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Electromagnetic Energies of Nuclei and the Nuclear Compton Amplitude*

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The electromagnetic energy of a nucleus is derived in perturbation theory, which relates this quantity to the amplitude for the forward scattering of virtual photons on a nucleus (nuclear Compton amplitude). Using the gauge invariance of this amplitude, the energy is separated into Coulomb and transverse components. Our formalism, although basically nonrelativistic, admits corrections of order $(v/c)^2$ to the nuclear charge operator. The energy is further separated into one-body terms, related to the n–p mass difference, and two-body terms which lead to the Breit interaction and the nuclear Lamb shift. These results are then related to electron scattering sum rules in the manner of Cottingham. Mesonic contributions to the electromagnetic energy are also discussed.

1. INTRODUCTION

The theory of electromagnetic contributions to the masses of composite systems constitutes an important area of study in physics. In atomic physics all the binding is electromagnetic in origin and this is the discipline where the most accurate calculations of masses and mass differences is possible [1–3]. The underlying reason for this precision is the relative simplicity of the electromagnetic interaction, which has been studied heavily in a variety of contexts. Regrettably the strong interaction problem is not well understood and the electromagnetic corrections to masses of hadronic systems are more difficult to calculate, although not quite as difficult as the hadronic problem itself. The electromagnetic contributions to nuclear masses were studied at an early stage in nuclear physics and still is an important field. Recently, strong interest has been shown in the electromagnetic mass shifts of elementary particles [4], such as the n–p system and the π^+–π^0–π^- system.

Many different approaches to essentially the same problem have been tried, and the purpose of the present work is to cast the nuclear physics electromagnetic

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problem into the form in which the problem has evolved in atomic physics and elementary particle physics. For a variety of reasons, the other approaches are different and not completely applicable to the nuclear problem. Nevertheless, we feel that a search for common ground sheds light on the basic physics and some of the more obscure aspects of the problem will be clarified. In addition, because electromagnetic physics is a unified subject, the subject of electromagnetic mass shifts is intimately related to other electromagnetic topics.

In Section 2 we develop, using time-dependent perturbation theory the change in energy of a system to first order in α, the fine structure constant. The hadronic part of the expression for the electromagnetic energy is the forward Compton amplitude for virtual photons of mass $t^{1/2}$. This expression, due to Feynman and Speisman [5], Cini, Ferrari and Gatto [6], Cottingham [7], and others, has proven useful in particle physics. In keeping with common usage in nuclear physics [8], terms are kept which are of order $(v/c)^2$, where v is some nuclear velocity. In the charge operator such terms are relativistic corrections and these terms play an important role in the result.

In Section 3 the results are simplified by using Coulomb gauge for the photon propagator, the Compton amplitude having been shown to be gauge invariant [9]. The resulting expression is split into three separate terms, the electromagnetic mass shift for the individual nucleons, the matrix element of the Barker–Glover [10] form of the Breit interaction [11], and the nuclear Lamb shift in the form originally derived by Bethe [12]. Following Feynman [13], we show that the electromagnetic energy shift for excited states has an imaginary part which is proportional to the electromagnetic decay rate of these states.

In Section 4 various approximate treatments of the Breit interaction in nuclear physics are discussed and inconsistencies are pointed out. The three-body system is discussed in some detail.

In Section 5 the electromagnetic energy shift is cast into a form similar to that developed by Cottingham [7], which involves sum rules obtained from elastic and inelastic electron scattering. Various technical problems associated with a practical implementation of this result are discussed.

Finally, in Section 6 various contributions to the electromagnetic energy from mesonic processes are discussed in a qualitative fashion.

2. Time-Dependent Perturbation Theory

The method we utilize for calculating the energy shift in nuclei due to the electromagnetic interactions between the nucleons is time-dependent perturbation theory. Although this technique is quite common in nuclear physics in bound state calculations, our basic approach is sufficiently different from these calculations to make a sketch of the central ideas useful. Stationary states of a system have the property that their time dependence is given by

$$\psi_N{}^\circ(t) = \psi_N{}^\circ(0) \exp(-iE_N{}^\circ t) \tag{1}$$

where N labels the state with an energy $E_N{}^\circ$. If we turn on (at $t = -\infty$) a small time-independent perturbation H' in the total Hamiltonian, the wavefunction becomes

$$\psi_N(t) = \psi_N(0) \exp(-i(E_N{}^\circ + \delta E_N)\, t) \cong \psi_N(0) \exp(-iE_N{}^\circ t)(1 - i\delta E_N t) \tag{2}$$

where δE_N is the shift in the energy eigenvalue $E_N{}^\circ$. If we develop a perturbation expansion of $\psi_N(t)$, using time-dependent perturbation theory in the usual fashion, by writing

$$\psi_N(t) = \sum_M a_{NM}(t)\, \psi_M{}^\circ(0) \exp(-iE_M{}^\circ t) \tag{3}$$

we find to lowest order in H', the perturbing Hamiltonian,

$$\begin{aligned} a_{NM}(t) &\equiv a_N{}^\circ \delta_{MN} + \delta a_{NM}(t) \\ \delta a_{NM}(t) &= -ia_N{}^\circ \int_{-T_i}^{t} \exp(i(E_M{}^\circ - E_N{}^\circ)\, t')\, H'_{MN}\, dt' \end{aligned} \tag{4}$$

where $-T_i$ is the time when the perturbation was turned on. If we now take t to be very large and positive, while making $-T_i$ large and negative, we find, assuming the state N is nondegenerate,

$$\begin{aligned} \delta a_{NM} &\to -ia_N{}^\circ H'_{NN} T \delta_{MN} \\ T &\equiv t + T_i \to \infty \end{aligned} \tag{5}$$

or equivalently that the fractional change in the wavefunction of the Nth state is

$$\delta a_{NN}/a_N{}^\circ = -iH'_{NN}T, \tag{6}$$

which is essentially Eq. (2). One finds the obvious result that $\delta E_N = H'_{NN}$, obtainable from time-independent perturbation theory.

Our approach to calculating electromagnetic energy shifts will emphasize the time-dependent approach, because retardation of the electromagnetic interaction is less easy to set up in other approaches [13]. Retardation is essentially time-dependence, but because these forces are *internal* to our nucleus, only *diagonal* matrix elements are nonzero, as in Eq. (6). Thus, we can still use Eq. (6) to identify δE_N in a time-dependent treatment. The reader is referred to Feynman's original work for additional details [13]; the beauty and clarity of these papers would be very difficult to improve upon. We adopt the notation of Bjorken and Drell [14] and develop, using Feynman's techniques, a perturbation expansion of the wave-function of our nucleus. Many details are given in [8, 9] and will be omitted here. We define our vertices (charges and currents) by means of a Foldy–Wouthuysen [15] reduction of the equations of motion of a single nucleon interacting with external electromagnetic potentials ϕ (scalar) and $\mathbf{A}$ (vector) which form a four-vector $A^\mu = (\phi, \mathbf{A})$. We limit ourselves to terms in the charge operator of order (v^2/c^2), where v is the nucleon velocity, or equivalently $1/m^2$, where m is the nucleon mass. We also limit the current operator to terms of order v/c and ignore until later any potential dependent terms in the charge operator $\hat{\rho}(\mathbf{x})$ and the current operator $\hat{\mathbf{J}}(\mathbf{x})$ (i.e., exchange currents and charges). The resulting reduction is performed in many places, and we merely quote the results. We write

$$H = \hat{H}_o + e \int \hat{J}_\mu(\mathbf{x}) A^\mu(\mathbf{x})\, d^3\mathbf{x} + e^2 \int A_\mu(\mathbf{x}) A_\nu(\mathbf{y}) \hat{f}^{\mu\nu}_{\mathrm{SG}}(\mathbf{x}, \mathbf{y})\, d^3\mathbf{x}\, d^3\mathbf{y} \tag{7}$$

where e is the proton charge and $\hat{J}^\mu(\mathbf{x}) \equiv (\hat{\rho}(\mathbf{x}), \hat{\mathbf{J}}(\mathbf{x}))$ form an approximate four-vector

$$\hat{\mathbf{J}}(\mathbf{x}) = \sum_{i=1}^{A} \int \frac{\hat{e}_i(\mathbf{x}-\mathbf{y})}{2m} \{\mathbf{p}_i, \delta^3(\mathbf{y}-\mathbf{x}_i)\}\, d^3\mathbf{y}$$

$$+ \sum_{i=1}^{A} \int \frac{\hat{\mu}_i(\mathbf{x}-\mathbf{y})}{2m} \boldsymbol{\sigma}(i) \times \boldsymbol{\nabla}_i \delta^3(\mathbf{y}-\mathbf{x}_i)\, d^3\mathbf{y} \tag{8a}$$

$$\hat{f}^{\mu\nu}_{\mathrm{SG}}(\mathbf{x}, \mathbf{y}) = \delta^{\mu\nu} \hat{f}_{\mathrm{SG}}(\mathbf{x}, \mathbf{y}) = \delta^{\mu\nu} \sum_{i=1}^{A} \int d^3\mathbf{z}\, \hat{e}_i(\mathbf{x}-\mathbf{z})\, \hat{e}_i(\mathbf{y}-\mathbf{z}) \frac{\delta^3(\mathbf{z}-\mathbf{x}_i)}{2m} \tag{8b}$$

$$\hat{\rho}(\mathbf{x}) = \left(1 + \frac{\boldsymbol{\nabla}_x^2}{8m^2}\right) \sum_{i=1}^{A} \int \hat{e}_i(\mathbf{x}-\mathbf{y})\, \delta^3(\mathbf{y}-\mathbf{x}_i)\, d^3\mathbf{y}$$

$$+ \sum_{i=1}^{A} \frac{(2\hat{\mu}_i(\mathbf{x}-\mathbf{y}) - \hat{e}_i(\mathbf{x}-\mathbf{y}))}{8m^2} \{\boldsymbol{\sigma}(i) \times \mathbf{p}_i\, ;\, \boldsymbol{\nabla}_y \delta^3(\mathbf{y}-\mathbf{x}_i)\} \tag{8c}$$

$$\hat{H}_o = \sum_{i=1}^{A} \left(\frac{\mathbf{p}_i^2}{2m} - \frac{\mathbf{p}_i^4}{8m^3}\right) + \sum_{i<j} V_{ij}\,. \tag{8d}$$

The purely nuclear (i.e., strong) Hamiltonian $\hat{H}_o$ is assumed to consist of a kinetic energy term and a sum of two-body potentials V_{ij}. For most of our present purposes we may ignore the relativistic correction terms in $\hat{H}_o$.

