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LOW-ENERGY THEOREMS FOR NUCLEAR POLARIZABILITIES

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Abstract: Low-energy theorems are derived for the nuclear polarizabilities of the photon scattering amplitude in a non-relativistic frame work. Center-of-mass recoil corrections are included in first order. We find, as a generalization of Friar's result, that EL contributions vanish to order k^{2L-2} for even L. For this result gauge conditions were crucial. To second order in the photon energy our results are equivalent to the low-energy theorem of Friar.

1. Introduction

Photon scattering as a pure electromagnetic process is a very interesting tool for the study of the internal structure of a system, because the photon is a clean and weak probe with a well-known interaction. Since the reaction proceeds in general via a two-step process, the whole internal dynamics of the system is participating in the intermediate propagation of the system.

Of special interest have been in the past low-energy theorems, which are a consequence of the gauge invariance of the electromagnetic interaction. Already in 1951 Sachs and Austern [1]) were the first to show by using gauge invariance and considering dipole radiation only that for nuclear photon scattering the low-energy expansion of the elastic photon scattering amplitude up to second order in the photon energy is completely determined by a constant and a quadratic term. The constant term is given by the classical Thomson amplitude of the nucleus as a whole determined by its charge and mass, whereas the quadratic term, often called Rayleigh amplitude, is governed by the static electric polarizability.

The same result has been obtained later by Silbar *et al.* [2]), using the explicit form of the electromagnetic interaction which follows from minimal substitution in the kinetic energy. This, however, means that they implicitly excluded exchange and momentum-dependent forces. Therefore, this approach cannot be considered satisfactory because the exchange character is an important ingredient of the nuclear interaction. In fact, their derivation would have failed in the presence of such forces.

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On the other hand, this demonstrates the power of gauge invariance arguments in the derivation of Sachs and Austern, where no assumption on the nuclear interaction was necessary.

In 1954 Low [3]) and independently Gell-Mann and Goldberger [4]) have shown that the low-energy theorem can be derived from very basic principles in quantum field theory. They found that due to Lorentz and again gauge invariance, the elastic photon scattering amplitude of a spin-$\frac{1}{2}$ particle is determined up to first order in the photon energy by its charge, mass and magnetic moment. In the light of their result it is evident that the low-energy expansion of refs. [1,2]) is not completely correct. Only for a spin-zero nucleus the term linear in the photon energy will be absent.

With respect to the Rayleigh amplitude Ericson and Hüfner [5]) pointed out that in addition to the electric polarizability para- and diamagnetic susceptibilities contribute as well as center-of-mass corrections. They emphasized that in order to obtain this result a correct treatment of the c.m. energies is necessary. However, their derivation was still lacking in rigor because the c.m. contribution to the current had been neglected [6]). Furthermore, exchange- and momentum-dependent forces had to be excluded specifically, because they had to rely on the explicit form of the seagull amplitude (A^2 amplitude) as obtained from the kinetic energy by minimal substitution. Again only spin-zero nuclei had been considered.

Finally, Friar [6]) gave a rigorous and complete low-energy expansion of the photon scattering amplitude up to second order including consistently all c.m. recoil corrections of order $1/M_A$, where M_A is the nuclear mass. His approach is in the spirit of Sachs and Austern emphasizing again the important rôle of gauge conditions for the electromagnetic current and the seagull amplitude. Because of these conditions it was not necessary to specify the form of the intrinsic nuclear current operator nor the seagull operator. The only additional assumptions were (i) analyticity of matrix elements with respect to the photon momenta allowing a power series expansion, and (ii) the explicit non-relativistic form of the c.m. convection current.

The purpose of this work is to generalize the low-energy expansion of the scattering amplitude by deriving low-energy theorems for the generalized polarizabilities [7–11]). They are derived from an expansion of the photon scattering amplitude into multipole fields of the incoming as well as the scattered photon. The polarizabilities are independent quantities and can in principle be determined from experiment. Our derivation of the low-energy theorem for them makes use of the same physical input as Friar's work [6]) and we will recover his result when restricting to terms up to second order only.

In sect. 2 we briefly review the definitions of generalized polarizabilities and derive relations to the partial wave helicity amplitudes [12]). The general non-relativistic form of the photon scattering amplitude including c.m. recoil contributions up to order M_A^{-1} is given in sect. 3. Furthermore, the implications of gauge invariance on the scattering amplitude are discussed. Then, in sect. 4, explicit expressions for the polarizabilities are derived. It will be shown that for electric contributions certain cancellations between the contributions from two-photon, resonance and c.m. recoil amplitude will occur due to the gauge conditions. Finally the general low-energy expansion for the polarizabilities is derived in sect. 5 with explicit expressions up to second order in the photon energy.

2. Polarizabilities and partial wave helicity amplitudes

For the basic formalism pertinent to nuclear photon scattering we will rely on earlier work [13]). Only those parts which are of some special importance will be briefly reviewed here. The transverse gauge is used throughout.

The scattering of a photon with momentum $\boldsymbol{k}$ and polarization λ into a photon with momentum $\boldsymbol{k}'$ and polarization λ' is described by the scattering amplitude

$$T^{\mathrm{fi}}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k}) = \langle \mathrm{f};\, I_{\mathrm{f}} M_{\mathrm{f}} |\, \hat{T}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k})\, | \mathrm{i};\, I_{\mathrm{i}} M_{\mathrm{i}} \rangle \tag{1}$$

with the scattering operator $\hat{T}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k})$. During the scattering process the nuclear target will undergo a transition from the initial state $|\mathrm{i}, \boldsymbol{P}_{\mathrm{i}}\rangle$ to the final state $|\mathrm{f}, \boldsymbol{P}_{\mathrm{f}}\rangle$. These states are characterized by a label for the intrinsic state (i, f) as well as the total momentum $(\boldsymbol{P}_{\mathrm{i}}, \boldsymbol{P}_{\mathrm{f}})$. The spins and spin projections on some chosen quantization axis of these states will be denoted by $(I_{\mathrm{i}}, M_{\mathrm{i}})$ and $(I_{\mathrm{f}}, M_{\mathrm{f}})$, respectively. Note that the scattering operator $\hat{T}$ acts on the intrinsic states only but might depend on initial and final nuclear c.m. momentum $\boldsymbol{P}_{\mathrm{i}}$ and $\boldsymbol{P}_{\mathrm{f}}$.

A convenient decomposition of eq. (1) into multipole components is provided in terms of an expansion into polarizabilities [7-11]),

$$\begin{aligned} T^{\mathrm{fi}}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k}) = (-)^{I_{\mathrm{f}}-M_{\mathrm{i}}} \sum_{LL'JMM'm} (-)^{L'+L} \hat{J}^2 \begin{pmatrix} I_{\mathrm{f}} & J & I_{\mathrm{i}} \\ -M_{\mathrm{f}} & m & M_{\mathrm{i}} \end{pmatrix} \begin{pmatrix} L & L' & J \\ M & M' & -m \end{pmatrix} \\ \times P_J^{L'L\lambda'\lambda}(k',k) D^{L'}_{M'-\lambda'}(R') D^{L}_{M\lambda}(R) \end{aligned} \tag{2}$$

with the general definition of the polarizabilities

$$\begin{aligned} P_J^{L'L\lambda'\lambda}(k',k) = \frac{(-)^{L'+L-I_{\mathrm{f}}}}{(8\pi^2)^2} \hat{L}'^2 \hat{L}^2 \sum_{M_{\mathrm{f}} M_{\mathrm{i}} MM'm} (-)^{M_{\mathrm{i}}} \begin{pmatrix} I_{\mathrm{f}} & J & I_{\mathrm{i}} \\ -M_{\mathrm{f}} & m & M_{\mathrm{i}} \end{pmatrix} \begin{pmatrix} L & L' & J \\ M & M' & -m \end{pmatrix} \\ \times \int \mathrm{d}R' \int \mathrm{d}R\, D^{L'*}_{M'-\lambda'}(R') D^{L*}_{M\lambda}(R) T^{\mathrm{fi}}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k}) \,. \end{aligned} \tag{3}$$

R and R' denote the rotations, which rotate the quantization axis into the direction of $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively (see fig. 1). For the rotation matrices we use the convention of Rose [14]).

Furthermore, it is useful to introduce polarizabilities with definite parity transfers for incoming and outgoing photon, namely

$$P_J(M^{\nu'}L', M^{\nu}L, k', k) = \tfrac{1}{4} \sum_{\lambda',\lambda=\pm 1} \lambda'^{\nu'} \lambda^{\nu} P_J^{L'L\lambda'\lambda}(k',k) \,, \tag{4}$$

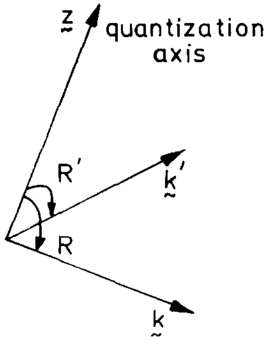

Fig. 1. Scattering geometry and the rotations R and R' carrying the quantization axis into k and k', respectively.

in terms of which one has

$$P_J^{L'L\lambda'\lambda}(k', k) = \sum_{\nu,\nu'=0,1} \lambda'^{\nu'}\lambda^{\nu} P_J(M^{\nu'}L', M^{\nu}L, k, k') , \tag{5}$$

with $M^0 = E$ ($\nu = 0$, electric) and $M^1 = M$ ($\nu = 1$, magnetic). A simple parity selection rule follows if parity is conserved, viz.

$$P_J(M^{\nu'}L', M^{\nu}L, k', k) = 0 , \qquad \text{if } (-)^{L'+\nu'+L+\nu} \neq \pi_i\pi_f , \tag{6}$$

where the parities of the initial and final states have been denoted by π_i and π_f, respectively.

Let us now briefly remind the reader of the physics contained in the decomposition (2) of the photon scattering amplitude. As can be immediately seen from eq. (3), the polarizabilities correspond to an expansion of the incoming and scattered photon into multipole fields of order L and L', respectively, while the total angular momentum transferred to the nucleus for given multipoles L' and L is denoted by J, which is constrained by the conditions $|L-L'| \leq J \leq L+L'$ and $|I_i - I_f| \leq J \leq I_i + I_f$. Polarizabilities with $J=0$ are called scalar, those with $J=1$ vector and with $J=2$ tensor polarizabilities. A useful graphical representation of the polarizabilities is given in fig. 2. We want to emphasize that the decomposition into polarizabilities as defined by eqs. (2) and (3) is independent of any assumptions about the nuclear dynamics involved. Furthermore, eq. (2) resembles a complete separation of the

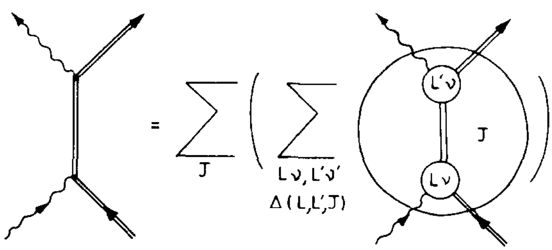

Fig. 2. Decomposition of the scattering amplitude according to angular momentum and parity transfer.

scattering geometry, contained in the rotation matrices from the internal dynamics of the system represented by the polarizabilities.

Another useful expansion of the photon scattering amplitude (eq. (1)) is provided in terms of partial wave helicity amplitudes [12]) (PWHA) defined in the photon-nucleus c.m. system

$$T_{\lambda'\lambda_f\lambda\lambda_i}(\boldsymbol{k}', \boldsymbol{k}) = \sum_j \hat{j}^2 T^j_{\lambda'\lambda_f\lambda\lambda_i}(k', k) D^j_{\lambda-\lambda_i, \lambda'-\lambda_f}(\tilde{R}) . \tag{7}$$

Here $\tilde{R} = \tilde{R}(\phi, \theta, -\phi)$ is the rotation of $\boldsymbol{k}'$ into $\boldsymbol{k}$, where (θ, ϕ) denote the spherical angles of $\boldsymbol{k}'$ in a coordinate system, in which the direction of $\boldsymbol{k}$ is chosen to be the z-axis. The helicities of the nuclear initial and final states, i.e. the spin projections along the momenta of the initial and final nucleus are denoted by λ_i and λ_f, respectively. Just as the polarizabilities the PWHA do not depend on the specific geometry of the scattering process, i.e. they are independent of the angle between $\boldsymbol{k}'$ and $\boldsymbol{k}$. They represent a different coupling scheme (see fig. 3).

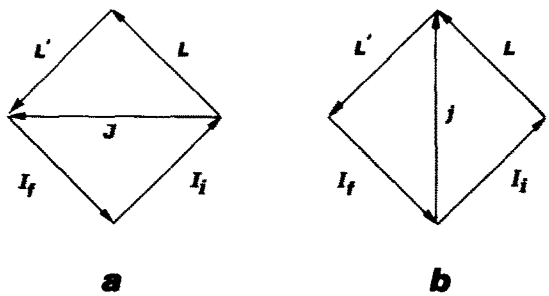

Fig. 3. The coupling schemes for the polarizabilities (a) and the partial wave helicity amplitudes (b).

Since both the polarizabilities as well as the PWHA have been used often in the literature (the latter mostly in high energy physics), it is useful to show how these quantities are related to each other.

