16.1 The force on a magnetic dipole of one Bohr magneton is

$$F = -\frac{dV}{dz} = -\frac{d}{dz}\left(-\boldsymbol{\mu}\bullet\mathbf{B}\right) = \mu_B \frac{dB}{dz}$$

The force deflects the atom transverse to its motion, so the deflection angle of an atom traveling a distance L through the field gradient is

$$\theta = \frac{\Delta p}{p} = \frac{F\Delta t}{Mv} = \frac{F\left(L/v\right)}{Mv} = \frac{\mu_B L}{Mv^2}\frac{dB}{dz}$$

A thermal atom has $Mv^2 = 2k_BT$, so the deflection angle is

$$\theta = \frac{\mu_B L}{2k_BT}\frac{dB}{dz} = L\frac{\left(1.4\,MHz/G\right)\left(4.14\times10^{-15}\,eVs\right)}{2\left(8.62\times10^{-5}\,eV/K\right)300K}100\,G/cm$$
$$= L\left(11.2\,\mu rad/cm\right)$$

16.2 The Larmor precession frequency in a field B_0 is

$$\omega_0 = \frac{eB_0}{m_e}$$

At a distance of 1 cm in a 100 G/cm trap, the field is 100G and the Larmor precession frequency is [use $\mu_B = e\hbar/2m_e = h\left(1.4\,MHz/G\right)$]

$$\omega_0 = \frac{eB_0}{m_e} = B_0\frac{2\mu_B}{\hbar} = 100G\frac{2h\left(1.4\,MHz/G\right)}{h/2\pi} = 1.76\times10^9\,rad/s$$

Newton's 2nd law for an atom moving in a circle of radius r in a magnetic trap with gradient dB/dz is

$$F = Ma$$
$$\mu_B\frac{dB}{dz} = M\frac{v^2}{r} = M\omega^2 r$$

giving a frequency of
(use $\mu_B = e\hbar/2m_e = \left(1.4\,MHz/G\right)\left(4.14\times10^{-15}\,eVs\right) = 5.79\times10^{-9}\,eV/G$)

$$\omega = \sqrt{\frac{\mu_B}{Mr}\frac{dB}{dz}} = \sqrt{\frac{\mu_B c^2}{Mc^2 r}\frac{dB}{dz}}$$
$$= \sqrt{\frac{\left(5.79\times10^{-9}\,eV/G\right)\left(3\times10^{10}\,cm/s\right)^2}{\left(85\times931\,MeV\right)1cm}100\,G/cm} = 81\,rad/s$$

For the magnetic moment to adiabatically follow the changing magnetic field direction, we require that the Larmor precession frequency be much larger than the frequency of motion in the trap, which is clearly the case for this position. To find the size of the "hole" in the trap where the Larmor precession frequency is smaller than the frequency of motion in the trap, equate the two frequencies and solve for the radius:

$$\omega_0 = \omega \quad \Rightarrow \quad \frac{eB_0}{m_e} = \sqrt{\frac{\mu_B}{Mr}\frac{dB}{dz}}$$

Setting $B_0 = r(dB/dz)$, we get
(use $\mu_B = e\hbar/2m_e = (1.4\,MHz/G)(4.14\times10^{-15}\,eVs) = 5.79\times10^{-9}\,eV/G$)

$$\left(\frac{er}{m_e}\frac{dB}{dz}\right)^2 = \frac{\mu_B}{Mr}\frac{dB}{dz}$$

$$r^3 = \left(\frac{m_e}{e}\right)^2 \frac{\mu_B}{M}\left(\frac{dB}{dz}\right)^{-1} = \left(\frac{\hbar c}{2\mu_B}\right)^2 \frac{\mu_B}{Mc^2}\left(\frac{dB}{dz}\right)^{-1} = \frac{(hc)^2}{16\pi^2\mu_B Mc^2}\left(\frac{dB}{dz}\right)^{-1}$$

$$r = \left[\frac{(1240\,eVnm)^2}{16\pi^2(5.79\times10^{-9}\,eV/G)(85\times931\;MeV)(100\,G/cm)(10^{-7}\,cm/nm)}\right]^{1/3}$$
$$= 0.13\,\mu m$$

16.3 The acceleration of a laser slowed rubidium atom is

$$a_{\max} = \frac{F_{\max}}{M} = \frac{\hbar k}{2M\tau} = \frac{h}{2M\lambda\tau} = \frac{(hc)c}{2Mc^2\lambda\tau}$$
$$= \frac{(1240\,eVnm)(3\times10^8\,m/s)}{2(85\times931\;MeV)(780\,nm)(27\times10^{-9}\,s)} = 1.12\times10^5\,m/s^2$$

The distance to stop the atom from kinematics is

$$v^2 = v_0^2 - 2az \quad \Rightarrow \quad z = \frac{v_0^2}{2a}$$

A thermal atom has $Mv^2 = 2k_BT$, so the stopping distance for a rubidium atom ($M = 85amu$, $\lambda = 780nm$, $\tau = 27ns$) is

$$z = \frac{v_0^2}{2a} = \frac{2k_BT}{2M}\frac{2M\lambda\tau}{h} = \frac{2k_BT\lambda\tau}{h}$$
$$= \frac{2(8.62\times10^{-5}\,eV/K)(300K)(780\times10^{-9}\,m)(27\times10^{-9}\,s)}{(4.14\times10^{-15}\,eVs)} = 0.26m$$