In addition, $\boldsymbol{\sigma}(i)$ is the usual Pauli spin operator, $\mathbf{p}_i$ is the momentum operator, and $\mathbf{x}_i$ is the coordinate of the ith particle. The sums extend over all A particles. We have introduced the nucleon charge and magnetic moment form factors, $\hat{e}_i$ and $\hat{\mu}_i$, which are defined by [8]

$$\begin{aligned} \hat{e}_i(\mathbf{x}-\mathbf{y}) &= G_E{}^p(\mathbf{x}-\mathbf{y})\left[\frac{1+\tau_3(i)}{2}\right] + G_E{}^n(\mathbf{x}-\mathbf{y})\left[\frac{1-\tau_3(i)}{2}\right] \\ \hat{\mu}_i(\mathbf{x}-\mathbf{y}) &= G_M{}^p(\mathbf{x}-\mathbf{y})\left[\frac{1+\tau_3(i)}{2}\right] + G_M{}^n(\mathbf{x}-\mathbf{y})\left[\frac{1-\tau_3(i)}{2}\right] \end{aligned} \tag{9}$$

where $\tau_3(i)$ is the third component of the ith particle's isospin operator. Equation (9) actually defines the particular combination of the Dirac and Pauli form factors F_1 and F_2 which we wish to call the charge and magnetic moment form factors. Our definition is the one introducted by Sachs [16]. We have defined, for example

$$\begin{aligned} G_E{}^p(\mathbf{q}^2) &= \int e^{i\mathbf{q}\cdot\mathbf{x}} G_E{}^p(\mathbf{x})\, d^3\mathbf{x} = F_1{}^p(\mathbf{q}^2) - \frac{\kappa_p \mathbf{q}^2 F_2{}^p(\mathbf{q}^2)}{4m^2} \\ G_M{}^p(\mathbf{q}^2) &= \int e^{i\mathbf{q}\cdot\mathbf{x}} G_M{}^p(\mathbf{x})\, d^3\mathbf{x} = F_1{}^p + \kappa_p F_2{}^p \\ G_E{}^p(0) &= 1 \qquad G_E{}^n(0) = 0 \\ G_M{}^p(0) &= \mu_p = 1 + \kappa_p \qquad G_M{}^n(0) = \mu_n = \kappa_n \end{aligned} \tag{10}$$

where μ_p and μ_n are the (static) nucleon magnetic moments in units of nuclear magnetons, and κ is the anomalous part of that moment. The current is composed of the usual convection and magnetization pieces. The seagull term (8c) has been discussed in detail in [9, 17]. The symbol $\delta_{\mu\nu}$ is an extended Kroneker delta δ_{mn} with $\delta_{oo} = \delta_{o\mu} = \delta_{\nu o} = 0$, where Greek indices indicate a four-vector and Latin indices denote a three-vector.

The terms of order $(1/m)^{\circ}$ in $\hat{\rho}$ and $(1/m)$ in $\hat{\mathbf{J}}$ satisfy the continuity equation

$$\nabla \cdot \hat{\mathbf{J}}(\mathbf{x}) = -i[\hat{H}_o, \hat{\rho}(\mathbf{x})] \tag{11}$$

and the Compton amplitude formed from $\hat{\mathbf{J}}$, $\hat{\rho}$, and $\hat{f}_{SG}$ is gauge-invariant, as was shown in [9]. The first term of order $(1/m^2)$ in Eq. (8a) is the Darwin–Foldy contribution to the effective charge density of the nucleon, while the second term is the spin–orbit contribution. The former is due to Zitterbewegung, while the latter is due to the Thomas precession and the electric dipole interaction of a moving magnetic dipole. The effect of these terms on electron scattering is thoroughly discussed in [8]. There is one additional term of order $1/m^2$ which could be included in

$\hat{f}_{SG}$. It arises from the gauge-invariant minimal substitution in the spin–orbit part of Eq. (8a). This makes a seagull of the form f_{om} or f_{no} (i.e., a term in the Hamiltonian of the form $\mathbf{A}\phi$). We will see that this term does not contribute to the processes we are interested in and we ignore it. All other terms of order $1/m^2$ in our model Hamiltonian are included. The Hamiltonian $\hat{H}_o$ has eigenstates $|n\mathbf{P}_n\rangle$ with energies E_n [52] (including relativistic corrections of order $(v/c)^2$)

$$\begin{aligned} \hat{H}_o\,|n\mathbf{P}_n\rangle &= E_n\,|n\mathbf{P}_n\rangle \\ E_n &= m_t + \omega_n + \frac{\mathbf{P}_n{}^2}{2m_t} - \frac{\mathbf{P}_n{}^4}{8m_t{}^3} - \frac{\mathbf{P}_n{}^2\omega_n}{2m_t{}^2} \end{aligned} \tag{12}$$

where $m_t = A \cdot m$ and ω_n is the intrinsic energy of the state $|n\mathbf{P}_n\rangle$ which moves with momentum $\mathbf{P}_n$. The center-of-mass (plane wave) part of the state is normalized to 1 in a volume V. In the absence of an external interaction, the nucleus propagates *forward* in time by means of a Green's function

$$\hat{G} = V \sum_n \int \frac{d^4P_n}{(2\pi)^4} \frac{|n\mathbf{P}_n\rangle\langle n\mathbf{P}_n|}{P_n{}^\circ - E_n + i\epsilon}\,. \tag{13}$$

All the time dependence of our system is carried by the nuclear operators. The photon Green's function, furthermore, is given by [14]

$$D_F(x - y) = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq\cdot(x-y)}}{q^2 + i\epsilon}\,. \tag{14}$$

We wish to derive Feynman rules for the electromagnetic interactions of our system and follow the treatment of [9]. The essential ingredients are the Hermiticity of transition currents and the fact that a photon exchanged between two nucleons can be thought to have originated on either nucleon. With this symmetry we find the following rules: for the S-matrix, associate a factor i with each nucleus or photon Green's function, and a factor $-i$ with each vertex (current). We associate a factor $-i$ with the seagull term $\hat{f}_{SG}$ and a factor of 2 as well, since the two photons at that vertex can be connected to other vertices in either order. These rules allow us to recover the results of [9] for electron–nucleus scattering to second order in α.

We wish to calculate the effect of internal photon exchange on the amplitude of a state i (Fig. 1a). Using the rules developed above we obtain

$$\delta a_{ii} = e^2 \int d^4x\, d^4y\, \langle i\mathbf{P}_i\,|\,\hat{J}^\mu(y)\,\hat{G}\hat{J}^\nu(x) + 2\hat{f}^{\mu\nu}_{SG}(y, x)\,|\,i\mathbf{P}_i\rangle\, g_{\mu\nu} D_F(x - y) \tag{15}$$

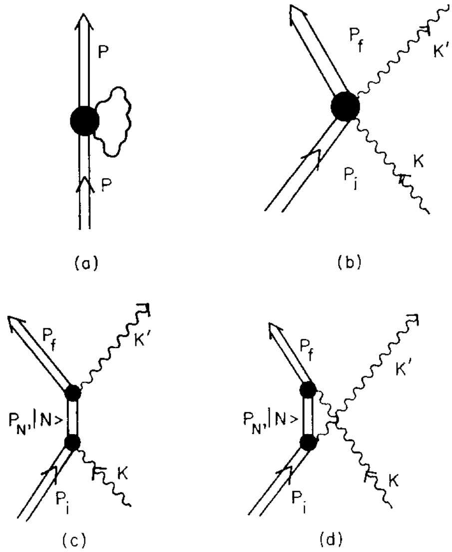

FIG. 1. Feynman diagrams representing (a) electromagnetic energy of a nucleus in lowest order; (b) Compton scattering from a nucleus; (c) direct part of the Compton amplitude; and (d) crossed-photon part of the Compton amplitude.

where $g^{\mu\nu}$ is the usual metric tensor and

$$\hat{J}_\mu(y) = e^{i\hat{H}_o y_o} J_\mu(\mathbf{y})\, e^{-i\hat{H}_o y_o}$$

$$\hat{f}^{\mu\nu}_{\mathrm{SG}}(x, y) = e^{i\hat{H}_o x_o} f^{\mu\nu}_{\mathrm{SG}}(\mathbf{x}, \mathbf{y})\, e^{-i\hat{H}_o x_o} \delta(x_o - y_o). \tag{16}$$

Using Eq. (6) we find (using $a_i^\circ = 1$) for the Coulomb energy shift ΔE_c,

$$\Delta E_c = i\delta a_{ii}/T \tag{17}$$

where δa_{ii} is actually infinite, being proportional to T, the length of time the interaction is allowed to occur, and thus ΔE_c is finite. The diagonal nature of the interaction term in Eq. (15) shows that the $\mu = o$, $\nu = n$ components of the seagull discussed earlier do not contribute and that if we wish the answer to be correct to order $(1/m^2)$ we must include contributions to $\hat{\rho}$ (i.e., $\mu = o$) of order $(1/m^2)$ while $\hat{\mathbf{J}}(\mu = n)$ can be approximated by terms of order $(1/m)$.

Using translation-invariance we can simplify ΔE_c. We find that an overall energy-momentum δ-function factors out, telling us that the energy and momentum of the final state in Eq. (15) must be identical to that of the initial state (we have already assumed this result in writing Eq. (15)). Taking the diagonal matrix element and choosing the lab frame ($\mathbf{P}_i = 0$) gives us a factor VT from the δ-function. In addition, in the first part of Eq. (15) there is a factor of V from the Green's function and a factor of $1/V^{1/2}$ from each of the two intermediate states and the initial and final states. The factors of V exactly cancel in this term as they do in the seagull term. Putting everything together we find

$$\Delta E_c = \frac{-i\alpha}{(2\pi)^3} \int d^4q \, \frac{M_{\mu\nu}(\mathbf{q}, q_o)}{q^2 + i\epsilon} \, g^{\mu\nu} \tag{18}$$

$$\begin{aligned} M^{\mu\nu}(\mathbf{q}, q_o) = {} & \sum_n \frac{\langle i \mid \hat{J}^\mu(0) \mid n\mathbf{q}\rangle\langle n\mathbf{q} \mid \hat{J}^\nu(0) \mid i\rangle}{E_i - E_n + q_o + i\epsilon} \\ & + \sum_n \frac{\langle i \mid \hat{J}^\nu(0) \mid n\mathbf{q}\rangle\langle n\mathbf{q} \mid \hat{J}^\mu(0) \mid i\rangle}{E_i - E_n - q_o + i\epsilon} + \delta^{\mu\nu} \frac{\langle i \mid \sum_i \hat{e}_i^{\,2}(\mathbf{q}^2) \mid i\rangle}{m} \end{aligned} \tag{19}$$

where Eq. (8c) has been used for the seagull term. Equation (18) is the Feynman–Cottingham formula [5–7] for the lowest-order (in α) electromagnetic energy shift. Equation (19) defines the forward (virtual) Compton amplitude, the amplitude for the scattering of a photon of energy q_o and momentum $\mathbf{q}$ in the forward direction from our semirelativistic system.

The S-matrix for Compton scattering of real photons is given by

$$S_{\text{comp}} = -i(2\pi)^4 \, \delta^4(k' + P_f - P_i - k) \, e^2 \epsilon^\mu(\mathbf{k}) \, \epsilon^\nu(\mathbf{k}') \, T_{\mu\nu}(k, k'; \mathbf{P}_i) \tag{20}$$

where a photon of momentum $\mathbf{k}$ and polarization $\epsilon^\mu(\mathbf{k})$ scatters from a nucleus moving with momentum $\mathbf{P}_i$ and becomes a photon of momentum $\mathbf{k}'$ and polarization $\epsilon^\nu(\mathbf{k}')$. The amplitude $T_{\mu\nu}$ has been demonstrated to be gauge invariant and crossing symmetric [9]. Denoting $k_0 = |\,\mathbf{k}\,|$ by k, and $k_o' = |\,\mathbf{k}'\,|$ by k' we have

$$\begin{aligned} T_{\mu\nu}(k, k'; \mathbf{P}_i) = {} & \sum_n \frac{\langle i\mathbf{P}_f \mid \hat{J}_\mu(0) \mid n\mathbf{k} + \mathbf{P}_i\rangle\langle n\mathbf{k} + \mathbf{P}_i \mid \hat{J}_\nu(0) \mid i\mathbf{P}_i\rangle}{E_1 - E_n + k + i\epsilon} \\ & + \sum_n \frac{\langle i\mathbf{P}_f \mid \hat{J}_\nu(0) \mid n - \mathbf{k}' + \mathbf{P}_i\rangle\langle n - \mathbf{k}' + \mathbf{P}_i \mid \hat{J}_\mu(0) \mid i\mathbf{P}_i\rangle}{E_i - E_n - k' + i\epsilon} \\ & + \delta_{\mu\nu}\langle i\mathbf{P}_f \mid \int d^3\mathbf{x} \, e^{i\mathbf{k}\cdot\mathbf{x}} \hat{f}_{\mathrm{SG}}(\mathbf{x}, 0) \mid i\mathbf{P}_i\rangle \end{aligned} \tag{21a}$$

where the energy of the intermediate states E_n depends on the momentum of those states because of Eq. (12). Alternatively, we can write $T_{\mu\nu}$ as a time-ordered product in the usual fashion [17]

$$\begin{aligned} T_{\mu\nu}(k, k'; \mathbf{P}_i) = {} & i \int d^4x \, e^{-ik\cdot x} \langle i\mathbf{P}_f \mid T(\hat{J}_\mu(x), \hat{J}_\nu(0)) \mid i\mathbf{P}_i\rangle \\ & + \text{equal time terms.} \end{aligned} \tag{21b}$$

The entire amplitude is represented graphically in Fig. 1b. This amplitude may be broken down into three separate contributions: the direct term shown in Fig. 1c and corresponding to the first term in Eq. 21a, the crossed-photon term in Fig. 1d which corresponds to the second term in Eq. (21a), and the seagull term which is graphically identical in structure to Fig. 1b and corresponds to the last term in Eq. (21a).