We will start by writing the scattering amplitude in the helicity basis in terms of the scattering amplitude of eq. (1)

$$T_{\lambda'\lambda_f\lambda\lambda_i}(\boldsymbol{k}', \boldsymbol{k}) = \sum_{M_f M_i} D^{I_f *}_{M_f, -\lambda_f}(R') D^{I_i}_{M_i, -\lambda_i}(R) \langle f; I_f M_f | \hat{T}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) | i; I_i M_i \rangle . \tag{8}$$

Then, we insert on the right-hand side the decomposition into polarizabilities from eq. (2) and take the quantization axis in the direction of $\boldsymbol{k}$. Therefore, R is the identity and R' can be identified with $\tilde{R}$. This gives then the helicity scattering amplitude in terms of the polarizabilities.

$$\begin{aligned} T_{\lambda'\lambda_f\lambda\lambda_i}(\boldsymbol{k}', \boldsymbol{k}) = (-)^{1-I_i} \sum_{L'LJj} (-)^{L'+L+j} \hat{j}^2 \hat{J}^2 \begin{pmatrix} L & I_i & j \\ \lambda & -\lambda_i & \lambda_i - \lambda \end{pmatrix} \\ \times \begin{pmatrix} I_f & L' & j \\ \lambda_f & -\lambda' & \lambda' - \lambda_f \end{pmatrix} \begin{Bmatrix} I_f & I_i & J \\ L & L' & j \end{Bmatrix} \\ \times P_J^{L'L\lambda'\lambda}(k', k) D^j_{\lambda_i - \lambda, \lambda_f - \lambda'}(\tilde{R}) , \end{aligned} \tag{9}$$

where we have used the addition theorem of the rotation matrices. Comparison of eq. (9) with (7) yields the PWHA as a linear combination of the polarizabilities.

$$T^j_{\lambda'\lambda_f\lambda\lambda_i}(k',k)=(-)^{1-\lambda_i-\lambda_f+I_i+j}\sum_{L'LJ}(-)^{L'+L}\hat{J}^2\begin{pmatrix}L & I_i & j\\ \lambda & -\lambda_i & \lambda_i-\lambda\end{pmatrix}$$
$$\times\begin{pmatrix}I_f & L' & j\\ \lambda_f & -\lambda' & \lambda'-\lambda_f\end{pmatrix}\begin{Bmatrix}I_f & I_i & J\\ L & L' & j\end{Bmatrix}P_J^{L'L\lambda'\lambda}(k',k)\,. \tag{10}$$

It shows the transition from one coupling scheme of fig. 3 to the other. Inverting eq. (10), we can write the polarizabilities as linear combinations of the PWHA

$$P_J^{L'L\lambda'\lambda}(k',k)=(-)^{1+L'+L-I_i}\hat{L}'^2\hat{L}^2\sum_j(-)^{-j}\hat{j}^2\begin{Bmatrix}I_f & I_i & J\\ L & L' & j\end{Bmatrix}$$
$$\times\sum_{\lambda_f\lambda_i}(-)^{\lambda_f+\lambda_i}\begin{pmatrix}L & I_i & j\\ \lambda & -\lambda_i & \lambda_i-\lambda\end{pmatrix}\begin{pmatrix}I_f & L' & j\\ \lambda_f & -\lambda' & \lambda'-\lambda_f\end{pmatrix}T^j_{\lambda'\lambda_f\lambda\lambda_i}\,. \tag{11}$$

While all the above results do not depend on any assumptions about the structure of the scattering amplitude, we now will derive a very useful relation, which holds if the scattering operator can be written in the form

$$\hat{T}_{\lambda'\lambda}(\boldsymbol{k}',\boldsymbol{k})=\sum_{l'l}\int d^3x'\,d^3x\,e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'}\varepsilon'^{*}_{\lambda',l'}\hat{T}_{l'l}(\boldsymbol{x}',\boldsymbol{x},k_0',k_0)\varepsilon_{\lambda,l}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{12}$$

In this case we may employ the well-known decomposition of the plane wave [15])

$$\boldsymbol{\varepsilon}_\lambda\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}=-\sqrt{2\pi}\sum_{LM,\nu=0,1}\lambda^\nu\hat{L}\boldsymbol{A}^{[L]}_{\nu,M}(k,\boldsymbol{x})D^L_{M\lambda}(R)\,, \tag{13}$$

with the electric ($\nu=0$) and magnetic ($\nu=1$) multipole fields

$$\boldsymbol{A}_0^{[L]}(k,\boldsymbol{x})=\frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{A}_1^{[L]}(k,\boldsymbol{x})\,, \tag{14}$$

$$\boldsymbol{A}_1^{[L]}(k,\boldsymbol{x})=i^Lj_L(kx)\boldsymbol{Y}_L^{[L]}(\hat{\boldsymbol{x}})\,. \tag{15}$$

Inserting eq. (12) into (1) and using (13), one easily finds

$$P_J^{fi}(M^{\nu'}L',M^{\nu}L,k',k)=\langle I_f\|\hat{P}^{[J]}(M^{\nu'}L',M^{\nu}L,k',k)\|I_i\rangle$$
$$=2\pi(-)^{L+J}\frac{\hat{L}'\hat{L}}{\hat{J}}\left\langle I_f\left\|\int d^3x'\,d^3x\sum_{l'l}[A^{[L']}_{\nu',l'}(k',\boldsymbol{x}')\right.\right.$$
$$\left.\left.\times A^{[L]}_{\nu,l}(k,\boldsymbol{x})]^{[J]}\times\hat{T}_{l'l}(\boldsymbol{x}',\boldsymbol{x},k_0',k_0)\right\|I_i\right\rangle\,. \tag{16}$$

This formula will be the starting point for our derivation of the low-energy theorems.

In closing this section, we would like to point out that the scattering operator $\hat{T}_{l'l}(\boldsymbol{x}',\boldsymbol{x})$ is in general non-local, because the excited system propagates from point

x, where the incoming photon is absorbed, to the point x', where the outgoing photon is emitted or vice versa.

3. Non-relativistic scattering operator and gauge conditions

We shall briefly review the non-relativistic form of the scattering operator and the gauge conditions. The latter will be instrumental for the derivation of low-energy expansions for the different polarizabilities. These results are model-independent as they do not require any specific assumptions about the nuclear dynamics involved.

In the following we will restrict ourselves to elastic scattering. The generalization to inelastic scattering is straightforward. We start out by separating the photon scattering operator into two parts, one which contributes to first order perturbation theory and is called two-photon operator (TPO), $\hat{B}_{l'l}(x', x)$, and another part, the resonance operator $\hat{R}_{l'l}(x', x)$, which emerges in second-order perturbation theory [13])

$$\hat{T}_{l'l}(x', x, k_0', k_0) = -\hat{B}_{l'l}(x', x) + \hat{R}_{l'l}(x', x, k_0', k_0) . \tag{17}$$

The TPO is independent from k_0 and k_0' in the non-relativistic domain. It consists of the TPO's of the individual nucleons plus a meson exchange contribution [16,17]). The resonance amplitude is determined by the electromagnetic current operator $\hat{J}_\mu(x) = (\hat{\rho}(x), \hat{J}(x))$. It contains c.m. contributions from the c.m. convection current and from c.m. energies. They are obtained by separating the c.m. contribution to the hamiltonian and the electromagnetic current operator

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{c.m.}} = \hat{H}_0 + \frac{1}{2M_A} P^2 , \tag{18}$$

$$\hat{J}(x) = \hat{j}(x) + \hat{j}^{\text{c.m.}}(x) , \tag{19}$$

where

$$\hat{j}^{\text{c.m.}}(x) = \frac{1}{2M_A} \{P, \hat{\rho}(x)\} \tag{20}$$

is the c.m. convection current operator. The total momentum operator is denoted by P and the target mass by M_A. The resonance operator is then expanded up to first order in $1/M_A$.

$$\hat{R}_{l'l}(x', x, k_0', k_0) = \hat{R}^0_{l'l}(x', x, k_0', k_0) + \hat{R}^{\text{c.m.}}_{l'l}(x', x, k_0', k_0) , \tag{21}$$

with

$$\hat{R}^0_{l'l}(x', x, k_0', k_0) = \hat{j}_{l'}(x')\hat{G}(k_0)\hat{j}_l(x) + \begin{pmatrix} x \leftrightarrow x' \\ l \leftrightarrow l' \\ k_0 \leftrightarrow -k_0' \end{pmatrix} , \tag{22}$$

where

$$\hat{G}(k_0) = (\hat{H}_0 - \varepsilon_i - k_0 - i\varepsilon)^{-1} \tag{23}$$

is the intrinsic propagator. The c.m. recoil term $\hat{R}^{\text{c.m.}}$ is of order M_A^{-1}.

It is evident that the explicit form of the c.m. recoil operator is frame dependent while the intrinsic operator $\hat{R}^0_{l'l}$ is independent, because the intrinsic wave functions do not depend on the c.m. motion in the non-relativistic limit. In view of our previous comparison with the helicity amplitude, we will now choose the photon-nucleus c.m. frame in contrast to ref. [13]), where we have used the Breit frame. As is shown in detail in appendix A, one finds ($k_0' = k_0$ for elastic scattering)

$$\hat{R}^{\text{c.m.}}_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0) = \frac{1}{M_A}\Big(\tfrac{1}{2}k_0^2 \frac{\partial}{\partial k_0} \hat{R}^0_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0) - \boldsymbol{\nabla}_{x'} \cdot \boldsymbol{\nabla}_x \hat{j}_l(\boldsymbol{x}) \hat{G}(-k_0)^2 \hat{j}_{l'}(\boldsymbol{x}') - i(\nabla_{l'} \hat{j}_l(\boldsymbol{x}) \hat{G}(-k_0) \hat{\rho}(\boldsymbol{x}') - \hat{\rho}(\boldsymbol{x}) \hat{G}(-k_0) \nabla'_l \hat{j}_{l'}(\boldsymbol{x}'))\Big), \tag{24}$$

The corresponding expression for the Breit frame is given in appendix A, too. For the derivation of the low-energy theorems it will be useful to separate the intermediate ground state contribution to $\hat{G}(k_0)$

$$\hat{G}(k_0) = -\frac{1}{k}\hat{P} + \hat{G}_Q(k_0), \tag{25}$$

$$\hat{G}_Q(k_0) = \hat{Q}\hat{G}(k_0)\hat{Q} = \hat{G}_Q(0) + k_0 \hat{G}^2_Q + k_0^2 \hat{G}^3_Q + \cdots, \tag{26}$$

where $\hat{P}$ projects on the ground state and $\hat{Q} = \hat{\mathbb{1}} - \hat{P}$ projects off the ground state. Then, one finds

$$\hat{R}^0_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0) = \hat{R}^0_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0)_P + \hat{R}^0_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0)_Q \tag{27}$$

for the intrinsic operator, where the first term describes the intermediate ground state contribution.

Correspondingly, one has for the c.m. recoil operator

$$\hat{R}^{\text{c.m.}}_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0) = \hat{R}^{\text{c.m.}}_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0)_P + \hat{R}^{\text{c.m.}}_{l'l}(\boldsymbol{x}', \boldsymbol{x}, k_0)_Q . \tag{28}$$

Now, we will study the consequences of gauge invariance. The most important one is that the above separation into two-photon and resonance part (eq. (17)) is not gauge independent, i.e., the operators $\hat{\boldsymbol{B}}_{l'l}$ and $\hat{\boldsymbol{R}}_{l'l}$ are related by gauge conditions. This connection is usually expressed by the condition [6,17])

$$\sum_{l'} \frac{\partial}{\partial x'_{l'}} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x}) = i[\hat{\rho}(\boldsymbol{x}'), \hat{J}_l(\boldsymbol{x})] \tag{29}$$

or by using the continuity equation

$$\boldsymbol{\nabla} \cdot \hat{\boldsymbol{J}}(\boldsymbol{x}) = -i[\hat{H}, \hat{\rho}(\boldsymbol{x})], \tag{30}$$

one finds

$$\sum_{l'l} \frac{\partial}{\partial x'_{l'}} \frac{\partial}{\partial x_l} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x}) = [\hat{\rho}(\boldsymbol{x}'), [\hat{H}, \hat{\rho}(\boldsymbol{x})]]. \tag{31}$$

Separating out the c.m. part of current and hamiltonian operators according to eqs. (18) through (20), we obtain the following gauge condition for the intrinsic operators

$$\sum_{l'} \frac{\partial}{\partial x'_{l'}} \hat{B}_{l'l}(x', x) = i[\hat{\rho}(x'), \hat{j}_l(x)] + \frac{1}{M_A} \frac{\partial}{\partial x'_l} \hat{\rho}(x')\hat{\rho}(x) , \tag{32}$$

where we have used

$$\begin{aligned}[\hat{\rho}(x'), \hat{j}_l^{\text{c.m.}}(x)] &= \frac{1}{2M_A}[\hat{\rho}(x'), \{\hat{P}_l, \hat{\rho}(x)\}] \\ &= \frac{1}{2M_A}\{[\hat{\rho}(x'), \hat{P}_l], \hat{\rho}(x)\} \\ &= -\frac{i}{M_A} \frac{\partial}{\partial x'_l} \hat{\rho}(x')\hat{\rho}(x) ,\end{aligned} \tag{33}$$

assuming that the charge density operators at different points commute. Similarly eq. (31) becomes

$$\sum_{l'l} \frac{\partial}{\partial x'_{l'}} \frac{\partial}{\partial x_l} \hat{B}_{l'l}(x', x) = [\hat{\rho}(x'), [\hat{H}_0, \hat{\rho}(x)]] + \frac{1}{M_A} \sum_l \frac{\partial}{\partial x'_l} \hat{\rho}(x') \frac{\partial}{\partial x_l} \hat{\rho}(x) . \tag{34}$$

