7.1 The two-body Hamiltonian is

$$H_{sys} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(\mathbf{r}_1, \mathbf{r}_2)$$

We assume that the potential is a function of the separation only, such that $V(\mathbf{r}_1,\mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) = V(r)$. Change to center of mass and relative coordinates using the relations

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \text{ , } \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \text{ , } \mathbf{p}_{rel} = \frac{m_1\mathbf{p}_2 - m_2\mathbf{p}_1}{m_1 + m_2}$$

Inverting these gives

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2}{m_1 + m_2}\mathbf{r} \text{ , } \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{p}_1 = -\mathbf{p}_{rel} + \frac{m_1}{m_1 + m_2}\mathbf{P} \text{ , } \mathbf{p}_2 = \mathbf{p}_{rel} + \frac{m_2}{m_1 + m_2}\mathbf{P}$$

Now substitute into the Hamiltonian:

$$\begin{aligned}
H_{sys} &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(\mathbf{r}_1, \mathbf{r}_2) \\
&= \frac{1}{2m_1}\left(-\mathbf{p}_{rel} + \frac{m_1}{m_1+m_2}\mathbf{P}\right)^2 + \frac{1}{2m_2}\left(\mathbf{p}_{rel} + \frac{m_2}{m_1+m_2}\mathbf{P}\right)^2 + V(r) \\
&= \frac{1}{2m_1}\left(\mathbf{p}_{rel}^2 + \left(\frac{m_1}{m_1+m_2}\right)^2\mathbf{P}^2 - \frac{2m_1}{m_1+m_2}\mathbf{P}\bullet\mathbf{p}_{rel}\right)^2 + \\
&\qquad \frac{1}{2m_2}\left(\mathbf{p}_{rel}^2 + \left(\frac{m_2}{m_1+m_2}\right)^2\mathbf{P}^2 + \frac{2m_2}{m_1+m_2}\mathbf{P}\bullet\mathbf{p}_{rel}\right)^2 + V(r) \\
&= \frac{\mathbf{p}_{rel}^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \left(\frac{1}{m_1+m_2}\right)\mathbf{P}^2 + V(r)
\end{aligned}$$

Using the definitions of the total mass $M = m_1 + m_2$ and the reduced mass $\mu = 1/(1/m_1 + 1/m_2)$, we get

$$H_{sys} = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}_{rel}^2}{2\mu} + V(r)$$

and the Hamiltonian separates into center-of-mass motion and relative motion.

(b) In the position representation, the Hamiltonian is

$$
\begin{aligned}
H_{sys} &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(\mathbf{r}_1, \mathbf{r}_2) \doteq \frac{1}{2m_1}\left(-i\hbar\nabla_1\right)^2 + \frac{1}{2m_2}\left(-i\hbar\nabla_2\right)^2 + V(\mathbf{r}_1, \mathbf{r}_2) \\
&\doteq -\frac{\hbar^2}{2m_1}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2}\right) - \frac{\hbar^2}{2m_2}\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial z_2^2}\right) + V(\mathbf{r}_1, \mathbf{r}_2)
\end{aligned}
$$

Using the definitons of the center of mass and relative coordinates, the derivatives are:

$$
\begin{aligned}
\frac{\partial}{\partial x_1} &= \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} + \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} = -\frac{\partial}{\partial x} + \frac{m_1}{m_1 + m_2}\frac{\partial}{\partial X} \\
\frac{\partial}{\partial x_2} &= \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} + \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} = \frac{\partial}{\partial x} + \frac{m_2}{m_1 + m_2}\frac{\partial}{\partial X}
\end{aligned}
$$

and likewise for the other coordinates. The cross terms in the second derivatives cancel, giving

$$
\begin{aligned}
H_{sys} &\doteq -\frac{\hbar^2}{2m_1}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2}\right) - \frac{\hbar^2}{2m_2}\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial z_2^2}\right) + V(\mathbf{r}_1, \mathbf{r}_2) \\
&\doteq -\frac{\hbar^2}{2m_1}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{m_1}{m_1 + m_2}\right)^2\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)\right) \\
&\qquad -\frac{\hbar^2}{2m_2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{m_2}{m_1 + m_2}\right)^2\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)\right) + V(r) \\
&\doteq -\frac{\hbar^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{\hbar^2}{2(m_1 + m_2)}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right) + V(r) \\
&\doteq -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right) + V(r) \\
&\doteq -\frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{\hbar^2}{2M}\nabla_R^2 + V(r)
\end{aligned}
$$

The Hamiltonian is separated into the center of mass and relative coordinates.

7.2 Step 1: Write the partial differential equation in the appropriate coordinate system:

$$
H_{sys}\psi_{sys} = E_{sys}\psi_{sys}
$$

$$
\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{\hbar^2}{2M}\nabla_R^2 + V(r)\right]\psi_{sys}(\mathbf{R}, \mathbf{r}) = E_{sys}\psi_{sys}(\mathbf{R}, \mathbf{r})
$$

Step 2: *Assume* that the solution can be written as the product of functions, at least one of which depends on only one variable. In this case we separate center of mass from relative coordinates:

$$\psi_{sys}(\mathbf{R},\mathbf{r}) = \psi_{CM}(\mathbf{R})\ \ \psi_{rel}(\mathbf{r})$$

Plug this assumed solution into the partial differential equation from step 1. Because of the special form of ψ, the derivatives each act on only one of the functions, yielding

$$-\psi_{CM}(\mathbf{R})\frac{\hbar^2}{2\mu}\nabla_r^2\psi_{rel}(\mathbf{r}) - \psi_{rel}(\mathbf{r})\frac{\hbar^2}{2M}\nabla_R^2\psi_{CM}(\mathbf{R}) + V(r)\psi_{CM}(\mathbf{R})\psi_{rel}(\mathbf{r}) = E_{sys}\psi_{CM}(\mathbf{R})\psi_{rel}(\mathbf{r})$$

<u>Step 3:</u> Divide both sides of the equation by the full function:

$$-\frac{\hbar^2}{2\mu\psi_{rel}(\mathbf{r})}\nabla_r^2\psi_{rel}(\mathbf{r}) - \frac{\hbar^2}{2M\psi_{CM}(\mathbf{R})}\nabla_R^2\psi_{CM}(\mathbf{R}) + V(r) = E_{sys}$$

<u>Step 4:</u> Isolate *all* of the dependence on one coordinate on one side of the equation:

$$-\frac{\hbar^2}{2\mu\psi_{rel}(\mathbf{r})}\nabla_r^2\psi_{rel}(\mathbf{r}) + V(r) = E_{sys} + \frac{\hbar^2}{2M\psi_{CM}(\mathbf{R})}\nabla_R^2\psi_{CM}(\mathbf{R})$$

<u>Step 5:</u> Now imagine changing the isolated variable $\mathbf{r}$ by a small amount. In principle, the left-hand side of the equation could change, but nothing on the right-hand side would. Therefore, if the equation is to be true for all values of $\mathbf{r}$, the particular combination of $\mathbf{r}$ dependences on the left-hand side must result in no overall dependence on $\mathbf{r}$ —*the left-hand side must be a constant*. We thus define a **separation constant**, which we call E_{rel} in this case

$$-\frac{\hbar^2}{2\mu\psi_{rel}(\mathbf{r})}\nabla_r^2\psi_{rel}(\mathbf{r}) + V(r) = E_{sys} + \frac{\hbar^2}{2M\psi_{CM}(\mathbf{R})}\nabla_R^2\psi_{CM}(\mathbf{R}) \equiv E_{rel}$$

<u>Step 6:</u> Write each equation in standard form by multiplying each equation by its unknown function to clear it from the denominator. Rearranging slightly, we obtain the relative and center of mass equations in the more standard forms

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right]\psi_{rel}(\mathbf{r}) = E_{rel}\psi_{rel}(\mathbf{r})$$
$$\left[-\frac{\hbar^2}{2M}\nabla_R^2\right]\psi_{CM}(\mathbf{R}) = E_{CM}\psi_{CM}(\mathbf{R})$$

Where

$$E_{sys} = E_{rel} + E_{CM}$$

We thus have the separated equations

$$H_{rel}\psi_{rel}(\mathbf{r}) = E_{rel}\psi_{rel}(\mathbf{r})$$
$$H_{CM}\psi_{CM}(\mathbf{R}) = E_{CM}\psi_{CM}(\mathbf{R})$$

7.3 <u>Step 1:</u> Write the partial differential equation in the appropriate coordinate system:

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)\psi_{CM}(X,Y,Z)=E_{CM}\psi_{CM}(X,Y,Z)$$

<u>Step 2:</u> *Assume* that the solution can be written as the product of functions, at least one of which depends on only one variable. In this case we separate all three coordinates:

$$\psi_{CM}(X,Y,Z)=f(X)g(Y)h(Z)$$

Plug this assumed solution into the partial differential equation from step 1. Because of the special form of ψ, the derivatives each act on only one of the functions. Any partial derivatives that act only on a function of a single variable may be rewritten as total derivatives, yielding

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)f(X)g(Y)h(Z)=E_{CM}f(X)g(Y)h(Z)$$

$$-\frac{\hbar^2}{2M}\left(g(Y)h(Z)\frac{d^2}{dX^2}f(X)+f(X)h(Z)\frac{d^2}{dY^2}g(Y)+f(X)g(Y)\frac{d^2}{dZ^2}h(Z)\right)=E_{CM}f(X)g(Y)h(Z)$$

<u>Step 3:</u> Divide both sides of the equation by the full function:

$$-\frac{\hbar^2}{2M}\left(\frac{1}{f(X)}\frac{d^2}{dX^2}f(X)+\frac{1}{g(Y)}\frac{d^2}{dY^2}g(Y)+\frac{1}{h(Z)}\frac{d^2}{dZ^2}h(Z)\right)=E_{CM}$$

<u>Step 4:</u> Isolate *all* of the dependence on one coordinate on one side of the equation:

$$-\frac{\hbar^2}{2M}\frac{1}{f(X)}\frac{d^2}{dX^2}f(X)=E_{CM}+\frac{\hbar^2}{2M}\left(\frac{1}{g(Y)}\frac{d^2}{dY^2}g(Y)+\frac{1}{h(Z)}\frac{d^2}{dZ^2}h(Z)\right)$$

<u>Step 5:</u> Now imagine changing the isolated variable X by a small amount. In principle, the left-hand side of the equation could change, but nothing on the right-hand side would. Therefore, if the equation is to be true for all values of X, the particular combination of X dependences on the left-hand side must result in no overall dependence on X —*the left-hand side must be a constant.* We thus define a **separation constant**, which we call E_X in this case

$$-\frac{\hbar^2}{2M}\frac{1}{f(X)}\frac{d^2}{dX^2}f(X)=E_{CM}+\frac{\hbar^2}{2M}\left(\frac{1}{g(Y)}\frac{d^2}{dY^2}g(Y)+\frac{1}{h(Z)}\frac{d^2}{dZ^2}h(Z)\right)\equiv E_X$$

Apply the same reasoning to the two parts of the right hand side to get:

$$-\frac{\hbar^2}{2M}\frac{1}{g(Y)}\frac{d^2}{dY^2}g(Y)\equiv E_Y$$

$$-\frac{\hbar^2}{2M}\frac{1}{h(Z)}\frac{d^2}{dZ^2}h(Z)\equiv E_Z$$

Where

$$E_{CM} = E_X + E_Y + E_Z$$

<u>Step 6:</u> Write each equation in standard form by multiplying each equation by its unknown function to clear it from the denominator. Rearranging slightly, we obtain

$$-\frac{\hbar^2}{2M}\frac{d^2}{dX^2}f(X) = E_X f(X)$$
$$-\frac{\hbar^2}{2M}\frac{d^2}{dY^2}g(Y) = E_Y g(Y)$$
$$-\frac{\hbar^2}{2M}\frac{d^2}{dZ^2}h(Z) \equiv E_Z h(Z)$$

We thus have three separated equations.

