To make this point clear, let us rewrite the integral in a way that emphasizes its dependence on the momentum variable p:

$$\langle q_{\mathrm{f}}|e^{-i\hat{H}t/\hbar}|q_{\mathrm{i}}\rangle = \int_{\substack{q(t)=q_{\mathrm{f}}\\ q(0)=q_{\mathrm{i}}}} Dq\ e^{-\frac{i}{\hbar}\int_0^t dt' V(q)} \int Dp\ e^{-\frac{i}{\hbar}\int_0^t dt'\left(\frac{p^2}{2m}-p\dot{q}\right)}. \tag{3.7}$$

The exponent is quadratic in p which means that we are dealing with the continuum generalization of a Gaussian integral. Carrying out the integration by means of Eq. (3.13) below, one obtains

$$\boxed{\langle q_{\mathrm{f}}|e^{-i\hat{H}t/\hbar}|q_{\mathrm{i}}\rangle = \int_{\substack{q(t)=q_{\mathrm{f}}\\ q(0)=q_{\mathrm{i}}}} Dq\ \exp\left[\frac{i}{\hbar}\int_0^t dt' L(q,\dot{q})\right],} \tag{3.8}$$

where $Dq = \lim_{N\to\infty}\left(\frac{Nm}{it2\pi\hbar}\right)^{N/2}\prod_{n=1}^{N-1} dq_n$ denotes the functional measure of the remaining q-integration, and $L(q,\dot{q}) = m\dot{q}^2/2 - V(q)$ represents the classical Lagrangian. Strictly speaking, the (finite-dimensional) integral formula (3.13) is not directly applicable to the infinite-dimensional Gaussian integral (3.7). This, however, does not represent a substantial problem as we can always rediscretize the integral (3.7), apply Eq. (3.13), and reinstate the continuum limit after integration (exercise).

Together Eq. (3.6) and (3.8) represent the central results of this section. A quantum mechanical transition amplitude has been expressed in terms of an infinite-dimensional integral extending over paths through phase space, Eq. (3.6), or coordinate space, Eq. (3.8). All paths begin (end) at the initial (final) coordinate of the matrix element. Each path is weighted by its *classical* action. Notice in particular that the quantum transition amplitude has been cast in a form that does not contain quantum mechanical operators. Nonetheless, quantum mechanics is still fully present! The point is that the integration extends over all paths and not just the subset of solutions of the classical equations of motion. (The distinguished role classical paths play in the path integral will be discussed below in Section 3.2.) The two forms of the path integral, Eq. (3.6) and Eq. (3.8), represent the formal implementation of the "alternative picture" of quantum mechanics proposed heuristically at the beginning of the chapter.

Johann Carl Friedrich Gauss
1777–1855

Worked in a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy, and optics. As well as several books, Gauss published a number of memoirs (reports of his experiences), mainly in the journal of the Royal Society of Göttingen. However, in general, he was unwilling to publish anything that could be regarded as controversial and, as a result, some of his most brilliant work was found only after his death.

INFO **Gaussian integration**: Apart from a few rare exceptions, all integrals encountered in this book will be of Gaussian form. In most cases the dimension of the integrals will be large if not infinite. Yet, after a bit of practice, it will become clear that high-dimensional representatives of Gaussian integrals are no more difficult to handle than their one-dimensional counterparts.

Therefore, considering the important role played by Gaussian integration in field theory, we will here derive the principal formulae once and for all. Our starting point is the one-dimensional integral (both real and complex). The ideas underlying the proofs of the one-dimensional formulae will provide the key to the derivation of more complex functional identities that will be used liberally throughout the remainder of the text.

One-dimensional Gaussian integral: The basic ancestor of all Gaussian integrals is the identity

$$\boxed{\int_{-\infty}^{\infty} dx\; e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}},\;\; \mathrm{Re}\, a > 0.} \tag{3.9}$$

In the following we will need various generalizations of Eq. (3.9). Firstly, we have $\int_{-\infty}^{\infty} dx\; e^{-ax^2/2}x^2 = \sqrt{2\pi/a^3}$, a result established either by substituting $a \to a + \epsilon$ in Eq. (3.9) and expanding both the left and the right side of the equation to leading order in ϵ, or by differentiating Eq. (3.9). Often one encounters integrals where the exponent is not purely quadratic from the outset but rather contains both quadratic and linear pieces. The generalization of Eq. (3.9) to this case reads

$$\int_{-\infty}^{\infty} dx\; e^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}}. \tag{3.10}$$

To prove this identity, one simply eliminates the linear term by means of the change of variables $x \to x + b/a$ which transforms the exponent $-ax^2/2 + bx \to -ax^2/2 + b^2/2a$. The constant factor scales out and we are left with Eq. (3.9). Note that Eq. (3.10) holds even for complex b. The reason is that as a result of shifting the integration contour into the complex plane no singularities are encountered, i.e. the integral remains invariant.

