are intertwined by first-order differential operators A_1, A_1^+ in the way

$$H_1 A_1^+ = A_1^+ H_0, \qquad H_0 A_1 = A_1 H_1,$$
 (24)

$$A_1 = \frac{1}{\sqrt{2}} \left[\frac{d}{dx} + \alpha_1(x) \right], \quad A_1^+ = \frac{1}{\sqrt{2}} \left[-\frac{d}{dx} + \alpha_1(x) \right].$$
 (25)

By employing that, at the operator level, it is valid

$$\frac{d}{dx}f(x) = f(x)\frac{d}{dx} + f'(x), \quad \frac{d^2}{dx^2}f(x) = f(x)\frac{d^2}{dx^2} + 2f'(x)\frac{d}{dx} + f''(x), \tag{26}$$

where f(x) is a multiplicative operator when acting on a generic wavefunction $\psi(x)$, the calculation of each member of the first intertwining relationship in Eq.(24) leads to

$$\sqrt{2}H_1A_1^+ = \frac{1}{2}\frac{d^3}{dx^3} - \frac{\alpha_1}{2}\frac{d^2}{dx^2} - (V_1 + \alpha_1')\frac{d}{dx} + \alpha_1V_1 - \frac{\alpha_1''}{2},\tag{27}$$

$$\sqrt{2}A_1^+ H_0 = \frac{1}{2} \frac{d^3}{dx^3} - \frac{\alpha_1}{2} \frac{d^2}{dx^2} - V_0 \frac{d}{dx} + \alpha_1 V_0 - V_0'.$$
 (28)

Thus:

$$V_1 = V_0 - \alpha_1', \tag{29}$$

$$\alpha_1 V_1 - \frac{\alpha_1''}{2} = \alpha_1 V_0 - V_0'. \tag{30}$$

By plugging the V_1 of Eq.(29) in Eq.(30) and integrating the result we get:

$$\alpha_1' + \alpha_1^2 = 2[V_0(x) - \epsilon]. \tag{31}$$

This nonlinear first-order differential equation for α_1 is the well-known *Riccati equation*. In terms of a seed Schrödinger solution $u^{(0)}$ such that $\alpha_1 = u^{(0)'}/u^{(0)}$ it turns out that:

$$-\frac{1}{2}u^{(0)"} + V_0 u^{(0)} = H_0 u^{(0)} = \epsilon u^{(0)}, \tag{32}$$

which is the initial stationary Schrödinger equation. These equations allow to establish the link between the intertwining technique and the factorization method, since it is valid:

$$H_0 = A_1 A_1^+ + \epsilon, \quad H_1 = A_1^+ A_1 + \epsilon.$$
 (33)

Suppose now that H_0 is a given solvable Hamiltonian, with known eigenfunctions $\psi_n^{(0)}$ and eigenvalues E_n , namely,

$$H_0\psi_n^{(0)} = E_n\psi_n^{(0)}, \quad n = 0, 1, \dots$$
 (34)

In addition, let us take a nodeless mathematical eigenfunction $u^{(0)}$ associated to $\epsilon \leq E_0$ in order to implement the intertwining procedure. The equations (24) and the resulting factorizations (33) indicate that, if $A_1^+\psi_n^{(0)}\neq 0$, then $\{\psi_n^{(1)}=A_1^+\psi_n^{(0)}/\sqrt{E_n-\epsilon}\}$ is an orthonormal set of eigenfunctions of H_1 with eigenvalues $\{E_n\}$. This is a basis if there is not a square-integrable function $\psi_{\epsilon}^{(1)}$ which is orthogonal to the full set. Thus, let us look for $\psi_{\epsilon}^{(1)}$ such that

$$0 = (\psi_{\epsilon}^{(1)}, \psi_n^{(1)}) \propto (\psi_{\epsilon}^{(1)}, A_1^+ \psi_n^{(0)}) = (A_1 \psi_{\epsilon}^{(1)}, \psi_n^{(0)}) \quad \Rightarrow \quad A_1 \psi_{\epsilon}^{(1)} = 0. \tag{35}$$

By solving this first-order differential equation we get:

$$\psi_{\epsilon}^{(1)} \propto e^{-\int_0^x \alpha_1(y)dy} = \frac{1}{u^{(0)}}.$$
 (36)

Since $H_1\psi_{\epsilon}^{(1)} = \epsilon\psi_{\epsilon}^{(1)}$ (compare Eq.(33)), then the spectrum of H_1 depends of the normalizability of $\psi_{\epsilon}^{(1)}$. We can identify three different situations.

(i) Let us choose $\epsilon = E_0$ and $u^{(0)} = \psi_0^{(0)}$, which is nodeless in the domain of V_0 , $\alpha_1 = \psi_0^{(0)'}/\psi_0^{(0)}$. Thus, $V_1 = V_0 - \alpha_1'$ is completely determined, and the corresponding Hamiltonian H_1 has eigenfunctions and eigenvalues given by

$$\psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - E_0}}, \quad \text{Sp}(H_1) = \{E_n, \ n = 1, 2, \dots\}.$$
 (37)

Notice that $E_0 \notin \operatorname{Sp}(H_1)$ since $\psi_{\epsilon}^{(1)} \propto 1/u^{(0)}$ is not square-integrable.

(ii) Let now a nodeless seed solution $u^{(0)}$ for $\epsilon < E_0$ be chosen so that $\alpha_1 = u^{(0)'}/u^{(0)}$ is non-singular. Note that $u^{(0)}$ diverges at both ends of the x-domain $\Rightarrow \psi_{\epsilon}^{(1)} \propto 1/u^{(0)}$ tends to zero at those ends. Hence, the full set of eigenfunctions and eigenvalues is

$$\psi_{\epsilon}^{(1)} \propto \frac{1}{u^{(0)}}, \quad \psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - \epsilon}}, \quad \text{Sp}(H_1) = \{\epsilon, E_n, \ n = 0, 1, 2, \dots\}.$$
 (38)

(iii) Let us employ a solution $u^{(0)}$ with a node at one of the ends of the x-domain for $\epsilon < E_0$. Thus, the transformation induced by $\alpha_1 = u^{(0)'}/u^{(0)}$ is non-singular, $\psi_{\epsilon}^{(1)} \propto 1/u^{(0)}$ diverges at the end where $u^{(0)}$ tends to zero, and then the eigenfunctions and eigenvalues of H_1 become

$$\psi_n^{(1)} = \frac{A_1^+ \psi_n^{(0)}}{\sqrt{E_n - \epsilon}}, \quad \text{Sp}(H_1) = \{ E_n, \ n = 0, 1, 2, \dots \}.$$
 (39)

This implies that H_1 and H_0 are isospectral Hamiltonians.

2.2 Supersymmetric quantum mechanics

Let us build up now the so-called supersymmetric quantum mechanics, by realizing the Witten supersymmetry algebra with two generators [110]

$$[Q_i, H_{ss}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij}H_{ss}, \quad i, j = 1, 2,$$
 (40)

in the way

$$Q_1 = \frac{Q^+ + Q}{\sqrt{2}}, \quad Q_2 = \frac{Q^+ - Q}{i\sqrt{2}}, \quad Q = \begin{pmatrix} 0 & 0 \\ A_1 & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A_1^+ \\ 0 & 0 \end{pmatrix},$$
 (41)

$$H_{\rm ss} = \{Q, Q^+\} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} = \begin{pmatrix} A_1^+ A_1 & 0 \\ 0 & A_1 A_1^+ \end{pmatrix} = \begin{pmatrix} H_1 - \epsilon & 0 \\ 0 & H_0 - \epsilon \end{pmatrix}. \tag{42}$$

Notice that there is a linear relationship between H_{ss} and $H^p = \text{Diag}\{H_1, H_0\}$:

$$H_{\rm ss} = (H^p - \epsilon). \tag{43}$$

There are two-fold degenerated levels of H_{ss} [$E_n - E_0$ with n = 1, 2, ... for the case (i) and $E_n - \epsilon$ with n = 0, 1, 2, ... for cases (ii) and (iii)] which have associated the two orthonormal eigenvectors

$$\begin{pmatrix} 0 \\ \psi_n^{(0)} \end{pmatrix}, \qquad \begin{pmatrix} \psi_n^{(1)} \\ 0 \end{pmatrix}. \tag{44}$$

The ground state energy of H_{ss} in cases (i) and (ii) is non-degenerated (and equal to 0), with the corresponding eigenstate being given respectively by

$$\begin{pmatrix} 0 \\ \psi_0^{(0)} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{\epsilon}^{(1)} \\ 0 \end{pmatrix}. \tag{45}$$

In this case it is said that the supersymmetry is unbroken. On the other hand, in case (iii) the ground state energy $(E_0 - \epsilon)$ is doubly degenerated, and then the supersymmetry is spontaneously broken.