7.4 Using the definitions in Eq. (7.47), we obtain

$$\begin{aligned}\left[L_x, L_y\right] &= \left[yp_z - zp_y, zp_x - xp_z\right] \\ &= yp_z zp_x - yp_z xp_z - zp_y zp_x + zp_y xp_z - zp_x yp_z + zp_x zp_y + xp_z yp_z - xp_z zp_y\end{aligned}$$

Now use the commutation relations to move commuting operators through each other (e.g., $yp_z zp_x = yp_x p_z z$) and cancel terms:

$$\begin{aligned}\left[L_x, L_y\right] &= yp_x p_z z - xyp_z p_z - zzp_x p_y + xp_y zp_z - yp_x zp_z + zzp_x p_y + xyp_z p_z - xp_y p_z z \\ &= yp_x p_z z + xp_y zp_z - yp_x zp_z - xp_y p_z z\end{aligned}$$

Collect terms and use the commutation relation $\left[z, p_z\right] = i\hbar$

$$\begin{aligned}\left[L_x, L_y\right] &= xp_y\left(zp_z - p_z z\right) - yp_x\left(zp_z - p_z z\right) \\ &= xp_y\left[z, p_z\right] - yp_x\left[z, p_z\right] = i\hbar\left(xp_y - yp_x\right) = i\hbar L_z\end{aligned}$$

Likewise, we get

$$\begin{aligned}\left[L_y, L_z\right] &= \left[zp_x - xp_z, xp_y - yp_x\right] \\ &= zp_x xp_y - zp_x yp_x - xp_z xp_y + xp_z yp_x - xp_y zp_x + xp_y xp_z + yp_x zp_x - yp_x xp_z\end{aligned}$$

Now use the commutation relations to move commuting operators through each other and cancel terms:

$$\begin{aligned}\left[L_y, L_z\right] &= zp_y p_x x - yzp_x p_x - xxp_y p_z + yp_z xp_x - zp_y xp_z + xxp_y p_z + yzp_x p_x - yp_z p_x x \\ &= zp_y p_x x + yp_z xp_x - zp_z xp_x - yp_z p_x x\end{aligned}$$

Collect terms and use the commutation relation $\left[x, p_x\right] = i\hbar$

$$\left[L_y,L_z\right]=yp_z\left(xp_x-p_xx\right)-zp_y\left(xp_x-p_xx\right)$$
$$=yp_z\left[x,p_x\right]-zp_y\left[x,p_x\right]=i\hbar\left(yp_z-zp_y\right)=i\hbar L_x$$

Finally, we get

$$\left[L_z,L_x\right]=\left[zp_y-yp_x,yp_z-zp_y\right]$$
$$=xp_yyp_z-xp_yzp_y-yp_xyp_z+yp_xzp_y-yp_zxp_y+yp_zyp_x+zp_yxp_y-zp_yyp_x$$

Now use the commutation relations to move commuting operators through each other and cancel terms:

$$\left[L_z,L_x\right]=xp_zp_yy-zxp_yp_y-yyp_zp_x+zp_xyp_y-xp_zyp_x+yyp_zp_x+zxp_yp_y-zp_xp_yy$$
$$=xp_zp_yy+zp_xyp_y-xp_xyp_y-zp_xp_yy$$

Collect terms and use the commutation relation $\left[y,p_y\right]=i\hbar$

$$\left[L_z,L_x\right]=zp_x\left(yp_y-p_yy\right)-xp_z\left(yp_y-p_yy\right)$$
$$=zp_x\left[y,p_y\right]-xp_z\left[y,p_y\right]=i\hbar\left(zp_x-xp_z\right)=i\hbar L_y$$

For the commutators with $\mathbf{L}^2$, we get

$$\left[\mathbf{L}^2,L_x\right]=\left[L_x^2+L_y^2+L_z^2,L_x\right]=\left[L_x^2,L_x\right]+\left[L_y^2,L_x\right]+\left[L_z^2,L_x\right]$$
$$=L_y^2L_x-L_xL_y^2+L_z^2L_x-L_xL_z^2$$

Add zero to this equation, but choose the terms that sum to zero cleverly so they help:

$$\left[\mathbf{L}^2,L_x\right]=L_yL_yL_x\underbrace{-L_yL_xL_y+L_yL_xL_y}_{=0}-L_xL_yL_y+L_zL_zL_x\underbrace{-L_zL_xL_z+L_zL_xL_z}_{=0}-L_xL_zL_z$$
$$=L_y\left[L_y,L_x\right]+\left[L_y,L_x\right]L_y+L_z\left[L_z,L_x\right]+\left[L_z,L_x\right]L_z$$
$$=-i\hbar L_yL_z-i\hbar L_zL_y+i\hbar L_zL_y+i\hbar L_yL_z=0$$

For the L_y component, we get

$$\left[\mathbf{L}^2,L_y\right]=\left[L_x^2+L_y^2+L_z^2,L_y\right]=\left[L_x^2,L_y\right]+\left[L_y^2,L_y\right]+\left[L_z^2,L_y\right]$$
$$=L_x^2L_y-L_yL_x^2+L_z^2L_y-L_yL_z^2$$

Add zero to this equation, but choose the terms that sum to zero cleverly so they help:

$$\left[\mathbf{L}^2,L_y\right]=L_xL_xL_y\underbrace{-L_xL_yL_x+L_xL_yL_x}_{=0}-L_yL_xL_x+L_zL_zL_y\underbrace{-L_zL_yL_z+L_zL_yL_z}_{=0}-L_yL_zL_z$$
$$=L_x\left[L_x,L_y\right]+\left[L_x,L_y\right]L_x+L_z\left[L_z,L_y\right]+\left[L_z,L_y\right]L_z$$
$$=i\hbar L_xL_z+i\hbar L_zL_x-i\hbar L_zL_x-i\hbar L_xL_z=0$$

For the L_z component, we get

$$\begin{aligned}\left[\mathbf{L}^2, L_z\right] &= \left[L_x^2 + L_y^2 + L_z^2, L_z\right] \\ &= \left[L_x^2, L_z\right] + \left[L_y^2, L_z\right] + \left[L_z^2, L_z\right] \\ &= L_x^2 L_z - L_z L_x^2 + L_y^2 L_z - L_z L_y^2\end{aligned}$$

Add zero to this equation, but choose the terms that sum to zero cleverly so they help:

$$\begin{aligned}\left[\mathbf{L}^2, L_z\right] &= L_x L_x L_z \underbrace{- L_x L_z L_x + L_x L_z L_x}_{=0} - L_z L_x L_x + L_y L_y L_z \underbrace{- L_y L_z L_y + L_y L_z L_y}_{=0} - L_z L_y L_y \\ &= L_x\left[L_x, L_z\right] + \left[L_x, L_z\right] L_x + L_y\left[L_y, L_z\right] + \left[L_y, L_z\right] L_y \\ &= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y = 0\end{aligned}$$

7.5 a) The possible results of a measurement of the angular momentum component L_z are always $+1\hbar,\ 0\hbar,\ -1\hbar$ for an $\ell = 1$ system. The probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \left|\langle 11|\psi\rangle\right|^2 = \left|\langle 11|\left[\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right]\right|^2 \\ &= \left|\tfrac{2}{\sqrt{29}}\langle 11|11\rangle + \tfrac{3i}{\sqrt{29}}\langle 11|10\rangle - \tfrac{4}{\sqrt{29}}\langle 11|1,-1\rangle\right|^2 = \left|\tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{4}{29}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_0 &= \left|\langle 10|\psi\rangle\right|^2 = \left|\langle 10|\left[\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right]\right|^2 \\ &= \left|\tfrac{2}{\sqrt{29}}\langle 10|11\rangle + \tfrac{3i}{\sqrt{29}}\langle 10|10\rangle - \tfrac{4}{\sqrt{29}}\langle 10|1,-1\rangle\right|^2 = \left|\tfrac{3i}{\sqrt{29}}\right|^2 = \tfrac{9}{29}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{-1} &= \left|\langle 1,-1|\psi\rangle\right|^2 = \left|\langle 1,-1|\left[\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right]\right|^2 \\ &= \left|\tfrac{2}{\sqrt{29}}\langle 1,-1|11\rangle + \tfrac{3i}{\sqrt{29}}\langle 1,-1|10\rangle - \tfrac{4}{\sqrt{29}}\langle 1,-1|1,-1\rangle\right|^2 = \left|-\tfrac{4}{\sqrt{29}}\right|^2 = \tfrac{16}{29}\end{aligned}$$

The three probabilities add to unity, as they must.

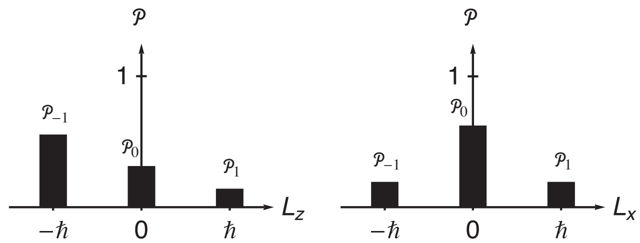
b) The possible results of a measurement of the angular momentum component L_x are always $+1\hbar,\ 0\hbar,\ -1\hbar$ for an $\ell = 1$ system. The probabilities are

$$\begin{aligned}\mathcal{P}_{1x} &= \left|{}_x\langle 11|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 11| + \tfrac{1}{\sqrt{2}}\langle 10| + \tfrac{1}{2}\langle 1,-1|\right)\left(\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{29}} + \tfrac{3i}{\sqrt{2}\sqrt{29}} - \tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{1}{58}\left|-\sqrt{2} + 3i\right|^2 = \tfrac{11}{58}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{0x} &= \left|{}_x\langle 10|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle 11| - \tfrac{1}{\sqrt{2}}\langle 1,-1|\right)\left(\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right)\right|^2 \\ &= \left|\tfrac{2}{\sqrt{2}\sqrt{29}} + \tfrac{4}{\sqrt{2}\sqrt{29}}\right|^2 = \tfrac{36}{58}\end{aligned}$$

$$\mathcal{P}_{-1x} = \left|{}_x\langle 1,-1|\psi_{in}\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 11| - \tfrac{1}{\sqrt{2}}\langle 10| + \tfrac{1}{2}\langle 1,-1|\right)\left(\tfrac{2}{\sqrt{29}}|11\rangle + \tfrac{3i}{\sqrt{29}}|10\rangle - \tfrac{4}{\sqrt{29}}|1,-1\rangle\right)\right|^2$$
$$= \left|\tfrac{1}{\sqrt{29}} - \tfrac{3i}{\sqrt{2}\sqrt{29}} - \tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{1}{58}\left|-\sqrt{2} - 3i\right|^2 = \tfrac{11}{58}$$

The three probabilities add to unity, as they must.
c) The histograms are shown below.



7.6 a) The possible results of a measurement of the angular momentum component L_z are always $+1\hbar, 0\hbar, -1\hbar$ for an $\ell = 1$ system. The probabilities are

$$\mathcal{P}_1 = \left|\langle 11|\psi\rangle\right|^2 = \left|\langle 11|\left[\tfrac{1}{\sqrt{14}}|11\rangle - \tfrac{3}{\sqrt{14}}|10\rangle + i\tfrac{2}{\sqrt{14}}|1,-1\rangle\right]\right|^2$$
$$= \left|\tfrac{1}{\sqrt{14}}\langle 11|11\rangle - \tfrac{3}{\sqrt{14}}\langle 11|10\rangle + i\tfrac{2}{\sqrt{14}}\langle 11|1,-1\rangle\right|^2 = \left|\tfrac{1}{\sqrt{14}}\right|^2 = \tfrac{1}{14}$$

$$\mathcal{P}_0 = \left|\langle 10|\psi\rangle\right|^2 = \left|\langle 10|\left[\tfrac{1}{\sqrt{14}}|11\rangle - \tfrac{3}{\sqrt{14}}|10\rangle + i\tfrac{2}{\sqrt{14}}|1,-1\rangle\right]\right|^2$$
$$= \left|\tfrac{1}{\sqrt{14}}\langle 10|11\rangle - \tfrac{3}{\sqrt{14}}\langle 10|10\rangle + i\tfrac{2}{\sqrt{14}}\langle 10|1,-1\rangle\right|^2 = \left|-\tfrac{3}{\sqrt{14}}\right|^2 = \tfrac{9}{14}$$

$$\mathcal{P}_{-1} = \left|\langle 1,-1|\psi\rangle\right|^2 = \left|\langle 1,-1|\left[\tfrac{1}{\sqrt{14}}|11\rangle - \tfrac{3}{\sqrt{14}}|10\rangle + i\tfrac{2}{\sqrt{14}}|1,-1\rangle\right]\right|^2$$
$$= \left|\tfrac{1}{\sqrt{14}}\langle 1,-1|11\rangle - \tfrac{3}{\sqrt{14}}\langle 1,-1|10\rangle + i\tfrac{2}{\sqrt{14}}\langle 1,-1|1,-1\rangle\right|^2 = \left|i\tfrac{2}{\sqrt{14}}\right|^2 = \tfrac{4}{14}$$

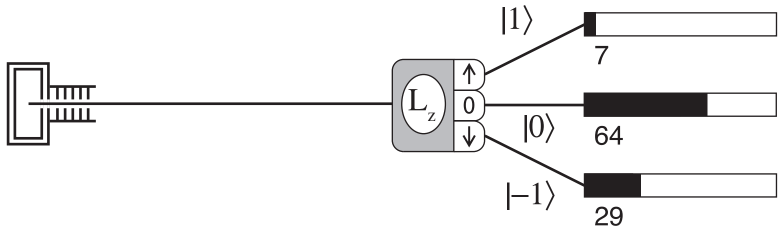
The three probabilities add to unity, as they must.
b) Now the initial state is $|\psi\rangle = |1,-1\rangle$. The possible results of a measurement of the angular momentum component L_x are always $+1\hbar, 0\hbar, -1\hbar$ for an $\ell = 1$ system. The probabilities are

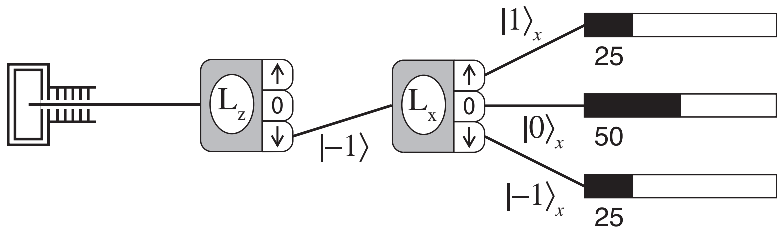
$$\mathcal{P}_{1x} = \left|{}_x\langle 11|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 11| + \tfrac{1}{\sqrt{2}}\langle 10| + \tfrac{1}{2}\langle 1,-1|\right)|1,-1\rangle\right|^2 = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$$\mathcal{P}_{0x} = \left|{}_x\langle 10|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle 11| - \tfrac{1}{\sqrt{2}}\langle 1,-1|\right)|1,-1\rangle\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{-1x} = \left|{}_x\langle 1,-1|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 11| - \tfrac{1}{\sqrt{2}}\langle 10| + \tfrac{1}{2}\langle 1,-1|\right)|1,-1\rangle\right|^2 = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

The three probabilities add to unity, as they must.
c) The schematic diagrams of these measurements are shown below.





