ANNALS OF PHYSICS 95, 170–201 (1975)

Low-Energy Theorems for Nuclear Compton and Raman Scattering and $0^+ \to 0^+$ Two-Photon Decays in Nuclei*

J. L. FRIAR

Department of Physics, Brown University, Providence, Rhode Island 02912

Received June 13, 1975

Low-energy theorems for elastic photon scattering (nuclear Compton scattering) from a nucleus of arbitrary spin are derived in the nonrelativistic approximation through terms quadratic in the photon frequency. The same derivation is made for the special case of $0^+ \to 0^+$ nuclear excitation by inelastic photon scattering (nuclear Raman scattering). Use is made of the general principle of gauge invariance, which bypasses the need to specify the form of the current operator explicitly. A general discussion of the contribution of mesonic exchanges is made and their effect is isolated. The center-of-mass correction to the nuclear diamagnetic susceptibility is calculated. The $0^+ \to 0^+$ two-photon decay amplitude is obtained from the nuclear Raman amplitude and the transition rate is calculated.

1. INTRODUCTION

Recently, Ericson and Hüfner (EH) have analyzed the scattering of low-frequency photons from spinless nuclei [1]. Using this approximation, the nuclear Compton amplitude consists of two terms, the frequency independent Thompson amplitude and the Rayleigh amplitude, which is quadratic in the photon frequency. The Thompson amplitude depends only on the charge and mass of the target, a result that follows from a few very basic assumptions about the nature of the electromagnetic interaction and the target, and is otherwise independent of the target's internal dynamics [2, 3]. For a target with nonzero spin, the scattering amplitude contains additional terms linear in the photon frequency that depend only on the target's charge, mass, and magnetic moment [2, 3], a result whose calculation depends only on basic principles. The reconciliation of these fundamental calculations with calculations of photon scattering from the constituents of a composite system has recently led to a greatly improved understanding of the effects

* This work supported in part by the U.S. Energy Research and Development Administration under Contract ERDA E(11-1)-3235.

of relativity on the wave functions of slowly moving composite systems [4–8]. The reader is referred to the particularly illuminating discussion in Ref. [8].

The Rayleigh amplitude contains terms that follow from fundamental principles, but in addition the amplitude contains structure dependent functions that depend on the dynamical details [9–15]. Within the usual nonrelativistic model, without potential-dependent currents, these structure functions were constructed explicitly by EH, and their interpretation was facilitated by a classical derivation of electromagnetic scattering.

Compton scattering is an example of a two-photon process, one photon being in the initial state and one in the final state. Other two-photon processes are possible, such as two-photon absorption or two-photon emission, but conservation of energy requires that the nuclear part of the amplitude be inelastic. The latter two processes, in fact, are intimately related to nuclear Raman scattering, that is, photon scattering with excitation (or deexcitation) of the nucleus. In atomic physics the two-photon absorption process is rather common, because lasers provide the intense light sources necessary for the realization of the process. Two-photon decays of the $2s$ states of hydrogenic ions have been observed and the transition rates have been determined for the following elements: He [16, 17], Fl [18], O [18], S [19], and Ar [19]. The angular distribution for the He decay was measured in [20, 21]. In addition, the (presumed two-photon) decay rates of the 2^1S_0 state of He [22] and the heliumlike ions of Li [23] and Ar [19] have been measured. These atomic measurements are in good agreement with theoretical predictions [29–31]. In addition, the general properties of three-photon decay rates have been calculated [32].

The situation in nuclear physics regarding inelastic two-photon processes is much less satisfactory. Nuclear Raman scattering has been observed in numerous instances and this work is summarized nicely by Arenhövel [33]. Two-photon decays in nuclei are extremely difficult to observe, and most experimental work has concentrated on nuclei with a 0^+ first excited state and a 0^+ ground state. The excited state cannot decay by real single-photon emission and proceeds primarily by electron–positron pair creation, which is a virtual single-photon process. Competing with the latter process is real two-photon decay. Among the very few nuclei with a 0^+ ground state and 0^+ first excited state are ^{16}O (6.05 MeV transition), ^{40}Ca (3.35 MeV) and ^{90}Zr (1.76 MeV). Measured two-photon decay rates for these nuclei are contradictory; in many cases the measured rates were larger than the upper limits established in other experiments. Recent measurements [34–36] of the ^{90}Zr rate are in mutual agreement and a similar number is reported [37] for the ^{40}Ca decay rate. Ref. [37] contains a complete list of references to the earlier experimental work.

Theoretical work on this problem [38–53] has centered to a large extent on the nuclear matrix elements, which result in various approximate treatments of the decay problem. A review of the older theoretical work is contained in [51]. Many of these papers disagree by factors of two or four, and this point will be discussed in Section 6. The recent work of Eichler [47] is similar in spirit, if not in approach [77] or scope, to the present work.

Other two-photon decay processes are possible where one or both photons are virtual and result in the emission of one or two atomic electrons from the atom. Theoretical work on this type of process was initiated by Goldberger [54], continued in [40, 45, 46], and more recently discussed by Grechukhin [55, 56]. The recent experiment of Jurčević, Ilakovac, and Krečak [57] measured the e–γ decay rate of ^{113}In, and their paper contains references to earlier work. No further consideration of this type of decay will be made here.

Our primary interest is in deriving low-energy theorems for nuclear Compton and Raman scattering, using gauge invariance as a tool rather than as a feature of a specific model. We will see that the cancellations of terms that resulted in the work of EH were the result of this principle and were not a fortuitous accident. At the same time a number of minor errors in EH will be corrected. In addition, we will treat Compton scattering for a nucleus of arbitrary spin. One specific new result will be the correct center-of-mass correction for the nuclear diamagnetic susceptibility. The two-photon decay amplitude will be derived for $0^+ \to 0^+$ transitions from the Raman amplitude.

In particular, Section 2 deals with the construction of the general two-photon amplitude for a nucleus and we discuss its relationship to the scattering and decay problem. Section 3 deals with gauge invariance and the restrictions this general principle places on the amplitude. The general two-photon amplitude is expanded in Section 4, assuming low-energy photons, and low-energy theorems are derived. The cross section for photon scattering and the transition probability for two-photon decays are derived in Section 5. A discussion of these results and our conclusions are presented in Section 6.

2. Two-Photon Amplitudes

Our treatment of two-photon amplitudes will necessarily be approximate, and many interesting physical phenomena will be ignored. In particular, we perform a nonrelativistic treatment of the nuclear physics, and the effects of relativity on nuclear wavefunctions and on nuclear current operators will be ignored. Specifically, we use charge and current operators that are the result of a (v/c) expansion of the complete current and charge operators. Since (v/c), the ratio of some nuclear velocity to the speed of light, is the same as momentum divided by some mass, a (v/c) expansion will be reckoned to give the same result as a $(1/m)$ expansion. Furthermore, the nucleus is a weakly bound system and the potential and kinetic energies are therefore roughly equal and opposite, so that potential-dependent quantities such as exchange currents will be considered as order $(1/m)$. We shall keep only terms through order $(1/m)$.

Many of the arguments we consider in this section have been treated in part by Friar and Rosen (FR), particularly in [58, Sect. 2 and Appendix A] and we will use these results where possible to avoid duplication. In particular in [58], the gauge invariance of the virtual two-photon amplitude was demonstrated, and a demonstration of translation invariance of the various currents was made; these results will be assumed. We will use the metric and notation of Bjorken and Drell [59] where possible.

Since one of the primary tasks of this work is to generalize the results of EH to include the effect of meson currents, we must ask the question of how this interesting phenomenon affects the electromagnetic interaction of a nucleus. In lowest-order perturbation theory, depicted in Fig. 1a, a nucleus in the state $|\, i\mathbf{P}_i\rangle$ with initial four-momentum P_i interacts with a photon bringing four-momentum $q \equiv (q_0, \mathbf{q})$ into the interaction vertex, whereupon the nucleus emerges in the state $|\, f\mathbf{P}_f\rangle$ with four-momentum P_f. The nuclear states are eigenfunctions of the purely nuclear Hamiltonian $\hat{H}_0$

$$\hat{H}_0 \,|\, f\mathbf{P}_f\rangle = E_f \,|\, f\mathbf{P}_f\rangle \tag{1a}$$

$$E_f = \mathbf{P}_f^2/2m_t + \omega_f , \tag{1b}$$

etc., where ω_f is the intrinsic (internal) energy of the final state with quantum numbers f and the remaining component is the recoil energy for a nucleus with nucleon number A and mass $m_t = A \cdot m$. All nucleons will be assumed to have equal masses m. Although Fig. 1a depicts only final states with no mesons present, such final states are possible if sufficient energy q_0 is available. In addition, mesonless final states can be created through an intermediate meson which, having been produced from a single nucleon, can land on any nucleon in the nucleus. If the latter nucleon is the same as the former, the process contributes to the nucleon form factor. If the meson attaches itself to a different nucleon, the processes are contributions to the exchange currents. Thus, the meson contributions to the production of real or virtual mesonless states can be treated by modification of the charge and current density operators. To the order in $(1/m)$ we are presently working, we can ignore the mesonic contributions to the charge operator, since an examination of pion-exchange contributions to this operator [66] indicates that it is of relativistic order $(1/m)^2$.

Inclusion of meson-exchange currents, without allowing for possible meson production, would be inconsistent. In Figs. 1b–1d, we depict processes of second order in the electromagnetic field. In Figs. 1b and 1c (virtual) intermediate nuclear states are excited, while Fig. 1d illustrates the so-called seagull diagram. An under-

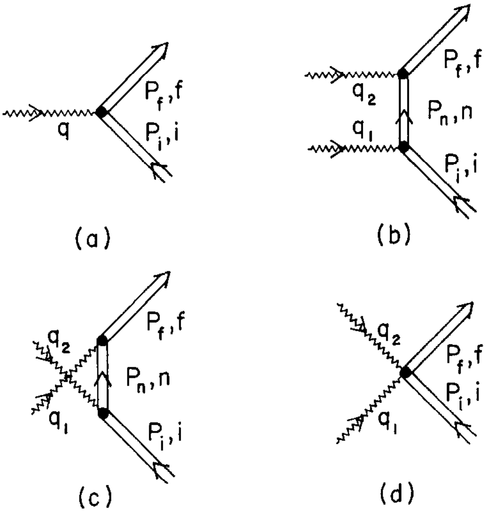

FIG. 1. Single photon absorption (or emission) by a nucleus is shown in (a) while direct two-photon absorption (or emission) by a nucleus and the crossed two-photon contribution are shown in (b) and (c), respectively. The seagull contribution to two-photon processes is shown in (d).

standing of the origin of the latter contribution is important and helpful in discerning the role of meson production. While we have included only nuclear intermediate states in Figs. 1b and 1c, we obviously must allow for the possibility of mesons in these states. The seagull diagram is the result of the nonrelativistic reduction process mentioned earlier, and arises because of the possibility of exciting and deexciting very high-mass states that contain nucleon–antinucleon pairs. This "pair" contribution, expanded to order $(1/m)$, generates the contribution shown in Fig. 1d, which is given by the $\mathbf{A}^2/2m$ term in the Schroedinger equation. Higher-order terms in the $(1/m)$-expansion are relativistic corrections, which we ignore. By means of this expansion, the very complicated "pair" contribution has been replaced by an excellent approximation, and we no longer need to consider anti-nucleon contributions to intermediate states. If we restrict ourselves to those low-energy processes that are not sensitive to high-mass intermediate states involving mesons, a similar procedure can be followed. We assume that the intermediate meson energy dominates each of the energy denominators we generate in perturbation theory and we ignore all nuclear energies in comparison to the meson energies. This closure approximation eliminates the nuclear component of nucleus-plus-meson intermediate states and what remains is an effective two-body seagull contribution similar to that of Eq. (1d). Using this approximation scheme, all intermediate states involve *no* mesons, and the sole effect of the mesonic degrees-of-freedom is contained in the exchange current operators and the effective "seagull" interaction.