The stopping distance for a sodium atom ($M = 23amu$, $\lambda = 589nm$, $\tau = 16ns$) is

$$z = \frac{2(8.62\times10^{-5}\,eV/K)(300K)(589\times10^{-9}\,m)(16\times10^{-9}\,s)}{(4.14\times10^{-15}\,eVs)} = 0.12m$$

16.4 The general expression for the scattering force is

$$F_{scatt}=\hbar k\frac{A_{21}}{2}\frac{I}{I_0}\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(\omega_{Laser}-\omega_{21}+kv\right)^2+\left(\frac{A_{21}}{2}\right)^2\left(1+\frac{I}{I_0}\right)}$$

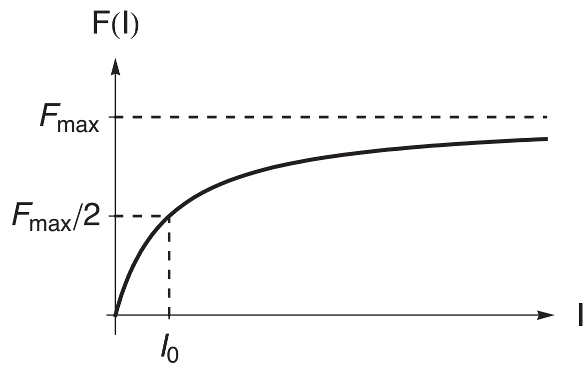
The variables are the laser intensity *I*, the laser frequency ω_{Laser}, and the atomic velocity v. As a function of the laser intensity, the force is maximal when the intensity is large enough that $I \gg I_0$. The force is maximal when the detuning term $\left(\omega_{Laser}-\omega_{21}+kv\right)$ is zero, which requires that the laser be resonant with the Doppler shifted atoms:

$$\omega_{Laser}=\omega_{21}-kv$$

Under these conditions, the maximal force is

$$F_{scatt}=\hbar k\frac{A_{21}}{2}\frac{I}{I_0}\frac{\left(\frac{A_{21}}{2}\right)^2}{(0)^2+\left(\frac{A_{21}}{2}\right)^2\left(1+\frac{I}{I_0}\right)}=\hbar k\frac{A_{21}}{2}\frac{\frac{I}{I_0}}{\left(1+\frac{I}{I_0}\right)}\underset{\lim\frac{I}{I_0}\to\infty}{=}\hbar k\frac{A_{21}}{2}=\frac{\hbar k}{2\tau}$$

in agreement with Eq. (16.8). The plot below shows the intensity dependence of the resonant force:



For small intensities, the force grows linearly with the intensity, but for intensities much larger than the intensity I_0, the force saturates to its maximum value. Hence, the intensity I_0 is called the saturation intensity.

16.5 The Doppler-shifted angular frequency of a laser beam with wave vector $\mathbf{k}$ as observed by an atom with velocity $\mathbf{v}$ is

$$\omega_{Atom}=\omega_{Laser}-\mathbf{k}\bullet\mathbf{v}$$

For the laser to be resonant with moving atoms, the laser detuning must be

$$\omega_{Laser}-\omega_{Atom}=\mathbf{k}\bullet\mathbf{v}$$
$$2\pi\left(f_{Laser}-f_{21}\right)=\mathbf{k}\bullet\mathbf{v}$$

For a laser beam collinear with the atoms, we get

$$f_{Laser} - f_{21} = \frac{1}{2\pi} kv = \frac{1}{2\pi} \frac{2\pi}{\lambda} v = \frac{v}{\lambda}$$

Thus, Rubidium atoms ($\lambda = 780nm$) moving at 350 m/s are resonant for a detuning

$$f_{Laser} - f_{21} = \frac{v}{\lambda} = \frac{350\, m/s}{780 \times 10^{-9} m} = 449\ \ MHz$$

16.6 In Problem 16.5, we found that the frequency of a laser resonant with moving atoms is

$$f_{Laser} - f_{21} = \frac{v}{\lambda}$$

The chirp rate is the rate of frequency change

$$\frac{d}{dt} f_{Laser} = \frac{d}{dt}\frac{v}{\lambda} = \frac{1}{\lambda}\frac{dv}{dt} = \frac{a}{\lambda}$$

The chirp rate is limited by the acceleration provided by the scattering force [see Eq. (16.9)]. Hence the maximum possible chirp rate is

$$\left(\frac{d}{dt} f_{Laser}\right)_{\max} = \frac{a_{max}}{\lambda} = \frac{h}{2M\lambda^2\tau} = \frac{(hc)c}{2Mc^2\lambda^2\tau}$$

For rubidium atoms, this maximum rate is

$$\begin{aligned}\left(\frac{d}{dt} f_{Laser}\right)_{\max} &= \frac{(1240\, eVnm)(3\times 10^{8}\ m/s)}{2(85\times 931\ \ MeV)(780\, nm)(780\times 10^{-9}\, m)(27\times 10^{-9}\, s)} \\ &= 1.43\times 10^{11} s^{-2} = 143\ \ MHz/ms\end{aligned}$$

The Doppler width of the Maxwellian velocity distribution is of order 600 MHz, so the time to cool the beam of atoms is about 4 ms.