7.7 The possible results of a measurement of $\mathbf{L}^2$ are always $\ell(\ell+1)\hbar^2$, where ℓ is the angular momentum quantum number. The possible results of a measurement of the angular momentum component L_z are always $m\hbar$, where $m=-\ell\ldots\ell$. By inspection we can see that this system has only $\ell=1$ and $\ell=2$, with possible m values of $m=0,1,2$.
a) The $\mathbf{L}^2$ probabilities are

$$\begin{aligned}\mathcal{P}_{\ell=1} &= \sum_{m=-\ell}^{\ell}\left|\langle 1m|\psi\rangle\right|^2 = \sum_{m=-\ell}^{\ell}\left|\langle 1m|\left(\tfrac{1}{\sqrt{10}}|11\rangle-\tfrac{2}{\sqrt{10}}|10\rangle+i\tfrac{2}{\sqrt{10}}|22\rangle+i\tfrac{1}{\sqrt{10}}|20\rangle\right)\right|^2\\ &=\left|\tfrac{1}{\sqrt{10}}\right|^2+\left|-\tfrac{2}{\sqrt{10}}\right|^2=\tfrac{5}{10}=\tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{\ell=2} &= \sum_{m=-\ell}^{\ell}\left|\langle 2m|\psi\rangle\right|^2 = \sum_{m=-\ell}^{\ell}\left|\langle 2m|\left(\tfrac{1}{\sqrt{10}}|11\rangle-\tfrac{2}{\sqrt{10}}|10\rangle+i\tfrac{2}{\sqrt{10}}|22\rangle+i\tfrac{1}{\sqrt{10}}|20\rangle\right)\right|^2\\ &=\left|i\tfrac{2}{\sqrt{10}}\right|^2+\left|i\tfrac{1}{\sqrt{10}}\right|^2=\tfrac{5}{10}=\tfrac{1}{2}\end{aligned}$$

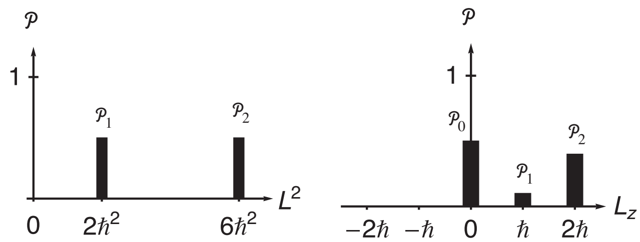
The two probabilities add to unity, as they must.
a) The L_z probabilities are

$$\begin{aligned}\mathcal{P}_{m=0} &= \sum_{\ell=0}^{\infty}\left|\langle \ell 0|\psi\rangle\right|^2 = \sum_{\ell=0}^{\infty}\left|\langle \ell 0|\left(\tfrac{1}{\sqrt{10}}|11\rangle-\tfrac{2}{\sqrt{10}}|10\rangle+i\tfrac{2}{\sqrt{10}}|22\rangle+i\tfrac{1}{\sqrt{10}}|20\rangle\right)\right|^2\\ &=\left|-\tfrac{2}{\sqrt{10}}\right|^2+\left|i\tfrac{1}{\sqrt{10}}\right|^2=\tfrac{5}{10}=\tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{m=1} &= \sum_{\ell=1}^{\infty}\left|\langle \ell 1|\psi\rangle\right|^2 = \sum_{\ell=1}^{\infty}\left|\langle \ell 1|\left(\tfrac{1}{\sqrt{10}}|11\rangle-\tfrac{2}{\sqrt{10}}|10\rangle+i\tfrac{2}{\sqrt{10}}|22\rangle+i\tfrac{1}{\sqrt{10}}|20\rangle\right)\right|^2\\ &=\left|\tfrac{1}{\sqrt{10}}\right|^2=\tfrac{1}{10}\end{aligned}$$

$$\mathcal{P}_{m=2} = \sum_{\ell=2}^{\infty} \left| \langle \ell 2 | \psi \rangle \right|^2 = \sum_{\ell=2}^{\infty} \left| \langle \ell 2 | \left(\tfrac{1}{\sqrt{10}} |11\rangle - \tfrac{2}{\sqrt{10}} |10\rangle + i \tfrac{2}{\sqrt{10}} |22\rangle + i \tfrac{1}{\sqrt{10}} |20\rangle \right) \right|^2$$
$$= \left| i \tfrac{2}{\sqrt{10}} \right|^2 = \tfrac{4}{10} = \tfrac{2}{5}$$

The three probabilities add to unity, as they must.
c) The histograms are shown below.



7.8 Use the chain rule to write the Cartesian derivatives in terms of spherical coordinates

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$
$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

We need the spherical coordinates in terms of the Cartesian cords:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$
$$\phi = \tan^{-1}\left(\frac{y}{x} \right)$$

The required partial derivatives are

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}=\sin\theta\cos\phi,\ \ \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta\sin\phi,\ \ \frac{\partial r}{\partial z}=\frac{z}{r}=\cos\theta$$

$$\frac{\partial\theta}{\partial x}=-\frac{1}{\sqrt{1-z^2/r^2}}\left(\frac{-zx}{r^3}\right)=\frac{\cos\theta\cos\phi}{r},\ \ \frac{\partial\theta}{\partial y}=-\frac{1}{\sqrt{1-z^2/r^2}}\left(\frac{-zy}{r^3}\right)=\frac{\cos\theta\sin\phi}{r}$$

$$\frac{\partial\theta}{\partial z}=-\frac{1}{\sqrt{1-z^2/r^2}}\left(\frac{1}{r}-\frac{z^2}{r^3}\right)=-\frac{\sin\theta}{r},\ \ \frac{\partial\phi}{\partial x}=\frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)=-\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial\phi}{\partial y}=\frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)=\frac{\cos\phi}{r\sin\theta},\ \ \frac{\partial\phi}{\partial z}=0$$

Putting these into the angular momentum equation for L_x gives

$$\begin{aligned}
L_x &\doteq -i\hbar\left(y\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)-z\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)\right)\\
&\doteq -i\hbar\left(y\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+0\frac{\partial}{\partial\phi}\right)-z\left(\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\\
&\doteq -i\hbar\left(\begin{array}{l}\left(r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r}-\sin^2\theta\sin\phi\frac{\partial}{\partial\theta}\right)\\ -\left(r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r}+\cos^2\theta\sin\phi\frac{\partial}{\partial\theta}+\frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\end{array}\right)\\
&\doteq i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}$$

Putting these into the angular momentum equation for L_y gives

$$\begin{aligned}
L_y &\doteq -i\hbar\left(z\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)-x\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)\right)\\
&\doteq -i\hbar\left(z\left(\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)-x\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+0\frac{\partial}{\partial\phi}\right)\right)\\
&\doteq -i\hbar\left(\begin{array}{l}\left(r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r}+\cos^2\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ -\left(r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r}-\sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)\end{array}\right)\\
&\doteq i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}$$

Putting these into the angular momentum equation for L_z gives

$$L_z \doteq -i\hbar\left(x\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)-y\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)\right)$$

$$\doteq -i\hbar\left(\begin{array}{l} x\left(\sin\theta\sin\phi\dfrac{\partial}{\partial r}+\dfrac{\cos\theta\sin\phi}{r}\dfrac{\partial}{\partial\theta}+\dfrac{\cos\phi}{r\sin\theta}\dfrac{\partial}{\partial\phi}\right)\\ -y\left(\sin\theta\cos\phi\dfrac{\partial}{\partial r}+\dfrac{\cos\theta\cos\phi}{r}\dfrac{\partial}{\partial\theta}-\dfrac{\sin\phi}{r\sin\theta}\dfrac{\partial}{\partial\phi}\right)\end{array}\right)$$

$$\doteq -i\hbar\left(\begin{array}{l} \left(r\sin^2\theta\sin\phi\cos\phi\dfrac{\partial}{\partial r}+\sin\theta\cos\theta\sin\phi\cos\phi\dfrac{\partial}{\partial\theta}+\cos^2\phi\dfrac{\partial}{\partial\phi}\right)\\ -\left(r\sin^2\theta\sin\phi\cos\phi\dfrac{\partial}{\partial r}+\sin\theta\cos\theta\sin\phi\cos\phi\dfrac{\partial}{\partial\theta}-\sin^2\phi\dfrac{\partial}{\partial\phi}\right)\end{array}\right)$$

$$\doteq -i\hbar\frac{\partial}{\partial\phi}$$

For the operator $\mathbf{L}^2$, we get

$$\mathbf{L}^2 = L_x^2+L_y^2+L_z^2$$

$$\doteq -\hbar^2\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)^2-\hbar^2\left(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)^2-\hbar^2\left(\frac{\partial}{\partial\phi}\right)^2$$

$$\doteq -\hbar^2\left(\begin{array}{l}\sin^2\phi\dfrac{\partial^2}{\partial\theta^2}-\dfrac{\sin\phi\cos\phi}{\sin^2\theta}\dfrac{\partial}{\partial\phi}+2\cot\theta\sin\phi\cos\phi\dfrac{\partial^2}{\partial\theta\partial\phi}\\ +\cot\theta\cos^2\phi\dfrac{\partial}{\partial\theta}+\cot^2\theta\cos^2\phi\dfrac{\partial^2}{\partial\phi^2}-\cot^2\theta\sin\phi\cos\phi\dfrac{\partial}{\partial\phi}\end{array}\right)$$

$$-\hbar^2\left(\begin{array}{l}\cos^2\phi\dfrac{\partial^2}{\partial\theta^2}+\dfrac{\sin\phi\cos\phi}{\sin^2\theta}\dfrac{\partial}{\partial\phi}-2\cot\theta\sin\phi\cos\phi\dfrac{\partial^2}{\partial\theta\partial\phi}\\ +\cot\theta\sin^2\phi\dfrac{\partial}{\partial\theta}+\cot^2\theta\sin^2\phi\dfrac{\partial^2}{\partial\phi^2}+\cot^2\theta\sin\phi\cos\phi\dfrac{\partial}{\partial\phi}\end{array}\right)-\hbar^2\left(\frac{\partial^2}{\partial\phi^2}\right)$$

$$\doteq -\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\cot^2\theta\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial\phi^2}\right)$$

which can be re written as

$$\mathbf{L}^2 \doteq -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

7.9 The central force Hamiltonian is

$$H \doteq -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right] + V(r)$$

The commutator with the angular momentum operator $\mathbf{L}^2$ is

$$\begin{aligned}\left[H,\mathbf{L}^2\right] &\doteq \left(-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right] + V(r)\right)\mathbf{L}^2 \\ &\quad -\mathbf{L}^2\left(-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right] + V(r)\right) \\ &\doteq -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\mathbf{L}^2 + \mathbf{L}^2\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + V(r)\mathbf{L}^2 - \mathbf{L}^2 V(r)\end{aligned}$$

$\mathbf{L}^2$ has derivatives only with respect to angular variables, so it's order with respect to radial variables can be changed, giving

$$\left[H,\mathbf{L}^2\right] \doteq -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\mathbf{L}^2 + \frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\mathbf{L}^2 + V(r)\mathbf{L}^2 - V(r)\mathbf{L}^2 = 0$$

so they commute. The commutator with the angular momentum operator L_z is

$$\begin{aligned}\left[H,L_z\right] &\doteq \left(-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right] + V(r)\right)L_z \\ &\quad -L_z\left(-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right] + V(r)\right) \\ &\doteq -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]L_z + L_z\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + V(r)L_z - L_z V(r)\end{aligned}$$

where we used $\left[\mathbf{L}^2, L_z\right] = 0$. L_z has derivatives with respect to angular variables only (ϕ), so it's order with respect to radial variables can be changed, giving

$$\left[H,L_z\right] \doteq -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]L_z + \frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]L_z + V(r)L_z - V(r)L_z = 0$$

so they commute.