4. The nuclear polarizabilities in the non-relativistic limit

We will now use the defining equation (16) and the non relativistic form of the scattering operator as given in sect. 3 in order to derive explicit expressions for the polarizability operators $\hat{P}^{[J]}(M^{\nu'}L', M^{\nu}L, k)$. For this purpose we rewrite the multipole fields as defined in eqs. (14) and (15) in the following form

$$\begin{aligned}A_0^{[L]}(k, x) &= A_{\text{el}}^{[L]}(k, x) \\ &= -\frac{1}{\hat{L}}(\sqrt{L+1}\, a_{L-1}^{[L]}(k, x) + \sqrt{L}\, a_{L+1}^{[L]}(k, x)) \\ &= \frac{i}{k}\sqrt{\frac{L+1}{L}}\, \nabla c^{[L]}(k, x) - \frac{\hat{L}}{\sqrt{L}}\, a_{L+1}^{[L]}(k, x) ,\end{aligned} \tag{35}$$

$$\begin{aligned}A_1^{[L]}(k, x) &= A_{\text{mag}}^{[L]}(k, x) \\ &= a_L^{[L]}(k, x) ,\end{aligned} \tag{36}$$

where we introduce

$$a_K^{[L]}(k, x) = i^K j_K(kx)\, Y_K^{[L]}(\hat{x}) , \tag{37}$$

$$c^{[L]}(k, x) = i^L j_L(kx)\, Y^{[L]}(\hat{x}) . \tag{38}$$

The second form of the electric multipole field is particularly useful, because one can transform the gradient term with the help of the gauge conditions. For the transverse electric multipole moment one obtains the well known expression (Siegert theorem)

$$\hat{T}_0^{[L]}(k) = \int d^3x\, \hat{\boldsymbol{j}}(\boldsymbol{x}) \cdot \boldsymbol{A}_0^{[L]}(k, \boldsymbol{x})$$

$$= -\frac{1}{k}\sqrt{\frac{L+1}{L}}\,[\hat{H}_0, \hat{M}^{[L]}(k)] - \frac{\hat{L}}{\sqrt{L}}\,\hat{\mathcal{T}}_{L+1}^{[L]}(k)\,, \tag{39}$$

where

$$\hat{M}^{[L]}(k) = \int d^3x\, \hat{\rho}(\boldsymbol{x}) c^{[L]}(k, \boldsymbol{x})\,, \tag{40}$$

$$\hat{\mathcal{T}}_K^{[L]}(k) = \int d^3x\, \hat{\boldsymbol{j}}(\boldsymbol{x}) \cdot \boldsymbol{a}_K^{[L]}(k, \boldsymbol{x})\,. \tag{41}$$

Furthermore, one finds for the two-photon operator using the gauge condition (36)

$$\sum_{l'} \int d^3x' \nabla'_{l'} c^{[L]}(k, \boldsymbol{x}) \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x})$$

$$= -i[\hat{M}^{[L]}(k), \hat{j}_l(\boldsymbol{x})] + \frac{1}{M_A}\int d^3x' \nabla'_l c^{[L]}(k, \boldsymbol{x}') \hat{\rho}(\boldsymbol{x}') \hat{\rho}(\boldsymbol{x})\,. \tag{42}$$

With the help of the gradient formula

$$\boldsymbol{\nabla} c^{[L]} = ik \sum_K c_{LK} \boldsymbol{a}_K^{[L]}\,, \tag{43}$$

where

$$c_{LK} = (-)^L \hat{K} \begin{pmatrix} K & L & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

one obtains for the second term of the r.h.s. of (42)

$$\int d^3x' \nabla'_l c^{[L]}(k, \boldsymbol{x}') \hat{\rho}(\boldsymbol{x}') = ik \sum_K c_{LK} \int d^3x' a_{K,l}^{[L]}(k, \boldsymbol{x}') \hat{\rho}(\boldsymbol{x}')$$

$$= ik \sum_K c_{LK} [\hat{M}^{[K]}(k) \times e_l^{[1]}]^{[L]}\,. \tag{45}$$

We will frequently use the following relation

$$\sum_l [O'^{[L']}_{K',l} \times O^{[L]}_{K,l}]^{[J]} = (-)^{K'+L+J} \hat{L}' \hat{L} \begin{Bmatrix} L & L' & J \\ K' & K & 1 \end{Bmatrix} \times [\Omega'^{[K']} \times \Omega^{[K]}]^{[J]} \tag{46}$$

for vector multipole operators of the type

$$\boldsymbol{O}_K^{[L]} = [\Omega^{[K]} \times \boldsymbol{e}^{[1]}]^{[L]}\,, \tag{47}$$

where $\Omega^{[K]}$ is a multipole operator and

$$e_\mu^{[1]} \quad (\mu = -1, 0, +1) \tag{48}$$

a set of spherical unit vectors.

This relation is easily proved by recoupling

$$\sum_l [[\Omega'^{[K']} \times e_l^{[1]}]^{[L']} \times [\Omega^{[K]} \times e_l^{[1]}]^{[L]}]^{[J]}$$

$$= \sum_{\Lambda S} \hat{L}'\hat{L}\hat{\Lambda}\hat{S} \begin{Bmatrix} K' & 1 & L' \\ K & 1 & L \\ \Lambda & S & J \end{Bmatrix} [[\Omega'^{[K']} \times \Omega^{[K]}]^{[\Lambda]} \times \sum_l [e_l^{[1]} \times e_l^{[1]}]^{[S]}]^{[J]} \tag{49}$$

and by use of the relation

$$\sum_l [e_l^{[1]} \times e_l^{[1]}]^{[S]} = -\sqrt{3}\, \delta_{S0} \,. \tag{50}$$

Now we will consider in detail the different contributions to the polarizabilities from TPO, intrinsic and c.m. recoil resonance operators.

4.1. THE TPO CONTRIBUTIONS

For the pure electric polarizabilities we start by using the second form of the electric multipole fields of (35):

$$\begin{aligned}
[A_{0,l'}^{[L']}(\boldsymbol{x}') \times A_{0,l}^{[L]}(\boldsymbol{x})]^{[J]} = &-\frac{1}{k^2}\sqrt{\frac{L'+1}{L'}}\sqrt{\frac{L+1}{L}}[\nabla'_{l'}c^{[L']}(\boldsymbol{x}') \times \nabla_l c^{[L]}(\boldsymbol{x})]^{[J]} \\
&-\frac{i}{k}\left(\sqrt{\frac{L'+1}{L'}}\frac{\hat{L}}{\sqrt{L}}[\nabla'_{l'}c^{[L']}(\boldsymbol{x}') \times a_{L+1,l}^{[L]}(\boldsymbol{x})]^{[J]}\right. \\
&\left.+\frac{\hat{L}'}{\sqrt{L}}\sqrt{\frac{L+1}{L}}[a_{L'+1,l'}^{[L']}(\boldsymbol{x}') \times \nabla_l c^{[L]}(\boldsymbol{x})]^{[J]}\right) \\
&+\frac{\hat{L}'\hat{L}}{\sqrt{L'L}}[a_{L'+1,l'}^{[L']}(\boldsymbol{x}') \times a_{L+1,l}^{[L]}(\boldsymbol{x})]^{[J]} .
\end{aligned} \tag{51}$$

For the gradient terms we will now utilize the gauge relation of (34) with (45).

$$\begin{aligned}
&\sum_{l'l}\int \mathrm{d}^3x'\,\mathrm{d}^3x[\nabla'_{l'}c^{[L']}(\boldsymbol{x}') \times \nabla_l c^{[L]}(\boldsymbol{x})]^{[J]}\hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x}) \\
&\quad = [\hat{M}^{[L']} \times [\hat{H}_0, \hat{M}^{[L]}]]_-^{[J]} \\
&\qquad -\frac{k^2}{M_A}\sum_l\sum_{K'K} c_{L'K'}c_{LK}[[\hat{M}^{[K']} \times e_l^{[1]}]^{[L']} \times [\hat{M}^{[K]} \times e_l^{[1]}]^{[L]}]^{[J]} \\
&\quad = [\hat{M}^{[L']} \times [\hat{H}_0, \hat{M}^{[L]}]]_-^{[J]} - \frac{k^2}{M_A}\sum_{K'K} c_{L'K'}c_{LK}\hat{M}_{K'K}^{L'L[J]} ,
\end{aligned} \tag{52}$$

where we have used (46) in the last step introducing

$$\hat{M}^{L'L[J]}_{K'K}(k) = \int \mathrm{d}^3x'\,\mathrm{d}^3x \sum_l [a^{[L']}_{K',l}(k, \boldsymbol{x}') \times a^{[L]}_{K,l}(k, \boldsymbol{x})]^{[J]} \hat{\rho}(\boldsymbol{x}')\hat{\rho}(\boldsymbol{x})$$

$$= (-)^{K'+L+J} \hat{L}'\hat{L} \begin{Bmatrix} L & L' & J \\ K' & K & 1 \end{Bmatrix} [\hat{M}^{[K']}(k) \times \hat{M}^{[K]}(k)]^{[J]} . \tag{53}$$

The coupled commutator in (52) is a special case ($\hat{\Omega} = \mathbb{1}$), of the general definition for coupled commutators or anti-commutators with an intermediate scalar operator $\hat{\Omega}$

$$[\hat{A}^{[L']} \times \hat{B}^{[L]}]^{[J]}_{\hat{\Omega},\pm} := [\hat{A}^{[L']} \times \hat{\Omega}\hat{B}^{[L]}]^{[J]} \pm (-)^{L'+L+J} [\hat{B}^{[L]} \times \hat{\Omega}\hat{A}^{[L']}]^{[J]} . \tag{54}$$

In a similar manner one finds for the mixed terms

$$\sum_{l'l} \int \mathrm{d}^3x'\,\mathrm{d}^3x [\nabla'_{l'} c^{[L']}(\boldsymbol{x}') \times a^{[L]}_{L+1,l}(\boldsymbol{x})]^{[J]} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x})$$

$$= -i[\hat{M}^{[L']} \times \hat{\mathcal{T}}^{[L]}_{L+1}]^{[J]}_{-} + \frac{ik}{M_A} \sum_{K'} c_{L'K'} \hat{M}^{L'L[J]}_{K'L+1} , \tag{55}$$

$$\sum_{l'l} \int \mathrm{d}^3x'\,\mathrm{d}^3x [a^{[L']}_{L'+1,l'}(\boldsymbol{x}') \times \nabla_l c^{[L]}(\boldsymbol{x})]^{[J]} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x})$$

$$= (-)^{L'+L+J} \left(-i[\hat{M}^{[L]} \times \hat{\mathcal{T}}^{[L']}_{L'+1}]^{[J]}_{-} + \frac{ik}{M_A} \sum_{K} c_{LK} \hat{M}^{LL'[J]}_{KL'+1} \right) . \tag{56}$$

For the last term of (51) we use the following recoupling scheme for the multipole fields of (37)

$$\hat{B}^{L'L[J]}_{K'K}(k) = \sum_{l'l} \int \mathrm{d}^3x'\,\mathrm{d}^3x [a^{[L']}_{K',l'}(\boldsymbol{x}') \times a^{[L]}_{K,l}(\boldsymbol{x})]^{[J]} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x})$$

$$= \hat{L}'\hat{L} \sum_{\Lambda S} \hat{\Lambda}\hat{S} \begin{Bmatrix} K' & 1 & L' \\ K & 1 & L \\ \Lambda & S & J \end{Bmatrix}$$

$$\times \int \mathrm{d}^3x'\,\mathrm{d}^3x [[c^{[K']}(\boldsymbol{x}') \times c^{[K]}(\boldsymbol{x})]^{[\Lambda]} \times \hat{B}^{[S]}(\boldsymbol{x}', \boldsymbol{x})]^{[J]} , \tag{57}$$

where

$$\hat{B}^{[S]}(\boldsymbol{x}', \boldsymbol{x}) = \sum_{l'l} [e^{[1]}_{l'} \times e^{[1]}_{l}]^{[S]} \hat{B}_{l'l}(\boldsymbol{x}', \boldsymbol{x}) . \tag{58}$$