Following FR, we write the Hamiltonian in the form

$$H = \hat{H}_0 + e \int \hat{J}_\mu(\mathbf{x})\, A^\mu(\mathbf{x})\, d^3\mathbf{x} + \frac{e^2}{2} \int A^\mu(\mathbf{x})\, A^\nu(\mathbf{y})\, \hat{B}_{\mu\nu}(\mathbf{x}, \mathbf{y})\, d^3\mathbf{x}\, d^3\mathbf{y}, \tag{2}$$

where the current operator is $\hat{J}_\mu(\mathbf{x}) = (\hat{\rho}(\mathbf{x}), \hat{\mathbf{J}}(\mathbf{x}))$ and $\hat{B}_{\mu\nu}$ is the seagull operator. We have assumed, in view of our previous assumption, that electromagnetic interactions in the seagull portion of H occur at the same time. Our notation for the charge and current reflects the relativistic transformation properties of the four-current, which is not an attribute of the approximate nonrelativistic quantities we will actually use. This notation is merely a convenience. In the absence of exchange currents, we recover the result of FR,

$$\hat{B}_{\mu\nu}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{\mu\nu}}{m} \hat{f}_{SG}(\mathbf{x}, \mathbf{y}) \tag{3}$$

where $\hat{f}_{SG}$ is given in [58, Eqs. 4 and 5] and $\delta_{\mu\nu}$ is the usual three-dimensional Kronecker delta δ_{mn} extended to four dimensions with $\delta_{00} = \delta_{0m} = 0$. We uniformly denote 3-vectors by Latin indices and 4-vectors by Greek indices.

The two-photon amplitudes, whether for emission or absorption of both photons or the scattering of a photon, may be calculated using Feynman rules or ordinary time-dependent perturbation theory in interaction representation with second-quantized photon fields. Adopting the latter viewpoint, we define

$$H_I = H_I^0 + H_I' \tag{4a}$$

$$H_I^0 = e \int A^\mu(x)\, \hat{J}_\mu(x)\, d^4x \tag{4b}$$

$$H_I' = \frac{e^2}{2} \int A^\mu(x)\, A^\nu(y)\, \hat{B}_{\mu\nu}(x, y)\, d^4y\, d^4x \tag{4c}$$

$$\hat{J}_\mu(x) = \exp(i\hat{H}_0 x_0)\, \hat{J}_\mu(\mathbf{x})\, \exp(-i\hat{H}_0 x_0) \tag{5a}$$

$$\hat{B}_{\mu\nu}(x, y) = \exp(i\hat{H}_0 x_0)\, \hat{B}_{\mu\nu}(\mathbf{x}, \mathbf{y})\, \exp(-i\hat{H}_0 x_0)\, \delta(x_0 - y_0). \tag{5b}$$

The amplitude for two-photon decay of the initial state to the final state can be obtained by calculating the contribution of H_I' in first order perturbation theory and H_I^0 through second order in perturbation theory. The single-photon decay amplitude can be obtained using first-order perturbation theory on H_I^0. If the polarization vectors of the two photons are denoted $\epsilon(k)$ and $\epsilon'(k')$ and correspond to momenta k and k', respectively, the two-photon decay S-matrix element, after calculating the photon matrix elements, is given by

$$S_{fi} = \frac{-ie^2}{(4\omega\omega')^{1/2}} \epsilon^\mu(k)\, \epsilon^\nu(k') \iint d^4x\, d^4y \exp(i(k\cdot x + k'\cdot y)) \\ \times \langle f\mathbf{P}_f \mid \hat{B}_{\mu\nu}(x, y) - iT(\hat{J}_\mu(x), \hat{J}_\nu(y)) \mid i\mathbf{P}_i\rangle, \tag{6}$$

where $k = (\omega, \mathbf{k})$, $k' = (\omega', \mathbf{k}')$, and $T(\,,\,)$ denotes a time-ordered product. We exploit the translation-invariance of the current and seagull operators, which was proven in FR for the special case of no exchange currents.

$$T(J_\mu(x), J_\nu(y)) = e^{i\hat{P}\cdot y} T(\hat{J}_\mu(x-y), \hat{J}_\nu(0))\, e^{-i\hat{P}\cdot y} \tag{7a}$$

$$B_{\mu\nu}(x, y) = e^{i\hat{P}\cdot y} \hat{B}_{\mu\nu}(x-y, 0)\, e^{-i\hat{P}\cdot y}. \tag{7b}$$

Changing variables and performing one integral, we obtain

$$S_{fi} = \frac{-ie^2}{(4\omega\omega')^{1/2}} (2\pi)^4\, \delta^4(P_f - P_i + k' + k)\, \epsilon^\mu(k)\, \epsilon^\nu(k')\, M_{\mu\nu} \tag{8a}$$

$$M_{\mu\nu} = \int d^4z\, e^{ik\cdot z} \langle f\mathbf{P}_f \mid \hat{B}_{\mu\nu}(z, 0) - iT(J_\mu(z), J_\nu(0)) \mid i\mathbf{P}_i\rangle. \tag{8b}$$

To determine the amplitude for scattering of a photon with momentum k and polarization ϵ, which emerges as a photon of momentum k' and polarization ϵ' after changing the nuclear state from i to f, an identical calculation shows that one need only make the substitution $k \to -k$ in Eq. (8b) and in the δ-function in Eq. (8a). In fact, this is just the substitution rule [59].

The remaining integrals can be performed if a complete set of states is inserted between the currents in the last term in Eq. (8b). The virtual Compton amplitude defined in FR and shown in Fig. 1 with photon momenta q_1 and q_2 flowing *into* the vertex can be trivially determined from this amplitude by means of the substitution rule. We obtain

$$T_{\mu\nu}(q_1, q_2) = \sum_n \frac{\langle f\mathbf{P}_f \mid \hat{J}_\mu(0) \mid n\mathbf{q}_1 + \mathbf{P}_i\rangle\langle n\mathbf{q}_1 + \mathbf{P}_i \mid \hat{J}_\nu(0) \mid i\mathbf{P}_i\rangle}{E_i - E_n + q_1^0 + i\epsilon} \\ + \sum_n \frac{\langle f\mathbf{P}_f \mid \hat{J}_\nu(0) \mid n\mathbf{q}_2 + \mathbf{P}_i\rangle\langle n\mathbf{q}_2 + \mathbf{P}_i \mid \hat{J}_\mu(0) \mid i\mathbf{P}_i\rangle}{E_i - E_n' + q_2^0 + i\epsilon} \\ + \langle f\mathbf{P}_f \mid \hat{B}_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) \mid i\mathbf{P}_i\rangle, \tag{9}$$

where

$$\hat{B}_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) = \int d^3\mathbf{z}\, e^{i\mathbf{q}_1\cdot\mathbf{z}} \hat{B}_{\mu\nu}(\mathbf{z}, 0) \tag{10}$$

depends on $\mathbf{q}_2$ through the momentum-conserving δ-function. In terms of this amplitude, which was used in FR to calculate dispersion corrections to electron scattering, the two-photon decay amplitude is given by

$$M_{\mu\nu}(k, k') = T_{\mu\nu}(-k', -k). \tag{11a}$$

The amplitude for photon scattering, which has the same form as Eq. (8a), is given by

$$N_{\mu\nu}(k, k') = T_{\mu\nu}(-k', k) \tag{11b}$$

where $N_{\mu\nu}$ is the analog of $M_{\mu\nu}$ for this process. As noted in FR, the prime on E_n in the second term in Eq. (9) indicates the recoil part of the nuclear energy is different from the first term.

For completeness, we calculate the amplitude for the emission of a single photon of four-momentum k:

$$S_{fi} = \frac{-ie(2\pi)^4}{(2\omega)^{1/2}} \delta^4(P_f - P_i + k)\, \epsilon^\mu(k) \langle f\mathbf{P}_f \mid \hat{J}_\mu(0) \mid i\mathbf{P}_i\rangle. \tag{12}$$

As noted in FR, the representation used above for the charge and current operators is not the usual one. The following identities were found to be useful in switching to the more conventional representation

$$\langle f\mathbf{P}_f \mid \hat{J}_\mu(0) \mid i\mathbf{P}_i\rangle \equiv \langle f \mid J_\mu(\mathbf{q}, \mathbf{S}) \mid i\rangle \tag{13a}$$

$$\langle f\mathbf{P}_f \mid \hat{B}_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) \mid i\mathbf{P}_i\rangle \equiv \langle f \mid B_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) \mid i\rangle \tag{13b}$$

$$\rho(\mathbf{q}) = \sum_{i=1}^{A} \hat{e}_i(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}_i'} \tag{14a}$$

$$\mathbf{J}(\mathbf{q}, \mathbf{S}) = \mathbf{J}(\mathbf{q}) + \mathbf{S}\rho(\mathbf{q})/2m_t \tag{14b}$$

$$\mathbf{q} = \mathbf{P}_f - \mathbf{P}_i \qquad \mathbf{S} = \mathbf{P}_f + \mathbf{P}_i, \tag{14c}$$

where $\mathbf{J}(\mathbf{q})$ is the S-independent current composed of an exchange part, a magnetization part, and a convection part. The latter two parts are treated in many places, including FR, while the exchange part arising from pions is discussed in detail in [60, 61]. The term proportional to S and ρ is the convection current of the nucleus as a whole, and its form is guaranteed to this order in (v/c) by Lorentz transformation arguments (see [62, Appendix A]). In terms of the definitions in Eq. (13), Eq. (9) becomes

$$\begin{aligned} T_{\mu\nu}(q_1, q_2) = {} & \sum_n \frac{\langle f \mid J_\mu(\mathbf{q}_2, 2\mathbf{P}_f - \mathbf{q}_2) \mid n\rangle\langle n \mid J_\nu(\mathbf{q}_1, 2\mathbf{P}_i + \mathbf{q}_1) \mid i\rangle}{E_i - E_n + q_1^0 + i\epsilon} \\ & + \sum_n \frac{\langle f \mid J_\nu(\mathbf{q}_1, 2\mathbf{P}_f - \mathbf{q}_1) \mid n\rangle\langle n \mid J_\mu(\mathbf{q}_2, 2\mathbf{P}_i + \mathbf{q}_2) \mid i\rangle}{E_i - E_n' + q_2^0 + i\epsilon} \\ & + \langle f \mid B_{\mu\nu}(\mathbf{q}_1, \mathbf{q}_2) \mid i\rangle, \end{aligned} \tag{15a}$$

where momentum and energy conservation state that

$$q_1{}^0 + q_2{}^0 + E_i = E_f \tag{15b}$$

$$\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}. \tag{15c}$$

Although we are only interested in using Eq. (15) for those cases in which the nucleus couples to real photons ($k^2 = k'^2 = 0$), we will find it convenient to work with the more general amplitude we have constructed. At the appropriate time the appropriate substitution for $\mathbf{q}_1$, $\mathbf{q}_2$, $q_1{}^0$, and $q_2{}^0$ will be made.