Expression (19) has a very simple structure in terms of the variable q_0, since q_0 and $\mathbf{q}$ are unrelated. The seagull is independent of q_0, while all the q_0-dependence of the other terms (which we will call dispersive terms) is located in the denominator. Because the remainder of the integrand of Eq. (18) is symmetric in q_0, a simple change of variables shows that the two dispersive terms are in fact equal. With a suitable redefinition of the various quantities, Eq. (19) (possibly without a seagull term) is a suitable expression for the forward Compton amplitude for the corresponding relativistic problem [4]. One merely uses the relativistic forms of the charge and current operators and the relativistic relationship between energy and momentum in the denominators, while summing over positive *and* negative energy intermediate states. The negative energy contributions reflect the possibility of nucleon pair production. Our seagull term, in fact, comes from approximating these pair contributions.

There are two different approaches we may use to evaluate the Coulomb energy shift ΔE_c using Eq. (18). The approach of Cottingham is to shift the q_0-integral into an integral from $-i\infty$ to $i\infty$ by using a contour rotation. Using a somewhat unconventional notation, we define $t \equiv q_0^2 - \mathbf{q}^2$, which becomes $-(q_0^2 + \mathbf{q}^2)$, a negative semidefinite quantity after the contour rotation. One then requires $M^{\mu\nu}(\mathbf{q}; iq_0)$, which can be decomposed into components using invariance arguments and these components can be obtained in terms of physically interesting quantities (electron-scattering form factors, which will be discussed later) by means of dispersion relations in the variable q_0 for fixed negative t, which is the physical region for electron scattering. The expression (18) can then be manipulated into an integral over the above-mentioned form factors. We have performed our calculation using a nonrelativistic reduction of the extremely complicated relativistic many-body problem. It has been shown recently [18] that the *nonrelativistic* or semirelativistic forward Compton amplitude (for real photons) does not have sufficiently good analytic properties in the upper half-plane of the (complex) photon energy variable to allow a dispersion relation to be written. The same arguments apply to the virtual Compton amplitude, and Cottingham's approach cannot be used directly on the *semirelativistic* expressions given by Eqs. (18) and (19). We refer to Cottingham's original paper for the treatment of the relativistic problem as it concerns the proton–neutron mass difference and to the review by Zee [4] for possible problems with the method. Although the nuclear problem presumably can be handled in this way, there seems to be no clear reason for attempting to do so. We comment further on this in Section 5.

Although we cannot write a fixed-t dispersion relation for Eq. (19), we note that it is very easy to write a fixed-$\mathbf{q}^2$ dispersion relation in the variable q_o. In effect we will exploit this property of Eq. (19) in what follows.

3. The Breit Interaction

It is convenient to exploit the gauge invariance of the Compton amplitude and adopt Coulomb gauge. This amounts to the replacement [9]

$$\frac{g^{\mu\nu}}{q^2 + i\epsilon} \to -\frac{g^{oo}}{\mathbf{q}^2} - \frac{(\delta_{mn} - \hat{q}_m \hat{q}_n)}{q^2 - i\epsilon} \tag{22}$$

where m and n are the space components of μ and ν and the term in parentheses is orthogonal to $(\mathbf{q})_m$ or $(\mathbf{q})_n$. This gauge, in other words, separates the energy into transverse and Coulomb pieces. Using Eq. (19), the q_0-integral in Eq. (18) may be performed. In the Coulomb term the only q_o-dependence lies in the denominator of $M_{\mu\nu}$; this integral is nominally logarithmically divergent. However, splitting the denominator into a principal value term and a δ-function term in the usual way, it is easily shown that the principal value integral vanishes if the infinite limits of the integral are handled symmetrically. Therefore only the δ-function pole term survives. We find that

$$\Delta E_c = E_c{}^{\circ} + \Delta E_T + \Delta E_{SG} \tag{23a}$$

$$\Delta E_c{}^{\circ} = \frac{\alpha}{4\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2} \sum_n |\langle n\mathbf{q} \mid \hat{\rho}(0) \mid i\rangle|^2 \tag{23b}$$

$$= \frac{\alpha}{4\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2} \langle i \mid \rho(-\mathbf{q})\, \rho(\mathbf{q}) \mid i\rangle \tag{23c}$$

$$\Delta E_T = \frac{-\alpha}{4\pi^2} \sum_n \int \frac{d^3\mathbf{q}\, |\langle n\mathbf{q} \mid \hat{\mathbf{J}}_\perp(0) \mid i\rangle|^2}{|\mathbf{q}|\,(\mathbf{q} + E_n - E_i - i\epsilon)} \tag{23d}$$

$$\Delta E_{SG} = \frac{\alpha}{4\pi^2 m} \int \frac{d^3\mathbf{q}}{|\mathbf{q}|} \langle i \mid \sum_i \hat{e}_i{}^2(\mathbf{q}) \mid i\rangle \tag{23e}$$

where the results of [9, Appendix A] have been used to relate $\hat{\rho}(0)$ to $\rho(\mathbf{q})$. Both $\rho(\mathbf{q})$ and $\hat{\mathbf{J}}_\perp$, the component of $\hat{\mathbf{J}}$ perpendicular to $\hat{\mathbf{q}}$, will be defined later. We may also use the identity

$$\frac{1}{|\mathbf{q}| + \omega_n - i\epsilon} = \frac{1}{|\mathbf{q}|} - \frac{\omega_n}{(|\mathbf{q}| + \omega_n - i\epsilon)\,|\mathbf{q}|} \tag{24}$$

with $\omega_n \equiv E_n - E_i$ to obtain

$$\Delta E_T = \Delta E_T^\circ + \Delta E_\beta \tag{25a}$$

$$\Delta E_T^\circ = -\frac{\alpha}{4\pi^2}\int \frac{d^3\mathbf{q}}{\mathbf{q}^2}\sum_n |\langle n\mathbf{q} \mid \hat{\mathbf{J}}_\perp(0) \mid i\rangle|^2$$

$$= -\frac{\alpha}{4\pi^2}\int \frac{d^3\mathbf{q}}{\mathbf{q}^2}\langle i \mid \mathbf{J}(\mathbf{q}) \cdot_\perp \mathbf{J}(-\mathbf{q}) \mid i\rangle \tag{25b}$$

$$\Delta E_\beta = \frac{\alpha}{4\pi^2}\sum_n \omega_n \int \frac{d^3\mathbf{q}\, |\langle n\mathbf{q} \mid \hat{\mathbf{J}}_\perp(0) \mid i\rangle|^2}{\mathbf{q}^2(|\mathbf{q}| + \omega_n - i\epsilon)} \tag{25c}$$

where

$$\mathbf{J} \cdot_\perp \mathbf{J} \equiv J_m(\delta_{mn} - \hat{q}_m\hat{q}_n)\, J_n\,. \tag{25d}$$

In arriving at these results we find that the two dispersive terms contribute equally and that the seagull term, ΔE_{SG}, is obtained solely from the transverse part of Eq. 22. The Coulomb term ΔE_c° and the transverse current term ΔE_T° have a very similar structure and may be written in the form

$$\Delta E_o = \Delta E_c^\circ + \Delta E_T^\circ = \frac{\alpha}{4\pi^2}\int \frac{d^3\mathbf{q}}{\mathbf{q}^2}\langle i \mid \rho(\mathbf{q})\,\rho(-\mathbf{q}) - \mathbf{J} \cdot_\perp \mathbf{J} \mid i\rangle. \tag{26}$$

The remaining term ΔE_β which we separated from ΔE_T is actually higher order in $(1/m)$ than the other terms, since formally it is of order $1/m^3$ (ω_n being of order $1/m$).

Matrix elements of the charge and current operators $\hat{\rho}(0)$ and $\hat{\mathbf{J}}(0)$ at the origin in Eqs. (23b) and (25a) have been replaced by matrix elements of the operators $\rho(\mathbf{q})$ and $\mathbf{J}(\mathbf{q})$ which involve only internal (intrinsic) wavefunctions according to the prescriptions of [8, 9, Appendix A]. The latter operators in the lab frame are given by

$$\rho(\mathbf{q}) = \sum_{i=1}^{A} e^{i\mathbf{q}\cdot\mathbf{x}_i'}\hat{e}_i(\mathbf{q}^2)\left(1 - \frac{\mathbf{q}^2}{8m^2}\right)$$

$$- i\sum_{i=1}^{A}\frac{(2\hat{\mu}_i(\mathbf{q}^2) - \hat{e}_i(\mathbf{q}^2))}{8m^2}\{\boldsymbol{\sigma}(i)\times\boldsymbol{\pi}_i\cdot\mathbf{q},\, e^{i\mathbf{q}\cdot\mathbf{x}_i'}\} \tag{27a}$$

$$\mathbf{J}(\mathbf{q}) = \frac{\mathbf{q}}{2m_t}\,\rho(\mathbf{q}) + i\sum_{i=1}^{A}\frac{\hat{\mu}_i(\mathbf{q}^2)(\boldsymbol{\sigma}(i)\times\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}_i'}}{2m} + \sum_{i=1}^{A}\{\boldsymbol{\pi}_i\,,\, e^{i\mathbf{q}\cdot\mathbf{x}_i'}\}\frac{\hat{e}_i}{2m} \tag{27b}$$

$$\boldsymbol{\pi}_i = \mathbf{p}_i - \mathbf{P}/A \qquad \mathbf{P} = \sum_i \mathbf{p}_i \tag{27c}$$

$$\mathbf{x}_i' = \mathbf{x}_i - \mathbf{R} \qquad \mathbf{R} = \sum_i \mathbf{x}_i/A. \tag{27d}$$

In these equations, $\mathbf{x}_i'$ is the relative coordinate, $\boldsymbol{\pi}_i$ is the relative momentum, the terms of order $1/m^2$ in ρ are the Darwin–Foldy and spin–orbit terms and the three components of the current operator are the nuclear convection current, spin magnetization current, and internal convection current.

