3.1 The Schrödinger equation for a general state is

$$i\hbar\frac{d}{dt}\sum_n c_n(t)|E_n\rangle = H\sum_n c_n(t)|E_n\rangle$$

For a two-state system, the Schrödinger equation in matrix form is

$$i\hbar\frac{d}{dt}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

This gives the two equations

$$i\hbar\frac{d}{dt}c_1(t) = E_1 c_1(t)$$
$$i\hbar\frac{d}{dt}c_2(t) = E_2 c_2(t)$$

When rewritten as

$$\frac{dc_1(t)}{dt} = -i\frac{E_1}{\hbar}c_1(t)$$
$$\frac{dc_2(t)}{dt} = -i\frac{E_2}{\hbar}c_2(t)$$

these are equivalent to Eq. (3.8).

3.2 The evolution of a general state subject to a time-independent Hamiltonian is

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle$$

The probability of an energy measurement is

$$\mathcal{P}_{E_m} = |\langle E_m|\psi(t)\rangle|^2 = \left|\langle E_m|\sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle\right|^2 = \left|\sum_n c_n e^{-iE_n t/\hbar}\langle E_m|E_n\rangle\right|^2$$
$$= \left|\sum_n c_n e^{-iE_n t/\hbar}\delta_{mn}\right|^2 = \left|c_m e^{-iE_m t/\hbar}\right|^2 = |c_m|^2$$

which is independent of time. The probability of measuring another observable A is

$$\mathcal{P}_{a_m} = |\langle a_m|\psi(t)\rangle|^2 = \left|\langle a_m|\sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle\right|^2 = \left|\sum_n c_n e^{-iE_n t/\hbar}\langle a_m|E_n\rangle\right|^2$$

If A and H commute, then they share eigenstates, and assuming that the index labels are the same, $\langle a_m|E_n\rangle = \langle E_m|E_n\rangle = \delta_{mn}$ and the probability

$$\mathcal{P}_{a_m} = \left|\sum_n c_n e^{-iE_n t/\hbar}\delta_{mn}\right|^2 = \left|c_m e^{-iE_m t/\hbar}\right|^2 = \left|c_m\right|^2$$

is also independent of time.

3.3 The Hamiltonian is

$$H \doteq \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{pmatrix}$$

Using the definition

$$\tan\theta = \frac{\omega_1}{\omega_0}$$

we treat ω_0 and ω_1 as the sides of a right triangle to obtain

$$\sin\theta = \frac{\omega_1}{\sqrt{\omega_0^2+\omega_1^2}}; \quad \cos\theta = \frac{\omega_0}{\sqrt{\omega_0^2+\omega_1^2}}$$

and write the Hamiltonian as

$$H \doteq \frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Diagonalizing the Hamiltonian gives the energy eigenvalues

$$\begin{vmatrix} \frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\cos\theta - \lambda & \frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\sin\theta \\ \frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\sin\theta & -\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\cos\theta - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\sin\theta\right)^2 = 0$$

$$-\frac{\hbar^2}{4}\left(\omega_0^2+\omega_1^2\right)\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\left(\omega_0^2+\omega_1^2\right)\sin^2\theta = 0$$

$$\lambda^2 - \frac{\hbar^2}{4}\left(\omega_0^2+\omega_1^2\right) = 0$$

$$\lambda = \pm\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}$$

as expected and the eigenstates:

$$\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\cos\theta + b\sin\theta = \pm a \quad \Rightarrow \quad b = a\frac{\pm 1-\cos\theta}{\sin\theta}$$

$$|a|^2 + |a|^2\left(\frac{\pm 1-\cos\theta}{\sin\theta}\right)^2 = 1$$

$$|a|^2 = \frac{\sin^2\theta}{2\mp 2\cos\theta} = \frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{4\begin{Bmatrix} \sin^2\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} \end{Bmatrix}} = \begin{Bmatrix} \cos^2\frac{\theta}{2} \\ \sin^2\frac{\theta}{2} \end{Bmatrix}$$

$$\left|+\right\rangle_n = \cos\tfrac{\theta}{2}\left|+\right\rangle + \sin\tfrac{\theta}{2}\left|-\right\rangle$$
$$\left|-\right\rangle_n = \sin\tfrac{\theta}{2}\left|+\right\rangle - \cos\tfrac{\theta}{2}\left|-\right\rangle$$

3.4 The Hamiltonian is

$$H \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The commutator with the S_z is

$$[H,S_z] \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\doteq \left(\frac{\hbar}{2}\right)^2\omega_0\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

The commutator with the S_x is

$$[H,S_x] \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\doteq \left(\frac{\hbar}{2}\right)^2\omega_0\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \left(\frac{\hbar}{2}\right)^2\omega_0\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \neq 0$$

The commutator with the S_y is

$$[H,S_y] \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\doteq \left(\frac{\hbar}{2}\right)^2\omega_0\left[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right] = \left(\frac{\hbar}{2}\right)^2\omega_0\begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} \neq 0$$

For spin precession of a spin-1/2 particle with a magnetic moment in a uniform magnetic field aligned with the z-axis, the probability of measuring the S_z component is time independent because S_z commutes with the Hamiltonian. Conversely, the probabilities of measuring the S_x or S_y components are time dependent because they do not commute with the Hamiltonian.

3.5 (a) The possible results of a measurement of the spin component S_x are always $\pm\hbar/2$ for a spin-½ particle. The probabilities are

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi(0)\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi(0)\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

(b) In a field aligned along the y-axis, the energy eigenstates are $|\pm\rangle_y$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}|\psi(0)\rangle = |+\rangle &= \left(|+\rangle_y\ {}_y\langle +| + |-\rangle_y\ {}_y\langle -|\right)|+\rangle = \left({}_y\langle +|+\rangle\right)|+\rangle_y + \left({}_y\langle -|+\rangle\right)|-\rangle_y \\ &= \tfrac{1}{\sqrt{2}}|+\rangle_y + \tfrac{1}{\sqrt{2}}|-\rangle_y\end{aligned}$$

The time evolved state is

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}e^{-iE_+t/\hbar}|+\rangle_y + \tfrac{1}{\sqrt{2}}e^{-iE_-t/\hbar}|-\rangle_y = \tfrac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|+\rangle_y + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 t/2}|-\rangle_y$$

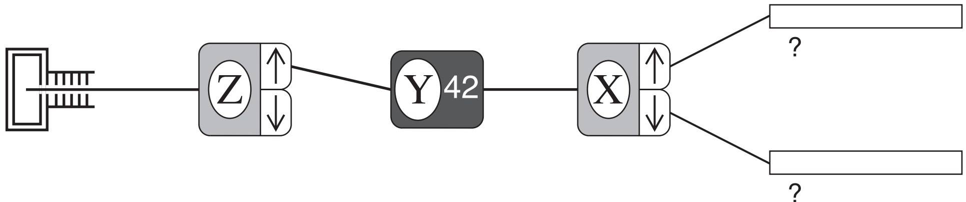
and when expressed in the S_z basis is

$$\begin{aligned}|\psi(t)\rangle &= \tfrac{1}{\sqrt{2}}e^{-i\omega_0 t/2}\left(\tfrac{1}{\sqrt{2}}|+\rangle + i\tfrac{1}{\sqrt{2}}|-\rangle\right) + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 t/2}\left(\tfrac{1}{\sqrt{2}}|+\rangle - i\tfrac{1}{\sqrt{2}}|-\rangle\right) \\ &= \tfrac{1}{2}\left(e^{-i\omega_0 t/2} + e^{+i\omega_0 t/2}\right)|+\rangle + i\tfrac{1}{2}\left(e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2}\right)|-\rangle \\ &= \cos(\omega_0 t/2)|+\rangle + \sin(\omega_0 t/2)|-\rangle\end{aligned}$$

(c) The probability of measuring S_x to be $+\hbar/2$ is

$$\begin{aligned}\mathcal{P}_{+x} &= \left|{}_x\langle +|\psi(t)\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\cos(\omega_0 t/2)|+\rangle + \sin(\omega_0 t/2)|-\rangle\right)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{2}}\cos(\omega_0 t/2) + \tfrac{1}{\sqrt{2}}\sin(\omega_0 t/2)\right|^2 = \tfrac{1}{2}\left(1 + 2\cos(\omega_0 t/2)\sin(\omega_0 t/2)\right) \\ &= \tfrac{1}{2}\left(1 + \sin\omega_0 t\right)\end{aligned}$$