7.10 <u>Step 1:</u> Write the partial differential equation in the appropriate coordinate system:

$$\mathbf{L}^2 Y(\theta,\phi) = A\ \hbar^2\ Y(\theta,\phi)$$

$$-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y(\theta,\phi) = A\ \hbar^2\ Y(\theta,\phi)$$

<u>Step 2:</u> *Assume* that the solution can be written as the product of functions, at least one of which depends on only one variable:

$$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$$

Plug this assumed solution into the partial differential equation from step 1. Because of the special form of Y, the derivatives each act on only one of the functions. Any partial derivatives that act only on a function of a single variable may be rewritten as total derivatives, yielding

$$-\hbar^2\left(\Phi(\phi)\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta(\theta)+\frac{1}{\sin^2\theta}\Theta(\theta)\frac{d^2}{d\phi^2}\Phi(\phi)\right)=A\ \hbar^2\Theta(\theta)\Phi(\phi)$$

<u>Step 3:</u> Divide both sides of the equation by the full function:

$$-\hbar^2\left(\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta(\theta)+\frac{1}{\Phi(\phi)\sin^2\theta}\frac{d^2}{d\phi^2}\Phi(\phi)\right)=A\ \hbar^2$$

<u>Step 4:</u> Isolate *all* of the dependence on one coordinate on one side of the equation:

$$\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta(\theta)+A\sin^2\theta=-\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)$$

<u>Step 5:</u> Now imagine changing the isolated variable θ by a small amount. In principle, the left-hand side of the equation could change, but nothing on the right-hand side would. Therefore, if the equation is to be true for all values of θ, the particular combination of θ dependences on the left-hand side must result in no overall dependence on θ —*the left-hand side must be a constant*. We thus define a **separation constant**, which we call B in this case

$$\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta(\theta)+A\sin^2\theta=-\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)\equiv B$$

<u>Step 6:</u> Write each equation in standard form by multiplying each equation by its unknown function to clear it from the denominator:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\Theta(\theta)-\frac{B}{\sin^2\theta}\Theta(\theta)=-A\Theta(\theta)$$

$$\frac{d^2}{d\phi^2}\Phi(\phi)=-B\,\Phi(\phi)$$

We thus have two separated equations.

7.11 The eigenstates are

$$\left|m\right\rangle \doteq \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

The inner product for $m \neq n$ is

$$\left\langle m \middle| n\right\rangle = \int_0^{2\pi}\Phi_m^*(\phi)\Phi_n(\phi)d\phi = \int_0^{2\pi}\frac{1}{\sqrt{2\pi}}e^{-im\phi}\frac{1}{\sqrt{2\pi}}e^{in\phi}d\phi = \frac{1}{2\pi}\int_0^{2\pi}e^{i(n-m)\phi}d\phi$$

$$= \frac{1}{2\pi}\left[\frac{e^{i(n-m)\phi}}{i(n-m)}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{e^{i(n-m)2\pi}-1}{i(n-m)}\right] = \frac{1}{2\pi}\left[\frac{1-1}{i(n-m)}\right] = 0$$

The inner product for $m = n$ is

$$\left\langle m \middle| m\right\rangle = \int_0^{2\pi}\Phi_m^*(\phi)\Phi_m(\phi)d\phi = \int_0^{2\pi}\frac{1}{\sqrt{2\pi}}e^{-im\phi}\frac{1}{\sqrt{2\pi}}e^{im\phi}d\phi = \frac{1}{2\pi}\int_0^{2\pi}d\phi = \frac{2\pi}{2\pi} = 1$$

Thus, we get

$$\left\langle m \middle| n\right\rangle = \delta_{mn}$$

and the states are orthonormal.

7.12 The observable L_z for a particle on a ring has possible measurement results $L_z = m\hbar$. For this state the probabilities are

$$\mathcal{P}_{L_z=+2\hbar} = \left|\left\langle 2 \middle| \psi\right\rangle\right|^2 = \left|\left\langle 2\right|\tfrac{1}{\sqrt{7}}\left(\left|0\right\rangle + 2\left|1\right\rangle + \left|-1\right\rangle + \left|2\right\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{7}}\right|^2 = \tfrac{1}{7} \simeq 0.14$$

$$\mathcal{P}_{L_z=+1\hbar} = \left|\left\langle 1 \middle| \psi\right\rangle\right|^2 = \left|\left\langle 1\right|\tfrac{1}{\sqrt{7}}\left(\left|0\right\rangle + 2\left|1\right\rangle + \left|-1\right\rangle + \left|2\right\rangle\right)\right|^2 = \left|\tfrac{2}{\sqrt{7}}\right|^2 = \tfrac{4}{7} \simeq 0.58$$

$$\mathcal{P}_{L_z=-1\hbar} = \left|\left\langle -1 \middle| \psi\right\rangle\right|^2 = \left|\left\langle -1\right|\tfrac{1}{\sqrt{7}}\left(\left|0\right\rangle + 2\left|1\right\rangle + \left|-1\right\rangle + \left|2\right\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{7}}\right|^2 = \tfrac{1}{7} \simeq 0.14$$

$$\mathcal{P}_{L_z=0\hbar} = \left|\left\langle 0 \middle| \psi\right\rangle\right|^2 = \left|\left\langle 0\right|\tfrac{1}{\sqrt{7}}\left(\left|0\right\rangle + 2\left|1\right\rangle + \left|-1\right\rangle + \left|2\right\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{7}}\right|^2 = \tfrac{1}{7} \simeq 0.14$$

These agree with the results shown in Fig. 7.9.

7.13 a) For this state the probabilities of an L_z measurement are

$$\mathcal{P}_{L_z=+2\hbar} = \left|\left\langle 2 \middle| \psi\right\rangle\right|^2 = \left|\left\langle 2\right|\left(\tfrac{\sqrt{3}}{2}\left|3\right\rangle + i\tfrac{1}{2}\left|-2\right\rangle\right)\right|^2 = 0$$

$$\mathcal{P}_{L_z=+3\hbar} = \left|\left\langle 3 \middle| \psi\right\rangle\right|^2 = \left|\left\langle 1\right|\left(\tfrac{\sqrt{3}}{2}\left|3\right\rangle + i\tfrac{1}{2}\left|-2\right\rangle\right)\right|^2 = \left|\tfrac{\sqrt{3}}{2}\right|^2 = \tfrac{3}{4}$$

$$\mathcal{P}_{L_z=-2\hbar} = \left|\left\langle -2 \middle| \psi\right\rangle\right|^2 = \left|\left\langle -1\right|\left(\tfrac{\sqrt{3}}{2}\left|3\right\rangle + i\tfrac{1}{2}\left|-2\right\rangle\right)\right|^2 = \left|i\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

b) The energy $2\hbar^2/I$ corresponds to $|m|=2$. For an energy measurement, the probabilities are

$$\mathcal{P}_{E_2} = \left|\langle 2|\psi\rangle\right|^2 + \left|\langle -2|\psi\rangle\right|^2 = \left|\langle 2|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 + \left|\langle -2|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|i\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$$\mathcal{P}_{E_3} = \left|\langle 3|\psi\rangle\right|^2 + \left|\langle -3|\psi\rangle\right|^2 = \left|\langle 3|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 + \left|\langle -3|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|\tfrac{\sqrt{3}}{2}\right|^2 = \tfrac{3}{4}$$

c) The expectation value of L_z is

$$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\,\mathcal{P}_{L_z=mh} = 3\hbar\mathcal{P}_{L_z=+3h} + (-2\hbar)\mathcal{P}_{L_z=-2h} = 3\hbar\tfrac{3}{4} + (-2\hbar)\tfrac{1}{4} = \tfrac{7}{4}\hbar$$

d) The expectation value of the energy is

$$\langle E\rangle = \langle\psi|H|\psi\rangle = \sum_{m=0}^{\infty} E_m\,\mathcal{P}_{E_m} = \sum_{m=0}^{\infty}\frac{m^2\hbar^2}{2I}\mathcal{P}_{E_m} = \frac{2^2\hbar^2}{2I}\mathcal{P}_{E_2} + \frac{3^2\hbar^2}{2I}\mathcal{P}_{E_3} = \frac{\hbar^2}{2I}\left(4\tfrac{1}{4} + 9\tfrac{3}{4}\right) = \frac{31\hbar^2}{8I}$$

7.14 a) The possible results of an energy measurement are $E_{|m|} = m^2\hbar^2/2I$. For this state, the probabilities of an energy measurement are

$$\begin{aligned}
\mathcal{P}_{E_2} &= \left|\langle 2|\psi\rangle\right|^2 + \left|\langle -2|\psi\rangle\right|^2 \\
&= \left|\langle 2|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 + \left|\langle -2|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 \\
&= \left|-2i\tfrac{1}{\sqrt{15}}\right|^2 + \left|3\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{13}{15} \\
\mathcal{P}_{E_1} &= \left|\langle 1|\psi\rangle\right|^2 + \left|\langle -1|\psi\rangle\right|^2 \\
&= \left|\langle 1|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 + \left|\langle -1|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 \\
&= \left|i\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{1}{15} \\
\mathcal{P}_{E_0} &= \left|\langle 0|\psi\rangle\right|^2 = \left|\langle 0|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{1}{15}
\end{aligned}$$

The expectation value of the energy is is

$$\begin{aligned}
\langle E\rangle = \langle\psi|H|\psi\rangle &= \sum_{m=0}^{\infty} E_m\,\mathcal{P}_{E_m} = \sum_{m=0}^{\infty}\frac{m^2\hbar^2}{2I}\mathcal{P}_{E_m} = \frac{0^2\hbar^2}{2I}\mathcal{P}_{E_0} + \frac{1^2\hbar^2}{2I}\mathcal{P}_{E_1} + \frac{2^2\hbar^2}{2I}\mathcal{P}_{E_2} \\
&= \frac{\hbar^2}{2I}\left(0\tfrac{1}{15} + 1\tfrac{1}{15} + 4\tfrac{13}{15}\right) = \frac{53\hbar^2}{30I}
\end{aligned}$$

b) The observable L_z has possible measurement results $L_z = m\hbar$. The probabilities of an L_z measurement are

$$\mathcal{P}_{L_z=+2\hbar} = \left|\langle 2|\psi\rangle\right|^2 = \left|\langle 2|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 = \left|-2i\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{4}{15}$$

$$\mathcal{P}_{L_z=+1\hbar} = \left|\langle 1|\psi\rangle\right|^2 = \left|\langle 1|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 = \left|i\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{1}{15}$$

$$\mathcal{P}_{L_z=0\hbar} = \left|\langle 0|\psi\rangle\right|^2 = \left|\langle 0|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{1}{15}$$

$$\mathcal{P}_{L_z=-2\hbar} = \left|\langle -2|\psi\rangle\right|^2 = \left|\langle -2|\tfrac{1}{\sqrt{15}}\left(|0\rangle + i|1\rangle - 2i|2\rangle + 3|-2\rangle\right)\right|^2 = \left|3\tfrac{1}{\sqrt{15}}\right|^2 = \tfrac{9}{15}$$

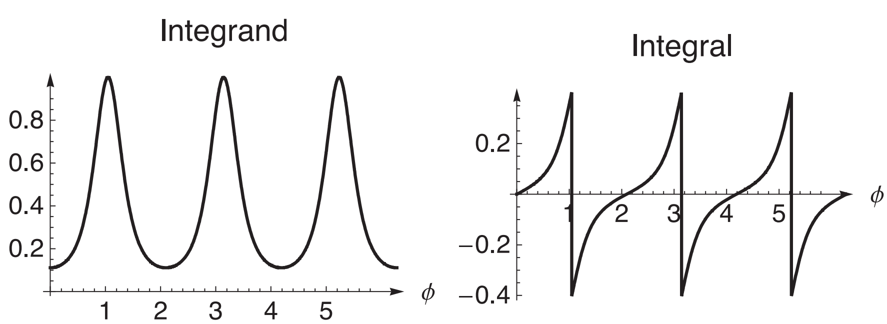
The expectation value of L_z is

$$\begin{aligned}\langle L_z\rangle = \langle\psi|L_z|\psi\rangle &= \sum_{m=-\infty}^{\infty} m\hbar\,\mathcal{P}_{L_z=m\hbar} = 2\hbar\mathcal{P}_{L_z=+2\hbar} + 1\hbar\mathcal{P}_{L_z=+1\hbar} + 0\hbar\mathcal{P}_{L_z=0\hbar} + (-2\hbar)\mathcal{P}_{L_z=-2\hbar} \\ &= 2\hbar\tfrac{4}{15} + 1\hbar\tfrac{1}{15} + 0\hbar\tfrac{1}{15} + (-2\hbar)\tfrac{9}{15} = -\tfrac{9}{15}\hbar = -\tfrac{3}{5}\hbar\end{aligned}$$

7.15 a) Normalize the wave function:

$$\begin{aligned}1 = \langle\psi|\psi\rangle = \int_0^{2\pi}\left|\frac{N}{2+\cos 3\phi}\right|^2 d\phi &= |N|^2\int_0^{2\pi}\frac{1}{4+4\cos 3\phi+\cos^2 3\phi}\,d\phi \\ &= |N|^2\left[\frac{4}{9\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{3\phi}{2}\right) - \frac{\sin 3\phi}{9(2+\cos 3\phi)}\right]_0^{2\pi}\end{aligned}$$

The arctan function is problematic because of its limited range. Looking at the integrand and the resultant indefinite integral:



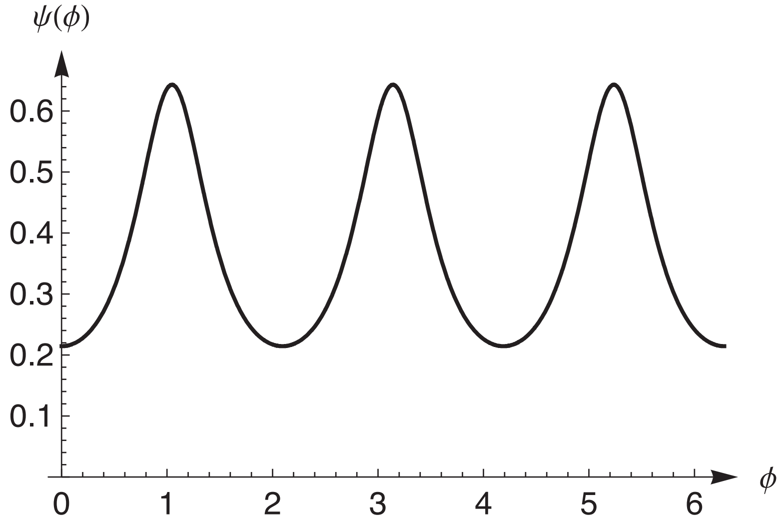
we see that we can do the integral from 0 to $2\pi/6$ and then multiply by 6 to avoid the arctan problem:

$$\begin{aligned}1 &= 6|N|^2\left[\frac{4}{9\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{3\phi}{2}\right) - \frac{\sin 3\phi}{9(2+\cos 3\phi)}\right]_0^{2\pi/6} \\ &= 6|N|^2\left[\frac{4}{9\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2}\right)\right] = 6|N|^2\left[\frac{4}{9\sqrt{3}}\frac{\pi}{2}\right] = |N|^2\left[\frac{4\pi}{3\sqrt{3}}\right]\end{aligned}$$