16.7 The scattering force from one beam is

$$\mathbf{F}_{scatt}(\mathbf{v}) = \hbar\mathbf{k}\frac{A_{21}}{2}\frac{I}{I_0}\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(\omega_{Laser} - \omega_{21} - \mathbf{k}\bullet\mathbf{v}\right)^2 + \left(\frac{A_{21}}{2}\right)^2}$$

For two counterpropagating beams, the total force is $\mathbf{F}_{+k\hat{\mathbf{x}}}(v)+\mathbf{F}_{-k\hat{\mathbf{x}}}(v)$:

$$
\begin{aligned}
F_x(v) &= F_{+k\hat{\mathbf{x}}}(v)+F_{-k\hat{\mathbf{x}}}(v) \\
&= \frac{A_{21}}{2}\frac{I}{I_0}\left[\hbar k\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(\omega_{Laser}-\omega_{21}-kv\right)^2+\left(\frac{A_{21}}{2}\right)^2}+\hbar(-k)\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(\omega_{Laser}-\omega_{21}+kv\right)^2+\left(\frac{A_{21}}{2}\right)^2}\right]
\end{aligned}
$$

Assume that the laser detuning $\omega_{Laser}-\omega_{21}\cong -A_{21}$ and factor

$$
\begin{aligned}
F_x(v) &= \hbar k\frac{A_{21}}{2}\frac{I}{I_0}\left[\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(-A_{21}-kv\right)^2+\left(\frac{A_{21}}{2}\right)^2}-\frac{\left(\frac{A_{21}}{2}\right)^2}{\left(-A_{21}+kv\right)^2+\left(\frac{A_{21}}{2}\right)^2}\right] \\
&= \hbar k\frac{A_{21}}{2}\frac{I}{I_0}\left[\frac{1}{1+4\left(1+kv/A_{21}\right)^2}-\frac{1}{1+4\left(1+kv/A_{21}\right)^2}\right]
\end{aligned}
$$

Assume that the velocity is much less than A_{21}/k to isolate the linear region near zero velocity where the force is linear in the velocity ($F=-\alpha v$):

$$
\begin{aligned}
F_x(v) &\cong \hbar k\frac{A_{21}}{2}\frac{I}{I_0}\left[\frac{1}{1+4\left(1+2\,kv/A_{21}\right)}-\frac{1}{1+4\left(1-2\,kv/A_{21}\right)}\right] \\
&\cong \hbar k\frac{A_{21}}{2}\frac{I}{I_0}\left[\frac{1}{5+8\,kv/A_{21}}-\frac{1}{5-8\,kv/A_{21}}\right] \\
&\cong \hbar k\frac{A_{21}}{10}\frac{I}{I_0}\left[\frac{1}{1+8\,kv/5A_{21}}-\frac{1}{1-8\,kv/5A_{21}}\right] \\
&\cong \hbar k\frac{A_{21}}{10}\frac{I}{I_0}\left[1-8\,kv/5A_{21}-\left(1+8\,kv/5A_{21}\right)\right] \\
&\cong -\left(\hbar k^2\frac{8}{25}\frac{I}{I_0}\right)v
\end{aligned}
$$

Thus the force at low velocity is linear in the velocity ($F=-\alpha v$), with a friction coefficient α given by

$$
\alpha=\hbar k^2\frac{8}{25}\frac{I}{I_0}
$$

16.8 For a system with 3 qubits, labeled *A*, *B*, and *C*, the basis states of this system are the uncoupled basis states

$$\begin{array}{ll} |000\rangle = |0\rangle_A |0\rangle_B |0\rangle_C & |100\rangle = |1\rangle_A |0\rangle_B |0\rangle_C \\ |001\rangle = |0\rangle_A |0\rangle_B |1\rangle_C & |101\rangle = |1\rangle_A |0\rangle_B |1\rangle_C \\ |010\rangle = |0\rangle_A |1\rangle_B |0\rangle_C & |110\rangle = |1\rangle_A |1\rangle_B |0\rangle_C \\ |011\rangle = |0\rangle_A |1\rangle_B |1\rangle_C & |111\rangle = |1\rangle_A |1\rangle_B |1\rangle_C \end{array}$$

In this 3-qubit system, a general superposition state, which includes entangled states, is

$$\begin{aligned} |\psi\rangle = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + c_{011}|011\rangle + \\ + c_{100}|100\rangle + c_{101}|101\rangle + c_{110}|110\rangle + c_{111}|111\rangle \end{aligned}$$

This single 3-qubit state contains $2^3 = 8$ pieces of information—the c_{ijk} coefficients. An example of a 3-qubit product state, which is not entangled, is

$$|\psi\rangle = \left(a_0|0\rangle_A + a_1|1\rangle_A\right)\left(b_0|0\rangle_B + b_1|1\rangle_B\right)\left(c_0|0\rangle_C + c_1|1\rangle_C\right)$$

This 3-qubit state contains $2\times 3 = 6$ pieces of information—the a_i, b_i, and c_{i} coefficients.