Note the symmetry relation

$$\hat{B}^{LL'[J]}_{KK'} = (-)^{L'+L+J} \hat{B}^{L'L[J]}_{K'K} . \tag{59}$$

Collecting all terms one obtains for the pure electric polarizabilities

$$\hat{P}^{[J]}_{\mathrm{TPA}}(\mathrm{E}L', \mathrm{E}L) = (-)^{L+1} \frac{a^J_{L'L}}{\sqrt{L'L}} \Bigg[\hat{L}'\hat{L}\hat{B}^{L'L[J]}_{L'+1,L+1} + \frac{1}{M_A}\Bigg(\sqrt{L'+1}\,\sqrt{L+1} \sum_{K'K} c_{L'K'}c_{LK}\hat{M}^{L'L[J]}_{K'K} + \hat{L}\sqrt{L'+1}\sum_{K'} c_{L'K'}\hat{M}^{L'L[J]}_{K'L+1} + \hat{L}'\sqrt{L+1}\sum_{K} c_{LK}\hat{M}^{LL'[J]}_{K,L'+1}\Bigg)\Bigg] + \hat{\mathscr{P}}^{[J]}(\mathrm{E}L', \mathrm{E}L) \tag{60}$$

where

$$a^J_{L'L} = (-)^J 2\pi \frac{\hat{L}'\hat{L}}{\hat{J}}. \tag{61}$$

We have separated

$$\hat{\mathscr{P}}^{[J]}(\mathrm{E}L', \mathrm{E}L) = (-)^{L} \frac{a^J_{L'L}}{\sqrt{L'L}} \Bigg(\frac{1}{k^2}\sqrt{L'+1}\,\sqrt{L+1}[\hat{M}^{[L']}\times[\hat{H}_0, \hat{M}^{[L]}]]^{[J]}_- + \frac{1}{k}\hat{L}\sqrt{L'+1}[\hat{M}^{[L']}\times\hat{\mathscr{T}}^{[L]}_{L+1}]^{[J]}_- + \frac{1}{k}\hat{L}'\sqrt{L+1}[\hat{M}^{[L]}\times\hat{\mathscr{T}}^{[L']}_{L'+1}]^{[J]}_- \Bigg), \tag{62}$$

which will be cancelled by a corresponding contribution to the intrinsic resonance polarizability (see eq. (70)). We would like to emphasize that this cancellation is the essential consequence of the gauge conditions. It demonstrates clearly the fact that the separate contributions of resonance and two-photon amplitude have no physical meaning, only the total contribution is a relevant quantity. As a side remark, this cancellation is also valid for inelastic scattering.

The evaluation of the pure magnetic polarizabilities is straight forward with the help of (57) yielding

$$P^{[J]}_{\mathrm{TPA}}(\mathrm{M}L', \mathrm{M}L) = (-)^{L+1} a^J_{L'L}\hat{B}^{L'L[J]}_{L'L}. \tag{63}$$

Finally for the mixed polarizabilities one finds using again the gauge relation (42)

$$P^{[J]}_{\mathrm{TPA}}(\mathrm{E}L', \mathrm{M}L) = (-)^{L} \frac{a^J_{L'L}}{\sqrt{L'}} \Bigg(\frac{\sqrt{L'+1}}{M_A}\sum_{K'} c_{L'K'}\hat{M}^{L'L[J]}_{K'L} + \hat{L}'\hat{B}^{L'L[J]}_{L'+1,L}\Bigg) + \mathscr{P}^{[J]}(\mathrm{E}L', \mathrm{M}L), \tag{64}$$

where

$$\mathscr{P}^{[J]}(\mathrm{E}L', \mathrm{M}L) = (-)^{L+1} a^J_{L'L}\frac{1}{k}\sqrt{\frac{L'+1}{L'}}[\hat{M}^{[L']}\times\hat{T}^{[L]}_{\mathrm{mag}}]^{[J]}_- \tag{65}$$

will be cancelled again by a part of the intrinsic resonance contribution. Furthermore from crossing symmetry

$$\hat{P}^{[J]}_{\mathrm{TPA}}(\mathrm{M}L', \mathrm{E}L) = (-)^J \hat{P}^{[J]}_{\mathrm{TPA}}(\mathrm{E}L, \mathrm{M}L') . \tag{66}$$

4.2. THE INTRINSIC RESONANCE CONTRIBUTION

According to the separation into intermediate ground and excited states (see (27)-(28)) one finds in a straightforward manner from (16)

$$\begin{aligned}\hat{P}^{[J]}_{\mathrm{in}}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L) = (-)^L a^J_{L'L}\Bigg(&\left[\hat{T}^{[L']}_{\nu'} \times \left(-\frac{1}{k}\hat{P} + \hat{G}_Q(k_0)\right)\hat{T}^{[L]}_{\nu}\right]^{[J]} \\ &+ (-)^{L'+L+J}\left[\hat{T}^{[L]}_{\nu} \times \left(\frac{1}{k}\hat{P} + \hat{G}_Q(-k_0)\right)\hat{T}^{[L']}_{\nu'}\right]^{[J]}\Bigg) .\end{aligned} \tag{67}$$

In view of the Siegert theorem (39) for the electric multipoles one may use the following identities which hold for any operator $\hat{\Omega}$

$$\begin{aligned}[\hat{H}_0, \hat{\Omega}]\hat{G}(k_0) &= [\hat{G}^{-1}(k_0), \hat{\Omega}]\hat{G}(k_0) \\ &= -\hat{\Omega} + \hat{G}^{-1}(k_0)\hat{\Omega}\hat{G}(k_0) ,\end{aligned} \tag{68}$$

$$\hat{G}(k_0)[\hat{H}_0, \hat{\Omega}] = \hat{\Omega} - \hat{G}(k_0)\hat{\Omega}\hat{G}^{-1}(k_0) . \tag{69}$$

Then, the pure electric intrinsic resonance polarizabilities become

$$\begin{aligned}\hat{P}^{[J]}_{\mathrm{in}}(\mathrm{E}L', \mathrm{E}L) = (-)^{L+1}\frac{a^J_{L'L}}{\sqrt{L'L}}\Bigg[&-\frac{1}{k^2}\sqrt{L'+1}\sqrt{L+1}(\hat{G}^{-1}(k_0)[\hat{M}^{[L']} \times \hat{M}^{[L]}]^{[J]} \\ &+ (-)^{L'+L+J}[\hat{M}^{[L]} \times \hat{M}^{[L']}]^{[J]}\hat{G}^{-1}(-k_0)) \\ &+ [\hat{\Omega}^{[L']}(k_0) \times \hat{G}(k_0)\hat{\tilde{\Omega}}^{[L]}(k_0)]^{[J]} \\ &+ (-)^{L'+L+J}[\hat{\Omega}^{[L]}(-k_0) \times \hat{G}(-k_0)\hat{\tilde{\Omega}}^{[L']}(-k_0)]^{[J]}\Bigg] \\ &- \hat{\mathscr{P}}^{[J]}(\mathrm{E}L', \mathrm{E}L) ,\end{aligned} \tag{70}$$

with $\mathscr{P}^{[J]}(\mathrm{E}L', \mathrm{E}L)$ given in (62), where we have introduced

$$\hat{\Omega}^{[L]}(k_0) = \frac{\sqrt{L+1}}{k}\hat{G}^{-1}(k_0)\hat{M}^{[L]} + \hat{L}\hat{\mathscr{T}}^{[L]}_{L+1} , \tag{71}$$

$$\hat{\tilde{\Omega}}^{[L]}(k_0) = \frac{\sqrt{L+1}}{k}\hat{M}^{[L]}\hat{G}^{-1}(k_0) - \hat{L}\hat{\mathscr{T}}^{[L]}_{L+1} . \tag{72}$$

In a similar manner one obtains for the mixed polarizabilities

$$\begin{aligned}\hat{P}^{[J]}_{\mathrm{in}}(\mathrm{E}L', \mathrm{M}L) = (-)^{L+1}\frac{a^J_{L'L}}{\sqrt{L'}}(&[\hat{\Omega}^{[L']}(k_0)\hat{G}(k_0) \times \hat{T}^{[L]}_{\mathrm{mag}}]^{[J]} \\ &+ (-)^{L'+L+J}[\hat{T}^{[L]}_{\mathrm{mag}} \times \hat{G}(-k_0)\hat{\tilde{\Omega}}^{[L']}(-k_0)]^{[J]}) \\ &- \mathscr{P}^{[J]}(\mathrm{E}L', \mathrm{M}L) ,\end{aligned} \tag{73}$$

where $\mathscr{P}^{[J]}(\mathrm{E}L', \mathrm{M}L)$ is given in (65). Furthermore, we have from crossing symmetry

$$\hat{P}^{[J]}_{\mathrm{in}}(\mathrm{M}L', \mathrm{E}L, k_0) = (-)^J \hat{P}^{[J]}_{\mathrm{in}}(\mathrm{E}L, \mathrm{M}L', -k_0) . \tag{74}$$

4.3. THE C.M. RECOIL CONTRIBUTION

The c.m. recoil scattering operator (24) inserted into (16) leads to

$$\begin{aligned}\hat{P}^{[J]}_{\mathrm{c.m.}}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L) = \frac{(-)^L}{M_A} a^J_{L'L}(\tfrac{1}{2}k^2([\hat{T}^{[L']}_{\nu'} \times \hat{G}^2(k_0)\hat{T}^{[L]}_{\nu}]^{[J]} \\ -(-)^{L'+L+J}[\hat{T}^{[L]}_{\nu} \times \hat{G}^2(-k_0)\hat{T}^{[L']}_{\nu'}]^{[J]}) \\ +(-)^{L'+L+J}\sum_l(-[\hat{T}'^{[L']}_{\nu,l} \times \hat{G}^2(-k_0)\hat{T}'^{[L]}_{\nu',l}]^{[J]} \\ +i[\hat{M}'^{[L]}_{\nu,l} \times \hat{G}(-k_0)\hat{T}'^{[L']}_{\nu',l}]^{[J]} \\ -i[\hat{T}'^{[L]}_{\nu,l} \times \hat{G}(-k_0)\hat{M}'^{[L']}_{\nu',l}]^{[J]}))\,,\end{aligned} \tag{75}$$

where we have introduced

$$\begin{aligned}\hat{M}'^{[L]}_{\nu,l} &= \int d^3x\, A^{[L]}_{\nu,l}(k, \boldsymbol{x})\hat{\rho}(\boldsymbol{x}) \\ &= (-)^L\sqrt{2}\sum_{K=L\pm(1-\nu)}\hat{K}\begin{pmatrix}1 & K & L\\ 1 & 0 & -1\end{pmatrix}[\hat{M}^{[K]} \times e^{[1]}_l]^{[L]}\end{aligned} \tag{76}$$

and

$$\begin{aligned}\hat{T}'^{[L]}_{\nu,l} &= \sum_{l'}\int d^3x\, A^{[L]}_{\nu,l'}(k, \boldsymbol{x})\nabla_l\hat{j}_{l'}(\boldsymbol{x}) \\ &= ik(-)^L\sum_K\hat{K}\begin{pmatrix}1 & K & L\\ 0 & 1 & -1\end{pmatrix}[T^{[K]}_{\bar{\nu}(\nu LK)} \times e^{[1]}_l]^{[L]}\,,\end{aligned} \tag{77}$$

with

$$\bar{\nu}(\nu LK) = \begin{cases}|\nu-1| & K=L\\ \nu & K=L\pm 1.\end{cases} \tag{78}$$

The latter relation (77) is derived in appendix B. It is then straighforward to insert these expressions into (75) and one finds

$$\begin{aligned}\hat{P}^{[J]}_{\mathrm{c.m.}}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L) = \frac{(-)^L}{M_A} a^J_{L'L}(\tfrac{1}{2}k^2([\hat{T}^{[L']}_{\nu'} \times \hat{G}^2(k_0)\hat{T}^{[L]}_{\nu}]^{[J]} \\ -(-)^{L'+L+J}[\hat{T}^{[L]}_{\nu} \times \hat{G}^2(-k_0)\hat{T}^{[L']}_{\nu'}]^{[J]}) \\ -(-)^{L'}k\sum_{K=L\pm(1-\nu)}(-)^K\sum_{K'}c^{LL'}_{KK'}[\hat{M}^{[K]} \times \hat{G}(-k_0)\hat{T}^{[K']}_{\bar{\nu}(\nu'L'K')}]^{[J]} \\ +(-)^{L'}k\sum_K(-)^K\sum_{K'=L'\pm(1-\nu')}c^{L'L}_{K'K}[\hat{T}^{[K]}_{\bar{\nu}(\nu LK)} \times \hat{G}(-k_0)\hat{M}^{[K']}]^{[J]} \\ +(-)^{L'}k^2\sum_{KK'}(-)^K\bar{c}^{LL'}_{KK'}[\hat{T}^{[K]}_{\bar{\nu}(\nu LK)} \\ \times\hat{G}^2(-k_0)\hat{T}^{[K']}_{\bar{\nu}(\nu'L'K')}]^{[J]})\,,\end{aligned} \tag{79}$$

where

$$c_{KK'}^{LL'}=\sqrt{2}\hat{L}'\hat{L}\hat{K}'\hat{K}\begin{pmatrix}1 & K & L\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & K' & L'\\ 0 & 1 & -1\end{pmatrix}\begin{Bmatrix}L & L' & J\\ K' & K & 1\end{Bmatrix}, \tag{80}$$

$$\bar{c}_{KK'}^{LL'}=\hat{L}'\hat{L}\hat{K}'\hat{K}\begin{pmatrix}1 & K & L\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & K' & L'\\ 0 & 1 & -1\end{pmatrix}\begin{Bmatrix}L & L' & J\\ K' & K & 1\end{Bmatrix}. \tag{81}$$

This completes the various contributions to the polarizabilities and we can now derive the low energy theorems.