3. Gauge Invariance

The virtual Compton amplitude (Eq. (15)) has several symmetries. It is invariant under the simultaneous interchange of the variables q_1 and q_2, and the indices μ and ν. This is crossing symmetry and arises in our amplitudes from the Bose statistics of the two identical photons. Equally important for our use are the requirements of gauge invariance for the one-and two-photon amplitudes

$$q^\mu \langle f \mid J_\mu(\mathbf{q}, \mathbf{S}) \mid i \rangle = 0 \tag{16a}$$

$$q_2{}^\mu T_{\mu\nu} = q_1{}^\nu T_{\mu\nu} = 0. \tag{16b}$$

Eq. (16a) is simply the requirement of current conservation. In terms of the operators $\mathbf{J}$ and ρ this can be cast into the form

$$\mathbf{q} \cdot \mathbf{J}(\mathbf{q}, \mathbf{S}) = [h_0, \rho(\mathbf{q})] + \mathbf{q} \cdot \mathbf{S}\rho(\mathbf{q})/2m_t \tag{17}$$

where h_0 is the part of the Hamiltonian that determines the internal structure, and the last term is proportional to the kinetic energy of recoil of the entire nucleus. Note that $\mathbf{q}$ and $\mathbf{S}$ are independent vectors, in general. According to Eq. (17), that component of $\mathbf{J}$ parallel to $\mathbf{q}$ is determinable from a knowledge of ρ and h_0, while in the absence of further information, we cannot determine the form of the remaining components of $\mathbf{J}$. Because we are interested in amplitudes that involve low-energy photons, we wish to expand Eq. (17) in a power series in $\mathbf{q}$. We first take derivatives of the equation with respect to q_m, $q_m q_n$, and $q_m q_n q_r$ keeping other quantities fixed and then set $\mathbf{q} = 0$. Defining the electric dipole, quadrupole, and octupole operators as

$$\begin{aligned} D_m &= -\frac{i\,\partial\rho(0)}{\partial q_m} \\ Q_{mn} &= -\frac{\partial}{\partial q_m}\frac{\partial\rho(0)}{\partial q_n} \\ O_{mnr} &= \frac{i\partial}{\partial q_m}\frac{\partial}{\partial q_n}\frac{\partial\rho(0)}{\partial q_r}, \end{aligned} \tag{18}$$

we find the following identities, using $J_m(0) \equiv J_m(0, \mathbf{S})$

$$J_m(0) = i[h_0, D_m] + \frac{S_m Z}{2m_t} \tag{19a}$$

$$\frac{\partial J_m(0)}{\partial q_n} + \frac{\partial J_n(0)}{\partial q_m} = -[h_0, Q_{mn}] + \frac{i(S_m D_n + S_n D_m)}{2m_t} \tag{19b}$$

$$\frac{\partial^2 J_m(0)}{\partial q_n \, \partial q_r} + \text{sym} = -i[h_0, O_{mnr}] - \frac{(Q_{mn} S_r + \text{sym})}{2m_t}, \tag{19c}$$

where Z is the proton number and sym indicates symmetrization with respect to all the indices. Gauge invariance relates the symmetric derivatives of $\mathbf{J}$ to the electric multipole operators. We are presently interested in the small-$\mathbf{q}$ expansion of J_m and it will be sufficient to expand $\mathbf{J}$ to first order in $\mathbf{q}$

$$J_m(\mathbf{q}, \mathbf{S}) \cong J_m(0) + \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} J_m(0) + \cdots. \tag{20}$$

The second term can be split into symmetric and antisymmetric combinations

$$q_n \frac{\partial}{\partial q_n} J_m(0) = \frac{q_n}{2} \left[\left(\frac{\partial J_m(0)}{\partial q_n} + \frac{\partial J_n(0)}{\partial q_m} \right) + \left(\frac{\partial J_m(0)}{\partial q_n} - \frac{\partial J_n(0)}{\partial q_m} \right) \right]. \tag{21}$$

The former piece is determined by Eq. (19b), while the latter piece, which is the curl of $\mathbf{J}$ with respect to $\mathbf{q}$, is determined by the *definition* of the magnetic moment

$$\frac{\partial J_m(0)}{\partial q_n} - \frac{\partial J_n(0)}{\partial q_m} = -2i\epsilon_{mnr}\mu_r, \tag{22}$$

where $\boldsymbol{\mu}$ is the magnetic moment operator. Collecting all terms, we get

$$\begin{aligned} J_m(\mathbf{q}, \mathbf{S}) = {} & i[h_0, D_m] + S_m Z/2m_t - (q_n/2)[h_0, Q_{mn}] \\ & + i(S_m \mathbf{q} \cdot \mathbf{D} + D_m \mathbf{q} \cdot \mathbf{S})/4m_t - i(\mathbf{q} \times \boldsymbol{\mu})_m + \cdots. \end{aligned} \tag{23}$$

Although these relations are obtained easily by other methods, the same technique may be used on the two photon amplitude $T_{\mu\nu}$. As shown in FR, the use of current conservation alone, Eq. (16a), allows us to cast Eq. (16*b*) into the form

$$\begin{aligned} J_\mu(\mathbf{q}_1, 2\mathbf{P}_f - \mathbf{q}_1)\, \rho(\mathbf{q}_2) - \rho(\mathbf{q}_2)\, J_\mu(\mathbf{q}_1, 2\mathbf{P}_i + \mathbf{q}_1) &= -q_2{}^\alpha B_{\alpha\mu}(\mathbf{q}_1, \mathbf{q}_2) \\ J_\mu(\mathbf{q}_2, 2\mathbf{P}_f - \mathbf{q}_2)\, \rho(\mathbf{q}_1) - \rho(\mathbf{q}_1)\, J_\mu(\mathbf{q}_2, 2\mathbf{P}_i + \mathbf{q}_2) &= -q_1{}^\alpha B_{\mu\alpha}(\mathbf{q}_1, \mathbf{q}_2), \end{aligned} \tag{24}$$

where we have removed the initial and final states for convenience. No assumption about the nature of the charge and current operators has been made.[1] Using Eq. (14b), the S-dependent pieces may be separated from $\mathbf{J}(q)$. In addition, because we are performing a nonrelativistic treatment, the $\mu = 0$ pieces on the left-hand side of Eq. (24) commute [62] and we find

$$B_{0\alpha} q_1^{\alpha} = B_{\alpha 0} q_2^{\alpha} = 0 \tag{25}$$

In the absence of exchange currents (Eq. (3)), $B_{0\alpha} \equiv 0$, in the nonrelativistic approximation, and we assume that these contributions also vanish in the mesonic seagull terms. If they do not, the inclusion of these terms is easily handled.[2] Note that the spin-orbit relativistic corrections [62] to ρ would negate the commutativity of the $\mu = 0$ pieces and would produce a seagull term of the form B_{0m}. Neglecting $B_{0\alpha}$ and $B_{\alpha 0}$ as higher-order (in $1/m$) terms, we find for the $\mu = m$ terms

$$[J_m(\mathbf{q}_1), \rho(\mathbf{q}_2)] = -\frac{q_{2m}}{m_t} \rho(\mathbf{q}_1)\, \rho(\mathbf{q}_2) + q_{2k} B_{km}(\mathbf{q}_1, \mathbf{q}_2) \tag{26a}$$

$$[J_m(\mathbf{q}_2), \rho(\mathbf{q}_1)] = -\frac{q_{1m}}{m_t} \rho(\mathbf{q}_1)\, \rho(\mathbf{q}_2) + q_{1k} B_{mk}(\mathbf{q}_1, \mathbf{q}_2). \tag{26b}$$

Differentiating Eq. (26a) with respect to $(\mathbf{q}_2)_n$ and then equating $\mathbf{q}_2$ to zero, and the analogous operation on Eq. (26b) yields

$$i[J_m(\mathbf{q}_1), D_n] = -\delta_{mn} Z \rho(\mathbf{q}_1)/m_t + B_{nm}(\mathbf{q}_1, 0) \tag{27a}$$

$$i[J_m(\mathbf{q}_2), D_n] = -\delta_{mn} Z \rho(\mathbf{q}_2)/m_t + B_{mn}(0, \mathbf{q}_2). \tag{27b}$$

Expansions obviously can be made with respect to the remaining momentum variable, if desired. For example, taking $\mathbf{q}_1 = 0$ in Eq. (27a) or $\mathbf{q}_2 = 0$ in Eq. (27b) yields

$$[[h_0, D_m], D_n] = \delta_{mn} Z^2/m_t - B_{nm}(0, 0). \tag{28}$$

[1] In writing Eq. (24), we have assumed implicitly that the charge, current, and seagull operators do not involve time derivatives. Such terms in the charge and current operators may be eliminated by modifying the definition of the charge and current operators and the seagull (equal time) terms. For simplicity, we ignore the possibility that B depends on q_1^0 or q_2^0. Should such terms occur, they may be separated from the rest of the B terms and treated separately. In the absence of meson exchange effects, such terms do not occur in nonrelativistic order, although they do occur as relativistic corrections [4–6, 8].

[2] Since q_2^0 and $\mathbf{q}_2$ as well as q_1^0 and $\mathbf{q}_1$ are independent variables, Eq. (25) shows that if B_{00} does not vanish, B_{0m} must depend on q_1^0 and q_2^0, and therefore B_{mn} must have terms that depend on q_1^0 and q_2^0 also. As discussed in the previous footnote, such terms in B_{mn} may be lumped together with the $B_{0\alpha}$ terms and treated separately. Such additional terms in B_{mn} would produce additional contributions to Eq. (58).

Taking two derivatives of $\rho(\mathbf{q}_2)$ and one of $J_m(\mathbf{q}_1)$ in Eq. (26a) yields

$$\left[\frac{1}{2}[h_0, Q_{mn}] + i\epsilon_{mnr}\mu_r, Q_{ab}\right] = \delta_{am}\frac{D_n D_b}{m_t} + \frac{\partial^2 B_{am}(0,0)}{\partial q_{2b}\,\partial q_{1n}} + a \leftrightarrow b \quad (29a)$$

$$\left[\frac{1}{2}[h_0, Q_{mn}] + i\epsilon_{mnr}\mu_r, Q_{ab}\right] = \delta_{am}\frac{D_n D_b}{m_t} + \frac{\partial^2 B_{ma}(0,0)}{\partial q_{1b}\,\partial q_{2n}} + a \leftrightarrow b. \quad (29b)$$

Note that the seagull contribution at zero momentum is given by a simple double commutator, while that part of B_{am} proportional to $q_{1n}q_{2b}$ which is symmetric in $a \leftrightarrow b$ is also related to a simple commutator. The remaining parts are independent, just as the antisymmetric derivatives of $\mathbf{J}$ were not specified by gauge invariance. We will find it necessary to separate the B-contributions in Eq. (29) according to their symmetry, and for this reason we define

$$A_{ab}^{mn} = \left.\frac{\partial^2 B_{mn}(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_{1b}\partial q_{2a}}\right|_{\mathbf{q}_1=\mathbf{q}_2=0}. \quad (30)$$

Crossing symmetry gives

$$A_{ab}^{mn} = A_{ba}^{nm}. \quad (31)$$

With four indices, A has three sets of pairs of indices, one set being fixed by Eq. (31). One set is irrelevant for our task, while the pairs (m, a) and (n, b) are very important because of Eq. (29). The symmetric combination $A_{ab}^{mn} + A_{mb}^{an}$ is determinable from the multipole operators, as is the symmetric combination $A_{ab}^{mn} + A_{an}^{mb}$. We divide the A-terms according to their symmetry in the pairs of indices (m, a) and (n, b),

$$A_{ab}^{mn} = A_{ab}^{mn}(s, s) + A_{ab}^{mn}(s, a) + A_{ab}^{mn}(a, s) + A_{ab}^{mn}(a, a), \quad (32)$$

where s or a in the first or second position indicates symmetrization or antisymmetrization of the sets (m, a) or (n, b), respectively. For example,

$$A_{ab}^{mn}(s, a) = (A_{ab}^{mn} - A_{an}^{mb} + A_{mb}^{an} - A_{mn}^{ab})/4.$$

We find therefore

$$[[h_0, Q_{mn}], Q_{ab}] = (\delta_{am}D_nD_b + \delta_{an}D_mD_b)/m_t + a \leftrightarrow b + 4A_{bn}^{am}(s, s) \quad (33a)$$

$$2i\epsilon_{mnr}[\mu_r, Q_{ab}] = (\delta_{am}D_nD_b - \delta_{an}D_mD_b)/m_t + a \leftrightarrow b + 4A_{bn}^{am}(s, a) \quad (33b)$$

$$A_{nb}^{ma}(a, s) = A_{bn}^{am}(s, a). \quad (33c)$$

The remaining term $A(a, a)$ can be cast into the form of a double curl

$$A_{ab}^{mn}(a, a) = \epsilon_{mar}\epsilon_{nbs}\chi_{rs}^{D}$$
$$\chi_{rs}^{D} = \epsilon_{rmn}\frac{\partial}{\partial q_{2m}}\epsilon_{stu}\frac{\partial}{\partial q_{1t}}B_{nu}(0, 0)/4, \tag{34}$$

which does not enter into any gauge-invariance relationship.

4. Low-Energy Theorems

We are primarily interested in three processes, which we will consider in the lab frame: photon scattering from a nucleus with no nuclear excitation (nuclear Compton effect), photon scattering with nuclear excitation (nuclear Raman effect), and two-photon decays of excited nuclear states. In each case, we will restrict our attention to those cases involving low-energy photons, which will allow us to simplify the general results of Section 2. For the latter two processes, which involve a change of nuclear state, we will consider only $0^+ \rightarrow 0^+$ nuclear transitions for simplicity. A further restriction is necessary since we have already made the nonrelativistic approximation in deriving Eq. (15), and any further results must be consistent with this approximation.