For a system with 4 qubits, labeled *A*, *B*, *C*, and *D*, the basis states of this system are the uncoupled basis states

$$\begin{array}{lll} |0000\rangle = |0\rangle_A |0\rangle_B |0\rangle_C |0\rangle_D & |0110\rangle = |0\rangle_A |1\rangle_B |1\rangle_C |0\rangle_D & |1100\rangle = |1\rangle_A |1\rangle_B |0\rangle_C |0\rangle_D \\ |0001\rangle = |0\rangle_A |0\rangle_B |0\rangle_C |1\rangle_D & |0111\rangle = |0\rangle_A |1\rangle_B |1\rangle_C |1\rangle_D & |1101\rangle = |1\rangle_A |1\rangle_B |0\rangle_C |1\rangle_D \\ |0010\rangle = |0\rangle_A |0\rangle_B |1\rangle_C |0\rangle_D & |1000\rangle = |1\rangle_A |0\rangle_B |0\rangle_C |0\rangle_D & |1110\rangle = |1\rangle_A |1\rangle_B |1\rangle_C |0\rangle_D \\ |0011\rangle = |0\rangle_A |0\rangle_B |1\rangle_C |1\rangle_D & |1001\rangle = |1\rangle_A |0\rangle_B |0\rangle_C |1\rangle_D & |1111\rangle = |1\rangle_A |1\rangle_B |1\rangle_C |1\rangle_D \\ |0100\rangle = |0\rangle_A |1\rangle_B |0\rangle_C |0\rangle_D & |1010\rangle = |1\rangle_A |0\rangle_B |1\rangle_C |0\rangle_D & \\ |0101\rangle = |0\rangle_A |1\rangle_B |0\rangle_C |1\rangle_D & |1011\rangle = |1\rangle_A |0\rangle_B |1\rangle_C |1\rangle_D & \end{array}$$

In this 4-qubit system, a general superposition state, which includes entangled states, is

$$\begin{aligned} |\psi\rangle = c_{0000}|0000\rangle + c_{0001}|0001\rangle + c_{0010}|0010\rangle + c_{0011}|0011\rangle + c_{0100}|0100\rangle + c_{0101}|0101\rangle + \\ + c_{0110}|0110\rangle + c_{0111}|0111\rangle + c_{1000}|1000\rangle + c_{1001}|1001\rangle + c_{1010}|1010\rangle \\ + c_{1011}|1011\rangle + c_{1100}|1100\rangle + c_{1101}|1101\rangle + c_{1110}|1110\rangle + c_{1111}|1111\rangle \end{aligned}$$

This single 4-qubit state contains $2^4 = 16$ pieces of information—the c_{ijkm} coefficients. An example of a 4-qubit product state, which is not entangled, is

$$|\psi\rangle = \left(a_0|0\rangle_A + a_1|1\rangle_A\right)\left(b_0|0\rangle_B + b_1|1\rangle_B\right)\left(c_0|0\rangle_C + c_1|1\rangle_C\right)\left(d_0|0\rangle_D + d_1|1\rangle_D\right)$$

This 4-qubit state contains $2\times 4 = 8$ pieces of information—the a_i, b_i, c_{i}, and d_{i} coefficients.

16.9 For a 1-qubit gate, the computational basis is

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The *NOT* gate is represented by

$$U_{NOT} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The action of the *NOT* gate on the computational basis is

$$|\psi_{out}\rangle = U_{NOT}|\psi_{in}\rangle$$
$$|\psi_{out}\rangle_0 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$|\psi_{out}\rangle_1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore

$$U_{NOT}|0\rangle = |1\rangle$$
$$U_{NOT}|1\rangle = |0\rangle$$

and a quantum *NOT* gate changes $|0\rangle \rightarrow |1\rangle$ and also $|1\rangle \rightarrow |0\rangle$ as advertised. For a general a superposition state $a|0\rangle + b|1\rangle$, the transformed state

$$|\psi_{out}\rangle = U_{NOT}|\psi_{in}\rangle \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

has a norm

$$\langle\psi_{out}|\psi_{out}\rangle = \begin{pmatrix} b^* & a^* \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = |b|^2 + |a|^2$$

that is the same as the norm of the input state:

$$\langle\psi_{in}|\psi_{in}\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2$$

so the transformation is unitary.

