Derivation of an inter-cluster potential from a two-fermion contact interaction

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The potential acting between a point particle and a bound state of A particles is related to the interaction between single particles. Specifically, the A-particle state is totally symmetric in coordinate space ("bosonic"), and the single particles should be amenable to a first-order description with two- and three-body contact/zero-range interactions. Furthermore, the internal space of the $A+1$ equal-mass particles is given as A-dimensional, which demands an $A+1$ body state of mixed symmetry.

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The inter-cluster potential In order to study the large-N limit, we develop a model inspired by the resonating-group formalism ([1],[2]). That entails the assumption of a frozen A-body core whose spatially symmetric wave function we parametrize with a single parameter a via

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}(\boldsymbol{r}_i-\boldsymbol{R}_A)^2} \quad ; \begin{array}{l}\boldsymbol{r}_i \ \text{: single-particle coordinates}\\ \boldsymbol{R}_A \ \text{: core centre of mass}\end{array} \quad . \tag{1}$$

The system is thereby reduced to only three degrees of freedom, namely the relative distance between core and the odd particle. The respective equation of motion reads in terms of the effective mass μ, the relative kinetic energy E between core and odd particle:

$$\int \left\{ \phi_A^* \left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E + \mathcal{V}\right)\mathcal{A}\left[\phi_A\psi(\boldsymbol{R})\right]\right\} d\boldsymbol{r}_{1\ldots A} = 0 \quad . \tag{2}$$

Antisymmetrization is required between two particles only, $\mathcal{A} = \mathbb{1} - P_{A,A+1}$, and the interaction is effective, only if it involves the odd particle:

$$\mathcal{V} = C_0(\Lambda)\sum_{i}^{A-1}\delta_\Lambda^{(3)}(\boldsymbol{r}_i - \boldsymbol{r}_{A+1}) \tag{3}$$

$$+ D_1(\Lambda)\sum_{i<j}^{A-1}\Big[\delta_\Lambda^{(3)}(\boldsymbol{r}_i-\boldsymbol{r}_j)\,\delta_\Lambda^{(3)}(\boldsymbol{r}_i-\boldsymbol{r}_{A+1}) \tag{4}$$

$$+\delta_\Lambda^{(3)}(\boldsymbol{r}_i-\boldsymbol{r}_{A+1})\,\delta_\Lambda^{(3)}(\boldsymbol{r}_j-\boldsymbol{r}_{A+1})\Big] \quad . \tag{5}$$

The contribution from the identical copy of the $(A+1)^{\text{th}}$ particle – without loss of generality identified with label A – interacting is excluded, because we anticipate its vanishing because of antisymmetrization. As in our calculations, the zero-range contact forces are smeared out in order to obtain regular solutions – in contrast to, *e.g.* , Bethe-Peierls [3] boundary conditions – this *a priori* exclusion introduces artefacts. We assume that those are insignificant relative to the other terms from the inter-fragment interaction.

The integration in (2) yields a three-dimensional Schrödinger equation with a non-local potential

$$\left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E\right)\psi(\boldsymbol{R}) + \sum_{n=1}^{3}\eta_n\, e^{-\kappa_n\boldsymbol{R}^2}\psi(\boldsymbol{R}) - \sum_{n=1}^{4}\zeta_n\int\left\{\mathcal{O}_n e^{-\alpha_n\boldsymbol{R}^2-\beta_n\boldsymbol{R}\cdot\boldsymbol{R}'-\gamma_n\boldsymbol{R}'^2}\right\}\psi(\boldsymbol{R}')d\boldsymbol{R}' = 0 \quad . \tag{6}$$

The coefficients $\alpha,\ldots,\kappa$ are functions of the number of core particles A, the core size a, the interaction regulator Λ, and the single-particle mass m, and

$$\begin{aligned}\mathcal{O}_1 = &-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E \qquad (7)\\ \rightarrow &-\frac{\hbar^2}{2\mu}\left(4\alpha_1^2\boldsymbol{R}^2 + \beta_1^2\boldsymbol{R}'^2 + 4\alpha_1\beta_1\boldsymbol{R}\cdot\boldsymbol{R}' - 2\alpha_1\right) - E \quad ,\end{aligned}$$

while $\mathcal{O}_{2,3,4} = \mathbb{1}$. The sign (+) of the first sum – the direct interaction – is that of the microscopic LEC. In contrast, the second, non-local part reverts the sign. For example, the effect of an attractive 2-body potential becomes a repulsive contribution due to antisymmetrization in this term.

