Universal aspects of the multi-channel 4-body scattering system

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We investigate the scattering system of 4 equal-mass quantum particles at energies which allow for open rearrangement channels with an interaction that approaches the unitary limit sustaining a 2-body bound state at threshold.

As a benchmark, we obtain the universal ratio of the elastic dimer-dimer scattering length to the scattering length between the two fermions which constitute each dimer. Subsequently, we extend the analysis to a 4-component-fermion system in which a new scale enters in form of a bound trimer. We commence the study with spin- and fermion-species-independent contact interactions between two and three particles with single 2- and 3-body bound states of non-zero energy. We find a hierarchy of dimer-dimer scattering lengths which follows from the different couplings of the three degenerate dimer-dimer channels with the two trimer-fermion channels. Counter-intuitively, we find this hierarchy reversed in the unitary limit when dimer-dimer, dimer-fermion-fermion, and 4-fermion thresholds merge.

Specifically, we obtain ratios of 3-1 and 2-2 scattering lengths as functions of the 3-body scale as unitarity is approached from a dimer resembling the deuteron. All observables are subjected to a renormalization-group analysis in form of a regulator-cutoff variation. Aside from a cusp close to the 3-1 threshold, we do not find signatures of resonant behaviour.

I. INTRODUCTION

The four-fermion quantum scattering problem exhibits most features any interaction theory which aspires to be useful for the description of low-energy nuclear and atomic few-body must include. A lot has been learned, in particular, about the nuclear force from decades of studies aiming for a theoretical description consistent with experimental data for resonances, coupled angular-momentum and rearrangement channels, and relatively closely spaced thresholds. While these endeavours aimed for a comprehensive description of low- and high-energy observables, an attempt to expand the nuclear interaction commences with a leading-order which allows for relatively large uncertainties which are reduced by considering higher-orders systematically. The arguably most significant success of the adaptation of this framework of an effective field theory (EFT) to bound-state observables of few-body systems has been the identification of certain properties which are universal to atoms, nuclei, hadrons, clusters, and other quantum particles whose short-distance structures differ but which exhibit a notable separation of scales between their interaction ranges (r) and typical 2-body scales (a), *e.g.* , the range of the nuclear force being relatively small compared to the spatial extend of the deuteron.

In the theoretical limit, $|r/a| \to 0$, the scale-free 2-body system is uncorrelated with observables of systems involving more than two particles which evolve subject to the interaction underlying this limit. For instance, a van der Waals inter-atomic potential and a zero-range contact interaction which both yield $|r/a| = 0$ will evoke identical few-body observables: the Efimov spectrum in cases which allow for a totally symmetric spatial wave function, and the amount of attenuation of the effective interaction between two dimers[†] compared with the force which binds the dimers. The latter was quantified in terms of the ration of the pertinent scattering lengths, $a_{dd} = 0.6\,a$, for 2-component fermions (2CF) which by themselves cannot realize the spatial symmetry required for Efimov spectra. These two examples of universal behaviour represent bound-state and scattering observables emerging in bosonic and fermionic systems.

In this article, we expand on the second phenomenon: scattering observables which are found in all systems which realize, or are close to the limit $|r/a| \to 0$, and their dependence on the particle statistics. In particular, we analyse features of the 4-body scattering system which result from bose statistics and are thus relevant for the prominent $J^\pi = 0^+$ helium-4/α channel. Without reference to a specific type of fermion, we characterize the problem as follows: What kind of 4-body scattering matrix follows from a spin- and species-independent[‡] 2-body interaction which approaches the *unitary* limit $|r/a| \to 0$ for any pair of 4-component (4CF), equimassive fermions?

Firstly, we will specify the structure of interaction we chose to realize the unitary limit including the method to assess the sensitivity of our finding with respect to variations of this choice. Secondly, we define the scattering problem approximated by five asymptotic two-fragment channels and its variational solution. The latter is presented separately for the uncoupled, fermionic case and

[†] We adopt the dimer-trimer-tetramer notion for bound states of two, three, and four particles.

[‡] We avoid the nuclear term *isospin* and thus refer to neutrons and protons in their respective spin up/down states as *species* with labels 1,2 in the 2CF case, and 1,2,3,4 for 4CF, such that, *e.g.* , proton-spin-up $\hat{=}$1.

the coupled scenario in subsequent sections followed by an out-looking conclusion.

