Derivation of an inter-cluster potential from a two-fermion contact interaction

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The potential acting between a point particle and a bound state of A particles is related to the interaction between single particles. Specifically, the A-particle state is totally symmetric in coordinate space ("bosonic"), and the single particles should be amenable to a first-order description with two- and three-body contact/zero-range interactions. Furthermore, the internal space of the $A+1$ equal-mass particles is given as A-dimensional, which demands an $A+1$ body state of mixed symmetry.

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The inter-cluster potential In order to study the large-N limit, we develop a model inspired by the resonating-group formalism ([1],[2]). That entails the assumption of a frozen A-body core whose spatially symmetric wave function we parametrize with a single parameter a via

$$\phi_A := e^{-\frac{a}{2}\sum_{i=1}^{A}(\boldsymbol{r}_i-\boldsymbol{R}_A)^2} \quad ; \begin{array}{l} \boldsymbol{r}_i \ : \text{single-particle coordinates} \\ \boldsymbol{R}_A \ : \text{core centre of mass} \end{array} \quad . \tag{1}$$

The system is thereby reduced to only three degrees of freedom, namely the relative distance between core and the odd particle. The respective equation of motion reads in terms of the effective mass μ, the relative kinetic energy E between core and odd particle:

$$\int \left\{ \phi_A^* \left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E + \mathcal{V}_{A,A+1} \right) \mathcal{A}\left[\phi_A \psi(\boldsymbol{R})\right] \right\} d\boldsymbol{r}_{1\ldots A} = 0 \tag{2}$$

Antisymmetrization is required between two particles only, $\mathcal{A} = \mathbb{1} - P_{A,A+1}$, and the interaction term is given by

$$\mathcal{V}_{A,A+1} = C_0(\Lambda)\sum_{i<j}^{X} \delta_\Lambda^{(3)}\big(\boldsymbol{r}_i - \boldsymbol{r}_j\big) + D_1(\Lambda) \sum_{\substack{i<j<k \\ \text{cyclic}}}^{X} \delta_\Lambda^{(3)}\big(\boldsymbol{r}_i - \boldsymbol{r}_j\big)\, \delta_\Lambda^{(3)}\big(\boldsymbol{r}_j - \boldsymbol{r}_k\big) \ . \tag{3}$$

The integration in (2) yields a three-dimensional Schrödinger equation with a non-local potential

$$\left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E \right)\psi(\boldsymbol{R}) + \sum_{n=1}^{3}\eta_n\ e^{-\kappa_n \boldsymbol{R}^2}\psi(\boldsymbol{R}) - \sum_{n=1}^{4}\zeta_n \int \left\{ \mathcal{O}_n e^{-\alpha_n \boldsymbol{R}^2 - \beta_n \boldsymbol{R}\cdot\boldsymbol{R}' - \gamma_n \boldsymbol{R}'^2} \right\} \psi(\boldsymbol{R}')d\boldsymbol{R}' = 0 \ . \tag{4}$$

The coefficients $\alpha,\ldots,\kappa$ are functions of the number of core particles A, the core size a, the interaction regulator Λ, and the single-particle mass m, and

$$\begin{aligned} \mathcal{O}_1 = & -\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}_R^2 - E \\ \to & -\frac{\hbar^2}{2\mu}\left(4\alpha_1^2\boldsymbol{R}^2 + \beta_1^2\boldsymbol{R}'^2 + 4\alpha_1\beta_1\boldsymbol{R}\cdot\boldsymbol{R}' - 2\alpha_1\right) - E \ , \end{aligned} \tag{5}$$

while $\mathcal{O}_{2,3,4} = \mathbb{1}$. The sign (+) of the first sum – the direct interaction – is that of the microscopic LEC. In contrast, the second, non-local part reverts the sign. For example, the effect of an attractive 2-body potential becomes a repulsive contribution due to antisymmetrization in this term.