A few comments are in order. Firstly, $\zeta_{1\ldots4} = 0$ if $\mathcal{A} = \mathbb{1}$, *i.e.* , the inter-cluster potential is local. The first non-local term encodes the so-called exchange interaction. It is non-zero even in the absence of inter-particle forces. The two- and three-body contact forces affect the inter-cluster potential structurally in the same way. However, the respective coefficients differ significantly in their dependence on A, a and Λ. It is the combination of both, the two- and three-body terms, which results in the changing character of the interaction, the formation of attractive and repulsive regions, and thereby the possibility to bind the odd particle to the core. We postpone a detailed analysis of the sensitivity of the inter-cluster potential, for now, and continue with the discussion of the emerging spectrum of (6).

Partial-wave projection We solve Eq.(6) by expanding the total wave function①

$$\psi(\boldsymbol{R}) = R^{-1}\sum_{lm}\phi_{lm}(R)Y_{lm}(\hat{\boldsymbol{R}}) \tag{8}$$

and a projection with the integral operator

$$\int d^2\hat{\boldsymbol{R}}\; Y^*_{lm}(\hat{\boldsymbol{R}}) \tag{9}$$

from the left. The uncoupling of different partial waves becomes explicit when

$$e^{-\beta\boldsymbol{R}\cdot\boldsymbol{R}'} = 4\pi\sum_{LM} i^L j_L(i\beta RR')Y^*_{LM}(\hat{\boldsymbol{R}})Y_{LM}(\hat{\boldsymbol{R}}') \tag{10}$$

is substituted. In the $n = 1$ part, which encodes the exchange effect on the free Hamiltonian, we apply the Laplacian before making the above and following substitutions:

$$\boldsymbol{R}\cdot\boldsymbol{R}' = -\sqrt{3}\left[\,\boldsymbol{R}_p \otimes \boldsymbol{R}'_{-p}\,\right]^{00} \tag{11}$$

with $\boldsymbol{r}_m = \sqrt{\frac{4\pi}{3}}rY_{1,m}(\hat{\boldsymbol{r}})$. Using Eq.(4.6.3) and Eq.(3.7.8) of [4], we obtain the equation of motion for a single partial wave:

$$\left(\frac{\hbar^2}{2\mu}\left[-\partial_R^2 + \frac{l(l+1)}{R^2}\right] - E\right)\phi_{lm}(R) \tag{12a}$$

$$-(-E)\int\left(4\pi\, i^l\cdot\zeta_1\cdot j_l(i\beta_1 RR')\right)\cdot e^{-\alpha_1\boldsymbol{R}^2-\gamma_1\boldsymbol{R}'^2}\phi_{lm}(R')\; R'dR' \tag{12b}$$

$$-\left(\frac{\hbar^2}{2\mu}\right)\int(4\pi\cdot\zeta_1)\cdot e^{-\alpha_n\boldsymbol{R}^2-\gamma_n\boldsymbol{R}'^2}\cdot\Bigg\{\left[-(4\alpha_1^2R^2+\beta_1^2R'^2-2\alpha_1)+\frac{l(l+1)}{R^2}\right]i^l j_l(i\beta_1 RR')$$
$$+(4\alpha_1\beta_1)\cdot RR'\cdot\left(i^{l-1}j_{l-1}(i\beta_1RR')(2l-3)\begin{pmatrix}1 & l-1 & l\\ 0 & 0 & 0\end{pmatrix}^2 + i^{l+1}j_{l+1}(i\beta_1RR')(2l-1)\begin{pmatrix}1 & l+1 & l\\ 0 & 0 & 0\end{pmatrix}^2\right)\Bigg\}\;\phi_{lm}(R')\; R'dR' \tag{12c}$$