16.10 For the spin to precess about the z-axis, we apply a magnetic field B_0 in the z-direction. The energy states in this applied field are $|\pm\rangle$ and the energies are $E_\pm = \pm\hbar\omega_0/2$, where $\omega_0 = eB_0/m_e$ is the Larmor precession frequency. To find how the state vector is changed by the applied magnetic field, we use the Schrödinger time-evolution recipe. The initial general state is

$$|\psi(0)\rangle = c_+|+\rangle + c_-|-\rangle$$

This state is already in the energy basis, so to find the time-evolved state, we insert the time-dependent phase factor for each energy basis state:

$$|\psi(t)\rangle = c_+ e^{-iE_+t/\hbar}|+\rangle + c_- e^{-iE_-t/\hbar}|-\rangle = c_+ e^{-i\omega_0 t/2}|+\rangle + c_- e^{+i\omega_0 t/2}|-\rangle$$

To have a π rotation about the z-axis requires that the field be applied long enough to have $\omega_0 t = \pi$. Thus the state vector after the time evolution is

$$|\psi(t)\rangle = c_+ e^{-i\pi/2}|+\rangle + c_- e^{+i\pi/2}|-\rangle = -i\left(c_+|+\rangle - c_-|-\rangle\right)$$

or in matrix notation:

$$\begin{pmatrix} c_+' \\ c_-' \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

The overall phase $e^{-i\pi/2} = -i$ does not produce any measurable effects, so we ignore it in defining the quantum gate transformation matrix

$$U_Z \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is the Pauli matrix σ_z.

16.11 For the spin to precess about the y-axis, we apply a magnetic field B_0 in the y-direction. The energy states in this applied field are $|\pm\rangle_y$ and the energies are $E_\pm = \pm\hbar\omega_0/2$, where $\omega_0 = eB_0/m_e$ is the Larmor precession frequency. To find how the state vector is changed by the applied magnetic field, we use the Schrödinger time-evolution recipe. The initial general state is

$$|\psi(0)\rangle = c_+|+\rangle + c_-|-\rangle$$

We must write this state in the energy basis, which is the S_y basis in this case:

$$\begin{aligned} |\psi(0)\rangle &= \left(|+\rangle_y\, {}_y\langle+| + |-\rangle_y\, {}_y\langle-|\right)|\psi(0)\rangle \\ &= c_+\left({}_y\langle+|+\rangle|+\rangle_y + {}_y\langle-|+\rangle|-\rangle_y\right) + c_-\left({}_y\langle+|-\rangle|+\rangle_y + {}_y\langle-|-\rangle|-\rangle_y\right) \\ &= \tfrac{1}{\sqrt{2}}\left(c_+ - ic_-\right)|+\rangle_y + \tfrac{1}{\sqrt{2}}\left(c_+ + ic_-\right)|-\rangle_y \end{aligned}$$

To find the time-evolved state, we insert the time-dependent phase factor for each energy basis state:

$$\begin{aligned}|\psi(t)\rangle &= \tfrac{1}{\sqrt{2}}(c_+ - ic_-)e^{-iE_+t/\hbar}|+\rangle_y + \tfrac{1}{\sqrt{2}}(c_+ + ic_-)e^{-iE_-t/\hbar}|-\rangle_y \\ &= \tfrac{1}{\sqrt{2}}(c_+ - ic_-)e^{-i\omega_0 t/2}|+\rangle_y + \tfrac{1}{\sqrt{2}}(c_+ + ic_-)e^{+i\omega_0 t/2}|-\rangle_y\end{aligned}$$

To have a π rotation about the *y*-axis requires that the field be applied long enough to have $\omega_0 t = \pi$. Thus the state vector after the time evolution is

$$\begin{aligned}|\psi(t)\rangle &= \tfrac{1}{\sqrt{2}}(c_+ - ic_-)e^{-i\pi/2}|+\rangle_y + \tfrac{1}{\sqrt{2}}(c_+ + ic_-)e^{+i\pi/2}|-\rangle_y \\ &= -i\tfrac{1}{\sqrt{2}}(c_+ - ic_-)\tfrac{1}{\sqrt{2}}\left(\,|+\rangle + i\,|-\rangle\,\right) + i\tfrac{1}{\sqrt{2}}(c_+ + ic_-)\tfrac{1}{\sqrt{2}}\left(\,|+\rangle - i\,|-\rangle\,\right) \\ &= -i\left(-ic_-|+\rangle + ic_+|-\rangle\right)\end{aligned}$$

or in matrix notation:

$$\begin{pmatrix} c'_+ \\ c'_- \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

The overall phase $e^{-i\pi/2} = -i$ does not produce any measurable effects, so we ignore it in defining the quantum *NOT* gate transformation matrix:

$$U_Y \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which is the Pauli matrix σ_y.

