$\boxed{\mathrm{I}}$

$$\partial_{\underline{r}}^2 \, e^{-\alpha \underline{r}^2 - \beta \underline{r}\cdot\underline{r}' - \gamma \underline{r}'^2} = \partial_{\underline{r}} \left(-2\alpha \underline{r} - \beta \underline{r}'\right) e^{-\cdots}$$
$$= \left(-2\alpha + 4\alpha^2 \underline{r}^2 + \beta^2 r'^2 + 4\alpha\beta \underline{r}\cdot\underline{r}'\right) e^{\cdots}$$

---

$$e^{-\beta \underline{r}\cdot\underline{r}'} = 4\pi \sum_{LM} i^L j_L(i\beta r r') Y^*_{LM}(\hat{r}') Y_{LM}(\hat{r})$$

$$\underline{r}\cdot\underline{r}' = -\sqrt{3}\left[r_p \otimes r'_q\right]^{00} = -\sqrt{3}\Big(\underbrace{(1010|00)}_{-\frac{1}{\sqrt{3}}} r_0 r'_0 + \underbrace{(111-1|00)}_{\frac{1}{\sqrt{3}}} r_+ r'_- + \underbrace{(1-111|00)}_{\frac{1}{\sqrt{3}}} r_- r'_+\Big)$$

$$= \frac{4\pi}{3} r r' \left(-Y_{10} Y'_{10} + Y_{1+1} Y'_{1-1} + Y_{1-1} Y'_{1+1}\right)$$

$\int d\hat{r}\, Y^*_{l_1 m_1} Y_{l_2 m_2} = \delta_{l_1 l_2} \delta_{m_1 m_2}$

$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3 j'_3} \delta_{m_3 m'_3} \delta(j_1 j_2 j_3)$

---

$$= \left[\left(4\alpha^2 \underline{r}^2 + \beta^2 \underline{r}'^2 - 2\alpha\right) + 4\alpha\beta \frac{4\pi}{3} r r' \left(Y_{10} Y'_{10} - Y_{11} Y'_{1-1} - Y_{1-1} Y'_{11}\right)\right] 4\pi i^L j_L(i\beta r r') Y^{*\prime}_{LM} Y_{LM}$$
$$\cdot e^{-\alpha r^2 - \beta r'^2}$$

$$= \sum_{LM} 4\pi i^L j_L(i\beta r r') e^{-\alpha r^2 - \beta r'^2} \; Y^{*\prime}_{LM} Y_{LM} \cdot \Big\{ \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right)$$
$$+ \left(Y'_{10} Y_{10} - Y'_{1+} Y_{1-} - Y'_{1-} Y_{1+}\right) \frac{16\pi}{3} \alpha\beta r r' \Big\}.$$

$$\int d\hat{r}' \int dr' r'^2 \cdot \int d\hat{r}\, Y^*_{l'm'} \quad \cdot \frac{1}{r'} \sum_{lm} Y'_{lm}$$

(a) $\sum_{\substack{LM \\ lm}} Y^{*\prime}_{LM} Y_{LM} Y^*_{l'm'} Y'_{lm} \longrightarrow \delta_{Ll} \delta_{Mm}$

(b) $\sum_{\substack{LM \\ lm}} Y^{*\prime}_{LM} Y^*_{l'm'} Y_{LM} Y'_{lm} \left(Y_{10} Y'_{10} - Y_{1+} Y'_{1-} - Y_{1-} Y'_{1+}\right)$

$$\int d\hat{r}'\, Y^{*\prime}_{LM} Y'_{lm} Y'_{1q} = (-)^M \int d\hat{r}\, Y'_{L-M} Y'_{lm} Y'_{1q} = \frac{\hat{L}\hat{l}\hat{1}}{\sqrt{4\pi}} \begin{pmatrix} L & l & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & l & 1 \\ -M & m & q \end{pmatrix} \cdot (-)^M$$

$$(-)^{m'} \int d\hat{r}\, Y^*_{l'-m'} Y_{LM} Y_{1-q} = \qquad = \frac{\hat{L}\hat{l}'\hat{1}}{\sqrt{4\pi}} \begin{pmatrix} l' & L & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l' & L & 1 \\ -m' & M & q \end{pmatrix} (-)^{m'}$$

$$= \frac{3(2L+1)\hat{l}\hat{l}'}{4\pi} \sum_{\substack{Ll \\ Mm}} \begin{pmatrix} L & l & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l' & L & 1 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} L & l & 1 \\ -M & m & q \end{pmatrix}\begin{pmatrix} l' & L & 1 \\ -m' & M & p \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & L & l \\ q & -M & m \end{pmatrix}\begin{pmatrix} 1 & L & l' \\ p & M & -m' \end{pmatrix} \cdot (-)^{1+L+l'}$$

