11.1 Use:

$$\left[J_x, J_y\right] = i\hbar J_z$$
$$\left[J_y, J_z\right] = i\hbar J_x$$
$$\left[J_z, J_x\right] = i\hbar J_y$$
$$\left[\mathbf{J}^2, J_x\right] = \left[\mathbf{J}^2, J_y\right] = \left[\mathbf{J}^2, J_z\right] = 0$$

For the ladder operators, we have

$$\begin{aligned}\left[J_+, J_-\right] &= [J_x + iJ_y, J_x - iJ_y] = \left[J_x, J_x\right] - i\left[J_x, J_y\right] + i\left[J_y, J_x\right] + \left[J_y, J_y\right] \\ &= 0 - i\left(i\hbar J_z\right) + i\left(-i\hbar J_z\right) + 0 = 2\hbar J_z\end{aligned}$$

and

$$\left[\mathbf{J}^2, J_\pm\right] = \left[\mathbf{J}^2, J_x \pm iJ_y\right] = \left[\mathbf{J}^2, J_x\right] \pm i\left[\mathbf{J}^2, J_y\right] = 0 \pm i0 = 0$$

Finally, we have

$$\begin{aligned}\left[J_z, J_\pm\right] &= \left[J_z, J_x \pm iJ_y\right] = \left[J_z, J_x\right] \pm i\left[J_z, J_y\right] = i\hbar J_y \pm i\left(-i\hbar J_x\right) \\ &= \pm\hbar J_x + i\hbar J_y = \pm\hbar\left(J_x \pm iJ_y\right) = \pm\hbar J_\pm\end{aligned}$$

11.2 Use

$$\left[J_z, J_\pm\right] = \pm\hbar J_\pm$$

To show that the lowering operator deserves its name, act with J_- on an angular momentum eigenstate $\left|jm_j\right\rangle$, where we assume that $\left|jm_j\right\rangle$ is a normalized eigenstate that satisfies the eigenvalue equation $J_z\left|jm_j\right\rangle = m_j\hbar\left|jm_j\right\rangle$. To learn about the angular momentum component of the new ket $J_-\left|jm_j\right\rangle$, consider what happens when the component operator J_z acts on the new ket $J_-\left|jm_j\right\rangle$:

$$J_z\left(J_-\left|jm_j\right\rangle\right) = J_zJ_-\left|jm_j\right\rangle$$

The commutator above tells us that $J_zJ_- = J_-J_z - \hbar J_-$, so we get

$$J_z\left(J_-\left|jm_j\right\rangle\right) = \left(J_-J_z - \hbar J_-\right)\left|jm_j\right\rangle = J_-J_z\left|jm_j\right\rangle - \hbar J_-\left|jm_j\right\rangle$$

Now use the angular momentum eigenvalue equation $J_z\left|jm_j\right\rangle = m_j\hbar\left|jm_j\right\rangle$ to obtain

$$J_z\left(J_-\left|jm_j\right\rangle\right) = J_-m_j\hbar\left|jm_j\right\rangle - \hbar J_-\left|jm_j\right\rangle = \left(m_j\hbar - \hbar\right)\left(J_-\left|jm_j\right\rangle\right) = \left(m_j - 1\right)\hbar\left(J_-\left|jm_j\right\rangle\right)$$

This equation tells us that when the new ket $J_-\left|jm_j\right\rangle$ is acted on by the component operator J_z, the result is the same ket $J_-\left|jm_j\right\rangle$ multiplied by the factor $\left(m_j - 1\right)\hbar$, which means that the new ket $J_-\left|jm_j\right\rangle$ is also an eigenstate of J_z, but with a magnetic quantum

number $\left(m_j - 1\right)$ that is smaller than the magnetic quantum number m_j of the original ket $\left|jm_j\right\rangle$ by one unit. Hence we have gone down the angular momentum ladder by one rung.
For the raising operator, we have a similar argument:

$$\begin{aligned} J_z\left(J_+\left|jm_j\right\rangle\right) &= J_zJ_+\left|jm_j\right\rangle = \left(J_+J_z + \hbar J_+\right)\left|jm_j\right\rangle = J_+J_z\left|jm_j\right\rangle + \hbar J_+\left|jm_j\right\rangle \\ &= J_+m_j\hbar\left|jm_j\right\rangle + \hbar J_+\left|jm_j\right\rangle = \left(m_j\hbar + \hbar\right)\left(J_+\left|jm_j\right\rangle\right) = \left(m_j + 1\right)\hbar\left(J_+\left|jm_j\right\rangle\right) \end{aligned}$$

Hence, the new ket $J_+\left|jm_j\right\rangle$ is also an eigenstate of J_z, but with a magnetic quantum number $\left(m_j + 1\right)$ that is larger than the magnetic quantum number m_j of the original ket $\left|jm_j\right\rangle$ by one unit. Hence we have gone up the angular momentum ladder by one rung.
To find the proportionality factor, write the new kets as:

$$J_\pm\left|jm_j\right\rangle = c_\pm\left|j,m_j \pm 1\right\rangle$$

and normalize them

$$\begin{aligned} \left|c_\pm\left|j,m_j \pm 1\right\rangle\right|^2 &= \left|J_\pm\left|jm_j\right\rangle\right|^2 \\ \left|c_\pm\right|^2 &= \left|\left\langle jm_j\right|J_\pm^\dagger J_\pm\left|jm_j\right\rangle\right|^2 = \left|\left\langle jm_j\right|J_\mp J_\pm\left|jm_j\right\rangle\right|^2 \\ &= \left|\left\langle jm_j\right|\left(J_x \mp iJ_y\right)\left(J_x \mp iJ_y\right)\left|jm_j\right\rangle\right|^2 = \left|\left\langle jm_j\right|J_x^2 + J_y^2 \pm i\left(J_xJ_y - J_yJ_x\right)\left|jm_j\right\rangle\right|^2 \\ &= \left|\left\langle jm_j\right|\mathbf{J}^2 - J_z^2 \pm i\left(i\hbar J_z\right)\left|jm_j\right\rangle\right|^2 = \left|\left\langle jm_j\right|j\left(j+1\right)\hbar^2 - m_j^2\hbar^2 \mp m_j\hbar^2\left|jm_j\right\rangle\right|^2 \\ &= \left(j\left(j+1\right) - m_j\left(m_j \pm 1\right)\right)\hbar^2 \end{aligned}$$

giving

$$J_\pm\left|jm_j\right\rangle = \hbar\sqrt{\left(j\left(j+1\right) - m_j\left(m_j \pm 1\right)\right)}\left|j,m_j \pm 1\right\rangle$$

11.3 The angular momentum eigenvalue equations are

$$\begin{aligned} \mathbf{J}^2\left|jm_j\right\rangle &= j\left(j+1\right)\hbar^2\left|jm_j\right\rangle \\ J_z\left|jm_j\right\rangle &= m_j\hbar\left|jm_j\right\rangle \end{aligned}$$

We define the "length" of $\mathbf{J}$ as the square root of the expectation value of $\mathbf{J}^2$:

$$\sqrt{\left\langle jm_j\right|\mathbf{J}^2\left|jm_j\right\rangle} = \sqrt{\left\langle jm_j\right|j\left(j+1\right)\hbar^2\left|jm_j\right\rangle} = \sqrt{j\left(j+1\right)}\hbar$$

The value of the angular momentum component J_z, $m_j\hbar$, must be less than or equal to the "length" of $\mathbf{J}$, so

$$m_j\hbar \le \sqrt{j\left(j+1\right)}\hbar \quad \Rightarrow m_j \le \sqrt{j\left(j+1\right)}$$

We know that the values of m_j form the rungs of the angular momentum ladder with unit steps between them. The rungs must lie on integers or half integers, so we have two possibilities for the ladder

$$\begin{array}{lll} \underline{\quad\quad} \; j+1 & & \underline{\quad\quad} \; j+\frac{3}{2} \\ \underline{\quad\quad} \; j & or & \underline{\quad\quad} \; j+\frac{1}{2} \\ \underline{\quad\quad} \; j-1 & & \underline{\quad\quad} \; j-\frac{1}{2} \end{array}$$

The question is: which is the top rung? Extending the inequality above,

$$j \le m_j \le \sqrt{j(j+1)} \le (j+1)$$

we see that m_j must lie at or below the $(j+1)$ rung of the left ladder. Now note that

$$\begin{aligned} & j(j+1) = j^2 + j \\ & \left(j+\tfrac{1}{2}\right)\left(j+\tfrac{1}{2}\right) = j^2 + j + \tfrac{1}{4} \\ & \Rightarrow j(j+1) < \left(j+\tfrac{1}{2}\right)\left(j+\tfrac{1}{2}\right) \end{aligned}$$

which implies that

$$\begin{aligned} m_j &\le \sqrt{j(j+1)} < \sqrt{\left(j+\tfrac{1}{2}\right)\left(j+\tfrac{1}{2}\right)} \\ m_j &\le \sqrt{j(j+1)} < \left(j+\tfrac{1}{2}\right) \end{aligned}$$