$$-\sum_{n=2}^{4}\zeta_n\int(4\pi\, i^l)\cdot j_l(i\beta_n RR')\cdot e^{-\alpha_n\boldsymbol{R}^2-\gamma_n\boldsymbol{R}'^2}\;\;\phi_{lm}(R')\; R'\,dR' \tag{12d}$$

$$+\sum_{n=1}^{3}\eta_n\; e^{-\kappa_n\boldsymbol{R}^2}\phi_{lm}(R) \tag{12e}$$

$$= 0\;\;.$$

This equation defines a generalized Eigenvalue problem

$$\int dR'\;\hat{\mathcal{D}}_{RR'}\;\phi_{lm}(R') = E\int dR'\;\left(\delta_{RR'}+\hat{\mathcal{K}}_{RR'}\right)\phi_{lm}(R') \tag{13}$$

whose solution we expand in a finite set of harmonic oscillator functions②.

② $\phi_{nl\nu}(R) = N_{nl\nu}R^{l+1}e^{-\nu R^2}L_n^{l+1/2}(2\nu R^2)$ in terms of a normalizing constant $N_{nl\nu} = \left(\left(\frac{2\nu^3}{\pi}\right)^{1/2}\cdot\frac{2^{2l+n+3}n!\nu^l}{(2n+2l+1)!!}\right)^{1/2}$ and gener-

n	η_n	κ_n
1		
2		
3		

n	ζ_n	α_n	β_n	γ_n
1				
2				
3				
4				

(14)

TABLE I: Numerical values which specify the core-particle motion via Eq. (12) for $\Lambda = 4\,\mathrm{fm}^{-1}$, $A = 4$, $m_N = 938$ MeV, $a = 0.56$ fm, $\hbar c = 197$ MeV·fm.

Example: 5-body system In the following, we analyse features of the effective interaction for the specific case of four internal degrees of freedom and a system of five particles with identical masses. With the numerical values as given in table 14, this example pertains to nuclear physics as described in leading order with the pionless effective field theory.

alized Laguerre polynomials $L_n^m(x)$.

APPENDIX: THE CORE WAVE FUNCTION

Our ansatz for an A-boson system in its ground state (symmetric in space *wrt.* particle exchange) is

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}\bar{\boldsymbol{r}}_{\mathrm{i}}^2} = e^{-a\sum^{A-1}\bar{\boldsymbol{r}}_{\mathrm{i}}^2 - a\sum_{i<j}^{A-1}\bar{\boldsymbol{r}}_{\mathrm{i}}\cdot\bar{\boldsymbol{r}}_{\mathrm{j}}} \quad . \tag{15}$$

Instead of single-particle coordinates, the usage of cluster coordinates identifies the centre of mass as the origin of a "harmonic", effective potential, in which the particles reside in their independent-particle/mean-field ground state. All observables of the system in this state depend thus on a single parameter, the (oscillator) width a. The root-mean-square radius, in particular, relates to a via

$$\mathfrak{r}^2 = \frac{\int_{\mathbb{R}^{3(A-1)}} d(\bar{\boldsymbol{r}}_1,\ldots,\bar{\boldsymbol{r}}_{\mathrm{A-1}})\ \sum_{i=1}^{A-1}\bar{\boldsymbol{r}}_{\mathrm{i}}^2\phi_A^2}{\int_{\mathbb{R}^{3(A-1)}} d(\bar{\boldsymbol{r}}_1,\ldots,\bar{\boldsymbol{r}}_{\mathrm{A-1}})\ \phi_A^2} \tag{16}$$

$$= \frac{3}{2}\cdot\frac{(A-1)^2}{A}\cdot a^{-1} \overset{A\gg 1}{\cong} \frac{3}{2}\cdot A\cdot a^{-1} \quad . \tag{17}$$

We assume the correlation between the radius $\mathfrak{r}$ and the ground-state wave function of the 3-nucleon system, to hold for any spatially symmetric A-boson system. However, the functional relation, $\mathfrak{r} = f\,[A, B(3), B(2)]$, is unknown (to the best of our knowledge).