The normalized wave function is

$$\psi(\phi)=\sqrt{\frac{3\sqrt{3}}{4\pi}}\,\frac{1}{2+\cos 3\phi}$$

b) Plot:



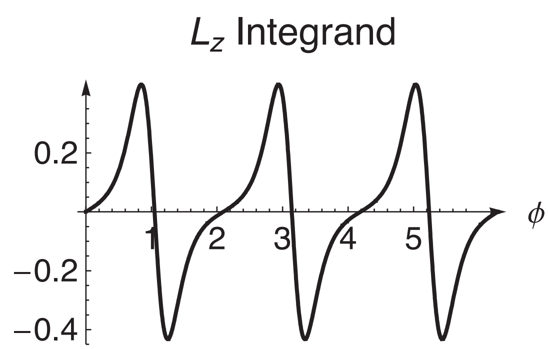
c) The expectation value of L_z is

$$\langle L_z\rangle=\langle\psi|L_z|\psi\rangle=\int_0^{2\pi}\psi^*(\phi)\;L_z\;\psi(\phi)\,d\phi=\int_0^{2\pi}\psi^*(\phi)\left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi(\phi)\,d\phi$$

Insert the wave function to get

$$\begin{aligned}\langle L_z\rangle&=\frac{3\sqrt{3}}{4\pi}\int_0^{2\pi}\frac{1}{2+\cos 3\phi}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\frac{1}{2+\cos 3\phi}\,d\phi\\&=-i\hbar\frac{3\sqrt{3}}{4\pi}\int_0^{2\pi}\frac{1}{2+\cos 3\phi}\;\frac{3\sin 3\phi}{\left(2+\cos 3\phi\right)^2}\,d\phi\\&=-i\hbar\frac{3\sqrt{3}}{4\pi}\int_0^{2\pi}\;\frac{3\sin 3\phi}{\left(2+\cos 3\phi\right)^3}\,d\phi\end{aligned}$$

Plot the integrand:



The integrand is clearly equally positive and negative, so the integral is zero. If it were not, then the expectation value would be imaginary, which is not physical.

$$\langle L_z \rangle = 0$$

7.16 The initial state is

$$|\psi\rangle = \sqrt{\tfrac{1}{5}}|2\rangle - i\sqrt{\tfrac{4}{5}}|-1\rangle$$

The initial wave function is

$$\psi(\phi,0) = \sqrt{\tfrac{1}{5}}\Phi_2(\phi) + i\sqrt{\tfrac{4}{5}}\Phi_{-1}(\phi) = \frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{1}{5}}e^{i2\phi} + i\frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{4}{5}}e^{-i\phi}$$

The time-evolved wave function is

$$\psi(\phi,t) = \frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{1}{5}}e^{i2\phi}e^{-i4\hbar t/2I} + i\frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{4}{5}}e^{-i\phi}e^{-i\hbar t/2I}$$

The probability density is

$$\begin{aligned}
\mathcal{P}(\phi,t) &= \left|\psi(\phi,t)\right|^2 = \psi^*(\phi,t)\psi(\phi,t) \\
&= \frac{1}{2\pi}\left(\sqrt{\tfrac{1}{5}}e^{-i2\phi}e^{i4\hbar t/2I} - i\sqrt{\tfrac{4}{5}}e^{i\phi}e^{i\hbar t/2I}\right)\left(\sqrt{\tfrac{1}{5}}e^{i2\phi}e^{-i4\hbar t/2I} + i\sqrt{\tfrac{4}{5}}e^{-i\phi}e^{-i\hbar t/2I}\right) \\
&= \frac{1}{2\pi}\left[1 + i\sqrt{\tfrac{4}{25}}e^{-i3\phi}e^{i3\hbar t/2I} - i\sqrt{\tfrac{4}{25}}e^{i3\phi}e^{-i3\hbar t/2I}\right] \\
&= \frac{1}{2\pi}\left[1 + \tfrac{4}{5}\sin\left(3\phi - 3\hbar t/2I\right)\right]
\end{aligned}$$

7.17 The probability density in general for a two-state superposition of the form [Eq. (7.114)]

$$\psi(\phi,t) = c_1\Phi_{m_1}(\phi)e^{-iE_{|m_1|}t/\hbar} + c_2e^{i\theta}\Phi_{m_2}(\phi)e^{-iE_{|m_2|}t/\hbar}$$

is [Eq. (7.115)]

$$\mathcal{P}(\phi,t) = \left|\psi(\phi,t)\right|^2 = \frac{1}{2\pi}\left[1 + 2c_1c_2\cos\left\{(m_1 - m_2)\phi - \theta - \left(E_{|m_1|} - E_{|m_2|}\right)t/\hbar\right\}\right]$$

For this case we have

$$\mathcal{P}(\phi,t) = \frac{1}{2\pi}\left[1 - \frac{12}{13}\sin\left(3\phi + \frac{3\hbar}{2I}t\right)\right]$$

So we have the parameters

$$
\begin{aligned}
c_1 c_2 &= \frac{6}{13} \\
m_1 - m_2 &= 3 \\
E_{|m_1|} - E_{|m_2|} &= m_1^2 \frac{\hbar^2}{2I} - m_2^2 \frac{\hbar^2}{2I} = -\frac{3\hbar^2}{2I} \\
\theta &= \frac{3\pi}{2}
\end{aligned}
$$

Using the normalization condition $c_1^2 + c_2^2 = 1$ gives

$$
\begin{aligned}
&1 = c_1^2 + c_2^2 = c_1^2 + \left(\frac{6}{13c_1}\right)^2 = c_1^2 + \frac{36}{169c_1^2} \\
&c_1^4 - c_1^2 + \tfrac{36}{169} = 0 \quad \Rightarrow \quad c_1^2 = \tfrac{1}{2}\left(1 \pm \sqrt{1 - 4\tfrac{36}{169}}\right) = \tfrac{1}{2}\left(1 \pm \tfrac{5}{13}\right) = \tfrac{9}{13},\ \tfrac{4}{13}
\end{aligned}
$$

So we get

$$
c_1 = \sqrt{\frac{9}{13}},\ c_2 = \sqrt{\frac{4}{13}} \qquad or \qquad c_1 = \sqrt{\frac{4}{13}},\ c_2 = \sqrt{\frac{9}{13}}
$$

Both cases produce the same physical result. They cannot be distinguished with this data. The quantum numbers are determined from

$$
\begin{aligned}
m_1 - m_2 &= 3 \\
m_1^2 - m_2^2 &= (m_1 - m_2)(m_1 + m_2) = -3 \\
&\Rightarrow (m_1 + m_2) = -1
\end{aligned}
$$

Add and subtract the first and last equations to get

$$
m_1 = 1, \quad m_2 = -2
$$

The negative index is larger, so the wave moves toward negative ϕ, as the probability density evidences.
Hence the initial state of the particle is

$$
\begin{aligned}
\psi(\phi,0) &= \sqrt{\tfrac{4}{13}}\Phi_1(\phi) - i\sqrt{\tfrac{9}{13}}\Phi_{-2}(\phi) = \frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{4}{13}}e^{i\phi} - i\frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{9}{13}}e^{-i2\phi} \\
or \qquad & \sqrt{\tfrac{9}{13}}\Phi_1(\phi) - i\sqrt{\tfrac{4}{13}}\Phi_{-2}(\phi) = \frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{9}{13}}e^{i\phi} - i\frac{1}{\sqrt{2\pi}}\sqrt{\tfrac{4}{13}}e^{-i2\phi}
\end{aligned}
$$

or in ket notation:

$$
|\psi(0)\rangle = \tfrac{1}{\sqrt{13}}\left(2|1\rangle - i3|-2\rangle\right) \qquad or \qquad \tfrac{1}{\sqrt{13}}\left(3|1\rangle - i2|-2\rangle\right)
$$

7.18 The moment of inertia is

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

To write this in terms of the reduced mass and the bond length, note that in the center of mass frame

$$\begin{aligned} m_1 r_1 - m_2 r_2 &= 0 \\ r_1 + r_2 &= r_0 \end{aligned}$$

Solving these two equations for r_1 and r_2 gives

$$\begin{aligned} r_1 &= r_0 \frac{m_2}{m_1 + m_2} = r_0 \frac{\mu}{m_1} \\ r_2 &= r_0 \frac{m_1}{m_1 + m_2} = r_0 \frac{\mu}{m_2} \end{aligned}$$

where the reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Substituting back into the moment of inertia gives

$$\begin{aligned} I = m_1 r_1^2 + m_2 r_2^2 &= m_1 \left(r_0 \frac{\mu}{m_1} \right)^2 + m_2 r_2^2 \left(r_0 \frac{\mu}{m_2} \right)^2 \\ &= r_0^2 \frac{\mu^2}{m_1} + r_0^2 \frac{\mu^2}{m_2} = \mu^2 r_0^2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \mu r_0^2 \end{aligned}$$

7.19 The parameters for hydrogen iodide are

$$m_1 = 1 amu, \quad m_2 = 127 amu, \quad r_0 = 0.1609 nm$$

The reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{127}{128} amu$$

The rotational constant is

$$\begin{aligned} \frac{\hbar^2}{2I} = \frac{\hbar^2}{2\mu r_0^2} = \frac{(hc/2\pi)^2}{2\mu c^2 r_0^2} &= \frac{(1240 eVnm/2\pi)^2}{2(127/128 amu)(931.5\, MeV/amu)(0.1609 nm)^2} \\ &= 0.814 meV = 6.56 cm^{-1} \end{aligned}$$

This agrees well with the published value of 6.42 cm^{-1}.

7.20 a) Let $n = p + 2$ (i.e., $p = n - 2$)

$$\sum_{n=0}^{3} n = \sum_{p=-2}^{1} (p+2)$$

and then change the dummy *p* back to *n*:

$$\sum_{n=0}^{3} n = \sum_{p=-2}^{1} (p+2) = \sum_{n=-2}^{1} (n+2)$$

Now write out the 2 expressions

$$\sum_{n=0}^{3} n = 0 + 1 + 2 + 3$$

$$\sum_{n=-2}^{1} (n+2) = 0 + 1 + 2 + 3$$

to see that they are the same.

b) Let $n = p + 2$ (i.e., $p = n - 2$)

$$\sum_{n=1}^{5} e^{in\phi} = \sum_{p=-1}^{3} e^{i(p+2)\phi}$$

and then change the dummy *p* back to *n*:

$$\sum_{n=1}^{5} e^{in\phi} = \sum_{p=-1}^{3} e^{i(p+2)\phi} = \sum_{n=-1}^{3} e^{i(n+2)\phi}$$

Now write out the 2 expressions

$$\sum_{n=1}^{5} e^{in\phi} = e^{i\phi} + e^{i2\phi} + e^{i3\phi} + e^{i4\phi} + e^{i5\phi}$$

$$\sum_{n=-1}^{3} e^{i(n+2)\phi} = e^{i\phi} + e^{i2\phi} + e^{i3\phi} + e^{i4\phi} + e^{i5\phi}$$

to see that they are the same.

c) Let $n = p + 2$ (i.e., $p = n - 2$)

$$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} = \sum_{p=-2}^{\infty} a_{p+2} (p+2)(p+1) z^{p}$$

and then change the dummy *p* back to *n*:

$$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} = \sum_{p=-2}^{\infty} a_{p+2} (p+2)(p+1) z^{p} = \sum_{n=-2}^{\infty} a_{n+2} (n+2)(n+1) z^{p}$$

Now write out the 2 expressions

$$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} = a_0 0(-1) z^{-2} + a_1 1(0) z^{-1} + a_2 2(1) z^0 + a_3 3(2) z^1 + a_4 4(5) z^2 + \ldots$$
$$= a_2 2 + a_3 6z + a_4 20 z^2 + \ldots$$
$$\sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1) z^n = a_0 0(-1) z^{-2} + a_1 1(0) z^{-1} + a_2 2(1) z^0 + a_3 3(2) z^1 + a_4 4(5) z^2 + \ldots$$
$$= a_2 2 + a_3 6z + a_4 20 z^2 + \ldots$$

to see that they are the same. In this case, we can change the lower summation limit because the 1st two terms are zero, yielding

$$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) z^n$$

as in Eq. (7.137).