This tells us that that m_j must lie below the $\left(j+\frac{1}{2}\right)$ rung of the right ladder, which is below the the $(j+1)$ rung of the left ladder, thus excluding the $(j+1)$ rung. The $\left(j+\frac{1}{2}\right)$ rung is above the *j* rung of the left ladder. So the top rung must be the *j* rung of the left ladder:

$$m_{j,\max} = j$$

11.4 a) A spin 3/2 system has *s* (or *j*) equal to 3/2 and so has four possible states with m_s (or m_j) equal to +3/2, +1/2, -1/2, and -3/2. The four eigenstates are

$$|sm_s\rangle = \left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{-1}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{-3}{2}\right\rangle$$

The eigenvalue equations are

$$\mathbf{S}^2 |sm_s\rangle = s(s+1)\hbar^2 |sm_s\rangle = \tfrac{15}{4}\hbar^2 \left|\tfrac{3}{2},m_s\right\rangle$$

$$S_z |sm_s\rangle = m_s\hbar |sm_s\rangle = \begin{cases} \tfrac{3}{2}\hbar\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle \\ \tfrac{1}{2}\hbar\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle \\ -\tfrac{1}{2}\hbar\left|\tfrac{3}{2},\tfrac{-1}{2}\right\rangle \\ -\tfrac{3}{2}\hbar\left|\tfrac{3}{2},\tfrac{-3}{2}\right\rangle \end{cases}$$

b) The matrices representing $\mathbf{S}^2$ and S_z are diagonal in their own basis—the $|sm_s\rangle$ basis, with the diagonal eigenvalues from above:

$$\mathbf{S}^2 \doteq \begin{pmatrix} \frac{15}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & \frac{15}{4}\hbar^2 & 0 & 0 \\ 0 & 0 & \frac{15}{4}\hbar^2 & 0 \\ 0 & 0 & 0 & \frac{15}{4}\hbar^2 \end{pmatrix}$$

$$S_z \doteq \begin{pmatrix} \frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & \frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}$$

c) Generate the matrices representing S_x and S_y by noting that $S_\pm = S_x \pm iS_y$, which gives

$$S_x = \tfrac{1}{2}\left(S_+ + S_-\right)$$
$$S_y = \tfrac{-i}{2}\left(S_+ - S_-\right)$$

First generate $S_\pm$ using

$$S_\pm |sm_s\rangle = \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\,|s, m_s \pm 1\rangle$$

which gives the matrix elements

$$\begin{aligned} \langle s'm_s'|S_\pm|sm_s\rangle &= \langle s'm_s'|\hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\,|s, m_s \pm 1\rangle \\ &= \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\,\langle s'm_s'|s, m_s \pm 1\rangle \\ &= \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\,\delta_{ss'}\delta_{m_s', m_s \pm 1} \end{aligned}$$

Hence

$$S_+ \doteq \begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_- \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}$$

This gives

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

d) To find the eigenvalues of S_x, diagonalize the matrix

$$\det\left(S_x - \lambda\mathbf{I}\right) = \begin{vmatrix} -\lambda & \sqrt{3}\hbar/2 & 0 & 0 \\ \sqrt{3}\hbar/2 & -\lambda & \hbar & 0 \\ 0 & \hbar & -\lambda & \sqrt{3}\hbar/2 \\ 0 & 0 & \sqrt{3}\hbar/2 & -\lambda \end{vmatrix}$$

$$= -\lambda\left(-\lambda\left(\lambda^2 - 3\hbar^2/4\right) - \hbar\left(-\lambda\hbar\right)\right) - \sqrt{3}\hbar/2\left(\sqrt{3}\hbar/2\left(\lambda^2 - 3\hbar^2/4\right) - \hbar(0)\right)$$

$$= \lambda^2\left(\lambda^2 - 3\hbar^2/4\right) - 3\hbar^2/4\left(\lambda^2 - 3\hbar^2/4\right) - \lambda^2\hbar^2$$

$$0 = \left(\lambda^2 - 3\hbar^2/4\right)^2 - \lambda^2\hbar^2$$

Solve to get

$$\left(\lambda^2 - 3\hbar^2/4\right) = \pm\lambda\hbar$$

$$\lambda^2 \mp \lambda\hbar - 3\hbar^2/4 = 0$$

$$\lambda = \pm\frac{\hbar}{2} \pm \sqrt{\left(\frac{\hbar}{2}\right)^2 + \frac{3\hbar^2}{4}} = \pm\frac{\hbar}{2} \pm \hbar$$

Which gives the four values

$$S_x = \frac{3\hbar}{2},\quad \frac{\hbar}{2},\quad -\frac{\hbar}{2},\quad -\frac{3\hbar}{2}$$

as expected.

11.5 The eigenvalue equations for S_z and I_z are

$$S_z\left|m_s m_I\right\rangle = m_s\hbar\left|m_s m_I\right\rangle$$

$$I_z\left|m_s m_I\right\rangle = m_I\hbar\left|m_s m_I\right\rangle$$

Act on the first equation with I_z and on the second with S_z

$$I_zS_z\left|m_sm_I\right\rangle = I_zm_s\hbar\left|m_sm_I\right\rangle = m_s\hbar I_z\left|m_sm_I\right\rangle = m_s\hbar\left(m_I\hbar\right)\left|m_sm_I\right\rangle = m_sm_I\hbar^2\left|m_sm_I\right\rangle$$
$$S_zI_z\left|m_sm_I\right\rangle = S_zm_I\hbar\left|m_sm_I\right\rangle = m_I\hbar S_z\left|m_sm_I\right\rangle = m_I\hbar\left(m_s\hbar\right)\left|m_sm_I\right\rangle = m_sm_I\hbar^2\left|m_sm_I\right\rangle$$

and then subtract the two equations

$$I_zS_z\left|m_sm_I\right\rangle - S_zI_z\left|m_sm_I\right\rangle = m_sm_I\hbar^2\left|m_sm_I\right\rangle - m_sm_I\hbar^2\left|m_sm_I\right\rangle = 0$$
$$\left(I_zS_z - S_zI_z\right)\left|m_sm_I\right\rangle = 0$$

and conclude that

$$\left[I_z, S_z\right] = I_zS_z - S_zI_z = 0$$

11.6 The uncoupled basis $\left|m_sm_I\right\rangle$ comprises four states:

$$\left|++\right\rangle,\ \left|+-\right\rangle,\ \left|-+\right\rangle,\ \left|--\right\rangle$$

The eigenvalue equations for S_z and I_z are

$$S_z\left|m_sm_I\right\rangle = m_s\hbar\left|m_sm_I\right\rangle$$
$$I_z\left|m_sm_I\right\rangle = m_I\hbar\left|m_sm_I\right\rangle$$

Find the matrix elements of S_z:

$$\left\langle m_s'm_I'\right|S_z\left|m_sm_I\right\rangle = \left\langle m_s'm_I'\right|m_s\hbar\left|m_sm_I\right\rangle = m_s\hbar\left\langle m_s'm_I'\middle|m_sm_I\right\rangle = m_s\hbar\delta_{m_sm_s'}\delta_{m_Im_I'}$$

So the S_z matrix is diagonal in the uncoupled basis:

$$S_z \doteq \frac{\hbar}{2}\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}\begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

Find the matrix elements of I_z:

$$\left\langle m_s'm_I'\right|I_z\left|m_sm_I\right\rangle = \left\langle m_s'm_I'\right|m_I\hbar\left|m_sm_I\right\rangle = m_I\hbar\left\langle m_s'm_I'\middle|m_sm_I\right\rangle = m_I\hbar\delta_{m_sm_s'}\delta_{m_Im_I'}$$

So the I_z is also diagonal:

$$I_z \doteq \frac{\hbar}{2}\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}\begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

The eigenvalue equations for $\mathbf{S}^2$ and $\mathbf{I}^2$ are

$$\mathbf{S}^2\left|m_s m_I\right\rangle = s(s+1)\hbar^2\left|m_s m_I\right\rangle$$
$$\mathbf{I}^2\left|m_s m_I\right\rangle = I(I+1)\hbar^2\left|m_s m_I\right\rangle$$

where $s = 1/2$ and $I = 1/2$. Now use these to find the matrix elements:

$$\left\langle m_s' m_I'\right|\mathbf{S}^2\left|m_s m_I\right\rangle = \left\langle m_s' m_I'\right|s(s+1)\hbar^2\left|m_s m_I\right\rangle = s(s+1)\hbar^2\left\langle m_s' m_I' \middle| m_s m_I\right\rangle = s(s+1)\hbar^2\delta_{m_s m_s'}\delta_{m_I m_I'}$$
$$\left\langle m_s' m_I'\right|\mathbf{I}^2\left|m_s m_I\right\rangle = \left\langle m_s' m_I'\right|I(I+1)\hbar^2\left|m_s m_I\right\rangle = I(I+1)\hbar^2\left\langle m_s' m_I' \middle| m_s m_I\right\rangle = I(I+1)\hbar^2\delta_{m_s m_s'}\delta_{m_I m_I'}$$

yielding

$$\mathbf{S}^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

$$\mathbf{I}^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

So each is proportional to the identity matrix.