16.12 As we found in Problem 16.10, a π rotation about the *z*-axis is represented by the quantum gate transformation matrix

$$U_Z \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For a π/2 rotation about the *y*-axis, the transformation matrix can be obtained by an extension of Problem 16.11. A π/2 rotation requires that the field be applied long enough to have $\omega_0 t = \pi/2$. Thus the state vector after the time evolution is

$$\begin{aligned}|\psi(t)\rangle &= \tfrac{1}{\sqrt{2}}(c_+ - ic_-)e^{-i\pi/4}|+\rangle_y + \tfrac{1}{\sqrt{2}}(c_+ + ic_-)e^{+i\pi/4}|-\rangle_y \\ &= \tfrac{1}{\sqrt{2}}(c_+ - ic_-)\tfrac{1}{\sqrt{2}}(1-i)\tfrac{1}{\sqrt{2}}\left(\,|+\rangle + i\,|-\rangle\,\right) + \tfrac{1}{\sqrt{2}}(c_+ + ic_-)\tfrac{1}{\sqrt{2}}(1+i)\tfrac{1}{\sqrt{2}}\left(\,|+\rangle - i\,|-\rangle\,\right) \\ &= \tfrac{1}{\sqrt{2}}\left[(c_+ - c_-)|+\rangle + (c_+ + c_-)|-\rangle\right]\end{aligned}$$

or in matrix notation:

$$\begin{pmatrix} c'_+ \\ c'_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

Thus, the transformation matrix is

$$U_{Y_{\pi/2}} \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Thus the Hadamard transformation matrix is

$$U_H = U_{Y_{\pi/2}} U_Z \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

16.13 The transformations of the *CNOT* gate are

$$\begin{aligned} U_{CNOT}\left|00\right\rangle &= \left|00\right\rangle \\ U_{CNOT}\left|01\right\rangle &= \left|01\right\rangle \\ U_{CNOT}\left|10\right\rangle &= \left|11\right\rangle \\ U_{CNOT}\left|11\right\rangle &= \left|10\right\rangle \end{aligned}$$

To find the matrix elements of U_{CNOT}, project these equations onto the basis states. There are only four nonzero elements:

$$\begin{aligned} \left\langle 00\right| U_{CNOT}\left|00\right\rangle &= 1 \\ \left\langle 01\right| U_{CNOT}\left|01\right\rangle &= 1 \\ \left\langle 11\right| U_{CNOT}\left|10\right\rangle &= 1 \\ \left\langle 10\right| U_{CNOT}\left|11\right\rangle &= 1 \end{aligned}$$

Thus the transformation matrix is

$$U_{CNOT} \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

16.14 (i) For the input state

$$\left|\psi_1\right\rangle = \left|00\right\rangle = \left|0\right\rangle_C \left|0\right\rangle_T$$

the action of the Hadamard gate yields

$$\left|\psi_2\right\rangle = U_{Had,C}\left|\psi_1\right\rangle = \left(U_{Had,C}\left|0\right\rangle_C\right)\left|0\right\rangle_T$$

The transformation of the single control qubit is

$$U_{Had,C}\left|0\right\rangle_C \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)$$

The resultant state of the 2-qubit system before the *CNOT* gate is

$$\left|\psi_2\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|0\right\rangle_T = \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|10\right\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The transformation of the *CNOT* gate is

$$\left|\psi_3\right\rangle = U_{CNOT}\left|\psi_2\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The output state is thus

$$\left|\psi_3\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)$$

This is the entangled Bell state $\left|\beta_{00}\right\rangle$ from Eq. (16.23).
(ii) For the input state

$$\left|\psi_1\right\rangle = \left|01\right\rangle = \left|0\right\rangle_C\left|1\right\rangle_T$$

the action of the Hadamard gate yields

$$\left|\psi_2\right\rangle = U_{Had,C}\left|\psi_1\right\rangle = \left(U_{Had,C}\left|0\right\rangle_C\right)\left|1\right\rangle_T$$

The transformation of the single control qubit is

$$U_{Had,C}\left|0\right\rangle_C \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)$$

The resultant state of the 2-qubit system before the *CNOT* gate is

$$\left|\psi_2\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|1\right\rangle_T = \tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|11\right\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The transformation of the *CNOT* gate is

$$\left|\psi_3\right\rangle = U_{CNOT}\left|\psi_2\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The output state is thus

$$\left|\psi_3\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right)$$

This is the entangled Bell state $\left|\beta_{01}\right\rangle$ from Eq. (16.23).
(iii) For the input state

$$\left|\psi_1\right\rangle = \left|10\right\rangle = \left|1\right\rangle_C \left|0\right\rangle_T$$

the action of the Hadamard gate yields

$$\left|\psi_2\right\rangle = U_{Had,C}\left|\psi_1\right\rangle = \left(U_{Had,C}\left|1\right\rangle_C\right)\left|0\right\rangle_T$$

The transformation of the single control qubit is

$$\begin{aligned} U_{Had,C}\left|1\right\rangle_C &\doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right) \end{aligned}$$

The resultant state of the 2-qubit system before the *CNOT* gate is

$$\left|\psi_2\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right)\left|0\right\rangle_T = \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|10\right\rangle\right) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

The transformation of the *CNOT* gate is

$$\left|\psi_3\right\rangle = U_{CNOT}\left|\psi_2\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The output state is thus

$$\left|\psi_3\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|11\right\rangle\right)$$

This is the entangled Bell state $\left|\beta_{10}\right\rangle$ from Eq. (16.23).