(d) Figure:



3.6 (a) The measurement collapses the state to

$$|\psi(0)\rangle = |+\rangle_x$$

(b) In a field aligned along the z-axis, the energy eigenstates are $|\pm\rangle$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}|\psi(0)\rangle &= |+\rangle_x = \left(|+\rangle\langle+| + |-\rangle\langle-|\right)|+\rangle_x = \left(\langle+|+\rangle_x\right)|+\rangle + \left(\langle-|+\rangle_x\right)|-\rangle \\ &= \tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle\end{aligned}$$

The time evolved state is

$$|\psi(T)\rangle = \tfrac{1}{\sqrt{2}}e^{-iE_+T/\hbar}|+\rangle + \tfrac{1}{\sqrt{2}}e^{-iE_-T/\hbar}|-\rangle = \tfrac{1}{\sqrt{2}}e^{-i\omega_0 T/2}|+\rangle + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 T/2}|-\rangle$$

(c) In this new field the energy eigenstates are $|\pm\rangle_y$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. We start the clock over for this new time evolution. The initial state vector written in this new energy basis is

$$\begin{aligned}|\psi(0)\rangle &= \tfrac{1}{\sqrt{2}}e^{-i\omega_0 T/2}|+\rangle + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 T/2}|-\rangle = \left(|+\rangle_y\,{}_y\langle+| + |-\rangle_y\,{}_y\langle-|\right)\left(\tfrac{1}{\sqrt{2}}e^{-i\omega_0 T/2}|+\rangle + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 T/2}|-\rangle\right) \\ &= \left(\tfrac{1}{\sqrt{2}}e^{-i\omega_0 T/2}\,{}_y\langle+|+\rangle + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 T/2}\,{}_y\langle+|-\rangle\right)|+\rangle_y + \left(\tfrac{1}{\sqrt{2}}e^{-i\omega_0 T/2}\,{}_y\langle-|+\rangle + \tfrac{1}{\sqrt{2}}e^{+i\omega_0 T/2}\,{}_y\langle-|-\rangle\right)|-\rangle_y \\ &= \tfrac{1}{2}\left(e^{-i\omega_0 T/2} - i\,e^{+i\omega_0 T/2}\right)|+\rangle_y + \tfrac{1}{2}\left(e^{-i\omega_0 T/2} + ie^{+i\omega_0 T/2}\right)|-\rangle_y\end{aligned}$$

The time evolution of this state is

$$\begin{aligned}|\psi(T)\rangle &= \tfrac{1}{2}\left(e^{-i\omega_0 T/2} - i\,e^{+i\omega_0 T/2}\right)e^{-iE_+t/\hbar}|+\rangle_y + \tfrac{1}{2}\left(e^{-i\omega_0 T/2} + ie^{+i\omega_0 T/2}\right)e^{-iE_-t/\hbar}|-\rangle_y \\ &= \tfrac{1}{2}\left(e^{-i\omega_0 T/2} - i\,e^{+i\omega_0 T/2}\right)e^{-i\omega_0 T/2}|+\rangle_y + \tfrac{1}{2}\left(e^{-i\omega_0 T/2} + ie^{+i\omega_0 T/2}\right)e^{+i\omega_0 T/2}|-\rangle_y \\ &= \tfrac{1}{2}\left(e^{-i\omega_0 T} - i\right)|+\rangle_y + \tfrac{1}{2}\left(1 + ie^{+i\omega_0 T}\right)|-\rangle_y\end{aligned}$$

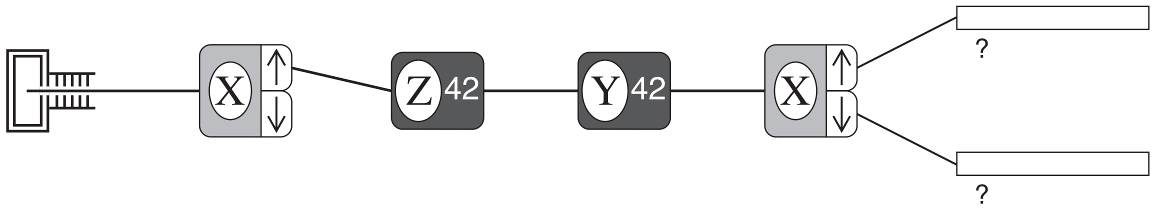
The probability of measuring S_x to be $+\hbar/2$ is

$$\mathcal{P}_{+x} = \left|{}_x\langle+|\psi(T)\rangle\right|^2 = \left|{}_x\langle+|\left(\tfrac{1}{2}\left(e^{-i\omega_0 T} - i\right)|+\rangle_y + \tfrac{1}{2}\left(1 + ie^{+i\omega_0 T}\right)|-\rangle_y\right)\right|^2$$

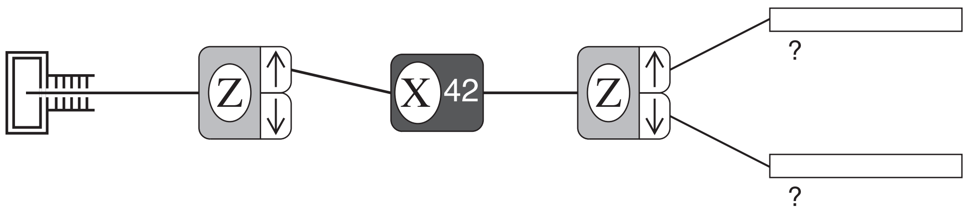
Using matrix notation, we have

$$\begin{aligned}\mathcal{P}_{+x} &= \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\left\{\tfrac{1}{2}\left(e^{-i\omega_0 T} - i\right)\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \tfrac{1}{2}\left(1 + ie^{+i\omega_0 T}\right)\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right\}\right|^2 \\ &= \tfrac{1}{16}\left|\left(e^{-i\omega_0 T} - i\right)(1+i) + \left(1 + ie^{+i\omega_0 T}\right)(1-i)\right|^2 \\ &= \tfrac{1}{16}\left|e^{-i\omega_0 T} + ie^{-i\omega_0 T} - i + 1 + 1 - i + ie^{+i\omega_0 T} + e^{+i\omega_0 T}\right|^2 \\ &= \tfrac{1}{4}\left|1 - i + \cos\omega_0 T + i\cos\omega_0 T\right|^2 = \tfrac{1}{2}\left(1 + \cos^2\omega_0 T\right) = \tfrac{1}{4}\left(3 + \cos 2\omega_0 T\right)\end{aligned}$$

Experiment schematic:



3.7 Experiment schematic:



The measurement at the first analyzer collapses the state to

$$\left|\psi(0)\right\rangle = \left|+\right\rangle$$

In the field aligned along the x-axis, the energy eigenstates are $\left|\pm\right\rangle_x$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}\left|\psi(0)\right\rangle = \left|+\right\rangle &= \left(\left|+\right\rangle_x {}_x\!\left\langle+\right| + \left|-\right\rangle_x {}_x\!\left\langle-\right|\right)\left|+\right\rangle = \left({}_x\!\left\langle+|+\right\rangle\right)\left|+\right\rangle_x + \left({}_x\!\left\langle-|+\right\rangle\right)\left|-\right\rangle_x \\ &= \tfrac{1}{\sqrt{2}}\left|+\right\rangle_x + \tfrac{1}{\sqrt{2}}\left|-\right\rangle_x\end{aligned}$$

The time evolved state is

$$\left|\psi(t)\right\rangle = \tfrac{1}{\sqrt{2}}e^{-iE_+t/\hbar}\left|+\right\rangle_x + \tfrac{1}{\sqrt{2}}e^{-iE_-t/\hbar}\left|-\right\rangle_x = \tfrac{1}{\sqrt{2}}e^{-i\omega_0t/2}\left|+\right\rangle_x + \tfrac{1}{\sqrt{2}}e^{+i\omega_0t/2}\left|-\right\rangle_x$$