3 Example: trigonometric Pöschl-Teller potentials

Let us apply the previous technique to the trigonometric Pöschl-Teller potentials [91,92]:

$$V_0(x) = \frac{(\lambda - 1)\lambda}{2\sin^2(x)} + \frac{(\nu - 1)\nu}{2\cos^2(x)}, \quad 0 \le x \le \pi/2, \quad \lambda, \nu > 1.$$
 (46)

First of all, we require the general solution of the Schrödinger equation (32) with the $V_0(x)$ given in (46), which is given by

$$u^{(0)}(x) = \sin^{\lambda}(x)\cos^{\nu}(x) \left\{ A_{2}F_{1} \left[\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}}; \lambda + \frac{1}{2}; \sin^{2}(x) \right] + B\sin^{1-2\lambda}(x) {}_{2}F_{1} \left[\frac{1+\nu-\lambda}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{1+\nu-\lambda}{2} - \sqrt{\frac{\epsilon}{2}}; \frac{3}{2} - \lambda; \sin^{2}(x) \right] \right\}, \quad (47)$$

where $\mu = \lambda + \nu$. The physical eigenfunctions $\psi_n^{(0)}(x)$ satisfy $\psi_n^{(0)}(0) = \psi_n^{(0)}(\pi/2) = 0$. If we ask that $\psi_n^{(0)}(0) = 0$, it turns out that B = 0. Moreover, in order to avoid that the divergent behavior at $x = \pi/2$ of the hypergeometric function of the remaining term dominates over the null one induced by $\cos^{\nu}(x)$ for E > 0 it has to happen that

$$\frac{\mu}{2} \pm \sqrt{\frac{E_n}{2}} = -n \quad \Rightarrow \quad E_n = \frac{(\mu + 2n)^2}{2}, \quad n = 0, 1, 2, \dots$$
(48)

The corresponding normalized eigenfunctions are given by

$$\psi_n^{(0)} = \left[\frac{2(\mu + 2n)n!\Gamma(\mu + n)(\lambda + \frac{1}{2})_n}{\Gamma(\lambda + \frac{1}{2})\Gamma^3(\nu + \frac{1}{2})(\nu + \frac{1}{2})_n} \right]^{\frac{1}{2}} \sin^{\lambda}(x) \cos^{\nu}(x) \, {}_{2}F_{1}[-n, n + \mu; \lambda + \frac{1}{2}; \sin^{2}(x)]. \tag{49}$$

It will be important later to know the number of zeros which $u^{(0)}$ has, according to the value taken by ϵ . This information can be obtained by comparing the asymptotic behavior of $u^{(0)}$ at $x \to 0$ and $x \to \pi/2$; by picking up B = 1 and A = -b/a + q in Eq.(47) with

$$a = \frac{\Gamma(\lambda + \frac{1}{2})\Gamma(\nu - \frac{1}{2})}{\Gamma(\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}})\Gamma(\frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}})}, \quad b = \frac{\Gamma(\frac{3}{2} - \lambda)\Gamma(\nu - \frac{1}{2})}{\Gamma(\frac{1+\nu-\lambda}{2} + \sqrt{\frac{\epsilon}{2}})\Gamma(\frac{1+\nu-\lambda}{2} - \sqrt{\frac{\epsilon}{2}})},$$
(50)

it turns out that:

- If $\epsilon < E_0$, $u^{(0)}$ has either 0 or 1 nodes for q > 0 or q < 0 respectively
- In general, if $E_{i-1} < \epsilon < E_i$, $u^{(0)}$ will have either i or i+1 nodes for q > 0 or q < 0 respectively, $i = 2, 3, \ldots$

Concerning the spectral modifications which can be induced through the standard first-order supersymmetric quantum mechanics, it is obtained the following.

(a) Deleting the ground state of H_0 . Let us choose $\epsilon = E_0$ and

$$u^{(0)} = \psi_0^{(0)} \propto \sin^{\lambda}(x) \cos^{\nu}(x). \tag{51}$$

The first-order SUSY partner of V_0 becomes (see Eq.(29))

$$V_1 = \frac{\lambda(\lambda+1)}{2\sin^2(x)} + \frac{\nu(\nu+1)}{2\cos^2(x)}, \quad \lambda, \nu > 1.$$
 (52)

Since $\psi_{\epsilon}^{(1)} \propto 1/\psi_0^{(0)}$ diverges at $x = 0, \pi/2 \Rightarrow E_0 \not\in \operatorname{Sp}(H_1) = \{E_n, n = 1, 2, \dots\}$. Note that V_1 can be obtained from V_0 by changing $\lambda \to \lambda + 1$, $\nu \to \nu + 1$; this property is known nowadays as *shape invariance* [111], although it was known since Infeld-Hull work [1].

(b) Creating a new ground state. Let us take now $\epsilon < E_0$ and a nodeless $u^{(0)}$ (Eq.(47) with B=1, A=-b/a+q, q>0). As $u^{(0)} \to \infty$ when $x \to 0, \pi/2 \Rightarrow \psi_{\epsilon}^{(1)}(0)=\psi_{\epsilon}^{(1)}(\pi/2)=0$, i.e., $\psi_{\epsilon}^{(1)}$ is an eigenfunction of H_1 with eigenvalue ϵ , and therefore $\operatorname{Sp}(H_1)=\{\epsilon, E_n, n=0, 1, \ldots\}$.

In order to deal with the singularities at $x = 0, \pi/2$ induced by $u^{(0)}$ on V_1 , it is expressed $u^{(0)} = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v$, where v is a nodeless bounded function in $[0, \pi/2]$. Hence

$$V_1 = \frac{(\lambda - 2)(\lambda - 1)}{2\sin^2(x)} + \frac{(\nu - 2)(\nu - 1)}{2\cos^2(x)} - [\ln v]'', \quad \lambda, \nu > 2.$$
 (53)

(c) Isospectral potentials. They arise from the previous case (for $\epsilon < E_0$) in the limit when $u^{(0)}$ acquires a node at one of the ends of the domain so that $\psi_{\epsilon}^{(1)}$ leaves to be an eigenstate of H_1 . We can take, for example, the solution given in Eq.(47) with A=1, $B=0 \Rightarrow u^{(0)}(0)=0$. We isolate the singularities induced on V_1 by expressing $u^{(0)}=\sin^{\lambda}(x)\cos^{1-\nu}(x)v$, where v is a nodeless bounded function in $[0, \pi/2]$. Hence

$$V_1 = \frac{\lambda(\lambda+1)}{2\sin^2(x)} + \frac{(\nu-2)(\nu-1)}{2\cos^2(x)} - [\ln v]'', \quad \lambda > 1, \nu > 2.$$
 (54)

Notice that now H_1 and H_0 are isospectral.

An illustration of the several first-order SUSY partner potentials $V_1(x)$ of $V_0(x)$ (see Eqs.(52-54)) is given in Figure 1.

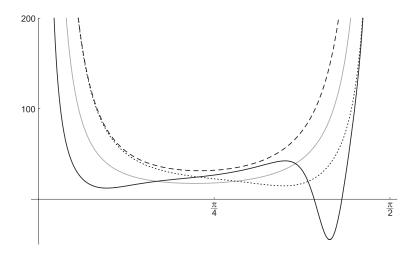


Figure 1: Trigonometric Pöschl-Teller potential $V_0(x)$ for $\lambda = 3, \nu = 4$ (gray curve) and its first-order SUSY partners $V_1(x)$ generated through the following spectral modifications: (a) deleting the ground state $E_0 = 24.5$ (dashed curve); (b) creating a new ground state at $\epsilon = 19$ (continuous curve); (c) isospectral transformation for $\epsilon = 19$ (dotted curve) [92]

4 Second-order Supersymmetric Quantum Mechanics

The second-order supersymmetric quantum mechanics can be implemented through the iteration of two first-order transformations. However, by means of this approach it is not revealed in full the rich set of spectral manipulations which is offered by the second-order

technique. This is avoided if we address the method directly, by assuming that the intertwining operator B_2^+ is of second-order in the derivatives, namely [67,68,83,112]

$$H_2B_2^+ = B_2^+ H_0, (55)$$

$$H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \quad i = 0, 2, \tag{56}$$

$$B_2^+ = \frac{1}{2} \left(\frac{d^2}{dx^2} - \eta(x) \frac{d}{dx} + \gamma(x) \right).$$
 (57)

A calculation similar as for the first-order case leads to the following set of equations [23]:

$$V_2 = V_0 - \eta', (58)$$

$$\frac{\eta''}{2} - \gamma' - \eta V_2 = 2V_0' - \eta V_0, \tag{59}$$

$$\gamma V_2 - \frac{\gamma''}{2} = V_0'' - \eta V_0' + \gamma V_0. \tag{60}$$

By decoupling this system, assuming that V_0 is a given solvable potential, we arrive at:

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V_0 + d,\tag{61}$$

$$\frac{\eta \eta''}{2} - \frac{\eta'^2}{4} + \frac{\eta^4}{4} + \eta^2 \eta' - 2V_0 \eta^2 + d\eta^2 + c = 0, \quad c, d \in \mathbb{R}.$$
 (62)

Notice that V_2 and γ become completely determined once it is found η ; this task is performed by assuming the ansätz

$$\eta' = -\eta^2 + 2\beta\eta + 2\xi, (63)$$

where the functions β and ξ are to be determined. Therefore

$$\xi^2 \equiv c, \quad \beta' + \beta^2 = 2[V_0 - \epsilon], \quad \epsilon = (d + \xi)/2, \tag{64}$$

which is once again a Riccati equation for β . By assuming that $\beta = u^{(0)'}/u^{(0)}$ it turns out that $u^{(0)}$ satisfies the stationary Schrödinger equation

$$-\frac{1}{2}u^{(0)''} + V_0 u^{(0)} = \epsilon u^{(0)}. \tag{65}$$

Since $\xi = \pm \sqrt{c}$, we get in general two different values for ϵ , $\epsilon_1 \equiv (d + \sqrt{c})/2$, $\epsilon_2 \equiv (d - \sqrt{c})/2$. We thus arrive at a natural classification of the second-order SUSY transformations, depending on the sign taken by the parameter c [112].