Later, we will be concerned with the generalization of the Gaussian integral to complex arguments. The extension of Eq. (3.9) to this case reads

$$\int d(\bar{z}, z)\; e^{-\bar{z}wz} = \frac{\pi}{w}, \quad \mathrm{Re}\, w > 0,$$

where $\bar{z}$ represents the complex conjugate of z. Here, $\int d(\bar{z}, z) \equiv \int_{-\infty}^{\infty} dx\; dy$ represents the independent integration over the real and imaginary parts of $z = x + iy$. The identity is easy to prove: owing to the fact that $\bar{z}z = x^2 + y^2$, the integral factorizes into two pieces, each of which is equivalent to Eq. (3.9) with $a = w$. Similarly, it may be checked that the complex generalization of Eq. (3.10) is given by

$$\int d(\bar{z}, z)\, e^{-\bar{z}wz+\bar{u}z+\bar{z}v} = \frac{\pi}{w}e^{\frac{\bar{u}v}{w}}, \quad \mathrm{Re}\, w > 0. \tag{3.11}$$

More importantly $\bar{u}$ and v may be independent complex numbers; they need not be related to each other by complex conjugation (exercise).

Gaussian integration in more than one dimension: All of the integrals above have higher-dimensional counterparts. Although the real and complex versions of the N-dimensional integral formulae can be derived in a perfectly analogous manner, it is better to discuss them separately in order not to confuse the notation.

(a) **Real case**: The multi-dimensional generalization of the prototype integral (3.9) reads

$$\int d\mathbf{v}\; e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = (2\pi)^{N/2}\det \mathbf{A}^{-1/2}, \tag{3.12}$$

where $\mathbf{A}$ is a positive definite real symmetric N-dimensional matrix and $\mathbf{v}$ is an N-component real vector. The proof makes use of the fact that $\mathbf{A}$ (by virtue of being symmetric) can be diagonalized by orthogonal transformation, $\mathbf{A} = \mathbf{O}^T\mathbf{D}\mathbf{O}$, where the matrix $\mathbf{O}$ is orthogonal, and all elements of the diagonal matrix $\mathbf{D}$ are positive. The matrix $\mathbf{O}$ can be absorbed into the integration vector by means of the variable transformation, $\mathbf{v} \mapsto \mathbf{O}\mathbf{v}$, which has unit Jacobian, $\det \mathbf{O} = 1$. As a result, we are left with a Gaussian integral with exponent $-\mathbf{v}^T\mathbf{D}\mathbf{v}/2$. Due to the diagonality of $\mathbf{D}$, the integral factorizes into N independent Gaussian integrals, each of which contributes a factor $\sqrt{2\pi/d_i}$, where $d_i, i = 1, \ldots, N$, is the ith entry of the matrix $\mathbf{D}$. Noting that $\prod_{i=1}^N d_i = \det \mathbf{D} = \det \mathbf{A}$, Eq. (3.12) is derived.
The multi-dimensional generalization of Eq. (3.10) reads

$$\boxed{\int d\mathbf{v}\; e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\cdot\mathbf{v}} = (2\pi)^{N/2}\det \mathbf{A}^{-1/2} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}},} \tag{3.13}$$

where $\mathbf{j}$ is an arbitrary N-component vector. Equation (3.13) is proven by analogy with Eq. (3.10), i.e. by shifting the integration vector according to $\mathbf{v} \to \mathbf{v} + \mathbf{A}^{-1}\mathbf{j}$, which does not change the value of the integral but removes the linear term from the exponent, $-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} + \mathbf{j}^T\cdot\mathbf{v} \to -\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} + \frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}$. The resulting integral is of the type (3.12), and we arrive at Eq. (3.13).
The integral (3.13) not only is of importance in its own right, but also serves as a "generator" of other useful integral identities. Applying the differentiation operation $\partial^2_{j_m j_n}|_{\mathbf{j}=0}$ to the left- and the right-hand side of Eq. (3.13), one obtains the identity[7] $\int d\mathbf{v}\; e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} v_m v_n = (2\pi)^{N/2}\det \mathbf{A}^{-1/2} A^{-1}_{mn}$. This result can be more compactly formulated as

