14.1 To keep track of orders of the perturbation, let's use the λ notation from Chapter 10. At the end, we'll set $\lambda = 1$. Thus the differential equation (14.10) is

$$i\hbar \frac{dc_k(t)}{dt} = \sum_n c_n(t) e^{i(E_k - E_n)t/\hbar} \langle k|\lambda H'(t)|n\rangle$$

Now expand the coefficient c_n in a perturbation series

$$c_n = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$$

and get

$$i\hbar \frac{d}{dt}\left(c_k^{(0)} + \lambda c_k^{(1)} + \lambda^2 c_k^{(2)} + \ldots\right) = \sum_n \left(c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots\right) e^{i(E_k - E_n)t/\hbar} \langle k|\lambda H'(t)|n\rangle$$

$$i\hbar \left(\frac{dc_k^{(0)}}{dt} + \lambda \frac{dc_k^{(1)}}{dt} + \lambda^2 \frac{dc_k^{(2)}}{dt} + \ldots\right) = \sum_n \left(\lambda c_n^{(0)} + \lambda^2 c_n^{(1)} + \lambda^3 c_n^{(2)} + \ldots\right) e^{i(E_k - E_n)t/\hbar} \langle k|H'(t)|n\rangle$$

Now equate terms from the two sides of the equation with the same order of the perturbation:

$$\lambda^0: \quad i\hbar \frac{dc_k^{(0)}(t)}{dt} = 0$$

$$\lambda^1: \quad i\hbar \frac{dc_k^{(1)}(t)}{dt} = \sum_n c_n^{(0)}(t) e^{i(E_k - E_n)t/\hbar} \langle k|H'(t)|n\rangle$$

$$\lambda^2: \quad i\hbar \frac{dc_k^{(2)}(t)}{dt} = \sum_n c_n^{(1)}(t) e^{i(E_k - E_n)t/\hbar} \langle k|H'(t)|n\rangle$$

The first two of these equations are Eqs. (14.12) and (14.13).

14.2 The perturbation is

$$H'(t) = V_0 x^2 e^{-t/\tau}$$

Time dependent perturbation theory tells us that the coefficient to be in a new state $|2\rangle$ at time *T* after starting in state $|1\rangle$ is :

$$c_2(T) = \frac{1}{i\hbar}\int_0^T \langle 2|H'(t')|1\rangle e^{i(E_2 - E_1)t'/\hbar}\, dt'$$

For this case, we get

$$c_2(T) = \frac{1}{i\hbar}\int_0^T \langle 2|H'(t')|1\rangle e^{i(E_2 - E_1)t'/\hbar}\, dt' = \frac{1}{i\hbar}\int_0^T \langle 2|V_0 x^2 e^{-t'/\tau}|1\rangle e^{i(E_2 - E_1)t'/\hbar}\, dt'$$

$$= \frac{V_0}{i\hbar}\langle 2|x^2|1\rangle \int_0^T e^{-t'/\tau} e^{i(E_2 - E_1)t'/\hbar}\, dt'$$

Let's tackle the spatial matrix element first

$$
\begin{aligned}
\langle 2|x^2|1\rangle &= \int_0^L \varphi_2^*(x)x^2\varphi_1(x)\,dx = \frac{2}{L}\int_0^L x^2 \sin\frac{2\pi x}{L}\sin\frac{\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^L x^2\,\frac{1}{2}\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)dx \\
&= \frac{1}{\pi^3 L}\left[\begin{array}{l} 2\pi L^2 x\cos\dfrac{\pi x}{L} + \left(\pi^2 L x^2 - 2L^3\right)\sin\dfrac{\pi x}{L} \\ -\dfrac{1}{27}\left\{6\pi L^2 x\cos\dfrac{3\pi x}{L} + \left(9\pi^2 L x^2 - 2L^3\right)\sin\dfrac{3\pi x}{L}\right\}\end{array}\right]_0^L \\
&= -\frac{16L^2}{9\pi^2}
\end{aligned}
$$

Putting this into the equation above and doing the time integral, we get

$$
\begin{aligned}
c_2(T) &= \frac{V_0}{i\hbar}\left(-\frac{16L^2}{9\pi^2}\right)\int_0^T e^{-t'/\tau}e^{i(E_2-E_1)t'/\hbar}\,dt' = -\frac{16L^2V_0}{i9\hbar\pi^2}\int_0^T e^{-t'/\tau}e^{i\omega_{21}t'}\,dt' \\
&= -\frac{16L^2V_0}{i9\hbar\pi^2}\left[\frac{e^{i\omega_{21}t'-t'/\tau}}{i\omega_{21}-1/\tau}\right]_0^T = -\frac{16L^2V_0}{i9\hbar\pi^2}\left[\frac{e^{i\omega_{21}T-T/\tau}-1}{i\omega_{21}-1/\tau}\right]
\end{aligned}
$$

The unperturbed well energies are:

$$
E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \Rightarrow \quad \omega_{21} = \frac{E_2-E_1}{\hbar} = \frac{3\pi^2\hbar}{2mL^2}
$$

Putting this together we get

$$
\begin{aligned}
\mathcal{P}_{1->2}(T) &= |c_2(T)|^2 = \left|-\frac{16L^2V_0}{i9\hbar\pi^2}\left[\frac{e^{i\omega_{21}T-T/\tau}-1}{i\omega_{21}-1/\tau}\right]\right|^2 \\
&= \left(\frac{16L^2V_0}{9\hbar\pi^2}\right)^2\frac{1}{\omega_{21}^2+1/\tau^2}\left(e^{i\omega_{21}T-T/\tau}-1\right)\left(e^{-i\omega_{21}T-T/\tau}-1\right) \\
&= \left(\frac{16L^2V_0}{9\hbar\pi^2}\right)^2\frac{1}{\omega_{21}^2+1/\tau^2}\left[e^{-2T/\tau}+1-e^{-T/\tau}\left(e^{-i\omega_{21}T}+e^{i\omega_{21}T}\right)\right] \\
&= \left(\frac{16L^2V_0}{9\hbar\pi^2}\right)^2\frac{1}{\omega_{21}^2+1/\tau^2}\left[1+e^{-2T/\tau}-2e^{-T/\tau}\cos\omega_{21}T\right]
\end{aligned}
$$

In the long time limit, we get the probability

$$
\mathcal{P}_{1->2}(\infty) = \left(\frac{16L^2V_0}{9\hbar\pi^2}\right)^2\frac{1}{\omega_{21}^2+1/\tau^2}
$$

14.3 The perturbation is

$$H'(t) = V_0 x e^{-\alpha t^2}$$

Time dependent perturbation theory tells us that the coefficient to be in a new state $|f\rangle$ at time $t = \infty$ after starting in state $|1\rangle$ is :

$$c_f(\infty) = \frac{1}{i\hbar}\int_0^\infty \langle f|H'(t')|1\rangle e^{i(E_f - E_1)t'/\hbar}\, dt'$$