$$\overset{p=-q}{=} \begin{pmatrix} 1 & L & l \\ q & -M & m \end{pmatrix}\begin{pmatrix} 1 & L & l' \\ -q & M & -m' \end{pmatrix} \cdot (-1)^{1+L+l'}$$

$$= \begin{pmatrix} 1 & L & l \\ q & -M & m \end{pmatrix}\begin{pmatrix} 1 & L & l' \\ q & -M & m' \end{pmatrix}$$

$$Y_{L-M}\, Y_{\ell' m'}\, Y_{1p} = \begin{pmatrix} L & 1 & \ell' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & \ell' \\ -M & p & m' \end{pmatrix}$$

$$Y'_{LM}\, Y'_{\ell m}\, Y'_{1-p} = \begin{pmatrix} L & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & \ell \\ M & -p & m \end{pmatrix}$$

$$\sum_{\substack{LM \\ \ell m}} \int d^2(\Omega_{\dot{\imath}\dot{\imath}})\, Y_{\ell' m'}\, Y^*_{LM}\, Y'_{LM}\, Y'_{\ell m}$$

$$Y^*_{LM}\, Y_{\ell' m'} = \delta_{L\ell'}\, \delta_{m' M}$$

$$\Rightarrow \int d^2\, Y'_{\ell' m'}\, Y'_{\ell m}$$

$$(-)^{m'}\, Y^*_{\ell' -m'}\, Y_{\ell m}$$

$$= (-)^{m}\, \delta_{\ell \ell'}\, \delta_{m -m'}$$

$\boxed{\text{II}}$

$$\partial_{\underline{r}}^2 \, e^{-\alpha \underline{r}^2 - \beta \underline{r}\cdot\underline{r}' - \gamma r'^2} = \left( 4\alpha^2 r^2 + \beta^2 r'^2 + 4\alpha\beta \underline{r}\cdot\underline{r}' - 2\alpha \right) e^{\cdots} = \boxed{N}$$

$$(\S 34) \text{ in LLQM: } \quad e^{-\beta \underline{r}\cdot\underline{r}'} = 4\pi \sum_{LM} i^L \underbrace{j_L(i\beta r r')}_{\text{spherical Bessel}} Y^*_{LM} Y'_{LM} \qquad j_0 = \frac{\sin x}{x} \; ; \; j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\underline{r}\cdot\underline{r}' = -\sqrt{3}\left[ r_p \otimes r_q \right]^{00} = -\frac{4\pi}{3} r r' \left( Y_{1+} Y'_{1-} + Y_{1-} Y'_{1+} - Y_{10} Y'_{10} \right)$$

$$= +\frac{4\pi}{3} r r' \sum_p Y_{1p} Y'_{1-p} (-1)^{+p}$$

$$\Rightarrow \boxed{N} \sum_{lm} \frac{1}{r'} Y'_{lm} \phi'_{lm} \cdot \int d\hat{r}\, Y^*_{l'm'} \int d\hat{r}'\, r'^2 dr$$

$$= \sum_{\substack{LM\\ lm}} \int dr'\, r'^2 \int d(\hat{r},\hat{r}') \frac{1}{r'} 4\pi i^L j_L(i\beta r r') Y^*_{LM} Y'_{LM} \left( 4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha + 4\alpha\beta \frac{4\pi}{3} r r' \sum_p Y_{1p} Y'_{1-p} (-)^p \right)$$

$$\cdot Y'_{lm} Y^*_{l'm'} \phi'_{lm}$$

$$= \sum_{\substack{LM\\ lm}} \int dr'\, r'\, 4\pi i^L \int d(\hat{r},\hat{r}')\, j_L \underbrace{Y^*_{LM} Y^*_{l'm'}}_{= Y_{LM} Y^*_{l'm'} (-)^M} Y^{*\prime}_{L-M} Y'_{lm} (-)^M \left( 4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha \right) \phi'_{lm}$$

$$+ \int dr'\, r' \int d(\hat{r},\hat{r}') \sum_{\substack{LM\\ lm\\ p}} \frac{(4\pi)}{3} i^l j_l\, 4\alpha\beta r r' (-)^{M+p} Y_{LM} Y'_{LM} Y'_{lm} \underbrace{Y^*_{l'm'}}_{Y_{l'-m'}(-1)^{m'}} Y_{1p} Y'_{1-p} \phi'_{lm}$$

$$= \int dr'\, r'\, 4\pi \sum_{\substack{LM\\ lm}} \delta_{Ll'} \delta_{Mm'} \delta_{Ll} \delta_{-Mm} (-)^{2M} \left( 4\alpha^2 r^2 - \beta^2 r'^2 - 2\alpha \right) \phi'_{lm}$$