In real photon scattering without excitation, it is always possible to choose the photon energy arbitrarily small compared to the excitation energy of the excited intermediate states, and a consistent expansion of the photon energy dependent term in the energy denominators in $T_{\mu\nu}$ is well-defined. If there is excitation, this is no longer possible and the minimum energy is the nuclear excitation energy plus the additional small amount of recoil energy. Similarly, for two-photon decay processes the sum of the energies of the two photons is fixed. Therefore, we must compromise and assume that in any cases of interest the energy difference of initial and final states can be considered small compared to the energies of any important intermediate states in the sum in Eq. (15a). If this situation does not exist, the expansion we will derive will be a poor approximation to the complete expression. In what follows, we will ignore all terms of order $(q_1{}^0)^3$, $(q_2{}^0)^3$ or mixed cubic terms involving $q_1{}^0$, $q_2{}^0$, $\mathbf{q}_1$, $\mathbf{q}_2$.

A special problem is posed by those contributions to Eq. (15a) from intermediate states corresponding to the initial or final state. In this instance, the denominators are very small. For the $0^+ \rightarrow 0^+$ case the numerator of these terms can be shown to vanish using Eq. (35) below. For this special case, therefore, no differentiation of the various intermediate state contributions to the amplitude need be made, and no distinction will be made between summing over all states

n and summing over all but the initial or final states. In the elastic scattering case for a nucleus of arbitrary spin (not 0^+), there is a contribution from the $n = i$ term. Such terms can be separated using dimensional arguments. The $n = i$ denominators are proportional to $q_1{}^0$ or $q_2{}^0$ plus recoil terms and these quantities can be made arbitrarily small, since they are photon energies. The currents in the numerators are each of order $(1/m)$, and if $n = i$, the terms independent of $\mathbf{q}_1$ and $\mathbf{q}_2$ vanish. The lowest-order terms in the amplitude, corresponding to a photon frequency ω, are proportional to ω/mass^2. It has been known for several years that the lowest-order relativistic corrections to the elastic amplitude are of this order. Therefore, we must disregard all such contributions. The $n \neq i$ terms have a different dependence on the mass. Neglecting $q_1{}^0$ and $q_2{}^0$, the denominators are equal to energy differences of nuclear states and these are of order (1/mass). Since the currents are also of this order, the amplitudes are of order 1/mass and ω^2/mass, as we shall see later. In one instance, to be discussed later, it will be necessary to consider explicitly the fact that the intermediate state sum is only over the excited states.

Consistency in the expansion scheme demands that we consider $q_1{}^0$ and $q_2{}^0$ in the denominators in Eq. (15) to be the same order in $(1/m)$ as excitation energies. In our final results, we will find a single term that is of the form $q_1{}^0 q_2{}^0 \chi$, where χ is inversely proportional to an excitation energy or, equivalently, proportional to a mass. The product, therefore, is of order (1/mass), and the nuclear physics dependent quantity χ can be quite large compared to nuclear quantities proportional to (1/mass).

Our first task is to expand the seagull using the results of Section 3. We will find it convenient to include certain higher-order (in $\mathbf{q}_1$ or $\mathbf{q}_2$) terms to simplify our expressions, and it should be borne in mind that this inclusion is inconsistent, since other terms of the same order are neglected. We also choose to work in transverse gauge, which simplifies the Lorentz condition for the photons

$$\begin{gathered} \epsilon_0 = \epsilon_0{}' = 0 \\ \boldsymbol{\epsilon}'(\mathbf{q}_1) \cdot \mathbf{q}_1 = \boldsymbol{\epsilon}(\mathbf{q}_2) \cdot \mathbf{q}_2 = 0 \end{gathered} \tag{35}$$

and because of the simplification that results, we will work with the combination $\epsilon_m(\mathbf{q}_2)\, T_{mn}(q_1, q_2)\, \epsilon_n{}'(\mathbf{q}_1)$. Denoting

$$B = \epsilon_m B_{mn} \epsilon_n{}',$$

we find

$$B(\mathbf{q}_1, \mathbf{q}_2) \cong B(0, \mathbf{q}_2) + B(\mathbf{q}_1, 0) - B(0, 0) + \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1}\, \mathbf{q}_2 \cdot \frac{\partial}{\partial \mathbf{q}_2}\, B(0, 0) \tag{36}$$

and the last term can be written

$$\mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} \mathbf{q}_2 \cdot \frac{\partial}{\partial \mathbf{q}_2} B(0, 0) = q_{2a} q_{1b} A_{ab}^{mn} \epsilon_m \epsilon_n{}' \tag{37}$$

and A may be decomposed according to the symmetry of its indices as in Eq. (32).

The current matrix elements also may be expanded simply and we define $J^{(n)}$ as that term in the expansion of $\boldsymbol{\epsilon} \cdot \mathbf{J}(\mathbf{q})$ which contains n powers of the photon momentum variable $\mathbf{q}$. The simplest contributions are the recoil terms, and in the lab frame, the recoil energies in the first and last energy denominators in Eq. (15) are, respectively,

$$\omega_R = \mathbf{q}_1{}^2/2m_t \qquad \omega_R{}' = \mathbf{q}_2{}^2/2m_t\,. \tag{38}$$

In addition, recoil affects the current operators as shown in Eq. (14b). Examining the S-terms in Eq. (15a) we get

$$\begin{aligned} T_s = {}& \sum_n \frac{\langle f \mid \boldsymbol{\epsilon} \cdot \mathbf{q}_1 \rho(\mathbf{q}_2)/m_t \mid n\rangle\langle n \mid \boldsymbol{\epsilon}' \cdot \mathbf{J}(\mathbf{q}_1) \mid i\rangle}{\omega_i - \omega_n + q_1{}^0} \\ & + \sum_n \frac{\langle f \mid \boldsymbol{\epsilon}' \cdot \mathbf{q}_2 \rho(\mathbf{q}_1)/m_t \mid n\rangle\langle n \mid \boldsymbol{\epsilon} \cdot \mathbf{J}(q_2) \mid i\rangle}{\omega_i - \omega_n + q_2{}^0}\,. \end{aligned} \tag{39}$$

The matrix elements proportional to ρ must be expanded further in powers of the photon momentum, since $\rho(0) = Z$ and the constant matrix element implies $n = f$. This is an example of the type of term we previously agreed to ignore. Expanding ρ to first order and $\mathbf{J}$ and the denominator to zeroth order, using Eq. (18) and (19a) yields

$$T_s = \langle f \mid \boldsymbol{\epsilon} \cdot \mathbf{q}_1 \boldsymbol{\epsilon}' \cdot \mathbf{D} \mathbf{q}_2 \cdot \mathbf{D} + \boldsymbol{\epsilon}' \cdot \mathbf{q}_2 \mathbf{q}_1 \cdot \mathbf{D} \boldsymbol{\epsilon} \cdot \mathbf{D} \mid i\rangle/m_t\,, \tag{40}$$

where parity conservation and the fact that our initial and final states have the same parity have been used. As noted by Breitenburger [63], the contributions from the left-most matrix element in T_s have a form reminiscent of the electric dipole interaction of a system moving in an external magnetic field.

The remaining terms in T can be simplified by expanding the $\mathbf{J}$'s to second order and by rearranging and expanding the energy denominators. The first term in Eq. (15) can be expanded and written as

$$T^{(1)} = \sum_n \frac{\langle f \mid J^{(0)}(\mathbf{q}_2) + J^{(1)} + J^{(2)} \mid n\rangle\langle n \mid J^{(0)\prime}(\mathbf{q}_1) + J^{(1)\prime} + J^{(2)\prime} \mid i\rangle}{\omega_i - \omega_n + \omega_R + q_1{}^0} \tag{41}$$

where $J' \equiv \boldsymbol{\epsilon}' \cdot \mathbf{J}(\mathbf{q}_1)$ and $J \equiv \boldsymbol{\epsilon} \cdot \mathbf{J}(\mathbf{q}_2)$. The numerator to second order will involve combinations similar to the expansion in Eq. (36). We find, using $D = \boldsymbol{\epsilon} \cdot \mathbf{D}$, $D' = \boldsymbol{\epsilon}' \cdot \mathbf{D}$, etc., and $\omega_0 \equiv \omega_i - \omega_f$

$$
\begin{aligned}
T^{(1)} = {} & i\langle f \mid DJ'(\mathbf{q}_1) - J(\mathbf{q}_2)\, D' \mid i\rangle \\
& + \langle f \mid D\, \frac{[h_0\,, D']}{2} - \frac{[h_0\,, D]}{2}\, D' - (q_1{}^0 + \omega_0/2 - \omega_R)\, DD' \mid i\rangle \\
& + \sum_n \left(\frac{1}{E_n - E_i - q_1{}^0}\right) (q_1{}^0 q_2{}^0 \langle f \mid D \mid n\rangle\langle n \mid D' \mid i\rangle \\
& - \langle f \mid J^{(1)} \mid n\rangle\langle n \mid J^{(1)\prime} \mid i\rangle - iq_2{}^0\langle f \mid D \mid n\rangle\langle n \mid J^{(1)\prime} \mid i\rangle \\
& - iq_1{}^0\langle f \mid J^{(1)} \mid n\rangle\langle n \mid D' \mid i\rangle)
\end{aligned} \tag{42}
$$

The second, or crossed, term produces considerable simplification when added to $T^{(1)}$. We now discuss the separate contributions to T shown above. The first term contains all the $J^{(2)}$ pieces and only dipole intermediate states contribute to this term for the spin-0^+ case. Thus, this term contains the retarded $E1$ contributions. Adding the corresponding crossed term gives

$$
i\langle f \mid [D, J'(\mathbf{q}_1)] + [D', J(\mathbf{q}_2)] \mid i\rangle \tag{43}
$$

which, when added to the B-terms $B(0, \mathbf{q}_2)$ and $B(\mathbf{q}_1\,, 0)$ from Eq. (36), produces a net result for these $E1$ contributions including retardation

$$
\begin{aligned}
& \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z}{m_t} \langle f \mid \rho(\mathbf{q}_1) + \rho(\mathbf{q}_2) \mid i\rangle \\
& \quad \cong \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z}{m_t} \left(2Z\delta_{fi} - \frac{(q_{2m}q_{2n} + q_{1n}q_{1m})}{2} \langle f \mid Q_{mn} \mid i\rangle\right)
\end{aligned} \tag{44}
$$

where Eqs. (27) and (18) have been used and a term proportional to $\langle f \mid \mathbf{D} \mid i\rangle$ has been dropped. The second term in Eq. (42) can be combined similarly with the crossed term to produce

$$
\left(\frac{\omega_R + \omega_R{}' - \mathbf{q}^2}{2m_t}\right) \langle f \mid \boldsymbol{\epsilon} \cdot \mathbf{D}\boldsymbol{\epsilon}' \cdot \mathbf{D} \mid i\rangle - \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \, \frac{Z^2\delta_{fi}}{m_t} \tag{45a}
$$

using Eq. (28). The recoil term proportional to $\mathbf{q}^2$ in parentheses is the recoil energy of the final nucleus. Using the definitions of ω_R and $\omega_R{}'$ and Eq. (15c), this can be put in the form

$$
(\mathbf{q}_1 \cdot \mathbf{q}_2 \langle f \mid \boldsymbol{\epsilon} \cdot \mathbf{D}\boldsymbol{\epsilon}' \cdot \mathbf{D} \mid i\rangle - \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z^2 \delta_{fi})/m_t\,. \tag{45b}
$$

The last two terms in Eq. (42) proportional to $J^{(1)}$, do not contribute to spinless initial and final states. In fact, they vanish for elastic scattering as well, since the parity of initial and final states must change. We will drop these terms. The first of the two remaining terms is actually the most important of all the terms we will consider. Because the numerator is second order in small quantities, the denominator can be written as $\omega_n - \omega_i - q_1{}^0$ or $E_n - E_i$ or $E_n - E_f$ with equal justification, and the same is true for the crossed term. For elastic photon scattering, consistency demands that we use $\epsilon_n \equiv \omega_n - \omega_i$. For two photon decays, where $q_1{}^0$ ranges from 0 to $-\omega_0$, the denominator ranges from $\omega_n - \omega_i$ to $\omega_n - \omega_f$. Some average value is appropriate and we call this $\bar{\epsilon}_n$. This term added to its crossed equivalent therefore becomes $q_1{}^0 q_2{}^0 \epsilon_m \alpha^E_{mn} \epsilon_m{}'$ where

$$\alpha^E_{mn} = \sum_{n'} \left[\frac{\langle f \mid D_m \mid n \rangle \langle n \mid D_n \mid i \rangle}{\bar{\epsilon}_n} + \frac{\langle f \mid D_n \mid n \rangle \langle n \mid D_m \mid i \rangle}{\bar{\epsilon}_n} \right], \tag{46a}$$

where the prime on n indicates that the intermediate states i and f are not summed over. For spinless initial and final states a simpler result for the complete contribution is obtained

$$T_\alpha = q_1{}^0 q_2{}^0 \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' \alpha_E \tag{46b}$$

$$\alpha_E = 2 \sum_{n'} \frac{\langle f \mid D_z \mid n \rangle \langle n \mid D_z \mid i \rangle}{\bar{\epsilon}_n}, \tag{46c}$$

where α_E is the static electric (dipole) polarizability of the nucleus for $f = i$, and the dynamic electric polarizability for the inelastic case. As in previous cases, only electric dipole intermediate states contribute.