For this case, we get

$$\begin{aligned} c_f(\infty) &= \frac{1}{i\hbar}\int_0^\infty \langle f|H'(t')|1\rangle e^{i(E_f - E_1)t'/\hbar}\, dt' = \frac{1}{i\hbar}\int_0^\infty \langle f|V_0 x e^{-\alpha t'^2}|1\rangle e^{i(E_f - E_1)t'/\hbar}\, dt' \\ &= \frac{V_0}{i\hbar}\langle f|x|1\rangle \int_0^\infty e^{-\alpha t'^2} e^{i(E_f - E_1)t'/\hbar}\, dt' \end{aligned}$$

The time integral can be done without knowing the final state, so let's do that first:

$$c_f(\infty) = \frac{V_0}{i\hbar}\langle f|x|1\rangle \int_0^\infty e^{-\alpha t'^2} e^{i\omega_{f1}t'}\, dt' = \frac{V_0}{i\hbar}\langle f|x|1\rangle \left[\frac{e^{i\omega_{f1}t' - \alpha t'^2}}{i\omega_{f1} - \alpha}\right]_0^\infty = \frac{V_0}{i\hbar}\langle f|x|1\rangle \left(\frac{1}{\alpha - i\omega_{f1}}\right)$$

a) For the case of $|f\rangle = |2\rangle$, the spatial matrix element is

$$\begin{aligned} \langle 2|x|1\rangle &= \int_0^L \varphi_2^*(x) x \varphi_1(x)\, dx = \frac{2}{L}\int_0^L x \sin\frac{2\pi x}{L} \sin\frac{\pi x}{L}\, dx \\ &= \frac{2}{L}\int_0^L x \frac{1}{2}\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right) dx \\ &= \frac{1}{\pi^2 L}\left[\begin{matrix} L^2 \cos\frac{\pi x}{L} + \pi L x \sin\frac{\pi x}{L} \\ -\frac{1}{9}\left\{L^2 \cos\frac{3\pi x}{L} + 3\pi L x \sin\frac{3\pi x}{L}\right\} \end{matrix}\right]_0^L = -\frac{16L}{9\pi^2} \end{aligned}$$

Putting this together we get

$$\mathcal{P}_{1->2}(\infty) = |c_2(\infty)|^2 = \left|-\frac{16LV_0}{i9\hbar\pi^2}\left(\frac{1}{\alpha - i\omega_{21}}\right)\right|^2 = \left(\frac{16LV_0}{9\hbar\pi^2}\right)^2 \left(\frac{1}{\alpha^2 + \omega_{21}^2}\right)$$

where

$$\omega_{21} = \frac{E_2 - E_1}{\hbar} = \frac{3\pi^2\hbar}{2mL^2}$$

b) For the case of $|f\rangle = |3\rangle$, the spatial matrix element is

$$\begin{aligned}
\langle 3|x|1\rangle &= \int_0^L \varphi_3^*(x)x\varphi_1(x)\,dx = \frac{2}{L}\int_0^L x\sin\frac{3\pi x}{L}\sin\frac{\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^L x\frac{1}{2}\left(\cos\frac{2\pi x}{L}-\cos\frac{4\pi x}{L}\right)dx \\
&= \frac{1}{4\pi^2 L}\left[\begin{array}{l} L^2\cos\frac{2\pi x}{L}+2\pi Lx\sin\frac{2\pi x}{L} \\ -\frac{1}{4}\left\{L^2\cos\frac{4\pi x}{L}+4\pi Lx\sin\frac{4\pi x}{L}\right\}\end{array}\right]_0^L = 0
\end{aligned}$$

The matrix element is zero, so the probability is zero:

$$\mathcal{P}_{1->3}(\infty) = |c_3(\infty)|^2 = 0$$

14.4 The perturbation is

$$H' = \begin{cases} V_0 \ , & 0 < x < \frac{L}{2} \\ 0 \ , & \frac{L}{2} < x < L \end{cases}$$

Time dependent perturbation theory tells us that the coefficient to be in a new state $|2\rangle$ after starting in state $|1\rangle$ is :

$$c_2(t) = \frac{1}{i\hbar}\int_0^t \langle 2|H'(t')|1\rangle e^{i(E_2-E_1)t'/\hbar}\,dt'$$

In this case, the matrix element is time independent and is given by

$$\begin{aligned}
\langle 2|H'(t')|1\rangle &= \int_0^L \varphi_2^*(x)H'\varphi_1(x)\,dx = \int_0^{L/2}\varphi_2^*(x)\ V_0\ \varphi_1(x)\,dx + \int_{L/2}^L \varphi_2^*(x)\ 0\ \varphi_1(x)\,dx \\
&= \frac{2}{L}V_0\int_0^{L/2}\sin\frac{2\pi x}{L}\sin\frac{\pi x}{L}\,dx = \frac{2}{L}V_0\left[\frac{L\sin\left(\frac{\pi}{2}\right)}{2\pi}-\frac{L\sin\left(\frac{3\pi}{2}\right)}{6\pi}\right] \\
&= \frac{4}{3\pi}V_0
\end{aligned}$$

For the time integral we get

$$\begin{aligned}
c_2(t) &= \frac{1}{i\hbar}\int_0^t \langle 2|H'(t')|1\rangle e^{i(E_2-E_1)t'/\hbar}\,dt' = \frac{1}{i\hbar}H_{21}\int_0^t e^{i\omega_{21}t'}\,dt' = \frac{1}{i\hbar}H_{21}\left[\frac{e^{i\omega_{21}t'}}{i\omega_{21}}\right]_0^t \\
&= \frac{1}{i\hbar}H_{21}\left[\frac{e^{i\omega_{21}t}-1}{i\omega_{21}}\right] = \frac{1}{i\hbar}H_{21}\frac{1}{i\omega_{21}}e^{i\omega_{21}t/2}\left[e^{i\omega_{21}t/2}-e^{-i\omega_{21}t/2}\right] \\
&= \frac{1}{i\hbar}H_{21}\frac{2}{\omega_{21}}e^{i\omega_{21}t/2}\sin(\omega_{21}t/2)
\end{aligned}$$

The unperturbed well energies are:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \Rightarrow \quad E_2 - E_1 = \frac{3\pi^2\hbar^2}{2mL^2}$$

Putting this together we get

$$\mathcal{P}_{1->2}(T) = |c_2(T)|^2 = \left| \frac{1}{i\hbar} H_{21} \frac{2}{\omega_{21}} e^{i\omega_{21}T/2} \sin\frac{\omega_{21}}{2}T \right|^2 = \left[\frac{1}{\hbar}\frac{4}{3\pi} V_0 \frac{2}{\omega_{21}} \sin\frac{\omega_{21}}{2}T \right]^2$$

$$= \left[\frac{16mL^2V_0}{9\pi^3\hbar^2} \sin\left(\frac{3\pi^2\hbar T}{4mL^2} \right) \right]^2 = \frac{2^8 m^2 L^4 V_0^2}{3^4\pi^6\hbar^4} \sin^2\left(\frac{3\pi^2\hbar T}{4mL^2} \right)$$