5. Low-energy theorems

The low-energy theorems for the elastic polarizabilities are obtained by a power series expansion with respect to k. As one can see from the explicit expressions the leading power will be at least $k^{L'+L-2}$ for $P^{[J]}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L, k)$. Therefore we write

$$P^{[J]}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L, k)=k^{L'+L-2}\sum_{n\geq 0} k^n p^{[J]}(\mathrm{M}^{\nu'}L', \mathrm{M}^{\nu}L)_{(n)}. \tag{82}$$

As we have shown in the previous section, gauge conditions lead to important cancellations between two-photon and resonance amplitudes. Therefore, these terms need not be considered explicitly here.

We present our results in two parts. The first one will contain the formal expressions for all polarizabilities of the low energy expansion up to $n=2$. In the second part we will specialize these results to those polarizabilities, which contribute to the scattering amplitude up to the second order in k.

5.1. DETERMINATION OF $p^{[J]}(\mathrm{M}^{\nu'}L, \mathrm{M}^{\nu}L)_{(n)}$ UP TO $n=2$

We will first consider separately the three contributions from TPA, intrinsic resonance and c.m. recoil operators. The starting point is the low energy behaviour of the charge and transverse multipoles of (40) and (41) which for the TPA part determines completely the low-energy expansion.

From the Taylor expansion of the spherical Bessel functions

$$j_L(x)=\sum_{n=0}^{\infty}\alpha_{2n}^{L}x^{L+2n}, \qquad \alpha_{2n}^{L}=\frac{(-)^n}{2^n n!(2L+2n+1)!!}, \tag{83}$$

one obtains for the multipole fields

$$c^{[L]}(k, \boldsymbol{x})=\sum_{n=0}^{\infty}k^{L+2n}c_{(2n)}^{[L]}(\boldsymbol{x}) \tag{84}$$

$$\boldsymbol{a}_K^{[L]}(k, \boldsymbol{x})=\sum_{n=0}^{\infty}k^{K+2n}\boldsymbol{a}_{K(2n)}^{[L]}(\boldsymbol{x}), \tag{85}$$

where

$$c^{[L]}_{(2n)}(\boldsymbol{x}) = i^L \alpha^L_{2n} x^{L+2n} Y^{[L]}(\hat{\boldsymbol{x}}) , \tag{86}$$

$$\boldsymbol{a}^{[L]}_{K(2n)}(\boldsymbol{x}) = i^K \alpha^K_{2n} x^{K+2n} \boldsymbol{Y}^{[L]}_K(\hat{\boldsymbol{x}}) . \tag{87}$$

With these expansions of the multipole fields one finds for the corresponding operators keeping terms up to second order beyond the leading one

$$\hat{M}^{[L]}(k) = k^L(\hat{M}^{[L]}_{(0)} + k^2 \hat{M}^{[L]}_{(2)} + \cdots) \tag{88}$$

for the charge multipoles, where

$$\hat{M}^{[L]}_{(n)} = \int \mathrm{d}^3x\, \hat{\rho}(\boldsymbol{x}) c^{[L]}_{(n)}(\boldsymbol{x}) , \tag{89}$$

$$\hat{T}^{[L]}_{\nu}(k) = k^{L+\nu-1}(\hat{T}^{[L]}_{\nu(0)} + k^2 \hat{T}^{[L]}_{\nu(2)} + \cdots) \tag{90}$$

for the transverse multipoles, where

$$\hat{T}^{[L]}_{\mathrm{el}(0)} = -\sqrt{\frac{L+1}{L}}\,[\hat{H}_0, \hat{M}^{[L]}_{(0)}] , \tag{91}$$

$$\hat{T}^{[L]}_{\mathrm{el}(2)} = -\sqrt{\frac{L+1}{L}}\,[\hat{H}_0, \hat{M}^{[L]}_{(2)}] - \frac{\hat{L}}{\sqrt{L}}\,\hat{\mathscr{T}}^{[L]}_{L+1(0)} , \tag{92}$$

$$\hat{T}^{[L]}_{\mathrm{mag}(n)} = \hat{\mathscr{T}}^{[L]}_{L(n)} \tag{93}$$

with

$$\hat{\mathscr{T}}^{[L]}_{K(n)} = \int \mathrm{d}^3x\, \hat{\boldsymbol{j}}(\boldsymbol{x}) \cdot \boldsymbol{a}^{[L]}_{K(n)} . \tag{94}$$

Noting the following low energy behaviour

$$\hat{M}^{L'L[J]}_{K'K}(k) \overset{k\to 0}{\sim} k^{K'+K} , \tag{95}$$

$$\hat{B}^{L'L[J]}_{K'K}(k) \overset{k\to 0}{\sim} k^{K'+K} , \tag{96}$$

one finds for the TPA part of the polarizabilities up to second order beyond $k^{L'+L-2}$:

(i) *Pure electric*

$$\hat{p}^{[J]}_{\mathrm{TPA}}(\mathrm{E}L', \mathrm{E}L)_{(0)} = (-)^{L'+J} \frac{a^J_{L'L}}{M_A} \sqrt{L'+1}\,\sqrt{L+1} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \times [\hat{M}^{[L'-1]}_{(0)} \times \hat{M}^{[L-1]}_{(0)}]^{[J]} + \hat{\not{k}}^{[J]}(\mathrm{E}L', \mathrm{E}L)_{(0)} , \tag{97}$$

$$\hat{p}^{[J]}_{\mathrm{TPA}}(\mathrm{E}L', \mathrm{E}L)_{(1)} = 0, \tag{98}$$

$$\begin{aligned}\hat{p}^{[J]}_{\mathrm{TPA}}(\mathrm{E}L', \mathrm{E}L)_{(2)} = \frac{(-)^{L+1}}{M_A} \frac{a^J_{L'L}}{\hat{L}'\hat{L}} \big(&\sqrt{L'+1}\,\sqrt{L+1}\, \hat{M}^{L'L[J]}_{L'-1\,L-1(2)} \\ &+ \sqrt{L'}\,\sqrt{L+1}\, \hat{M}^{L'L[J]}_{L'+1\,L-1(0)} \\ &+ \sqrt{L'+1}\,\sqrt{L}\, \hat{M}^{L'L[J]}_{L'-1\,L+1(0)}\big) \\ &+ \hat{\not{k}}^{[J]}(\mathrm{E}L', \mathrm{E}L)_{(2)} .\end{aligned} \tag{99}$$

Here $\hat{\not{p}}^{[J]}(EL', EL)_{(n)}$ are the expansion coefficients for $\hat{P}^{[J]}(EL', EL)$ which will be cancelled by the resonance contributions.

(ii) *Pure magnetic*

$$\hat{p}^{[J]}_{\mathrm{TPA}}(ML', ML)_{(0/1)} = 0 \tag{100}$$

$$\hat{p}^{[J]}_{\mathrm{TPA}}(ML', ML)_{(2)} = (-)^{L+1} a^J_{L'L} \hat{B}^{L'L[J]}_{L'L(0)}\,, \tag{101}$$

where $\hat{B}^{L'L[J]}_{L'L(0)}$ is obtained by replacing in (57) the $c^{[K]}$ multipole fields by the lowest order $c^{[K]}_{(0)}$.

(iii) *Mixed electric-magnetic*

$$\hat{p}^{[J]}_{\mathrm{TPA}}(EL', ML)_{(0/2)} = 0 \tag{102}$$

$$\begin{aligned}\hat{p}^{[J]}_{\mathrm{TPA}}(EL', ML)_{(1)} = {} & \frac{(-)^{L'+J+1}}{M_A} a^J_{L'L} \hat{L} \sqrt{L'+1} \begin{Bmatrix} L & L' & J \\ L'-1 & L & 1 \end{Bmatrix} [\hat{M}^{[L'-1]}_{(0)} \times \hat{M}^{[L]}_{(0)}]^{[J]} \\ & + \hat{\not{p}}^{[J]}(EL', ML)_{(2)}\,. \end{aligned} \tag{103}$$

For the intrinsic resonance and c.m. recoil contributions the expansion in powers of k will be a little more complicated due to the k-dependence of the propagators. Using the expansion of (25) and (26) one finds for the intrinsic resonance contribution:

(i) *Pure electric*

$$\hat{p}^{[J]}_{\mathrm{res}}(EL', EL)_{(0)} = -\hat{\not{p}}^{[J]}(EL', EL)_{(0)}\,, \tag{104}$$

$$\hat{p}^{[J]}_{\mathrm{res}}(EL', EL)_{(1)} = (-)^{L+1} a^J_{L'L} \sqrt{\frac{L'+1}{L}} \sqrt{\frac{L+1}{L}} [\hat{M}^{[L']}_{(0)} \times \hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{Q},-}\,, \tag{105}$$

$$\begin{aligned}\hat{p}^{[J]}_{\mathrm{res}}(EL', EL)_{(2)} = {} & \frac{(-)^L}{\sqrt{L'L}} a^J_{L'L} \big(-\sqrt{L'+1}\sqrt{L+1}\, [M^{[L']}_{(0)} \times M^{[L]}_{(0)}]^{[J]}_{\hat{G}_Q(0),+} \\ & + \hat{L}' \sqrt{L+1} [\hat{\mathcal{T}}^{[L']}_{L'+1(0)} \times \hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{P},-} \\ & - \hat{L} \sqrt{L'+1} [\hat{M}^{[L']}_{(0)} \times \hat{\mathcal{T}}^{[L]}_{L+1(0)}]^{[J]}_{\hat{P},-} \big) \\ & - \hat{\not{p}}^{[J]}(EL', EL)_{(2)}\,. \end{aligned} \tag{106}$$

(ii) *Pure magnetic*

$$\hat{p}^{[J]}_{\mathrm{in}}(ML', ML)_{(0)} = 0, \tag{107}$$

$$\hat{p}^{[J]}_{\mathrm{in}}(ML', ML)_{(1)} = (-)^{L+1} a^J_{L'L} [\hat{T}^{[L']}_{\mathrm{mag}(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{P},-}\,, \tag{108}$$

$$\hat{p}^{[J]}_{\mathrm{in}}(ML', ML)_{(2)} = (-)^{L} a^J_{L'L} [\hat{T}^{[L']}_{\mathrm{mag}(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{G}_Q(0),+}\,. \tag{109}$$

(iii) *Mixed electric-magnetic*

$$\hat{p}^{[J]}_{\mathrm{in}}(EL', ML)_{(0/2)} = 0, \tag{110}$$

$$\hat{p}^{[J]}_{\mathrm{in}}(EL', ML)_{(1)} = (-)^{L+1} a^J_{L'L} \sqrt{\frac{L'+1}{L'}} [\hat{M}^{[L']}_{(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{P},-}\,. \tag{111}$$

Finally, for the c.m. recoil contributions one finds more complicated expressions. For this reason we will list here the lowest order terms only and refer to appendix C, where the higher order terms are contained in the complete expressions:

(i) *Pure electric*

$$\hat{p}^{[J]}_{\text{c.m.}}(\mathrm{E}L', \mathrm{E}L)_{(0)} = \frac{(-)^{L+1}}{M_A} a^J_{L'L} \sqrt{L'+1}\sqrt{L+1} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \times [\hat{M}^{[L-1]}_{(0)} \times \hat{Q}\hat{M}^{[L'-1]}_{(0)}]^{[J]}, \tag{112}$$

(ii) *Pure magnetic*

$$\hat{p}^{[J]}_{\text{c.m.}}(\mathrm{M}L', \mathrm{M}L)_{(0)} = \frac{(-)^{L+1}}{M_A} a^J_{L'L} \sqrt{\frac{L'^2-1}{L'}} \sqrt{\frac{L^2-1}{L}} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \times [\hat{T}^{[L-1]}_{\text{mag}(0)} \times \hat{P}\hat{T}^{[L'-1]}_{\text{mag}(0)}]^{[J]}, \tag{113}$$

(iii) *Mixed electric-magnetic*

$$\hat{p}^{[J]}_{\text{c.m.}}(\mathrm{E}L', \mathrm{M}L)_{(0)} = \frac{(-)^{L+1}}{M_A} a^J_{L'L} \sqrt{L'+1} \sqrt{\frac{L^2-1}{L}} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \times [\hat{T}^{[L-1]}_{\text{mag}(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}. \tag{114}$$

Collecting all contributions we find for the low-energy expansion of the polarizabilities the following lowest order coefficients.

(i) *Pure electric*

$$\hat{p}^{[J]}(\mathrm{E}L', \mathrm{E}L)_{(0)} = \frac{(-)^{L}}{M_A} a^J_{L'L} \sqrt{L'+1}\sqrt{L+1} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \times [\hat{M}^{[L-1]}_{(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}. \tag{115}$$

As already mentioned, large cancellations between the various contributions due to the gauge conditions have led to this simple result.