4.1 (i) Real case with c > 0

For c > 0 we have that $\epsilon_1, \epsilon_2 \in \mathbb{R}$, $\epsilon_1 \neq \epsilon_2$; denoting by β_1, β_2 the corresponding solutions of the Riccati equation it turns out that (compare Eq.(63))

$$\eta' = -\eta^2 + 2\beta_1 \eta + 2(\epsilon_1 - \epsilon_2), \tag{66}$$

$$\eta' = -\eta^2 + 2\beta_2 \eta + 2(\epsilon_2 - \epsilon_1). \tag{67}$$

By making the difference of both equations it is obtained:

$$\eta = -\frac{2(\epsilon_1 - \epsilon_2)}{\beta_1 - \beta_2} = \frac{2(\epsilon_1 - \epsilon_2)u_1^{(0)}u_2^{(0)}}{W(u_1^{(0)}, u_2^{(0)})} = \frac{W'(u_1^{(0)}, u_2^{(0)})}{W(u_1^{(0)}, u_2^{(0)})}.$$
 (68)

Notice that now the new potential V_2 has no extra singularities with respect to V_0 if $W(u_1^{(0)}, u_2^{(0)})$ is nodeless (compare Eq.(58)).

The spectrum of H_2 depends on the normalizability of the two eigenfunctions $\psi_{\epsilon_{1,2}}^{(2)}$ of H_2 associated to $\epsilon_{1,2}$ which belong as well to the kernel of B_2 ,

$$B_2 \psi_{\epsilon_i}^{(2)} = 0, \quad H_2 \psi_{\epsilon_i}^{(2)} = \epsilon_j \psi_{\epsilon_i}^{(2)}, \quad j = 1, 2.$$
 (69)

Their explicit expressions in terms of $u_1^{(0)}$ and $u_2^{(0)}$ become [23]

$$\psi_{\epsilon_1}^{(2)} \propto \frac{u_2^{(0)}}{W(u_1^{(0)}, u_2^{(0)})}, \quad \psi_{\epsilon_2}^{(2)} \propto \frac{u_1^{(0)}}{W(u_1^{(0)}, u_2^{(0)})}.$$
(70)

We have observed several possible spectral modifications induced on the Pöschl-Teller potentials.

(a) Deleting 2 neighbor levels. By taking as seeds two physical eigenfunctions of H_0 associated to a pair of neighbor eigenvalues, namely $\epsilon_1 = E_i$, $\epsilon_2 = E_{i-1}$, $u_1^{(0)} = \psi_i^{(0)}$, $u_2^{(0)} = \psi_{i-1}^{(0)}$ it is straightforward to show that

$$W(u_1, u_2) \propto \sin^{2\lambda + 1}(x) \cos^{2\nu + 1}(x) \mathcal{W}, \tag{71}$$

where

$$W = \frac{W\{{}_{2}F_{1}[-i, i + \mu; \lambda + \frac{1}{2}; \sin^{2}(x)], {}_{2}F_{1}[-i + 1, i - 1 + \mu; \lambda + \frac{1}{2}; \sin^{2}(x)]\}}{\sin(x)\cos(x)}$$
(72)

is a nodeless bounded function in $[0, \pi/2]$. Hence,

$$V_2 = \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu+1)(\nu+2)}{2\cos^2(x)} - (\ln \mathcal{W})'', \quad \lambda, \nu > 1.$$
 (73)

Notice that the two mathematical eigenfunctions $\psi_{\epsilon_1}^{(2)}$, $\psi_{\epsilon_2}^{(2)}$ of H_2 associated to $\epsilon_1 = E_i$, $\epsilon_2 = E_{i-1}$ do not vanish at $0, \pi/2$,

$$\lim_{x \to 0, \frac{\pi}{2}} |\psi_{\epsilon_{1,2}}^{(2)}| = \infty. \tag{74}$$

Thus, $E_{i-1}, E_i \notin \operatorname{Sp}(H_2) = \{E_0, \dots E_{i-2}, E_{i+1}, \dots\}$, i.e., we have deleted the levels E_{i-1}, E_i of H_0 in order to generate $V_2(x)$.

(b) Creating two new levels. Let us choose now $E_{i-1} < \epsilon_2 < \epsilon_1 < E_i, i = 0, 1, 2, \ldots$, $E_{-1} \equiv -\infty$, with the corresponding seeds $u^{(0)}$ as given in Eq.(47) with $B_{1,2} = 1$, $A_{1,2} = -b_{1,2}/a_{1,2} + q_{1,2}$, $q_2 < 0$, $q_1 > 0$. This choice ensures that $u_2^{(0)}$ and $u_1^{(0)}$ have i + 1 and i

alternating nodes, which produces a nodeless Wronskian [23, 96, 113]. The divergence of the $u^{(0)}$ solutions for $x \to 0, \pi/2$ is isolated by expressing

$$u_{1,2}^{(0)} = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v_{1,2}, \tag{75}$$

 $v_{1,2}$ being bounded for $x \in [0, \pi/2]$, $v_{1,2}(0) \neq 0 \neq v_{1,2}(\pi/2)$. Since the second term of the Taylor series expansion of $v_{1,2}$ is proportional to $\sin^2(x) \Rightarrow v'_{1,2}$ tends to zero as $\sin(x)$ for $x \to 0$ and as $\cos(x)$ for $x \to \pi/2$. Thus,

$$W(u_1^{(0)}, u_2^{(0)}) = \sin^{3-2\lambda}(x)\cos^{3-2\nu}(x)\mathcal{W}, \tag{76}$$

where $W = W(v_1, v_2)/[\sin(x)\cos(x)]$ is nodeless bounded in $[0, \pi/2]$. Hence

$$V_2 = \frac{(\lambda - 3)(\lambda - 2)}{2\sin^2(x)} + \frac{(\nu - 3)(\nu - 2)}{2\cos^2(x)} - (\ln \mathcal{W})'', \quad \lambda, \nu > 3.$$
 (77)

Since $\lim_{x\to 0,\frac{\pi}{2}} \psi_{\epsilon_{1,2}}^{(2)} = 0 \Rightarrow \operatorname{Sp}(H_2) = \{E_0,\ldots,E_{i-1},\epsilon_2,\epsilon_1,E_i,\ldots\}$, i.e., we have created two new levels ϵ_1,ϵ_2 between the originally neighbor ones E_{i-1},E_i for generating $V_2(x)$.

It is worth to mention some other spectral modifications which can be straightforwardly implemented [92]:

- Isospectral transformations
- Creating a new level
- Moving an arbitrary physical level
- Deleting an original physical level

An illustration of the potentials $V_2(x)$ induced by some of the previously discussed spectral modifications is given in Figure 2.

4.2 (ii) Confluent case with c = 0

For $\xi = 0$ we have that $\epsilon \equiv \epsilon_1 = \epsilon_2 \in \mathbb{R}$. It this case, once the solution β to the Riccati equation for ϵ is found, one has to solve the Bernoulli equation resultant for η ,

$$\eta' = -\eta^2 + 2\beta\eta \quad \Rightarrow \quad \eta = \frac{w'(x)}{w(x)},$$
 (78)

$$w(x) = \tilde{w}_0 + \int \exp[2\int \beta(x)dx]dx = w_0 + \int_{x_0}^x [u^{(0)}(y)]^2 dy, \tag{79}$$

$$u^{(0)} \propto \exp[\int \beta(x) \, dx],$$
 (80)

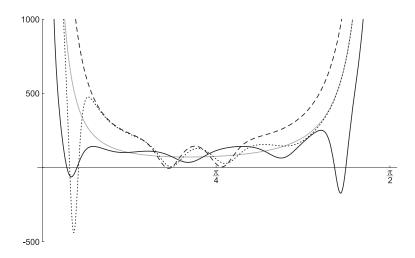


Figure 2: Trigonometric Pöschl-Teller potential $V_0(x)$ for $\lambda = 5, \nu = 8$ (gray curve) and some second-order SUSY partners $V_2(x)$ in the real case generated by producing the following spectral modifications: (a) deleting the levels $E_2 = 144.5$, $E_3 = 180.5$ (dashed curve); (b) creating two new levels at $\epsilon_1 = 128$, $\epsilon_2 = 115.52$ (continuous curve); (e) moving the level $E_2 = 144.5$ up to $\epsilon_1 = 169.28$ (dotted curve) [92]

with $w_0 \in \mathbb{R}$. Notice that now w(x) has to be nodeless, which is obtained for seed solutions $u^{(0)}$ such that

$$\lim_{x \to x_r} u^{(0)} = 0, \quad \int_{x_0}^{x_r} [u^{(0)}(y)]^2 \, dy < \infty \quad \text{or}$$
 (81)

$$\lim_{x \to x_l} u^{(0)} = 0, \quad \int_{x_l}^{x_0} [u^{(0)}(y)]^2 \, dy < \infty, \tag{82}$$

and adjusting as well w_0 , where $[x_l, x_r]$ is the x-domain of the involved problem, $x_0 \in [x_l, x_r]$. There is one eigenfunction, $\psi_{\epsilon}^{(2)} \propto u^{(0)}/w$, of H_2 associated to ϵ belonging as well to the kernel of B_2 , which implies that $\operatorname{Sp}(H_2)$ depends of the normalizability of $\psi_{\epsilon}^{(2)}$.

For the Pöschl-Teller potentials, several possibilities for modifying the initial spectrum have been observed.