14.5 The decaying field has time dependence

$$E(t) = \begin{cases} 0 & t < 0 \\ E_0 e^{-t/2\tau} e^{-i\omega_{21}t} & t > 0 \end{cases}$$

The Fourier transform of this is

$$\tilde{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} E_0 e^{-t/2\tau} e^{i\omega_{21}t} e^{-i\omega t}\, dt = \frac{E_0}{\sqrt{2\pi}} \int_0^{\infty} e^{-i(\omega-\omega_{21})t - t/2\tau}\, dt$$

$$= \frac{E_0}{\sqrt{2\pi}} \left[\frac{e^{-i(\omega-\omega_{21})t - t/2\tau}}{-i(\omega-\omega_{21}) - 1/2\tau} \right]_0^{\infty} = \frac{E_0}{\sqrt{2\pi}} \left[\frac{1}{i(\omega-\omega_{21}) + 1/2\tau} \right]$$

The radiated power is

$$P(\omega) = (factors)\left|\tilde{E}(\omega)\right|^2 = (factors)\frac{E_0^2}{2\pi} \left| \frac{1}{i(\omega-\omega_{21}) + 1/2\tau} \right|^2$$

$$= (factors)\left(\frac{1}{(\omega-\omega_{21})^2 + (1/2\tau)^2} \right)$$

The inverse of the decay time is the Einstein *A* coefficient, so the frequency spectrum has the form

$$g(\omega) \propto \frac{1}{(\omega-\omega_{21})^2 + (A_{21}/2)^2}$$

Converting to energy gives

$$g(E) \propto \frac{1}{(E-\hbar\omega_{21})^2 + (\hbar A_{21}/2)^2}$$

Now normalize:

$$1=\int_0^\infty g(E)\,dE=\int_0^\infty \frac{C}{\left(E-\hbar\omega_{21}\right)^2+\left(\hbar A_{21}/2\right)^2}\,dE$$

Change variables to $x=E-\hbar\omega_{21}$:

$$1=C\int_{-\hbar\omega_{21}}^{\infty}\frac{1}{x^2+\left(\hbar A_{21}/2\right)^2}\,dx=C\left[\frac{\tan^{-1}\left(2x/\hbar A_{21}\right)}{\hbar A_{21}/2}\right]_{-\hbar\omega_{21}}^{\infty}=\frac{2C}{\hbar A_{21}}\left[\frac{\pi}{2}-\tan^{-1}\left(-\frac{\omega_{21}}{A_{21}}\right)\right]$$

Now assume that the line width is much less than the resonance frequency ($A_{21}\ll\omega_{21}$), which effectively makes the integral limits $-\infty\rightarrow+\infty$. The result is

$$1=\frac{2\pi C}{\hbar A_{21}}\quad\Rightarrow\quad C=\frac{\hbar A_{21}}{2\pi}$$

The resulting energy spectrum is

$$g(E)=\frac{\hbar A_{21}/2\pi}{\left(E-\hbar\omega_{21}\right)^2+\left(\hbar A_{21}/2\right)^2}$$

14.6 The perturbation Hamiltonian is

$$H'=-\mathbf{d}\bullet\mathbf{E}=-(-e\mathbf{r})\bullet\mathcal{E}_0\left(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}\right)e^{-t/\tau}=e\mathcal{E}_0e^{-t/\tau}\left(x+y+z\right)$$

Time dependent perturbation theory tells us that the coefficient to be in a new state $|f\rangle$ at time $t=\infty$ after starting in state $|1\rangle$ is :

$$c_f(\infty)=\frac{1}{i\hbar}\int_0^\infty\langle f|H'(t')|1\rangle e^{i\left(E_f-E_1\right)t'/\hbar}\,dt'$$

For this case, we get

$$\begin{aligned}c_{2\ell m}(\infty)&=\frac{1}{i\hbar}\int_0^\infty\langle 2\ell m|H'(t')|100\rangle e^{i(E_2-E_1)t'/\hbar}\,dt'\\&=\frac{1}{i\hbar}\int_0^\infty\langle 2\ell m|e\mathcal{E}_0e^{-t'/\tau}\left(x+y+z\right)|100\rangle e^{i(E_2-E_1)t'/\hbar}\,dt'\\&=\frac{e\mathcal{E}_0}{i\hbar}\langle 2\ell m|\left(x+y+z\right)|100\rangle\int_0^\infty e^{-t'/\tau}e^{i(E_2-E_1)t'/\hbar}\,dt'\end{aligned}$$

The time integral is independent of the final state, so let's do that first:

$$\begin{aligned}c_{2\ell m}(\infty)&=\frac{e\mathcal{E}_0}{i\hbar}\langle 2\ell m|\left(x+y+z\right)|100\rangle\int_0^\infty e^{-t'/\tau}e^{i\omega_{21}t'}\,dt'\\&=\frac{e\mathcal{E}_0}{i\hbar}\langle 2\ell m|\left(x+y+z\right)|100\rangle\left[\frac{e^{i\omega_{21}t'-t'/\tau}}{i\omega_{21}-1/\tau}\right]_0^\infty\\&=\frac{e\mathcal{E}_0}{i\hbar}\langle 2\ell m|\left(x+y+z\right)|100\rangle\left(\frac{-1}{i\omega_{21}-1/\tau}\right)\end{aligned}$$

In Problem 14.7, we calculate the spatial matrix elements

$$\langle 100|\vec{\mathbf{r}}|200\rangle = 0$$

$$\langle 100|\vec{\mathbf{r}}|210\rangle = \frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\hat{\mathbf{k}}$$

$$\langle 100|\vec{\mathbf{r}}|21,\pm1\rangle = \frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\frac{1}{\sqrt{2}}\left[\mp\hat{\mathbf{i}} - i\hat{\mathbf{j}}\right]$$

Thus, of the 12 possible spatial matrix elements for the 4 $n=2$ states, only 5 are nonzero:

$$\langle 100|z|210\rangle = \frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}$$

$$\langle 100|x|21,\pm1\rangle = \mp\frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\frac{1}{\sqrt{2}}$$

$$\langle 100|y|21,\pm1\rangle = -\frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\frac{1}{\sqrt{2}}$$

The resultant probabilities are

$$\mathcal{P}_{200}(\infty) = |c_{200}(\infty)|^2 = 0$$

$$\mathcal{P}_{210}(\infty) = |c_{210}(\infty)|^2 = \left(\frac{ea_0\mathcal{E}_0}{\hbar}\right)^2\left(\frac{1}{\omega_{21}^2 + 1/\tau^2}\right)\frac{2^{15}}{3^{10}}$$

$$\mathcal{P}_{21,\pm1}(\infty) = |c_{21,\pm1}(\infty)|^2 = \left(\frac{ea_0\mathcal{E}_0}{\hbar}\right)^2\left(\frac{1}{\omega_{21}^2 + 1/\tau^2}\right)\frac{2^{16}}{3^{10}}$$