$$\langle v_m v_n \rangle = A^{-1}_{mn}, \tag{3.14}$$

where we have introduced the shorthand notation

$$\langle \cdots \rangle \equiv (2\pi)^{-N/2}\det \mathbf{A}^{1/2} \int d\mathbf{v}\; e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} (\cdots), \tag{3.15}$$

suggesting an interpretation of the Gaussian weight as a probability distribution.
Indeed, the differentiation operation leading to Eq. (3.14) can be iterated. Differentiating four times, one obtains $\langle v_m v_n v_q v_p \rangle = A^{-1}_{mn}A^{-1}_{qp} + A^{-1}_{mq}A^{-1}_{np} + A^{-1}_{mp}A^{-1}_{nq}$. One way of memorizing the structure of this – important – identity is that the Gaussian "expectation" value $\langle v_m v_n v_p v_q \rangle$ is given by all "pairings" of type (3.14) that can be formed from the four components v_m. This rule generalizes to expectation values of arbitrary order: $2n$-fold differentiation of Eq. (3.13) yields

$$\boxed{\langle v_{i_1} v_{i_2} \ldots v_{i_{2n}} \rangle = \sum_{\substack{\text{pairings of} \\ \{i_1,\ldots,i_{2n}\}}} A^{-1}_{i_{k_1} i_{k_2}} \ldots A^{-1}_{i_{k_{2n-1}} i_{k_{2n}}}.} \tag{3.16}$$

This result is the mathematical identity underlying **Wick's theorem** (for real bosonic fields), to be discussed in more physical terms below.

[7] Note that the notation A^{-1}_{mn} refers to the mn-element of the matrix $\mathbf{A}^{-1}$.

(b) **Complex case**: The results above are straightforwardly extended to multi-dimensional complex Gaussian integrals. The complex version of Eq. (3.12) is given by

$$\int d(\mathbf{v}^\dagger, \mathbf{v})\ e^{-\mathbf{v}^\dagger \mathbf{A}\mathbf{v}} = \pi^N \det \mathbf{A}^{-1}, \tag{3.17}$$

where $\mathbf{v}$ is a complex N-component vector, $d(\mathbf{v}^\dagger, \mathbf{v}) \equiv \prod_{i=1}^N d\ \mathrm{Re}\ v_i\ d\ \mathrm{Im}\ v_i$, and $\mathbf{A}$ is a complex matrix with positive definite Hermitian part. (Remember that every matrix can be decomposed into a Hermitian and an anti-Hermitian component, $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\dagger) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^\dagger)$.) For Hermitian $\mathbf{A}$, the proof of Eq. (3.17) is analogous to that of Eq. (3.12), i.e. $\mathbf{A}$ is unitarily diagonalizable, $\mathbf{A} = \mathbf{U}^\dagger \mathbf{A}\mathbf{U}$, the matrices $\mathbf{U}$ can be transformed into $\mathbf{v}$, the resulting integral factorizes, etc. For non-Hermitian $\mathbf{A}$ the proof is more elaborate, if unedifying, and we refer to the literature for details. The generalization of Eq. (3.17) to exponents with linear contributions reads

$$\boxed{\int d(\mathbf{v}^\dagger, \mathbf{v})\ e^{-\mathbf{v}^\dagger \mathbf{A}\mathbf{v} + \mathbf{w}^\dagger \cdot \mathbf{v} + \mathbf{v}^\dagger \cdot \mathbf{w}'} = \pi^N \det \mathbf{A}^{-1} e^{\mathbf{w}^\dagger \mathbf{A}^{-1} \mathbf{w}'}.} \tag{3.18}$$

Note that $\mathbf{w}$ and $\mathbf{w}'$ may be independent complex vectors. The proof of this identity mirrors that of Eq. (3.13), i.e. by effecting the shift $\mathbf{v}^\dagger \to \mathbf{v}^\dagger + \mathbf{w}^\dagger$, $\mathbf{v} \to \mathbf{v} + \mathbf{w}'$.[8] As with Eq. (3.13), Eq. (3.18) may also serve as a generator of integral identities. Differentiating the integral twice according to $\partial^2_{w'_m, \bar{w}_n}|_{\mathbf{w}=\mathbf{w}'=0}$ gives

$$\langle \bar{v}_m v_n \rangle = A^{-1}_{nm},$$

where $\langle \cdots \rangle \equiv \pi^{-N} \det \mathbf{A} \int d(\mathbf{v}^\dagger, \mathbf{v})\ e^{-\mathbf{v}^\dagger \mathbf{A}\mathbf{v}} (\cdots)$. The iteration to more than two derivatives gives $\langle \bar{v}_n \bar{v}_m v_p v_q \rangle = A^{-1}_{pm} A^{-1}_{qn} + A^{-1}_{pn} A^{-1}_{qm}$ and, eventually,

$$\boxed{\langle \bar{v}_{i_1} \bar{v}_{i_2} \cdots \bar{v}_{i_n} v_{j_1} v_{j_2} \cdots v_{j_n} \rangle = \sum_P A^{-1}_{j_1 i_{P1}} \cdots A^{-1}_{j_n i_{Pn}},}$$

where $\sum_P$ represents for the sum over all permutations of n integers.