(a) Creating a new level. Let us choose $\mathbb{R} \ni \epsilon \neq E_i$ and $u^{(0)}$ as given in Eq.(47) with $A=1,\ B=0$:

$$u^{(0)} = \sin^{\lambda}(x)\cos^{\nu}(x) {}_{2}F_{1}\left(\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}}; \lambda + \frac{1}{2}; \sin^{2}(x)\right)$$
(83)

$$= \sin^{\lambda}(x)\cos^{1-\nu}(x)v(x), \tag{84}$$

where v is bounded for $x \in [0, \pi/2]$, $v(0) \neq 0 \neq v(\pi/2)$. By calculating the involved integral with $x_0 = 0$ it is obtained:

$$w = w_0 + \sum_{m=0}^{\infty} \frac{(\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}})_m (\frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}})_m \sin^{2\lambda + 2m + 1}(x)}{(\lambda + \frac{1}{2})_m m! (2\lambda + 2m + 1)} \times_3 F_2 \left(\frac{1 + \lambda - \nu}{2} - \sqrt{\frac{\epsilon}{2}}, \frac{1 + \lambda - \nu}{2} + \sqrt{\frac{\epsilon}{2}}, \lambda + m + \frac{1}{2}; \lambda + \frac{1}{2}, \lambda + m + \frac{3}{2}; \sin^2(x) \right).$$
(85)

Notice that w is nodeless in $[0, \pi/2]$ for $w_0 > 0$ and it has one node for $w_0 < 0$. By choosing now the w of Eq.(85) with $w_0 > 0$, we have to isolate anyways a divergence of kind $\cos^{3-2\nu}(x)$ arising for $x \to \pi/2$ through the factorization

$$w = \cos^{3-2\nu}(x) \mathcal{W}, \tag{86}$$

 \mathcal{W} being a nodeless bounded function for $x \in [0, \pi/2]$. From Eq.(58) we finally get

$$V_2 = \frac{(\lambda - 1)\lambda}{2\sin^2(x)} + \frac{(\nu - 3)(\nu - 2)}{2\cos^2(x)} - (\ln \mathcal{W})'', \quad \lambda > 1, \nu > 3.$$
 (87)

Since $\lim_{x\to 0,\frac{\pi}{2}} \psi_{\epsilon}^{(2)} = 0$, it turns out that $\operatorname{Sp}(H_2) = \{\epsilon, E_n, n = 0, 1, \dots\}$. It is important to remark that the procedure can be implemented for solutions associated to $\epsilon > E_0$, i.e., by means of the confluent second-order transformation one is able to modify the levels above the ground state energy of H_0

Some additional possibilities of spectral manipulation are now just pointed out [92].

- Isospectral transformations
- Deletion of an arbitrary level

The potentials $V_2(x)$ induced by some of the mentioned spectral modifications are illustrated in Figure 3.

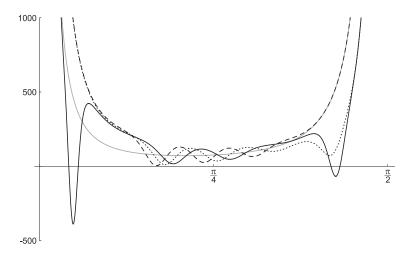


Figure 3: Trigonometric Pöschl-Teller potential $V_0(x)$ for $\lambda = 5, \nu = 8$ (gray curve) and its second-order SUSY partners $V_2(x)$ in the confluent case, generated through the following spectral modifications: (a) creating a new level at $\epsilon = 147.92$ (continuous curve); (b) isospectral transformation with $\epsilon = 162$ (dotted curve); (c) deleting the level $E_3 = 180.5$ (dashed curve) [92]

4.3 (iii) Complex case with c < 0

For c < 0 it happens that $\epsilon_{1,2} \in \mathbb{C}$, $\epsilon_2 = \bar{\epsilon}_1$. As we want that V_2 to be real, it has to be chosen $\beta_2 = \bar{\beta}_1$. Thus,

$$\eta = -\frac{2\operatorname{Im}(\epsilon_1)}{\operatorname{Im}[\beta_1]} = \frac{w'}{w}, \quad w = \frac{W(u_1^{(0)}, \bar{u}_1^{(0)})}{2(\epsilon_1 - \bar{\epsilon}_1)}, \tag{88}$$

where $u_1^{(0)}$ is a complex solution of the Schrödinger equation associated to $\epsilon_1 \in \mathbb{C}$. As in the previous cases, w has to be nodeless in order to avoid the arising of extra singularities in V_2 . For this to happens, it is sufficient that

$$\lim_{x \to x_r} u_1^{(0)} = 0 \quad \text{or} \quad \lim_{x \to x_l} u_1^{(0)} = 0. \tag{89}$$

In both cases we get that $Sp(H_2) = Sp(H_0)$.

We illustrate now the technique through the trigonometric Pöschl-Teller potentials, for which we take $\epsilon \in \mathbb{C}$, $u^{(0)}$ as in Eq.(47) with $A=1,\ B=0 \Rightarrow u^{(0)}(0)=0,\ |u^{(0)}| \to \infty$ when $x \to \pi/2$. It is appropriate to express now

$$u^{(0)} = \sin^{\lambda}(x)\cos^{1-\nu}(x)v.$$
 (90)

Hence, it is obtained:

$$w = \sin^{2\lambda+1}(x)\cos^{3-2\nu}(x)\mathcal{W}, \tag{91}$$

$$W = \frac{W(\mathbf{v}, \bar{\mathbf{v}})}{2(\epsilon - \bar{\epsilon})\sin(x)\cos(x)},\tag{92}$$

$$V_2 = \frac{(\lambda + 1)(\lambda + 2)}{2\sin^2(x)} + \frac{(\nu - 3)(\nu - 2)}{2\cos^2(x)} - (\ln \mathcal{W})'', \quad \lambda > 1, \nu > 3.$$
 (93)

The potentials V_0 and V_2 turn out to be isospectral. An illustration of $V_0(x)$ and the $V_2(x)$ of Eq.(93) is given in Figure 4.

A general conclusion, which can be inferred of the previously discussed first and secondorder SUSY QM but it is valid for an arbitrary order, is that the spectra of the initial and new Hamiltonians differ little. Thus, one would expect that some properties which are somehow related to the corresponding spectral problems would be essentially the same. We will see next that this is true, in particular, for the algebraic structure ruling the two involved Hamiltonians.

5 Algebra of the SUSY partner Hamiltonians

Given an initial Hamiltonian H_0 , which is characterized by a certain algebraic structure, it is natural to pose the following questions:

- What is the algebra ruling the potentials generated through SUSY QM?
- What other properties of H_0 are inherited by its SUSY partner Hamiltonians?

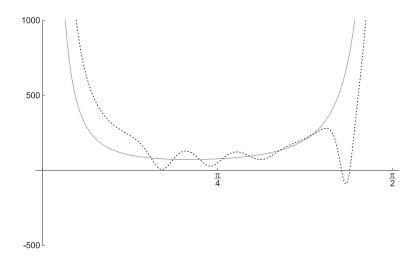


Figure 4: Trigonometric Pöschl-Teller potential $V_0(x)$ for $\lambda = 5, \nu = 8$ (gray curve) and its second-order SUSY partner $V_2(x)$ in the complex case generated by using $\epsilon = 176.344 + 1.5i$ and a solution $u^{(0)}$ vanishing at the origin (dotted curve) [92]

These questions will be analyzed here through a simple system which algebraic structure generalizes the standard Heisenberg-Weyl (oscillator) algebra. Thus, let us consider a onedimensional Schrödinger Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x),\tag{94}$$

whose eigenvectors $|\psi_n^{(0)}\rangle$ and eigenvalues E_n satisfy (for calculational convenience we switch now to the Dirac bra and ket notation)

$$H_0 |\psi_n^{(0)}\rangle = E_n |\psi_n^{(0)}\rangle, \quad E_0 < E_1 < E_2 < \dots$$
 (95)

$$H_0|\psi_n^{(0)}\rangle = E_n|\psi_n^{(0)}\rangle, \quad E_0 < E_1 < E_2 < \dots$$

$$\langle \psi_m^{(0)}|\psi_n^{(0)}\rangle = \delta_{mn}, \qquad \sum_{m=0}^{\infty} |\psi_m^{(0)}\rangle\langle\psi_m^{(0)}| = 1.$$
(95)

It is supposed that there is an analytic dependence between the eigenvalues E_n and the index labeling them, namely,

$$E_n \equiv E(n). \tag{97}$$

Let us define now the *intrinsic algebra* of H_0 , by introducing a pair of annihilation and creation operators a_0^{\pm} whose action onto the eigenvector of H_0 is given by

$$a_0^-|\psi_n^{(0)}\rangle = r_I(n)|\psi_{n-1}^{(0)}\rangle, \quad a_0^+|\psi_n^{(0)}\rangle = \bar{r}_I(n+1)|\psi_{n+1}^{(0)}\rangle,$$

$$r_I(n) = e^{i\alpha[E(n)-E(n-1)]} \sqrt{E(n)-E_0}, \quad \alpha \in \mathbb{R}.$$
(98)

$$r_I(n) = e^{i\alpha[E(n) - E(n-1)]} \sqrt{E(n) - E_0}, \quad \alpha \in \mathbb{R}.$$
(99)

With these definitions it is straightforward to show that:

$$a_0^+ a_0^- = H_0 - E_0. (100)$$

It is introduced the number operator N_0 through its action onto the eigenvectors of H_0 :

$$N_0|\psi_n^{(0)}\rangle = n|\psi_n^{(0)}\rangle, \quad [N_0, a_0^{\pm}] = \pm a_0^{\pm}.$$
 (101)

Now, the intrinsic algebra of the systems is characterized by

$$a_0^+ a_0^- = E(N_0) - E_0, \quad a_0^- a_0^+ = E(N_0 + 1) - E_0,$$
 (102)

$$[a_0^-, a_0^+] = E(N_0 + 1) - E(N_0) \equiv f(N_0). \tag{103}$$

It is important to realize that for general E(n) the intrinsic algebra becomes nonlinear (since $f(N_0)$ is not necessarily a linear function of N_0). However, this intrinsic nonlinear algebra can be linearized by deforming the corresponding annihilation and creation operators in the way

$$a_{0_L}^- = \frac{r_L(N_0 + 1)}{r_I(N_0 + 1)} a_0^-, \quad a_{0_L}^+ = \frac{r_L(N_0)}{r_I(N_0)} a_0^+, \quad r_L(n) = e^{i\alpha f(n-1)} \sqrt{n},$$
(104)

$$[N_0, a_{0_L}^{\pm}] = \pm a_{0_L}^{\pm}, \quad a_{0_L}^{+} a_{0_L}^{-} = N_0, \quad a_{0_L}^{-} a_{0_L}^{+} = N_0 + 1, \quad [a_{0_L}^{-}, a_{0_L}^{+}] = 1.$$
 (105)