14.7 The lifetime is inversely related to the Einstein A coefficient:

$$\tau = \frac{1}{A}$$

$$A_{ba} = \frac{4e^2\omega_{ba}^3}{3\hbar c^3}\left|\langle b|\vec{\mathbf{r}}|a\rangle\right|^2$$

To calculate the matrix elements we need the following.

$$R_{10}(r) = \frac{2}{a_0^{\frac{3}{2}}}e^{-r/a_0}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}(a_0)^{\frac{3}{2}}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}(a_0)^{\frac{3}{2}}}\frac{r}{a_0}e^{-r/2a_0}$$

$$Y_{00}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta=Y_{10}^{*}(\theta,\phi)$$

$$Y_{1\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}=-Y_{1\pm1}^{*}(\theta,\phi)$$

$$z=r\cos\theta=r\sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi)=r\sqrt{\frac{4\pi}{3}}Y_{10}^{*}(\theta,\phi)$$

$$x=r\sin\theta\cos\phi=r\sqrt{\frac{2\pi}{3}}\left[-Y_{11}(\theta,\phi)+Y_{1-1}(\theta,\phi)\right]=r\sqrt{\frac{2\pi}{3}}\left[-Y_{11}^{*}(\theta,\phi)+Y_{1-1}^{*}(\theta,\phi)\right]$$

$$y=r\sin\theta\cos\phi=ir\sqrt{\frac{2\pi}{3}}\left[Y_{11}(\theta,\phi)+Y_{1-1}(\theta,\phi)\right]=-ir\sqrt{\frac{2\pi}{3}}\left[Y_{11}^{*}(\theta,\phi)+Y_{1-1}^{*}(\theta,\phi)\right]$$

$$\int_0^\infty x^n e^{-ax}dx=\frac{n!}{a^{n+1}}$$

The matrix element can be broken into radial and angular parts:

$$\begin{aligned}\langle b|\vec{\mathbf{r}}|a\rangle&=\langle n_2\ell_2m_2|r\,\hat{\mathbf{r}}|n_1\ell_1m_1\rangle\\&=\int_0^\infty R_{n_2\ell_2}^{*}(r)\ r\ R_{n_1\ell_1}(r)\ r^2dr\int Y_{\ell_2m_2}^{*}(\theta,\phi)\hat{\mathbf{r}}Y_{\ell_1m_1}(\theta,\phi)d\Omega\\&=F_r(n_2,\ell_2,n_1,\ell_1)\vec{\mathbf{G}}_\Omega(\ell_2,m_2,\ell_1,m_1)\end{aligned}$$

The angular part gives the selection rules. Note that you cannot simply pull the radial unit vector out of the integral because it depends on the variables of integration (i.e. it is not a constant!). If you do this you will get different (but wrong) selection rules.

$$\begin{aligned}\vec{\mathbf{G}}_\Omega(\ell_2,m_2,\ell_1,m_1)&=\int Y_{\ell_2m_2}^{*}(\theta,\phi)\hat{\mathbf{r}}Y_{\ell_1m_1}(\theta,\phi)d\Omega\\&=\int Y_{\ell_2m_2}^{*}(\theta,\phi)\left[\frac{x}{r}\hat{\mathbf{i}}+\frac{y}{r}\hat{\mathbf{j}}+\frac{z}{r}\hat{\mathbf{k}}\right]Y_{\ell_1m_1}(\theta,\phi)d\Omega\\&=\int Y_{\ell_2m_2}^{*}(\theta,\phi)\sqrt{\frac{4\pi}{3}}\left[\begin{array}{l}\frac{1}{\sqrt{2}}\left[-Y_{11}^{*}(\theta,\phi)+Y_{1-1}^{*}(\theta,\phi)\right]\hat{\mathbf{i}}+\\-i\frac{1}{\sqrt{2}}\left[Y_{11}^{*}(\theta,\phi)+Y_{1-1}^{*}(\theta,\phi)\right]\hat{\mathbf{j}}+\\Y_{10}^{*}(\theta,\phi)\hat{\mathbf{k}}\end{array}\right]Y_{\ell_1m_1}(\theta,\phi)d\Omega\end{aligned}$$

For the 2->1 transition, this simplifies considerably to:

$$\vec{\mathbf{G}}_\Omega(0,0,\ell,m)=\int Y_{00}^*(\theta,\phi)\sqrt{\frac{4\pi}{3}}\begin{bmatrix}\frac{1}{\sqrt{2}}\left[-Y_{11}^*(\theta,\phi)+Y_{1-1}^*(\theta,\phi)\right]\hat{\mathbf{i}}+\\ -i\frac{1}{\sqrt{2}}\left[Y_{11}^*(\theta,\phi)+Y_{1-1}^*(\theta,\phi)\right]\hat{\mathbf{j}}+\\ Y_{10}^*(\theta,\phi)\hat{\mathbf{k}}\end{bmatrix}Y_{\ell m}(\theta,\phi)\,d\Omega$$

$$=\frac{1}{\sqrt{4\pi}}\sqrt{\frac{4\pi}{3}}\int\begin{bmatrix}\frac{1}{\sqrt{2}}\left[-Y_{11}^*(\theta,\phi)+Y_{1-1}^*(\theta,\phi)\right]\hat{\mathbf{i}}+\\ -i\frac{1}{\sqrt{2}}\left[Y_{11}^*(\theta,\phi)+Y_{1-1}^*(\theta,\phi)\right]\hat{\mathbf{j}}+\\ Y_{10}^*(\theta,\phi)\hat{\mathbf{k}}\end{bmatrix}Y_{\ell m}(\theta,\phi)\,d\Omega$$

$$=\frac{1}{\sqrt{4\pi}}\sqrt{\frac{4\pi}{3}}\begin{bmatrix}\frac{1}{\sqrt{2}}\left[-\delta_{\ell 1}\delta_{m1}+\delta_{\ell 1}\delta_{m-1}\right]\hat{\mathbf{i}}+\\ -i\frac{1}{\sqrt{2}}\left[\delta_{\ell 1}\delta_{m1}+\delta_{\ell 1}\delta_{m-1}\right]\hat{\mathbf{j}}+\\ \delta_{\ell 1}\delta_{m0}\hat{\mathbf{k}}\end{bmatrix}$$

$$=\frac{\delta_{\ell 1}}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\left[-\delta_{m1}+\delta_{m-1}\right]\hat{\mathbf{i}}-i\frac{1}{\sqrt{2}}\left[\delta_{m1}+\delta_{m-1}\right]\hat{\mathbf{j}}+\delta_{\ell 1}\delta_{m0}\hat{\mathbf{k}}\right]$$