To obtain useful expressions for the operators in Eq. (26), we exploit the fact that each of the operators is a sum over individual nucleons and the product of operators is a double sum. This is illustrated by the charge operator term:

$$\begin{aligned}\rho(\mathbf{q})\,\rho(-\mathbf{q}) \cong{} & \left(1-\frac{\mathbf{q}^2}{4m^2}\right)\sum_{i\neq j}^{A}\hat{e}_i(\mathbf{q}^2)\,\hat{e}_j(\mathbf{q}^2)\exp(i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)) \\ & +\sum_{i=1}^{A}\hat{e}_i^2(\mathbf{q}^2)\left(1-\frac{\mathbf{q}^2}{4m^2}\right) \\ & -i\sum_{i\neq j}^{A}\frac{\hat{e}_j(2\hat{\mu}_i-\hat{e}_i)}{4m^2}\{\boldsymbol{\sigma}(i)\times\boldsymbol{\pi}_i\cdot\mathbf{q},\exp(i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j))\}\end{aligned} \tag{28}$$

where a spin–orbit term involving a single sum has been dropped because it vanishes upon integration in Eq. (26). The middle term may be simplified using Eq. (9)

$$\sum_{i=1}^{A}\hat{e}_i^2(\mathbf{q}^2)=Z[G_E^{p}(\mathbf{q}^2)]^2+N[G_E^{n}(\mathbf{q}^2)]^2 \tag{29}$$

where Z and N are the proton and neutron numbers, respectively. The matrix element of Eq. (29) is completely independent of the nuclear dynamics. The same type of term arises in the current matrix element as well.

$$\begin{aligned}\mathbf{J}(\mathbf{q})\underset{\perp}{\cdot}\mathbf{J}(-\mathbf{q}) ={} & \sum_{i\neq j}^{A}\frac{\hat{\mu}_i(\mathbf{q}^2)\,\hat{\mu}_j(\mathbf{q}^2)}{4m^2}\,\boldsymbol{\sigma}(i)\times\mathbf{q}\cdot\boldsymbol{\sigma}(j)\times\mathbf{q}\exp(i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j)) \\ & +i\sum_{i\neq j}^{A}\frac{\hat{\mu}_i\hat{e}_j}{2m^2}\{\boldsymbol{\sigma}(i)\times\mathbf{q}\cdot\boldsymbol{\pi}_j,\exp(i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j))\} \\ & +\sum_{i\neq j}^{A}\frac{\hat{e}_i\hat{e}_j}{m^2}\exp(i\mathbf{q}\cdot(\mathbf{x}_i-\mathbf{x}_j))(\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_j-(\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_i)(\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_j)) \\ & +\sum_{i=1}^{A}\frac{\hat{\mu}_i^2\mathbf{q}^2}{2m^2}+\sum_{i=1}^{A}\frac{\hat{e}_i^2}{m^2}(\boldsymbol{\pi}_i^2-(\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_i)^2)\end{aligned} \tag{30}$$

where the last two terms may be simplified to

$$\frac{\mathbf{q}^2}{2m^2}\sum_{i=1}^{A}\hat{\mu}_i^2 = \frac{\mathbf{q}^2}{2m^2}(Z[G_M{}^p]^2 + N[G_M{}^n]^2) \tag{31a}$$

$$\sum_{i=1}^{A}\frac{\hat{e}_i{}^2}{m^2}(\boldsymbol{\pi}_i{}^2 - (\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_i)^2) = \frac{[G_E{}^p]^2}{m^2}\sum_{\text{protons}}(\boldsymbol{\pi}_i{}^2 - (\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_i)^2)$$
$$+\frac{[G_E{}^n]^2}{m^2}\sum_{\text{neutrons}}(\boldsymbol{\pi}_i{}^2 - (\hat{\mathbf{q}}\cdot\boldsymbol{\pi}_i)^2). \tag{31b}$$

The angular integral in Eq. (26) simplifies the results in Eq. (31b). Gathering together all the single-sum terms, including the seagull, which we call ΔE_{SP} and all the double sum terms in Eq. (29) and (30), which we call ΔE_{BR} we find

$$\Delta E_C = \Delta E_{\text{SP}} + \Delta E_{\text{BR}} + \Delta E_\beta \tag{32a}$$

$$\Delta E_{\text{SP}} = \frac{\alpha}{\pi}\int_0^\infty dq\,\Big\{\Big(1 + \frac{q}{m} - \frac{q^2}{4m^2}\Big)(Z[G_E{}^p]^2 + N[G_E{}^n]^2)$$
$$- \frac{q^2}{2m^2}(Z[G_M{}^p]^2 + N[G_M{}^n]^2)$$
$$- \frac{2}{3m^2}\langle i \mid [G_E{}^p]^2 \sum_{\text{protons}} \boldsymbol{\pi}_p{}^2 + [G_E{}^n]^2 \sum_{\text{neutrons}} \boldsymbol{\pi}_n{}^2 \mid i\rangle\Big\} \tag{32b}$$

where only the last term involves the nuclear physics. Obviously, E_{SP} is the single-particle electromagnetic energy shift. The first two terms represent the mass shift for a free particle within the framework of our model and are divergent. We have no pretensions about our ability to calculate the electromagnetic self-energy of the individual nucleons. While individual self-mass graphs may be calculated using standard [14] covariant Q.E.D. techniques, our approximations in developing nonrelativistic effective charge, current, and seagull operators have mutilated the intricate structure of these graphs. All our approximations, in fact, attempt to deal with slowly-moving nucleons, either real or virtual. The self-mass graphs on the other hand have ultraviolet divergences, which correspond to very high momenta. Our ruthless approximation procedure, while adequate for most tasks, actually makes the self-mass contributions more divergent than they should be.

The first term in Eq. (32b) represents three separate contributions to the self-mass. The (1) term is the Coulomb term and the $-q^2/4m^2$ term is the Darwin–Foldy term, which also arises from the Coulomb interaction. Since the latter term is produced by smearing of the nucleon charge distribution by Zittebewegung, one would expect this to soften the divergence in the complete Coulomb contribution.

That this is not so is due to the approximations discussed above. These terms are represented graphically by Fig. 2a. The q/m term is the contribution of the seagull graph and is a transverse contribution. The main Coulomb term is linearly divergent if we ignore the form factors, and this reflects the linear divergence of a classical ball of charge of radius R, $\Delta E \sim e^2/R$, as the radius is shrunk to zero. In the Q.E.D. case, the pair graph shown in Fig. 2b, which led to the seagull term, was shown by Weisskopf [14] to cancel the linear divergence, leaving a logarithmic divergence. Our seagull is quadratically divergent.

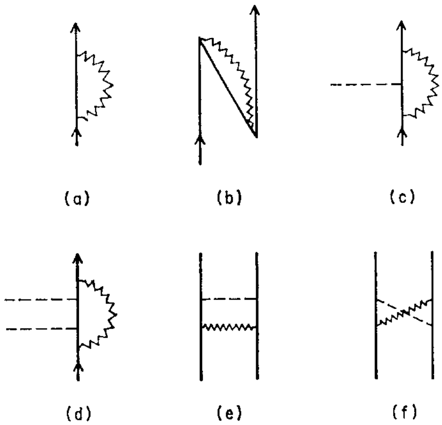

FIG. 2. Positive energy (a) and pair (b) intermediate state contributions to the energy of a nucleon, radiative corrections to the one (c) and two (d) pion–nucleon vertices, direct (e) and crossed (f) meson–photon exchange diagrams.

It is interesting to examine the contribution of the pair term *before* this contribution is approximated as a seagull. The energy denominators for the pair terms vanish at $q_0 \sim \pm 2m$, and thus, produce a pole term similar to others we have seen, but at much higher energy. After approximating the pair term as a seagull, the pole is eliminated. The second term in Eq, (32b) is the magnetic contribution and this has the opposite sign. If one includes the nucleon form factors, the integrals are finite since the form factors act as a cutoff. A covariant calculation of the neutron–proton mass difference using an ad hoc momentum cutoff was performed by Feynman and Speisman [5], who observed that the opposite sign of the magnetic interaction could make the neutron heavier than the proton. This is in agreement with experiment but opposite to naive expectations based on the Coulomb interaction. Cini, Ferrari, and Gato [6] pointed out that the phenomenological charge and current form factors act as a cutoff, just as they do in Eq. (32b). Unfortunately, these calculations did not include nucleon-plus-meson intermediate states, which appear to be important [20]. In the two-body pieces, ΔE_{BR}, the mediating effect of the nuclear wavefunctions eliminates any serious dependence on high-q, or hard photon, parts of the integral in Eq. (26).

Our procedure will be to use physical nucleon masses and drop the nucleon mass-shift terms. The remaining part of Eq. (32b) has an interesting origin discussed by Bethe [12]. Since the terms in the first line of Eq. (32b) proportional to Z and N are the electromagnetic mass shifts of the protons δm_p and neutrons δm_n, respectively, we would expect that the mass change would affect the kinetic energy of the nucleons in the nucleus. In the absence of the electromagnetic self-interaction, the kinetic energy of a nonrelativistic nucleus at rest is $\sum_i \pi_i^2/2m$, where m is the bare nucleon mass. When the electromagnetic interactions are turned on, the physical nucleon mass becomes $m_{\mathrm{ph}} = m + \delta m$, and m_{ph} instead of m produces, using the protons as an example

$$\sum_{\text{protons}} \frac{\pi_p^2}{2m_{\mathrm{ph}}} \cong \sum_p \frac{\pi_p^2}{2m} - \frac{\delta m_p}{2m^2} \sum_p \pi_p^2 \tag{33}$$

which has the same form as the last line in Eq. (32b). Thus, if we use *physical* nucleon masses in the equation of motion, and in the kinetic energy of the nucleus in particular, we can ignore *all* of ΔE_{SP}.

The most important contribution is ΔE_{BR}, which may be evaluated by performing the $\mathbf{q}$-integral in Eq. (26). The following identities are useful

$$\int \frac{d^3\mathbf{q}}{\mathbf{q}^2}\, e^{i\mathbf{q}\cdot\mathbf{x}} = \frac{2\pi^2}{|\mathbf{x}|} \tag{34a}$$

$$\int \frac{d^3\mathbf{q}}{\mathbf{q}^4}\, e^{i\mathbf{q}\cdot\mathbf{x}} = -\pi^2\, |\mathbf{x}| \tag{34b}$$

$$\int \frac{d^3\mathbf{q}}{\mathbf{q}^2}\, q_i q_j e^{i\mathbf{q}\cdot\mathbf{x}} = -\nabla_i \nabla_j \left(\frac{2\pi^2}{|\mathbf{x}|}\right) \tag{34c}$$

and other trivial extensions of Eqs. (34a) and (34b) which may be obtained by taking derivatives, Using Eq. (27), the two-body pieces may be written (using an obvious notation)

$$V_{\mathrm{BR}} = \frac{1}{2} \sum_{i \neq j} V_{ij}^{\mathrm{BR}} \tag{35a}$$

$$V_{ij}^{\mathrm{BR}} = \frac{\alpha}{2\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2} \left(\rho_i(\mathbf{q})\, \rho_j(-\mathbf{q}) - \mathbf{J}_{i\,\perp} \mathbf{J}_j\right). \tag{35b}$$

We see that the individual contributions to V_{ij}^{BR} each contain a product of nucleon form factors and that each of these may be written as a convolution. We define, for example,

$$\hat{e}_i \otimes I(\mathbf{x}_i - \mathbf{x}_j) \otimes \hat{e}_j \equiv \int d^3\mathbf{x}\, \hat{e}_i(\mathbf{x}) \int d^3\mathbf{y}\, \hat{e}_j(\mathbf{y})\, I(\mathbf{x} - \mathbf{y} + \mathbf{x}_i - \mathbf{x}_j). \quad (35\text{c})$$