11.7 The ladder operators are

$$S_+ = S_x + iS_y \quad I_+ = I_x + iI_y$$
$$S_- = S_x - iS_y \quad I_- = I_x - iI_y$$

Solve these for the Cartesian components:

$$S_x = \tfrac{1}{2}\left(S_+ + S_-\right) \quad I_x = \tfrac{1}{2}\left(I_+ + I_-\right)$$
$$S_y = \tfrac{-i}{2}\left(S_+ - S_-\right) \quad I_y = \tfrac{-i}{2}\left(I_+ + I_-\right)$$

and substitute to get

$$\begin{aligned}\mathbf{S}\bullet\mathbf{I} &= S_x I_x + S_y I_y + S_z I_z = \tfrac{1}{2}\left(S_+ + S_-\right)\tfrac{1}{2}\left(I_+ + I_-\right) + \tfrac{-i}{2}\left(S_+ - S_-\right)\tfrac{-i}{2}\left(I_+ - I_-\right) + S_z I_z \\ &= \tfrac{1}{4}\left(S_+ I_+ + S_- I_+ + S_+ I_- + S_- I_-\right) - \tfrac{1}{4}\left(S_+ I_+ - S_- I_+ - S_+ I_- + S_- I_-\right) + S_z I_z \\ &= \tfrac{1}{2}\left(S_- I_+ + S_+ I_-\right) + S_z I_z\end{aligned}$$

11.8 The ladder operators yield zero when acting on the extreme states

$$S_+|++\rangle = S_+|+-\rangle = S_-|-+\rangle = S_-|--\rangle = 0$$
$$I_+|++\rangle = I_+|-+\rangle = I_-|+-\rangle = I_-|--\rangle = 0$$

For the other states, use the ladder operator equation

$$J_\pm\left|j,m_j\right\rangle = \hbar\left[j(j+1)-m_j(m_j\pm 1)\right]^{1/2}\left|j,m_j\pm 1\right\rangle$$

which gives

$$\begin{aligned} S_+|-+\rangle &= \hbar\left[s(s+1)-m_s(m_s+1)\right]^{1/2}|++\rangle = \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}|++\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|++\rangle \\ &= \hbar|++\rangle \end{aligned}$$

The other results are

$$\begin{aligned}
S_+|--\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}|+-\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|+-\rangle = \hbar|+-\rangle \\
S_-|++\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}|-+\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|-+\rangle = \hbar|-+\rangle \\
S_-|+-\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}|--\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|--\rangle = \hbar|--\rangle \\
I_+|+-\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}|++\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|++\rangle = \hbar|++\rangle \\
I_+|--\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(-\tfrac{1}{2}\right)(-\tfrac{1}{2}+1)\right]^{1/2}|-+\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|-+\rangle = \hbar|-+\rangle \\
I_-|++\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}|+-\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|+-\rangle = \hbar|+-\rangle \\
I_-|-+\rangle &= \hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\left(\tfrac{1}{2}\right)(\tfrac{1}{2}-1)\right]^{1/2}|--\rangle = \hbar\left[\tfrac{3}{4}+\tfrac{1}{4}\right]^{1/2}|--\rangle = \hbar|--\rangle
\end{aligned}$$

The matrix elements of the hyperfine Hamiltonian are

$$\begin{aligned} \left\langle m_s' m_I' \middle| H_{hf}' \middle| m_s m_I \right\rangle &= \frac{A}{\hbar^2}\left\langle m_s' m_I' \middle| \mathbf{S}\bullet\mathbf{I} \middle| m_s m_I \right\rangle \\ &= \frac{A}{\hbar^2}\left\langle m_s' m_I' \middle| \left\{\tfrac{1}{2}\left(S_- I_+ + S_+ I_-\right) + S_z I_z\right\} \middle| m_s m_I \right\rangle \end{aligned}$$

The action of $\mathbf{S}\bullet\mathbf{I}$ on the basis states $\left|m_s m_I\right\rangle$ is

$$\begin{aligned}
\mathbf{S}\bullet\mathbf{I}|++\rangle &= \left\{\tfrac{1}{2}\left(S_+I_- + S_-I_+\right) + S_zI_z\right\}|++\rangle = \left\{\tfrac{1}{2}(0+0) + \tfrac{1}{2}\hbar\tfrac{1}{2}\hbar\right\}|++\rangle = \tfrac{1}{4}\hbar^2|++\rangle \\
\mathbf{S}\bullet\mathbf{I}|--\rangle &= \left\{\tfrac{1}{2}(0+0) + \left(\tfrac{-1}{2}\right)\hbar\left(\tfrac{-1}{2}\right)\hbar\right\}|--\rangle = \tfrac{1}{4}\hbar^2|--\rangle \\
\mathbf{S}\bullet\mathbf{I}|+-\rangle &= 0 + \tfrac{1}{2}\hbar\hbar|-+\rangle + \tfrac{1}{2}\hbar\left(\tfrac{-1}{2}\right)\hbar|+-\rangle = \tfrac{1}{4}\hbar^2\left(2|-+\rangle - |+-\rangle\right) \\
\mathbf{S}\bullet\mathbf{I}|-+\rangle &= 0 + \tfrac{1}{2}\hbar\hbar|+-\rangle + \tfrac{1}{2}\hbar\left(\tfrac{-1}{2}\right)\hbar|-+\rangle = \tfrac{1}{4}\hbar^2\left(2|+-\rangle - |-+\rangle\right)
\end{aligned}$$

The resultant Hamiltonian in the uncoupled basis is

$$H'_{hf} \doteq \frac{A}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

11.9 Recall the angular momentum commutation relations

$$\left[J_x, J_y\right] = i\hbar J_z \ ; \ \left[J_y, J_z\right] = i\hbar J_x \ ; \ \left[J_z, J_x\right] = i\hbar J_y$$

$$\left[\mathbf{J}^2, J_x\right] = \left[\mathbf{J}^2, J_y\right] = \left[\mathbf{J}^2, J_z\right] = 0$$

Taking commutators with the hyperfine Hamiltonian gives (recall that the electron and proton spin operators commute)

$$\left[H'_{hf}, \mathbf{S}^2\right] = \left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}, \mathbf{S}^2\right] = \frac{A}{\hbar^2}\mathbf{I}\bullet\left[\mathbf{S}, \mathbf{S}^2\right] = \frac{A}{\hbar^2}\mathbf{I}\bullet\left[S_x\mathbf{i} + S_y\mathbf{j} + S_z\mathbf{k}, \mathbf{S}^2\right] =$$

$$= \frac{A}{\hbar^2}\mathbf{I}\bullet\left\{\mathbf{i}\left[S_x, \mathbf{S}^2\right] + \mathbf{j}\left[S_y, \mathbf{S}^2\right] + \mathbf{k}\left[S_z, \mathbf{S}^2\right]\right\} = 0$$

$$\left[H'_{hf}, \mathbf{I}^2\right] = \left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}, \mathbf{I}^2\right] = \frac{A}{\hbar^2}\mathbf{S}\bullet\left[\mathbf{I}, \mathbf{I}^2\right] = \frac{A}{\hbar^2}\mathbf{S}\bullet\left[I_x\mathbf{i} + I_y\mathbf{j} + I_z\mathbf{k}, \mathbf{I}^2\right] =$$

$$= \frac{A}{\hbar^2}\mathbf{S}\bullet\left\{\mathbf{i}\left[I_x, \mathbf{I}^2\right] + \mathbf{j}\left[I_y, \mathbf{I}^2\right] + \mathbf{k}\left[I_z, \mathbf{I}^2\right]\right\} = 0$$

$$\left[H'_{hf}, S_z\right] = \left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}, S_z\right] = \frac{A}{\hbar^2}\left[S_xI_x + S_yI_y + S_zI_z, S_z\right]$$

$$= \frac{A}{\hbar^2}\left\{I_x\left[S_x, S_z\right] + I_y\left[S_y, S_z\right] + I_z\left[S_z, S_z\right]\right\}$$

$$= \frac{A}{\hbar^2}\left\{-i\hbar S_yI_x + i\hbar S_xI_y\right\} = \frac{-iA}{\hbar}\left\{S_yI_x - S_xI_y\right\} \neq 0$$

$$\left[H'_{hf}, I_z\right] = \left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}, I_z\right] = \frac{A}{\hbar^2}\left[S_xI_x + S_yI_y + S_zI_z, I_z\right]$$

$$= \frac{A}{\hbar^2}\left\{S_x\left[I_x, I_z\right] + S_y\left[I_y, I_z\right] + S_z\left[I_z, I_z\right]\right\}$$

$$= \frac{A}{\hbar^2}\left\{-i\hbar I_yS_x + i\hbar I_xS_y\right\} = \frac{-iA}{\hbar}\left\{I_yS_x - I_xS_y\right\} \neq 0$$

11.10 The matrix for $\mathbf{F}^2$ is

$$\mathbf{F}^2 \doteq \hbar^2 \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

Diagonalize:

$$\begin{vmatrix} 2\hbar^2-\lambda & 0 & 0 & 0 \\ 0 & \hbar^2-\lambda & \hbar^2 & 0 \\ 0 & \hbar^2 & \hbar^2-\lambda & 0 \\ 0 & 0 & 0 & 2\hbar^2-\lambda \end{vmatrix} = 0$$

$$\left(2\hbar^2-\lambda\right)\left(2\hbar^2-\lambda\right)\left\{\left(\hbar^2-\lambda\right)\left(\hbar^2-\lambda\right)-\hbar^4\right\}=0$$

$$\left(2\hbar^2-\lambda\right)^2\left\{\left(\hbar^2-\lambda\right)^2-\hbar^4\right\}=0$$