In particular, for the harmonic oscillator Hamiltonian the intrinsic algebraic structure reduces to the standard Heisenberg-Weyl algebra:

$$E(n) = n + \frac{1}{2}, \quad f(n) = E(n+1) - E(n) = 1,$$
 (106)

$$[N_0, a_0^{\pm}] = \pm a_0^{\pm}, \quad [a_0^-, a_0^+] = 1.$$
 (107)

On the other hand, for the trigonometric Pöschl-Teller potentials we recover the su(1,1) algebra:

$$E(n) = \frac{(\mu + 2n)^2}{2}, \quad f(n) = E(n+1) - E(n) = 4n + 2\mu + 2, \tag{108}$$

$$[N_0, a_0^{\pm}] = \pm a_0^{\pm}, \quad [a_0^-, a_0^+] = 4N_0 + 2\mu + 2.$$
 (109)

Let us characterize now the algebraic structure of the SUSY partner Hamiltonians of H_0 [78]. In order to illustrate the procedure, we restrict ourselves just to a second-order SUSY transformation which creates two new levels at the positions ϵ_1 , ϵ_2 for H_2 , namely,

$$Sp(H_2) = \{\epsilon_1, \epsilon_2, E_n, n = 0, 1, \dots\}$$
 (110)

Notice that the eigenstates of H_2 , $|\psi_n^{(2)}\rangle$, associated to E_n , $n=0,1,\ldots$ are obtained from $|\psi_n^{(0)}\rangle$ and vice versa through the action of the intertwining operators B_2^+ , B_2 in the way:

$$|\psi_n^{(2)}\rangle = \frac{B_2^+|\psi_n^{(0)}\rangle}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}, \qquad |\psi_n^{(0)}\rangle = \frac{B_2|\psi_n^{(2)}\rangle}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}.$$
 (111)

The eigenstates of H_2 constitute a complete orthonormal set,

$$H_2|\psi_n^{(2)}\rangle = E_n|\psi_n^{(2)}\rangle, \quad H_2|\psi_{\epsilon_i}^{(2)}\rangle = \epsilon_i|\psi_{\epsilon_i}^{(2)}\rangle,$$
 (112)

$$\langle \psi_{\epsilon_i}^{(2)} | \psi_n^{(2)} \rangle = 0, \ \langle \psi_m^{(2)} | \psi_n^{(2)} \rangle = \delta_{mn}, \ \langle \psi_{\epsilon_i}^{(2)} | \psi_{\epsilon_j}^{(2)} \rangle = \delta_{ij}, \ n, m = 0, 1, \dots, \ i, j = 1, 2,$$
 (113)

$$\sum_{l=1}^{2} |\psi_{\epsilon_{l}}^{(2)}\rangle\langle\psi_{\epsilon_{l}}^{(2)}| + \sum_{m=0}^{\infty} |\psi_{m}^{(2)}\rangle\langle\psi_{m}^{(2)}| = 1.$$
 (114)

It is important to notice that the two new levels ϵ_1 , ϵ_2 can be placed at positions essentially arbitrary (either both below the initial ground state E_0 or in between two neighbor physical levels E_{i-1} , E_i). This fact somehow breaks the symmetry defined by E(n). Thus, it should be clear that we need to isolate, in a sense, the two new levels ϵ_1 , ϵ_2 . With this aim, it is defined now the number operator N_2 through the following action onto the eigenstates of H_2 :

$$N_2|\psi_n^{(2)}\rangle = n|\psi_n^{(2)}\rangle, \quad N_2|\psi_{\epsilon_i}^{(2)}\rangle = 0, \quad n = 0, 1, \dots \ i = 1, 2.$$
 (115)

Departing now from the operators a_0^{\pm} of the intrinsic algebra of H_0 , let us construct those of the *natural algebra* of H_2 [78], which represent a generalization of the operators introduced previously for the SUSY partners of the harmonic oscillator [30, 73, 74, 77],

$$a_{2_N}^{\pm} = B_2^{+} a_0^{\pm} B_2. \tag{116}$$

An schematic representation of these operators is given in Figure 5. Their action onto the eigenvectors of H_2 is given by

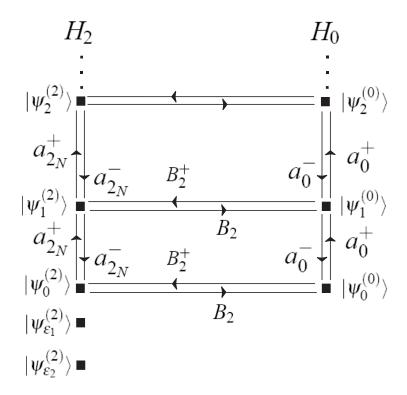


Figure 5: Representation of the annihilation and creation operators $a_{2_N}^{\pm}$ of the natural algebra of H_2 .

$$a_{2_N}^{\pm}|\psi_{\epsilon_i}^{(2)}\rangle = 0, \quad a_{2_N}^{-}|\psi_n^{(2)}\rangle = r_N(n) |\psi_{n-1}^{(2)}\rangle, \quad a_{2_N}^{+}|\psi_n^{(2)}\rangle = \bar{r}_N(n+1) |\psi_{n+1}^{(2)}\rangle,$$
 (117)

$$r_N(n) = \left\{ \prod_{i=1}^2 [E(n) - \epsilon_i] [E(n-1) - \epsilon_i] \right\}^{\frac{1}{2}} r_I(n), \quad i = 1, 2, \ n = 0, 1, \dots$$
 (118)

Using these equations it is found that

$$[a_{2_N}^-, a_{2_N}^+] = [\bar{r}_N(N_2 + 1)r_N(N_2 + 1) - \bar{r}_N(N_2)r_N(N_2)] \sum_{m=0}^{\infty} |\psi_m^{(2)}\rangle\langle\psi_m^{(2)}|. \tag{119}$$

Let us build up now the annihilation and creation operators a_2^{\pm} which generate the intrinsic algebra of H_2 ,

$$a_{2}^{-} = \frac{r_{I}(N_{2}+1)}{r_{N}(N_{2}+1)} a_{2_{N}}^{-}, \quad a_{2}^{+} = \frac{r_{I}(N_{2})}{r_{N}(N_{2})} a_{2_{N}}^{+}, \quad [a_{2}^{-}, a_{2}^{+}] = f(N_{2}) \sum_{m=0}^{\infty} |\psi_{m}^{(2)}\rangle\langle\psi_{m}^{(2)}|. \quad (120)$$

Their action onto the eigenstates of H_2 reads

$$a_2^{\pm}|\psi_{\epsilon_i}^{(2)}\rangle = 0, \quad a_2^{-}|\psi_n^{(2)}\rangle = r_I(n)|\psi_{n-1}^{(2)}\rangle,$$
 (121)

$$a_2^+|\psi_n^{(2)}\rangle = \bar{r}_I(n+1)|\psi_{n+1}^{(2)}\rangle, \quad i = 1, 2, \ n = 0, 1, \dots$$
 (122)

Notice that this algebra of H_2 is the same as the intrinsic algebra of H_0 when we work on the restriction to the subspace spanned by the eigenvectors $|\psi_n^{(2)}\rangle$ associated to the isospectral part of the spectrum.

Let us construct, finally, the annihilation and creation operators $a_{2_L}^{\pm}$ of the linear algebra of H_2 ,

$$a_{2L}^{-} = \frac{r_L(N_2+1)}{r_I(N_2+1)} a_2^{-}, \quad a_{2L}^{+} = \frac{r_L(N_2)}{r_I(N_2)} a_2^{+},$$
 (123)

$$[N_2, a_{2_L}^{\pm}] = \pm a_{2_L}^{\pm}, \qquad [a_{2_L}^-, a_{2_L}^+] = \sum_{m=0}^{\infty} |\psi_m^{(2)}\rangle\langle\psi_m^{(2)}|.$$
 (124)

Their action onto the eigenstates of H_2 is given by

$$a_{2_I}^{\pm} |\psi_{\epsilon_i}^{(2)}\rangle = 0, \quad a_{2_I}^{-} |\psi_n^{(2)}\rangle = r_L(n)|\psi_{n-1}^{(2)}\rangle,$$
 (125)

$$a_{2L}^{+}|\psi_{n}^{(2)}\rangle = \bar{r}_{L}(n+1)|\psi_{n+1}^{(2)}\rangle, \quad i=1,2, \ n=0,1,\dots$$
 (126)

Once again, this algebra of H_2 coincides with the corresponding one of H_0 on the subspace spanned by $\{|\psi_n^{(2)}\rangle, n=0,1,\ldots\}$.

6 Coherent states

Once identified the algebraic structures associated to H_0 and H_2 , it would be important to generate the corresponding coherent states. It is well know that there are several definitions of coherent states [114–117]; here we will derive them as eigenstates with complex eigenvalues of the annihilation operators of the systems. Since there are available several annihilation operators for H_0 and H_2 , we will call them with the same name as the corresponding associated algebra.

6.1 Coherent states of H_0

Since for H_0 we identified two different algebraic structures, intrinsic and linear one, we will look for the two corresponding families of coherent states.