This gives the Coulomb potential V_{ij}^c obtained from $\rho_i \rho_j$

$$V_{ij}^c = \alpha \Big(\hat{e}_i \otimes \frac{1}{|\mathbf{x}_{ij}|} \otimes \hat{e}_j - \frac{\pi}{m^2}\, \hat{e}_i \otimes \delta^3(\mathbf{x}_{ij}) \otimes \hat{e}_j$$

$$+ (2\hat{\mu}_i - \hat{e}_i) \otimes \Big(\boldsymbol{\sigma}(i) \cdot \nabla_{ij} \frac{1}{|\mathbf{x}_{ij}|} \times \boldsymbol{\pi}_i\Big) \otimes \hat{e}_j\Big) \quad (36)$$

where $\mathbf{x}_{ij} \equiv \mathbf{x}_i - \mathbf{x}_j$ and $\nabla_{ij} = \partial/\partial\mathbf{x}_{ij}$. The three terms are, respectively, the static Coulomb, Darwin–Foldy, and spin–orbit potentials. In addition to these contributions, the transverse components of the current produce

$$V_{ij}^T = -\frac{\alpha}{2m^2} \Big\{\hat{e}_i \otimes \Big[\frac{(\boldsymbol{\pi}_i \cdot \mathbf{x}_{ij})(\mathbf{x}_{ij} \cdot \boldsymbol{\pi}_j)}{|\mathbf{x}_{ij}|^3} + \frac{\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_j}{|\mathbf{x}_{ij}|}\Big] \otimes \hat{e}_j$$

$$+ \hat{\mu}_i \otimes \Big[\frac{4\pi}{3}\, \boldsymbol{\sigma}(i) \cdot \boldsymbol{\sigma}(i)\, \delta^3(\mathbf{x}_{ij})$$

$$+ \frac{3\boldsymbol{\sigma}(i) \cdot \mathbf{x}_{ij} \boldsymbol{\sigma}(j) \cdot \mathbf{x}_{ij}}{2\,|\mathbf{x}_{ij}|^5} - \frac{1}{2}\frac{\boldsymbol{\sigma}(i) \cdot \boldsymbol{\sigma}(j)}{|\mathbf{x}_{ij}|^3}\Big] \otimes \hat{\mu}_j$$

$$\times\, 2\hat{e}_i \otimes \Big[\frac{\boldsymbol{\sigma}(j) \cdot \mathbf{x}_{ij} \times \boldsymbol{\pi}_i}{|\mathbf{x}_{ij}|^3}\Big] \otimes \hat{\mu}_j\Big\}. \quad (37)$$

Specialization to the case of point charges and magnetic moments is made by the substitution $\hat{e}_i(\mathbf{x}) \to \hat{e}_i\, \delta^3(\mathbf{x})$, etc., in which case the definition (35c) reduces to a simple product. The three contributions to V_{ij}^T are the orbit–orbit, spin–spin, and spin-other-orbit potentials, respectively. Together, Eqs. (36) and (37) give the Barker–Glover [10] form of the Breit [11] interaction, including the static Coulomb potential. This interaction in various forms has been extensively used in atomic physics [21–24], and has seen limited use in nuclear physics [25–27]. The energy shift ΔE_{BR} can therefore be written

$$\Delta E_{\mathrm{BR}} = \langle i \mid V_{\mathrm{BR}} \mid i\rangle = \langle i \mid \tfrac{1}{2} \sum_{i \neq j} V_{ij}^{\mathrm{BR}} \mid i\rangle. \quad (38)$$

Of some interest is the more general case of unequal mass particles. If we specialize to two particles of mass m_1 and m_2, the m^2 denominator in V_{ij}^T becomes $m_1 m_2$, while the $e_i e_j/m^2$ factor in the Darwin term in V_{ij}^c becomes $(e_i e_j/2)((1/m_1^2) +$

$(1/m_2^2)$) and a similar result for the spin–orbit term. If we take the limit, $m_2 \to \infty$, the potential V_{ij}^T vanishes, while a single term in each of the Darwin and spin–orbit contributions survives. This is a reasonable result, since the V_T terms are essentially magnetic in nature. Furthermore, the surviving terms are just the terms that result when the Dirac equation for a particle interacting with a static Coulomb potential is reduced to order $(1/m^2)$ using the Foldy–Wouthuysen procedure [15]. This result is actually a general theorem, proven for electrons by Deser [28]. It is interesting to note that the electromagnetic Bethe–Salpeter equation in the *ladder* approximation does not satisfy this theorem [29].

Of some recent interest is the fact that the interaction we have developed, V_{BR}, is not Galilean invariant [24, 26] although this is not altogether surprising in view of our inclusion of relativistic corrections in the derivation. Galilean invariant potentials would depend on momentum differences $(\boldsymbol{\pi}_i - \boldsymbol{\pi}_j)$ and not on the sums $(\boldsymbol{\pi}_i + \boldsymbol{\pi}_j)$. A non-Galilean invariant potential between two interacting nucleons will depend on their total momentum relative to the fixed nucleus. This must be taken into account when evaluating matrix elements [26] of V_{BR}.

The remaining term in the electromagnetic energy shift is the term ΔE_β which we will evaluate in the manner of Bethe [12]. This term vanishes if we take $\omega_n = 0$, and is a retardation correction within the framework of our formalism. In fact, it is also part of the Lamb-shift of the energy levels when applied to an atom. If the integrand in Eq. (25c) did not depend on ω_n, we could eliminate the sum over n by means of an identity often used in evaluating sum-rules. Since there is a complicated dependence on ω_n in the denominators, we use the closure approximation and replace ω_n by $\bar{\omega}$ in the denominator *alone*. Clearly, we are not allowed to take $\bar{\omega} = 0$. This allows us to write

$$\Delta E_\beta = \frac{\alpha}{8\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2(|\mathbf{q}| + \bar{\omega})} \langle i \mid [\mathbf{J}_\perp, [\hat{H}_o, \mathbf{J}]] \mid i \rangle. \tag{39}$$

The effective excitation energy $\bar{\omega}$ is defined as that energy which makes the appropriate pieces of ΔE_β, defined below, equal to the exact result for these pieces, similar to the way dispersion corrections [9] to electron scattering are handled. Assuming a momentum and isospin-independent potential we write

$$\hat{H}_o = T + V \tag{40a}$$

$$V = \tfrac{1}{2} \sum_{i \neq j} V_{ij} \tag{40b}$$

and we find that the T term in Eq. (39) generates quadratically divergent single-particle terms, which we drop as higher-order contributions to ΔE_{SP}. In Bethe's nonrelativistic treatment of the Lamb-shift the effects of retardation in the currents were dropped (i.e., $\mathbf{q} \to 0$), and with this approximation T commutes with $\mathbf{J}$. The approximation may be justified in a careful calculation. In the remaining term involving the potential there are two-body terms involving currents connecting different nucleons and one-body terms connecting a nucleon to itself. The latter terms involve the spin magnetization current and the convection current. The spin terms are again highly divergent and we neglect them as before; these terms do not arise if the potential is spin-independent. After performing some angular integrals we find

$$\Delta E_\beta \equiv \Delta E_\beta^{(1)} + \Delta E_\beta^{(2)} \tag{41a}$$

$$\Delta E_\beta^{(1)} = \frac{\alpha}{3\pi m} \int \frac{dq}{q+\bar{\omega}} \langle i \mid \sum_i \hat{e}_i^2[\mathbf{p}_i\,;\,[V,\mathbf{p}_i]] \mid i \rangle \tag{41b}$$

$$\Delta E_\beta^{(2)} = \frac{\alpha}{8\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2(\mid \mathbf{q} \mid + \bar{\omega})} \langle i \mid \sum_{i \neq j} [\mathbf{J}_i(-\mathbf{q})_\perp\,,\,[V,\mathbf{J}_j(\mathbf{q})]] \mid i \rangle \tag{41c}$$

where we have written

$$\begin{aligned} \mathbf{J}(\mathbf{q}) &= \sum_i \mathbf{J}_i(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}_i{}'} \\ \mathbf{J}_i(\mathbf{q}) &= \frac{i\hat{\mu}_i(\mathbf{q}^2)}{2m}\,(\boldsymbol{\sigma}(i) \times \mathbf{q}) + \frac{\boldsymbol{\pi}_i \hat{e}_i(\mathbf{q}^2)}{m}\,. \end{aligned} \tag{41d}$$

The two-body terms in Eq. (41c) are finite because the nuclear wavefunctions cut off the hard-photon part of the integral. We ignore the spin-dependent part of **J** for simplicity. The remaining part of both commutators is essentially the same and easy to evaluate. The one-body term in Eq. (41b) has a divergent integral, but the divergence is only logarithmic, if we ignore the nucleon structure. In the notation of Bjorken and Drell [14] the upper limit of the integral is taken to be $k_{\min}$, which is much larger than $\bar{\omega}$, and using $\hat{e}_i(\mathbf{q}^2) \simeq \hat{e}_i(0) \equiv e_i$, we obtain

$$\Delta E_\beta^{(1)} = \frac{\alpha}{3\pi m^2} \log\left[\frac{k_{\min}}{\bar{\omega}}\right] \langle i \mid \sum_i e_i(\nabla_i^2 V) \mid i \rangle. \tag{42a}$$

Graphically this contribution to ΔE_β is represented by the soft photon parts of Figs. 2c and 2d and graphs which involve more meson insertions (dashed lines). The contribution of these graphs is divergent but the logarithmic divergence is cancelled in Q.E.D. by the hard-photon part which is calculated covariantly. Stevens [30] has calculated the radiative correction for the pion potential corresponding to Fig. 2c, and his result for the vertex in a particular model for the pion–nucleon interaction contains a part which is exactly the same as the corresponding term in Q.E.D. Assuming that this result holds in general, when the term is added to Eq. (42a) the result is to replace $k_{\min}$ by $m/2$ and we find

$$\Delta E_\beta^{(1)} = \frac{\alpha}{3\pi m^2} \log \left[\frac{m}{2\bar{\omega}}\right] \langle i \mid \sum_i e_i(\nabla_i^2 V) \mid i\rangle \tag{42b}$$

which is the usual [14] Lamb-shift result. We have made numerous approximations which we have no hope of justifying, but rather we appeal to the analogous Q.E.D. calculation where justification is available. We note, however, that there are obvious differences between the two cases, one of which is the appearance of an electromagnetic correction to the $pp\pi^0$ coupling constant, found by Stevens [30], and which was shown by Morrison [31] to be model dependent. The Ward identity precludes such a term in Q.E.D. Finally the two-body term becomes

$$\Delta E_\beta^{(2)} = \frac{\alpha}{2\pi m^2} \langle i \mid \sum_{i \neq j} e_i e_j (f_{ij} \nabla_i \cdot \nabla_j + (\nabla_\alpha \nabla_\beta g_{ij}) \nabla_{i\alpha} \nabla_{j\beta}) V \mid i\rangle \tag{42c}$$

$$f_{ij} = \int_0^\infty \frac{dq}{q + \bar{\omega}} J_0(\mid \mathbf{x}_{ij} \mid q) \qquad g_{ij} = \int_0^\infty \frac{dq}{q^2(q + \bar{\omega})} J_0(\mid \mathbf{x}_{ij} \mid q) \tag{42d}$$

and we have defined $\nabla_\alpha g_{ij} = [(\partial/\partial \mathbf{x}_{ij}) g(\mid \mathbf{x}_{ji} \mid)]_\alpha$. Recall that we are not allowed to drop $\bar{\omega}$ in Eq. (42d), since the functions f and g then have spurious infrared difficulties (small q). The contribution $\Delta E_\beta^{(2)}$ arises from the soft photon part of Fig. 2e and similar graphs with the photon line crossing additional meson lines. As mentioned earlier, ΔE_β arises because of retardation. The infrared divergence in $\Delta E_\beta^{(2)}$ for $(\bar{\omega} = 0)$ was noted by Morrison. Recently, Riska and Chu [32] have performed a careful calculation of the residual contribution of Figs. 2e and 2f after subtracting out the iteration of the single photon-exchange and single pion-exchange forces.