Thus we get decay only from the 2*p* state, the 2*s* state cannot decay via electric dipole radiation. The radial part of the matrix element is

$$F_r(n_2,\ell_2,n_1,\ell_1)=\int_0^\infty R_{n_2\ell_2}^*(r)\,r\,R_{n_1\ell_1}(r)r^2dr$$

$$F_r(1,0,2,1)=\int_0^\infty\frac{2}{a_0^{\frac{3}{2}}}e^{-r/a_0}r\frac{1}{2\sqrt{6}(a_0)^{\frac{3}{2}}}\frac{r}{a_0}e^{-r/2a_0}r^2dr=\frac{1}{\sqrt{6}}\int_0^\infty\frac{r^4}{a_0^4}e^{-3r/2a_0}dr$$

$$=\frac{a_0}{\sqrt{6}}\int_0^\infty x^4e^{-3x/2}dx=\frac{a_0}{\sqrt{6}}\frac{4!}{\left(\frac{3}{2}\right)^5}=\frac{a_0}{\sqrt{6}}\frac{2^8}{3^4}$$

Thus we get

$$\langle 100|\vec{\mathbf{r}}|200\rangle=0$$

$$\langle 100|\vec{\mathbf{r}}|210\rangle=\frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\hat{\mathbf{k}}$$

$$\langle 100|\vec{\mathbf{r}}|21\pm 1\rangle=\frac{a_0}{\sqrt{2}}\frac{2^8}{3^5}\frac{1}{\sqrt{2}}\left[\mp\hat{\mathbf{i}}-i\hat{\mathbf{j}}\right]$$

$$\left|\langle 100|\vec{\mathbf{r}}|21m\rangle\right|^2=a_0^2\frac{2^{15}}{3^{10}}$$

Note that the magnitude of the $2p$-$1s$ matrix element is independent of the m-state. The decay rate and lifetime are thus

$$A_{ba} = \frac{4e^2\omega_{ba}^3}{3\hbar c^3}\left|\langle b|\vec{\mathbf{r}}|a\rangle\right|^2$$

$$A_{2p-1s} = \frac{4e^2\omega_{21}^3}{3\hbar c^3}a_0^2\frac{2^{15}}{3^{10}} = \frac{4e^2}{3\hbar c^3}\frac{1}{\hbar^3}\left(\frac{3}{4}\frac{1}{2}\alpha^2 mc^2\right)^3\left(\frac{\hbar^2}{me^2}\right)^2\frac{2^{15}}{3^{10}}$$

$$= \frac{1}{\hbar}\left(\frac{2}{3}\right)^8\alpha^5 mc^2 = \frac{1}{6.5821\times10^{-16}eVs}\left(\frac{2}{3}\right)^8\left(\frac{1}{137.04}\right)^3 27.2eV = 6.265\times10^8 s^{-1}$$

$$\tau_{2p-1s} = \frac{1}{A_{2p-1s}} = 1.596\times10^{-9}s = 1.6\ ns$$

14.8 a,b) The coefficient for the n^{th} state, assuming we start in the 1^{st} state is:

$$c_n(t) = \frac{1}{i\hbar}\int_0^t\langle n|H'(t')|1\rangle e^{i(E_n-E_1)t'/\hbar}\,dt' = \frac{1}{i\hbar}\int_0^t\langle n|V_0\sin\frac{\pi x}{L}e^{-\gamma t}|1\rangle e^{i(E_n-E_1)t'/\hbar}\,dt'$$

$$= \frac{1}{i\hbar}\langle n|V_0\sin\frac{\pi x}{L}|1\rangle\int_0^t e^{-\gamma t}e^{i(E_n-E_1)t'/\hbar}\,dt'$$

To find the matrix element, we integrate over the well:

$$V_{n1} = \langle n|V_0\sin\frac{\pi x}{L}|1\rangle = \int_0^L\varphi_n^*(x)V_0\sin\frac{\pi x}{L}\varphi_1(x)dx$$

$$= \frac{2V_0}{L}\int_0^L\sin\frac{n\pi x}{L}\sin\frac{\pi x}{L}\sin\frac{\pi x}{L}dx = \frac{2V_0}{L}\int_0^L\sin\frac{n\pi x}{L}\sin^2\frac{\pi x}{L}dx$$

$$= \frac{2V_0}{L}\int_0^L\sin\left(\frac{n\pi x}{L}\right)\frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{V_0}{L}\int_0^L\left[\sin\frac{n\pi x}{L}-\sin\frac{n\pi x}{L}\cos\frac{2\pi x}{L}\right]dx$$

$$= \frac{V_0}{\pi}\int_0^L\left[\sin ny-\sin ny\cos 2y\right]dy$$

$$= \frac{V_0}{\pi}\left[-\frac{1}{n}\cos ny+\frac{\cos[(n-2)y]}{2(n-2)}+\frac{\cos[(n+2)y]}{2(n+2)}\right]_0^\pi$$

$$= \frac{V_0}{\pi}\left[-\frac{1}{n}[\cos n\pi-1]+\frac{\cos[n\pi-2\pi]-1}{2(n-2)}+\frac{\cos[n\pi+2\pi]-1}{2(n+2)}\right]$$

$$= \frac{V_0}{\pi}\left[-\frac{1}{n}\left[(-1)^n-1\right]+\frac{\left[(-1)^n-1\right]}{2(n-2)}+\frac{\left[(-1)^n-1\right]}{2(n+2)}\right]$$

$$= \frac{4V_0}{\pi n(n^2-4)}\left[(-1)^n-1\right] = \frac{8V_0}{\pi n(4-n^2)}\delta_{n,odd}$$

The time integral is

$$c_n(t) = \frac{1}{i\hbar} V_{n1} \int_0^t e^{-\gamma t'} e^{i(E_n - E_1)t'/\hbar}\, dt' = \frac{1}{i\hbar} V_{n1} \int_0^t e^{-\gamma t'} e^{i\omega_{n1} t'}\, dt' = \frac{1}{i\hbar} V_{n1} \int_0^t e^{i\omega_{n1} t' - \gamma t'}\, dt'$$

$$= \frac{1}{i\hbar} V_{n1} \left[\frac{e^{i\omega_{n1} t' - \gamma t'}}{i\omega_{n1} - \gamma} \right]_0^t = \frac{1}{i\hbar} V_{n1} \left[\frac{e^{i\omega_{n1} t - \gamma t} - 1}{i\omega_{n1} - \gamma} \right]$$