The numerical results obtained for $B(3) = 1.5, 3, 4$ MeV at $B(2) = 1$ MeV yield an almost linear increase: $\mathfrak{r} \propto A$ for $A = 3\ldots 6$. The rate of this increase is independent of $B(3)$, while its starting point is set by $B(3)$ – the $\mathfrak{r}$ curve for some $B(3)$ is parallel and above all curves of smaller $B(3)$.

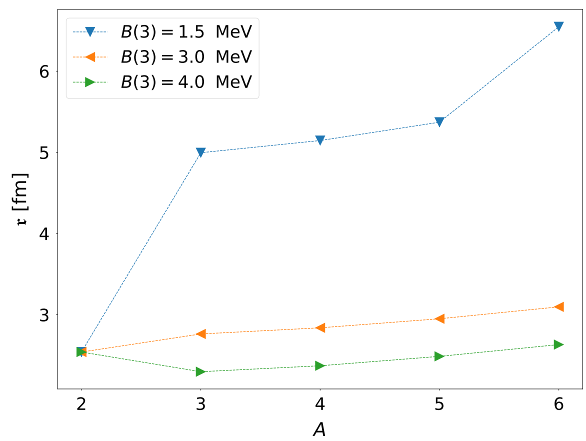

FIG. 1: Root-mean-square radii (defined in (16)) of a bound, A-boson system for various 3-body constraints at fixed $B(2) = 1$ MeV and $\Lambda = 2\,\mathrm{fm}^{-1}$.

Assuming instead A-incompressible spheres deformed into a large sphere yields $V = \frac{4}{3}\pi\mathfrak{r}^3 \propto A$ and a slower with A growth of the systems.

For an appropriate parametrization of the core wave function we use the numerical values for $A = 6$ at $\Lambda = 2\,\mathrm{fm}^{-1}$: $a = \frac{25}{4}\ \mathfrak{r}^{-2} = 0.15^{③}, 0.65, 0.90\ \mathrm{fm}^{-2}$.

$$a = \begin{cases} \frac{3}{2\ r_{\mathrm{drop}}} A^{2/3} & \text{liquid drop} \\ \frac{3}{2\ r_{\mathrm{chain}}} A^{-1} & \text{linear chain} \end{cases}$$

APPENDIX: INTER-CLUSTER POTENTIAL

Consider two compound systems, whose relative motion is much slower compared with the motion of the particles within each of these fragments. Within an arbitrarily small time interval, the probability of a given particle to interact/hit/overlap with another particle is then enhanced for the partner belonging to the same compound.

$$P_{dt}(\mathrm{intra}) \gg P_{dt}(\mathrm{inter})$$

or

$$\#(\text{internal collisions}) \gg \#(\text{inter-cluster collisions})\ \ .$$

The relative motion of particles within each cluster is then decoupled from, *one*, the relative motion *wrt.* the other cluster(s), *and two*, the internal motion in those. In turn, the internal structure affects cluster-relative motion. This reasoning underlies the single-channel resonating-group approximation and also foreshadows the non-hermitian character of the effective interaction.

$$\widehat{M}_{\mathrm{D}} = \begin{pmatrix} 4a & & & \\ & 4a & & (2a)_{\triangledown} \\ (2a)_{\triangle} & & \ddots & \\ & & & 4a \end{pmatrix}\ ;\ \boldsymbol{\mathcal{S}}_{\mathrm{D}} = \mathbf{0}\ ;\ B_{\mathrm{D}} = (R - R') \tag{18}$$

③ $\mathfrak{r}(A = 6, B(3) = 1.5)$ was calculated at $\Lambda = 1\,\mathrm{fm}^{-1}$ due to numerical hindrances at larger cutoffs. The converged value is expected to be significantly smaller, as suggested by the outlier in Fig.1.