II. INTERACTION THEORY

The minimal theory from which the wealth of nuclear, atomic, and all other approximately unitary systems can be developed systematically uses a Hamiltonian of the form

$$H^{(0)} = \sum_i T_1(\boldsymbol{r}_{\rm i}, m) + \sum_{\{i,j\}} V_2(\boldsymbol{r}_{\rm ij}, \lambda) + \sum_{\{i,j,k\}} V_3(\boldsymbol{r}_{\rm ij}, \boldsymbol{r}_{\rm ik}, \lambda) \tag{1}$$

comprising a 1-particle kinetic energy and 2- and 3-particle potentials depending on single-particle and one or two relative coordinates, respectively. λ differentiates between different potential forms which all yield a two body system at/close-to unitarity. The appearance of a 3-body potential is due to our choice for the 2-body potential which reduces to a zero-range contact interaction and thus is characterized by a single strength that is tuned to the unitarity condition $|a| \approx \infty$. Hence, the 3-body scale needs to enter differently in form of a 3-particle operator. The superscript identifies the hamiltonian as the leading order (LO) of an EFT expansion, and we limit the study to particles with equal masses m. For practical reasons related to the numerical method (see sec. III), we limit the renormalization-group (RG) transformation to those interactions which are represented by

$$V_2 = c(\lambda)\, e^{-\lambda\,(\boldsymbol{r}_i - \boldsymbol{r}_j)^2} \quad , \tag{2}$$

and

$$V_3 = d(\lambda)\, e^{-\lambda\,\left[(\boldsymbol{r}_i - \boldsymbol{r}_j)^2 + (\boldsymbol{r}_i - \boldsymbol{r}_k)^2\right]} \qquad . \tag{3}$$

The RG parameter is thus identical to a cutoff, each value of which corresponding to an interaction which, by definition, must furnish a scattering length a much larger than the interaction range $r \sim \lambda^{-1}$. Instead of renormalizing the low-energy constant (LEC) $c(\lambda)$ to such an a, we constrain it by a sequence of 2-body binding energies $B(2) \to 0$ in order to take the limit to unitarity.

Here, in addition to the spatial location, we consider single-particle states occupying one of two or one out of four internal states, *e.g.* , the two spin orientations of a pure neutron system, or the four states of a nucleon: neutron/proton spin-up/down. From the invariance of $H^{(0)}$ with respect to any transposition $i \leftrightarrow j$ and the anti-symmetry of the system's wave function thereunder, physical matrix elements of V_3 vanish for states comprised solely of 2CF. Hence, such systems do not form trimers. The formation of trimers, the non-vanishing V_3, the emergence of a unique 3-body scale, can be understood as consequences of the continuous scale symmetry which characterizes the unitary 2-body system being broken and realized discretely in the 3- and more-body sector of 4CF. The existence of trimers and of four degenerate dimer states (see sec. III) defines five coupled channels for the 4CF 4-body system which we will describe in the next section.

Succinctly stated, the problem of identifying properties of an arbitrary N-fermion systems which exhibits an approximately scale-free 2-body sector is encoded in the solution to

$$H^{(0)} \hat{\mathcal{A}}_N \mid \{\boldsymbol{r}_{\rm i}, \tau_i\} \,\rangle_n = E_n \hat{\mathcal{A}}_N \mid \{\boldsymbol{r}_{\rm i}, \tau_i\} \,\rangle_n \tag{4}$$

with $\tau_i \in \{1, \dots, \text{nbr. of species}\}$, the anti-symmetrizer $\hat{\mathcal{A}} = \sum_{\mathcal{P} \in \mathcal{S}_N} \text{sign}(\mathcal{P})\, \mathcal{P}$, and a $H^{(0)}$ properly renormalized to the unitary 2-body system and a 3-body scale of choice. To access the bound-state spectrum, the boundary condition

$$\lim_{|\boldsymbol{r}_{\rm ij}| \to \infty} \underbrace{\langle \{\boldsymbol{r}_{\rm i}\} \mid \{\boldsymbol{r}_{\rm i}, \tau_i\} \,\rangle_n}_{=:\Psi_n} = 0 \tag{5}$$

must be met $\forall i \neq j$ and $E_n < 0$.

III. THE SCATTERING PROBLEM

The scattering problem imposes boundary conditions pertinent to specific incoming and outgoing states which are defined in the infinite past and future, respectively. A complete set of these asymptotic states includes all partitions of the N particles into infinitely separated, bound fragments. However, if the interest is on reactions between two fragments at energies which are relatively small compared with their lowest separation energies, a 2-fragment approximation of the asymptotic wave function

$$\Psi_n \to \phi^{(1)}\, \chi(\boldsymbol{r}_{\rm rel})\, \phi^{(2)} \tag{6}$$

suffices.

A. Asymptotic states

For an interaction which does not depended on the intrinsic state of a particle, *i.e.* , acts alike between all pairs of elements of a spin-isospin multiplet, a scattering channel is defined solely by: total number of each particle species in the system (relevant for obtaining the correct (anti)symmetrization), orbital-angular-momentum structure (consistent with the order of the EFT expansion of the interaction), a fragmentation of the particles into bound clusters (any selection of species to furnish a cluster which allows for a totally antisymmetric internal wave function is legitimate. A discrimination between various (iso)spin coupling schemes only produces redundant channels).