$\neq 0 \Leftrightarrow M-p=-m$ and $-M+p=m'$ $\Rightarrow M+m'=p$

$$+ \sqrt{\frac{\hat{L}\hat{l}'\hat{1}}{4\pi}} \sqrt{\frac{\hat{L}\hat{l}\hat{1}}{4\pi}} \int dr'\, r' \sum_{\substack{LM\\ lm\\ p}} \frac{(4\pi)^2}{3} i^l j_L\, 4\alpha\beta r r' (-)^{p+M+m'} \begin{pmatrix} L & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & l \\ M & -p & m \end{pmatrix} \underbrace{\begin{pmatrix} L & 1 & l' \\ -M & p & -m' \end{pmatrix}}_{= (-)^{L+l'+1} \begin{pmatrix} L & 1 & l' \\ M & -p & m' \end{pmatrix}} \phi'_{lm}$$

$$= \int dr'\, 4\pi r'\, \cancel{(-)^{m'}} \phi'_{l'm'} \left( 4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha \right) j_{l'}\, i^{l'}$$

$$+ \int dr'\, r'\, 4\pi i^l \sum_{\substack{lm\\ L\\ M,p}} \hat{L}\, \hat{l}' \hat{l}\, j_L\, 4\alpha\beta r r' (-)^{L+l'+1} \begin{pmatrix} L & 1 & l \\ & 0 & \end{pmatrix} \begin{pmatrix} L & 1 & l' \\ & 0 & \end{pmatrix} \begin{pmatrix} L & 1 & l \\ M & -p & m \end{pmatrix} \begin{pmatrix} L & 1 & l' \\ M & -p & m' \end{pmatrix} \phi'_{lm}$$

<u>note</u> $\sum_p f(-p) = \sum_p f(p)$

because $\sum_p$ is over a symmetric interval $-1, 0, 1$ ✓

III

$$= \int dr' \, 4\pi r' i^{\ell'} j_{\ell'} \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right) \phi'_{\ell' m'}$$

$$+ \int dr' 4\pi r' \sum_{\substack{\ell m \\ L}} i^L j_L \, \hat{L} \hat{\ell}' \hat{\ell} \, 4\alpha\beta r r' (-)^{L+\ell'+1} \begin{pmatrix} L & 1 & \ell \\ & 0 & \end{pmatrix} \begin{pmatrix} L & 1 & \ell' \\ & 0 & \end{pmatrix} \frac{1}{\hat{\ell}'} \delta_{\ell\ell'} \delta_{mm'} \, \delta(L1\ell') \, \phi'_{\ell m}$$

$$\sum_{p=0}^{a} f(-p) = \sum$$

$$= \int dr' \, 4\pi r' \Big[ i^{\ell'} j_{\ell'} \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right) \phi'_{\ell' m'}$$

$$+ \sum_L i^L j_L \, \hat{L} \hat{\ell}' \, 4\alpha\beta r r' (-)^{L+\ell'+1} \underbrace{\begin{pmatrix} L & 1 & \ell' \\ & 0 & \end{pmatrix}^2}_{\neq 0 \iff L+1+\ell' = \text{even}} \delta(L1\ell') \, \phi'_{\ell' m'} \Big]$$

---

$$= \int dr' \, 4\pi r' \Big[ i^{\ell'} j_{\ell'} \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right) + \sum_{L} i^L j_L \, \hat{L} \hat{\ell}' \, 4\alpha\beta r r' \begin{pmatrix} L & 1 & \ell' \\ & 0 & \end{pmatrix} \delta(L1\ell') \Big] \phi'_{\ell' m'}$$

$\uparrow$ s.t. $L+1+\ell' = \text{even}$

$\ell' = |L-1|, \dots, L+1$

---

S wave ($\ell' = 0$)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{1}{\sqrt{3}}$$

$$\int dr' \, 4\pi r' \Big[ j_0(i\beta r r') \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right) + i j_1(i\beta r r') (\sqrt{3})^{-1} 4\alpha\beta r r' \Big] \phi'_{00}$$

$L = 0, 2, 4$

P wave ($\ell' = 1$)

$1 = L = 0$ ✓

$L = 2$ ✓

$$\int dr' \, 4\pi r' \Big[ i j_1(i\beta r r') \left(4\alpha^2 r^2 + \beta^2 r'^2 - 2\alpha\right) + \left( j_0(i\beta r r') \sqrt{3}^{-1} - j_2(i\beta r r') \, 2\sqrt{3}^{-1} \sqrt{5}^{-1} \right) 4\alpha\beta r r' \Big] \phi'_{1m'}$$