7.21 Rodrigues' formula is

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}\left(z^2-1\right)^\ell$$

Work with the z form and then substitute $z = \cos\theta$ at the end. Starting with $\ell = 0$:

$$P_0(z) = \frac{1}{2^0 0!}\frac{d^0}{dz^0}\left(z^2-1\right)^0 = 1$$

Continuing with

$$P_1(z) = \frac{1}{2^1 1!}\frac{d^1}{dz^1}\left(z^2-1\right)^1 = \frac{1}{2}2z = z$$
$$P_2(z) = \frac{1}{2^2 2!}\frac{d^2}{dz^2}\left(z^2-1\right)^2 = \frac{1}{8}\frac{d}{dz}\left[2\left(z^2-1\right)2z\right] = \frac{1}{2}\left(3z^2-1\right)$$

Now they get a little messier

$$P_3(z) = \frac{1}{2^3 3!}\frac{d^3}{dz^3}\left(z^2-1\right)^3 = \frac{1}{48}\frac{d^2}{dz^2}\left[3\left(z^2-1\right)^2 2z\right] = \frac{1}{8}\frac{d}{dz}\left[2\left(z^2-1\right)2z^2 + \left(z^2-1\right)^2\right]$$
$$= \frac{1}{8}\frac{d}{dz}\left(5z^4 - 6z^2 + 1\right) = \frac{1}{2}\left(5z^3 - 3z\right)$$

and finally

$$
\begin{aligned}
P_4(z) &= \frac{1}{2^4 4!}\frac{d^4}{dz^4}\left(z^2-1\right)^4 = \frac{1}{384}\frac{d^3}{dz^3}\left[4\left(z^2-1\right)^3 2z\right] = \frac{1}{48}\frac{d^2}{dz^2}\left[3\left(z^2-1\right)^2 2z^2 + \left(z^2-1\right)^3\right] \\
&= \frac{1}{48}\frac{d}{dz}\left[12\left(z^2-1\right)2z^3 + 6\left(z^2-1\right)^2 2z + 3\left(z^2-1\right)^2 2z\right] \\
&= \frac{1}{48}\left[48z^4 + 24\left(z^2-1\right)3z^2 + 36\left(z^2-1\right)2z^2 + 18\left(z^2-1\right)^2\right] \\
&= \frac{1}{8}\left(35z^4 - 30z^2 + 3\right)
\end{aligned}
$$

The θ forms are

$$
\begin{aligned}
P_0(\theta) &= 1 \\
P_1(\theta) &= \cos\theta \\
P_2(\theta) &= \frac{1}{2}\left(3\cos^2\theta - 1\right) \\
P_3(\theta) &= \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) \\
P_4(\theta) &= \frac{1}{8}\left(35\cos^4\theta - 30\cos^2\theta + 3\right)
\end{aligned}
$$

Now check orthogonality:

$$
\begin{aligned}
\int_0^\pi P_2(\theta) P_4(\theta)\sin\theta\, d\theta &= \int_0^\pi \frac{1}{2}\left(3\cos^2\theta - 1\right)\frac{1}{8}\left(35\cos^4\theta - 30\cos^2\theta + 3\right)\sin\theta\, d\theta \\
&= \frac{1}{16}\int_0^\pi \left(105\cos^6\theta - 125\cos^4\theta + 39\cos^2\theta - 3\right)\sin\theta\, d\theta \\
&= -\frac{1}{16}\left[15\cos^7\theta - 25\cos^5\theta + 13\cos^3\theta - 3\cos\theta\right]_0^\pi \\
&= -\frac{1}{16}\left[-\left(15 - 25 + 13 - 3\right) - \left(15 - 25 + 13 - 3\right)\right] = 0
\end{aligned}
$$

and normalization

$$
\begin{aligned}
\int_0^\pi P_2(\theta) P_2(\theta)\sin\theta\, d\theta &= \int_0^\pi \frac{1}{2}\left(3\cos^2\theta - 1\right)\frac{1}{2}\left(3\cos^2\theta - 1\right)\sin\theta\, d\theta \\
&= \frac{1}{4}\int_0^\pi \left(9\cos^4\theta - 6\cos^2\theta + 1\right)\sin\theta\, d\theta \\
&= -\frac{1}{4}\left[\frac{9}{5}\cos^5\theta - 2\cos^3\theta + \cos\theta\right]_0^\pi \\
&= -\frac{1}{4}\left[-\left(\frac{9}{5} - 2 + 1\right) - \left(\frac{9}{5} - 2 + 1\right)\right] = \frac{2}{5} = \frac{2}{2\bullet 2 + 1}
\end{aligned}
$$

which agrees with Eq. (7.148).

7.22 Using Eq. (7.151):

$$P_\ell^m(z) = \left(1-z^2\right)^{m/2} \frac{d^m}{dz^m} P_\ell(z)$$

We have

$$P_2^1(z) = \left(1-z^2\right)^{1/2} \frac{d^1}{dz^1} P_2(z) = \left(1-z^2\right)^{1/2} \frac{d}{dz}\frac{1}{2}\left(3z^2-1\right) = \left(1-z^2\right)^{1/2}\frac{1}{2}6z = 3z\left(1-z^2\right)^{1/2}$$

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta$$

and

$$P_3^3(z) = \left(1-z^2\right)^{3/2} \frac{d^3}{dz^3} P_3(z) = \left(1-z^2\right)^{3/2} \frac{d^3}{dz^3}\frac{1}{2}\left(5z^3-3z\right) = \left(1-z^2\right)^{3/2}\frac{1}{2}\frac{d^2}{dz^2}\left(15z^2-3\right)$$

$$= \left(1-z^2\right)^{3/2}\frac{1}{2}\frac{d}{dz}(30z) = 15\left(1-z^2\right)^{3/2}$$

$$P_3^3(\cos\theta) = 15\sin^3\theta$$

7.23 Using Eq. (7.161):

$$Y_\ell^m(\theta,\phi) = (-1)^{(m+|m|/2)}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^m(\cos\theta)e^{im\phi}$$

we get

$$Y_1^0(\theta,\phi) = (-1)^{(0+|0|)/2}\sqrt{\frac{(2\bullet 1+1)}{4\pi}\frac{(1-|0|)!}{(1+|0|)!}}P_1^0(\cos\theta)e^{i0\phi} = \sqrt{\frac{3}{4\pi}}P_1^0(\cos\theta)$$

Using Eq. (7.151) for the associated Legendre function gives

$$P_1^0(z) = \left(1-z^2\right)^{0/2}\frac{d^0}{dz^0}P_1(z) = P_1(z) = z = 3z\left(1-z^2\right)^{1/2}$$

$$P_1^0(\cos\theta) = \cos\theta$$

So the resulting spherical harmonic is

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Similarly:

$$Y_2^{-2}(\theta,\phi)=(-1)^{(-2+|-2|)/2}\sqrt{\frac{(2\bullet 2+1)}{4\pi}\frac{(2-|-2|)!}{(2+|-2|)!}}P_2^{-2}(\cos\theta)e^{-i2\phi}=\sqrt{\frac{5}{4\pi\bullet 4!}}P_1^0(\cos\theta)e^{-i2\phi}$$

$$P_2^{-2}(z)=\left(1-z^2\right)^{2/2}\frac{d^2}{dz^2}P_2(z)=\left(1-z^2\right)\frac{d^2}{dz^2}\frac{1}{2}\left(3z^2-1\right)=3\left(1-z^2\right)$$

$$P_2^{-2}(\cos\theta)=3\sin^2\theta$$

yielding

$$Y_2^{-2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-i2\phi}$$

Check normalization and orthogonality

$$\begin{aligned}\langle 10|10\rangle &= \int_0^{2\pi}\int_0^{\pi}Y_1^{0*}(\theta,\phi)Y_1^0(\theta,\phi)\sin\theta d\theta d\phi\\ &=\int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{3}{4\pi}}\cos\theta\sqrt{\frac{3}{4\pi}}\cos\theta\sin\theta d\theta d\phi\\ &=\frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin\theta d\theta d\phi=\frac{3}{4\pi}2\pi\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi}=\frac{3}{2}\left(\frac{1}{3}+\frac{1}{3}\right)=1\end{aligned}$$

$$\begin{aligned}\langle 2,-2|2,-2\rangle &= \int_0^{2\pi}\int_0^{\pi}Y_2^{-2*}(\theta,\phi)Y_2^{-2}(\theta,\phi)\sin\theta d\theta d\phi\\ &=\int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{+i2\phi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-i2\phi}\sin\theta d\theta d\phi\\ &=\frac{15}{32\pi}\int_0^{2\pi}\int_0^{\pi}\sin^5\theta d\theta d\phi\\ &=\frac{15}{32\pi}2\pi\left[-\frac{1}{80}\cos 5\theta+\frac{5}{48}\cos 3\theta-\frac{5}{8}\cos\theta\right]_0^{\pi}=\frac{15}{16}\left(\frac{512}{480}\right)=1\end{aligned}$$

$$\begin{aligned}\langle 10|2,-2\rangle &= \int_0^{2\pi}\int_0^{\pi}Y_1^{0*}(\theta,\phi)Y_2^{-2}(\theta,\phi)\sin\theta d\theta d\phi\\ &=\int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{3}{4\pi}}\cos\theta\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-i2\phi}\sin\theta d\theta d\phi\\ &=\frac{3}{8\pi}\sqrt{\frac{5}{2}}\int_0^{2\pi}\int_0^{\pi}\cos\theta\sin^3\theta e^{-i2\phi}d\theta d\phi\\ &=\frac{3}{8\pi}\sqrt{\frac{5}{2}}\left[\frac{e^{-i2\phi}}{-2i}\right]_0^{2\pi}\left[\frac{1}{4}\sin^4\theta\right]_0^{\pi}=0\end{aligned}$$

7.24 The position representation of L_z is

$$L_z \doteq -i\hbar \frac{\partial}{\partial \phi}$$

Apply this to the spherical harmonics:

$$\begin{aligned}
L_z Y_\ell^m(\theta,\phi) &= -i\hbar \frac{\partial}{\partial \phi} Y_\ell^m(\theta,\phi) \\
&= -i\hbar \frac{\partial}{\partial \phi}\left[(-1)^{(m+|m|/2)}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^m(\cos\theta)e^{im\phi}\right] \\
&= -i\hbar\left[(-1)^{(m+|m|/2)}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^m(\cos\theta)\right]\frac{\partial}{\partial \phi}\left[e^{im\phi}\right] \\
&= -i\hbar\left[(-1)^{(m+|m|/2)}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^m(\cos\theta)\right]\left[ime^{im\phi}\right] \\
&= m\hbar\left[(-1)^{(m+|m|/2)}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^m(\cos\theta)e^{im\phi}\right] \\
&= m\hbar Y_\ell^m(\theta,\phi)
\end{aligned}$$

Thus the spherical harmonics are eigenstates of L_z with eigenvalues $m\hbar$.

7.25 The position representation of $\mathbf{L}^2$ is

$$\boxed{\mathbf{L}^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}$$

Apply this to the spherical harmonics (using a shorthand for the normalization constant):

$$
\begin{aligned}
\mathbf{L}^2 Y_\ell^m(\theta,\phi) &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_\ell^m(\theta,\phi)\\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]N_{\ell m}P_\ell^m(\cos\theta)e^{im\phi}\\
&= -\hbar^2 N_{\ell m}\left[e^{im\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)P_\ell^m(\cos\theta)+P_\ell^m(\cos\theta)\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}e^{im\phi}\right]\\
&= -\hbar^2 N_{\ell m}\left[e^{im\phi}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)P_\ell^m(\cos\theta)+P_\ell^m(\cos\theta)\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}e^{im\phi}\right]\\
&= -\hbar^2 N_{\ell m}\left[e^{im\phi}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)P_\ell^m(\cos\theta)+P_\ell^m(\cos\theta)\frac{1}{\sin^2\theta}(im)^2 e^{im\phi}\right]\\
&= -\hbar^2 N_{\ell m}\left[e^{im\phi}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)P_\ell^m(\cos\theta)-\frac{m^2}{\sin^2\theta}P_\ell^m(\cos\theta)e^{im\phi}\right]
\end{aligned}
$$

The associated Legendre functions obey the differential equation (7.151)

$$
\begin{aligned}
&\left(\left(1-z^2\right)\frac{d^2}{dz^2}-2z\frac{d}{dz}+\ell(\ell+1)-\frac{m^2}{\left(1-z^2\right)}\right)P_\ell^m(z)=0\\
&\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)P_\ell^m(\cos\theta)=0\\
\Rightarrow\qquad &\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)P_\ell^m(\cos\theta)=\left(-\ell(\ell+1)+\frac{m^2}{\sin^2\theta}\right)P_\ell^m(\cos\theta)
\end{aligned}
$$

Substituting gives

$$
\begin{aligned}
\mathbf{L}^2 Y_\ell^m(\theta,\phi) &= -\hbar^2 N_{\ell m}\left[e^{im\phi}\left(-\ell(\ell+1)+\frac{m^2}{\sin^2\theta}\right)P_\ell^m(\cos\theta)-\frac{m^2}{\sin^2\theta}P_\ell^m(\cos\theta)e^{im\phi}\right]\\
&= -\hbar^2\left[-\ell(\ell+1)+\frac{m^2}{\sin^2\theta}-\frac{m^2}{\sin^2\theta}\right]N_{\ell m}P_\ell^m(\cos\theta)e^{im\phi}\\
&= \ell(\ell+1)\hbar^2 Y_\ell^m(\theta,\phi)
\end{aligned}
$$

Thus the spherical harmonics are eigenstates of $\mathbf{L}^2$ with eigenvalues $\ell(\ell+1)\hbar^2$.