The remaining term in $T^{(1)}$, can be decomposed using Eq. (23), which splits $J^{(1)}$ into magnetic dipole and electric quadrupole terms. For spinless initial and final states only 1^+ and 2^+ intermediate states contribute, respectively, and there is no interference between the two types of terms. For the general case, we get

$$\begin{aligned} T^{(1)} &= \langle f \mid [Q, [h_0, Q']] \mid i \rangle / 4 \\ &+ \sum_{n'} \frac{\langle f \mid \mu' \mid n \rangle \langle n \mid \mu \mid i \rangle}{\bar{\epsilon}_n} + \frac{\langle f \mid \mu \mid n \rangle \langle n \mid \mu' \mid i \rangle}{\bar{\epsilon}_n} \\ &+ \frac{i}{2} \langle i \mid [Q, \mu'] + [Q', \mu] \mid i \rangle + P \\ &\quad Q = Q_{mn} q_{2n} \epsilon_m \qquad Q' = Q_{mn} q_{1n} \epsilon_m{}' \\ &\quad \mu = \boldsymbol{\epsilon} \times \mathbf{q}_2 \cdot \boldsymbol{\mu} \qquad \mu' = \boldsymbol{\epsilon}' \times \mathbf{q}_1 \cdot \boldsymbol{\mu}. \end{aligned} \tag{47}$$

The commutator terms are the same as those considered previously in Eq. (33) and must be combined with appropriate terms from Eq. (37). The contribution labeled P is special in that it vanishes for $0^+ \to 0^+$ transitions and involves only ground-state matrix elements of the quadrupole and magnetic moment operators for the elastic scattering case. As we discussed at the beginning of this section, in virtually all the terms we have considered for elastic scattering it is not necessary to distinguish between sums over all intermediate states and sums over all but the ground state. Indeed, adding in the $n = i$ term was necessary in order to use closure, and this resulted in the commutator terms we derived in Eq. (42). In that case, the additional $n = i$ term vanished because the matrix element $\langle i \mid \mathbf{D} \cdot \boldsymbol{\epsilon} \mid i \rangle$ vanishes due to parity conservation. Only in the magnetic-dipole electric-quadrupole product terms (both positive parity) does the added term fail to vanish. Therefore, it must be subtracted from the complete sum and this is the origin of the term P. We only consider P for the elastic scattering case and we find

$$\begin{aligned} P = &- (i/2) \sum_{i'} [\langle i_f \mid Q \mid i' \rangle \langle i' \mid \mu' \mid i \rangle + \langle i_f \mid Q' \mid i' \rangle \langle i' \mid \mu \mid i \rangle] \\ &+ (i/2) \sum_{i'} [\langle i_f \mid \mu \mid i' \rangle \langle i' \mid Q' \mid i \rangle + \langle i_f \mid \mu' \mid i' \rangle \langle i' \mid Q \mid i \rangle] \end{aligned} \tag{48}$$

where the state labels i, i', i_f refer to the different magnetic substates of the initial, intermediate, and final states, respectively. Simplification of Eq. (48) can be obtained by examining the spin structure of the matrix elements. The matrix element of $\boldsymbol{\mu}$ must be proportional to $\mathbf{S}$, the total spin, a relationship that follows from the Wigner–Eckart theorem. Therefore, we can replace $\boldsymbol{\mu}$ effectively by the total spin operator of the nucleus.

$$\boldsymbol{\mu} \to (\mu/S)\mathbf{S} \equiv \bar{\mu}\mathbf{S} \qquad \mu \equiv \langle SS \mid \mu_z \mid SS \rangle \tag{49a}$$

where the notation $\mid SS \rangle$ indicates the magnetic substate with the largest z-projection for a nucleus with spin S. Similarly,

$$Q_{mn} \to \frac{\bar{Q}}{2} S_{mn} + \frac{\langle \mathbf{r}^2 \rangle}{3} \delta_{mn} \tag{49b}$$

$$S_{mn} = \{S_m , S_n\} - \frac{2S(S+1)}{3} \delta_{mn} \tag{49c}$$

$$\langle \mathbf{r}^2 \rangle = \langle f \mid \sum_i e_i \mathbf{x}_i'^2 \mid i \rangle + (Z\langle \mathbf{r}^2 \rangle_P + N\langle \mathbf{r}^2 \rangle_N)\, \delta_{fi} \tag{49d}$$

$$\bar{Q} = \langle SS \mid \sum_i e_i(3z_i'^2 - \mathbf{x}_i'^2) \mid SS \rangle/(2S^2 - S) \equiv Q/(2S^2 - S), \tag{49e}$$

where S_{mn} is traceless and $\langle \mathbf{r}^2 \rangle_P$, $\langle \mathbf{r}^2 \rangle_N$ refer to the intrinsic radii of the proton and neutron, respectively. Because of Eq. (35), the δ_{mn} parts of Q_{mn} do not contribute to Eq. (48). Using Eq. (49), the matrix elements of Q_{mn} and $\boldsymbol{\mu}$ may be replaced by the equivalent spin operators and closure used to eliminate the sum over intermediate spin states. The resulting combination of terms involves three spin operators. Because these terms may be written as a commutator, the expression may be reduced to a form involving a product of two spin operators. Dropping the initial and final states and writing P as an operator in spin space we have

$$P = (\bar{\mu}\bar{Q}/2)[\{\mathbf{S}\cdot\boldsymbol{\epsilon}, \mathbf{S}\cdot\boldsymbol{\epsilon}'\}\, \mathbf{q}_1\cdot\mathbf{q}_2 - \{\mathbf{S}\cdot\mathbf{q}_1, \mathbf{S}\cdot\mathbf{q}_2\}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}']. \tag{50}$$

Most of the remaining parts of Eq. (47) may be evaluated by combining them with the appropriate parts of the remaining B-terms. Decomposing A in Eqs. (36) and (37) into s–s, s–a, a–s, and a–a combinations, we combine the s–s part with the double commutator term (which we denote T_{QQ}), the s–a part with the $[Q, \mu']$ term and the a–s part with the $[Q', \mu]$ term (denoted $T_{Q\mu}$) and then use Eq. (33). These terms are then given by

$$\begin{aligned} T_{QQ} = -\frac{1}{4m_t}\,[&(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')\,\langle f|\,\mathbf{D}\cdot\mathbf{q}_1\,\mathbf{D}\cdot\mathbf{q}_2\,|\,i\rangle + (\mathbf{q}_1\cdot\mathbf{q}_2)\,\langle f|\,\mathbf{D}\cdot\boldsymbol{\epsilon}\,\mathbf{D}\cdot\boldsymbol{\epsilon}'\,|\,i\rangle \\ &+ (\boldsymbol{\epsilon}\cdot\mathbf{q}_1)\,\langle f|\,\mathbf{D}\cdot\boldsymbol{\epsilon}'\,\mathbf{D}\cdot\mathbf{q}_2\,|\,i\rangle + (\boldsymbol{\epsilon}'\cdot\mathbf{q}_2)\,\langle f|\,\mathbf{D}\cdot\boldsymbol{\epsilon}\,\mathbf{D}\cdot\mathbf{q}_1\,|\,i\rangle] \end{aligned} \tag{51a}$$

$$T_{Q\mu} = -\frac{1}{2m_t}\,[\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'\,\langle f|\,\mathbf{D}\cdot\mathbf{q}_1\,\mathbf{D}\cdot\mathbf{q}_2\,|\,i\rangle - (\mathbf{q}_1\cdot\mathbf{q}_2)\,\langle f|\,\mathbf{D}\cdot\boldsymbol{\epsilon}\,\mathbf{D}\cdot\boldsymbol{\epsilon}'\,|\,i\rangle]. \tag{51b}$$

For spinless initial and final states $T_{Q\mu}$ vanishes and only 2^+ intermediate states contribute to T_{QQ}. The remaining term in $T^{(1)}$ can be written in the form

$$T_{\mu\mu} = (\boldsymbol{\epsilon}\times\mathbf{q}_2)_m\,(\boldsymbol{\epsilon}'\times\mathbf{q}_1)_n\,\chi^P_{mn} \tag{51c}$$

$$\chi^P_{mn} = \sum_{n'} \frac{\langle f\,|\,\mu_m\,|\,n\rangle\langle n\,|\,\mu_n\,|\,i\rangle}{\bar{\epsilon}_n} + \frac{\langle f\,|\,\mu_n\,|\,n\rangle\langle n\,|\,\mu_m\,|\,i\rangle}{\bar{\epsilon}_n}. \tag{51d}$$

Using Eq. (34) the (a–a) part of A, the only remaining B-term, can be cast into the form

$$T^D = (\boldsymbol{\epsilon}\times\mathbf{q}_2)_m\,(\boldsymbol{\epsilon}'\times\mathbf{q}_1)_n\,\chi^D_{mn}\,. \tag{52}$$

The two quantities χ^P and χ^D are the nuclear paramagnetic and diamagnetic susceptibility tensors, respectively, and were discussed in detail by Ericson and Hufner [1, 64].

It is convenient to combine the terms that are proportional to $1/m_t$. We add Eq. (40), the nuclear convection current contribution, to Eq. (51), the $E2$–$E2$

and $E2$–$M1$ contributions, and the first part of Eq. (45b). The resulting expression may be simplified using the identity

$$(\boldsymbol{\epsilon} \times \mathbf{q}_2)_m(\boldsymbol{\epsilon}' \times \mathbf{q}_1)_n = (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}')(\mathbf{q}_1 \cdot \mathbf{q}_2)\,\delta_{mn} - (\boldsymbol{\epsilon}' \cdot \boldsymbol{\epsilon})\,q_{1m}q_{2n} - (\boldsymbol{\epsilon} \cdot \mathbf{q}_1)(\boldsymbol{\epsilon}' \cdot \mathbf{q}_2)\,\delta_{mn} - \epsilon_m{}'\epsilon_n(\mathbf{q}_1 \cdot \mathbf{q}_2) + \epsilon_m{}'q_{2n}(\boldsymbol{\epsilon} \cdot \mathbf{q}_1) + \epsilon_n q_{1m}(\boldsymbol{\epsilon}' \cdot \mathbf{q}_2) \tag{53}$$

and we get

$$(3/(4m_t))[\langle f \mid (\mathbf{D} \cdot \boldsymbol{\epsilon} \times \mathbf{q}_2)(\mathbf{D} \cdot \boldsymbol{\epsilon}' \times \mathbf{q}_1) \mid i\rangle - \langle f \mid \mathbf{D}^2 \mid i\rangle(\boldsymbol{\epsilon} \times \mathbf{q}_2 \cdot \boldsymbol{\epsilon}' \times \mathbf{q}_1)] \tag{54}$$

which has the same form as Eq. (52). The matrix element of D_mD_n can be written as an operator in spin space using a procedure similar to that used in Eq. (48).

$$D_mD_n \rightarrow D'S_{mn} + \langle \mathbf{D}^2\rangle\,\delta_{mn}/3 \tag{55a}$$

$$D' = (1/2)[\langle 3D_zD_z\rangle_{\max} - \langle \mathbf{D}^2\rangle]/(2S^2 - S) \tag{55b}$$

where the subscript max indicates a matrix element using states with the maximum magnetic quantum number. The quadrupole term in Eq. (44) may also be rewritten using Eq. (49), which yields

$$-\frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z}{6m_t}(\mathbf{q}_1{}^2 + \mathbf{q}_2{}^2)\langle \mathbf{r}^2\rangle - \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z}{2m_t}\,\bar{Q}\left((\mathbf{S} \cdot \mathbf{q}_1)^2 + (\mathbf{S} \cdot \mathbf{q}_2)^2 - \frac{S(S+1)(\mathbf{q}_1{}^2 + \mathbf{q}_2{}^2)}{3}\right) \tag{56}$$

where this quantity is an operator in spin-space. Similarly, the polarizability and susceptibility tensors can be written as spin-space operators

$$\begin{aligned} \chi_{mn} &\rightarrow \delta_{mn}\chi + \chi' S_{mn} \\ \chi &= \chi_{mm}/3 \\ \chi' &= (1/2)(3\langle\chi_{33}\rangle_{\max} - \chi)/(2S^2 - S), \end{aligned} \tag{57}$$

where the subscript again indicates that particular matrix element where both initial and final spin states have the maximum projection on the z-axis.