In particular we note for $L = L'$

$$\hat{p}^{[J]}(\mathrm{E}L, \mathrm{E}L)_{(0)} = \frac{(-)^{L}}{M_A} (L+1) a^J_{LL} \begin{Bmatrix} L & L & J \\ L-1 & L-1 & 1 \end{Bmatrix} [\hat{M}^{[L-1]}_{(0)} \times \hat{P}\hat{M}^{[L-1]}_{(0)}]^{[J]}, \tag{116}$$

which vanishes for even L if parity is conserved, because then the ground state expectation value of $\hat{M}^{[L-1]}$ is zero. This is a generalization of the result that there is no E2 contribution to the scattering amplitude of order k^2. For E1 one has

$$\hat{p}^{[J]}(\mathrm{E}1, \mathrm{E}1)_{(0)} = -\sqrt{3} \frac{(Ze)^2}{M_A} \delta_{J0}, \tag{117}$$

the well-known Thomson amplitude of the whole nucleus.

(ii) *Pure magnetic*
Here no cancellations occur. The lowest order term is completely determined by the c.m. recoil contribution (see (113))

$$\hat{p}^{[J]}(\mathrm{M}L',\mathrm{M}L)_{(0)}=\hat{p}^{[J]}_{\mathrm{c.m.}}(\mathrm{M}L',\mathrm{M}L)_{(0)}\,, \tag{118}$$

where L and L' must be greater than one. Since only the static ground state magnetic multipole moments of order $L'-1$ and $L-1$ are involved, this expression vanishes for odd L' and L, if parity is conserved.

(iii) *Mixed electric-magnetic*
Also here the lowest order term is given by the c.m. recoil contribution alone (see (114)).

$$\hat{p}^{[J]}(\mathrm{E}L',\mathrm{M}L)_{(0)}=\hat{p}^{[J]}_{\mathrm{c.m.}}(\mathrm{E}L',\mathrm{M}L)_{(0)}\,, \tag{119}$$

where L must be greater than one. Because of the appearance of ground state charge and magnetic multipole moments this term vanishes for odd L' and even L.

The first and second order contributions are listed in appendix C.

5.2. LOW-ENERGY EXPANSION OF POLARIZABILITIES UP TO ORDER k^2

With respect to the low energy expressions of charge and transverse multipole operators, we note the following well-known relations

$$\hat{M}^{[0]}_{(0)}=\frac{Ze}{\sqrt{4\pi}}\,, \tag{120}$$

$$\hat{M}^{[1]}_{(0)}=\frac{i}{\sqrt{12\pi}}\hat{D}^{[1]}\,,\qquad \hat{\boldsymbol{D}}=\int \mathrm{d}^3x\,\boldsymbol{x}\hat{\rho}(\boldsymbol{x})\,, \tag{121}$$

$$\hat{M}^{[2]}_{(0)}=-\tfrac{1}{30}\sqrt{\frac{5}{4\pi}}\,\hat{Q}^{[2]}\,,\qquad \hat{Q}^{[2]}=\sqrt{\tfrac{16}{5}\pi}\int \mathrm{d}^3x\,x^2\hat{\rho}(\boldsymbol{x})\,Y^{[2]}(\hat{\boldsymbol{x}})\,, \tag{122}$$

$$\hat{M}^{[2]}_{(2)}=-\frac{1}{6\sqrt{4\pi}}\int \mathrm{d}^3x\,x^2\hat{\rho}(\boldsymbol{x})\,, \tag{123}$$

$$\hat{T}^{[1]}_{\mathrm{mag}(0)}=-\frac{1}{\sqrt{6\pi}}\hat{\mu}^{[1]}\,,\qquad \hat{\boldsymbol{\mu}}=\tfrac{1}{2}\int \mathrm{d}^3x\,\boldsymbol{x}\times\boldsymbol{j}\,, \tag{124}$$

where Ze denotes the charge of the nucleus and $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{D}}$ and $\hat{Q}^{[2]}$ are the conventional magnetic dipole, electric dipole and quadrupole operators, respectively.

According to (16) the polarizabilities are defined as reduced matrix elements of the polarizability operators. Explicitly one finds for all those, which contribute to elastic scattering up to order k^2

(i) *Scalar polarizabilities* ($J=0$)

$$P_0(\mathrm{E1},\mathrm{E1})=-\sqrt{3}\hat{I}\left(\frac{e^2Z^2}{M_A}-\left(\alpha_E^{(0)}+\frac{e^2Z}{3M_A}\langle r^2\rangle-\frac{\bar{\mu}^2}{6M_A}I(I+1)\right)k^2\right) \tag{125}$$

$$P_0(\mathrm{M1},\mathrm{M1})=\sqrt{3}\hat{I}\left(\chi_P^{(0)}+\chi_D^{(0)}-\frac{\langle \boldsymbol{D}^2\rangle}{2M_A}\right)k^2 \tag{126}$$

$$P_0(\mathrm{E2},\mathrm{E2})=0 \tag{127}$$

$$P_0(\mathrm{M2},\mathrm{M2})=-\frac{\sqrt{5}}{6M_A}\,c_1(I)\bar{\mu}^2k^2 \tag{128}$$

(ii) Vector polarizabilities ($J=1$)

$$P_1(\mathrm{E1},\mathrm{E1})=\sqrt{\tfrac{2}{3}}c_1(I)\,\frac{eZ\bar{\mu}}{M_A}\,k\,, \tag{129}$$

$$P_1(\mathrm{M1},\mathrm{M1})=\sqrt{\tfrac{2}{3}}c_1(I)\bar{\mu}^2k\,, \tag{130}$$

$$P_1(\mathrm{E1},\mathrm{E2})=-\tfrac{1}{3}\sqrt{\tfrac{5}{2}}c_1(I)\,\frac{eZ\bar{\mu}}{M_A}\,k\,. \tag{131}$$

(iii) *Tensor polarizabilities* ($J=2$)

$$P_2(\mathrm{E1},\mathrm{E1})=-\frac{c_2(I)}{\sqrt{5}}\left(2\alpha_E^{(2)}+\frac{\bar{\mu}^2}{4M_A}-\frac{eZ\bar{Q}}{5M_A}\right)k^2\,, \tag{132}$$

$$P_2(\mathrm{M1},\mathrm{M1})=-\frac{c_2(I)}{\sqrt{5}}\left(2\chi_P^{(2)}+2\chi_D^{(2)}+\frac{3}{4M_A}\,D^{(2)}\right)k^2\,, \tag{133}$$

$$P_2(\mathrm{E1},\mathrm{M2})=\sqrt{\frac{3}{5}}\,\frac{c_2(I)}{2M_A}\left(\frac{eZ\bar{Q}}{3}+\frac{\bar{\mu}^2}{2}\right)k^2\,, \tag{134}$$

$$P_2(\mathrm{E2},\mathrm{M1})=-\tfrac{1}{2}\sqrt{\tfrac{3}{5}}c_2(I)\bar{Q}\bar{\mu}k^2\,, \tag{135}$$

$$P_2(\mathrm{E2},\mathrm{E2})=0\,, \tag{136}$$

$$P_2(\mathrm{M2},\mathrm{M2})=-\frac{1}{4}\sqrt{\frac{7}{15}}\,\frac{c_2(I)}{M_A}\,\bar{\mu}^2k^2\,, \tag{137}$$

$$P_2(\mathrm{E1},\mathrm{E3})=\frac{1}{5}\sqrt{\frac{7}{15}}\,\frac{eZ\bar{Q}}{M_A}\,c_2(I)k^2\,. \tag{138}$$

Here we have used the following notations denoting the ground state spin by I

$$\bar{\mu}=\langle I\,\|\,\mu^{[1]}\,\|\,I\rangle/c_1(I)\,, \tag{139}$$

$$\langle r^2\rangle=\frac{1}{e}\left\langle\int \mathrm{d}^3x\,x^2\hat{\rho}(\boldsymbol{x})\right\rangle, \tag{140}$$

$$\bar{Q}=\langle I\,\|\,Q^{[2]}\,\|\,I\rangle/(\sqrt{6}c_2(I))\,, \tag{141}$$

$$c_1(I)=\langle I\,\|\,S^{[1]}\,\|\,I\rangle=\hat{I}I(I+1)\,, \tag{142}$$

$$c_2(I)=\langle I\,\|\,[S^{[1]}\times S^{[1]}]^{[2]}\,\|\,I\rangle\,, \tag{143}$$

where $\boldsymbol{S}$ is the ground-state spin operator. Furthermore one has the static electric polarizabilities

$$\alpha_{\mathrm{E}}^{(0)} = \tfrac{2}{3}\langle \hat{\boldsymbol{D}} \cdot \hat{G}_Q(0)\hat{\boldsymbol{D}}\rangle\,, \tag{144}$$

$$\alpha_{\mathrm{E}}^{(2)} = \langle I \,\|\, [\hat{D}^{[1]} \times \hat{G}_Q(0)\hat{D}^{[1]}]^{[2]} \,\|\, I\rangle / c_2(I)\,, \tag{145}$$

and the para- and diamagnetic susceptibilities

$$\chi_{\mathrm{P}}^{(0)} = \tfrac{2}{3}\langle \hat{\boldsymbol{\mu}} \cdot \hat{G}_Q(0)\hat{\boldsymbol{\mu}}\rangle\,, \tag{146}$$

$$\chi_{\mathrm{P}}^{(2)} = \langle I \,\|\, [\hat{\mu}^{[1]} \times \hat{G}_Q(0)\hat{\mu}^{[1]}]^{[2]} \,\|\, I\rangle / c_2(I)\,, \tag{147}$$

$$\chi_{\mathrm{D}}^{(0)} = \tfrac{1}{2}\sum_S (-)^S \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & S \end{Bmatrix} \left\langle \int \mathrm{d}^3x'\,\mathrm{d}^3x [[x'^{[1]} \times x^{[1]}]^{[S]} \hat{B}^{[S]}(\boldsymbol{x}', \boldsymbol{x})]^{[0]} \right\rangle, \tag{148}$$

$$\chi_{\mathrm{D}}^{(2)} = \tfrac{3}{4}\sum_{S'S} \hat{S}\hat{S}' \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ S & S' & 2 \end{Bmatrix} \times \left\langle I \left\| \int \mathrm{d}^3x'\,\mathrm{d}^3x\, [[x'^{[1]} \times x^{[1]}]^{[S']} \hat{B}^{[S]}(\boldsymbol{x}', \boldsymbol{x})]^{[2]} \right\| I \right\rangle / c_2(I)\,. \tag{149}$$

For the calculation of the diamagnetic susceptibilities one needs a model for $B_{I'I}$. As a simple example we evaluate the susceptibilities for the kinetic TPA

$$B_{I'I}^{\mathrm{kin}}(\boldsymbol{x}', \boldsymbol{x}) = \delta_{I'I} \sum_j \frac{e_j^2}{M}\, \delta(\boldsymbol{x}' - \boldsymbol{r}_j)\delta(\boldsymbol{x} - \boldsymbol{r}_j) \tag{150}$$

and get

$$\chi_{\mathrm{D}}^{\mathrm{kin}\,(0)} = -\frac{1}{6M}\left\langle \sum_j e_j^2 r_j^2 \right\rangle \tag{151}$$

$$\chi_{\mathrm{D}}^{\mathrm{kin}\,(2)} = -\tfrac{1}{6}\sqrt{\tfrac{1}{10}\pi}\left\langle I \left\| \sum_j \frac{e_j^2}{M}\, r_j^2 Y^{[2]}(\hat{\boldsymbol{r}}_j) \right\| I \right\rangle \Big/ c_2(I)\,. \tag{152}$$

The results of eqs. (125)–(138), which have been reported before [18]), are complete to order k^2 in the photon energy and therefore are equivalent to the low energy theorem of Friar [6]). We would like to stress again that due to a cancellation between the TPA and the recoil corrections of the resonance amplitude the (E2, E2) polarizability vanishes to this order.

In conclusion we note that, by a consequent application of the gauge conditions, we have been able to set up formulas for the polarizabilities, which explicitly exhibit the consequences of the gauge conditions. The important observation is that, for all pure and mixed electric polarizabilities, a partial cancellation between TPA and resonance amplitude takes place. This partial cancellation occurs not only at low energy, but holds for all energies and is a consequence of the fact that the TPA and the resonance amplitude are not independent.