The use of the electric dipole part of the current operator in deriving Eq. (42b) (and to some extent Eq. (42c)) suggests that a good guess is to take $\bar{\omega}$ to be the energy of the giant (dipole) resonance, which exhausts an appreciable part of the dipole photonuclear sum-rule. The result is almost certainly a very tiny contribution to ΔE_c.

In deriving our basic formulas for the electromagnetic energy shift, we have appealed to basic principles which have aided us in simplifying rather formal expressions. These expressions, as emphasized by Feynman [13], contain even more information than we have indicated. Although our basic interest was in obtaining electromagnetic mass shifts for ground states, the method clearly works for excited states as well. In this case, however, the same electromagnetic interaction which leads to an energy shift also induces the nucleus to decay to states of lower energy. This is reflected in an imaginary part of ΔE_c. It is clear that ΔE_c^0, ΔE_{SG}, and

$\Delta E_T{}^0$ are real and that only ΔE_β has a denominator which can vanish, and this will only happen if $\omega_n < 0$, and requires states of lower energy than the state i. We find

$$\begin{aligned}\operatorname{Im}(\Delta E_c) &= -\frac{\alpha}{4\pi}\sum_{n<i}\int\frac{d^3\mathbf{q}}{|\mathbf{q}|}\,|\langle n\mathbf{q}\mid\hat{\mathbf{J}}_\perp(0)\mid i\rangle|^2\,\delta(|\mathbf{q}|+\omega_n)\\ &= -\frac{\alpha}{4\pi}\sum_{n<i}(E_i-E_n)\int d\Omega_{\mathbf{q}}\,|\langle n\mid\mathbf{J}_\perp(\hat{\mathbf{q}},-\omega_n)\mid i\rangle|^2.\end{aligned}\tag{43}$$

Calculating the total decay rate for photon emission [17], τ, we find that

$$\operatorname{Im}(\Delta E_c) = -\tau/2 \tag{44a}$$

or

$$\Delta E_c = \Delta E_\beta + \Delta E_{\mathrm{BR}} - i\tau/2. \tag{44b}$$

This is completely consistent with the exponentially decreasing time dependence $\exp(-iE_n{}^0t - i\Delta E_ct)$ of a decaying state with a lifetime $1/\tau$ and total width at half-maximum, τ. The entire contribution comes from the transverse current since Coulomb gauge is also transverse gauge for real photons.

4. Discussion

The results we previously obtained from Eq. (26) should be no surprise since this expression may also be derived semiclassically. Indeed, part of our result is just the analogue of the classical Darwin interaction [33]. Various pieces of the complete Breit interaction, in particular the spin–orbit interaction, have been calculated for a number of nuclei [34–36].

Of particular interest are those terms in Eqs. (36) and (37) which are magnetic in origin, since the neutron's contribution from these terms is comparable to the proton's contribution. This is reflected even in the dominant static Coulomb potential, the first term in Eq. (36). As discussed in [37, 38], the neutron's charge form factor is dominated by the anomalous magnetic moment term for small momentum transfers, indicating that $F_1{}^N$ is quite small. Thus, the neutron's charge form factor can be regarded to a good approximation both as "magnetic" in origin and, simultaneously, a relativistic correction. The Darwin term, the second term in Eq. (36), has the same form as the κ terms in $\hat{e}_i$, and its inclusion in Eq. (36) is the result of *defining* the nucleon form factor in terms of G_E, the Sachs form factor. If one uses instead a different charge form factor $\tilde{G}_E$

$$\tilde{G}_E = \frac{G_E}{(1+\mathbf{q}^2/4m^2)^{1/2}} \simeq G_E\left(1-\frac{\mathbf{q}^2}{8m^2}\right) \tag{45}$$

as argued in [8], the Darwin terms can be included in the static Coulomb term. The Darwin terms have led to some inconsistency in the past because of the various definitions of the charge form factor. Schwinger's form [39] of the interaction for s-states is valid only for point particles, but includes the anomalous magnetic moment terms. Schnieder and Thaler [40] used G_E and thus included the κ terms, but not the additional Darwin term in Eq. (45). It is not correct, however, to use the point nucleon form of the interaction *and* the Coulomb interaction in Eq. (36) with G_E, since this double counts the κ terms. This has been done explicitly or implicitly in most instances [27, 41, 42]. The remaining piece of the Coulomb part of the Breit potential is the spin–orbit interaction which is produced partly by the Thomas precession and partly by a magnetic effect. This term has been shown to be important in analyzing the charge distribution difference of isotopes [37, 38], since the neutrons make a substantial contribution to this difference.

Particular interest has been shown in the Coulomb energy differences of mirror nuclei [64], and, in particular, the ^3He–^3H Coulomb energy difference is still an important and unresolved theoretical problem. A review of this problem has recently been made by Okamoto and Pask [41]. A large variety of phenomenological calculations of this energy difference, as well as variational and nonvariational calculations of wavefunctions used to calculate ΔE_c, have been discussed in that paper. All present results indicate that calculations using wavefunctions which produce electromagnetic form factors in agreement with experiment produce Coulomb energies in the range 620–660 keV, while the experimental binding energy difference is 764 keV. The reason for the relatively small spread in calculated energies was stressed by the present author [43] earlier, who used Eq. (35b) without the current and relativistic correction terms to calculate ΔE_c for the trinucleon system. Both the electromagnetic charge form factor and the expectation value of the Coulomb energy depend sensitively on the size of a system, and allowing the form factor to determine the size roughly fixes the Coulomb energy. Two simple model wavefunctions of different types were used in [43] to calculate ΔE_c with identical results. Realistic forms for $G_E{}^p$ and $G_E{}^n$ were used, and the calculated Coulomb energy difference was 640 keV. It is important in all calculations of this type to include the nucleon charge distribution, since this makes an appreciable *reduction* of ΔE_c and simultaneously acts as a cutoff of the high momentum components of Eq. (35b), reducing the model dependence of the result.

Calculations of the transverse interaction's contribution to the mass difference in the trinucleon system have been reviewed by Okamoto and Pask. Crude estimates of the spin–spin force matrix element, assuming a completely symmetric s-state space wavefunction for ^3He and ^3H, yield approximately a 10 keV difference for ^3He–^3H, with both contributions positive. This is approximately 2% of the static Coulomb energy. An interesting contrast is the n–p system, using the symmetric quark model. In this case a crude calculation of the spin–spin magnetic interaction yields about 35% of the static Coulomb contribution to the nucleon mass difference and is greater for the neutron. The reason why the $T_z = \frac{1}{2}$ component of the nuclear iso-doublet gets a larger magnetic contribution than the $T_z = -\frac{1}{2}$ component and the opposite is true for the nucleon iso-doublet can be traced to the different structure of the totally symmetric and totally antisymmetric representations of the $SU(4)$ spin–isospin wavefunctions for the three-body systems. The relative sizes of the magnetic and Coulomb contributions can be traced to the fact that the magnetic interaction in Eq. (37) is much shorter-ranged than the static Coulomb interaction and the proton is relatively small. Alternatively, the structure in momentum space (Eq. (26)) shows that the magnetic interaction is much more dependent on the high momentum components of the wavefunction than the Coulomb interaction, and the uncertainty principle shows that these components should be larger for a small object than a large one. It is the large magnetic contribution which is often credited [5, 6] with producing a neutron heavier than a proton, since the charged object is normally expected to be the heavier one.

We mentioned earlier that the relativistic corrections to the strong interaction Hamiltonian were of no particular importance for *most* of our considerations. Nevertheless, corrections of order $(v/c)^2$ to the Hamiltonian which defines our basis will also affect our matrix elements to the same order as the relativistic corrections which we explicitly included in $\hat{\rho}$. To see this, for simplicity we write the complete interaction Hamiltonian for two equal mass particles in the center-of-mass system of the two particles

$$H = 2(\mathbf{p}^2 + m^2)^{1/2} + V_{\mathrm{BR}} + V \qquad H\psi = E\psi \tag{46a}$$

where $\mathbf{p}$ is the momentum of each particle of mass m and V is the strong potential. We have written the exact form of the kinetic energy instead of the approximation in Eq. (8d). Squaring this equation and defining $\epsilon \equiv (E^2 - 4m^2)/4m$ we find to order $(v/c)^2$

$$H'\psi = \epsilon\psi \qquad H' = H_{\mathrm{ST}} + V_c' \tag{46b}$$

$$H_{\mathrm{ST}} = \mathbf{p}^2/m + V + V^2/4m + \{\mathbf{p}^2, V\}/4m^2 \tag{46c}$$

$$V_c' = V_{\mathrm{BR}} + V_c^2/4m + \{V, V_c\}/4m + \{\mathbf{p}^2, V_c\}/4m^2 \tag{46d}$$

where V_c is the static Coulomb potential. This equation (46b) is equivalent to the original one except that it has been cast into nonrelativistic *form* using the trick of Coester *et al.* [44]. This manipulation produces extra potential terms in both H_{ST} and V_c'. The sum of all potential terms in (46c) is what Coester calls the "phenomenological potential." Our interest lies in the relativistic correction terms in V_c'. Note the appearance of the $\mathbf{p}^2 V_c$ term similar to that of Banerjee [45], and the *additional* VV_c term which should be equally important. This illustrates that no part of the Hamiltonian can be defined without reference to the remaining parts of the Hamiltonian. The Breit interaction we have derived is to be used with the Hamiltonian (46a). It is not at all clear that the trick used in deriving Eq. (46b) is especially useful in the many-body problem, where many-body forces are possible and the particular representation one uses will specify these forces. In particular, the usual representation (Eq. (46a)) generates weak purely electromagnetic three-body forces [46] in atoms; it is not clear whether the same situation holds for the other representation [44] (Eq. (46b)).

5. Sum Rules

In performing the q_o-integral in Eq. (18), the entire contribution of the Coulomb part in the Coulomb gauge arose from the pole in the dispersion terms in Eq. (19) at $q_o = \pm(E_n - E_i)$. Since a simple change of sign renders the second term in the equation equal to the first, we need only examine the positive sign case. The condition $q_o = E_n - E_i$ is the requirement for conservation of energy in elastic and inelastic electron scattering with an energy transfer q_o. With this in mind, we define the Coulomb and transverse current response functions [47–50]:

$$\begin{aligned} R_c(\mathbf{q}^2, q_o) &= \sum_n |\langle n\mathbf{q} \mid \hat{\rho}(0) \mid i\rangle|^2\, \delta(q_o - E_n + E_i) \\ &= \sum_{n'} |\langle n\mathbf{P}_f \mid \hat{\rho}(0) \mid i\rangle|^2\, \delta^4(q - P_f + P_i) \end{aligned} \tag{47a}$$

$$R_T(\mathbf{q}^2, q_o) = \sum_n |\langle n\mathbf{q} \mid \hat{\mathbf{J}}_\perp{}^2(0) \mid i\rangle|^2\, \delta(q_o - E_n + E_i) \tag{47b}$$

where a second form of R_c has been given for completeness. The primed sum includes an integral over $d^3\mathbf{P}_f$ in addition to a sum over internal states, n, and $P_i = (E_i, 0)$, etc., in an obvious notation. Integrating R_c and R_T over q_o at *fixed* $\mathbf{q}$ we find

$$\begin{aligned} \int_o^\infty dq_o\, R_c(\mathbf{q}^2, q_o) &= \langle i \mid \rho(\mathbf{q})\, \rho(-\mathbf{q}) \mid i\rangle \\ \int_o^\infty dq_o\, R_T(\mathbf{q}^2, q_o) &= \langle i \mid \mathbf{J}(\mathbf{q}) \perp \mathbf{J}(-\mathbf{q}) \mid i\rangle \end{aligned} \tag{47c}$$

which allows us to write ΔE_o in the form

$$\Delta E_o = \frac{\alpha}{4\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2} \int_o^\infty dq_o\, [R_c(\mathbf{q}^2, q_o) - R_T(\mathbf{q}^2, q_o)]. \tag{48}$$