7.26 These new operators are

$$
\begin{aligned}
L_\pm = L_x \pm iL_y &\doteq i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\pm i\bullet i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\\
&\doteq \hbar\left[\left(i\sin\phi\pm\cos\phi\right)\frac{\partial}{\partial\theta}+\cot\theta\left(i\cos\phi\mp\sin\phi\right)\frac{\partial}{\partial\phi}\right]\\
&\doteq \hbar\left[\pm\left(\cos\phi\pm i\sin\phi\right)\frac{\partial}{\partial\theta}+i\cot\theta\left(\cos\phi\pm i\sin\phi\right)\frac{\partial}{\partial\phi}\right]\\
&\doteq \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Acting on the spherical harmonics gives

$$
\begin{aligned}
L_\pm Y_1^0(\theta,\phi) &= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{4\pi}}\cos\theta\\
&= \hbar e^{\pm i\phi}\sqrt{\frac{3}{4\pi}}\left(\pm\left(-\sin\theta\right)+0\right)=\mp\hbar\sqrt{\frac{3}{4\pi}}\sin\theta e^{\pm i\phi}=\hbar\sqrt{2}Y_1^{\pm1}(\theta,\phi)\\
L_\pm Y_1^1(\theta,\phi) &= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)(-1)\sqrt{\frac{3}{8\pi}}\sin\theta e^{+i\phi}\\
&= -\hbar e^{\pm i\phi}\sqrt{\frac{3}{8\pi}}\left(\pm\cos\theta e^{+i\phi}+i\cot\theta\sin\theta i e^{+i\phi}\right)=-\hbar e^{\pm i\phi}\sqrt{\frac{3}{8\pi}}e^{+i\phi}\left(\pm\cos\theta-\cos\theta\right)\\
&= \begin{cases} 0 & ,\ + \\ \hbar 2\sqrt{\frac{3}{8\pi}}\cos\theta & ,\ - \end{cases} = \begin{cases} 0 & ,\ + \\ \hbar\sqrt{2}Y_1^0(\theta,\phi) & ,\ - \end{cases}\\
L_\pm Y_1^{-1}(\theta,\phi) &= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\\
&= \hbar e^{\pm i\phi}\sqrt{\frac{3}{8\pi}}\left(\pm\cos\theta e^{-i\phi}-i\cot\theta\sin\theta i e^{-i\phi}\right)=\hbar e^{\pm i\phi}\sqrt{\frac{3}{8\pi}}e^{-i\phi}\left(\pm\cos\theta+\cos\theta\right)\\
&= \begin{cases} \hbar 2\sqrt{\frac{3}{8\pi}}\cos\theta & ,\ + \\ 0 & ,\ - \end{cases} = \begin{cases} \hbar\sqrt{2}Y_1^0(\theta,\phi) & ,\ + \\ 0 & ,\ - \end{cases}
\end{aligned}
$$

In Dirac notation these are

$$L_+|11\rangle = 0 \qquad L_-|11\rangle = \hbar\sqrt{2}|10\rangle$$
$$L_+|10\rangle = \hbar\sqrt{2}|11\rangle \qquad L_-|10\rangle = \hbar\sqrt{2}|1,-1\rangle$$
$$L_+|1,-1\rangle = \hbar\sqrt{2}|10\rangle \qquad L_-|1,-1\rangle = 0$$

These operators raise or lower the magnetic quantum number, so are called raising or lowering operators. See Chapter 11.

7.27 The $\ell = 1$ spherical harmonics are

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} = \sqrt{\frac{3}{4\pi}}\frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{+i\phi} = -\sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi + i\sin\phi) = -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{\sqrt{x^2+y^2+z^2}}$$
$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} = \sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi - i\sin\phi) = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{\sqrt{x^2+y^2+z^2}}$$

Combine the $m = 1$ functions to get x or y in the numerator:

$$\frac{1}{\sqrt{2}}\left[-Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right] = \sqrt{\frac{3}{4\pi}}\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$\frac{i}{\sqrt{2}}\left[Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right] = \sqrt{\frac{3}{4\pi}}\frac{y}{\sqrt{x^2+y^2+z^2}}$$

Together with the $m = 1$ function, these are known as the "real" spherical harmonics. These are the p_x, p_y, p_z orbitals.

7.28 The expansion of a general function in terms of Legendre polynomials is

$$f(z) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(z)$$

To find the expansion coefficients, use Fourier's trick, noting that the Legendre polynomials are not normalized to unity

$$\int_{-1}^{1} P_k(z) f(z)\,dz = \int_{-1}^{1} P_k(z)\sum_{\ell=0}^{\infty} c_\ell P_\ell(z)\,dz = \sum_{\ell=0}^{\infty}\int_{-1}^{1} P_k(z) c_\ell P_\ell(z)\,dz$$
$$= c_k \frac{2}{2k+1}$$

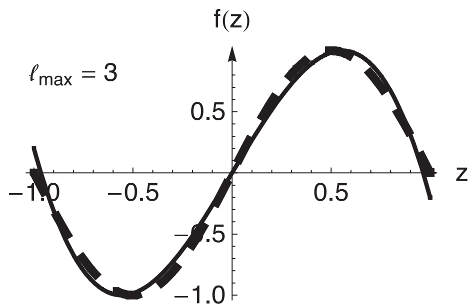
This gives the coefficients

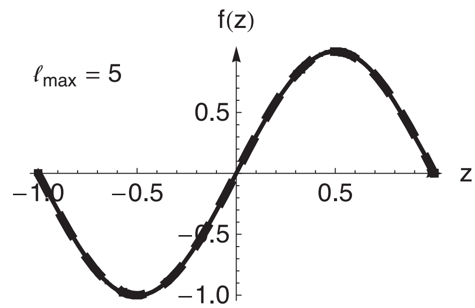
$$c_\ell = \left(\ell + \tfrac{1}{2}\right)\int_{-1}^{1} P_\ell(z) f(z)\,dz$$

For the function $f(z) = \sin \pi z$, Mathematica gives the coefficients

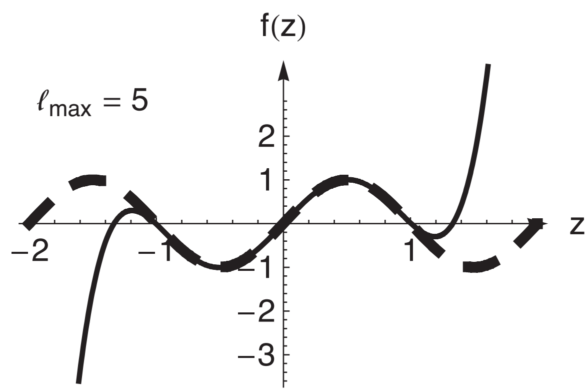
$$c_\ell = 0,\ 0.955,\ \ 0,\ \ -1.158,\ \ 0,\ \ 0.219,\ \ 0,\ \ -0.017,\ \ 0,\ \ 0.0007,\ \ \ldots$$

The 7th term is 1% of the 1st and 3rd terms (all even terms are zero), so a sum to $\ell = 5$ should provide a good approximation at the 1% level. The plots below confirm that.





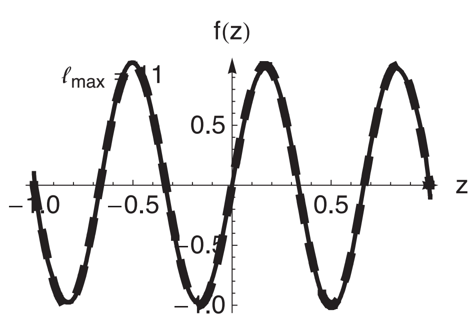
Over the range $-2 < z < 2$, the fit is terrible (see below) because the Legendre polynomials are designed to be used over the range $-1 < z < 1$.

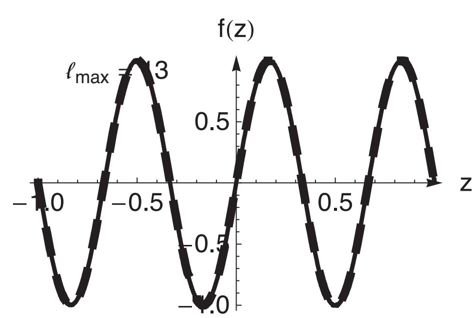


For the function $f(z) = \sin 3\pi z$, Mathematica gives the coefficients

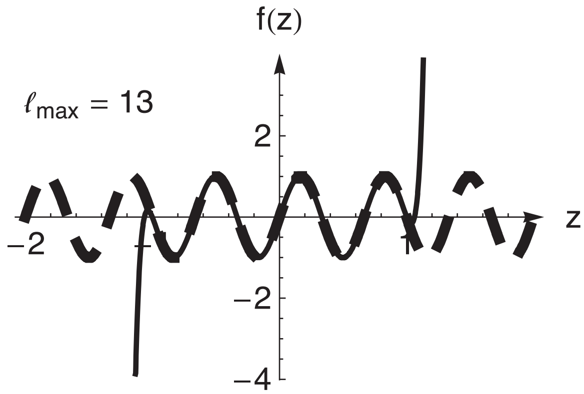
$$c_\ell = 0,\ 0.318,\ 0,\ 0.617,\ 0,\ -0.073,\ 0,\ -1.99,\ 0,\ 1.59,\ 0,\ -0.557,\ 0,\ 0.116,\ 0,\ -0.016,\ \ldots$$

The 15th term is 1% of the 7th and 9th terms (all even terms are zero), so a sum to $\ell = 13$ should provide a good approximation at the 1% level. The plots below confirm that.





Again, the fit over the range $-2 < z < 2$ is terrible (see below).



7.29 a) For this state the probabilities of an L_z measurement are

$$\mathcal{P}_{L_z=+2h} = \sum_{\ell=2}^{\infty}\left|\langle \ell 2|\psi\rangle\right|^2 = \sum_{\ell=2}^{\infty}\left|\left\langle \ell 2\left|\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right.\right\rangle\right|^2 = 0$$

$$\mathcal{P}_{L_z=-1h} = \sum_{\ell=1}^{\infty}\left|\langle \ell,-1|\psi\rangle\right|^2 = \sum_{\ell=1}^{\infty}\left|\left\langle \ell,1\left|\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right.\right\rangle\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{L_z=0h} = \sum_{\ell=0}^{\infty}\left|\langle \ell 0|\psi\rangle\right|^2 = \sum_{\ell=0}^{\infty}\left|\left\langle \ell 0\left|\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right.\right\rangle\right|^2 = \tfrac{1}{3}+\tfrac{1}{6} = \tfrac{1}{2}$$

b) The expectation value of L_z is

$$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\,\mathcal{P}_{L_z=mh} = 0\hbar\mathcal{P}_{L_z=0h} + (-1\hbar)\mathcal{P}_{L_z=-1h} = 0\hbar\tfrac{1}{2} + (-1\hbar)\tfrac{1}{2} = -\tfrac{1}{2}\hbar$$

c) The expectation value of $\mathbf{L}^2$ is

$$\begin{aligned}\langle \mathbf{L}^2\rangle &= \langle\psi|\mathbf{L}^2|\psi\rangle = \sum_{\ell m} c_{\ell m}^{*}\langle \ell m|\mathbf{L}^2\sum_{\ell' m'} c_{\ell' m'}|\ell' m'\rangle = \sum_{\ell m}|c_{\ell m}|^2\ell(\ell+1)\hbar^2\\ &= \tfrac{1}{2}2\hbar^2 + \tfrac{1}{3}2\hbar^2 + \tfrac{1}{6}0\hbar^2 = \tfrac{5}{3}\hbar^2\end{aligned}$$

d) The expectation value of the energy is

$$\begin{aligned}\langle E\rangle &= \langle\psi|H|\psi\rangle = \sum_{\ell m} c_{\ell m}^{*}\langle \ell m|\frac{\mathbf{L}^2}{2I}\sum_{\ell' m'} c_{\ell' m'}|\ell' m'\rangle = \sum_{\ell m}|c_{\ell m}|^2\ell(\ell+1)\frac{\hbar^2}{2I}\\ &= \tfrac{1}{2}2\frac{\hbar^2}{2I} + \tfrac{1}{3}2\frac{\hbar^2}{2I} + \tfrac{1}{6}0\frac{\hbar^2}{2I} = \tfrac{5}{6}\frac{\hbar^2}{I}\end{aligned}$$

e) The expectation value of the angular momentum component L_y is

$$\langle L_y\rangle = \langle\psi|L_y|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\,\mathcal{P}_{L_y=mh}$$

To find the probabilities of L_y measurements, use the L_y eigenstates in the calculations. Note that the $\ell = 0$ single eigenstate $|00\rangle$ is the same in all bases ($|00\rangle = |00\rangle_x = |00\rangle_y$). The $\ell = 1$ eigenstates are the same as the spin 1 system (see p. 60):

$$\begin{aligned}
|11\rangle_y &= \tfrac{1}{2}|11\rangle + i\tfrac{1}{\sqrt{2}}|10\rangle - \tfrac{1}{2}|1,-1\rangle \\
|10\rangle_y &= \tfrac{1}{\sqrt{2}}|11\rangle + \tfrac{1}{\sqrt{2}}|1,-1\rangle \\
|1,-1\rangle_y &= \tfrac{1}{2}|11\rangle - i\tfrac{1}{\sqrt{2}}|10\rangle - \tfrac{1}{2}|1,-1\rangle
\end{aligned}$$