The entire amplitude may now be written as an effective operator in spin-space. Collecting terms, the Compton amplitude becomes

$$\begin{aligned} T_c = {} & \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z^2}{m_t}\left(1 - \left(\frac{\mathbf{q}_1{}^2 + \mathbf{q}_2{}^2}{6}\right)\frac{\langle \mathbf{r}^2\rangle}{Z}\right) - \frac{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}' Z\bar{Q}}{2m_t}S_Q + q_1{}^0q_2{}^0(\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'\alpha_E + \alpha_E{}'S_E) \\ & + (\boldsymbol{\epsilon} \times \mathbf{q}_2) \cdot (\boldsymbol{\epsilon}' \times \mathbf{q}_1)\left(\chi_M - \frac{\langle \mathbf{D}^2\rangle}{2m_t}\right) + S_M\left(\chi_M{}' + \frac{3D'}{4m_t}\right) \\ & + \frac{\bar{\mu}\bar{Q}}{2}[S_E(\mathbf{q}_1 \cdot \mathbf{q}_2) - S_Q\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}'] \end{aligned} \tag{58}$$

where

$$\chi_M = \chi^D + \chi^P \qquad \chi_M' = \chi'^D + \chi'^P \tag{59a}$$

$$\begin{aligned} S_E &= \{\mathbf{S}\cdot\boldsymbol{\epsilon}, \mathbf{S}\cdot\boldsymbol{\epsilon}'\} - \tfrac{2}{3}S(S+1)\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' \\ S_M &= \{\mathbf{S}\cdot\boldsymbol{\epsilon}\times\mathbf{q}_2, \mathbf{S}\cdot\boldsymbol{\epsilon}'\times\mathbf{q}_1\} - \tfrac{2}{3}S(S+1)(\boldsymbol{\epsilon}\times\mathbf{q}_2)\cdot(\boldsymbol{\epsilon}'\times\mathbf{q}_1) \\ S_q &= \{\mathbf{S}\cdot\mathbf{q}_1, \mathbf{S}\cdot\mathbf{q}_2\} - \tfrac{2}{3}S(S+1)(\mathbf{q}_1\cdot\mathbf{q}_2) \\ S_Q &= (\mathbf{S}\cdot\mathbf{q}_1)^2 + (\mathbf{S}\cdot\mathbf{q}_2)^2 - S(S+1)(\mathbf{q}_1^2+\mathbf{q}_2^2)/3. \end{aligned} \tag{59b}$$

The Raman amplitude for spinless states becomes

$$\begin{aligned} T_R = &-\frac{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'Z}{6m_t}(\mathbf{q}_1^2+\mathbf{q}_2^2)\langle f\,|\,\mathbf{r}^2\,|\,i\rangle + q_1^0q_2^0\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'\alpha_E \\ &+ (\boldsymbol{\epsilon}\times\mathbf{q}_2)\cdot(\boldsymbol{\epsilon}'\times\mathbf{q}_1)\left(\chi_M - \frac{\langle\mathbf{D}^2\rangle}{2m_t}\right). \end{aligned} \tag{60}$$

Equations (58) and (60) are our primary results. All the terms except the electric polarizability term and the magnetic moment and susceptibility terms explicitly exhibit the $1/m$ dependence we discussed earlier, and the latter terms can be seen to be of the same type. The Compton and Raman amplitudes may be written more conventionally in terms of the initial and final photon energies, ω and ω', and momenta $\mathbf{k}$ and $\mathbf{k}'$, respectively, by the substitution $q_1^0 \to -\omega'$, $q_2^0 \to +\omega$, $\mathbf{q}_1 \to -\mathbf{k}'$, $\mathbf{q}_2 \to +\mathbf{k}$. To the order we have worked, we may take $\omega = \omega'$. Certain of the terms in Eq. (58) have been exhibited explicitly before, and the other terms are implicit in the results of numerous authors. The first two terms are the Thompson amplitude and the finite size correction to it. The next term is the quadrupole theorem term of Pais [11], Lin [15], and Bardakci and Pagels [12]. The electric-dipole polarizability arises because the nucleus can be deformed by the electric field of the incident photon and the electric dipole moment of the deformed nucleus interacts with the electric field which produced the deformation. Similarly, the paramagnetic susceptibility terms arises from deformations caused by the magnetic field of the incident photon. The last term, which reflects a mixed magnetic dipole–electric quadrupole interaction, was derived by Lin [14, 15]. Lin's results show that the nonrelativistic approximation eliminates a large number of terms of order $1/m^2$ and $1/m^3$ in the amplitude, which we have consistently ignored.

We wish to emphasize that the derivation of Eq. (58) was accomplished by assuming and using three basic ingredients: the nonrelativistic approximation to simplify the results and classify terms, analyticity of matrix elements so that expansions can be made, and gauge invariance. Similar assumptions, with Lorentz invariance replacing our first assumption, are made in more general treatments [2, 3, 15]. We have not assumed a gauge-invariant model and then used the model, but rather we have used the basic principle itself. We did not even need to specify the form of the current, and this quantity may contain meson exchange contributions as well as the more conventional elements. Similarly, the seagull term may have mesonic contributions. Such mesonic effects enter into the calculation of χ_M and μ.

The $\langle \mathbf{D}^2 \rangle$ term in the Rayleigh part of the amplitude in Eq. (60) serves an interesting purpose. A slightly different term of this form was derived by EH. These authors neglected the nuclear convection current contribution to the amplitude and made an algebraic error in combining the various pieces of the amplitude together. Making the required changes in their work [65] produces the terms in Eq. (58). The diamagnetic susceptibility has not been exhibited here and for completeness we calculate this quantity using the usual nonrelativistic model without exchange effects. In this case B_{mn} has the form

$$B_{mn} = \delta_{mn} \langle f | \sum_{i=1}^{A} \frac{\hat{e}_i(\mathbf{q}_1^2)\, \hat{e}_i(\mathbf{q}_2^2)}{m_i} \exp(i(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{x}_i') | i \rangle \tag{61}$$

and from this expression χ_{mn}^D may be derived, using Eq. (34),

$$D_{mn} = \frac{1}{4} (Q'_{mn} - Q'_{rr}\delta_{mn}) \tag{62a}$$

$$Q'_{mn} = \langle f | \sum_{i=1}^{A} \frac{e_i^2}{m_i} x_m' x_n' | i \rangle \tag{62b}$$

$$\chi^D = -\frac{1}{6} \langle f | \sum_i \frac{e_i^2 \mathbf{x}_i'^2}{m_i} | i \rangle \tag{62c}$$

where e_i is the total charge of the ith particle with mass m_i. For a single bound particle of charge e and mass m we have

$$\chi^D \to -\frac{e^2}{6m} \langle \mathbf{r}^2 \rangle. \tag{63}$$

For two particles with masses m_1 and m_2, charges e and $-e$, and a relative coordinate $\mathbf{r}$, the $\langle \mathbf{D}^2 \rangle$ part of Eq. (60) and χ_D in Eq. (62c) may be combined to give

$$\chi_D \to -\frac{e^2}{6\mu} \langle \mathbf{r}^2 \rangle \qquad \mu^{-1} \equiv m_1^{-1} + m_2^{-1}. \tag{64}$$

Comparison with Eq. (63) exhibits the fact that the $\mathbf{D}^2$-term serves as a center-of-mass correction to the diamagnetic susceptibility. A result similar to Eq. (64) was derived by Breitenberger [63], who investigated the reduced mass effect in the quadratic Zeeman effect for two oppositely charged spinless particles. The center-of-mass effect for the paramagnetic susceptibility of two spinless oppositely charged particles is already contained in Eq. (51). In this case, as noted by Breitenberger for the analogous Zeeman effect calculation, the magnetic moment must vanish for equal mass particles and a simple calculation verifies this. Finally, we note that in a nucleus each particle has mass m and $e_i^2 = e_i$, so that $Q'_{mn} = Q_{mn}/m$ and $\chi^D = -\langle \mathbf{r}^2 \rangle / 6m$.

Two-photon decay amplitudes may be obtained from Eq. (60) using the more conventional photon variables by substituting $\mathbf{q}_1 \to -\mathbf{k}'$, $\mathbf{q}_2 \to -\mathbf{k}$, $\mathbf{q}_1{}^0 \to -\omega'$, $q_2{}^0 \to -\omega$. Our approach has emphasized the similarity between the two-photon decay amplitudes and Raman amplitudes and that the form of both is determined by general principles, if one has low-energy photons.

For purposes of comparison, we note that the standard approach [2, 3] to low-energy theorems for Compton scattering is similar to our approach in some respects. If one forms the quantity $q_{2m} T_{mn} q_{1n}$ (summing only over spatial indices) use of Eq. (16b) shows that this quantity equals $q_2{}^0 q_1{}^0 T_{00}$. The amplitude T_{00} can be split into $n = i$ terms (intermediate state the same as initial or final state) and $n \neq i$ terms. The former terms are handled rather easily and separately, while the latter contribution can be easily shown to be quadratic in the photon frequency and obviously does not involve magnetic terms. In this way, the electric polarizability appears, and further arguments based on symmetry are needed to specify the magnetic terms.

5. KINEMATICS

The amplitudes for Compton and Raman scattering may be used to calculate cross sections and two-photon decay rates. We proceed first with Compton scattering and note that the Rayleigh part of the amplitude is quadratic in the photon frequency. We have ignored terms that are quartic in this frequency and would interfere with the Thompson term to produce a quartic term in the cross section. For the sake of consistency, we must neglect the square of the Rayleigh term, and only keep its cross term with the Thompson amplitude. This is a considerable simplification because all the spin dependence is in the former amplitude. In the process of calculating the cross section we must average over the initial spin of the nucleus and sum over the final spin, since we are not interested in the polarization of nuclear spin. This double sum is a trace with respect to the spin operators and the cross term mentioned above involved the trace of the spin operators S_E, S_M, S_q, and S_Q, which are traceless. The spin-dependent parts of the amplitude, therefore, do not contribute to the cross section to this order.

In addition, we need to average over initial photon polarization and sum over final photon polarization. This requires use of the identities

$$\overline{\epsilon_m(\mathbf{k})\, \epsilon_n{}^*(\mathbf{k})} = \delta_{mn} - \hat{k}_m \hat{k}_n \tag{65a}$$

$$\overline{|\, \epsilon(\mathbf{k}) \cdot \epsilon'(\mathbf{k}')|^2} = 1 + \cos^2 \theta \tag{65b}$$

$$\overline{\epsilon^* \cdot \epsilon'^*(\epsilon' \times \hat{\mathbf{k}}') \cdot (\epsilon \times \hat{\mathbf{k}})} = 2 \cos \theta \tag{65c}$$

$$\overline{|(\epsilon \times \hat{\mathbf{k}}) \cdot (\epsilon' \times \hat{\mathbf{k}}')|^2} = 1 + \cos^2 \theta \tag{65d}$$

where $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' = \cos \theta$ and the bar indicates a sum over photon polarizations. Equations (65b)–(65d) follow from Eq. (65a), which follows from Eq. (35). The cross section is formed in the usual way.

$$\frac{d\sigma}{d\Omega} \cong \frac{\alpha^2}{2} \left(1 - \frac{4\omega}{m_t} \sin^2 \frac{\theta}{2}\right) \overline{|\, T_c{}^2 \,|} \tag{66a}$$

$$T_c \cong \alpha' \epsilon \cdot \epsilon' + \beta'(\epsilon' \times \hat{\mathbf{k}}') \cdot (\epsilon \times \hat{\mathbf{k}})\, \omega^2 \tag{66b}$$

where the bar again indicates a sum over photon polarization, α is the fine structure constant, and the factor in parentheses arises from a combination of phase space factors. Performing the average we get

$$\overline{|\, T_c\,|^2} = (1 + \cos^2 \theta)\, \alpha'^2 + 4\omega^2 \alpha' \beta' \cos \theta + \text{Order}(\omega^4) \tag{67}$$

and keeping terms of the appropriate order we get

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{2m_t} \left\{ (1 + \cos^2 \theta) \left(\frac{Z^2}{m_t} - 2\omega^2 \left(\alpha_E + \frac{Z\langle \mathbf{r}^2 \rangle}{3m_t} \right) \right) \right.$$
$$\left. - \frac{4\omega Z^2 \sin^2 (\theta/2)}{m_t{}^2} - 4\omega^2 \cos \theta \left(\chi_M - \frac{\langle \mathbf{D}^2 \rangle}{2m_t} \right) \right\}. \tag{68}$$

The term in braces, proportional to $\omega/m_t{}^2$, is the phase space factor and is of the same order as terms we have neglected. These neglected terms in the amplitude of order (ω/m^2) are structure-independent and proportional to a single spin operator. The cross-term in the cross section between this term and the Thompson amplitude therefore vanishes, because it is traceless, and the only term in the cross section of this order is the kinematical phase space term in Eq. (68).