6.1.1 Intrinsic coherent states $|z,\alpha\rangle_0$ of H_0

The intrinsic coherent states $|z,\alpha\rangle_0$ of H_0 satisfy:

$$a_0^-|z,\alpha\rangle_0 = z|z,\alpha\rangle_0, \quad z \in \mathbb{C}.$$
 (127)

Expanding $|z,\alpha\rangle_0$ in the basis of eigenstates of H_0 ,

$$|z,\alpha\rangle_0 = \sum_{n=0}^{\infty} c_n |\psi_n^{(0)}\rangle, \tag{128}$$

and substituting in Eq.(127), it is obtained a recurrence relationship for c_n ,

$$c_n = \frac{z}{r_I(n)}c_{n-1} = \frac{z^n}{r_I(n)\dots r_I(1)}c_0 = \frac{e^{-i\alpha(E_n - E_0)}z^n}{\sqrt{(E_n - E_0)\dots (E_1 - E_0)}}c_0.$$
(129)

By choosing $c_0 \in \mathbb{R}^+$ so that $|z,\alpha\rangle_0$ is normalized we finally get

$$|z,\alpha\rangle_0 = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\rho_m}} |\psi_m^{(0)}\rangle, \tag{130}$$

$$\rho_m = \begin{cases} 1 & \text{for } m = 0, \\ (E_m - E_0) \dots (E_1 - E_0) & \text{for } m > 0. \end{cases}$$
 (131)

An important property that should be satisfied is the so-called completeness relationship, namely,

$$\int |z,\alpha\rangle_0 \,_0\langle z,\alpha \,|d\mu(z) = 1, \tag{132}$$

where the positive defined measure $d\mu(z)$ can be expressed as

$$d\mu(z) = \frac{1}{\pi} \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m} \right) \rho(|z|^2) d^2 z.$$
 (133)

Working in polar coordinates and changing variables $y = |z|^2$, it turns out that $\rho(y)$ must satisfy

$$\int_{0}^{\infty} y^{m} \rho(y) \, dy = \rho_{m}, \quad m = 0, 1, \dots$$
 (134)

This is a moment problem depending on ρ_n , i.e., on the spectral function E(n) [73,77,118–122]. If one is able to solve Eq.(134) for $\rho(y)$, the completeness relation becomes valid.

Once Eq.(132) is satisfied, an arbitrary quantum state can be expressed in terms of coherent states. In particular, an intrinsic coherent state $|z', \alpha\rangle_0$ admits the decomposition,

$$|z',\alpha\rangle_0 = \int |z,\alpha\rangle_0 \,_0\langle z,\alpha|z',\alpha\rangle_0 \,d\mu(z),\tag{135}$$

where the reproducing kernel $_0\langle z,\alpha|z',\alpha\rangle_0$ is given by

$${}_{0}\langle z, \alpha | z', \alpha \rangle_{0} = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_{m}}\right)^{-1/2} \left(\sum_{m=0}^{\infty} \frac{|z'|^{2m}}{\rho_{m}}\right)^{-1/2} \left(\sum_{m=0}^{\infty} \frac{(\bar{z}z')^{m}}{\rho_{m}}\right). \tag{136}$$

As $|z=0,\alpha\rangle_0 = |\psi_0^{(0)}\rangle$, it turns out that the eigenvalue z=0 is non-degenerate. If the system is initially in an intrinsic coherent state $|z,\alpha\rangle_0$, then it will be at any time in an intrinsic coherent state,

$$U_0(t)|z,\alpha\rangle_0 = e^{-itE_0}|z,\alpha+t\rangle_0,\tag{137}$$

where $U_0(t) = \exp(-iH_0t)$ is the evolution operator of the system.

6.1.2 Linear coherent states $|z,\alpha\rangle_{0_L}$ of H_0

Now, let us look for the linear coherent states such that

$$a_{0_L}^-|z,\alpha\rangle_{0_L} = z|z,\alpha\rangle_{0_L}, \quad z \in \mathbb{C}.$$
 (138)

A procedure similar to the previous one leads straightforwardly to

$$|z,\alpha\rangle_{0_L} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{m!}} |\psi_m^{(0)}\rangle,$$
 (139)

which, up to the phases, has the form of the standard coherent states. Now the completeness relationship is automatically satisfied,

$$\frac{1}{\pi} \int |z, \alpha\rangle_{0_L} \langle z, \alpha | d^2 z = 1. \tag{140}$$

Therefore, an arbitrary linear coherent state $|z', \alpha\rangle_{0_L}$ is expressed in terms of linear coherent states,

$$|z',\alpha\rangle_{0_L} = \frac{1}{\pi} \int |z,\alpha\rangle_{0_L} \,_{0_L} \langle z,\alpha|z',\alpha\rangle_{0_L} \, d^2z, \tag{141}$$

where the reproducing kernel is given by

$$_{0_L}\langle z, \alpha | z', \alpha \rangle_{0_L} = \exp\left(-|z|^2/2 + \bar{z}z' - |z'|^2/2\right).$$
 (142)

Notice that, once again, the only eigenstate of H_0 which is also a linear coherent state is the ground state, $|z=0,\alpha\rangle_{0_L}=|\psi_0^{(0)}\rangle$. Moreover, since $[a_{0_L}^-,a_{0_L}^+]=1$ then the linear coherent states also result from the action of the displacement operator $D_L(z)$ onto the ground state $|\psi_0^{(0)}\rangle$:

$$|z,\alpha\rangle_{0_L} = D_L(z)|\psi_0^{(0)}\rangle = \exp(za_{0_L}^+ - \bar{z}a_{0_L}^-)|\psi_0^{(0)}\rangle.$$
 (143)

Finally, an initial linear coherent state evolves in time as a linear coherent state, namely,

$$U_0(t)|z,\alpha\rangle_{0_L} = e^{-itE_0}|z,\alpha+t\rangle_{0_L}.$$
(144)

6.2 Coherent states of H_2

Let us remember that we have analyzed three different algebraic structures for H_2 , namely, natural, intrinsic and linear ones; thus three families of coherent states will be next constructed.

6.2.1 Natural coherent states $|z,\alpha\rangle_{2_N}$ of H_2

The natural coherent states $|z,\alpha\rangle_{2_N}$ of H_2 are defined by

$$a_{2_N}^-|z,\alpha\rangle_{2_N} = z|z,\alpha\rangle_{2_N}. (145)$$

A procedure similar to the previously used leads to

$$|z,\alpha\rangle_{2_N} = \left[\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m}\right]^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\widetilde{\rho}_m}} |\psi_m^{(2)}\rangle, \tag{146}$$

where

$$\widetilde{\rho}_{m} = \begin{cases} 1 & \text{for } m = 0, \\ \rho_{m} \prod_{i=1}^{2} (E_{m} - \epsilon_{i})(E_{m-1} - \epsilon_{i})^{2} \dots (E_{1} - \epsilon_{i})^{2}(E_{0} - \epsilon_{i}) & \text{for } m > 0. \end{cases}$$
(147)

Notice that now the completeness relationship has to be modified to include the projector onto the subspace spanned by the eigenvectors of H_2 associated to the two isolated levels ϵ_1 , ϵ_2 , namely,

$$\sum_{i=1}^{2} |\psi_{\epsilon_i}^{(2)}\rangle\langle\psi_{\epsilon_i}^{(2)}| + \int |z,\alpha\rangle_{2_N} \langle z,\alpha| \, d\widetilde{\mu}(z) = 1, \tag{148}$$

where

$$d\widetilde{\mu}(z) = \frac{1}{\pi} \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m} \right) \widetilde{\rho}(|z|^2) d^2 z, \tag{149}$$

$$\int_{0}^{\infty} y^{m} \widetilde{\rho}(y) \, dy = \widetilde{\rho}_{m}, \quad m \ge 0.$$
 (150)

It is clear that the moment problem characterized by Eq.(150) is more complicated than the one associated to Eq.(134) (compare the moments given by Eqs.(131) and (147)). Since $B_2|\psi_{\epsilon_i}^{(2)}\rangle = a_{2N}^-|\psi_{\epsilon_i}^{(2)}\rangle = 0$, i=1,2 and $a_{2N}^-|\psi_0^{(2)}\rangle = 0$, then the eigenvalue z=0 of a_{2N}^- is 3-fold degenerate. Finally, as for the coherent states of H_0 , it turns out that the natural coherent states $|z,\alpha\rangle_{2N}$ evolve coherently with t,

$$U_2(t)|z,\alpha\rangle_{2_N} = e^{-itE_0}|z,\alpha+t\rangle_{2_N},\tag{151}$$

where $U_2(t) = \exp(-iH_2t)$ is the evolution operator of the system characterized by H_2 .

6.2.2 Intrinsic coherent states $|z, \alpha\rangle_2$ of H_2

The intrinsic coherent states $|z,\alpha\rangle_2$ of H_2 are eigenstates with complex eigenvalue $z \in \mathbb{C}$ of the intrinsic annihilation operator a_2^- of H_2 . A treatment similar to the previous one leads to

$$|z,\alpha\rangle_2 = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\rho_m}} |\psi_m^{(2)}\rangle.$$
 (152)

The modified completeness relationship reads now

$$\sum_{i=1}^{2} |\psi_{\epsilon_i}^{(2)}\rangle\langle\psi_{\epsilon_i}^{(2)}| + \int |z,\alpha\rangle_{2} \langle z,\alpha| \, d\mu(z) = 1, \tag{153}$$

where $d\mu(z)$ is the same as for H_0 . Since $a_2^-|\psi_{\epsilon_i}^{(2)}\rangle = 0$, i = 1, 2 and $|z = 0, \alpha\rangle_2 = |\psi_0^{(2)}\rangle$, the eigenvalue z = 0 of a_2^- turns out to be 3-fold degenerate. As for the natural coherent states of H_2 , the intrinsic coherent states $|z, \alpha\rangle_2$ evolve in time in a coherent way.

