**Aperitif**

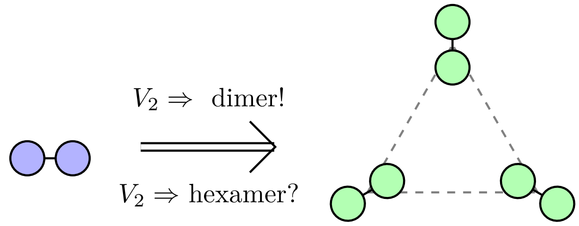
—"*The structure of dynamical equations is desceptively simple, obfuscating the marvelous phenomena which they point us to in nature decades and conturies post their conception.*"

a) The goal is the investigation of $N$-body systems and how linked/disentangled they behave with respect to their subsystems

b) The elementary instance we want to analyze–empirically, at fisrt–is the celestial $N$-body problem with the trajectories of $M < N$ of its constituents being fixed (we aim from the start to generalize the text-book problem [1, 2] which deals with 3 objects of which 2 are fixed).

c) Let us tackle the restricted 5-body problem and commence by formulating the pertinent equations of motion. We *chose* to fix the orbits of 4 of the 5 objects to circular trajectories about the origin, which is, the centre of mass.

d) The coordinates and the velocity of the 5th object comprise the only degrees of freedom of the system, and we choose to start from a Lagrangean formulated in cartesian coordinates as we expect this approach to generalize straight-forwardly to larger $N$ and $M$.




a)

b) $N$ point masses with $\{(m_i\ ,\ \mathbf{r}_i)\quad \text{for}\quad i = 1, \ldots, N\}$ and a fixed time dependence for $M$ of those objects: $\mathbf{r}_{i=1,\ldots,M}(t) \equiv \boldsymbol{\rho}_i(t)$

c) I suggest to use centre-of-mass coordinates:

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i\,\mathbf{r}_i \quad \text{and} \quad \tilde{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{R} \quad \text{for} \quad i = 1, \ldots, N \ , \tag{CM}$$

and

$$\tilde{\boldsymbol{\rho}}_i(t) = \rho_i \begin{pmatrix} \cos(\omega_i t) \\ \sin(\omega_i t) \\ 0 \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, M \quad . \tag{FC}$$

d) $\mathfrak{L} = T - V = \frac{m_5}{2}\left(\dot{\tilde{r}}_5\right)^2 - \gamma \sum_{i=1}^{4} \frac{m_5 m_i}{|\tilde{\mathbf{r}}_5 - \tilde{\rho}_i|}$

Derive the Hamiltonian as the Legendre transform of Lagrangean

$$\mathfrak{H} = \dot{\tilde{\mathbf{r}}}_5 \cdot \mathbf{p}_5 - \mathfrak{L}$$

and express it in terms of coordinates and conjugate momenta

$$\mathbf{p}_5 = \frac{\partial \mathfrak{L}}{\partial \dot{\tilde{\mathbf{r}}}_5} \quad .$$

Subsequently, derive the equations of motion:

$$\dot{\tilde{\mathbf{r}}}_5 = \frac{\partial \mathfrak{H}}{\partial \mathbf{p}_5} \quad \text{and} \quad \dot{\mathbf{p}}_5 = -\frac{\partial \mathfrak{H}}{\partial \tilde{\mathbf{r}}_5} \quad . \tag{EQM}$$

Ecce, these six equations will look quite ugly but thankfully it will be the computer who will do the heavy lifting.

**Primo piatto**

For the general $(N-M)$-body problem, the equations of motion are conveniently(?) written in matrix form. The vectors $\mathbf{y}$ and $\mathbf{f}$ have $2 \times \dim_{\text{space}} \times A$ components and we chose the letters to conform with [1]. Also note, that in the summation over $j$ we need to consider all objects, i.e., the constrained ones, too, because they still exert a force on the unconstrained masses. For $\dim_{\text{space}} = 3$, the equations of motion read explicitly