16.15 (i) For the Bell state $\left|\beta_{00}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)$, the action of the *CNOT* gate yields

$$\left|\psi_2\right\rangle = U_{CNOT}\left|\beta_{00}\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The action of the Hadamard gate on this intermediate state $\left|\psi_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|10\right\rangle\right)$ yields

$$\begin{aligned}\left|\psi_3\right\rangle &= U_{Had,C}\,\tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|10\right\rangle\right) = \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|0\right\rangle_C\right)\left|0\right\rangle_T + \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|1\right\rangle_C\right)\left|0\right\rangle_T \\ &= \tfrac{1}{2}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|0\right\rangle_T + \tfrac{1}{2}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right)\left|0\right\rangle_T = \left|0\right\rangle_C\left|0\right\rangle_T = \left|00\right\rangle\end{aligned}$$

Hence, the Bell state $\left|\beta_{00}\right\rangle$ is transformed into the computational basis state $\left|00\right\rangle$.
(ii) For the Bell state $\left|\beta_{01}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right)$, the action of the *CNOT* gate yields

$$\left|\psi_2\right\rangle = U_{CNOT}\left|\beta_{01}\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The action of the Hadamard gate on this intermediate state $\left|\psi_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|11\right\rangle\right)$ yields

$$\begin{aligned}\left|\psi_3\right\rangle &= U_{Had,C}\,\tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|11\right\rangle\right) = \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|0\right\rangle_C\right)\left|1\right\rangle_T + \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|1\right\rangle_C\right)\left|1\right\rangle_T \\ &= \tfrac{1}{2}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|1\right\rangle_T + \tfrac{1}{2}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right)\left|1\right\rangle_T = \left|0\right\rangle_C\left|1\right\rangle_T = \left|01\right\rangle\end{aligned}$$

Hence, the Bell state $\left|\beta_{01}\right\rangle$ is transformed into the computational basis state $\left|01\right\rangle$.
(iii) For the Bell state $\left|\beta_{10}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|11\right\rangle\right)$, the action of the *CNOT* gate yields

$$\left|\psi_2\right\rangle = U_{CNOT}\left|\beta_{00}\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

The action of the Hadamard gate on this intermediate state $\left|\psi_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|10\right\rangle\right)$ yields

$$\begin{aligned}\left|\psi_3\right\rangle &= U_{Had,C}\,\tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle - \left|10\right\rangle\right) = \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|0\right\rangle_C\right)\left|0\right\rangle_T - \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|1\right\rangle_C\right)\left|0\right\rangle_T \\ &= \tfrac{1}{2}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|0\right\rangle_T - \tfrac{1}{2}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right)\left|0\right\rangle_T = \left|1\right\rangle_C\left|0\right\rangle_T = \left|10\right\rangle\end{aligned}$$

Hence, the Bell state $\left|\beta_{10}\right\rangle$ is transformed into the computational basis state $\left|10\right\rangle$.
(iv) For the Bell state $\left|\beta_{11}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right)$, the action of the *CNOT* gate yields

$$\left|\psi_2\right\rangle = U_{CNOT}\left|\beta_{01}\right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

The action of the Hadamard gate on this intermediate state $\left|\psi_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|11\right\rangle\right)$ yields

$$\begin{aligned}\left|\psi_3\right\rangle &= U_{Had,C}\,\tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|11\right\rangle\right) = \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|0\right\rangle_C\right)\left|1\right\rangle_T - \tfrac{1}{\sqrt{2}}\left(U_{Had,C}\left|1\right\rangle_C\right)\left|1\right\rangle_T \\ &= \tfrac{1}{2}\left(\left|0\right\rangle_C + \left|1\right\rangle_C\right)\left|1\right\rangle_T - \tfrac{1}{2}\left(\left|0\right\rangle_C - \left|1\right\rangle_C\right)\left|1\right\rangle_T = \left|1\right\rangle_C\left|1\right\rangle_T = \left|11\right\rangle\end{aligned}$$

Hence, the Bell state $\left|\beta_{11}\right\rangle$ is transformed into the computational basis state $\left|11\right\rangle$.

16.16 The state vector for the complete three-qubit system is

$$\begin{aligned}\left|\psi_{ABC}\right\rangle &= \left|\beta_{00}\right\rangle_{AB}\left|\psi_{\text{secret}}\right\rangle_C \\ &= \left(\tfrac{1}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_B + \tfrac{1}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_B\right)\left(\alpha_0\left|0\right\rangle_C + \alpha_1\left|1\right\rangle_C\right) \\ &= \tfrac{\alpha_0}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_B\left|0\right\rangle_C + \tfrac{\alpha_0}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_B\left|0\right\rangle_C + \tfrac{\alpha_1}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_B\left|1\right\rangle_C + \tfrac{\alpha_1}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_B\left|1\right\rangle_C\end{aligned}$$