6.2.3 Linear coherent states $|z, \alpha\rangle_{2_L}$ of H_2

Let us consider, finally, the linear coherent states $|z,\alpha\rangle_{2_L}$ of H_2 as eigenstates of $a_{2_L}^-$ with eigenvalue $z\in\mathbb{C}$. A standard procedure produces now

$$|z,\alpha\rangle_{2L} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{m!}} |\psi_m^{(2)}\rangle.$$
 (154)

The completeness relationship is expressed as

$$\sum_{i=1}^{2} |\psi_{\epsilon_{i}}^{(2)}\rangle\langle\psi_{\epsilon_{i}}^{(2)}| + \frac{1}{\pi} \int |z,\alpha\rangle_{2_{L}} \langle z,\alpha| \, d^{2}z = 1.$$
 (155)

Once again, the eigenvalue z = 0 of a_{2L}^- is 3-fold degenerate. Moreover, the linear coherent states $|z,\alpha\rangle_{2L}$ appear from the action of the displacement operator $D_{2L}(z)$ onto the extremal state $|\psi_0^{(2)}\rangle$:

$$|z,\alpha\rangle_{2_L} = D_{2_L}(z)|\psi_0^{(2)}\rangle = \exp(za_{2_L}^+ - \bar{z}a_{2_L}^-)|\psi_0^{(2)}\rangle.$$
 (156)

As in the previous cases, the linear coherent states of H_2 evolve in time coherently.

Let us end up these two sections, about the algebraic structure of the SUSY partner Hamiltonians and the corresponding coherent states, with the following conclusions.

- The intrinsic and linear algebraic structures of H_0 are inherited by H_2 on the subspace associated to the levels of the isospectral part of the spectrum
- This reflects as well in the corresponding coherent states of H_2 , linear and intrinsic ones, whose expressions on the same subspace are equal to the corresponding ones of H_0
- We generalized successfully the construction of the natural algebra of H_2 when the initial potential $V_0(x)$ is more general than the harmonic oscillator

7 SUSY partners of periodic potentials

The exactly solvable models are important since the relevant physical information is encoded in a few expressions. Moreover, they are ideal either to test the accuracy (and convergence) of numerical techniques, or to implement some analytic approximate methods, as perturbation theory. Unfortunately, the number of exactly solvable periodic potentials is small, so it is important to enlarge this class. As we saw in the previous sections, a simple method to do the job is supersymmetric quantum mechanics. Before applying the technique to periodic potentials, however, let us review first the way in which it is classified the energy axis in allowed and forbidden energy bands.

For one-dimensional periodic potentials such that $V_0(x+T) = V_0(x)$, the Schrödinger equation

$$-\frac{1}{2}\psi''(x) + V_0(x)\psi(x) = E\psi(x), \tag{157}$$

can be conveniently expressed in matrix form:

$$\frac{d}{dx}\Psi(x) = \Lambda(x)\Psi(x), \quad \Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}, \tag{158}$$

$$\Lambda(x) = \begin{pmatrix} 0 & 1\\ 2[V_0(x) - E] & 0 \end{pmatrix}. \tag{159}$$

Therefore:

$$\Psi(x) = b(x)\Psi(0),\tag{160}$$

where the 2×2 transfer matrix b(x) is symplectic, which implies that its determinant is equal to 1. The transfer matrix can be expressed in terms of two real solutions $v_{1,2}(x)$ such that $v_1(0) = 1$, $v'_1(0) = 0$, $v_2(0) = 0$, $v'_2(0) = 1$, as follows

$$b(x) = \begin{pmatrix} \mathbf{v}_1(x) & \mathbf{v}_2(x) \\ \mathbf{v}_1'(x) & \mathbf{v}_2'(x) \end{pmatrix}. \tag{161}$$

The general behavior of ψ , and the kind of spectrum of the Hamiltonian H_0 , depends on the eigenvalues β_{\pm} of the Floquet matrix b(T), which in turn are determined by the so-called discriminant D = D(E) = Tr[b(T)]:

$$\beta^2 - D\beta + 1 = 0 \quad \Rightarrow \quad \beta_{\pm} = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} - 1}, \quad \beta_{+}\beta_{-} = 1.$$
 (162)

By picking up $\Psi(0)$ as one of the eigenvectors of the Floquet matrix b(T) with eigenvalue β , it turns out that

$$\Psi(T) = b(T)\Psi(0) = \beta\Psi(0). \tag{163}$$

Moreover, for these vectors in general we have that

$$\Psi(x + nT) = \beta^n \Psi(x). \tag{164}$$

The corresponding $\psi(x)$ are known as the Bloch functions, and they exist for any $E \in \mathbb{R}$. Depending on the value of the discriminant, the Bloch functions acquire three different asymptotic behaviors:

• If |D(E)| < 2 for a given E, it turns out that

$$\beta_{\pm} = \exp(\pm ikT),\tag{165}$$

where $k \in \mathbb{R}$ is called the *quasimomentum*. Equation (164) implies that the Bloch solutions $\psi^{\pm}(x)$ in this case are bounded for $|x| \to \infty$, and hence they can be equipped with a physical meaning. Therefore, E belong to an allowed energy band.

• The equation |D(E)| = 2 defines the band edges, denoted

$$E_0 < E_1 \le E_{1'} < E_2 \le E_{2'} < \dots < E_j \le E_{j'} < \dots$$
 (166)

For these energy values it turns out that $\beta_+ = \beta_- = \pm 1$, the two Bloch functions tend just to one which becomes periodic or antiperiodic. Thus, these energy values belong as well to the spectrum of H_0 . Moreover, the Bloch eigenfunctions ψ_j , $\psi_{j'}$ associated to E_j , $E_{j'}$ are real, they have the same number j of nodes, and both are either periodic or antiperiodic

• If |D(E)| > 2 for a given E, it turns out that

$$\beta_{+} = \beta \neq 0 \in \mathbb{R}, \quad \beta_{-} = \frac{1}{\beta}. \tag{167}$$

Then, the Bloch solutions ψ^{\pm} are unbounded either for $x \to \infty$ or for $x \to -\infty$ so that they cannot be equipped with a physical interpretation. Therefore, E belongs to a forbidden energy gap

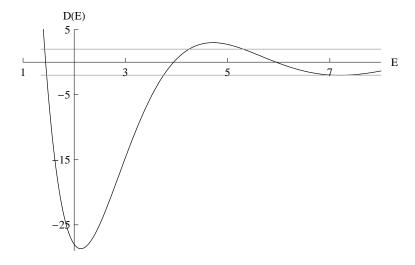


Figure 6: The discriminant D(E) versus E for $V_0(x) = 5\sin^2(x)$

As an illustration, we have plotted in Figure 6 the discriminant D(E) as a function of E for $V_0(x) = 5\sin^2(x)$.

7.1 Lamé potentials

A very special class of periodic systems are given by the Lamé potentials

$$V_0(x) = \frac{1}{2}n(n+1) m \operatorname{sn}^2(x|m), \quad n \in \mathbb{N},$$
(168)

where $\operatorname{sn}(x|m)$ is a Jacobi elliptic function. These are doubly periodic special functions defined by [99]

$$\operatorname{sn}(x|m) = \sin \varphi, \quad \operatorname{cn}(x|m) = \cos \varphi, \quad \operatorname{dn}(x|m) = \frac{d\varphi}{dx},$$
 (169)

with $x = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$. The real and imaginary periods are respectively (4K, 2iK'), (4K, 4iK'), (2K, 4iK'), where

$$K \equiv K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \ K' = K(m_1), \ m_1 = 1 - m.$$
 (170)

The special nature of the Lamé potentials is due to the following properties:

- They have 2n + 1 band edges, which define n + 1 allowed energy bands (n of them finite and 1 infinite) and n + 1 energy gaps (n finite and 1 infinite)
- It is possible to determine the general solution of the Schrödinger equation for any real value of the energy parameter
- In particular, there exist explicit analytic expressions for the band edge eigenfunctions

 Let us illustrate this behavior through the simplest example.

7.2 The Lamé potential with n = 1

In this case the Bloch functions for a real arbitrary energy value ϵ are given by

$$u^{(0)^{+}}(x) \propto \frac{\sigma(x+\omega'+a)}{\sigma(x+\omega')} e^{-x\zeta(a)}, \quad u^{(0)^{-}}(x) \propto \frac{\sigma(x+\omega'-a)}{\sigma(x+\omega')} e^{x\zeta(a)}, \quad (171)$$

$$\beta = e^{2a\zeta(\omega) - 2\omega\zeta(a)},\tag{172}$$

where $\zeta'(z) = -\wp(z)$, $[\ln \sigma(z)]' = \zeta(z)$, $\omega = K$, $\omega' = iK'$ and

$$2\epsilon = \frac{2}{3}(m+1) - \wp(a). \tag{173}$$

Note that \wp is the Weierstrass p-function, which is related to the Jacobi elliptic function sn through [123]

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(z\sqrt{e_1 - e_3})}.$$
(174)

There are now three band edges, and the explicit expression for the corresponding eigenvalues and eigenfunctions are given by

$$E_0 = \frac{m}{2}, \quad \psi_0^{(0)}(x) = \operatorname{dn} x,$$
 (175)

$$E_1 = \frac{1}{2}, \quad \psi_1^{(0)}(x) = \operatorname{cn} x,$$
 (176)

$$E_{1'} = \frac{1+m}{2}, \quad \psi_{1'}^{(0)}(x) = \operatorname{sn} x.$$
 (177)

The energy axis is classified as follows: the energy interval $(-\infty, m/2)$ represents the infinite energy gap (forbidden energies), then it comes the finite energy band [m/2, 1/2] (allowed energies), then it arises the finite energy gap (1/2, 1/2 + m/2), and finally it emerges the infinite energy band $[1/2 + m/2, \infty)$.