The transition probability for two-photon decays can be formed in an analogous way. We get

$$d\tau = (4\pi\alpha)^2 \sum_f \frac{d^3\mathbf{P}_f}{(2\pi)^3} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{(2\pi)^4\, \delta^4(P_f - P_i + k + k')}{(4\omega\omega')} \overline{|\, T_R\,|^2}, \tag{69}$$

where the sum is over appropriate final states. We first integrate over $\mathbf{P}_f$ and ω' and sum over photon polarizations

$$\frac{d^3\tau}{d\Omega_{\mathbf{k}}\, d\Omega_{\mathbf{k}'}\, d\omega} \cong \frac{\omega\omega'\alpha^2}{(2\pi)^3}\left(1 - \frac{(\omega' + \omega\cos\theta)}{m_t}\right) \times [(1 + \cos^2\theta)(\alpha'^2 + \chi'^2) + 4\alpha'\chi'\cos\theta] \tag{70a}$$

where

$$\omega' = \omega_0 - (\mathbf{k} + \mathbf{k}')^2/2m_t - \omega \tag{70b}$$

and

$$\alpha' = \omega\omega'\alpha_E - Z\langle\mathbf{r}^2\rangle(\omega^2 + \omega'^2)/6m_t \tag{70c}$$

$$\chi' = (\chi_M - \langle\mathbf{D}^2\rangle/2m_t)\,\omega\omega'. \tag{70d}$$

One of the angles is clearly extraneous and we measure the emission angle θ of the photon with energy ω with respect to the emission direction of the other photon. Defining $x = \cos\theta$, we have

$$\frac{d^2\tau}{dx\, d\omega} = \frac{\alpha^2}{\pi}\,\omega\omega'\left(1 - \frac{(\omega' + \omega x)}{m_t}\right)((1 + x^2)(\alpha'^2 + \chi'^2) + 4x\alpha'\chi'). \tag{71}$$

We see that the electric-magnetic cross term is responsible for front–back asymmetry in the angle between the two photons. This asymmetry is approximately

$$A \equiv \frac{d\tau(0°) - d\tau(180°)}{d\tau(0°) + d\tau(180°)} \cong \frac{2\alpha'\chi'}{\alpha'^2 + \chi'^2}, \tag{72}$$

which can be expected to be small. As noted earlier, the various nuclear quantities in Eqs. (70c) and (70d) can depend on different powers of the nucleon mass. The magnetic terms are explicitly of order $1/m$, while α_E is of order m. For this reason, we surmise α_E to be considerably larger than the magnetic susceptibilities. For $0^+ \to 0^+$ transitions, collective states will play an important role in determining the size of α_E and χ_M and detailed calculations are needed to establish whether the surmise is correct.

Integrating Eq. (71) over the relative angle x and ignoring the recoil term in ω' (Eq. (70b)), we obtain

$$\frac{d\tau}{d\omega} = \frac{8\alpha^2}{3\pi}\,\omega\omega'\left[(\alpha'^2 + \chi'^2)\left(1 - \frac{\omega'}{m_t}\right) - \frac{\omega}{m_t}\,\alpha'\chi'\right]. \tag{73}$$

Inserting the explicit dependence on $\omega\omega'$ contained in α' and χ', the total rate is obtained by integrating Eq. (73) over all photon energies. We must be careful, however, since the photons are indistinguishable and integrating over both photons' phase space would effectively count the decay twice. Having arbitrarily picked one member of the pair to deal with in Eqs. (70–73), we must divide by 2 when summing the transition probability per energy interval over the complete range of energy available. We get

$$\tau = \frac{1}{2} \int_0^{\omega_0} \frac{d\tau}{d\omega}\, d\omega = \frac{\alpha^2 \omega_0^{\,7}}{105\pi} \left[\alpha_E^{\,2} - \frac{8Z\langle \mathbf{r}^2 \rangle\, \alpha_E}{9m_t} + \frac{13Z^2 \langle \mathbf{r}^2 \rangle^2}{54m_t^{\,2}} + \chi'^2 \right], \quad (74)$$

where small recoil terms of relative size ω_0/m_t have been dropped. Eq. (74) is the complete rate to the order we have calculated.

6. Discussion and Summary

The literature on $0^+ \to 0^+$ two-photon decays can be classified into groups of papers depending on the form their equivalents of Eqs. (70) and (74) take. None of these papers contain the small recoil factor in Eq. (70). Of the papers that expand the amplitude for small photon energies, only Refs. [49] and [51] consider the magnetic susceptibility contribution, and these two papers consider only the paramagnetic part. The remaining papers consider only the electric polarizability contribution and none consider the $E1$ retardation contribution proportional to $Z\langle \mathbf{r}^2 \rangle / m_t$. In view of our expectation that $\alpha_E^{\,2}$ is the largest term in the brackets in Eq. (74), none of these omissions can be considered serious. More serious are the different numerical factors that occur in front of these brackets, some papers having an additional factor of $\frac{1}{2}$ [38, 40, 48] and others a factor of 2 [43, 49, 51]. Only [53] agrees with Eq. (74). The difference of a factor of 4 between the results in the references above presumably stems from neglecting the factor of 2, which is contained in α_E in Eq. (46c). The results that are too large by a factor of 2 (and the others as well) appear not to have included the factor of $\frac{1}{2}$ needed to avoid double counting each decay, an omission that has *not* occurred in the atomic two-photon decay literature. Equation (74) is consistent with [25, Eq. 6.2].

Experimentally, the absolute two-photon decay rate usually is not determined, but rather the ratio of this rate to e^+e^- pair decay rate or the ratio to the sum of the pair decay rate plus the e^--internal conversion rate. The signature of the pair decay is the two-photon annihilation of the positron and this signal is recorded at the same time as the signal from genuine two-photon decays. The ratio of rates is therefore a convenient quantity to determine. To compare calculations with experimental results we must also calculate the pair rate. Because of discrepancies in the literature and approximations which have been used, we present a brief calculation of this rate.

The process we wish to calculate is shown in Fig. 2b and is obtained by using the substitution rule [54] on Fig. 2a, which shows electron scattering from a

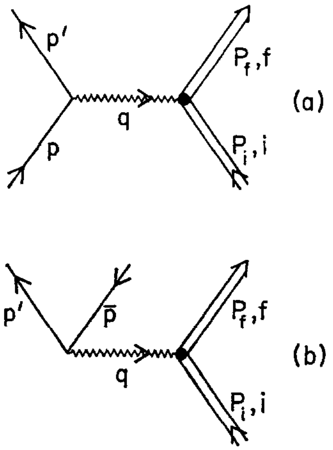

FIG. 2. Electron scattering with nuclear excitation (or deexcitation) is shown in (a), while (b) illustrates the electron–positron pair decay mode of a nuclear state obtained from (a) by means of the substitution rule.

spin-0 object in state i that makes a transition to state f (also spin-0). The amplitude for the latter process is given in [58, Eq. 18], and making the substitution $p \to -\bar{p}$ we get

$$S_{\text{pair}} = i\,\frac{(4\pi\alpha)}{q^2}\left[\frac{m_e^2}{EE'}\right]^{1/2}(2\pi)^4\,\delta^4(q + P_f - P_i)\,\bar{U}(\mathbf{p}')\,\gamma^\mu V(\bar{\mathbf{p}})\,\langle f\mathbf{P}_f \mid \hat{J}_\mu(0) \mid i\mathbf{P}_i\rangle \tag{75a}$$

$$q = -(p' + \bar{p}) \tag{75b}$$

where we identify $\bar{p}_0 = E$, $p_0' = E'$, p' and $\bar{p}$ are the electron and positron four-momenta, m_e is the electron mass, V is the positron spinor, and the nuclear matrix element involved in pair decay is essentially the same one involved in inelastic electron scattering. This matrix element can be expanded in terms of invariants using its Lorentz transformation properties and gauge invariance

$$\langle f\mathbf{P}_f \mid \hat{J}^\mu(0) \mid i\mathbf{P}_i\rangle = \frac{F_0'(q^2)}{m_t}\left(P_i^\mu - q^\mu\,\frac{P_i \cdot q}{q^2}\right). \tag{76}$$

We ignore all corrections of relativistic order, $(v/c)^2$, and identify [66] $F_0'(q^2)$ as the matrix element of $\rho(\mathbf{q})$ in Eq. (14a). Expanded to second order in q, as we have done throughout, we get

$$F_0'(q^2) \cong -(q^2/6)\langle f \mid \mathbf{r}^2 \mid i\rangle. \tag{77}$$

Because of current conservation the term proportional to q^μ does not contribute and P_i^{μ}/m_t has only a nonvanishing time component in the lab frame. Forming a transition rate and summing over the spins of the electron and positron we get

$$d\tau_{\text{pair}} = \frac{(4\pi\alpha)^2}{q^4EE'} \frac{d^3\mathbf{p}'\, d^3\bar{\mathbf{p}}\, d^3\mathbf{P}_f}{(2\pi)^9} (2\pi)^4\, \delta^4(p' + \bar{p} + P_f - P_i) \cdot \left[\frac{q^4}{36} |\langle f | \mathbf{r}^2 | i\rangle|^2\right] (EE' + p'\bar{p}x - m_e^2). \tag{78}$$

The first bracket is the square of the nuclear current matrix element and the second bracket is the spin-summed square of the electron–positron matrix element. We have denoted $\hat{\mathbf{p}}' \cdot \hat{\bar{\mathbf{p}}} = \cos\theta \equiv x$. We neglect all recoil contributions and integrate over the positron and nucleus momenta, agreeing to measure the electron's direction with respect to the positron's direction.

$$\frac{d^2\tau_{\text{pair}}}{dE\, dx} = \frac{\alpha^2}{9\pi} |\langle f | \mathbf{r}^2 | i\rangle|^2\, p'\bar{p}(EE' + p'\bar{p}x - m_e^2). \tag{79}$$

By integrating over the angle θ we get

$$\frac{d\tau_{\text{pair}}}{dE} = \frac{2\alpha^2}{9\pi} |\langle f | \mathbf{r}^2 | i\rangle|^2\, ((EE' - m_e^2)\, p\bar{p} \tag{80}$$

and since the total amount of energy available is ω_0 we obtain

$$\tau_{\text{pair}} = \frac{2\alpha^2}{9\pi} |\langle f | \mathbf{r}^2 | i\rangle|^2 \int_{m_e}^{\omega_0 - m_e} p'\bar{p}(EE' - m_e^2)\, dE. \tag{81}$$

This expression agrees with Dalitz [67] and Akhiezer and Berestetskii [68], but disagrees with Grechukhin [43] in the sign of the m_e^2 term. The integral is not trivial [67] and is only easy to evaluate when $m_e = 0$. In this limit, which should be an excellent approximation if $\omega_0 \gg m_e$, we get $\omega_0^5/30$ for the value of the integral and this gives

$$\tau_{\text{pair}}(m_e = 0) = \frac{\alpha^2}{135\pi}\, \omega_0^5\, |\langle f | \mathbf{r}^2 | i\rangle|^2 \tag{82}$$

which agrees with Oppenheimer and Schwinger [38]. Electrodynamic corrections of order α to τ have been calculated by Dalitz [67].