The probability of measuring S_z to be $-\hbar/2$ is

$$\begin{aligned}\mathcal{P}_- &= \left|\left\langle-|\psi(t)\right\rangle\right|^2 = \left|\left\langle-\right|\left(\tfrac{1}{\sqrt{2}}e^{-i\omega_0t/2}\left|+\right\rangle_x + \tfrac{1}{\sqrt{2}}e^{+i\omega_0t/2}\left|-\right\rangle_x\right)\right|^2 \\ &= \left|\tfrac{1}{\sqrt{2}}e^{-i\omega_0t/2}\left\langle-|+\right\rangle_x + \tfrac{1}{\sqrt{2}}e^{+i\omega_0t/2}\left\langle-|-\right\rangle_x\right|^2 = \left|\tfrac{1}{2}e^{-i\omega_0t/2} - \tfrac{1}{2}e^{+i\omega_0t/2}\right|^2 = \sin^2\frac{\omega_0t}{2}\end{aligned}$$

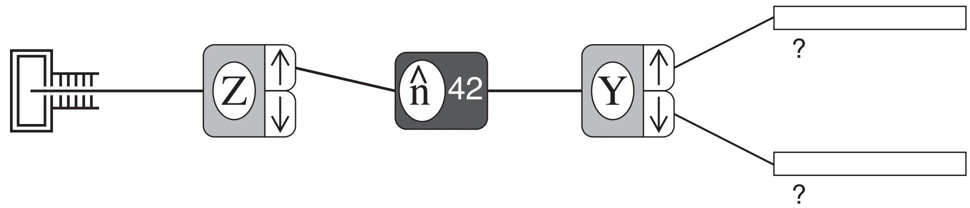
To have this probability equal to 25% requires

$$\mathcal{P}_- = \sin^2\frac{eB_0t}{2m_e} = \frac{1}{4} \quad\Rightarrow\quad \frac{eB_0t}{2m_e} = \frac{\pi}{6}$$

The time to traverse the distance d is $t = d/v$, yielding

$$\frac{eB_0d}{2m_ev} = \frac{\pi}{6} \quad\Rightarrow\quad d = \frac{\pi m_ev}{3eB_0}$$

3.8 Experiment schematic:



The measurement at the first analyzer collapses the state to

$$|\psi(0)\rangle = |+\rangle$$

In the field aligned in the *xz*-plane, the energy eigenstates are $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$ and $|-\rangle_n = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}|\psi(0)\rangle &= |+\rangle = \left(|+\rangle_{n\ n}\langle+| + |-\rangle_{n\ n}\langle-|\right)|+\rangle = \left({}_n\langle+|+\rangle\right)|+\rangle_n + \left({}_n\langle-|+\rangle\right)|-\rangle_n \\ &= \cos\tfrac{\theta}{2}|+\rangle_n + \sin\tfrac{\theta}{2}|-\rangle_n\end{aligned}$$

The time evolved state is

$$|\psi(t)\rangle = \cos\tfrac{\theta}{2}e^{-iE_+t/\hbar}|+\rangle_n + \sin\tfrac{\theta}{2}e^{-iE_-t/\hbar}|-\rangle_n = \cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}|+\rangle_n + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}|-\rangle_n$$

The probability of measuring S_y to be $+\hbar/2$ is

$$\mathcal{P}_{+y} = \left|{}_y\langle+|\psi(t)\rangle\right|^2 = \left|{}_y\langle+|\left(\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}|+\rangle_n + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}|-\rangle_n\right)\right|^2$$

Using matrix notation, we have

$$\begin{aligned}\mathcal{P}_{+y} &= \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\left\{\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}\right\}\right|^2 \\ &= \tfrac{1}{2}\left|\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}\left(\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\right) + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}\left(\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2}\right)\right|^2 \\ &= \tfrac{1}{2}\left|\cos^2\tfrac{\theta}{2}e^{-i\omega_0 t/2} - i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2} + \sin^2\tfrac{\theta}{2}e^{+i\omega_0 t/2} + i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{+i\omega_0 t/2}\right|^2 \\ &= \tfrac{1}{2}\left|\cos\tfrac{\omega_0 t}{2} - i\cos\theta\sin\tfrac{\omega_0 t}{2} - \sin\theta\sin\tfrac{\omega_0 t}{2}\right|^2 = \tfrac{1}{2}\left\{\left(\cos\tfrac{\omega_0 t}{2} - \sin\theta\sin\tfrac{\omega_0 t}{2}\right)^2 + \cos^2\theta\sin^2\tfrac{\omega_0 t}{2}\right\} \\ &= \tfrac{1}{2}\left\{\cos^2\tfrac{\omega_0 t}{2} + \sin^2\theta\sin^2\tfrac{\omega_0 t}{2} - 2\sin\theta\sin\tfrac{\omega_0 t}{2}\cos\tfrac{\omega_0 t}{2} + \cos^2\theta\sin^2\tfrac{\omega_0 t}{2}\right\} \\ &= \tfrac{1}{2}\left\{1 - \sin\theta\sin\omega_0 t\right\}\end{aligned}$$

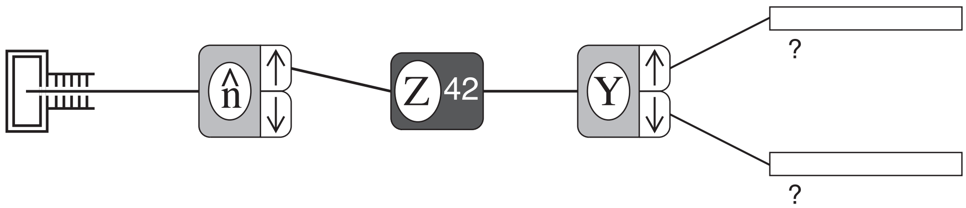
For $\theta = 0$, the precession is around the *z*-axis, which causes no change to the original $|\psi(0)\rangle = |+\rangle$ state:

$$\mathcal{P}_{+y}(\theta = 0) = \tfrac{1}{2}$$

For $\theta = \frac{\pi}{2}$, the precession is around the *x*-axis, which causes the original $|\psi(0)\rangle = |+\rangle$ state to precess to full alignment and anti-alignment along the *y*-axis:

$$\mathcal{P}_{+y}\left(\theta=\tfrac{\pi}{2}\right)=\tfrac{1}{2}\left\{1-\sin\omega_0 t\right\}$$

3.9 Experiment schematic:



The measurement at the first analyzer collapses the state to

$$\left|\psi(0)\right\rangle=\left|+\right\rangle_n=\cos\tfrac{\theta}{2}\left|+\right\rangle+e^{i\phi}\sin\tfrac{\theta}{2}\left|-\right\rangle$$

The angles for the first analyzer are $\theta=\frac{\pi}{2}$ and $\phi=\frac{\pi}{4}$, so the initial state is

$$\left|\psi(0)\right\rangle=\left|+\right\rangle_n=\tfrac{1}{\sqrt{2}}\left|+\right\rangle+e^{i\pi/4}\tfrac{1}{\sqrt{2}}\left|-\right\rangle$$

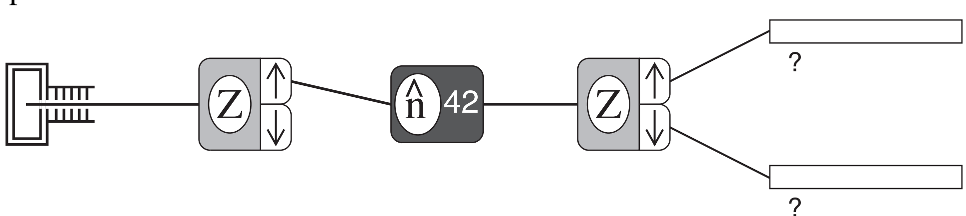
In the field aligned along the z-axis, the energy eigenstates are $\left|\pm\right\rangle$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0=eB_0/m_e$. The time evolved state is

$$\left|\psi(t)\right\rangle=\tfrac{1}{\sqrt{2}}e^{-iE_+t/\hbar}\left|+\right\rangle+e^{i\pi/4}\tfrac{1}{\sqrt{2}}e^{-iE_-t/\hbar}\left|-\right\rangle=\tfrac{1}{\sqrt{2}}e^{-i\omega_0t/2}\left|+\right\rangle+e^{i\pi/4}\tfrac{1}{\sqrt{2}}e^{+i\omega_0t/2}\left|-\right\rangle$$