The solutions are

$$\left(2\hbar^2-\lambda\right)^2=0 \quad \Rightarrow \quad \lambda = 2\hbar^2, 2\hbar^2$$

$$\left(\hbar^2-\lambda\right)^2-\hbar^4=0 \quad \Rightarrow \quad \lambda = 0\hbar^2, 2\hbar^2$$

Find the eigenvectors for $\lambda = 2\hbar^2$

$$\mathbf{F}^2\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$$

$$\hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2\hbar^2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} 2a = 2a \\ b+c = 2b \\ b+c = 2c \\ 2d = 2d \end{array} \quad \Rightarrow \quad b = c$$

This gives three solutions

$$\left|\psi\right\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

For the $\lambda = 0\hbar^2$ state, we get

$$\hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0\hbar^2\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad\Rightarrow\quad \begin{array}{c} 2a = 0a \\ b+c = 0b \\ b+c = 0c \\ 2d = 0d \end{array} \quad\Rightarrow\quad a = d = 0 \text{ , } b = -c$$

$$|\psi\rangle \doteq \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

The $\lambda = 2\hbar^2$ eigenvalues correspond to $F = 1$ and the $\lambda = 0\hbar^2$ eigenvalue corresponds to $F = 0$. The M_F eigenvalues are obtained from $M_F = m_s + m_I$, giving the four states

$$\begin{aligned} |F=1, M_F=1\rangle &= |++\rangle \\ |F=1, M_F=-1\rangle &= |--\rangle \\ |F=1, M_F=0\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |F=0, M_F=0\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \end{aligned}$$

11.11 The coupled basis states are

$$\begin{aligned} |F=1, M_F=1\rangle &= |++\rangle \\ |F=1, M_F=-1\rangle &= |--\rangle \\ |F=1, M_F=0\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |F=0, M_F=0\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \end{aligned}$$

All the uncoupled states have the same quantum numbers $s = 1/2$ and $I = 1/2$, so all the coupled states have these same quantum numbers and hence are eigenstates of $\mathbf{S}^2$ and $\mathbf{I}^2$ with eigenvalues $3\hbar^2/4$. The matrices are thus proportional to the identity matrix (as they are in the uncoupled basis)

$$\mathbf{S}^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{array}{c} 11 \\ 10 \\ 1,-1 \\ 00 \end{array}$$

$$\mathbf{I}^2 \doteq \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{array}{c} 11 \\ 10 \\ 1,-1 \\ 00 \end{array}$$

The matrix for $\mathbf{F}^2$ is obtained from the eigenvalue equation $\mathbf{F}^2 \left| FM_F \right\rangle = F(F+1)\hbar^2 \left| FM_F \right\rangle$.

$$\mathbf{F}^2 \doteq \hbar^2 \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 11 \\ 10 \\ 1,-1 \\ 00 \end{array}$$

The hyperfine perturbation is $H'_{hf} = \left(A/\hbar^2\right)\mathbf{S}\cdot\mathbf{I} = \left(A/2\hbar^2\right)\left(\mathbf{F}^2 - \mathbf{S}^2 - \mathbf{I}^2\right)$, resulting in

$$H'_{hf} = \frac{A}{2\hbar^2}\left\{ \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

$$= \frac{A}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{array}{c} 11 \\ 10 \\ 1,-1 \\ 00 \end{array}$$

11.12 Particle #1 has spin 1 (s_1 = 1) and particle #2 has spin 1/2 (s_2 = 1/2).
a) To find the possible uncoupled basis states $\left| s_1 s_2 m_1 m_2 \right\rangle$ note that:
There are 3 states with s_1 = 1, each with a different z-projection: m_1 = 1,0,-1
There are 2 states with s_2 = 1/2, each with a different z-projection m_2 = 1/2, -1/2.
There are 6 possible states in the uncoupled basis states $\left| s_1 s_2 m_1 m_2 \right\rangle$. These are

$$\boxed{\begin{array}{cc} \left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle \\ \left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle \\ \left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle \end{array}}$$

b) The stretched state with maximal projections is

$$\boxed{\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle}$$

This is the state $\left| s_1 = 1, s_2 = \tfrac{1}{2}, S = \tfrac{3}{2}, M = \tfrac{3}{2} \right\rangle = \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle$ in the coupled basis, so we have

$$\boxed{\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle}$$

c) Operate on this state with the lowering operator $S_- = S_{1-} + S_{2-}$ for the TOTAL spin to generate the next eigenstate of the total spin and total z-projection. To get the factors right, we must do the operation in both bases.

$$S_-\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \left(S_{1-}+S_{2-}\right)\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle = S_{1-}\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle + S_{2-}\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle$$

$$\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\hbar\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \sqrt{1(1+1)-1(1-1)}\hbar\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle$$

$$\sqrt{\tfrac{12}{4}}\hbar\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \sqrt{2}\hbar\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle + \hbar\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle$$

$$\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle$$

This is the second column in Table 11.3. Now apply the lowering operator to $\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$

$$S_-\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle = \left(S_{1-}+S_{2-}\right)\left(\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle\right)$$

$$\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \sqrt{1(1+1)-0(0-1)}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle$$
$$+\sqrt{1(1+1)-1(1-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle$$

$$\sqrt{\tfrac{16}{4}}\hbar\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \sqrt{2}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle + 2\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle$$

$$\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle$$

This is the third column in Table 11.3. Now apply the lowering operator to $\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle$

$$S_-\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \left(S_{1-}+S_{2-}\right)\left(\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle\right)$$

$$\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{-1}{2}(\tfrac{-1}{2}-1)}\hbar\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle = \sqrt{1(1+1)-0(0-1)}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle$$

$$\sqrt{\tfrac{12}{4}}\hbar\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \sqrt{2}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle + \hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle$$

$$\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle = \left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle$$

This is the fourth column in Table 11.3. This is just the opposite stretched state, which is an eigenstate in both bases. Now use orthogonality to generate $\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$ from $\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$.

$$0 = \left\langle\tfrac{3}{2}\tfrac{1}{2}\middle|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$
$$0 = \left(\sqrt{\tfrac{2}{3}}\left\langle 1\tfrac{1}{2}0\tfrac{1}{2}\right| + \sqrt{\tfrac{1}{3}}\left\langle 1\tfrac{1}{2}1\tfrac{-1}{2}\right|\right)\left(a\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle + b\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle\right)$$
$$0 = \sqrt{\tfrac{2}{3}}a + \sqrt{\tfrac{1}{3}}b$$
$$b = -a\sqrt{2}$$

a and b are both real and $\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$ is normalized, so (with standard choice of convention)

$$\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle$$

This is the fifth column in the table in the text. Now apply the lowering operator to $\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$

$$S_-\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left(S_{1-}+S_{2-}\right)\left(\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle\right)$$

$$\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{1(1+1)-1(1-1)}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{1(1+1)-0(0-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$
$$-\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle$$
$$\hbar\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

This is the sixth column in Table 11.3.

d) Using the results from (c), we get the Clebsch-Gordan table:

$j_1=1$		j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$j_2=\frac{1}{2}$	m	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
m1	m2							
1	$\frac{1}{2}$		1	0	0	0	0	0
1	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$\sqrt{\frac{2}{3}}$	0
0	$\frac{1}{2}$		0	$\sqrt{\frac{2}{3}}$	0	0	$-\frac{1}{\sqrt{3}}$	0
0	$-\frac{1}{2}$		0	0	$\sqrt{\frac{2}{3}}$	0	0	$\frac{1}{\sqrt{3}}$
−1	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$-\sqrt{\frac{2}{3}}$
−1	$-\frac{1}{2}$		0	0	0	1	0	0

11.13 In this system, particle #1 has spin 1 (j_1 = 1) and particle #2 has spin 1/2 (j_2 = 1/2). The possible uncoupled basis states $\left|j_1 j_2 m_1 m_2\right\rangle$ are:

There are 3 states with j_1 = 1, each with a different *z*-projection: m_1 = 1, 0, -1

There are 2 states with j_2 = 1/2, each with a different *z*-projection m_2 = 1/2, -1/2.