Appendix A

THE C.M. RECOIL SCATTERING OPERATOR

We start from the general form of the non-relativistic resonance scattering amplitude

$$R^{\mathrm{fi}}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) = \sum_n \frac{\langle \mathrm{f}|\boldsymbol{\varepsilon}'^*_{\lambda'} \cdot \hat{\boldsymbol{j}}(-\boldsymbol{k}', 2\boldsymbol{P}_\mathrm{f}+\boldsymbol{k}')|n\rangle\langle n|\boldsymbol{\varepsilon}_{\lambda'} \cdot \hat{\boldsymbol{j}}(\boldsymbol{k}, 2\boldsymbol{P}_\mathrm{i}+\boldsymbol{k})|\mathrm{i}\rangle}{E_n - E_i - k_0 - i\varepsilon}$$
$$+\sum_n \frac{\langle \mathrm{f}|\boldsymbol{\varepsilon}_\lambda \cdot \hat{\boldsymbol{j}}(\boldsymbol{k}, 2\boldsymbol{P}_\mathrm{f}-\boldsymbol{k})|n\rangle\langle n|\boldsymbol{\varepsilon}'^*_{\lambda'} \cdot \hat{\boldsymbol{j}}(-\boldsymbol{k}', 2\boldsymbol{P}_\mathrm{i}-\boldsymbol{k}')|\mathrm{i}\rangle}{E'_n - E_i + k'_0 - i\varepsilon}, \tag{A.1}$$

where

$$\hat{\boldsymbol{j}}(\boldsymbol{k}, \boldsymbol{P}) = \int \mathrm{d}^3x\, \hat{\boldsymbol{j}}(x)\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \frac{\boldsymbol{P}}{2M_A}\int \mathrm{d}^3x\, \hat{\rho}(\boldsymbol{x})\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$= \hat{\boldsymbol{j}}(\boldsymbol{k}) + \frac{1}{2M_A}\hat{\rho}(\boldsymbol{k})\boldsymbol{P}, \tag{A.2}$$

$$E_i = \varepsilon_i + \frac{\boldsymbol{P}_\mathrm{i}^2}{2M_A}, \tag{A.3}$$

$$E_n = \varepsilon_n + \frac{(\boldsymbol{P}_\mathrm{i}+\boldsymbol{k})^2}{2M_A}, \tag{A.4}$$

$$E'_n = \varepsilon_n + \frac{(\boldsymbol{P}_\mathrm{i}-\boldsymbol{k}')^2}{2M_A}. \tag{A.5}$$

We expand the energy denominators in powers of $1/M_A$

$$(E_n - E_i - k_0 - i\varepsilon)^{-1} = N_{ni}(k_0)\left(1 - \frac{1}{2M_A}(2\boldsymbol{P}_\mathrm{i}+\boldsymbol{k})\cdot\boldsymbol{k}N_{ni}(k_0) + \cdots\right), \tag{A.6}$$

$$(E'_n - E_i + k'_0 - i\varepsilon)^{-1} = N_{ni}(-k'_0)\left(1 - \frac{1}{2M_A}(-2\boldsymbol{P}_\mathrm{i}+\boldsymbol{k}')\cdot\boldsymbol{k}'N_{ni}(-k_0) + \cdots\right), \tag{A.7}$$

$$N_{ni}(k_0) = (\varepsilon_n - \varepsilon_i - k_0 - i\varepsilon)^{-1}. \tag{A.8}$$

Keeping terms of order $1/M_A$ only, one obtains for the c.m. recoil amplitude

$$R^{\mathrm{fi,c.m.}}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) = \frac{1}{2M_A}\sum_n \Bigg[(2\boldsymbol{\varepsilon}'^*_{\lambda'}\cdot\boldsymbol{P}_\mathrm{f}\langle \mathrm{f}|\hat{\rho}(-\boldsymbol{k}')|n\rangle\langle n|\boldsymbol{\varepsilon}_\lambda\cdot\hat{\boldsymbol{j}}(\boldsymbol{k})|\mathrm{i}\rangle$$
$$+2\langle \mathrm{f}|\boldsymbol{\varepsilon}'^*_{\lambda'}\cdot\hat{\boldsymbol{j}}(-\boldsymbol{k}')|n\rangle\langle n|\hat{\rho}(\boldsymbol{k})|\mathrm{i}\rangle\boldsymbol{\varepsilon}_\lambda\cdot\boldsymbol{P}_\mathrm{i})N_{ni}(k_0)$$
$$-(2\boldsymbol{P}_\mathrm{i}+\boldsymbol{k})\cdot\boldsymbol{k}\langle \mathrm{f}|\boldsymbol{\varepsilon}'^*_{\lambda'}\cdot\hat{\boldsymbol{j}}(-\boldsymbol{k}')|n\rangle\langle n|\boldsymbol{\varepsilon}_\lambda\cdot\hat{\boldsymbol{j}}(\boldsymbol{k})|\mathrm{i}\rangle N^2_{ni}(k_0)$$
$$+\left(\begin{matrix}\boldsymbol{\varepsilon}_\lambda & & \boldsymbol{\varepsilon}'^*_{\lambda'}\\ \boldsymbol{k} & \leftrightarrow & -\boldsymbol{k}'\\ k_0 & & -k'_0\end{matrix}\right\}\Bigg]. \tag{A.9}$$

In the c.m. system, where one has

$$P_{\rm i} = -k, \qquad P_{\rm f} = -k', \qquad k_0' = k_0 \qquad \text{(elastic scattering)} \tag{A.10}$$

one finds then

$$R^{\rm fi;\,c.m.}_{\lambda'\lambda}(k', k) = \frac{k^2}{2M_A}\frac{\partial R^{\rm fi,0}_{\lambda'\lambda}}{\partial k_0} - \frac{1}{M_A}\sum_n \{k'\cdot k\langle {\rm f}|\varepsilon_\lambda\cdot\hat{j}(k)|n\rangle\langle n|\varepsilon'^*_{\lambda'}\cdot\hat{j}(-k')|{\rm i}\rangle N^2_{ni}(-k_0)$$

$$+(\varepsilon_\lambda\cdot k'\langle {\rm f}|\hat{\rho}(k)|n\rangle\langle n|\varepsilon'^*_{\lambda'}\cdot\hat{j}(-k')|{\rm i}\rangle$$

$$+\varepsilon'^*_{\lambda'}\cdot k\langle {\rm f}|\varepsilon_\lambda\cdot\hat{j}(k)|n\rangle\langle n|\rho(-k')|{\rm i}\rangle)N_{ni}(-k_0)\}\,, \tag{A.11}$$

Where $R^{\rm fi,0}$ denotes the intrinsic resonance amplitude. Inserting the Fourier transforms, using

$$\varepsilon_\lambda\cdot k'\varepsilon'^*_{\lambda'}\cdot\hat{j}(-k') = -i\int {\rm d}^3x'\,{\rm e}^{-ik'\cdot x'}\varepsilon_\lambda\cdot\nabla'\varepsilon'^*_{\lambda'}\cdot\hat{j}(x')\,, \tag{A.12}$$

and analogous expressions for the other terms one obtains finally

$$R^{\rm fi;\,c.m.}_{\lambda'\lambda}(k', k) = \sum_{l'l}\int {\rm d}^3x'\,{\rm d}^3x\,\varepsilon'^*_{\lambda',l'}\varepsilon_{\lambda,l}\,{\rm e}^{-ik'\cdot x'+ik\cdot x}$$

$$\times\langle {\rm f}|\frac{1}{M_A}\Big(\tfrac{1}{2}k^2\frac{\partial}{\partial k_0}\hat{R}^0_{l'l}(x', x) - \nabla_{x'}\cdot\nabla_x\hat{j}_l(x)\hat{G}^2(-k_0)\hat{j}_{l'}(x')$$

$$-i(\nabla_{l'}\hat{j}_l(x)\hat{G}(-k_0)\hat{\rho}(x) - \hat{\rho}(x)\hat{G}(-k_0)\nabla'_l\hat{j}_{l'}(x'))\Big)|{\rm i}\rangle\,, \tag{A.13}$$

which is the c.m. recoil operator as given in (24).

In a similar manner one finds in the Breit frame, where

$$k' = k\,, \qquad P_{\rm i} = -P_{\rm f} = \tfrac{1}{2}(k' - k)\,, \tag{A.14}$$

$$R^{\rm c.m.}_{l'l}(x', x)_{\rm Breit} = \frac{1}{2M_A}\Bigg[\,i(\hat{\rho}(x')\hat{G}(k_0)\nabla_{l'}\hat{j}_l(x) - \nabla'_l\hat{j}_{l'}(x')\hat{G}(k_0)\hat{\rho}(x))$$

$$-\nabla_{x'}\cdot\nabla_x\hat{j}_{l'}(x')\hat{G}^2(k_0)\hat{j}_l(x)$$

$$+\begin{pmatrix} x & & x' \\ l & \leftrightarrow & l' \\ k_0 & & -k_0\end{pmatrix}\Bigg]\,. \tag{A.15}$$

Appendix B

DERIVATION OF THE RELATION (77)

Defining for $n = \pm 1$

$$b_n^{[L]}(k, x) = \sum_K \hat{K} \begin{pmatrix} 1 & K & L \\ n & 0 & -n \end{pmatrix} a_K^{[L]}(k, x) \tag{B.1}$$

it is straightforward to show that the transverse electric and magnetic multipole fields can be expressed as

$$A_\nu^{[L]}(k, x) = \sqrt{\tfrac{1}{2}}(-)^L (b_1^{[L]}(k, x) + (-)^\nu b_{-1}^{[L]}(k, x)), \qquad \nu = 0, 1 . \tag{B.2}$$

Furthermore, we introduce

$$\hat{\tau}_n^{[L]}(k) = \int d^3x\, b_n^{[L]}(k, x) \cdot j(x), \tag{B.3}$$

$$\hat{\tau}_n'^{[L]}(k) = \sum_l \int d^3x\, b_{n,l}^{[L]}(k, x) \nabla \hat{j}_l(x), \tag{B.4}$$

and analogously one finds for the transverse multipole operators

$$\hat{T}_\nu^{[L]}(k) = \sqrt{\tfrac{1}{2}}(-)^L (\hat{\tau}_1^{[L]} + (-)^\nu \hat{\tau}_{-1}^{[L]}), \qquad \nu = 0, 1, \tag{B.5}$$

and for the operators defined in (77)

$$\hat{T}_\nu'^{[L]}(k) = \sqrt{\tfrac{1}{2}}(\hat{\tau}_1'^{[L]} + (-)^\nu \hat{\tau}_{-1}'^{[L]}) . \tag{B.6}$$

After these introductory remarks we will evaluate

$$\hat{T}_K'^{[L]}(k) = \sum_l \int d^3x\, a_{K,l}^{[L]} \nabla \hat{j}_l(x) . \tag{B.7}$$

Partial integration and use of the gradient formula (43) yields.

$$\begin{aligned} \hat{T}_{K'}'^{[L]}(k) &= -\int d^3x [\nabla c^{[K']}(k, x) \times \hat{j}^{[1]}(x)]^{[L]} \\ &= -ik \sum_\lambda c_{K'\lambda} \int d^3x [a_\lambda^{[K']}(k, x) \times \hat{j}^{[1]}(x)]^{[L]}. \end{aligned} \tag{B.8}$$

By recoupling

$$\begin{aligned} [a_\lambda^{[K']} \times \hat{j}^{[1]}]^{[L]} &= [[c^{[\lambda]} \times e^{[1]}]^{[K']} \times \hat{j}^{[1]}]^{[L]} \\ &= (-)^{K'} \hat{K}' \sum_K (-)^K \hat{K} \begin{Bmatrix} 1 & \lambda & K \\ 1 & L & K' \end{Bmatrix} [[c^{[\lambda]} \times \hat{j}^{[1]}]^{[K]} \times e^{[1]}]^{[L]} \end{aligned} \tag{B.9}$$

one finds

$$\hat{T}'^{[L]}_{K'}(k) = -ik\hat{K}' \sum_{K\lambda} (-)^K \hat{\lambda}\hat{K} \begin{pmatrix} \lambda & K' & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 1 & \lambda & K' \\ 1 & L & K \end{Bmatrix} [\hat{T}^{[K]}_{\lambda}(k) \times e^{[1]}]^{[L]} . \tag{B.10}$$

Inserting this expression in (B.4) one obtains

$$\begin{aligned} \hat{\tau}'^{[L]}_n(k) &= \sum_{K'} \hat{K}' \begin{pmatrix} 1 & K' & L \\ n & 0 & -n \end{pmatrix} \hat{T}'^{[L]}_{K'}(k) \\ &= -ik \sum_{K\lambda} (-)^K \hat{\lambda}\hat{K} S(\lambda, K, L, n) [\hat{T}^{[K]}_{\lambda}(k) \times e^{[1]}]^{[L]} , \end{aligned} \tag{B.11}$$

where

$$\begin{aligned} S(\lambda, K, L, n) &= \sum_{K'} \hat{K}'^2 \begin{pmatrix} 1 & K' & L \\ n & 0 & -n \end{pmatrix} \begin{pmatrix} \lambda & K' & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 1 & \lambda & K' \\ 1 & L & K \end{Bmatrix} \\ &= \sum_m (-)^{L+K+1+m} \begin{pmatrix} \lambda & K & 1 \\ 0 & m & m \end{pmatrix} \begin{pmatrix} L & K & 1 \\ m & -m & 0 \end{pmatrix} \\ &\quad \times \sum_{K'} \hat{K}'^2 \begin{pmatrix} 1 & K' & L \\ n & 0 & -n \end{pmatrix} \begin{pmatrix} L & K' & 1 \\ -m & 0 & m \end{pmatrix} \\ &= (-)^n \begin{pmatrix} 1 & \lambda & K \\ n & 0 & -n \end{pmatrix} \begin{pmatrix} 1 & K & L \\ 0 & n & -n \end{pmatrix} . \end{aligned} \tag{B.12}$$