A few comments are in order. Firstly, $\zeta_{1\ldots4} = 0$ if $\mathcal{A} = \mathbb{1}$, *i.e.* , the inter-cluster potential is local. The first non-local term encodes the so-called exchange interaction. It is non-zero even in the absence of inter-particle forces. The two- and three-body contact forces affect the inter-cluster potential structurally in the same way. However, the respective coefficients differ significantly in their dependence on A, a and Λ. It is the combination of both, the two- and three-body terms, which results in the changing character of the interaction, the formation of attractive and repulsive regions, and thereby the possibility to bind the odd particle to the core. We postpone a detailed analysis of the sensitivity of the inter-cluster potential, for now, and continue with the discussion of the emerging spectrum of (4).

Partial-wave projection We solve Eq.(4) by expanding

$$\psi(\boldsymbol{R}') = R'^{-1} \sum_{l'm'} \psi_{l'm'}(R') Y_{l'm'}(\hat{\boldsymbol{R}}') \tag{6}$$

and a projection with the integral operator

$$R \int d^2 \hat{\boldsymbol{R}}\; Y^*_{lm}(\hat{\boldsymbol{R}}) \tag{7}$$

from the left. The uncoupling of different partial waves becomes explicit when

$$e^{-\beta \boldsymbol{R}\cdot\boldsymbol{R}'} = 4\pi \sum_{LM} i^L j_L(i\beta R R') Y^*_{LM}(\hat{\boldsymbol{R}}) Y_{LM}(\hat{\boldsymbol{R}}') \tag{8}$$

is substituted. In the $n = 1$ part, which encodes the exchange effect on the free Hamiltonian, we apply the Laplacian before making the above and following substitutions:

$$\boldsymbol{R}\cdot\boldsymbol{R}' = -\sqrt{3}\left[\,\boldsymbol{R}_p \otimes \boldsymbol{R}'_{-p}\,\right]^{00} \tag{9}$$

with $\boldsymbol{r}_m = \sqrt{\frac{4\pi}{3}} r Y_{1,m}(\hat{\boldsymbol{r}})$. Using Eq.(4.6.3) and Eq.(3.7.8) of [3], we obtain the equation of motion for a single partial wave:

$$\left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}^2_R - E\right)\psi_{lm}(R) + \sum_{n=1}^{3} \eta_n\; e^{-\kappa_n \boldsymbol{R}^2}\psi_{lm}(R) \tag{10}$$

$$-\sum_{n=2}^{4} \zeta_n \int\int (4\pi i^l)\cdot RR'\cdot j_l(i\beta_n RR')\cdot e^{-\alpha_n \boldsymbol{R}^2 - \gamma_n \boldsymbol{R}'^2}\;\psi(R')\; d(R,R') \tag{11}$$

$$-(-E)\zeta_1 \int\int (4\pi i^l)\cdot RR'\cdot j_l(i\beta_1 RR')\cdot e^{-\alpha_1 \boldsymbol{R}^2 - \gamma_1 \boldsymbol{R}'^2}\;\psi(R')\; d(R,R') \tag{12}$$

$$-\left(-\frac{\hbar^2}{2\mu}\right)\zeta_1 \int\int (4\pi) RR'\cdot e^{-\alpha_n \boldsymbol{R}^2 - \gamma_n \boldsymbol{R}'^2}\;\psi_{lm}(R')\cdot\Bigg\{(4\alpha_1^2 R^2 + \beta_1^2 R'^2 - 2\alpha_1) i^l j_l(i\beta_1 RR') \tag{13}$$

$$+(4\alpha_1\beta_1)\cdot RR'\cdot\left(i^{l-1} j_{l-1}(i\beta_1 RR')(2l-3)\begin{pmatrix}1 & l-1 & l\\ 0 & 0 & 0\end{pmatrix}^2 + i^{l+1} j_{l+1}(i\beta_1 RR')(2l-1)\begin{pmatrix}1 & l+1 & l\\ 0 & 0 & 0\end{pmatrix}^2\right)\Bigg\}\; d(R,R') \tag{14}$$

$$= 0\;. \tag{15}$$

APPENDIX: INTER-CLUSTER POTENTIAL

[1] John A. Wheeler. On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure. *Phys. Rev.*, 52:1107–1122, 1937.