There are 6 possible states in the uncoupled basis states $\left|j_1 j_2 m_1 m_2\right\rangle$. These are

$$\boxed{\begin{matrix} \left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle \\ \left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle \\ \left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle & \left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle \end{matrix}}$$

The stretched state with maximal projections is

$$\boxed{\left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle}$$

This is the state $\left|j_1=1, j_2=\tfrac{1}{2}, J=\tfrac{3}{2}, M=\tfrac{3}{2}\right\rangle = \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle$ in the coupled basis, so we have

$$\boxed{\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle = \left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle},$$

which is the first column in Table 11.3. Now operate on this state with the lowering operator $J_- = J_{1-} + J_{2-}$ for the total angular momentum to generate the next eigenstate of the total angular momentum and total z-projection. To get the factors right, we must do the operation in both bases.

$$
\begin{aligned}
J_-\left|\tfrac32\,\tfrac32\right\rangle &= \left(J_{1-}+J_{2-}\right)\left|1\,\tfrac12\,1\,\tfrac12\right\rangle = J_{1-}\left|1\,\tfrac12\,1\,\tfrac12\right\rangle + J_{2-}\left|1\,\tfrac12\,1\,\tfrac12\right\rangle \\
\sqrt{\tfrac32(\tfrac32+1)-\tfrac32(\tfrac32-1)}\,\hbar\left|\tfrac32\,\tfrac12\right\rangle &= \sqrt{1(1+1)-1(1-1)}\,\hbar\left|1\,\tfrac12\,0\,\tfrac12\right\rangle + \sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,\hbar\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle \\
\sqrt{\tfrac{12}{4}}\,\hbar\left|\tfrac32\,\tfrac12\right\rangle &= \sqrt2\,\hbar\left|1\,\tfrac12\,0\,\tfrac12\right\rangle + \hbar\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle \\
\left|\tfrac32\,\tfrac12\right\rangle &= \sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac12\right\rangle + \sqrt{\tfrac13}\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle
\end{aligned}
$$

This is the second column in Table 11.3. Now apply the lowering operator to $\left|\tfrac32\,\tfrac12\right\rangle$

$$
\begin{aligned}
J_-\left|\tfrac32\,\tfrac12\right\rangle &= \left(J_{1-}+J_{2-}\right)\left(\sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac12\right\rangle + \sqrt{\tfrac13}\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle\right) \\
\sqrt{\tfrac32(\tfrac32+1)-\tfrac12(\tfrac12-1)}\,\hbar\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \sqrt{1(1+1)-0(0-1)}\,\hbar\sqrt{\tfrac23}\left|1\,\tfrac12,-1\,\tfrac12\right\rangle + \sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,\hbar\sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac{-1}{2}\right\rangle \\
&\quad + \sqrt{1(1+1)-1(1-1)}\,\hbar\sqrt{\tfrac13}\left|1\,\tfrac12\,0\,\tfrac{-1}{2}\right\rangle \\
\sqrt{\tfrac{16}{4}}\,\hbar\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \sqrt2\,\hbar\sqrt{\tfrac23}\left|1\,\tfrac12,-1\,\tfrac12\right\rangle + 2\hbar\sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac{-1}{2}\right\rangle \\
\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \sqrt{\tfrac13}\left|1\,\tfrac12,-1\,\tfrac12\right\rangle + \sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac{-1}{2}\right\rangle
\end{aligned}
$$

This is the third column in Table 11.3. Now apply the lowering operator to $\left|\tfrac32\,\tfrac{-1}{2}\right\rangle$

$$
\begin{aligned}
J_-\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \left(J_{1-}+J_{2-}\right)\left(\sqrt{\tfrac13}\left|1\,\tfrac12,-1\,\tfrac12\right\rangle + \sqrt{\tfrac23}\left|1\,\tfrac12\,0\,\tfrac{-1}{2}\right\rangle\right) \\
\sqrt{\tfrac32(\tfrac32+1)-\tfrac{-1}{2}(\tfrac{-1}{2}-1)}\,\hbar\left|\tfrac32\,\tfrac{-3}{2}\right\rangle &= \sqrt{1(1+1)-0(0-1)}\,\hbar\sqrt{\tfrac23}\left|1\,\tfrac12,-1\,\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,\hbar\sqrt{\tfrac13}\left|1\,\tfrac12,-1\,\tfrac{-1}{2}\right\rangle \\
\sqrt{\tfrac{12}{4}}\,\hbar\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \sqrt2\,\hbar\sqrt{\tfrac23}\left|1\,\tfrac12,-1\,\tfrac{-1}{2}\right\rangle + \hbar\sqrt{\tfrac13}\left|1\,\tfrac12,-1\,\tfrac{-1}{2}\right\rangle \\
\left|\tfrac32\,\tfrac{-1}{2}\right\rangle &= \left|1\,\tfrac12,-1\,\tfrac{-1}{2}\right\rangle
\end{aligned}
$$

This is the fourth column in Table 11.3. This is just the opposite stretched state, which is an eigenstate in both bases. Now use orthogonality to generate $\left|\tfrac12\,\tfrac12\right\rangle$ from $\left|\tfrac32\,\tfrac12\right\rangle$.

$$
\begin{aligned}
0 &= \left\langle\tfrac32\,\tfrac12\middle|\tfrac12\,\tfrac12\right\rangle \\
0 &= \left(\sqrt{\tfrac23}\left\langle1\,\tfrac12\,0\,\tfrac12\right| + \sqrt{\tfrac13}\left\langle1\,\tfrac12\,1\,\tfrac{-1}{2}\right|\right)\left(a\left|1\,\tfrac12\,0\,\tfrac12\right\rangle + b\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle\right) \\
0 &= \sqrt{\tfrac23}a + \sqrt{\tfrac13}b \\
b &= -a\sqrt2
\end{aligned}
$$

a and b are both real and $\left|\tfrac12\,\tfrac12\right\rangle$ is normalized, so (with standard choice of convention)

$$\left|\tfrac12\,\tfrac12\right\rangle = \sqrt{\tfrac23}\left|1\,\tfrac12\,1\,\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac13}\left|1\,\tfrac12\,0\,\tfrac12\right\rangle$$

This is the fifth column in the table in the text. Now apply the lowering operator to $\left|\tfrac12\,\tfrac12\right\rangle$

$$J_-\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left(J_{1-}+J_{2-}\right)\left(\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle\right)$$

$$\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{1(1+1)-1(1-1)}\hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{1(1+1)-0(0-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

$$-\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle$$

$$\hbar\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \hbar\sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \hbar\sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle$$

This is the sixth column in Table 11.3. Using these results, we get the Clebsch-Gordan table:

$j_1=1$		j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$j_2=\frac{1}{2}$	m	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
m1	m2							
1	$\frac{1}{2}$		1	0	0	0	0	0
1	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$\sqrt{\frac{2}{3}}$	0
0	$\frac{1}{2}$		0	$\sqrt{\frac{2}{3}}$	0	0	$-\frac{1}{\sqrt{3}}$	0
0	$-\frac{1}{2}$		0	0	$\sqrt{\frac{2}{3}}$	0	0	$\frac{1}{\sqrt{3}}$
−1	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$-\sqrt{\frac{2}{3}}$
−1	$-\frac{1}{2}$		0	0	0	1	0	0

11.14 For a system with given angular momenta j_1 and j_2, the allowed z-components are

$$m_1 = -j_1, -j_1+1, \ldots, j_1-1, j_1$$
$$m_2 = -j_2, -j_2+1, \ldots, j_2-1, j_2$$

There are $(2j_1+1)$ m_1 states ($|j_1m_1\rangle$) and $(2j_2+1)$ m_2 states ($|j_2m_2\rangle$), so the number of uncoupled basis states $|j_1j_2m_1m_2\rangle = |j_1m_1\rangle|j_2m_2\rangle$ is the product $N=(2j_1+1)(2j_2+1)$. In the coupled basis, the allowed quantum numbers are

$$J = j_1+j_2,\ j_1+j_2-1, \ldots, |j_1-j_2|$$
$$M = -J, -J+1, \ldots, J-1, J$$

To find the number of coupled basis states $|JM\rangle$, we sum over all possible states:

$$N = \sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{-J}^{J}1 = \sum_{J=|j_1-j_2|}^{j_1+j_2}(2J+1)$$

Get rid of the absolute value sign by arranging j_1 and j_2 such that $j_1 > j_2$:

$$N = 2\sum_{J=j_1-j_2}^{j_1+j_2} J + \sum_{J=j_1-j_2}^{j_1+j_2} 1$$

and use

$$\sum_{i=1}^{n} i = \frac{i(i+1)}{2}$$

to get

$$\begin{aligned}
N &= 2\sum_{J=j_1-j_2}^{j_1+j_2} J + \sum_{J=j_1-j_2}^{j_1+j_2} 1 = 2\left(\sum_{J=1}^{j_1+j_2} J - \sum_{J=1}^{j_1-j_2-1} J\right) + \left(\sum_{J=1}^{j_1+j_2} 1 - \sum_{J=1}^{j_1-j_2-1} 1\right) \\
&= 2\left(\frac{(j_1+j_2)(j_1+j_2+1)}{2} - \frac{(j_1-j_2-1)(j_1-j_2)}{2}\right) + (j_1+j_2) - (j_1-j_2-1) \\
&= (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2) + (j_1+j_2) - (j_1-j_2-1) \\
&= 4j_1j_2 + 2(j_1+j_2) + (j_1-j_2) - (j_1-j_2-1) = 4j_1j_2 + 2j_1 + 2j_2 + 1 \\
&= (2j_1+1)(2j_2+1)
\end{aligned}$$

11.15 Particle #1 has angular momentum 1 (j_1 = 1) and particle #2 has angular momentum 1/2 (j_2 = 1/2).

a) The possible uncoupled basis states $|j_1 j_2 m_1 m_2\rangle$ are:

There are 3 states with j_1 = 1, each with a different *z*-projection: m_1 = 1,0,-1

There are 2 states with j_2 = 1/2, each with a different *z*-projection m_2 = 1/2, -1/2.

