Strong magnetic fields and contact interactions in few-fermion systems

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Technical manual detailing the implementation of a variational solution of the non-relativistic few-body problem in an external, *i.e.* , static magnetic field.

a. The symmetric Gauge

$$\boldsymbol{A}_i = \frac{B_0}{2}(-y(i), x(i), 0) \tag{1}$$

b. The Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{\mathtt{N}}\left\{\boldsymbol{\nabla}(i)^2 + i\left(\frac{\hbar^2}{2m}\right)\left(\frac{q(i)B_0}{\hbar}\right)L_z(i) + \left(\frac{\hbar^2}{2m}\right)\left(\frac{q(i)B_0}{\hbar}\right)^2\frac{1}{4}\left(x^2(i)+y^2(i)\right) - g(i)\left(\frac{\hbar^2}{2m}\right)\left(\frac{q(i)B_0}{\hbar}\right)\sigma_z(i)\right\} \tag{2}$$

$$+ \sum_{i<j}^{\mathtt{N}}\left(C_a + C_b\left[\sigma^+(i)\sigma^-(j) + \sigma^-(i)\sigma^+(j) - \sigma^z(i)\sigma^z(j)\right]\right)\ e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(j)]^2} + \sum_{\text{cyc.}}\sum_{i<j<k} D\cdot e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(j)]^2}e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(k)]^2} \tag{3}$$

c. The variational basis

$$\mid A,\alpha\ \rangle := e^{-\frac{1}{2}\boldsymbol{x}^T A_x \boldsymbol{x}}\ e^{-\frac{1}{2}\boldsymbol{y}^T A_y \boldsymbol{y}}\ e^{-\frac{1}{2}\boldsymbol{z}^T A_z \boldsymbol{z}}\cdot\sum_{n=1}^{\mathtt{ncmp}} C_\alpha^n \mid s_1^n,\ldots,s_{\mathtt{N}}^n\ ;\ t_1^n,\ldots,t_{\mathtt{N}}^n\ \rangle \tag{4}$$

with

$$\alpha = 1,\ \ldots,\ \mathtt{nts\_states}$$

d. The generic matrix element

$$I_{\mathcal{O}}(A',\alpha',A,\alpha;P) := \left\langle\ A',\alpha'\ \middle|\ \hat{\mathcal{O}}\otimes\hat{\mathcal{G}}\ \middle|\ \hat{P}(A),\hat{P}(\alpha)\ \right\rangle = \left\langle\ A'\ \middle|\ \hat{\mathcal{O}}\ \middle|\ \hat{P}(A)\ \right\rangle\cdot\left\langle\ \alpha'\ \middle|\ \hat{\mathcal{G}}\ \middle|\ \hat{P}(\alpha)\ \right\rangle \tag{5}$$

with $\hat{P}\in\mathcal{A}$, hence,

$$\hat{P}(A) = T_P^{\intercal}\,A\,T_P := A^P \quad . \tag{6}$$

$$\hat{\mathcal{O}} \in \left\{\mathbb{1}\ ;\ -\sum_{i=1}^{\mathtt{N}}\prod_{c=x,y,z}\boldsymbol{\partial}_c^{\intercal}(i)\mathbb{1}\boldsymbol{\partial}_c(i)\ ;\ \sum_{i=1}^{\mathtt{N}}q(i)L^z(i)\ ;\ \sum_{i=1}^{\mathtt{N}}q(i)(\omega_x x^2(i)+\omega_y y^2(i)+\omega_z z^2(i))\ ;\ \sum_{i=1}^{\mathtt{N}}q(i)\sigma^z(i)\ ;\ \sum_{i<j}^{\mathtt{N}}e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(j)]^2}\right\} \tag{7}$$

