Nucleons in magnetic field with Stochastic Variational Method

Moti Eliyahu

1 LO EFT

We need to solve the equation

$$H\Psi = E\Psi$$

The LO EFT Hamiltonian H, contain magnetic field strength B, point in the z direction, can be written as, see [1].

$$H = -\frac{\hbar^2}{2m_N}\sum_i \boldsymbol{\nabla}_i^2 + \frac{\hbar^2}{2m_N}\left(\frac{eB}{\hbar}\right)^2 \sum_i \mathrm{y}_i^2 - \frac{\hbar^2}{2m_N}\left(\frac{eB}{\hbar}\right)\sum_i g_i\sigma_{zi}$$
$$+\sum_{i<j}\left(C_1 + C_3 P_{ij}^{\sigma}\right) e^{-a r_{ij}^2} + \sum_{i<j<k}\sum_{cyc} D_1 e^{-a\left(r_{ij}^2 + r_{jk}^2\right)}$$

Where $\frac{\hbar^2}{m_N} = 41.47\ Mev \cdot fm^2$ and $\frac{eB}{\hbar}$ is input parameter for the magnetic field. $r_{ij}^2 = x_{ij}^2 + y_{ij}^2 + z_{ij}^2$ are the distance between the pair ij , $x_{ij} = x_i - x_j$, and $P_{ij}^{\sigma} = \frac{1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{2}$. The LEC C_1, C_3, D [2] and the cutoff a are from our Effective field theory without magnetic field.

The basis function are $\Psi = \psi^x\psi^y\psi^z$ where $\psi^x = e^{-\frac{1}{2}\boldsymbol{x}^T A^x \boldsymbol{x}}$ etc. and $\boldsymbol{x} = (x_1 \ldots x_N)$.

In SVM we choose randomly distance between particles d_{ij} and A^x is symmetrical matrix translated from $\psi^x = exp\left(-\sum_{i<j}^N \frac{(x_i - x_j)^2}{2d_{ij}^2} - \sum_i^N \varepsilon_i x_i^2\right)$.

The analytical expressions of the matrix element of the spatial basis function of all part of the Hamiltonian are: (same expression for y, z coordinats of course) [3][4][5][6][7]

$$\langle \psi'^x | \psi^x \rangle = \sqrt{\frac{(2\pi)^N}{\det(A^x + A'^x)}}$$

$$\left\langle \psi'^x \middle| \frac{\partial^2}{\partial x_i^2} \middle| \psi^x \right\rangle = -\langle \psi'^x | \psi^x \rangle \sum_k^N A'^x_{ik} A^x_{ik} \left(A^x + A'^x\right)^{-1}_{kk}$$

$$\left\langle \psi'^x \middle| x_i^2 \middle| \psi^x \right\rangle = \langle \psi'^x | \psi^x \rangle \left(A^x + A'^x\right)^{-1}_{ii}$$

$$\left\langle \psi'^x \middle| e^{-a x_{ij}^2} \middle| \psi^x \right\rangle = \langle \psi'^x | \psi^x \rangle (2as + 1)^{-1/2}$$

$$\left\langle \psi'^x \middle| e^{-a\left(x_{ik}^2 + x_{jk}^2\right)} \middle| \psi^x \right\rangle = \langle \psi'^x | \psi^x \rangle (2aB + I)^{-1/2}$$

Where $s = \boldsymbol{C}_{ij}^T (A^x + A'^x)^{-1} \boldsymbol{C}_{ij}$ and $B = \begin{pmatrix} \boldsymbol{C}_{ik}^T (A^x + A'^x)^{-1} \boldsymbol{C}_{ik} & \boldsymbol{C}_{ik}^T (A^x + A'^x)^{-1} \boldsymbol{C}_{jk} \\ \boldsymbol{C}_{jk}^T (A^x + A'^x)^{-1} \boldsymbol{C}_{ik} & \boldsymbol{C}_{jk}^T (A^x + A'^x)^{-1} \boldsymbol{C}_{jk} \end{pmatrix}$

I is 2×2 unit matrix and $\boldsymbol{C}_{ij}^{T} = (0, .., \underbrace{1}_{i}, ... 0, ..., \underbrace{-1}_{j}, ..0)$.

[1] Let us assume the field, of strength B, point in the z direction. There are various choices for **A**, we choose here $\boldsymbol{A} = \left(-\frac{1}{2}By, \frac{1}{2}Bx, 0\right)$ we get

$$\frac{\hbar^2}{2m_N}\left(-i\boldsymbol{\nabla} - \frac{e}{\hbar}\boldsymbol{A}\right)^2 = -\frac{\hbar^2}{2m_N}\boldsymbol{\nabla}^2 + i\left(\frac{\hbar^2}{2m_N}\right)\left(\frac{eB}{\hbar}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \left(\frac{\hbar^2}{2m_N}\right)\left(\frac{eB}{\hbar}\right)^2\frac{1}{4}(x^2 + y^2)$$

The matrix element of the $y\frac{\partial}{\partial x}$ are zero because Gaussian function are symmetric.

$$\left\langle \Psi' \middle| y\frac{\partial}{\partial x} \middle| \Psi \right\rangle = \left\langle \psi'^x \psi'^y \psi'^z \middle| y\frac{\partial}{\partial x} \middle| \psi^x \psi^y \psi^z \right\rangle = \left\langle \psi'^x \middle| \frac{\partial}{\partial x} \middle| \psi^x \right\rangle \underbrace{\langle \psi'^y | y | \psi^y \rangle}_{=0} \langle \psi'^z | \psi^z \rangle = 0$$

[2] The energy of spin magnetic term is

$$E = -\gamma \boldsymbol{B} \cdot \boldsymbol{S} = -\gamma B \hbar \sigma_z$$

Where $\sigma_z = \pm\frac{1}{2}$ and $\gamma = \frac{g\mu}{\hbar}$ and $\mu = \frac{e\hbar}{2m_p}$ and $g_p = 5.586$, $g_n = -3.826$.