For times longer than $1/\gamma$, the exponential term goes to zero, giving

$$c_n\left(t >> \tfrac{1}{\gamma}\right) = \frac{1}{i\hbar} V_{n1} \left[\frac{-1}{i\omega_{n1} - \gamma} \right]$$

and the long-time probability:

$$\mathcal{P}_n(t) = \left|c_n(t)\right|^2 = \frac{1}{\hbar^2} \left|V_{n1}\right|^2 \left[\frac{1}{i\omega_{n1} - \gamma} \cdot \frac{1}{-i\omega_{n1} - \gamma} \right] = \frac{1}{\hbar^2} \frac{1}{\omega_{n1}^2 + \gamma^2} \left[\frac{8V_0}{\pi n (4 - n^2)} \right]^2 \delta_{n,odd}$$

$$= \frac{64 V_0^2}{\hbar^2 \pi^2 n^2 (n^2 - 4)^2} \frac{1}{\omega_{n1}^2 + \gamma^2} \delta_{n,odd}$$

The energy difference between states gives us the Bohr frequency

$$\omega_{n1} = \frac{E_n - E_1}{\hbar} = \frac{n^2 \dfrac{\hbar^2 \pi^2}{2mL} - \dfrac{\hbar^2 \pi^2}{2mL}}{\hbar} = \frac{\hbar \pi^2}{2mL}\left(n^2 - 1\right)$$

yielding the probability

$$\mathcal{P}_n(t) = \frac{64 V_0^2}{\hbar^2 \pi^2 n^2 (n^2 - 4)^2} \frac{1}{\dfrac{\hbar^2 \pi^4}{4 m^2 L^2}\left(n^2 - 1\right)^2 + \gamma^2} \delta_{n,odd}$$

This is valid for times longer than $1/\gamma$

b) The Kronecker delta function from the matrix element tells us the selection rule: <u>The upper state must be an odd state</u>.

14.9 a,b) Time-dependent perturbation theory gives us the probability amplitude for the final state:

$$c_{n_f}(t) = \frac{1}{i\hbar} \int_0^t \langle n_f | \hat{H}'(t') | n_i \rangle e^{i(E_f - E_i)t'/\hbar}\, dt' = \frac{1}{i\hbar} \int_0^t \langle n | A x^3 e^{-\gamma t'} | 0 \rangle e^{i(E_f - E_i)t'/\hbar}\, dt'$$

$$= \frac{1}{i\hbar} \langle n | A x^3 | 0 \rangle \int_0^t e^{-\gamma t'} e^{i(E_n - E_0)t'/\hbar}\, dt' = \frac{A}{i\hbar} x_{n0}^3 \int_0^t e^{-\gamma t'} e^{i(E_n - E_0)t'/\hbar}\, dt'$$

To find the matrix elements use ladder operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$x^3 = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger + a\right)^3$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + a a a\right)$$

When we take the matrix elements, half of these will vanish because $a|0\rangle = 0$, so any term with an *a* rightmost vanishes. The $aaa^\dagger$ term is also zero since it tries to lower the ground state on the last ladder operation. Thus we are left with

$$\langle n|x^3|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\langle n|\left(a^\dagger a^\dagger a^\dagger + a^\dagger a a^\dagger + a a^\dagger a^\dagger\right)|0\rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\sqrt{1}\sqrt{2}\sqrt{3}\delta_{n3} + \sqrt{1}\sqrt{1}\sqrt{1}\delta_{n1} + \sqrt{1}\sqrt{2}\sqrt{2}\delta_{n1}\right) = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\sqrt{6}\delta_{n3} + 3\delta_{n1}\right)$$

Now do the time integral, noting that the harmonic oscillator energies are $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

$$c_n(t) = \frac{A}{i\hbar}x_{n0}^3\int_0^t e^{-\gamma t'}e^{i(E_n - E_0)t'/\hbar}\,dt' = \frac{A}{i\hbar}x_{n0}^3\int_0^t e^{-\gamma t'}e^{in\omega t'}\,dt' = \frac{A}{i\hbar}x_{n0}^3\int_0^t e^{in\omega t' - \gamma t'}\,dt'$$

$$= \frac{A}{i\hbar}x_{n0}^3\left[\frac{e^{in\omega t' - \gamma t'}}{in\omega - \gamma}\right]_0^t = \frac{A}{i\hbar}x_{n0}^3\left[\frac{e^{in\omega t - \gamma t} - 1}{in\omega - \gamma}\right]$$

For times longer than 1/γ, the exponential term goes to zero, giving

$$c_n\left(t >> \tfrac{1}{\gamma}\right) = \frac{A}{i\hbar}x_{n0}^3\left[\frac{-1}{in\omega - \gamma}\right]$$

$$\mathcal{P}_n(t) = \left|c_n(t)\right|^2 = \frac{A^2}{\hbar^2}\left|x_{n0}^3\right|^2\left[\frac{1}{in\omega - \gamma}\cdot\frac{1}{-in\omega - \gamma}\right]$$

$$= \frac{A^2}{\hbar^2}\frac{1}{n^2\omega^2 + \gamma^2}\left(\frac{\hbar}{2m\omega}\right)^3\left(\sqrt{6}\delta_{n3} + 3\delta_{n1}\right)^2 = \frac{3A^2\hbar}{8m^3\omega^3}\frac{1}{n^2\omega^2 + \gamma^2}\left[2\delta_{n3} + 3\delta_{n1}\right]$$

The Kronecker delta functions tells us the selection rules:

$$\boxed{\delta_{n3}, \delta_{n1} \Rightarrow n = 1, 3}$$

14.10 The angular integral is

$$I_{\ell_i m_i \to \ell_f m_f} = \int Y_{\ell_f}^{m_f *}(\theta,\phi)\,Y_1^m(\theta,\phi)\,Y_{\ell_i}^{m_i}(\theta,\phi)\,d\Omega$$

For a $p \to s$ transition, we must have $m_f = 0$, so the selection rule $m = m_f - m_i$ requires $m = -m_i$. If we choose $m = -m_i = 0$, then the integral is

$$\begin{aligned}
I_{p_0\to s_0} &= \int Y_0^{0*}(\theta,\phi)Y_1^0(\theta,\phi)Y_1^0(\theta,\phi)\,d\Omega \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{4\pi}}\cos\theta\sqrt{\frac{3}{4\pi}}\cos\theta\sin\theta\,d\theta\,d\phi \\
&= \frac{1}{\sqrt{4\pi}}\frac{3}{4\pi}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi = \frac{2\pi}{\sqrt{4\pi}}\frac{3}{4\pi}\left(\frac{2}{3}\right) = \frac{1}{\sqrt{4\pi}}
\end{aligned}$$

The result in Eq. (14.84) gives the integral as

$$\begin{aligned}
I_{p_0\to s_0} &= \left[\frac{3(2\ell_i+1)}{4\pi(2\ell_f+1)}\right]^{\frac{1}{2}}\langle \ell_i 1 m_i m|\ell_f m_f\rangle\langle \ell_i 100|\ell_f 0\rangle \\
&= \left[\frac{9}{4\pi}\right]^{\frac{1}{2}}\langle 1100|00\rangle\langle 1100|00\rangle = \frac{3}{\sqrt{4\pi}}\left(\langle 1100|00\rangle\right)^2
\end{aligned}$$

From Table 11.5 we get $\langle 1100|00\rangle = -1/\sqrt{3}$, giving

$$I_{p_0\to s_0} = \frac{3}{\sqrt{4\pi}}\left(-1/\sqrt{3}\right)^2 = \frac{1}{\sqrt{4\pi}}$$

So the two agree.