Thus the probabilities are

$$\begin{aligned}
\mathcal{P}_{L_y=+1\hbar} &= \sum_{\ell=1}^{\infty} \left| {}_y\langle \ell 1|\psi\rangle \right|^2 = \left| {}_y\langle 11|\psi\rangle \right|^2 = \\
&= \left| \left(\tfrac{1}{2}\langle 11| - i\tfrac{1}{\sqrt{2}}\langle 10| - \tfrac{1}{2}\langle 1,-1| \right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle \right) \right|^2 = \left| -i\tfrac{1}{\sqrt{6}} - \tfrac{1}{2\sqrt{2}} \right|^2 = \tfrac{1}{6} + \tfrac{1}{8} = \tfrac{7}{24} \\
\mathcal{P}_{L_y=0\hbar} &= \sum_{\ell=0}^{\infty} \left| {}_y\langle \ell 0|\psi\rangle \right|^2 = \left| {}_y\langle 00|\psi\rangle \right|^2 + \left| {}_y\langle 10|\psi\rangle \right|^2 = \\
&= \left| \langle 00|\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle \right) \right|^2 + \left| \left(\tfrac{1}{\sqrt{2}}\langle 11| + \tfrac{1}{\sqrt{2}}\langle 1,-1| \right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle \right) \right|^2 \\
&= \tfrac{1}{6} + \tfrac{1}{4} = \tfrac{10}{24} \\
\mathcal{P}_{L_y=-1\hbar} &= \sum_{\ell=1}^{\infty} \left| {}_y\langle \ell,-1|\psi\rangle \right|^2 = \left| {}_y\langle 1,-1|\psi\rangle \right|^2 = \\
&= \left| \left(\tfrac{1}{2}\langle 11| + i\tfrac{1}{\sqrt{2}}\langle 10| - \tfrac{1}{2}\langle 1,-1| \right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle \right) \right|^2 = \tfrac{1}{6} + \tfrac{1}{8} = \tfrac{7}{24}
\end{aligned}$$

The three probabilities add to unity, as they must. The expectation value of the angular momentum component L_y is

$$\begin{aligned}
\langle L_y \rangle &= \langle \psi|L_y|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\,\mathcal{P}_{L_y=m\hbar} = 1\hbar\mathcal{P}_{L_y=1\hbar} + 0\hbar\mathcal{P}_{L_y=0\hbar} + (-1\hbar)\mathcal{P}_{L_y=-1\hbar} \\
&= 1\hbar\tfrac{7}{24} + 0\hbar\tfrac{10}{24} + (-1\hbar)\tfrac{7}{24} = 0\hbar
\end{aligned}$$

7.30 a) Find the coefficients with overlap integrals

$$c_{\ell m} = \langle \ell m|\psi\rangle = \int_0^{2\pi}\int_0^{\pi} Y_\ell^{m*}(\theta,\phi)\,\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

For the first four spherical harmonics, we get

$$c_{00} = \langle 00|\psi\rangle = \int_0^{2\pi}\int_0^{\pi} Y_0^{0*}(\theta,\phi)\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{\sqrt{4\pi}} N\left(\frac{\pi^2}{4}-\theta^2\right)\sin\theta\, d\theta\, d\phi$$
$$= \frac{2\pi N}{\sqrt{4\pi}}\int_0^{\pi/2}\left(\frac{\pi^2}{4}-\theta^2\right)\sin\theta\, d\theta = \frac{2\pi N}{\sqrt{4\pi}}\left[-\frac{\pi^2}{4}\cos\theta - 2\theta\sin\theta + \left(\theta^2-2\right)\cos\theta\right]_0^{\pi/2}$$
$$= \frac{2\pi}{\sqrt{4\pi}}\left[\frac{\pi^2}{4}-\pi+2\right]\frac{1}{\sqrt{\frac{\pi^5}{8}+2\pi^3-24\pi^2+48\pi}} = \frac{\frac{\pi^2}{4}-\pi+2}{\sqrt{\frac{\pi^4}{8}+2\pi^2-24\pi+48}} \cong 0.6238$$

$$c_{10} = \langle 10|\psi\rangle = \int_0^{2\pi}\int_0^{\pi} Y_1^{0*}(\theta,\phi)\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\sqrt{\frac{3}{4\pi}}\cos\theta N\left(\frac{\pi^2}{4}-\theta^2\right)\sin\theta\, d\theta\, d\phi = \frac{2\pi N\sqrt{3}}{\sqrt{4\pi}}\int_0^{\pi/2}\left(\frac{\pi^2}{4}-\theta^2\right)\cos\theta\sin\theta\, d\theta$$
$$= \frac{2\pi N\sqrt{3}}{\sqrt{4\pi}}\left[\frac{\pi^2}{8}\sin^2\theta - \frac{1}{4}\theta\sin 2\theta + \frac{1}{8}\left(2\theta^2-1\right)\cos 2\theta\right]_0^{\pi/2}$$
$$= \frac{\pi\sqrt{3}}{\sqrt{\pi}}\left[\frac{\pi^2}{8}-\frac{1}{16}\pi^2+\frac{1}{4}\right]\frac{1}{\sqrt{\frac{\pi^5}{8}+2\pi^3-24\pi^2+48\pi}}$$
$$= \frac{\sqrt{3}\left(\frac{\pi^2}{16}+\frac{1}{4}\right)}{\sqrt{\frac{\pi^4}{8}+2\pi^2-24\pi+48}} \cong 0.7064$$

$$c_{11} = \langle 11|\psi\rangle = \int_0^{2\pi}\int_0^{\pi} Y_1^{1*}(\theta,\phi)\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi/2}(-1)\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} N\left(\frac{\pi^2}{4}-\theta^2\right)\sin\theta\, d\theta\, d\phi$$
$$= (-1)N\sqrt{\frac{3}{8\pi}}\left\{\int_0^{\pi/2}\sin\theta\left(\frac{\pi^2}{4}-\theta^2\right)\sin\theta\, d\theta\right\}\left\{\int_0^{2\pi}\left(\cos\phi + i\sin\phi\right)d\phi\right\}$$
$$= 0$$

Likewise $c_{1,-1} = \langle 1,-1|\psi\rangle = 0$.

b) The probability of an $\mathbf{L}^2$ measurement is

$$\mathcal{P}_{\mathbf{L}^2=\ell(\ell+1)\hbar^2} = \sum_{m=-\ell}^{\ell}\left|\langle \ell m|\psi\rangle\right|^2 = \sum_{m=-\ell}^{\ell}\left|c_{\ell m}\right|^2$$

For $\mathbf{L}^2 = 2\hbar^2$, $\ell = 1$, so the probability is

$$\mathcal{P}_{\ell=1} = \sum_{m=-1}^{1} \left|\langle 1m|\psi\rangle\right|^2 = \sum_{m=-1}^{1} \left|c_{1m}\right|^2 = \left|c_{1,-1}\right|^2 + \left|c_{10}\right|^2 + \left|c_{11}\right|^2$$

$$\cong 0 + \left(0.7064\right)^2 + 0 = 0.499$$

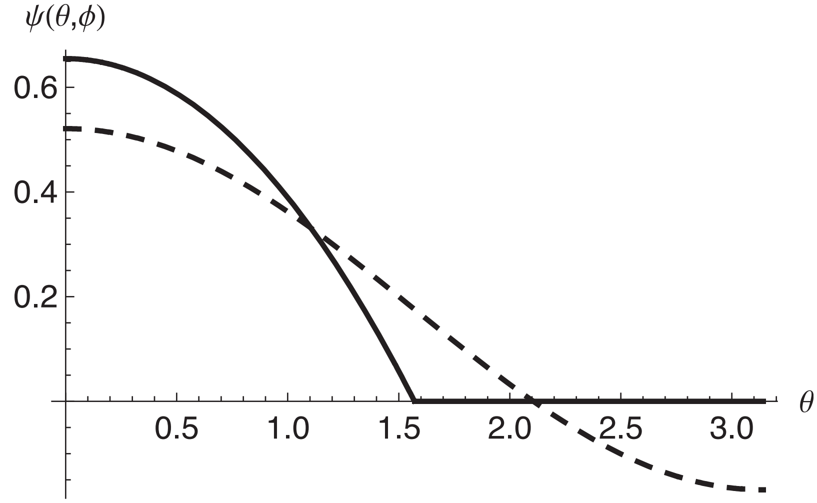
For $\mathbf{L}^2 = 0\hbar^2$, $\ell = 0$, so the probability is

$$\mathcal{P}_{\ell=0} = \sum_{m=-0}^{0} \left|\langle 0m|\psi\rangle\right|^2 = \sum_{m=-0}^{0} \left|c_{0m}\right|^2 = \left|c_{00}\right|^2 \cong \left(0.6238\right)^2 = 0.389$$

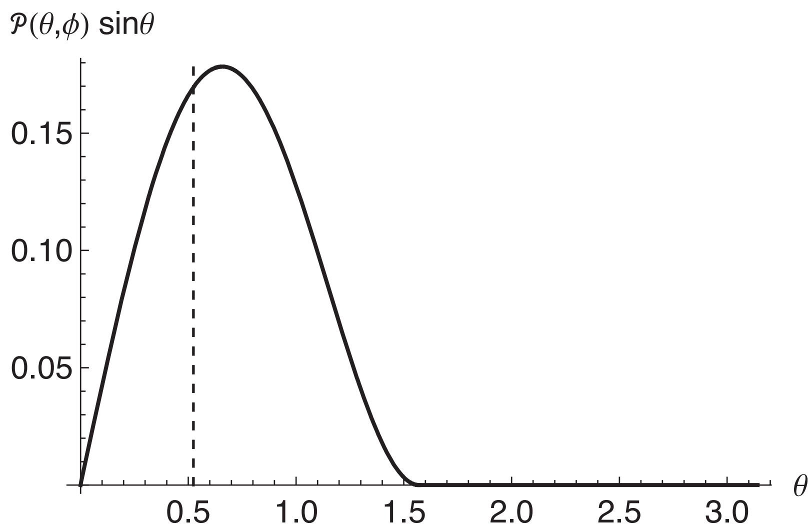
c) The probability of measuring the particle to be in the angular region $0 < \theta < \pi/6,\, 0 < \phi < \pi/6$ is

$$\begin{aligned}
\mathcal{P}_{0<\theta<\pi/6,0<\phi<\pi/6} &= \int_0^{2\pi}\int_0^{\pi} \mathcal{P}(\theta,\phi)\sin\theta d\theta d\phi = \int_0^{2\pi}\int_0^{\pi} \left|\psi(\theta,\phi)\right|^2 \sin\theta d\theta d\phi \\
&= \int_0^{\pi/6}\int_0^{\pi/6} \left|N\left(\frac{\pi^2}{4} - \theta^2\right)\right|^2 \sin\theta d\theta d\phi \\
&= N^2 \int_0^{\pi/6}\int_0^{\pi/6} \left(\frac{\pi^4}{16} - \frac{\pi^2}{2}\theta^2 + \theta^4\right)\sin\theta d\theta d\phi \\
&= \frac{\pi N^2}{6}\left[\begin{array}{l} -\dfrac{\pi^4}{16}\cos\theta - \pi^2\theta\sin\theta + \dfrac{\pi^2}{2}\left(\theta^2 - 2\right)\cos\theta \\ +4\theta\left(\theta^2 - 6\right)\sin\theta - \left(\theta^4 - 12\theta^2 + 24\right)\cos\theta \end{array}\right]_0^{\pi/6} \\
&= \frac{\pi}{6}\frac{1}{\dfrac{\pi^5}{8} + 2\pi^3 - 24\pi^2 + 48\pi}\left[\begin{array}{l} \dfrac{\pi^4}{16}\left(1 - \dfrac{\sqrt{3}}{2}\right) - \dfrac{\pi^3}{12} + \dfrac{\pi^2}{2}\left(\dfrac{\pi^2}{36} - 2\right)\dfrac{\sqrt{3}}{2} + \pi^2 \\ +\dfrac{\pi}{3}\left(\dfrac{\pi^2}{36} - 6\right) - \left(\dfrac{\pi^4}{1296} - \dfrac{\pi^2}{3} + 24\right)\dfrac{\sqrt{3}}{2} + 24 \end{array}\right] \\
&= \frac{1}{6}\frac{\pi^4\left(\dfrac{1}{16} - \dfrac{\sqrt{3}}{2}\right) - \pi^3\dfrac{2}{27} + \pi^2\left(1 - \dfrac{\sqrt{3}}{3}\right) - 2\pi + \left(24 - 12\sqrt{3}\right)}{\dfrac{\pi^4}{8} + 2\pi^2 - 24\pi + 48} \\
&\cong 0.0269
\end{aligned}$$

The plot below shows the wave function and the approximation (dashed) with the two spherical harmonic terms from part (a). Those two terms account for only 89% of the probability, so the match is OK, but not great.



The plot below shows the probability density times $\sin\theta$ as a function of the polar angle. The region between 0 and $\pi/6$ looks like about 1/3 of the probability. But the azimuthal fraction is only 1/12, so the total probability is expected to be $1/36 \simeq 0.03$, which agrees with the calculation above.



The probability of measuring the particle to be in the angular region $5\pi/6 < \theta < \pi, 0 < \phi < \pi/6$ is zero because the wave function is zero in the southern hemisphere ($\theta > \pi/2$).