The probability of measuring S_y to be $+\hbar/2$ is

$$\begin{aligned}
\mathcal{P}_{+y}&=\left|\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\tfrac{1}{\sqrt{2}}e^{-i\omega_0t/2}\\ \tfrac{1}{\sqrt{2}}e^{i\pi/4}e^{+i\omega_0t/2}\end{pmatrix}\right|^2\\
&=\tfrac{1}{4}\left|e^{-i\omega_0t/2}-ie^{+i\omega_0t/2}e^{i\pi/4}\right|^2=\tfrac{1}{2}\left|e^{i\pi/8}\left(e^{-i\omega_0t/2}e^{-i\pi/8}-ie^{+i\omega_0t/2}e^{i\pi/8}\right)\right|^2\\
&=\tfrac{1}{4}\left(1+1+ie^{-i\omega_0t}e^{-i\pi/4}-ie^{+i\omega_0t}e^{+i\pi/4}\right)\\
&=\tfrac{1}{2}\left(1+\sin\left(\omega_0t+\tfrac{\pi}{4}\right)\right)=\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{2}}\sin\omega_0t+\tfrac{1}{\sqrt{2}}\cos\omega_0t\right)
\end{aligned}$$

3.10 Experiment schematic:



The measurement at the first analyzer collapses the state to

$$\left|\psi(0)\right\rangle=\left|+\right\rangle$$

In the field aligned at 45° in the xz-plane, the energy eigenstates are $\left|+\right\rangle_n=\cos\frac{\theta}{2}\left|+\right\rangle+\sin\frac{\theta}{2}\left|-\right\rangle$ and $\left|-\right\rangle_n=\sin\frac{\theta}{2}\left|+\right\rangle-\cos\frac{\theta}{2}\left|-\right\rangle$ with $\theta=\pi/4$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}\left|\psi(0)\right\rangle &= \left|+\right\rangle = \left(\left|+\right\rangle_n {}_n\!\left\langle+\right| + \left|-\right\rangle_n {}_n\!\left\langle-\right|\right)\left|+\right\rangle = \left({}_n\!\left\langle+\middle|+\right\rangle\right)\left|+\right\rangle_n + \left({}_n\!\left\langle-\middle|+\right\rangle\right)\left|-\right\rangle_n \\ &= \cos\tfrac{\theta}{2}\left|+\right\rangle_n + \sin\tfrac{\theta}{2}\left|-\right\rangle_n\end{aligned}$$

The time evolved state is

$$\left|\psi(t)\right\rangle = \cos\tfrac{\theta}{2}e^{-iE_+t/\hbar}\left|+\right\rangle_n + \sin\tfrac{\theta}{2}e^{-iE_-t/\hbar}\left|-\right\rangle_n = \cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}\left|+\right\rangle_n + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}\left|-\right\rangle_n$$

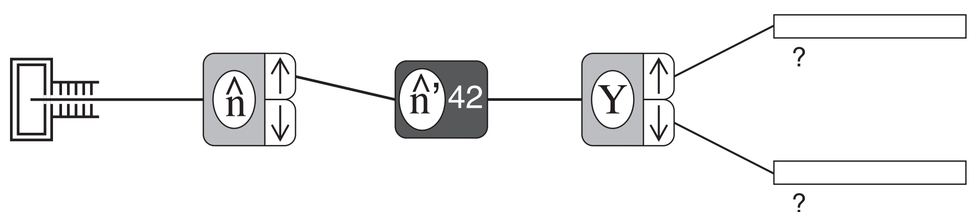
The probability of measuring S_z to be $-\hbar/2$ is

$$\mathcal{P}_- = \left|\left\langle-\middle|\psi(t)\right\rangle\right|^2 = \left|\left\langle-\right|\left(\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}\left|+\right\rangle_n + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}\left|-\right\rangle_n\right)\right|^2$$

Using matrix notation, we have

$$\begin{aligned}\mathcal{P}_- &= \left|\begin{pmatrix}0 & 1\end{pmatrix}\left\{\cos\tfrac{\theta}{2}e^{-i\omega_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} + \sin\tfrac{\theta}{2}e^{+i\omega_0 t/2}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}\right\}\right|^2 \\ &= \left|\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{-i\omega_0 t/2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{+i\omega_0 t/2}\right|^2 \\ &= \left|-2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\omega_0 t}{2}\right|^2 = \sin^2\theta\sin^2\tfrac{\omega_0 t}{2} = \tfrac{1}{2}\sin^2\tfrac{\omega_0 t}{2}\end{aligned}$$

3.11 Experiment schematic:



The measurement at the first analyzer collapses the state to

$$\left|\psi(0)\right\rangle = \left|+\right\rangle_n = \cos\tfrac{\theta}{2}\left|+\right\rangle + e^{i\phi}\sin\tfrac{\theta}{2}\left|-\right\rangle$$

The angles for the first analyzer are $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$, so the initial state is

$$\left|\psi(0)\right\rangle = \left|+\right\rangle_n = \tfrac{1}{\sqrt{2}}\left|+\right\rangle + e^{i\pi/4}\tfrac{1}{\sqrt{2}}\left|-\right\rangle$$

In the field aligned at 45° in the xz-plane, the energy eigenstates are $\left|+\right\rangle_{n'} = \cos\frac{\theta'}{2}\left|+\right\rangle + \sin\frac{\theta'}{2}\left|-\right\rangle$ and $\left|-\right\rangle_{n'} = \sin\frac{\theta'}{2}\left|+\right\rangle - \cos\frac{\theta'}{2}\left|-\right\rangle$ with $\theta' = \pi/4$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned}\left|\psi(0)\right\rangle = \left|+\right\rangle_n &= \left(\left|+\right\rangle_{n'} {}_{n'}\!\left\langle+\right| + \left|-\right\rangle_{n'} {}_{n'}\!\left\langle-\right|\right)\left|+\right\rangle_n = \left({}_{n'}\!\left\langle+\middle|+\right\rangle_n\right)\left|+\right\rangle_{n'} + \left({}_{n'}\!\left\langle-\middle|+\right\rangle_n\right)\left|-\right\rangle_{n'} \\ &= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2} + \tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)\left|+\right\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2} - \tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)\left|-\right\rangle_{n'}\end{aligned}$$

The time evolved state is

$$\begin{aligned}\left|\psi(t)\right\rangle &= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2}+\tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-iE_+t/\hbar}\left|+\right\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2}-\tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{-iE_-t/\hbar}\left|-\right\rangle_{n'}\\ &= \left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2}+\tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0t/2}\left|+\right\rangle_{n'} + \left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2}-\tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0t/2}\left|-\right\rangle_{n'}\end{aligned}$$

The probability of measuring S_y to be $+\hbar/2$ is

$$\begin{aligned}\mathcal{P}_{+y} &= \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\left\{\begin{matrix}\left(\tfrac{1}{\sqrt{2}}\cos\tfrac{\theta'}{2}+\tfrac{1}{\sqrt{2}}e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0t/2}\begin{pmatrix}\cos\tfrac{\theta'}{2}\\ \sin\tfrac{\theta'}{2}\end{pmatrix}\\ +\left(\tfrac{1}{\sqrt{2}}\sin\tfrac{\theta'}{2}-\tfrac{1}{\sqrt{2}}e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0t/2}\begin{pmatrix}\sin\tfrac{\theta'}{2}\\ -\cos\tfrac{\theta'}{2}\end{pmatrix}\end{matrix}\right\}\right|^2\\
&= \tfrac{1}{4}\left|\begin{matrix}\left(\cos\tfrac{\theta'}{2}+e^{i\pi/4}\sin\tfrac{\theta'}{2}\right)\left(\cos\tfrac{\theta'}{2}-i\sin\tfrac{\theta'}{2}\right)e^{-i\omega_0t/2}+\\ +\left(\sin\tfrac{\theta'}{2}-e^{i\pi/4}\cos\tfrac{\theta'}{2}\right)\left(\sin\tfrac{\theta'}{2}+i\cos\tfrac{\theta'}{2}\right)e^{+i\omega_0t/2}\end{matrix}\right|^2\\
&= \tfrac{1}{4}\left|\begin{matrix}\cos^2\tfrac{\theta'}{2}e^{-i\omega_0t/2}+\sin^2\tfrac{\theta'}{2}e^{+i\omega_0t/2}-2\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\omega_0t}{2}+\\ -2ie^{i\pi/4}\sin\tfrac{\theta'}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\omega_0t}{2}-ie^{i\pi/4}\sin^2\tfrac{\theta'}{2}e^{-i\omega_0t/2}-ie^{i\pi/4}\cos^2\tfrac{\theta'}{2}e^{+i\omega_0t/2}\end{matrix}\right|^2\\
&= \tfrac{1}{4}\left|\begin{matrix}\left(1+ie^{i\pi/4}\right)e^{+i\omega_0t/2}-2i\left(1+ie^{i\pi/4}\right)\cos^2\tfrac{\theta'}{2}\sin\tfrac{\omega_0t}{2}+\\ -2ie^{i\pi/4}\cos\tfrac{\omega_0t}{2}-\left(1+ie^{i\pi/4}\right)\sin\theta'\sin\tfrac{\omega_0t}{2}\end{matrix}\right|^2\\
&= \tfrac{1}{4}\left|\begin{matrix}\left(\tfrac{\sqrt{2}-1}{\sqrt{2}}+i\tfrac{1}{\sqrt{2}}\right)\left(\cos\tfrac{\omega_0t}{2}+i\sin\tfrac{\omega_0t}{2}\right)+\left(\tfrac{1}{\sqrt{2}}-i\tfrac{\sqrt{2}-1}{\sqrt{2}}\right)\left(1+\cos\theta'\right)\sin\tfrac{\omega_0t}{2}+\\ +\left(\tfrac{1-i}{\sqrt{2}}\right)2\cos\tfrac{\omega_0t}{2}-\left(\tfrac{\sqrt{2}-1}{\sqrt{2}}+i\tfrac{1}{\sqrt{2}}\right)\sin\theta'\sin\tfrac{\omega_0t}{2}\end{matrix}\right|^2\\
&= \tfrac{1}{8}\left|\begin{matrix}\left\{\left(1+\sqrt{2}\right)\cos\tfrac{\omega_0t}{2}+\left(\cos\theta'-\left(\sqrt{2}-1\right)\sin\theta'\right)\sin\tfrac{\omega_0t}{2}\right\}+\\ +i\left\{-\cos\tfrac{\omega_0t}{2}-\left(\left(\sqrt{2}+1\right)\cos\theta'+\sin\theta'\right)\sin\tfrac{\omega_0t}{2}\right\}\end{matrix}\right|^2\end{aligned}$$