The two-photon and pair decay rates depend only on five quantities, which are determined by the nuclear wavefunctions in the approximation scheme we have used. To the extent that the complete magnetic susceptibility is unimportant, only the electric polarizability α_E and the mean-square transition radius $\langle f | \mathbf{r}^2 | i\rangle$ are needed. The latter quantity is determined by inelastic electron scattering at low momentum transfer. Both quantities are difficult to calculate [52, 53].

One of the problems associated with inelastic processes is the energy denominator in α_E. We previously argued that if the dipole states that make a substantial contribution to α_E are much more energetic than ω_0, it does not make much difference what we use for $\bar{\epsilon}_n$ in Eq. (47). One test for sensitivity of the approximation would be to calculate α_E using $\epsilon_n = \omega_n - \omega_i$ and $\epsilon_n = \omega_n - \omega_f$ and compare the results. Unfortunately, the product of matrix elements is not positive definite and cancellations can take place. An example of this is given by the energy-weighted sum rule, for the case $f \neq i$, in the absence of exchange potentials.

$$\sum_n (2\omega_n - \omega_i - \omega_f) \langle f \mid D_z \mid n \rangle \langle n \mid D_z \mid i \rangle \equiv 0. \tag{83}$$

We can only hope that severe cancellations of this type do not occur for the inverse-energy-weighted sum rule. The very common approximation of assuming that most of the transition strength occurs in a narrow band of energies for which $\bar{\epsilon}_n \approx \bar{\omega}$ leads to the approximation for α_E

$$\alpha_E \cong \frac{2}{\bar{\omega}} \langle f \mid D_z^2 \mid i \rangle = \frac{2}{3\bar{\omega}} \langle f \mid \mathbf{D}^2 \mid i \rangle \tag{84}$$

which obviously does not work for the energy-weighted sum rule, unless $\langle \mathbf{D}^2 \rangle$ is an anomalously small quantity. Calculation of α_E and $\langle \mathbf{r}^2 \rangle$ remains a serious challenge. Virtually no serious estimates of χ^P exist, and the size of χ_M in relation to α_E is not known.

The Compton cross section in our nonrelativistic approximation depends on the same five quantities, calculated for elastic scattering, of course. Crude estimates of α_E, χ^D, and χ^P were made by EH. These quantities occur in the Rayleigh amplitude, which is quadratic in the photon frequency, and this has been exploited by EH and [64] to relate the sum of χ_M and $\alpha_E + Z\langle \mathbf{r}^2 \rangle / 3m_t$ to an integral over photonuclear cross sections.

$$\chi_m - \langle \mathbf{D}^2 \rangle / 2m_t + \alpha_E + \frac{Z\langle \mathbf{r}^2 \rangle}{3m_t} = \frac{1}{2\pi^2} \int_{\omega_{\text{th}}}^{\infty} \frac{\sigma(\omega)}{\omega^2}\, d\omega. \tag{85}$$

This follows immediately from the spin-averaged forward Compton amplitude, obtained from Eq. (58), and a dispersion relation for the Compton amplitude (e.g., [69, Eq. 1] where the left-hand side should read $\operatorname{Re} f(\omega)$). Since this amplitude usually is written in once-subtracted form, the Thompson amplitude is a constraint. Note, however, that the nonrelativistic Compton amplitude is not sufficiently analytic to allow a dispersion relation to be written [70], although the relativistic form of Eq. (85) still has meaning. In this regard, the fine analysis by Damashek and Gilman [71] of photoproton total cross sections using dispersion relations is recommended reading.

Recently, interest has been directed at the two-photon radiative capture of neutrons on protons [72–76], where it appears that crude estimates of transition rates [72–74] are much smaller than the experimental value [76]. Since this process involves a change of spin of the nuclear states, we have not discussed it, and we await confirmation of the experiment.

In summary, we have analyzed the nonrelativistic nuclear Compton and Raman amplitudes through terms second order in the photon's energy (Rayleigh amplitude) for a nucleus of arbitrary spin. No assumptions about the existence of exchange currents have been made and the derivation has stressed analyticity of the matrix elements and gauge invariance. The proper center-of-mass correction for the diamagnetic susceptibility has been derived.

The two-photon $0^+ \rightarrow 0^+$ decay amplitude has been obtained from the Raman amplitude and partial and total decay rates have been calculated. Numerous discrepancies in the earlier literature have been identified.

Acknowledgments

The author would like to thank Prof. Ben Chertok for introducing him to this topic and to Prof. Stavros Fallieros for numerous discussions of many aspects of this subject.

References

1. T. E. O. Ericson and J. Hüfner, *Nucl. Phys.* **B57** (1973), 604.
2. F. E. Low, *Phys. Rev.* **96** (1954), 1428.
3. M. Gell-Mann and M. L. Goldberger, *Phys. Rev.* **96** (1954), 1433.
4. H. Osborn, *Phys. Rev.* **176** (1968), 1514.
5. F. E. Close and L. A. Copley, *Nucl. Phys.* **B19** (1970), 477.
6. R. A. Krajcik and L. L. Foldy, *Phys. Rev. Letters* **24** (1970), 545.
7. S. J. Brodsky and J. R. Primack, *Ann. Phys. (N.Y.)* **52** (1969), 315.
8. R. A. Krajcik and L. L. Foldy, *Phys. Rev.* **D10** (1974), 1777.
9. A. M. Baldin, *Nucl. Phys.* **18** (1960), 310.
10. V. A. Petrun'kin, *Sov. Phys.-JETP* **13** (1961), 808.
11. A. Pais, *Nuovo Cim.* **53A** (1968), 433.
12. K. Bardakci and H. Pagels, *Phys. Rev.* **166** (1968), 1783.
13. S. Saito, *Phys. Rev.* **184** (1969), 1894.
14. K. Y. Lin, *Nuovo Cim.* **64A** (1970), 188.
15. K. Y. Lin, *Nuovo Cim.* **2A** (1971), 695.
16. M. H. Prior, *Phys. Rev. Letters* **29** (1972), 611.
17. C. A. Kocher, J. E. Clendenin, and R. Novick, *Phys. Rev. Letters* **29** (1972), 615.
18. C. L. Cocke, B. Curnutte, J. R. Macdonald, J. A. Bednar, and R. Marrus, *Phys. Rev.* **A9** (1974), 2242.
19. R. Marrus and R. W. Schmieder, *Phys. Rev.* **A5** (1972), 1160.

20. M. Lipeles, R. Novick, and N. Tolk, *Phys. Rev. Letters* **15** (1965), 690.
21. C. J. Artura, N. Tolk, and R. Novick, *Astrophys. J.* **157** (1969), L181.
22. R. S. Van Dyck, Jr., C. E. Johnson, and H. A. Shugart, *Phys. Rev. Letters* **25** (1970), 1403.
23. M. H. Prior and H. A. Shugart, *Phys. Rev. Letters* **27** (1971), 902.
24. M. Göppert-Mayer, *Ann. Physik (Ser. 5)* **9** (1931), 273.
25. G. Breit and E. Teller, *Astrophys. J.* **91** (1940), 215.
26. J. Shapiro and G. Breit, *Phys. Rev.* **113** (1959), 179.
27. B. A. Zon and L. P. Rapoport, *Sov. Phys.—JETP Letters* **7** (1968), 52
28. L. Spitzer, Jr. and J. L. Greenstein, *Astrophys. J.* **114** (1951), 407.
29. S. Klarsfeld, *Nuovo Cim. Letters* **1** (1969), 682.
30. S. Klarsfeld, *Phys. Letters* **30A** (1969), 382.
31. W. R. Johnson, *Phys. Rev. Letters* **29** (1972), 1123.
32. R. W. Schmieder, *Phys. Rev.* A7 (1973), 1458.
33. H. Arenhövel, *in* "Proceedings of the International Conference on Photonuclear Reactions and Applications," Asilomar, Calif. (B. L. Berman, Ed.), p. 449, USAEC Office of Information Services, 1973.
34. Y. Nakayama, *Phys. Rev.* C7 (1973), 322.
35. Y. Asano and C. S. Wu, *Nucl. Phys.* **A215** (1973), 557.
36. W. J. Braithwaite, A. E. Obst, and C. H. King, *Bull. Amer. Phys. Soc.* **19** (1974), 472.
37. E. Beardsworth, R. Hensler, J. W. Tape, N. Benczer-Koller, W. Darcey, and J. R. MacDonald, *Phys. Rev.* **C8** (1973), 216.
38. J. R. Oppenheimer and J. S. Schwinger, *Phys. Rev.* **56** (1939), 1066.
39. R. G. Sachs, *Phys. Rev.* **57** (1940), 194.
40. D. P. Grechukhin, *Sov. Phys.—JETP* **5** (1957), 846.
41. D. P. Grechukhin, *Nucl. Phys.* **35** (1962), 98.
42. D. P. Grechukhin, *Nucl. Phys.* **47** (1963), 273.
43. D. P. Grechukhin, *Nucl. Phys.* **62** (1965), 273.
44. A. G. W. Cameron, *Bull. Amer. Phys. Soc.* **3** (1958), 269.
45. J. Eichler and G. Jacob, *Z. Phys.* **157** (1959), 286.
46. J. Eichler, *Z. Phys.* **160** (1960), 333.
47. J. Eichler, *Phys. Rev.* **A9** (1974), 1762.
48. B. Margolis, *Nucl. Phys.* **28** (1961), 524.
49. J. Yoccoz, *J. Phys. Radium* **22** (1961), 685.
50. Z. Fried, *Bull. Amer. Phys. Soc.* **17** (1972), 57.
51. G. Sutter, *Ann. Phys. (Paris)* **8** (1963), 323.
52. G. F. Bertsch, *Phys. Letters* **21** (1966), 70.
53. G. F. Bertsch, *Part. Nucl.* **4** (1972), 237.
54. M. L. Goldberger, *Phys. Rev.* **73** (1948), 1119.
55. D. P. Grechukhin, *Sov. Nucl. Phys.* **4** (1967), 30.
56. D. P. Grechukhin, *Sov. J. Nucl. Phys.* **4** (1967), 354.
57. M. Jurčević, K. Ilakovac, and Z. Krečak, *Phys. Rev.* **C9** (1974), 1611.
58. J. L. Friar and M. Rosen, *Ann. Phys. (N.Y.)* **87** (1974), 289.
59. J. D. Bjorken and S. D. Drell, "Relativistic Quantum Mechanics," McGraw–Hill, New York, 1964.
60. M. Chemtob and M. Rho, *Nucl. Phys.* **A163** (1971), 1.
61. R. H. Thompson and L. Heller, *Phys. Rev.* **C7** (1973), 2355.
62. J. L. Friar, *Ann. Phys. (N.Y.)* **81** (1973), 332.
63. E. Breitenberger, *J. Opt. Soc. Amer.* **58** (1968), 1315.
64. T. E. O. Ericson and J. Hüfner, *Nucl. Phys.* **B47** (1972), 205.

65. J. HÜFNER, private communication, confirms these corrections.
66. J. L. FRIAR, to appear.
67. R. H. DALITZ, *Proc. Roy. Soc. (London) Ser. A.* **206** (1951), 521.
68. A. I. AKHIEZER AND V. B. BERESTETSKII, "Quantum Electrodynamics," Interscience, New York, 1965.
69. J. L. FRIAR AND S. FALLIEROS, *Phys. Rev.* **C11** (1975), 274.
70. J. L. FRIAR AND S. FALLIEROS, *Phys. Rev.* **C11** (1975), 277.
71. M. DAMASHEK AND F. J. GILMAN, *Phys. Rev.* **D1** (1970), 1319.
72. R. J. ADLER, *Phys. Rev.* **C6** (1972), 1964.
73. D. P. GRECHUKHIN, *Sov. J. Nucl. Phys.* **14** (1972), 62.
74. J. BLOMQVIST AND T. ERICSON, *Phys. Lett.* **57B** (1975), 117.
75. R. G. ARNOLD, B. T. CHERTOK, I. G. SCHRÖDER, AND J. L. ALBERI, *Phys. Rev.* **C8** (1973), 1179.
76. W. B. DRESS, C. GUET, P. PERRIN, AND P. D. MILLER, *Phys. Rev. Letters* **34** (1975), 752.
77. R. G. SACHS AND N. AUSTERN, *Phys. Rev.* **81** (1951), 705.