For a $p\to d$ transition, let's choose $m_i = 1$ and $m_f = 2$, so the selection rule $m = m_f - m_i$ requires $m = 1$. Then the integral is

$$\begin{aligned}
I_{p_1\to d_2} &= \int Y_2^{2*}(\theta,\phi)Y_1^1(\theta,\phi)Y_1^1(\theta,\phi)\,d\Omega \\
&= \int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-i2\phi}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)\sin\theta\,d\theta\,d\phi \\
&= \sqrt{\frac{15}{32\pi}}\frac{3}{8\pi}\int_0^{\pi}\sin^5\theta\,d\theta\int_0^{2\pi}d\phi = \sqrt{\frac{15}{32\pi}}\frac{6\pi}{8\pi}\left(\frac{16}{15}\right) = \sqrt{\frac{3}{10\pi}}
\end{aligned}$$

The result in Eq. (14.84) gives the integral as

$$\begin{aligned}
I_{p_1\to d_2} &= \left[\frac{3(2\ell_i+1)}{4\pi(2\ell_f+1)}\right]^{\frac{1}{2}}\langle \ell_i 1 m_i m|\ell_f m_f\rangle\langle \ell_i 100|\ell_f 0\rangle \\
&= \left[\frac{9}{20\pi}\right]^{\frac{1}{2}}\langle 1111|22\rangle\langle 1100|20\rangle
\end{aligned}$$

From Table 11.5 we get $\langle 1110|22\rangle = 1$ and $\langle 1100|20\rangle = \sqrt{2/3}$, giving

$$I_{p_1\to d_2} = \left[\frac{9}{20\pi}\right]^{\frac{1}{2}}1\left(\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{3}{10\pi}}$$

So the two agree.

14.11 The result in Eq. (14.84) gives the integral

$$\int Y_{\ell_f}^{m_f *}(\theta,\phi) Y_1^m(\theta,\phi) Y_{\ell_i}^{m_i}(\theta,\phi)\, d\Omega = \left[\frac{3(2\ell_i+1)}{4\pi(2\ell_f+1)}\right]^{\frac{1}{2}} \left\langle \ell_i 1 m_i m \middle| \ell_f m_f \right\rangle \left\langle \ell_i 100 \middle| \ell_f 0 \right\rangle$$

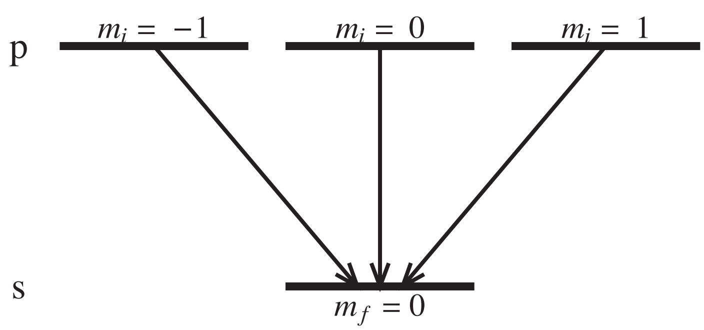
For a transition from an initial *p* state, the integral is

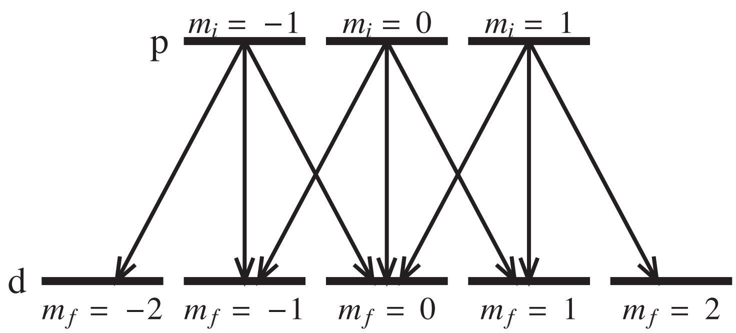
$$\int Y_{\ell_f}^{m_f *}(\theta,\phi) Y_1^m(\theta,\phi) Y_1^{m_i}(\theta,\phi)\, d\Omega = \left[\frac{9}{4\pi(2\ell_f+1)}\right]^{\frac{1}{2}} \left\langle 11 m_i m \middle| \ell_f m_f \right\rangle \left\langle 1100 \middle| \ell_f 0 \right\rangle$$

The parity selection rule $\ell_f \neq \ell_i$ comes from the Clebsch-Gordan coefficent $\langle 1100|10\rangle = 0$ and means that none of the nonzero Clebsch-Gordan coefficents $\left\langle 11 m_i m \middle| 1 m_f \right\rangle$ in Table 11.5 correspond to allowed transitions. All the other nonzero Clebsch-Gordan coefficients correspond to allowed transitions as follows:

$$\begin{aligned}
\left\langle 11 m_i m \middle| \ell_f m_f \right\rangle &: \quad p_{m_i} \to (s,d)_{m_f} \\
\langle 1111|22\rangle &: \quad p_1 \to d_2 \\
\langle 1110|21\rangle &: \quad p_1 \to d_1 \\
\langle 111,-1|20\rangle &: \quad p_1 \to d_0 \\
\langle 1101|21\rangle &: \quad p_0 \to d_1 \\
\langle 1100|20\rangle &: \quad p_0 \to d_0 \\
\langle 110,-1|2,-1\rangle &: \quad p_0 \to d_{-1} \\
\langle 11,-11|20\rangle &: \quad p_{-1} \to d_0 \\
\langle 11,-10|2,-1\rangle &: \quad p_{-1} \to d_{-1} \\
\langle 11,-1,-1|2,-2\rangle &: \quad p_{-1} \to d_{-2} \\
\langle 111,-1|00\rangle &: \quad p_1 \to s_0 \\
\langle 1100|00\rangle &: \quad p_0 \to s_0 \\
\langle 11,-11|00\rangle &: \quad p_{-1} \to s_0
\end{aligned}$$