Therefore we get

$$\begin{aligned} \hat{\tau}'^{[L]}_n &= ik \sum_K (-)^K \hat{K} \begin{pmatrix} 1 & K & L \\ 0 & n & -n \end{pmatrix} \left[\left(\sum_\lambda \hat{\lambda} \begin{pmatrix} 1 & \lambda & K \\ n & 0 & -n \end{pmatrix} \hat{T}^{[K]}_{\lambda} \times e^{[1]} \right]^{[L]} \\ &= ik \sum_K (-)^K \hat{K} \begin{pmatrix} 1 & K & L \\ 0 & n & -n \end{pmatrix} [\hat{\tau}^{[K]}_n \times e^{[1]}]^{[L]} , \end{aligned} \tag{B.13}$$

and finally from (B.6) the desired relation

$$\begin{aligned} \hat{T}'^{[L]}_{\nu}(k) &= \sqrt{\tfrac{1}{2}} (-)^L ik \sum_K (-)^K \hat{K} \begin{pmatrix} 1 & K & L \\ 0 & 1 & -1 \end{pmatrix} [(\hat{T}^{[K]}_1 - (-)^{K+L+\nu} \hat{T}^{[K]}_{-1}) \times e^{[1]}]^{[L]} \\ &= (-)^L ik \sum_K (-)^K \hat{K} \begin{pmatrix} 1 & K & L \\ 0 & 1 & -1 \end{pmatrix} [\hat{T}^{[K]}_{\bar{\nu}(\nu L K)} \times e^{[1]}]^{[L]} , \end{aligned} \tag{B.14}$$

where

$$(-)^{\bar{\nu}} = (-)^{K+L+\nu+1} , \tag{B.15}$$

i.e.

$$\bar{\nu}=\begin{cases}\nu & K=L\pm 1\\ |\nu-1| & K=L\end{cases}. \tag{B.16}$$

Appendix C

FIRST AND SECOND ORDER CONTRIBUTIONS OF THE POLARIZABILITIES

(i) *Pure electric*

$$\begin{aligned}\hat{p}^{[J]}(\mathrm{E}L',\mathrm{E}L)_{(1)}=&(-)^{L+1}a^J_{L'L}\sqrt{\frac{L'+1}{L'}}\sqrt{\frac{L+1}{L}}\Bigg[[\hat{M}^{[L']}_{(0)}\times\hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{Q},-}\\&-\frac{1}{M_A}\Bigg(\begin{Bmatrix}L & L' & J\\ L'-1 & L-1 & 1\end{Bmatrix}(\sqrt{2L-1}\sqrt{L'}[\hat{\mathscr{T}}^{[L-1]}_{L(0)}\\&\times\hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}-\sqrt{2L'-1}\sqrt{L}[\hat{M}^{[L-1]}_{(0)}\times\hat{P}\hat{\mathscr{T}}^{[L'-1]}_{L'(0)}]^{[J]})\\&+\frac{\hat{L}'\sqrt{L}}{L'+1}\begin{Bmatrix}L & L' & J\\ L' & L-1 & 1\end{Bmatrix}[\hat{M}^{[L-1]}_{(0)}\times\hat{P}\hat{T}^{[L']}_{\mathrm{mag}(0)}]^{[J]}\\&+\frac{\hat{L}\sqrt{L'}}{L+1}\begin{Bmatrix}L & L' & J\\ L'-1 & L & 1\end{Bmatrix}[\hat{T}^{[L]}_{\mathrm{mag}(0)}\times\hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}\Bigg)\Bigg],\end{aligned} \tag{C.1}$$

$$\begin{aligned}\hat{p}^{[J]}(\mathrm{E}L',\mathrm{E}L)_{(2)}=&(-)^{L}a^J_{L'L}\Bigg[-\sqrt{\frac{L'+1}{L'}}\sqrt{\frac{L+1}{L}}\Bigg([\hat{M}^{[L']}_{(0)}\times\hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{G}_Q(0),+}\\&+\frac{\hat{L}}{\sqrt{L+1}}[\hat{M}^{[L']}_{(0)}\times\hat{\mathscr{T}}^{[L]}_{L+1(0)}]^{[J]}_{\hat{P},-}+\frac{\hat{L}'}{\sqrt{L'+1}}[\hat{\mathscr{T}}^{[L']}_{L'+1(0)}\times\hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{P},-}\Bigg)\\&+\frac{1}{M_A}\Bigg(\sqrt{L'+1}\sqrt{L+1}\begin{Bmatrix}L & L' & J\\ L'-1 & L-1 & 1\end{Bmatrix}\\&\times\Bigg(-2[\hat{M}^{[L-1]}_{(0)}\times\hat{G}^2_{Q(0)}\hat{M}^{[L'-1]}_{(0)}]^{[J]}\\&+[\hat{M}^{[L-1]}_{(0)}\times\hat{P}\hat{M}^{[L'-1]}_{(2)}]^{[J]}+[\hat{M}^{[L-1]}_{(2)}\times\hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}\\&-\frac{\sqrt{2L'-1}\sqrt{2L-1}}{\sqrt{L'L}}[\hat{\mathscr{T}}^{[L-1]}_{L(0)}\times\hat{P}\hat{\mathscr{T}}^{[L'-1]}_{L'(0)}]^{[J]}\Bigg)\\&+\sqrt{L'(L+1)}\begin{Bmatrix}L & L' & J\\ L'+1 & L-1 & 1\end{Bmatrix}[\hat{M}^{[L-1]}_{(0)}\times\hat{P}\hat{M}^{[L'+1]}_{(0)}]^{[J]}\\&+\sqrt{L(L'+1)}\begin{Bmatrix}L & L' & J\\ L'-1 & L+1 & 1\end{Bmatrix}[\hat{M}^{[L+1]}_{(0)}\times\hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}\end{aligned}$$

$$+\hat{L}'\sqrt{2L-1}\sqrt{\frac{L+1}{L'L(L'+1)}}\begin{Bmatrix} L & L' & J \\ L' & L-1 & 1 \end{Bmatrix}[\hat{\mathcal{T}}^{[L-1]}_{L(0)} \times \hat{P}\hat{T}^{[L']}_{\mathrm{mag}(0)}]^{[J}$$

$$+\hat{L}\sqrt{2L'-1}\sqrt{\frac{L'+1}{L'L(L+1)}}\begin{Bmatrix} L & L' & J \\ L'-1 & L & 1 \end{Bmatrix}[\hat{T}^{[L]}_{\mathrm{mag}(0)} \times \hat{P}\hat{\mathcal{T}}^{[L'-1]}_{L'(0)}]^{[\cdot}$$

$$+\frac{\hat{L}'\hat{L}}{\sqrt{L'L(L'+1)(L+1)}}\begin{Bmatrix} L & L' & J \\ L' & L & 1 \end{Bmatrix}[\hat{T}^{[L]}_{\mathrm{mag}(0)} \times \hat{P}\hat{T}^{[L']}_{\mathrm{mag}(0)}]^{[J]}$$

$$-\frac{1}{2}\sqrt{\frac{L'+1}{L'}}\sqrt{\frac{L+1}{L}}[\hat{M}^{[L']}_{(0)} \times \hat{M}^{[L]}_{(0)}]^{[J]}_{\hat{Q},-}\Bigg)\Bigg]. \tag{C.2}$$

(ii) *Pure magnetic*

$$p^{[J]}(\mathrm{M}L',\mathrm{M}L)_{(1)} = (-)^{L+1}a^J_{L'L}\Bigg[[\hat{T}^{[L']}_{\mathrm{mag}(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{P},-}$$

$$-\frac{1}{M_A}\Bigg(\hat{L}'\sqrt{\frac{L^2-1}{L}}\begin{Bmatrix} L & L' & J \\ L' & L-1 & 1 \end{Bmatrix}[\hat{T}^{[L-1]}_{\mathrm{mag}(0)} \times \hat{P}\hat{M}^{[L']}_{(0)}]^{[J]}$$

$$+\hat{L}\sqrt{\frac{L'^2-1}{L'}}\begin{Bmatrix} L & L' & J \\ L'-1 & L & 1 \end{Bmatrix}[M^{[L]}_{(0)} \times \hat{P}\hat{T}^{[L'-1]}_{\mathrm{mag}(0)}]^{[J]}\Bigg)\Bigg], \tag{C.3}$$

$$p^{[J]}(\mathrm{M}L',\mathrm{M}L)_{(2)} = (-)^{L+1}a^J_{L'L}\Bigg[\hat{B}^{L'L[J]}_{L'L(0)} - [\hat{T}^{[L']}_{\mathrm{mag}(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{G}_Q(0),+}$$

$$-\frac{1}{M_A}\Bigg(\tfrac{1}{2}[\hat{T}^{[L']}_{\mathrm{mag}(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{P},-}$$

$$+\hat{L}'\frac{L'+1}{L'}\sqrt{\frac{L^2-1}{L}}\begin{Bmatrix} L & L' & J \\ L' & L-1 & 1 \end{Bmatrix}[\hat{T}^{[L-1]}_{\mathrm{mag}(0)} \times \hat{G}_Q(0)\hat{M}^{[L']}_{(0)}]^{[J]}$$

$$+\hat{L}\frac{L+1}{L}\sqrt{\frac{L'^2-1}{L'}}\begin{Bmatrix} L & L & J \\ L'-1 & L & 1 \end{Bmatrix}[\hat{M}^{[L]}_{(0)} \times \hat{G}_Q(0)\hat{T}^{[L'-1]}_{\mathrm{mag}(0)}]^{[J]}$$

$$-\frac{\hat{L}'\hat{L}}{L'L}(L'+L+1)\begin{Bmatrix} L & L' & J \\ L' & L & 1 \end{Bmatrix}[\hat{M}^{[L]}_{(0)} \times \hat{Q}\hat{M}^{[L']}_{(0)}]^{[J]}\Bigg)\Bigg]. \tag{C.4}$$

(iii) *Mixed electric–magnetic*

$$\hat{p}^{[J]}(\mathrm{E}L',\mathrm{M}L)_{(1)} = (-)^{L+1}a^J_{L'L}\sqrt{\frac{L'+1}{L'}}\Bigg[[\hat{M}^{[L']}_{(0)} \times \hat{T}^{[L]}_{\mathrm{mag}(0)}]^{[J]}_{\hat{P},-}$$

$$-\frac{1}{M_A}\Bigg(\hat{L}\sqrt{L'+1}\begin{Bmatrix} L & L' & J \\ L'-1 & L & 1 \end{Bmatrix}[\hat{M}^{[L]}_{(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]}$$

$$-\hat{L}'\sqrt{\frac{L^2-1}{L'L(L'+1)}}\begin{Bmatrix} L & L & J \\ L' & L-1 & 1 \end{Bmatrix}$$

$$\times[\hat{T}^{[L-1]}_{\mathrm{mag}(0)} \times \hat{P}\hat{T}^{[L']}_{\mathrm{mag}(0)}]^{[J]}\Bigg)\Bigg], \tag{C.5}$$

$$
\begin{aligned}
\hat{p}^{[J]}(\mathrm{E}L', \mathrm{M}L)_{(2)} = \frac{(-)^{L+1}}{M_A} a^J_{L'L} \Bigg[& \sqrt{L'+1}\sqrt{\frac{L^2-1}{L}} \begin{Bmatrix} L & L' & J \\ L'-1 & L-1 & 1 \end{Bmatrix} \\
& \times ([\hat{T}^{[L-1]}_{\mathrm{mag}(2)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]} + [\hat{T}^{[L-1]}_{\mathrm{mag}(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(2)}]^{[J]}) \\
& + \hat{L}\sqrt{L'+1} \begin{Bmatrix} L & L' & J \\ L'-1 & L & 1 \end{Bmatrix} \Bigg(\frac{\hat{L}}{L\sqrt{L+1}} [\hat{\mathscr{T}}^{[L]}_{L+1(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]} \\
& - \sqrt{2L'-1}\sqrt{\frac{L'+1}{L'}} [M^{[L]}_{(0)} \times \hat{P}\hat{\mathscr{T}}^{[L'-1]}_{L'(0)}]^{[J]} \Bigg) \\
& - \frac{\hat{L}'\hat{L}}{\sqrt{L'(L'+1)}} \begin{Bmatrix} L & L' & J \\ L' & L & 1 \end{Bmatrix} [\hat{M}^{[L]}_{(0)} \times \hat{P}\hat{T}^{[L']}_{\mathrm{mag}(0)}]^{[J]} \\
& + \sqrt{L'}\sqrt{\frac{L^2-1}{L}} \begin{Bmatrix} L & L' & J \\ L'+1 & L-1 & 1 \end{Bmatrix} [\hat{T}^{[L-1]}_{\mathrm{mag}(0)} \times \hat{P}\hat{M}^{[L'+1]}_{(0)}]^{[J]} \\
& - \sqrt{L'+1}\sqrt{\frac{L(L+2)}{L+1}} \begin{Bmatrix} L & L' & J \\ L'-1 & L+1 & 1 \end{Bmatrix} \\
& \times [\hat{T}^{[L+1]}_{\mathrm{mag}(0)} \times \hat{P}\hat{M}^{[L'-1]}_{(0)}]^{[J]} \Bigg] . \qquad \text{(C.6)}
\end{aligned}
$$

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