There are 6 possible states in the uncoupled basis states $|j_1 j_2 m_1 m_2\rangle$. These are

$$\boxed{\begin{array}{cc} |1\tfrac{1}{2}1\tfrac{1}{2}\rangle & |1\tfrac{1}{2}1\tfrac{-1}{2}\rangle \\ |1\tfrac{1}{2}0\tfrac{1}{2}\rangle & |1\tfrac{1}{2}0\tfrac{-1}{2}\rangle \\ |1\tfrac{1}{2},-1\tfrac{1}{2}\rangle & |1\tfrac{1}{2},-1\tfrac{-1}{2}\rangle \end{array}}$$

b) For any angular momentum addition, the possible values are $J = j_1 + j_2\ ,\ j_1 + j_2 - 1\ ,\ j_1 + j_2 - 2\ , \ldots |j_1 - j_2|$. In this case, we get

$$\boxed{J = \tfrac{3}{2}\ ,\ \tfrac{1}{2}}$$

The allowed values of *M* are always –*J* to *J*, giving

$$\boxed{\begin{array}{ll} J = \tfrac{3}{2}: & M = \tfrac{3}{2}\ ,\ \tfrac{1}{2}\ ,\ \tfrac{-1}{2}\ ,\ \tfrac{-3}{2} \\ J = \tfrac{1}{2}: & M = \tfrac{1}{2}\ ,\ \tfrac{-1}{2} \end{array}}$$

c) The coupled basis states are

$$\boxed{\begin{array}{llll} \left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle & \left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle \\ \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle & \left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle & & \end{array}}$$

d) Using the Clebsch-Gordan Table 11.3 gives

$$\begin{aligned}
\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle &= \left|1\tfrac{1}{2}1\tfrac{1}{2}\right\rangle \\
\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle \\
\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle &= \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}0\tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle \\
\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle &= \left|1\tfrac{1}{2},-1\tfrac{-1}{2}\right\rangle \\
\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2}1\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2}0\tfrac{1}{2}\right\rangle \\
\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle
\end{aligned}$$

11.16 The Hamiltonian from Eq. (11.9) is $H'_{hf} = A\mathbf{I}\bullet\mathbf{S}/\hbar^2$. The spin of the deuteron is $I=1$, so the allowed values of the coupled angular momentum $\mathbf{F}=\mathbf{S}+\mathbf{I}$ with $s=1/2$ for the electron are $F=\tfrac{1}{2},\tfrac{3}{2}$. Do the calculation in the coupled basis, like we did in Eq. (11.61). The hyperfine Hamiltonian from Eq. (11.60) is

$$H'_{hf} = \frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I} = \frac{A}{2\hbar^2}\left(\mathbf{F}^2 - \mathbf{S}^2 - \mathbf{I}^2\right)$$

The matrix elements are

$$\begin{aligned}
\left\langle F'M'_F\right|H'_{hf}\left|FM_F\right\rangle &= \frac{A}{2\hbar^2}\left\langle F'M'_F\right|\vec{\mathbf{F}}^2 - \vec{\mathbf{J}}^2 - \vec{\mathbf{I}}^2\left|FM_F\right\rangle \\
&= \frac{A}{2\hbar^2}\left[F(F+1) - s(s+1) - I(I+1)\right]\hbar^2\left\langle F'M'_F\middle|FM_F\right\rangle \\
&= \frac{A}{2}\left[F(F+1) - s(s+1) - I(I+1)\right]\delta_{F'F}\delta_{M'_F M_F}
\end{aligned}$$

For deuterium, $I=1$, $s=1/2$, and $F=\tfrac{1}{2},\tfrac{3}{2}$, giving the diagonal elements

$$\left\langle FM_F\right|H'_{hf}\left|FM_F\right\rangle = \begin{cases} \tfrac{1}{2}A; & F=\dfrac{3}{2} \\ -A; & F=\dfrac{1}{2} \end{cases}$$

and the matrix

$$H'_{hf} \doteq \begin{array}{c} \begin{pmatrix} A/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & A/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & A/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A & 0 \\ 0 & 0 & 0 & 0 & 0 & -A \end{pmatrix} \end{array} \begin{array}{c} \frac{3}{2},\frac{3}{2} \\ \frac{3}{2},\frac{1}{2} \\ \frac{3}{2},\frac{-1}{2} \\ \frac{3}{2},\frac{-3}{2} \\ \frac{1}{2},\frac{1}{2} \\ \frac{1}{2},\frac{-1}{2} \end{array}$$

The two hyperfine levels are split by the amount

$$\Delta E = \frac{3}{2}A$$

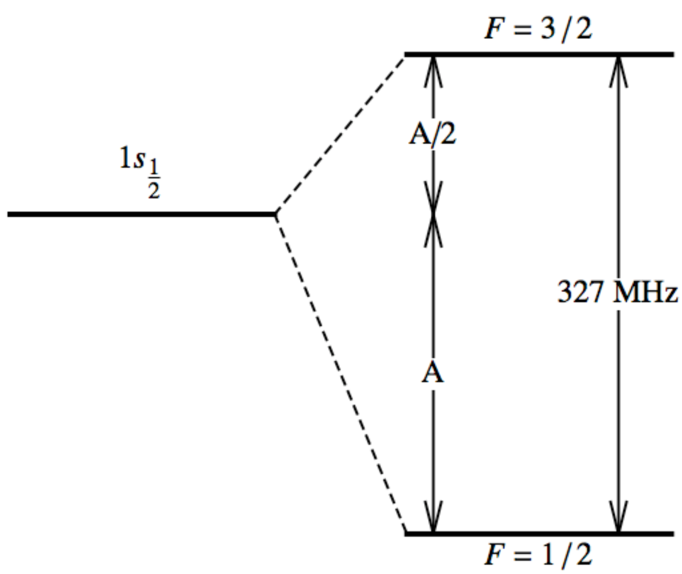
For deuterium $g_D = 0.857$, compared to $g_p = 5.59$. Hence, for deuterium

$$A_D = \frac{g_D}{g_p}A_H = \frac{0.857}{5.59}1420.4\ \ MHz = 217.8\ \ MHz$$

and the splitting is

$$\boxed{\Delta E = \frac{3}{2}A_D = 326.6\ \ MHz}$$

The ground state hyperfine structure of deuterium is shown below.



11.17 a) The hydrogen energy levels are given by $E_n = -\alpha^2 mc^2 / 2n^2$, where the m is really the reduced mass. For positronium, the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}$$

is approximately half of what it is in hydrogen. Hence the energy levels in positronium are half of what they are in hydrogen. Hence the positronium ground state energy is

$$E_1 = -\frac{13.6eV}{2} = -6.8eV$$

b) The uncoupled basis states are $\left| m_{s_1} m_{s_2} \right\rangle$

$$|++\rangle\,,\ |+-\rangle\,,\ |-+\rangle\,,\ |--\rangle$$

The coupled states are $|SM\rangle$

$$\left.\begin{aligned} |11\rangle &= |++\rangle \\ |10\rangle &= \frac{1}{\sqrt{2}}\big[|+-\rangle + |-+\rangle\big] \\ |1,-1\rangle &= |--\rangle \end{aligned}\right\} \textit{Triplet state}$$

$$\left.|00\rangle = \frac{1}{\sqrt{2}}\big[|+-\rangle - |-+\rangle\big]\right\} \textit{Singlet state}$$

c) The hyperfine Hamiltonian can be written as

$$\begin{aligned} \vec{\mathbf{S}} &= \vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_2 \\ \vec{\mathbf{S}}^2 &= \left(\vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_2\right)^2 = \vec{\mathbf{S}}_1^2 + \vec{\mathbf{S}}_2^2 + \vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 + \vec{\mathbf{S}}_2 \bullet \vec{\mathbf{S}}_1 = \vec{\mathbf{S}}_1^2 + \vec{\mathbf{S}}_2^2 + 2\vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 \\ \vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 &= \tfrac{1}{2}\left(\vec{\mathbf{S}}^2 - \vec{\mathbf{S}}_1^2 - \vec{\mathbf{S}}_2^2\right) = \tfrac{1}{2}\hbar^2\left(S(S+1) - \tfrac{3}{4}\right) \\ H' &= A\vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 = \tfrac{1}{2}A\hbar^2\left(S(S+1) - \tfrac{3}{4}\right) \end{aligned}$$

The matrix in the coupled basis is

$$H' = A\vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 = \tfrac{1}{2}A\hbar^2\left(S(S+1) - \tfrac{3}{2}\right)$$

$$H' \doteq A\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}\begin{array}{l} 11 \\ 10 \\ 1,-1 \\ 00 \end{array}$$

The matrix in the uncoupled basis is

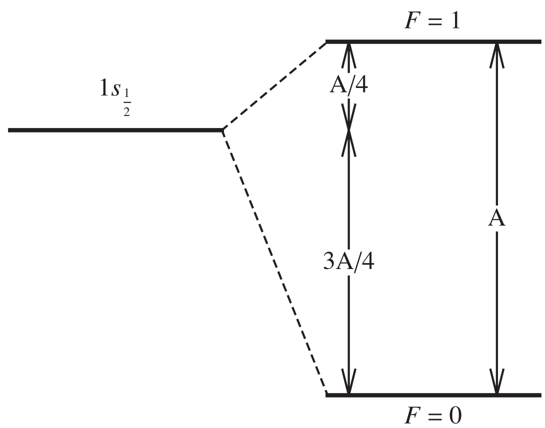
$$\vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} = \tfrac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right) + S_{1z}S_{2z}$$

$$H' = A\left[\tfrac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right) + S_{1z}S_{2z}\right] \doteq A\frac{\hbar^2}{4}\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}\begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

d) Since the matrix is already diagonal in the coupled basis, we can use that to get

$$H' = A\vec{\mathbf{S}}_1 \bullet \vec{\mathbf{S}}_2 = \tfrac{1}{2}A\hbar^2\left(S(S+1) - \tfrac{3}{2}\right)$$

$$\langle S'M'|H'|SM\rangle = \tfrac{1}{2}A\hbar^2\left(S(S+1) - \tfrac{3}{2}\right)\delta_{SS'}\delta_{MM'} = \begin{cases} -\tfrac{3}{4}A\hbar^2 & S = 0 \\ \tfrac{1}{4}A\hbar^2 & S = 1 \end{cases}$$