Squaring and simplifying gives

$$\begin{aligned}\mathcal{P}_{+y} &= \tfrac{1}{8}\left\{\left(4+2\sqrt{2}\right)\cos^2\tfrac{\omega_0t}{2}+\left(4-2\sqrt{2}\right)\sin^2\tfrac{\omega_0t}{2}+4\sqrt{2}\cos\theta'\cos\tfrac{\omega_0t}{2}\sin\tfrac{\omega_0t}{2}\right\}\\ &= \tfrac{1}{4}\left\{2+\sqrt{2}\cos\omega_0t+\sqrt{2}\cos\theta'\sin\omega_0t\right\}\\ &= \tfrac{1}{4}\left\{2+\sqrt{2}\cos\omega_0t+\sin\omega_0t\right\}\end{aligned}$$

3.12 The eigenstates and eigenvalues of H are, by inspection:

$$E=E_1,E_2;\quad \left|E_1\right\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix},\quad \left|E_2\right\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

We need to find the eigenvectors of A:

$$A \doteq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} -\lambda & a \\ a & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a)^2 = 0 \quad \Rightarrow \quad \lambda = \pm a$$

Find the eigenvectors:

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = +a\begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \quad v = u$$

yielding

$$|+a\rangle = \tfrac{1}{\sqrt{2}}\left(|E_1\rangle + |E_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -a\begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \quad v = -u$$

$$|-a\rangle = \tfrac{1}{\sqrt{2}}\left(|E_1\rangle - |E_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The initial state is

$$|\psi(0)\rangle = |+a\rangle = \tfrac{1}{\sqrt{2}}\left(|E_1\rangle + |E_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The time evolved state is

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}\left(e^{-iE_1t/\hbar}|E_1\rangle + e^{-iE_2t/\hbar}|E_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \end{pmatrix}$$

The expectation value of A is

$$\langle A\rangle=\langle\psi(t)|A|\psi(t)\rangle=\tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{+iE_1t/\hbar} & e^{+iE_2t/\hbar}\end{pmatrix}\begin{pmatrix} 0 & a\\ a & 0\end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} e^{-iE_1t/\hbar}\\ e^{-iE_2t/\hbar}\end{pmatrix}$$

$$=\tfrac{1}{2}\begin{pmatrix} e^{+iE_1t/\hbar} & e^{+iE_2t/\hbar}\end{pmatrix}\begin{pmatrix} ae^{-iE_2t/\hbar}\\ ae^{-iE_1t/\hbar}\end{pmatrix}=\tfrac{a}{2}\left(e^{+i(E_1-E_2)t/\hbar}+e^{+i(E_2-E_1)t/\hbar}\right)$$

$$=a\cos\left(\frac{E_2-E_1}{\hbar}t\right)=a\cos(\omega_{Bohr}t)$$

Hence the Bohr frequency is determined the energy difference:

$$\omega_{Bohr}=\frac{E_2-E_1}{\hbar}$$

3.13 First we need to find the energy eigenvalue and eigenstates. Diagonalizing H yields the eigenvalues

$$\begin{vmatrix} E_0-\lambda & 0 & A\\ 0 & E_1-\lambda & 0\\ A & 0 & E_0-\lambda\end{vmatrix}=0 \quad\Rightarrow\quad (E_0-\lambda)^2(E_1-\lambda)-A^2(E_1-\lambda)=0$$

$$\Rightarrow\quad (E_1-\lambda)\left\{(E_0-\lambda)^2-A^2\right\}=0 \quad\Rightarrow\quad \lambda=E_1,E_0+A,E_0-A$$

and the eigenvectors

$$\begin{pmatrix} E_0 & 0 & A\\ 0 & E_1 & 0\\ A & 0 & E_0\end{pmatrix}\begin{pmatrix} u\\ v\\ w\end{pmatrix}=E_1\begin{pmatrix} u\\ v\\ w\end{pmatrix} \quad\Rightarrow\quad \begin{array}{c} E_0u+Aw=E_1u\\ E_1v=E_1v\\ Au+E_0w=E_1w\end{array} \quad\Rightarrow\quad u=w=0$$

$$|u|^2+|v|^2+|w|^2=1 \quad\Rightarrow\quad |v|^2=1 \quad\Rightarrow\quad u=0,v=1,w=0 \quad\Rightarrow\quad |E_1\rangle=|2\rangle\doteq\begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}$$

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (E_0 \pm A)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \Rightarrow \begin{matrix} E_0 u + Aw = (E_0 \pm A)u \\ E_1 v = (E_0 \pm A)v \\ Au + E_0 w = (E_0 \pm A)w \end{matrix} \Rightarrow v = 0, u = \pm w$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \Rightarrow 2|u|^2 = 1 \Rightarrow u = \tfrac{1}{\sqrt{2}}, v = 0, w = \pm\tfrac{1}{\sqrt{2}}$$

$$\Rightarrow |E_0 \pm A\rangle = \tfrac{1}{\sqrt{2}}|1\rangle \pm \tfrac{1}{\sqrt{2}}|3\rangle \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix}$$

(a) The initial state is

$$|\psi(0)\rangle = |2\rangle = |E_1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The time evolved state is

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}|E_1\rangle \doteq \begin{pmatrix} 0 \\ e^{-iE_1 t/\hbar} \\ 0 \end{pmatrix}$$

The probability of measuring the system to be in state $|2\rangle$ is

$$\mathcal{P}_2 = |\langle 2|\psi(t)\rangle|^2 = |\langle 2|e^{-iE_1 t/2\hbar}|2\rangle|^2 = |e^{-iE_1 t/2\hbar}|^2 = 1$$

(b) The initial state is

$$|\psi(0)\rangle = |3\rangle = \tfrac{1}{\sqrt{2}}\left(|E_0 + A\rangle - |E_0 - A\rangle\right) \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The time-evolved state is

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}\left(e^{-i(E_0+A)t/\hbar}|E_0 + A\rangle - e^{-i(E_0-A)t/\hbar}|E_0 - A\rangle\right)$$

$$\doteq \tfrac{1}{\sqrt{2}} e^{-i(E_0+A)t/\hbar}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{\sqrt{2}} e^{-i(E_0-A)t/\hbar}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\doteq \tfrac{1}{2} e^{-iE_0 t/\hbar}\begin{pmatrix} -2i\sin(At/\hbar) \\ 0 \\ 2\cos(At/\hbar) \end{pmatrix}$$