$$
\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} := \frac{\mathrm{d}}{\mathrm{d}t}
\begin{pmatrix}
r_{x1} \\ r_{y1} \\ r_{z1} \\ \vdots \\ r_{xA} \\ r_{yA} \\ r_{zA} \\ p_{x1} \\ p_{y1} \\ p_{z1} \\ \vdots \\ p_{xA} \\ p_{yA} \\ p_{zA}
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial \mathfrak{H}}{\partial p_{x1}} \\ \\ \\ \vdots \\ \\ \\ \frac{\partial \mathfrak{H}}{\partial p_{zA}} \\ -\frac{\partial \mathfrak{H}}{\partial r_{x1}} \\ \\ \\ \vdots \\ \\ \\ -\frac{\partial \mathfrak{H}}{\partial r_{zA}}
\end{pmatrix}
=
\begin{pmatrix}
{}^{p_{x1}}\!/_{m_1} \\ {}^{p_{y1}}\!/_{m_1} \\ {}^{p_{z1}}\!/_{m_1} \\ \vdots \\ {}^{p_{xN}}\!/_{m_A} \\ {}^{p_{yN}}\!/_{m_A} \\ {}^{p_{zN}}\!/_{m_A} \\
\gamma m_1 \sum_{j\neq 1}^{N} \frac{(r_{x1}-r_{xj})m_j}{|\mathbf{r}_1-\mathbf{r}_j|^3} \\
\gamma m_1 \sum_{j\neq 1}^{N} \frac{(r_{y1}-r_{yj})m_j}{|\mathbf{r}_1-\mathbf{r}_j|^3} \\ \\ \vdots \\ \\ \\
\gamma m_N \sum_{j\neq A} \frac{(r_{zA}-r_{zj})m_j}{|\mathbf{r}_A-\mathbf{r}_j|^3}
\end{pmatrix}
=
\begin{pmatrix}
{}^{y_{3A+1}}\!/_{m_1} \\ {}^{y_{3A+2}}\!/_{m_1} \\ {}^{y_{3A+3}}\!/_{m_1} \\ \\ \\ \vdots \\ \\
\gamma m_1 \sum_{j\neq 1}^{N} \frac{(y_1-y_{3j-2})m_j}{\left|(y_1-y_{3j-2})^2+(y_2-y_{3j-1})^2+(y_3-y_{3j})^2\right|^{3/2}} \\ \\ \\ \vdots \\ \\ \\
\gamma m_1 \sum_{j\neq A} \frac{(y_{3A}-y_{3j})m_j}{\left|(y_{3A-2}-y_{3j-2})^2+(y_{3A-1}-y_{3j-1})^2+(y_{3A}-y_{3j})^2\right|^{3/2}}
\end{pmatrix}
=: \mathbf{f}(\mathbf{y},t)
\tag{EQM}
$$

Obtain $\mathbf{y}(t)$ for a temporal grid $t \in \{h, 2h, \ldots, nh\}$ using a computer program which takes as input:

- the masses $m_i$ of the $N$ objects,
- the initial positions $\mathbf{r}_i(0)$ and velocities $\dot{\mathbf{r}}_i(0)$ of the $A$ unconstrained objects,
- the fixed orbits $\{\mathbf{y}_i(t)\}$ with $i \in \{3A+1, \ldots, \}$ of the $M$ constrained objects, and
- the time-grid parameters $h$ and $n$.

# Nomenclature

$\tilde{\mathbf{r}}$ Bold symbols denote, if not stated otherwise, three-dimensional vectors, e.g., $\mathbf{r} = (x, y, z)$, the length/magnitude of a vector $r := |\mathbf{r}|$, and the tilde refers to vectors in the centre-of-mass frame.

$N, M, A$ Of $N$ objects, the orbits of $M$ are fixed which leaves $N - M := A$ trajectories for which we seek a solution.

dimer With *dimer*, we refer to a pair of particles which are *loosely* bound, i.e., a small disturbance suffices to induce a transition from a trajectory on which the relative distance between them does not diverge when $t \to \infty$ to one on which it does.

M total mass of the system, $M = \sum_{i=1}^{N} m_i$

# References


[1] E. W. Schmid, G. Spitz, and W. Lösch, "The celestial mechanics three-body problem," in *Theoretical Physics on the Personal Computer* (Springer Berlin Heidelberg, Berlin, Heidelberg, 1990) pp. 81–89.

[2] H. Geiges, "The three-body problem," in *The Geometry of Celestial Mechanics*, London Mathematical Society Student Texts (Cambridge University Press, 2016) p. 77–100.