[2] K. Wildermuth and Y. C. Tang. *A Unified Theory of the Nucleus.* Vieweg+Teubner Verlag, Wiesbaden, 1977.

[3] A. R. EDMONDS. *Angular Momentum in Quantum Mechanics.* Princeton University Press, 1985.

TABLE I: Defining parameters of the effective potential between a Gaussian A-body core, characterized via the width a (1), and one *odd* particle (see (4)). The 2- and 3-body LECs C_0^Λ and D_1^Λ are calibrated to a 2- and 3-body symmetric bound state (see table **??**). $A' = A-1, A'' = A-2, \ldots$. The coefficients η_i, ζ_i **do** consider non-zero interacting pair and triplet contributions from $\sum_{i<A}$ and $\sum_{\substack{i<A\\ j<A-1}}$ but **not** $(-1)^{\mathfrak{p}}$.

i	η_i	κ_i		
1	$8\ C_0^\Lambda \ \frac{A'}{\left(4+\frac{A'}{A}\frac{\Lambda^2}{a}\right)^{3/2}}$	$\frac{A\Lambda^2}{4A+A'\frac{\Lambda^2}{a}}$		
2	$\frac{32D_1^\Lambda a^3 A'' A'}{\left(16a^2A+4a(3A-1)\Lambda^2+A''\Lambda^4\right)^{3/2}}$	$\frac{\Lambda^2\left(4a^2A+2aA\Lambda^2\right)}{16a^2A+4a(3A''+5)\Lambda^2+A''\Lambda^4}$		
3	$\frac{32D_1^\Lambda A''A'}{\left(\frac{\left(4a+\Lambda^2\right)\left(4aA+A''\Lambda^2\right)}{a^2A}\right)^{3/2}}$	$\frac{2aA\Lambda^2}{4aA+A''\Lambda^2}$		
n	ζ_i	α_n	β_n	γ_n
1	$\frac{2\sqrt{2}}{\left(\frac{A'(A+1)^2}{aA^3}\right)^{3/2}}$	$\frac{a\left(A^3+A\right)}{2A'(A+1)^2}$	$\frac{2aA^2}{A'(A+1)^2}$	$\frac{a\left(A^3+A\right)}{2A'(A+1)^2}$
2	$\frac{8C_0^\Lambda a^3 A' A^{9/2}}{\pi^{3/2}(A+1)^3\left(4aA'+A''\Lambda^2\right)^{3/2}}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(3A^2+A+2\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A''\Lambda^2\right)}$	$\frac{4aA^2\left(2a+\Lambda^2\right)}{(A+1)^2\left(4aA'+A''\Lambda^2\right)}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(A^2-A+2\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A''\Lambda^2\right)}$
3	$\frac{32D_1^\Lambda A''A'(aA)^{9/2}}{\pi^{3/2}(A+1)^3\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)^{3/2}}$	$\frac{aA\left(16a^2\left(A^2+1\right)+4a\left(5A^2+A+4\right)\Lambda^2+\left(5A^2+2A+3\right)\Lambda^4\right)}{2(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$	$\frac{2aA^2\left(16a^2+16a\Lambda^2+3\Lambda^4\right)}{(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$	$\frac{aA\left(16a^2\left(A^2+1\right)+4a\left(3A^2-A+4\right)\Lambda^2+\left(A^2-2A+3\right)\Lambda^4\right)}{2(A+1)^2\left(16a^2A'+4a(3A-4)\Lambda^2+A'''\Lambda^4\right)}$
4	$\frac{32D_1^\Lambda A''A'}{\pi^{3/2}\left(\frac{(A+1)^2\left(4a+\Lambda^2\right)\left(4aA'+A'''\Lambda^2\right)}{a^3A^3}\right)^{3/2}}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(5A^2+2A+3\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$	$\frac{2aA^2\left(4a+3\Lambda^2\right)}{(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$	$\frac{aA\left(4a\left(A^2+1\right)+\left(A^2-2A+3\right)\Lambda^2\right)}{2(A+1)^2\left(4aA'+A'''\Lambda^2\right)}$