11.18 We have two particles, each with orbital angular momentum 1 ($\ell_1 = 1; \ell_2 = 1$) and spin angular momentum 1/2 ($s_1 = \tfrac{1}{2}; s_2 = \tfrac{1}{2}$). For any angular momentum addition, the possible values are $J = j_1 + j_2$, $j_1 + j_2 - 1$, $j_1 + j_2 - 2$, … $|j_1 - j_2|$.

a) $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$: Possible values are $L = 2,1,0$

b) $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$: Possible values are $S = 1,0$

c) Now add the total orbital angular momentum to the total spin angular momentum to get the total angular momentum: $\mathbf{J} = \mathbf{L} + \mathbf{S}$. The possible values are

		J		
$L = 2; S = 1$	3	2	1	
$L = 1; S = 1$		2	1	0
$L = 0; S = 1$			1	
$L = 2; S = 0$		2		
$L = 1; S = 0$			1	
$L = 0; S = 0$				0

This scheme is known as Russell-Saunders coupling in which we first add together all the individual orbital angular momenta to form a total orbital angular momentum, and all the individual orbital angular momenta to form a total spin angular momentum:

d) In the alternate scheme of J-J coupling, we add individual angular momenta together first, then add all the totals. For each particle we add the orbital and spin angular momentum together:

$$\mathbf{J}_1 = \mathbf{L}_1 + \mathbf{S}_1 \text{ with } j_1 = 1 + \tfrac{1}{2} = \tfrac{3}{2} \text{ or } j_1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$\mathbf{J}_2 = \mathbf{L}_2 + \mathbf{S}_2 \text{ with } j_2 = 1 + \tfrac{1}{2} = \tfrac{3}{2} \text{ or } j_2 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

e) Now we add the total individual angular momentum for each electron to form the total angular momentum: $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ with several possibilities for J (each of which has $2J+1$ possible substates).

	J			
$j_1 = \tfrac{3}{2}; j_2 = \tfrac{3}{2}$	3	2	1	0
$j_1 = \tfrac{1}{2}; j_2 = \tfrac{3}{2}$		2	1	
$j_1 = \tfrac{3}{2}; j_2 = \tfrac{1}{2}$		2	1	
$j_1 = \tfrac{1}{2}; j_2 = \tfrac{1}{2}$			1	0

The possible values of J are the same as in (c). Also note that the total number of states is the same in each scheme, as it must be.

11.19 The angular momentum operators in the position representation are (orbital only, see Problem 7.8)

$$L_x \doteq i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_y \doteq i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

so the ladder operators are

$$L_+ = L_x + iL_y \doteq +\hbar\left(e^{i\phi}\frac{\partial}{\partial\theta} + ie^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_- = L_x - iL_y \doteq -\hbar\left(e^{-i\phi}\frac{\partial}{\partial\theta} - ie^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}\right)$$

Now act with the raising operator:

$$
\begin{aligned}
Y_2^0(\theta,\phi) &= \sqrt{\tfrac{5}{16\pi}}\left(3\cos^2\theta-1\right) \\
L_+Y_2^0(\theta,\phi) &= +\hbar\left(e^{i\phi}\frac{\partial}{\partial\theta}+ie^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\right)\left(\sqrt{\tfrac{5}{16\pi}}\left(3\cos^2\theta-1\right)\right) \\
&= -\hbar e^{i\phi}\sqrt{\tfrac{5}{16\pi}}6\cos\theta\sin\theta = \hbar\sqrt{6}\left(-\sqrt{\tfrac{15}{8\pi}}\cos\theta\sin\theta e^{i\phi}\right)
\end{aligned}
$$

Noting that

$$
Y_2^1(\theta,\phi) = -\sqrt{\tfrac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}
$$

we have:

$$
L_+Y_2^0(\theta,\phi) = \hbar\sqrt{6}\left(Y_2^1(\theta,\phi)\right)
$$

This is what we expect from the raising operation:

$$
\begin{aligned}
J_\pm\left|jm_j\right\rangle &= \hbar\sqrt{\left(j(j+1)-m_j\left(m_j\pm1\right)\right)}\left|j,m_j\pm1\right\rangle \\
L_+\left|20\right\rangle &= \hbar\sqrt{6}\left|21\right\rangle
\end{aligned}
$$

11.20 Particle #1 has spin 1 (s_1 = 1) and particle #2 has spin 1/2 (s_2 = 1/2). Use the Clebsch-Gordan coefficients in Table 11.3 to find the coupled states in terms of the uncoupled states. The state with total spin ½ and *z*-component $-\hbar/2$ is in the sixth column in the table in the text:

$$
\left|\psi\right\rangle = \left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1\tfrac{1}{2},0\tfrac{-1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1\tfrac{1}{2},-1\tfrac{1}{2}\right\rangle
$$

The possible measurements of the *z*-component of the spin of particle 1 are $1\hbar,\ 0\hbar,\ -1\hbar$, which correspond to $m_1 = +1,\ 0,\ -1$. To find the probability of any one result we must sum over all the possible results of the spin component of particle 2, which are of $\hbar/2,\ -\hbar/2$ (corresponding to $m_2 = +1/2,\ -1/2$). Thus we get

$$\mathcal{P}_{m_1=+1} = \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=+1,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\left\langle s_1 = 1, s_2 = \tfrac{1}{2,} m_1 = +1, m_2 \middle| \psi \right\rangle\right|^2$$

$$= \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2} 1 m_2 \middle| \psi \right\rangle\right|^2 = \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2} 1 m_2 \middle| \left(\sqrt{\tfrac{1}{3}} \left| 1 \tfrac{1}{2} 0 \tfrac{-1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| 1 \tfrac{1}{2}, -1 \tfrac{1}{2} \right\rangle \right) \right\rangle\right|^2 = 0$$

$$\mathcal{P}_{m_1=0} = \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=0,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2} 0 m_2 \middle| \psi \right\rangle\right|^2$$

$$= \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2} 0 m_2 \middle| \left(\sqrt{\tfrac{1}{3}} \left| 1 \tfrac{1}{2} 0 \tfrac{-1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| 1 \tfrac{1}{2}, -1 \tfrac{1}{2} \right\rangle \right) \right\rangle\right|^2 = \tfrac{1}{3}$$

$$\mathcal{P}_{m_1=-1} = \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=-1,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2}, -1 m_2 \middle| \psi \right\rangle\right|^2$$

$$= \sum_{m_2=-1/2}^{1/2} \left|\left\langle 1 \tfrac{1}{2}, -1 m_2 \middle| \left(\sqrt{\tfrac{1}{3}} \left| 1 \tfrac{1}{2} 0 \tfrac{-1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| 1 \tfrac{1}{2}, -1 \tfrac{1}{2} \right\rangle \right) \right\rangle\right|^2 = \tfrac{2}{3}$$

The three probabilities add to unity, as they must. The probabilities of the measurement of the spin component of particle 2 are

$$\mathcal{P}_{m_2=+1/2} = \sum_{m_1=-1}^{1} \mathcal{P}_{m_1,m_2=+1/2} = \sum_{m_1=-1}^{1} \left|\left\langle s_1 = 1, s_2 = \tfrac{1}{2,} m_1, m_2 = +\tfrac{1}{2,} \middle| \psi \right\rangle\right|^2$$

$$= \sum_{m_1=-1}^{1} \left|\left\langle 1 \tfrac{1}{2} m_1 \tfrac{1}{2} \middle| \psi \right\rangle\right|^2 = \sum_{m_1=-1}^{1} \left|\left\langle 1 \tfrac{1}{2} m_1 \tfrac{1}{2} \middle| \left(\sqrt{\tfrac{1}{3}} \left| 1 \tfrac{1}{2} 0 \tfrac{-1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| 1 \tfrac{1}{2}, -1 \tfrac{1}{2} \right\rangle \right) \right\rangle\right|^2 = \tfrac{2}{3}$$

$$\mathcal{P}_{m_2=-1/2} = \sum_{m_1=-1}^{1} \mathcal{P}_{m_1,m_2=-1/2} = \sum_{m_1=-1}^{1} \left|\left\langle 1 \tfrac{1}{2} m_1 \tfrac{-1}{2} \middle| \psi \right\rangle\right|^2$$

$$= \sum_{m_1=-1}^{1} \left|\left\langle 1 \tfrac{1}{2} m_1 \tfrac{-1}{2} \middle| \left(\sqrt{\tfrac{1}{3}} \left| 1 \tfrac{1}{2} 0 \tfrac{-1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| 1 \tfrac{1}{2}, -1 \tfrac{1}{2} \right\rangle \right) \right\rangle\right|^2 = \tfrac{1}{3}$$

Again, the three probabilities add to unity, as they must.