For simplicitly, we have ignored the small contribution to ΔE_c from ΔE_β, which may also be cast into this form. The integrals in Eq. (47) are sum rules [48–50] which may be obtained in principle from electron scattering. It should be borne in mind that we are calculating ΔE_c in lowest-order perturbation theory and this is a good approximation only for the lightest nuclei. For such nuclei the cross section for electron scattering may be calculated using the Born approximation. The cross section for scattering electrons of incident energy E from a nucleus through an angle θ may be cast into the following form [50] in the lab frame (taking the electron mass to be zero)

$$\frac{d\sigma}{d\Omega\, dE} = \sigma_{\text{mott}} \left\{ \frac{q^4}{\mathbf{q}^4} R_c(\mathbf{q}^2, q_o) + \left(\frac{q^2}{2\mathbf{q}^2} + \tan^2 \left(\frac{\theta}{2}\right) \right) R_T(\mathbf{q}^2, q_o) \right\} \tag{49a}$$

$$\sigma_{\text{mott}} = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \tag{49b}$$

$$-t = q^2 \equiv \mathbf{q}^2 - q_o^2 = 4EE' \sin^2 \left(\frac{\theta}{2}\right) \qquad q_o = E - E' \tag{49c}$$

where E' is the final electron energy. By varying E and θ while keeping q_o and q^2 fixed, the functions R_c and R_T may be separated. Unfortunately, this is not an easy task, since data are taken at a fixed electron scattering angle as a function of q_o. Each q_o-point corresponds therefore to a different $\mathbf{q}^2$ and q^2. Nevertheless, it is possible with sufficient labor to perform the separation.

Much more serious than these particular difficulties are difficulties in principle associated with the fact that electron scattering is confined to positive q^2 according to Eq. (49c) (as well as positive q_o). The integrals in Eq. (47) at fixed $\mathbf{q}^2$ over all q_o necessarily involve negative values of q^2, which are unphysical. Three separate contributions are possible: elastic scattering, discrete inelastic scattering, and continum inelastic scattering. These regions are defined by the following conditions developed using relativistic kinematics for a target of mass m_t

$$q_o = \frac{q^2}{2m_t} \qquad \text{(elastic)} \tag{50a}$$

$$q_o = \frac{q^2 + \omega_o^2}{2m_t} + \omega_o \qquad \text{(discrete inelastic)} \tag{50b}$$

$$q_o \geqslant \frac{q^2 + \omega_T^2}{2m_t} + \omega_T \qquad \text{(continuum inelastic)} \tag{50c}$$

where ω_T is the threshold for inelastic breakup. The integrals over $d^3\mathbf{q}dq_o$ in Eq. (48) may be written in the form $d\mathbf{q}^2dq_o$ or dq^2dq_o. In terms of the former choice, the region of integration is defined in Fig. 3a by the shading and corresponds to

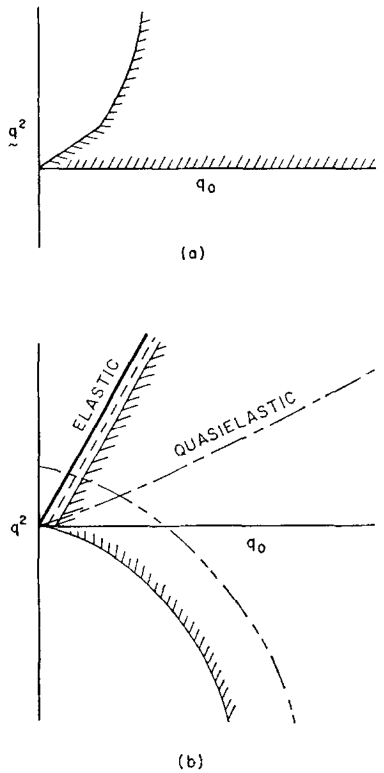

FIG. 3. Region of integration over response functions for variables $\mathbf{q}^2$ and q^2.

$\mathbf{q}^2 \geqslant 0$, $q_o^2 + 2m_t q_o \geqslant \mathbf{q}^2$. The latter condition (for elastic scattering) has been depicted in the figure with an exaggerated q_o^2 term. If the kinematical conditions in Eq. (50) are determined nonrelativistically, one has $q_o \geqslant \mathbf{q}^2/2m_t$.

For the latter choice of integration variables, the region of integration is again indicated by the shaded area in Fig. 3b, where the lower boundary is determined by the condition $q^2 \geqslant -q_o^2$ and the upper boundary by Eq. (50c). In addition, the elastic scattering contribution is indicated by a solid line labelled elastic (Eq. (50a)) and a single discrete excited state is indicated by the dashed line. The long dash–two short dash line corresponds to $\mathbf{q}^2$ = constant. The dominant feature of inelastic electron scattering at moderate to large momentum transfers is the quasi-elastic peak, which corresponds roughly to $q_o = q^2/2m$, and this curve is illustrated by the long dash–short dash line in Fig. 3b. This peak is due to essentially free scattering from the individual nucleons in the nucleus, and exerts a dominant influence on sum rules if the momentum transfer is not too small. Although the negative-q^2 region is not experimentally accessible, it is to be expected that the contribution from this region is small compared to the positive q^2-region which contains the large quasielastic contribution. In addition, the negative q^2–positive q_0 quadrant contains the small $\mathbf{q}^2$-region, where the form factors for individual states are small because of threshold effects. If indeed the negative q^2-region is unimportant, Eq. [48] may be used to calculate the Coulomb energy of a nucleus. In addition to the theoretical problem associated with the negative-q^2 region, we have already discussed the practical problem associated with fixed-$\mathbf{q}^2$ or fixed-q^2 sumrules. We must also consider the fact that part of the sum rule consists of single-particle mass terms which contribute only to the n–p mass difference. Because we have included the nucleon electromagnetic form factors, these contributions are finite, and after subtracting these terms from Eq. (48) the residual quantity is simply ΔE_{BR}. Nevertheless, subtracting a theoretical quantity from an experimental one is not easy to do in a clear and convincing way. Another difficulty is associated with the upper limits of integration, since as a practical matter it is necessary to have upper limits when doing numerical integrations. Ignoring the spin–orbit interaction the Coulomb part of Eq. (26) can be written

$$\Delta E_c' = \frac{\alpha}{4\pi^2} \int \frac{d^3\mathbf{q}}{\mathbf{q}^2} \left\{ \langle i \mid \sum_{i \neq j} \hat{e}_i \hat{e}_j e^{i\mathbf{q}\cdot\mathbf{x}_{ij}} \mid i \rangle + Z[G_E{}^p]^2 + N[G_E{}^n]^2 \right\}. \tag{51}$$

The physically important two-body contribution is proportional to a nucleon–nucleon correlation function (the nuclear-matrix element in Eq. (51)). This quantity can be expected to vanish rapidly with increasing $\mathbf{q}^2$. The remaining term in Eq. (51) is the subtraction term. Therefore, we expect the *subtracted* response function to vanish rapidly as a function of $\mathbf{q}^2$. Because of experimental errors, a practical upper limit in $\mathbf{q}^2$ is that value where R is within a few percent (error bars) of the subtraction term. This will make an error in $\Delta E_c'$ of that order, or perhaps somewhat less. This value of $\mathbf{q}^2$ is probably quite small, less than 5.0 fm^{-2}, for light nuclei. A graph of the appropriate correlation function obtained in a model calculation for ^{4}He is shown in [51] and confirms this estimate of the upper limit. This is fortunate, since the quasielastic energy is 100 MeV for $q^2 = 5.0$ fm^{-2}. Using the techniques of this chapter, it is a practical and theoretical necessity to cut off the integration below the meson production threshold or to estimate and eliminate the meson production part of the inelastic spectrum if the upper limit is continued above this point. The mesonic contributions to ΔE_c will be discussed later. If any significant nuclear contribution to Eq. (51) lies above the meson threshold the method will therefore fail.

A further question is the proper argument, $\mathbf{q}^2$ or q^2, in $R(\mathbf{q}^2, q_0)$. We have been performing a semirelativistic treatment of the photon–nucleon interaction by including terms of order $(1/m^2)$, such as the spin–orbit interaction. There remains a class of relativistic corrections which we have not taken into account, and this is the effect of relativity on the wavefunctions of the recoiling nucleus in the lab frame when a single photon interacts. It was shown recently that these corrections to the wavefunction can be put in the form [52]

$$\psi_{\mathbf{P}} = (1 - i\hat{\chi}(\mathbf{P}))\, \psi_o e^{i\mathbf{P}\cdot\mathbf{R}} \tag{52}$$

when using the variables of Eq. (27). In this equation, $\psi_{\mathbf{P}}$ is the approximate (to order $(1/m^2)$) wavefunction of the system moving with momentum $\mathbf{P}$, ψ_o is the internal wavefunction of the system (in its center-of-mass frame), and $\hat{\chi}(\mathbf{P})$ is an Hermitian function which vanishes when $\mathbf{P}$ vanishes. In addition, the wavefunction (52) is an eigenfunction of the semirelativistic Hamiltonian, Eq. (8d), which includes corrections of order $(1/m^3)$. The use of relativistic energies does not modify in any substantial way the results we have derived. Only the kinematics is slightly changed to accomodate the relativistic energy–momentum relationship.

In the lab frame one effect of the $\hat{\chi}$-term on charge form factors is to replace the nonrelativistic transition charge form factor $\langle n \mid \rho(\mathbf{q}) \mid i \rangle$, corresponding to an intrinsic excitation energy ω, by the same function with $\mid \mathbf{q} \mid \to \bar{q}$, an effective, invariant momentum transfer [53]

$$\bar{q}^2 = q^2 + \omega^2 = \mathbf{q}^2 - q_o{}^2 + \omega^2. \tag{53}$$

The resulting matrix elements have the same threshold dependence on $\mathbf{q}^2$ as before. When performing the sum over states at fixed $\mathbf{q}^2$ in Eq. (23), however, it is possible to show explicitly that the additional terms from $\hat{\chi}$ cancel completely. This is also obvious from the representation in Eq. (52); since $\hat{\chi}(\mathbf{q})$ does not depend on the state n, the use of closure eliminates $\hat{\chi}(\mathbf{q})$ from Eq. (23a), leaving only the terms we previously considered. We therefore do not have to explicitly consider additional "covariance" corrections to the nonrelativistic matrix elements. This incidentally proves that these relativistic corrections do not contribute to the energy-unweighted sum rule which we evaluated in Eq. (47). Nevertheless, in summing over $R(\mathbf{q}^2, q_o)$ we are fully entitled to write this as $R(q^2, q_o)$ or $R(\bar{q}^2, q_o)$.

Our discussion has centered on the nuclear matrix elements and has ignored the nucleon charge form factor. In setting up the problem we have glossed over the question of whether this should be evaluated at $\mathbf{q}^2$ or q^2. This is nontrivial, since the structure of the nucleon implies that the nucleon has an excited state spectrum, which we have ignored completely and which will be discussed later. Furthermore, the nucleon is not free, but bound. Nevertheless, the entire four-momentum transfer is absorbed by a single nucleon in our model and we therefore expect that for any specific transition in the nucleus the appropriate argument of G_E is q^2, rather than $\mathbf{q}^2$. Equation (53) also suggests that this is correct as does the work of Coester and Ostebee [54] and Gross [55] on the elastic form factors of the deuteron. However, the fact that the sum over the complete nuclear spectrum of states eliminates the $\hat{\chi}$-factors which produced an approximately covariant argument suggests that the argument $\mathbf{q}^2$ is just as likely to be correct as any other *in the sum-rules*, if the meson degrees-of-freedom are introduced. For consistency we use $G_E(\mathbf{q}^2)$ in this calculation.