7.2.1 First-order SUSY using Bloch functions

Let us choose in the first place as seed one real nodeless Bloch function $u^{(0)}$ with $\epsilon \leq E_0$. It is straightforward to check that $u^{(0)'}/u^{(0)}$ is periodic, which implies that $V_1(x)$ is also periodic (see Eq.(29)). The SUSY transformation maps bounded (unbounded) eigenfunctions of H_0 into bounded (unbounded) ones of H_1 . Since for $\epsilon < E_0$ the chosen Bloch function $u^{(0)}$ diverges either for $x \to \infty$ or $x \to -\infty$ ($\epsilon \notin \operatorname{Sp}(H_0)$), then $1/u^{(0)}$ will diverge at the opposite limits and therefore $\epsilon \notin \operatorname{Sp}(H_1)$. On the other hand, for $\epsilon = E_0 \in \operatorname{Sp}(H_0)$, the corresponding Bloch solution $u^{(0)}$ is nodeless periodic and therefore $1/u^{(0)}$ is also a nodeless periodic eigenfunction of H_1 with eigenvalue E_0 . In conclusion, for first-order SUSY transformations involving a nodeless Bloch function with $\epsilon \leq E_0$ it turns out that $\operatorname{Sp}(H_1) = \operatorname{Sp}(H_0)$, i.e., the SUSY mapping is isospectral.

In particular, for the Lamé potential with n=1 the first-order SUSY transformation which employs the lowest band edge eigenfunction $\psi_0^{(0)}(x) = \operatorname{dn} x$ leads to [124, 125]

$$V_1(x) = V_0\left(x + \frac{T}{2}\right),\tag{178}$$

and it is said that $V_1(x)$ is self-isospectral to $V_0(x)$. This property was discovered, for the first time, by Braden and McFarlane [124], although the name is due to Dunne and Feinberg [125]. Moreover, by using as seed the Bloch eigenfunction $u^{(0)^{\pm}}(x)$ with $\epsilon < E_0$ is is straightforward to show that [95]

$$V_1(x) = V_0(x \pm a), (179)$$

where a is related to ϵ through Eq.(173).

7.2.2 First-order SUSY using general solutions

Let us choose now a real nodeless linear combination $u^{(0)}$ of the two Bloch functions for $\epsilon < E_0$. Although now $u^{(0)'}/u^{(0)}$ is not strictly periodic, it is asymptotically periodic when $|x| \to \infty$. This property is transferred as well to the new potential $V_1(x)$ which is asymptotically periodic for $|x| \to \infty$. As in the previous case, the bounded (unbounded) eigenfunctions

of H_0 are mapped into bounded (unbounded) eigenfunctions of H_1 . Since $\lim_{|x|\to\infty} |u^{(0)}| = \infty$, it turns out that $1/u^{(0)}$ is square-integrable in $(-\infty,\infty)$. We conclude that, when appropriate linear combinations of two Bloch functions of H_0 associated to $\epsilon < E_0$ are used the implement the first-order SUSY transformation, we create a bound state for H_1 , i.e., $\operatorname{Sp}(H_1) = \{\epsilon\} \cup \operatorname{Sp}(H_0)$.

We apply now this kind of transformation to the Lamé potential with n=1, using as seed a real nodeless linear combination of the two Bloch solutions $u^{(0)^{\pm}}(x)$ given in Eq.(171) with $\epsilon < E_0$ [95,96]. In this way it is obtained an asymptotically periodic potential, as illustrated in Figure 7, where the corresponding bound state of H_1 is as well shown.

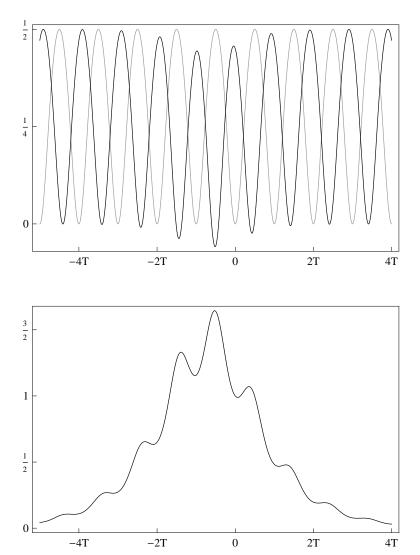


Figure 7: Up. First-order SUSY partner V_1 (black curve) of the Lamé potential (gray curve) for m = 0.5, $\epsilon = 0.22$. Down: The corresponding (unnormalized) bound state of H_1

7.2.3 Second-order SUSY using Bloch functions

Let us apply now a second-order SUSY transformation using two real Bloch solutions $u_{1,2}^{(0)}$ associated to $\epsilon_{1,2} \in (-\infty, E_0]$ or $\epsilon_{1,2} \in [E_j, E_{j'}], j = 1, 2, ...$ so that $W(u_1^{(0)}, u_2^{(0)})$ is nodeless [93]- [96]. For this choice it turns out that $W'(u_1^{(0)}, u_2^{(0)})/W(u_1^{(0)}, u_2^{(0)})$ as well as $V_2(x)$ are periodic (see Eqs.(58,68)). The SUSY transformation maps bounded (unbounded) eigenfunctions of H_0 into bounded (unbounded) ones of H_2 . Thus, H_0 and H_2 have the same band structure, namely, $Sp(H_2) = Sp(H_0)$.

For the Lamé potential with n=1, if we employ the two band edge eigenfunctions $\psi_1^{(0)}(x), \psi_{1'}^{(0)}(x)$ bounding the finite gap, we get once again [93]

$$V_2(x) = V_0\left(x + \frac{T}{2}\right). \tag{180}$$

Moreover, by using as seeds two Bloch functions $u_{1,2}^{(0)^+}(x)$ such that $\epsilon_{1,2} \in (E_1, E_{1'})$ it is straightforward to show that [95]

$$V_2(x) = V_0(x + \delta + \delta'), \quad a_1 = \delta + \omega', \quad a_2 = \delta' + \omega'.$$
 (181)

7.2.4 Second-order SUSY using general solutions

Let us choose now as seeds two real linear combinations of Bloch function for $\epsilon_{1,2} \in (-\infty, E_0]$ or $\epsilon_{1,2} \in [E_j, E_{j'}], j = 1, 2, \ldots$ so that $W(u_1^{(0)}, u_2^{(0)})$ is nodeless [95, 96]. Once again, $W'(u_1^{(0)}, u_2^{(0)})/W(u_1^{(0)}, u_2^{(0)})$ and consequently $V_2(x)$ are asymptotically periodic for $|x| \to \infty$, the second-order SUSY transformation maps bounded (unbounded) eigenfunctions of H_0 into bounded (unbounded) ones of H_2 . Since now the two eigenfunctions of H_2 , $u_2^{(0)}/W(u_1^{(0)}, u_2^{(0)})$, $u_1^{(0)}/W(u_1^{(0)}, u_2^{(0)})$ associated to ϵ_1 , ϵ_2 , can be made square-integrable for $x \in (-\infty, \infty)$, in turns out that $\operatorname{Sp}(H_2) = \{\epsilon_1, \epsilon_2\} \cup \operatorname{Sp}(H_0)$.

Let us illustrate the procedure for the Lamé potential with n=1, using as seeds two real linear combinations $u_{1,2}^{(0)}$ of $u_{1,2}^{(0)^+}$ and $u_{1,2}^{(0)^-}$ for $\epsilon_{1,2} \in (E_1, E_{1'})$. As in the general case, it is obtained a potential $V_2(x)$ which is asymptotically periodic for $|x| \to \infty$. The corresponding Hamiltonian has two bound states at the positions ϵ_1, ϵ_2 . An illustration of these results is shown in Figure 8.

Our conclusions concerning SUSY transformations applied to periodic potentials are the following.

- It was employed successfully the SUSY QM for generating new solvable potentials departing from a periodic initial one
- For seed solutions chosen as Bloch functions associated to the band edges or to factorization energies inside the spectral gaps, the new potential is periodic, isospectral to the initial one
- For seed solutions which are real linear combinations of Bloch functions, the new potential is asymptotically periodic, and the spectrum of the corresponding Hamiltonian will include a finite number of bound states embedded into the gaps

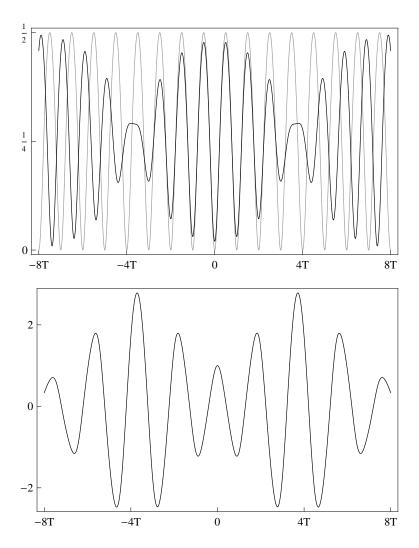


Figure 8: Up. Second-order SUSY partner V_2 (black curve) of the Lamé potential (gray curve) for $m=0.5,\ \epsilon_1=0.64,\ \epsilon_2=0.65$. Down: The (unnormalized) bound state of H_2 associated to $\epsilon_2=0.65$.

• The technique was clearly illustrated by means of the Lamé potential with n=1

8 Conclusions

- Along this paper it has been shown that supersymmetric quantum mechanics is a simple but powerful tool for generating potentials with known spectra departing from a given initial solvable one. Indeed, SUSY QM can be used to implement the spectral manipulation in quantum mechanics
- One of our aims when writing this paper is to present the development of the factorization method, in particular the further advances arising after Mielnik's paper about generalized factorizations [30]

• Supersymmetric quantum mechanics has interrelations with several interesting areas of mathematical physics such as solutions of non-linear differential equations [79–81,126–130], deformation of Lie algebras and coherent states [73–75,77,78,131,132], among others. In my opinion, a further exploration of these interrelations is needed, which represents the future of SUSY QM. We will pursue these issues in the next years with the hope of getting as many interesting results as we were obtaining previously.

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