All together gives

$$-\frac{\hbar^2}{2m_N}\left(\frac{eB}{\hbar}\right) g\sigma_z$$

So if we set the minimal coupling and the contact potential to zero and choose $\left(\frac{eB}{\hbar}\right) = \left(\frac{\hbar^2}{m}\right)^{-1}$ then the energy of deuteron (np system) for example need to be

$$\begin{aligned} -0.5(0.5g_p + 0.5g_n) = -0.439 \qquad & for\ |\uparrow\uparrow\rangle \\ 0 \qquad & for \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ -0.5(-0.5g_p - 0.5g_n) = 0.439 \qquad & for\ |\downarrow\downarrow\rangle \end{aligned}$$

[3] The connection between our LEC to $C_{S,T}$ LEC are $C_1 = \frac{C_{1,0}+C_{0,1}}{2}$, $C_3 = \frac{C_{1,0}-C_{0,1}}{2}$.

[4]

[5]

[6]

[7]

[8]

[9] The dimension of the parameter $\frac{eB}{\hbar}$ is $length^{-2}$. So if $B = 1T$, the numerical value of $\frac{e}{\hbar}B$ in unit of fm^{-2} will be $\frac{1 \cdot 1.6 \cdot 10^{-19} \cdot 1}{6.6 \cdot 10^{-34}}(10^{-15})^2 = 2.4 \cdot 10^{-16}\, fm^{-2}$ so just large magnetic field like $10^{15}T$ will be significant. **$\boldsymbol{eB}/\hbar$ is $\boldsymbol{eB}$ in the input files**

[10] Identity

$$\sum_{i}^{n} y_i^2 = \frac{1}{n}\left[\left(\sum_{i}^{n} y_i\right)^2 + \sum_{i<j}\left(y_j - y_i\right)^2\right]$$

Or

$$\sum_{i}^{n} y_i^2 = nY^2 + \frac{1}{n}\sum_{i<j} y_{ij}^2$$

Where $y_{ij} = y_j - y_i$ *and* $Y = \frac{1}{n}\sum_i^n y_i$

In the same way

$$\sum_{i}\left(p_y\right)_i^2 = \frac{1}{n}P_y^2 + \frac{4}{n}\sum_{i<j} p_{ij}^2$$

Where $p_{ij} = \frac{p_j - p_i}{2}$ *and* $P_y = \sum_i^n p_i$

Such that $\left[Y, P_y\right] = i\hbar$ *and* $\left[y_{ij}, p_{ij}\right] = i\hbar$

Write $\frac{\hbar^2}{2m}\left(\frac{eB}{\hbar}\right)^2 = \frac{1}{2}m\omega^2$ *where* $\omega = \frac{eB}{m}$

We get for our Hamiltonian,

$$\frac{1}{2m}\sum_{i}\left(p_y\right)_i^2 + \frac{\hbar^2}{2m}\left(\frac{eB}{\hbar}\right)^2\sum_{i} y_i^2 = \frac{1}{2m}\left(\frac{1}{n}P_y^2 + \frac{4}{n}\sum_{i<j} p_{ij}^2\right) + \frac{1}{2}m\omega^2\left(nY^2 + \frac{1}{n}\sum_{i<j} y_{ij}^2\right)$$

$$= \frac{1}{2M}P_y^2 + \frac{1}{2}M\omega^2 Y^2 + \sum_{i<j}\left(\frac{1}{2\left(\frac{M}{4}\right)}p_{ij}^2 + \frac{1}{2}\left(\frac{M}{4}\right)\left(\frac{2\omega}{n}\right)^2 y_{ij}^2\right)$$

The ground state energy of the $\frac{1}{2M}P_y^2 + \frac{1}{2}M\omega^2 Y^2$ *is* $\frac{1}{2}\hbar\omega$ *and for each* $\frac{1}{2\left(\frac{M}{4}\right)}p_{ij}^2 + \frac{1}{2}\left(\frac{M}{4}\right)\left(\frac{2\omega}{n}\right)^2 y_{ij}^2$ *is* $\frac{1}{2}\hbar\left(\frac{2\omega}{n}\right)$ *and for all pair is* $(n-1)\frac{1}{2}\hbar\omega$ *or* $(\frac{n-1}{2})\left(\frac{\hbar^2}{m}\right)\left(\frac{eB}{\hbar}\right)$ *in terms of our parametes.*

So if we calculate the ground state for two, three, and four systems without the term $\frac{1}{2}M\omega^2 Y^2$*(the center of mass vanish..) in magnetic field along z direction and with zero contact interaction, if we choose* $\left(\frac{eB}{\hbar}\right) = \left(\frac{\hbar^2}{m}\right)^{-1}$ *we need to get 0.5, 1, 1.5 for deuteron triton and helium.*

If our Hamiltonian will be just

$$\frac{1}{2m}\sum_{i}(p_x)_i^2+\frac{1}{2m}\sum_{i}(p_z)_i^2+\frac{1}{2M}P_y^2+\sum_{i<j}\left(\frac{1}{2\left(\frac{M}{4}\right)}p_{ij}^2+\frac{1}{2}\left(\frac{M}{4}\right)\left(\frac{2\omega}{n}\right)^2y_{ij}^2\right)$$