The probability of measuring the system to be in state $|3\rangle$ is

$$\mathcal{P}_3 = \left|\langle 3|\psi(t)\rangle\right|^2 = \left|\tfrac{1}{2}e^{-iE_0t/\hbar}2\cos\left(At/\hbar\right)\right|^2 = \cos^2\left(At/\hbar\right) = \tfrac{1}{2}\left(1+\cos\left(2At/\hbar\right)\right)$$

3.14 a) Before we do any calculations we must normalize the state.

$$\begin{aligned}
&|\psi(0)\rangle = C\left[3|a_1\rangle + 4|a_2\rangle\right] \\
&\quad 1 = \langle\psi|\psi\rangle = C^*\left[3\langle a_1| + 4\langle a_2|\right]C\left[3|a_1\rangle + 4|a_2\rangle\right] = |C|^2\left[9\langle a_1|a_1\rangle + 16\langle a_2|a_2\rangle\right] = |C|^2\,25 \\
&\quad C = \frac{1}{5}
\end{aligned}$$

We are free to choose C to be real and positive since an overall phase is not physical. Before we find the time dependent state vector we must find the energies and energy eigenstates so that we can write the initial state in the energy basis.

$$\begin{aligned}
&H \doteq E_0\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\
&\begin{vmatrix} 2E_0-\lambda & E_0 \\ E_0 & 2E_0-\lambda \end{vmatrix} = 0 \\
&\left(2E_0-\lambda\right)^2 - E_0^2 = 0 \\
&2E_0 - \lambda = \pm E_0 \\
&\lambda = 2E_0 \pm E_0 = E_0\ ,\ 3E_0
\end{aligned}$$

$$\begin{aligned}
&H|E_i\rangle = E_i|E_i\rangle \\
&E_0\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = E_i\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \\
&E_0\left(2\alpha_i + \beta_i\right) = E_i\alpha_i \\
&E_1 = E_0:\ \ E_0\left(2\alpha_i + \beta_i\right) = E_0\alpha_i \ \Rightarrow\ \alpha_i = -\beta_i \\
&E_1 = 3E_0:\ \ E_0\left(2\alpha_i + \beta_i\right) = 3E_0\alpha_i \ \Rightarrow\ \alpha_i = +\beta_i \\
&|E_1\rangle = \tfrac{1}{\sqrt{2}}\left(|a_1\rangle - |a_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&|E_2\rangle = \tfrac{1}{\sqrt{2}}\left(|a_1\rangle + |a_2\rangle\right) \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

Now write the initial state in the energy basis:

$$\left|\psi(0)\right\rangle = \sum_i \left|E_i\right\rangle\left\langle E_i\middle|\psi(0)\right\rangle$$
$$\left|\psi(0)\right\rangle = \sum_i \left|E_i\right\rangle\left\langle E_i\right|\tfrac{1}{5}\left[3\left|a_1\right\rangle + 4\left|a_2\right\rangle\right]$$
$$\left|\psi(0)\right\rangle = \tfrac{1}{5}\left|E_1\right\rangle\left[3\left\langle E_1\middle|a_1\right\rangle + 4\left\langle E_1\middle|a_2\right\rangle\right] + \tfrac{1}{5}\left|E_2\right\rangle\left[3\left\langle E_2\middle|a_1\right\rangle + 4\left\langle E_2\middle|a_2\right\rangle\right]$$
$$\left|\psi(0)\right\rangle = \tfrac{1}{5}\left|E_1\right\rangle\left[3\tfrac{1}{\sqrt{2}} - 4\tfrac{1}{\sqrt{2}}\right] + \tfrac{1}{5}\left|E_2\right\rangle\left[3\tfrac{1}{\sqrt{2}} + 4\tfrac{1}{\sqrt{2}}\right]$$
$$\left|\psi(0)\right\rangle = -\tfrac{1}{5\sqrt{2}}\left|E_1\right\rangle + \tfrac{7}{5\sqrt{2}}\left|E_2\right\rangle$$

Since these basis states are energy eigenstates, we simply multiply by the appropriate phase factor to find the time evolved state:

$$\left|\psi(0)\right\rangle = -\tfrac{1}{5\sqrt{2}}\left|E_1\right\rangle + \tfrac{7}{5\sqrt{2}}\left|E_2\right\rangle$$
$$\left|\psi(t)\right\rangle = \tfrac{1}{5\sqrt{2}}\left[-e^{-i\frac{E_1}{\hbar}t}\left|E_1\right\rangle + 7e^{-i\frac{E_2}{\hbar}t}\left|E_2\right\rangle\right] = \tfrac{1}{5\sqrt{2}}\left[-e^{-i\frac{E_0}{\hbar}t}\left|E_1\right\rangle + 7e^{-i\frac{3E_0}{\hbar}t}\left|E_2\right\rangle\right]$$
$$= \tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-\left|E_1\right\rangle + 7e^{-i\frac{2E_0}{\hbar}t}\left|E_2\right\rangle\right]$$

Since there are two energy eigenstates in the state vector, there are two possible energies that can be measured: $E_1 = E_0$, $E_2 = 3E_0$. The probabilities are

$$\mathcal{P}_{E_n} = \left|\left\langle E_n\middle|\psi(t)\right\rangle\right|^2$$
$$\mathcal{P}_{E_n} = \left|\left\langle E_n\right|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-\left|E_1\right\rangle + 7e^{-i\frac{2E_0}{\hbar}t}\left|E_2\right\rangle\right]\right|^2$$
$$\boxed{\begin{aligned}\mathcal{P}_{E_0} &= \left(-\tfrac{1}{5\sqrt{2}}\right)^2 = \tfrac{1}{50} = 0.02\\ \mathcal{P}_{3E_0} &= \left(\tfrac{7}{5\sqrt{2}}\right)^2 = \tfrac{49}{50} = 0.98\end{aligned}}$$

Since the energy states are stationary states, this result is time independent.

b) Now find the expectation value of the operator *A*. We have two methods to do this, the operator/matrix method and the probability weighting method. Since we don't have the matrix for *A*, the probability weighting method will be used.

$$\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_n a_n \mathcal{P}_{a_n}$$

$$\mathcal{P}_{a_n} = \left|\langle a_n|\psi(t)\rangle\right|^2 = \left|\langle a_n|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-|E_1\rangle + 7e^{-i\frac{2E_0}{\hbar}t}|E_2\rangle\right]\right|^2$$

$$\mathcal{P}_{a_1} = \left|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-\langle a_1|E_1\rangle + 7e^{-i\frac{2E_0}{\hbar}t}\langle a_1|E_2\rangle\right]\right|^2 = \left|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-\tfrac{1}{\sqrt{2}} + e^{-i\frac{2E_0}{\hbar}t}\tfrac{7}{\sqrt{2}}\right]\right|^2$$

$$\mathcal{P}_{a_1} = \tfrac{1}{100}\left|-1 + 7e^{-i\frac{2E_0}{\hbar}t}\right|^2 = \tfrac{1}{100}\left[50 - 14\cos\left(\tfrac{2E_0}{\hbar}t\right)\right]$$

$$\mathcal{P}_{a_2} = \left|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[-\langle a_2|E_1\rangle + 7e^{-i\frac{2E_0}{\hbar}t}\langle a_2|E_2\rangle\right]\right|^2 = \left|\tfrac{1}{5\sqrt{2}}e^{-i\frac{E_0}{\hbar}t}\left[\tfrac{1}{\sqrt{2}} + e^{-i\frac{2E_0}{\hbar}t}\tfrac{7}{\sqrt{2}}\right]\right|^2$$

$$\mathcal{P}_{a_2} = \tfrac{1}{100}\left|1 + 7e^{-i\frac{2E_0}{\hbar}t}\right|^2 = \tfrac{1}{100}\left[50 + 14\cos\left(\tfrac{2E_0}{\hbar}t\right)\right]$$

$$\langle A\rangle = \sum_n a_n \mathcal{P}_{a_n} = a_1\tfrac{1}{100}\left[50 - 14\cos\left(\tfrac{2E_0}{\hbar}t\right)\right] + a_2\tfrac{1}{100}\left[50 + 14\cos\left(\tfrac{2E_0}{\hbar}t\right)\right]$$

$$\langle A\rangle = \frac{a_1 + a_2}{2} + \frac{7(a_2 - a_1)}{50}\cos\left(\tfrac{2E_0}{\hbar}t\right)$$

3.15 Substituting the general energy state superposition $|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$ into the Schrödinger equation for a time-independent Hamiltonian yields

$$i\hbar\frac{d}{dt}\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle = H\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$$

$$i\hbar\sum_n c_n(-iE_n/\hbar)e^{-iE_nt/\hbar}|E_n\rangle = \sum_n c_n e^{-iE_nt/\hbar}(E_n)|E_n\rangle$$

$$\sum_n c_n E_n e^{-iE_nt/\hbar}|E_n\rangle = \sum_n c_n E_n e^{-iE_nt/\hbar}|E_n\rangle$$

So the Schrödinger equation is satisfied. Substituting the general energy state superposition $|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$ into the energy eigenvalue equation $H|\psi\rangle = E|\psi\rangle$ yields

$$H\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle = E\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$$

$$\sum_n c_n E_n e^{-iE_nt/\hbar}|E_n\rangle = E\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$$

There is no value of E that will satisfy this equation, so the general energy state superposition does not satisfy the energy eigenvalue equation.