11.21 Particle #1 has spin 1/2 (s_1 = 1/2), particle #2 has spin 1/2 (s_2 = 1/2), and particle #3 has spin 1/2 (s_3 = 1/2).

a) To find the possible uncoupled basis states $\left| s_1 s_2 s_3 m_1 m_2 m_3 \right\rangle$ note that:
There are 2 states with s_1 = 1/2, each with a different z-projection m_1 = 1/2, -1/2.
There are 2 states with s_2 = 1/2, each with a different z-projection m_2 = 1/2, -1/2.
There are 2 states with s_3 = 1/2, each with a different z-projection m_3 = 1/2, -1/2.
There are 8 possible states (2^3) in the uncoupled basis states $\left| s_1 s_2 s_3 m_1 m_2 m_3 \right\rangle$. These are

$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{-1}{2} \right\rangle$
$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{1}{2} \tfrac{1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{-1}{2} \tfrac{1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{1}{2} \tfrac{-1}{2} \right\rangle$	$\left\| \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2} \tfrac{-1}{2} \tfrac{-1}{2} \tfrac{-1}{2} \right\rangle$

b) The stretched state with maximal projections is

$$\boxed{\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle}$$

This is the state $\left|s_1=\tfrac{1}{2},s_2=\tfrac{1}{2},s_3=\tfrac{1}{2},S=\tfrac{3}{2},M=\tfrac{3}{2}\right\rangle=\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle$ in the coupled basis, so we have

$$\boxed{\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle=\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle}$$

Operate on this state with the lowering operator $S_-=S_{1-}+S_{2-}+S_{3-}$ for the TOTAL spin to generate the next eigenstate of the total spin and total z-projection. To get the factors right, we must do the operation in both bases.

$$\begin{aligned}
S_-\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle &= \left(S_{1-}+S_{2-}+S_{3-}\right)\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle\\
&= S_{1-}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+S_{2-}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+S_{3-}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle\\
\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\hbar\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle\\
\sqrt{\tfrac{12}{4}}\hbar\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle\\
\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle
\end{aligned}$$

To get the next state, operate with the lowering operator again:

$$\begin{aligned}
S_-\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \left(S_{1-}+S_{2-}+S_{3-}\right)\left(\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle\right)\\
\sqrt{\tfrac{16}{4}}\hbar\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left(2\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+2\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle+2\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle\right)\\
\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle+\sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle
\end{aligned}$$

To get the next state, operate with the lowering operator again:

$$\begin{aligned}
S_-\left|\tfrac{3}{2}\tfrac{-1}{2}\right\rangle &= \left(S_{1-}+S_{2-}+S_{3-}\right)\sqrt{\tfrac{1}{3}}\left(\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle+\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle\right)\\
\sqrt{\tfrac{12}{4}}\hbar\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left(\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle\right)\\
\left|\tfrac{3}{2}\tfrac{-3}{2}\right\rangle &= \left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle
\end{aligned}$$

This is the opposite stretched state, as expected. Now we must use the orthogonalization step to find states with the next lower value of the total spin *S*. Thus we expect that the coupled state $\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$ is orthogonal to and is composed of the same uncoupled states as coupled state $\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$. This gives

$$\begin{aligned}
0 &= \left\langle\tfrac{3}{2}\tfrac{1}{2}\middle|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\\
&= \sqrt{\tfrac{1}{3}}\left(\left\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right|+\left\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right|+\left\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right|\right)\left(a\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\tfrac{1}{2}\right\rangle+b\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{1}{2}\right\rangle+c\left|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{-1}{2}\right\rangle\right)\\
&= \sqrt{\tfrac{1}{3}}\left(a+b+c\right)
\end{aligned}$$

But there are three unknowns and only two equations (including the normalization equation). This ambiguity is also reflected in the fact that we need to find 8 coupled basis states, but the procedure we are following would only produce 2 more ($\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$ and $\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$)

to make 6 states. To solve the ambiguity and produce 2 more states we need to find 2 states that are orthogonal to $\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle$ and then act with the lowering operator to generate the other 2 states. There are different ways to do this and the solution is not unique (it is like specifying the z-axis and then asking where the x- and y-axes are). We will use Gram-Schmidt orthogonalization, which says you can make an orthonormal basis by choosing arbitrary "trial" states and then making them orthogonal to all previously determined states by subtracting off the projections along the previous basis states. In our case, we already have one basis state comprising the three uncoupled states, so we try a trial state for the second one

$$\left|\psi_{trial}\right\rangle = \left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle$$

and then subtract off the projection along the previous basis state

$$\begin{aligned}
\left|\psi_{new}\right\rangle &= \left|\psi_{trial}\right\rangle - \left|\psi_1\right\rangle\left\langle\psi_1\middle|\psi_{trial}\right\rangle \\
&= \left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \sqrt{\tfrac13}\left(\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right)\bullet \\
&\qquad \bullet\sqrt{\tfrac13}\left(\left\langle\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right| + \left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right| + \left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right|\right)\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle \\
&= \left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \tfrac13\left(\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right) \\
&= \tfrac23\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \tfrac13\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac13\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle
\end{aligned}$$

But that's not normalized. The new normalized state is

$$\left|\psi_2\right\rangle = \left|\tfrac12\tfrac12\right\rangle_a = \tfrac{2}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \tfrac{1}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac{1}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle$$

This would be different if we had chosen a different trial state. Now find the third state by choosing a new trial state and subtracting off the projections along the previous two basis states:

$$\begin{aligned}
\left|\psi_{trial}\right\rangle &= \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle \\
\left|\psi_{new}\right\rangle &= \left|\psi_{trial}\right\rangle - \left|\psi_1\right\rangle\left\langle\psi_1\middle|\psi_{trial}\right\rangle - \left|\psi_2\right\rangle\left\langle\psi_2\middle|\psi_{trial}\right\rangle \\
&= \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \sqrt{\tfrac13}\left(\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right)\bullet \\
&\qquad \bullet\sqrt{\tfrac13}\left(\left\langle\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right| + \left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right| + \left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right|\right)\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle \\
&\qquad -\left(\tfrac{2}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \tfrac{1}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac{1}{\sqrt6}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right)\bullet \\
&\qquad \bullet\left(\tfrac{2}{\sqrt6}\left\langle\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right| - \tfrac{1}{\sqrt6}\left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right| - \tfrac{1}{\sqrt6}\left\langle\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right|\right)\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle \\
&= \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac13\left(\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle + \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right) \\
&\qquad +\tfrac16\left(2\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle - \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle\right) \\
&= 0\left|\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\tfrac12\right\rangle + \tfrac12\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac12\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle
\end{aligned}$$

Normalizing this gives

$$\left|\psi_3\right\rangle = \left|\tfrac12\tfrac12\right\rangle_b = \tfrac{1}{\sqrt2}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\tfrac12\right\rangle - \tfrac{1}{\sqrt2}\left|\tfrac12\tfrac12\tfrac12\tfrac12\tfrac12\tfrac{-1}{2}\right\rangle$$

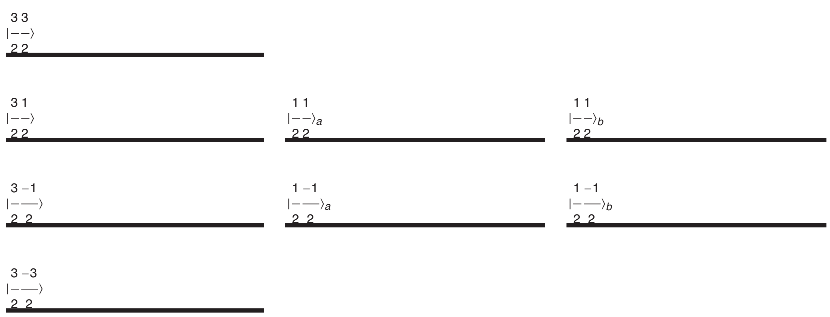
It is clear that $\left|\psi_2\right\rangle$ and $\left|\psi_3\right\rangle$ both have total spin projection ½. It turns out that they also both have total spin ½, which you can show by acting with $\mathbf{S}^2$. Now act with the lowering operator on these to states to generate the last two states:

$$\begin{aligned}
S_-\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_a &= \left(S_{1-}+S_{2-}+S_{3-}\right)\left(\tfrac{2}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\right)\\
\hbar\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle_a &= \tfrac{1}{\sqrt{6}}\left(\hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle+\hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle-2\hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\right\rangle\right)\\
\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle_a &= \tfrac{1}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle+\tfrac{1}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle-\tfrac{2}{\sqrt{6}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
S_-\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_b &= \left(S_{1-}+S_{2-}+S_{3-}\right)\left(\tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\right)\\
\hbar\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle_b &= \tfrac{1}{\sqrt{2}}\left(\hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle-\hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\right)\\
\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle_b &= \tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\right\rangle-\tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\,\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle
\end{aligned}$$

c) The manifold of states is



and the Clebsch-Gordan table for this system (with our arbitrary choices) is

$s_1=\frac{1}{2}$, $s_2=\frac{1}{2}$, $s_3=\frac{1}{2}$			S	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
			M	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
m_1	m_2	m_3									
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		1	0	0	0	0	0	0	0
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$\frac{2}{\sqrt{6}}$	0	0	0
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$-\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{2}}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$-\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{2}}$	0
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{2}}$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$-\frac{2}{\sqrt{6}}$	0	0
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	0	1	0	0	0	0