Gaussian functional integration: With this preparation, we are in a position to investigate the main practice of quantum and statistical field theory – the method of Gaussian functional integration. Turning to Eq. (3.13), let us suppose that the components of the vector $\mathbf{v}$ parameterize the weight of a real scalar field on the sites of a one-dimensional lattice. In the continuum limit, the set $\{v_i\}$ translates to a function $v(x)$, and the matrix A_{ij} is replaced by an **operator** kernel or **propagator** $A(x, x')$. In this limit, the natural generalization of Eq. (3.13) is

$$\begin{aligned} \int Dv(x)\ \exp\left[-\frac{1}{2}\int dx\ dx'\ v(x)A(x,x')v(x') + \int dx\ j(x)v(x)\right] \\ \propto (\det A)^{-1/2} \exp\left[\frac{1}{2}\int dx\ dx'\ j(x)A^{-1}(x,x')j(x')\right], \end{aligned} \tag{3.19}$$

[8] For an explanation of why $\mathbf{v}$ and $\mathbf{v}^\dagger$ may be shifted independently of each other, cf. the analyticity remarks made in connection with Eq. (3.11).

where the inverse kernel $A^{-1}(x, x')$ satisfies the equation

$$\boxed{\int dx' \; A(x, x')A^{-1}(x', x'') = \delta(x - x''),} \tag{3.20}$$

i.e. $A^{-1}(x, x')$ can be interpreted as the **Green function** of the operator $A(x, x')$. The notation $Dv(x)$ is used to denote the measure of the functional integral. Although the constant of proportionality, $(2\pi)^N$, left out of Eq. (3.19) is formally divergent in the thermodynamic limit $N \to \infty$, it does not affect averages that are obtained from derivatives of such integrals. For example, for Gaussian distributed functions, Eq. (3.14) has the generalization

$$\boxed{\langle v(x)v(x')\rangle = A^{-1}(x, x').}$$

Accordingly, Eq. (3.16) assumes the form

$$\boxed{\langle v(x_1)v(x_2)\cdots v(x_{2n})\rangle = \sum_{\substack{\text{pairings of}\\ \{x_1,\ldots,x_{2n}\}}} A^{-1}(x_{k_1}, x_{k_2})\cdots A^{-1}(x_{k_{2n-1}}, x_{k_{2n}}).} \tag{3.21}$$

The generalization of the other Gaussian averaging formulae discussed above should be obvious.

To make sense of Eq. (3.19) one must interpret the meaning of the determinant, $\det A$. When the variables entering the Gaussian integral were discrete, the integral simply represented the determinant of the (real symmetric) matrix. In the present case, one must interpret A as a Hermitian operator having an infinite set of eigenvalues. The determinant simply represents the product over this infinite set (see, e.g., Section 3.3).

Before turning to specific applications of the Feynman path integral, let us stay with the general structure of the formalism and identify two fundamental connections of the path integral to *classical point mechanics* and *classical and quantum statistical mechanics.*

Path integral and statistical mechanics

The path integral reveals a connection between quantum mechanics and classical (and quantum) statistical mechanics whose importance to all areas of field theory and statistical physics can hardly be exaggerated. To reveal this link, let us for a moment forget about quantum mechanics and consider, by way of an example, a perfectly classical, one-dimensional continuum model describing a “flexible string.” We assume that our string is held under constant tension, and confined to a “gutter-like potential” (as shown in Fig. 3.2). For simplicity, we also assume that the mass density of the string is pretty high, so that its fluctuations are “asymptotically slow” (the kinetic contribution to its energy is negligible). Transverse fluctuations of the string are then penalized by its line tension, and by the external potential.

Assuming that the transverse displacement of the string $u(x)$ is small, the potential energy stored in the string separates into two parts. The first arises from the line tension stored in the string, and the second comes from the external potential. Starting with the former, a transverse fluctuation of a line segment of length dx by an amount du leads to a potential