3.16 (a) The on-resonance time-evolution equations (3.95) are

$$i\hbar\dot{\alpha}_+(t) = \frac{\hbar\omega_1}{2}\alpha_-(t); \qquad i\hbar\dot{\alpha}_-(t) = \frac{\hbar\omega_1}{2}\alpha_+(t)$$

Substituting one into the other gives

$$\ddot{\alpha}_+(t) = -\frac{\omega_1^2}{4}\alpha_+(t)$$

with general solutions

$$\alpha_+(t) = Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}; \qquad \alpha_-(t) = -Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}$$

The state vector solutions Eqs. (3.92) and (3.94) for arbitrary initial conditions are

$$\left|\tilde{\psi}(t)\right\rangle = \left(Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}\right)\left|+\right\rangle + \left(-Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}\right)\left|-\right\rangle \doteq \begin{pmatrix} Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2} \\ -Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2} \end{pmatrix}$$

$$\begin{aligned} \left|\psi(t)\right\rangle &= \left(Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}\right)e^{-i\omega t/2}\left|+\right\rangle + \left(-Ae^{+i\omega_1 t/2} + Be^{-i\omega_1 t/2}\right)e^{i\omega t/2}\left|-\right\rangle \\ &\doteq \begin{pmatrix} Ae^{+i(\omega_1-\omega)t/2} + Be^{-i(\omega_1+\omega)t/2} \\ -Ae^{+i(\omega_1+\omega)t/2} + Be^{-i(\omega_1-\omega)t/2} \end{pmatrix} \end{aligned}$$

The initial state vectors are

$$\left|\tilde{\psi}(0)\right\rangle \doteq \begin{pmatrix} A+B \\ -A+B \end{pmatrix}; \quad \left|\psi(0)\right\rangle \doteq \begin{pmatrix} A+B \\ -A+B \end{pmatrix}$$

For a spin-1/2 system, the general initial state is the spin up state in the direction $\hat{\mathbf{n}}$ defined by the angles θ and ϕ:

$$\left|\psi(0)\right\rangle = \left|+\right\rangle_n \doteq \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

This gives

$$A = \tfrac{1}{2}\left(\cos\tfrac{\theta}{2} - e^{i\phi}\sin\tfrac{\theta}{2}\right); \quad B = \tfrac{1}{2}\left(\cos\tfrac{\theta}{2} + e^{i\phi}\sin\tfrac{\theta}{2}\right)$$

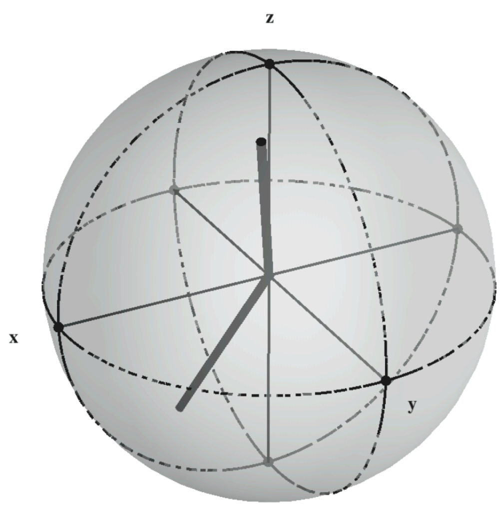
(b) A π-pulse ($\omega_1 t = \pi$) gives the state vectors

$$\begin{aligned} \left|\tilde{\psi}(t)\right\rangle &\doteq \begin{pmatrix} Ae^{+i\pi/2} + Be^{-i\pi/2} \\ -Ae^{+i\pi/2} + Be^{-i\pi/2} \end{pmatrix} = -i\begin{pmatrix} -A+B \\ A+B \end{pmatrix} = -i\begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \\ &\doteq -ie^{i\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} = -ie^{i\phi}\begin{pmatrix} \cos\frac{\pi-\theta}{2} \\ e^{-i\phi}\sin\frac{\pi-\theta}{2} \end{pmatrix} = -ie^{i\phi}\left|+\right\rangle_{n'(\pi-\theta,-\phi)} \end{aligned}$$

$$\left|\psi(t)\right\rangle \doteq \begin{pmatrix} iAe^{-i\omega t/2} - iBe^{-i\omega t/2} \\ -iAe^{+i\omega t/2} - iBe^{+i\omega t/2} \end{pmatrix} = -i\begin{pmatrix} e^{-i\omega t/2}\left(-A+B\right) \\ e^{+i\omega t/2}\left(A+B\right) \end{pmatrix} = -i\begin{pmatrix} e^{-i\omega t/2}e^{i\phi}\sin\frac{\theta}{2} \\ e^{+i\omega t/2}\cos\frac{\theta}{2} \end{pmatrix}$$

$$\doteq -ie^{-i\omega t/2}e^{i\phi}\begin{pmatrix} \cos\frac{\pi-\theta}{2} \\ e^{-i\phi+i\omega t}\sin\frac{\pi-\theta}{2} \end{pmatrix} = -ie^{-i\omega t/2}e^{i\phi}\left|+\right\rangle_{n''(\pi-\theta,-\phi+\omega t)}$$

The transformed vector $\left|\tilde{\psi}(t)\right\rangle$ in the rotating frame is "flipped" from $\left|+\right\rangle_n$ defined by the angles θ and ϕ to $\left|+\right\rangle_{n'}$ (modulo an overall phase) defined by the angles $\pi-\theta$ and $-\phi$. This is consistent with the precession in the rotating frame shown below:



In the nonrotating frame, the state $\left|\psi(t)\right\rangle$ is flipped from one z hemisphere to the other just as the $\left|\tilde{\psi}(t)\right\rangle$ state does. As a consequence the spin up and down probabilities are flipped:

$$\mathcal{P}_+(t) = \left|\left\langle + \middle| \psi(t)\right\rangle\right|^2 = \left|-ie^{-i\omega t/2}e^{i\phi}\sin\tfrac{\theta}{2}\right|^2 = \sin^2\tfrac{\theta}{2} = \mathcal{P}_-(0)$$

$$\mathcal{P}_-(t) = \left|\left\langle - \middle| \psi(t)\right\rangle\right|^2 = \left|-ie^{+i\omega t/2}\cos\tfrac{\theta}{2}\right|^2 = \cos^2\tfrac{\theta}{2} = \mathcal{P}_+(0)$$

But the frame rotation causes the initial state in the nonrotating frame $\left|+\right\rangle_n$ defined by the angles θ and ϕ to evolve to the state $\left|+\right\rangle_{n''}$ (modulo an overall phase) defined by the angles $\pi-\theta$ and $-\phi+\omega t$. Hence, the probabilities of measurements of the x and y spin components depend on the Larmor precession frequency ($\omega=\omega_0$).