The subtracted response function leads to the expression

$$\Delta E_{\mathrm{BR}} = \frac{\alpha}{2\pi} \int_o^\infty \frac{d\mathbf{q}^2}{|\mathbf{q}|} \Big\{ \int_o^\infty dq_o\, R_c(\mathbf{q}^2, q_o) - Z[\tilde{G}_E{}^p(\mathbf{q}^2)]^2 - N[(\tilde{G}_E{}^n(\mathbf{q}^2)]^2$$
$$- \int_o^\infty dq_o\, R_T(\mathbf{q}^2, q_o) + \frac{\mathbf{q}^2}{2m^2} (Z[G_M{}^p]^2 + N[G_M{}^n]^2)$$
$$+ \frac{2}{3m^2} \langle i \mid [G_E{}^p]^2 \sum_{\text{protons}} \pi_p{}^2 + [G_E{}^n]^2 \sum_{\text{neutrons}} \pi_n{}^2 \mid i\rangle \Big\}. \tag{54}$$

To the best of our knowledge, the representation (54) for the static Coulomb part of the electromagnetic energy was first derived by Efros [56] for a general nucleus and Rothleitner [57] for the deuteron. Recently, it has been rederived by O'Connell and Lightbody [58]. The derivations by Efros and O'Connell and Lightbody were obtained by manipulating identities for electron scattering sum rules expressed in terms of correlation functions (Eq. (47)) and the matrix element of the Fourier transform of the ordinary Coulomb potential, Eq. (34a). The latter two derivations were nonrelativistic, while Efros incorrectly removed the Darwin terms from his expression for the Coulomb energy.

In our future discussion we will ignore the small transverse contribution. The integral over $\mathbf{q}^2$ may be changed to an integral over q^2 with the result

$$\Delta E_{\mathrm{BR}} = \frac{\alpha}{2\pi} \int_o^\infty dq^2 \int_{q^2/2m}^\infty dq_o \frac{R_c(q^2, q_o)}{(q^2 + q_o{}^2)^{1/2}} - \frac{\alpha}{\pi} \int_o^\infty dq\, \{Z[\tilde{G}_E{}^p]^2 + N[\tilde{G}_E{}^n]^2\}$$
$$+ \frac{\alpha}{2\pi} \int_{-\infty}^{o} dq^2 \int_{(-q^2)^{1/2}}^\infty dq_o \frac{R_c(q^2, q_o)}{(q^2 + q_o{}^2)^{1/2}} . \tag{55}$$

The first term is the integral over the physical q^2-region and the lower integral is over the unphysical region (upper and lower half-plane, respectively, in Fig. 3b). As an example, we evaluate the contribution to ΔE_{BR} from elastic scattering alone for the case of a spin -0^+ nucleus. This arises from a part of R_c given by

$$R_c^{\mathrm{el}} = Z^2 F_o{}^2(q^2)\, \delta(q_o - q^2/2m_t) \tag{56a}$$

which produces

$$\Delta E_{\mathrm{el}} = \frac{Z^2\alpha}{2\pi} \int_o^\infty \frac{dq^2\, F_o{}^2(q^2)}{q(1 + q^2/4m_t{}^2)^{1/2}}$$
$$= \frac{Z^2\alpha}{\pi} \int_o^\infty \frac{dq\, F_o{}^2}{(1 + q^2/4m_t{}^2)^{1/2}} . \tag{56b}$$

We note that if we use q^2 with $q_0 = q^2/2m$, which is the condition for free nucleon scattering, as the argument of $\tilde{G}_E$ in Eq. (54) we arrive at a form for the subtraction term different from the one in Eq. (55).

$$\Delta E_{\text{sub}} = \frac{\alpha}{\pi} \int_0^\infty dq \, \frac{(1 + q^2/2m^2)}{(1 + q^2/4m^2)^{1/2}} \{Z[\tilde{G}_E{}^p]^2 + N[\tilde{G}_E{}^n]^2\}. \tag{57}$$

The additional terms in $(1/m^2)$ produce a change of approximately 7 %, using a dipole form factor for the proton and ignoring the neutron. This is probably an upper limit, but indicates the amount of uncertainty present in the form factor terms.

In view of the many small uncertainties, the sum-rule method for determining Coulomb energy shifts in light nuclei may be impractical. Using the first two terms in Eq. (55) may be satisfactory in a few cases where the nuclear structure theory is sufficiently well developed to calculate small corrections to sufficient accuracy. In particular, the ^{3}He–^{3}H case would be an obvious choice, since the ^{3}H electromagnetic energy is very small and may be calculated to sufficient accuracy [41]. In addition, the experimental determination of the sum rule for ^{3}H would allow a check of the subtraction procedure. Furthermore, the difference of the ^{3}He-integrated response function and twice the hydrogen one equals $E_c(^3\text{He}) - 2E_c(^3\text{H})$ plus very small neutron subtraction terms, and this quantity could perhaps be constructed more accurately and corrected theoretically for the small additional ^{3}H energy.

The procedure we have followed in writing the Breit part of the total electromagnetic energy of a nucleus was dictated by the semirelativistic dynamical framework we adopted. In particular, our original expressions for ΔE_c contained terrible ultraviolet divergences and these had to be removed as a first step in isolating the relevant two-body contributions. In contrast, Cottingham's relativistic calculation can be expected to have a much better ultraviolet behavior, but the question of possible subtractions in the dispersion relation he used still remains [4]. If one subtracts the Compton amplitudes of the individual nucleons from the amplitude of the whole nucleus, the electromagnetic mass shifts of the nucleons is accounted for, and the residual amplitude presumably describes the electromagnetic energy of the nucleus due to binding (i.e., the presence of other nucleons). The response function for each individual nucleons treated according to Cottingham's method presumably displays the cancellation of the linear divergence between the direct and pair pieces discussed earlier. This raises the question of how large a region of integration over q^2 and q_0 is needed to produce this cancellation. Similar considerations apply to the one-body terms in Eq. (33), which are the result of the *dynamical* effects of the n–p mass difference. Over how large a region in the residual (subtracted) response function is this term spread? In addition, how important would any logarithmic divergences be in the numerical result? If there are any such weak divergences or "almost" divergent terms, does the order in which the integrations are performed make any difference.

All of these questions are relevant because any method which fails to yield Coulomb energies within 10% of experimental values is not particularly worthwhile [64]. Working with subtracted response functions is a serious handicap because of experimental considerations. The method itself is limited to very low-Z nuclei, since Coulomb corrections [38] to the electron scattering determination of nuclear sizes are of the order $(Z\alpha)$. Furthermore, normalization problems and difficulties in making a q^2, q_0 separation of the response function will produce errors of at least a few percent. If the method depends on extracting reliable information far above meson production threshold, uncertainty in the radiative corrections will also limit the accuracy. Finally, if most of the contribution to ΔE_c comes from the low-q, low-q_0 region, it probably makes little difference what technique is used; in effect, one is simply using inelastic electron scattering to determing the correlation function, the nuclear physics information needed to compute ΔE_c in the absence of sizable mesonic effects.

Recently, after completion of this work, Billoire and Chemtob [65] advocated the use of the relativistic Cottingham formula as a practical method of "calculating" the Coulomb energy difference of ^{3}He–^{3}H and ^{41}Sc–^{41}Ca. While the latter case undoubtedly has too large a charge to allow us to work in first Born approximation, the former case is barely feasible as we have discussed, although serious questions remain to be answered about the technique.

6. Mesonic Corrections

Apart from the introduction of nucleon form factors, and the use of nuclear forces, no explicit introduction of mesons into the problem has been made. These contributions arise from two sources in our formalism. Mesons may participate in the excitation (virtual or real) of mesonless nuclear states through the mechanism of exchange-charge and exchange-current operators. These contributions arise from mesons produced on one nucleon by the electromagnetic interaction which land on a different nucleon. The form factor contributions arise from mesons which land on the same nucleon line they started from. In addition, the production of (virtual) states which include mesons will also contribute. In general, both contributions are needed to preserve gauge invariance. Recently, the structure of the nuclear Compton amplitude for soft photons was discussed [17] in detail with special emphasis on the mesonic effects. In dealing with photons of low energy and momentum, it was convenient to approximate the intermediate states involving mesons as effective seagull operators, in the same way the pair intermediate states were handled. It is virtually certain that the same technique is not useful in the present context, although it is very convenient to work in a representation with no *explicit* meson states.

The meson-exchange currents are known to make substantial contributions ($\leqslant 10\%$) to magnetic moments [59], and are needed for a consistent current continuity equation. Exchange-charge contributions to charge form factors have been calculated recently and may be important at large momentum transfers [60–63]. Within the current context, the effect of the excitation of intermediate states containing mesons is unknown. In general, the Feynman diagrams of a given order must be broken down according to their singularities to fit within our framework. Typical examples are shown in Fig. 4. Virtual meson production in Fig. 4a also may be regarded as an electromagnetic correction to the meson mass. Figures 4b and 4c depict exchange and catastrophic contributions to the exchange-current [59]

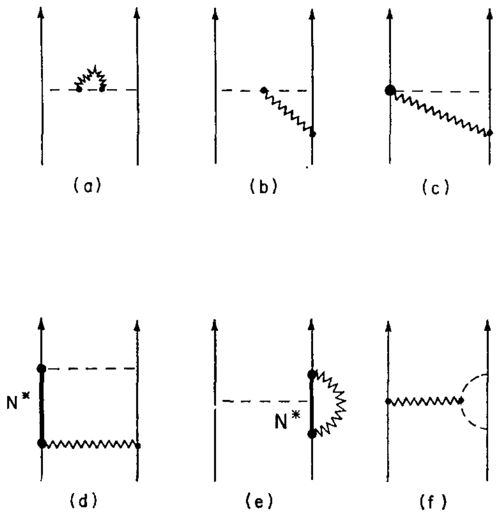

FIG. 4. Mesonic and isobaric contributions to the electromagnetic energy of a nucleus.

part of ΔE_c. Other mesonic effects are better described as excitations of nucleon isobars, and typical examples are shown in Figs. 4d and 4e. Finally, Fig. 4f shows a typical form factor term. There are obviously numerous two-body diagrams which must be considered and the reader is referred to Henley and Okamoto and Pask for a review of many aspects of this problem.

One class of diagrams which seems to have been overlooked is the class of three-body electromagnetic forces which arise from the exchange-currents and exchange-charges. Typical examples are shown in Figs. 5a and 5b. These are analogous to

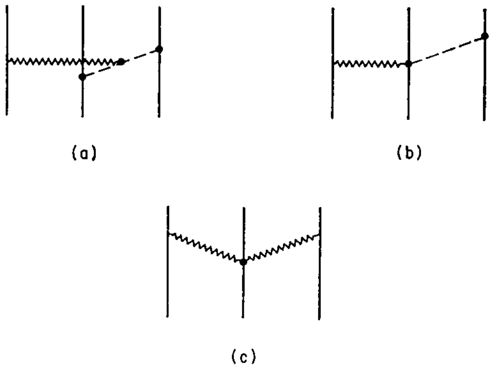

FIG. 5. Three-body contributions to the nuclear electromagnetic energy are shown in (a) and (b) where the dashed line depicts a meson, while (c) depicts the purely electromagnetic three-body force.

the purely electromagnetic three-body force depicted in Fig. 5c and discussed recently by Chanmugam and Schweber [46]. The latter force arises from the $\mathbf{A}^2/2m$ term in the electromagnetic Hamiltonian.

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