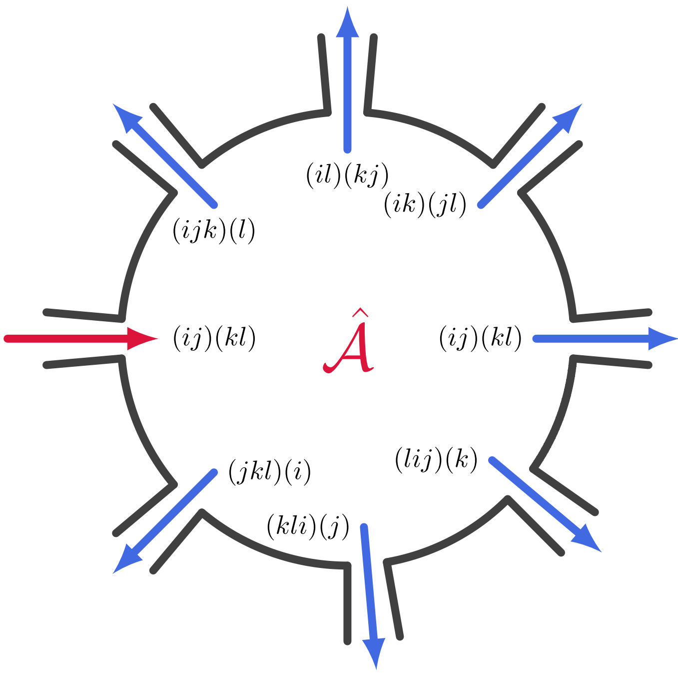
The zero-range, momentum-independent LO interaction supports only bound states with a totally symmetric spatial wave function. Hence, component particles of the state cannot occupy identical fermionic states, and the asymptotic fragments are to be taken from the following lists[†].

$$
\begin{aligned}
(2\text{CF}) &\;:\; \phi^{(1,2)} \in \{i, (i,j) : i,j \in \{1,2\}\} \\
(4\text{CF}) &\;:\; \phi^{(1,2)} \in \{i, (i,j), (i,j,k) : i,j,k \in \{1,2,3,4\}\}
\end{aligned}
$$

Relative S-wave scattering between two fragments composed of 2CF is thus a single-channel problem:

$(ij)(ij) \longrightarrow (ij)(ij)$

The 4CF problem, in turn,



In more familiar terms, the 2CF dimer could be a spin-singlet isospin-triplet dineutron[‡]. The four dimers composable from 4CF could be the deuteron, the spin-singlet neutron-proton, the di-neutron, and the di-proton. The two trimers would then represent the triton/hydrogen-3 and the helion/helium-3. As any (iso)spin dependence, including electromagnetism, enters at a higher order of the EFT expansion, the binding energies off all dimers are degenerate and approach zero in the unitary limit. The two trimers are degenerate, too, but their energy is set by the specific choice of the 3-body scale.

There is only one set of four 2CF which allows for two bound fragments, namely, (1212). For 4CF, we focus on that set which finds all four particles in different internal states, *i.e.* , (1234), pertinent to the nuclear 0^+ channel. Hence, there is only one single elastic channel for 2CF, while the 4CF case allows for five rearrangement channels. Both cases are covered with the following, general structure of the scattering wave function:

$$
\begin{aligned}
\Psi_{c'} &= \hat{\mathcal{A}}\left\{\textstyle\sum_{\textbf{channels}} \phi_c^{(1)}\phi_c^{(2)} \cdot \left(\delta_{c'c}u_0(kR) + a_{c'c}w_0^{(+)}(kR) + \sum_d b_{c'd}\Phi_d\right)\right\} \\
&= \left\{ \begin{array}{lcl} 2\text{CF} & : & \hat{\mathcal{A}}\left\{\phi^{(1)}(\boldsymbol{r}_{12})\phi^{(2)}(\boldsymbol{r}_{34})\chi_0(R)\right\} \\ 4\text{CF} & : & \hat{\mathcal{A}}\left\{\phi^{(1)}(\boldsymbol{r}_{12})\phi^{(2)}(\boldsymbol{r}_{34})\chi_0(R)\right\} \end{array} \right.
\end{aligned}
$$

IV. RESULTS (SINGLE-CHANNEL)

V. RESULTS (COUPLED-CHANNEL)

VI. CONCLUSIONS

† Roman particle-species labels on the same object are understood, here, to assume different values, only, *e.g.* , (i, j) can mean $(1, 3)$ but never $(2, 2)$.

‡ The LO of the nuclear contact EFT without pions binds two neutrons and lifts this degeneracy with the the spin-triplet state at higher orders.