The possible transitions are shown below

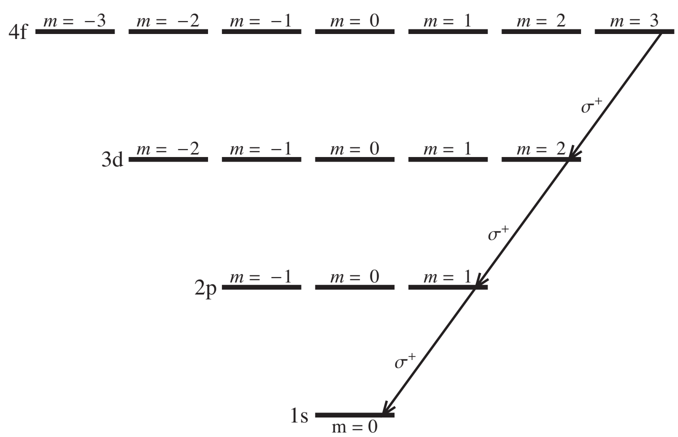




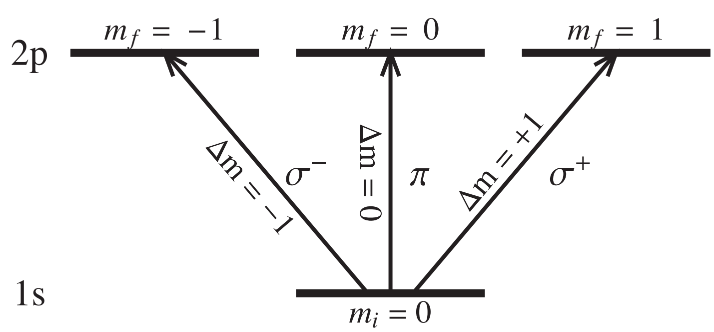
14.12 A hydrogen atom starts in the state $n = 4,\ \ell = 3,\ m_\ell = 3$. As it decays the principal quantum number n will decrease. The angular momentum quantum numbers obey the selection rules $\Delta\ell = \pm 1,\ \Delta m = 0, \pm 1$. The initial state is a stretched state with maximal values of the angular momentum quantum numbers, so they can only decrease as the atoms decays. Because they can only decrease by 1, the decay chain must be

$$4f_3 \rightarrow 3d_2 \rightarrow 2p_1 \rightarrow 1s_0$$

For each step of the decay, the magnetic quantum number changes by -1. The loss of angular momentum component by the atom must be compensated by a gain by the light field, so the emitted polarizations of the photons are σ^+ for each decay step. The full decay chain is shown below (energy not to scale).



14.13 The transition diagram for the $1s \rightarrow 2p$ absorption is shown below.



Compared to Fig. 14.10, the sign of Δm changes for each transition because the order of the initial and final states changes. Each transition obeys the magnetic quantum number selection rule $\Delta m = m_f - m_i = 0, \pm 1$ that describes the change in the *z*-component of the angular momentum of the atom. This change in angular momentum must come from the absorbed photon, so the photon helicity is determined by Δm. In other words, conservation of angular momentum of the whole system requires that

$$\begin{aligned} m_{total,f} - m_{total,i} &= 0 \\ \left(m_{atom,f} + m_{photon,f}\right) - \left(m_{atom,i} + m_{photon,i}\right) &= 0 \\ \left(m_{atom,f} - m_{atom,i}\right) + \left(m_{photon,f} - m_{photon,i}\right) &= 0 \\ \Delta m_{atom} = -\Delta m_{photon} &= -\left(m_{photon,f} - m_{photon,i}\right) \end{aligned}$$

In the absorption case, there is only an initial photon, so

$$\Delta m_{atom} = m_{photon,i}$$

and the photon helicity matches is equal to Δm. In the emission case depicted in Fig. 14.10, there is only a final photon, so

$$\Delta m_{atom} = -m_{photon,f}$$

so the sign of the emitted photon helicity is opposite to Δm. The net result is that the photon helicity is the same for emission and absorption.

14.14 For a one-dimensional system, the spontaneous emission rate [using Eq. (14.63)] from an excited state $\left|n_i\right\rangle$ to a lower state $\left|n_f\right\rangle$ is

$$A_{n_i n_f} = \frac{q^2 \omega_{n_i n_f}^3}{3\pi\varepsilon_0 \hbar c^3} \left|\left\langle n_i \middle| x \middle| n_f \right\rangle\right|^2$$

a) The selection rules come from examining the matrix element. For the harmonic oscillator, the matrix element is

$$\begin{aligned} \left\langle n_i \middle| x \middle| n_f \right\rangle &= \sqrt{\frac{\hbar}{2m\omega}} \left\langle n_i \middle| \left(a^\dagger + a\right) \middle| n_f \right\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\left\langle n_i \middle| a^\dagger \middle| n_f \right\rangle + \left\langle n_i \middle| a \middle| n_f \right\rangle \right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(\left[\left\langle n_i \middle| \sqrt{n_f + 1} \middle| n_f + 1 \right\rangle + \left\langle n_i \middle| \sqrt{n_f} \middle| n_f - 1 \right\rangle \right] \right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n_f + 1} \left\langle n_i \middle| n_f + 1 \right\rangle + \sqrt{n_f} \left\langle n_i \middle| n_f - 1 \right\rangle \right) \\ &= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n_f + 1}\, \delta_{n_i, n_f + 1} + \sqrt{n_f}\, \delta_{n_i, n_f - 1} \right) \end{aligned}$$

The state $\left|n_f\right\rangle$ must be lower that the state $\left|n_i\right\rangle$ for emission, so the selection rule is

$$\Delta n = n_f - n_i = -1$$
$$n_f = n_i - 1$$

b) The selection rule tells us that only the $n=1$ state can decay to the $n=0$ ground state.

c) The spontaneous emission rate for the first excited state is

$$A_{10} = \frac{q^2\omega_{10}^3}{3\pi\varepsilon_0\hbar c^3}\left|\langle 1|x|0\rangle\right|^2 = \frac{q^2\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\sqrt{\frac{\hbar}{2m\omega}}\right|^2 = \frac{q^2\omega^3}{3\pi\varepsilon_0\hbar c^3}\frac{\hbar}{2m\omega} = \frac{q^2\omega^2}{6\pi\varepsilon_0 mc^3}$$

d) For an electron bound in a potential with $\omega = 10^{15}\ rad/s$, the spontaneous emission rate for the first excited state is

$$A_{10} = \frac{e^2\omega^2}{6\pi\varepsilon_0 mc^3} = \frac{e^2}{4\pi\varepsilon_0\hbar c}\frac{2\hbar\omega^2}{3mc^2} = \frac{1}{137}\frac{2}{3}\frac{\left(6.582\times 10^{-16}\,eVs\right)\left(10^{15}\,s^{-1}\right)^2}{5.11\times 10^5\,eVs} = 6.27\times 10^6\,s^{-1}$$

The lifetime is

$$\tau_{10} = \frac{1}{A_{10}} = \frac{1}{6.27\times 10^6\,s^{-1}} = 0.16\,\mu s$$

Note: Eq. (14.63) was derived assuming that the blackbody radiation is isotropic and the polarization vector is random, which resulted in the 1/3 factor from Eq. (14.52). For a one-dimensional system, we might reasonably assume that the applied polarization is aligned with the oscillator, and hence not include the 1/3 factor. The result would be a spontaneous emission rate 3 times larger and a lifetime 3 times shorter than those above.