[1] John A. Wheeler. On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure. *Phys. Rev.*, 52:1107–1122, 1937.

[2] K. Wildermuth and Y. C. Tang. *A Unified Theory of the Nucleus*. Vieweg+Teubner Verlag, Wiesbaden, 1977.

[3] H. Bethe, R. Peierls, and Douglas Rayner Hartree. Quantum theory of the diplon. *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences*, 148(863):146–156, 1935.

[4] A. R. EDMONDS. *Angular Momentum in Quantum Mechanics*. Princeton University Press, 1985.

TABLE II: Defining parameters of the effective potential between a Gaussian A-body core, characterized via the width a (1), and one *odd* particle (see (6)). The 2- and 3-body LECs C_0^Λ and D_1^Λ are calibrated to a 2- and 3-body symmetric bound state (see table **??**). $A' = A-1, A'' = A-2, \ldots$. The coefficients η_i, ζ_i **do** consider non-zero interacting pair and triplet contributions from $\sum_{i<A}$ and $\sum_{\substack{i<A\\ j<A-1}}$ but **not** $(-1)^{\mathfrak{p}}$.

i	η_i	κ_i		
1	$8\,C_0^\Lambda \frac{A'}{\left(4+\frac{A'}{A}\frac{\Lambda^2}{a}\right)^{3/2}}$	$\frac{A\Lambda^2}{4A+A'\frac{\Lambda^2}{a}}$		
2	$\frac{32D_1^\Lambda a^3 A''A'A^{3/2}}{\left(16a^2A+4a(3A-1)\Lambda^2+A''\Lambda^4\right)^{3/2}}$	$\frac{\Lambda^2\left(4a^2A+2aA\Lambda^2\right)}{16a^2A+4a(3A-1)\Lambda^2+A''\Lambda^4}$		
3	$\frac{32D_1^\Lambda A''A'}{\left(\frac{\left(4a+\Lambda^2\right)\left(4aA+A''\Lambda^2\right)}{a^2A}\right)^{3/2}}$	$\frac{2aA\Lambda^2}{4aA+A''\Lambda^2}$		
n	ζ_i	α_n	β_n	γ_n
1	$\frac{2\sqrt{2}}{\left(\frac{A'(A+1)^2}{aA^3}\right)^{3/2}}$	$\frac{a\left(A^3+A\right)}{2A'(A+1)^2}$	$\frac{2aA^2}{A'(A+1)^2}$	$\frac{a\left(A^3+A\right)}{2A'(A+1)^2}$
2	$\frac{8C_0^\Lambda a^3A'A^{9/2}}{\pi^{3/2}(A+1)^3\left(4aA'+A''\Lambda^2\right)^{3/2}}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(3A^2+A+2\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A''\Lambda^2\right)}$	$\frac{4aA^2\left(2a+\Lambda^2\right)}{(A+1)^2\left(4aA'+A''\Lambda^2\right)}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(A^2-A+2\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A''\Lambda^2\right)}$
3	$\frac{32D_1^\Lambda A''A'(aA)^{9/2}}{\pi^{3/2}(A+1)^3\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)^{3/2}}$	$\frac{aA\left(16a^2\left(A^2+1\right)+4a\left(5A^2+A+4\right)\Lambda^2+\left(5A^2+2A+3\right)\Lambda^4\right)}{2(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$	$\frac{2aA^2\left(16a^2+16a\Lambda^2+3\Lambda^4\right)}{(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$	$\frac{aA\left(16a^2\left(A^2+1\right)+4a\left(3A^2-A+4\right)\Lambda^2+\left(A^2-2A+3\right)\Lambda^4\right)}{2(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$
4	$\frac{32D_1^\Lambda A''A'}{\pi^{3/2}\left(\frac{(A+1)^2\left(4a+\Lambda^2\right)\left(4aA'+A'''\Lambda^2\right)}{a^3A^3}\right)^{3/2}}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(5A^2+2A+3\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$	$\frac{2aA^2\left(4a+3\Lambda^2\right)}{(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(A^2-2A+3\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$