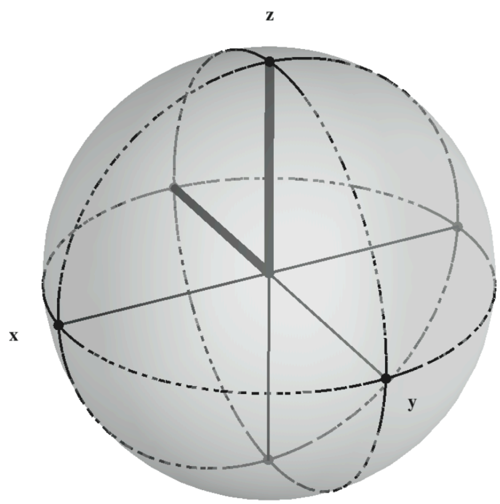
(c) For an initial state $\left|+\right\rangle$, $A=\frac{1}{2}$; $B=\frac{1}{2}$, and a π/2-pulse ($\omega_1 t=\pi/2$) yields

$$\left|\tilde{\psi}(t)\right\rangle \doteq \begin{pmatrix} \frac{1}{2}e^{+i\pi/4}+\frac{1}{2}e^{-i\pi/4} \\ -\frac{1}{2}e^{+i\pi/4}+\frac{1}{2}e^{-i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \left|-\right\rangle_y$$

$$\left|\psi(t)\right\rangle \doteq \begin{pmatrix} \frac{1}{2}e^{-i\omega t/2}e^{+i\pi/4} + \frac{1}{2}e^{-i\omega t/2}e^{-i\pi/4} \\ -\frac{1}{2}e^{+i\omega t/2}e^{+i\pi/4} + \frac{1}{2}e^{+i\omega t/2}e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\omega t/2} \\ -i\frac{1}{\sqrt{2}}e^{+i\omega t/2} \end{pmatrix}$$

$$\doteq e^{-i\omega t/2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{-i\pi/2+i\omega t} \end{pmatrix} = e^{-i\omega t/2}\left|+\right\rangle_{n(\theta=\pi/2,\phi=-\pi/2+\omega t)}$$

The transformed vector $\left|\tilde{\psi}(t)\right\rangle$ in the rotating frame is "rotated" by $\pi/2$ from $\left|+\right\rangle$ to $\left|-\right\rangle_y$. This is consistent with the precession in the rotating frame shown below.



But the frame rotation causes the initial state in the nonrotating frame $\left|+\right\rangle$ to "rotate" down to the *x-y* plane and then rotate around the *z*-axis at frequency ω.
(d) In the rotating frame, the state vector is fixed, while in the original frame the state vector rotates at the precession frequency.

3.17 The probability that an electron neutrino changes to a muon neutrino is

$$\mathcal{P}_{\nu_e\to\nu_\mu} = \sin^2\theta\sin^2\left(\frac{\left(m_1^2-m_2^2\right)Lc^3}{4E\hbar}\right)$$

The probability oscillates from 0 to a maximum value of $\sin^2\theta$, which occurs when the argument of the second $\sin^2$ term is $\pi/2$. Thus the distance is

$$L = \frac{2\pi E\hbar}{\left(m_1^2-m_2^2\right)c^3}$$

Using $m_1^2 - m_2^2 \simeq 8\times10^{-5}\ eV^2/c^4$, we get

$$L = \frac{2\pi E\hbar c}{\left(m_1^2-m_2^2\right)c^4} = \frac{8\ MeV\bullet 1240\ eVnm}{8\times10^{-5}\ eV^2} = 1.24\times10^5\,m = 124\ km$$

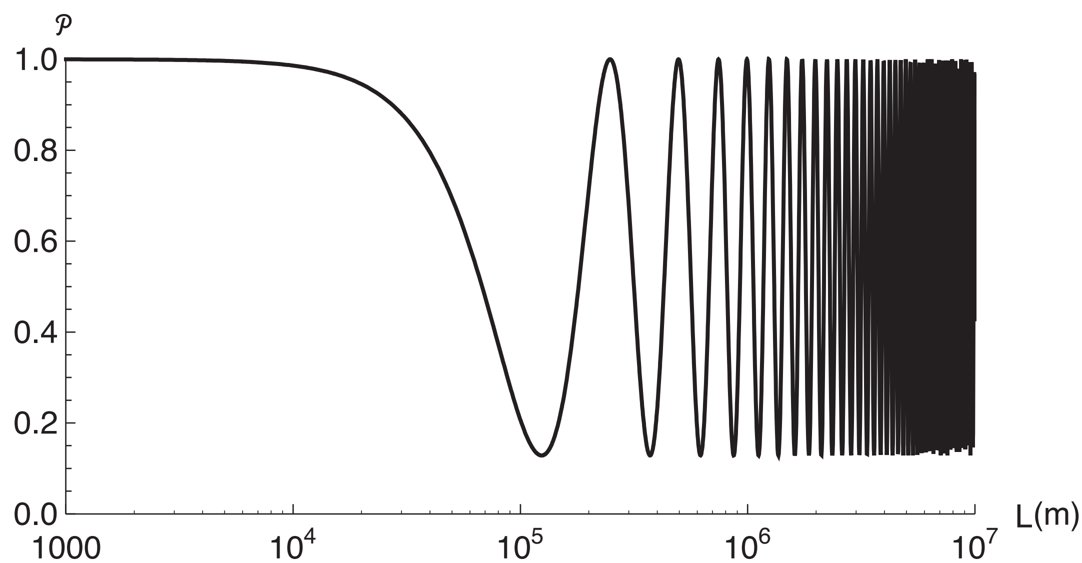
This is one half of an oscillation, so the full oscillation length is 2 times this. The number of oscillations is then

$$N = d/2L$$
$$N_{sun \to earth} = \frac{1.5\times10^{11}\,m}{2.48\times10^{5}\,m} = 6.04\times10^{5}$$
$$N_{earth} = \frac{2\times6.37\times10^{6}\,m}{2.48\times10^{5}\,m} = 51.4$$

3.18 The probability that an electron neutrino is detected is 1 minus the probability that a muon neutrino is detected:

$$\mathcal{P}_{\nu_e\to\nu_e} = 1 - \mathcal{P}_{\nu_e\to\nu_\mu} = 1 - \sin^2\theta \sin^2\left(\frac{\left(m_1^2 - m_2^2\right)Lc^3}{4E\hbar}\right)$$

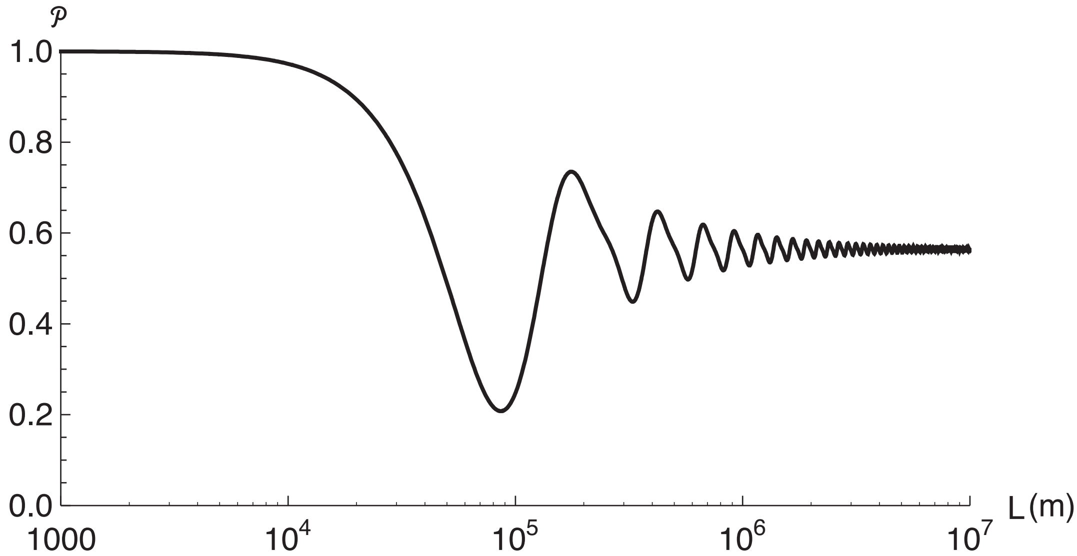
For an 8 MeV electron neutrino at the origin, the probability as a function of distance is shown below:



For a beam with a range of energies, we must average over the independent probabilities

$$\mathcal{P}_{\nu_e\to\nu_e} = \int_{E_{lower}}^{E_{upper}} \left[1 - \sin^2\theta \sin^2\left(\frac{\left(m_1^2 - m_2^2\right)Lc^3}{4E\hbar}\right)\right] f(E)\,dE$$

Assuming a uniform energy distribution from 4 MeV to 8 MeV gives $f(E) = 1/(4\ MeV)$. Integrating in Mathematica gives the plot:



The oscillations wash out with a spread in energy in the beam.