e. *The matrix elements*

$\hat{\mathcal{O}}$	$\left\langle A' \,\vert\, \hat{\mathcal{O}} \,\vert\, \hat{P}(A) \right\rangle$	$\left\langle \alpha' \,\vert\, \hat{\mathcal{G}} \,\vert\, \hat{P}(\alpha) \right\rangle$
$\mathbb{1} := \mathbb{1}^P_{\boldsymbol{r}} \cdot \mathbb{1}^P_{\boldsymbol{s}}$	$\left(\frac{(2\pi)^{3\mathtt{N}}}{\det \mathbb{A}_x \det \mathbb{A}_y \det \mathbb{A}_z}\right)^{\frac{1}{2}}$	$\sum_{n,n'}^{\mathtt{ncmp}(\alpha),\mathtt{ncmp}(\alpha')} C^n_\alpha C^{n'}_{\alpha'} \left\langle \boldsymbol{s}^{n'};\boldsymbol{t}^{n'} \,\vert\, \hat{P}(\boldsymbol{s}^n);\hat{P}(\boldsymbol{t}^n) \right\rangle$
$-\sum_{i=1}^{\mathtt{N}} \prod_{c=x,y,z} \boldsymbol{\partial}^\intercal_c(i)\mathbb{1}\boldsymbol{\partial}_c(i) \cdot \mathbb{1}^P_{\boldsymbol{s}}$	$\mathbb{1}^P_{\boldsymbol{r}} \cdot \prod_{c=x,y,z} (A_c)_{im}(\mathbb{A}_c^{-1})_{mn}(A^P_c)_{ni}$	$\mathbb{1}^P_{\boldsymbol{s}}$
$\sum_{i=1}^{\mathtt{N}} q(i)L_z(i) = q(i)\left[x(i)\partial_y(i) - y(i)\partial_x(i)\right]$	0	$\mathbb{Q}^P_{\boldsymbol{s}} := \sum_{i=1}^{\mathtt{N}} \sum_{n,n'}^{\mathtt{ncmp}(\alpha),\mathtt{ncmp}(\alpha')} \left[2 - \boldsymbol{t}^n(\hat{P}(i))\right] C^n_\alpha C^{n'}_{\alpha'} \left\langle \boldsymbol{s}^{n'};\boldsymbol{t}^{n'} \,\vert\, \hat{P}(\boldsymbol{s}^n);\hat{P}(\boldsymbol{t}^n) \right\rangle$
$\sum_{i=1}^{\mathtt{N}} q(i)(\omega_x x^2(i) + \omega_y y^2(i) + \omega_z z^2(i))$	$\mathbb{1}^P_{\boldsymbol{r}} \cdot \prod_{c=x,y,z} \omega_c \sum_{i=1}^{\mathtt{N}} (\mathbb{A}_c^{-1})_{ii}$	$\mathbb{Q}^P_{\boldsymbol{s}}$
$\sum_{i=1}^{\mathtt{N}} q(i)\sigma^z(i)$	$\mathbb{1}^P_{\boldsymbol{r}}$	$\sum_{i=1}^{\mathtt{N}} \sum_{n,n'}^{\mathtt{ncmp}(\alpha),\mathtt{ncmp}(\alpha')} \left[2 - \boldsymbol{t}^n(\hat{P}(i))\right] \left[3 - 2\boldsymbol{s}^n(\hat{P}(i))\right] C^n_\alpha C^{n'}_{\alpha'} \left\langle \boldsymbol{s}^{n'};\boldsymbol{t}^{n'} \,\vert\, \hat{P}(\boldsymbol{s}^n);\hat{P}(\boldsymbol{t}^n) \right\rangle$
$\sum_{i<j}^{\mathtt{N}} e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(j)]^2}$		
$\sum_{\text{cyc.}} \sum_{i<j<k} e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(j)]^2} e^{-\frac{\Lambda^2}{4}[\boldsymbol{r}(i)-\boldsymbol{r}(k)]^2}$		

(8)

with

$$\mathbb{A}_x = A'_x + A^P_x \tag{9}$$

$$\frac{\Lambda^2}{4} = \mathtt{apot} \tag{10}$$

Spin-isospin wave function for the deuteron:

$$\left| \alpha \hat{=} {}^3S_1; m_S = \begin{Bmatrix} 0 \\ \pm 1 \end{Bmatrix} \right\rangle = \left[\begin{Bmatrix} \vert\, \mathtt{12} \,\rangle + \vert\, \mathtt{21} \,\rangle \\ \vert\, \mathtt{11/22} \,\rangle + \vert\, \mathtt{11/22} \,\rangle \end{Bmatrix} \right] \otimes \left[\vert\, \mathtt{pn} \,\rangle - \vert\, \mathtt{np} \,\rangle \right] \tag{11}$$

$$\boldsymbol{s}^n(i) = \begin{cases} 1 \;\Rightarrow\; \sigma^z(i) = \uparrow \\ 2 \;\Rightarrow\; \sigma^z(i) = \downarrow \end{cases} \quad \text{and} \quad \boldsymbol{t}^n(i) = \begin{cases} 1 \;\Rightarrow\; \tau^z(i) = \mathtt{p} \\ 2 \;\Rightarrow\; \tau^z(i) = \mathtt{n} \end{cases} \tag{12}$$

#	O1b_#	O2b_#
0	$\left\langle \boldsymbol{\xi}_1^{n'}, \ldots, \boldsymbol{\xi}_N^{n'} \;\middle|\; \boldsymbol{\xi}_{\mathfrak{P}_1}^{n}, \ldots, \boldsymbol{\xi}_{\mathfrak{P}_N}^{n} \right\rangle$	$\left\langle \boldsymbol{\xi}_1^{n'}, \ldots, \boldsymbol{\xi}_N^{n'} \;\middle|\; \boldsymbol{\xi}_{\mathfrak{P}_1}^{n}, \ldots, \boldsymbol{\xi}_{\mathfrak{P}_N}^{n} \right\rangle$
1	$--$	$\left\langle \boldsymbol{\xi}_1^{n'}, \ldots, \boldsymbol{\xi}_i^{n'}, \ldots, \boldsymbol{\xi}_j^{n'}, \ldots, \boldsymbol{\xi}_N^{n'} \;\middle|\; \boldsymbol{\xi}_{\mathfrak{P}_1}^{n}, \ldots, \boldsymbol{\xi}_{\mathfrak{P}_j}^{n}, \ldots, \boldsymbol{\xi}_{\mathfrak{P}_i}^{n}, \ldots, \boldsymbol{\xi}_{\mathfrak{P}_N}^{n} \right\rangle$

TABLE I. Two basic operations are performed by the interaction Hamiltinian Eq.(2), namely, the identity, and the exchange.

`Operators.cpp`

$$\texttt{iop} = 0: \quad \mathbb{1} \qquad \text{(Wigner)} \tag{13}$$

$$\texttt{iop} = 1: \quad -\frac{1}{4}\left[\mathbb{1} + \boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)\right]\left[\mathbb{1} + \boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j)\right] \qquad \text{(Majorana)} \tag{14}$$

$$\texttt{iop} = 2: \quad \frac{1}{2}\left[\mathbb{1} + \boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j)\right] \qquad \text{(Bartlett)} \tag{15}$$

$$\texttt{iop} = 3: \quad -\frac{1}{2}\left[\mathbb{1} + \boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)\right] \qquad \text{(Heisenberg)} \tag{16}$$