The basis of entangled Bell states $\left|\beta_{ij}\right\rangle_{AC}$ of the two qubits *A* and *C* is

$$\begin{aligned}\left|\beta_{00}\right\rangle_{AC} &= \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle_{AC} + \left|11\right\rangle_{AC}\right) = \tfrac{1}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_C + \tfrac{1}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_C \\ \left|\beta_{01}\right\rangle_{AC} &= \tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle_{AC} + \left|10\right\rangle_{AC}\right) = \tfrac{1}{\sqrt{2}}\left|0\right\rangle_A\left|1\right\rangle_C + \tfrac{1}{\sqrt{2}}\left|1\right\rangle_A\left|0\right\rangle_C \\ \left|\beta_{10}\right\rangle_{AC} &= \tfrac{1}{\sqrt{2}}\left(\left|00\right\rangle_{AC} - \left|11\right\rangle_{AC}\right) = \tfrac{1}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_C - \tfrac{1}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_C \\ \left|\beta_{11}\right\rangle_{AC} &= \tfrac{1}{\sqrt{2}}\left(\left|01\right\rangle_{AC} - \left|10\right\rangle_{AC}\right) = \tfrac{1}{\sqrt{2}}\left|0\right\rangle_A\left|1\right\rangle_C - \tfrac{1}{\sqrt{2}}\left|1\right\rangle_A\left|0\right\rangle_C\end{aligned}$$

Regrouping states in the 3-qubit state, we get

$$\begin{aligned}\left|\psi_{ABC}\right\rangle &= \tfrac{\alpha_0}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_B\left|0\right\rangle_C + \tfrac{\alpha_0}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_B\left|0\right\rangle_C + \tfrac{\alpha_1}{\sqrt{2}}\left|0\right\rangle_A\left|0\right\rangle_B\left|1\right\rangle_C + \tfrac{\alpha_1}{\sqrt{2}}\left|1\right\rangle_A\left|1\right\rangle_B\left|1\right\rangle_C \\ &= \tfrac{1}{\sqrt{2}}\left\{\alpha_0\left|00\right\rangle_{AC}\left|0\right\rangle_B + \alpha_0\left|10\right\rangle_{AC}\left|1\right\rangle_B + \alpha_1\left|01\right\rangle_{AC}\left|0\right\rangle_B + \alpha_1\left|11\right\rangle_{AC}\left|1\right\rangle_B\right\}\end{aligned}$$

Now write each term twice and rearrange to get

$$\begin{aligned}\left|\psi_{ABC}\right\rangle = \tfrac{1}{2\sqrt{2}}\{&\alpha_0\left|00\right\rangle_{AC}\left|0\right\rangle_B + \alpha_1\left|11\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_1\left|01\right\rangle_{AC}\left|0\right\rangle_B + \alpha_0\left|10\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_0\left|00\right\rangle_{AC}\left|0\right\rangle_B + \alpha_1\left|11\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_1\left|01\right\rangle_{AC}\left|0\right\rangle_B + \alpha_0\left|10\right\rangle_{AC}\left|1\right\rangle_B\}\end{aligned}$$

Add and subtract 4 more terms to get

$$\begin{aligned}\left|\psi_{ABC}\right\rangle = \tfrac{1}{2\sqrt{2}}\{&\alpha_0\left|00\right\rangle_{AC}\left|0\right\rangle_B + \alpha_0\left|11\right\rangle_{AC}\left|0\right\rangle_B + \alpha_1\left|00\right\rangle_{AC}\left|1\right\rangle_B + \alpha_1\left|11\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_1\left|01\right\rangle_{AC}\left|0\right\rangle_B + \alpha_1\left|10\right\rangle_{AC}\left|0\right\rangle_B + \alpha_0\left|01\right\rangle_{AC}\left|1\right\rangle_B + \alpha_0\left|10\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_0\left|00\right\rangle_{AC}\left|0\right\rangle_B - \alpha_0\left|11\right\rangle_{AC}\left|0\right\rangle_B - \alpha_1\left|00\right\rangle_{AC}\left|1\right\rangle_B + \alpha_1\left|11\right\rangle_{AC}\left|1\right\rangle_B \\ &+\alpha_1\left|01\right\rangle_{AC}\left|0\right\rangle_B - \alpha_1\left|10\right\rangle_{AC}\left|0\right\rangle_B - \alpha_0\left|01\right\rangle_{AC}\left|1\right\rangle_B + \alpha_0\left|10\right\rangle_{AC}\left|1\right\rangle_B\}\end{aligned}$$

Then factor into Bell states to get

$$\begin{aligned}\left|\psi_{ABC}\right\rangle = \tfrac{1}{2\sqrt{2}}\Big\{&\left(|00\rangle_{AC}+|11\rangle_{AC}\right)\left(\alpha_0|0\rangle_B+\alpha_1|1\rangle_B\right)\\ &+\left(|01\rangle_{AC}+|10\rangle_{AC}\right)\left(\alpha_1|0\rangle_B+\alpha_0|1\rangle_B\right)\\ &+\left(|00\rangle_{AC}-|11\rangle_{AC}\right)\left(\alpha_0|0\rangle_B-\alpha_1|1\rangle_B\right)\\ &+\left(|01\rangle_{AC}-|10\rangle_{AC}\right)\left(\alpha_1|0\rangle_B-\alpha_0|1\rangle_B\right)\Big\}\end{aligned}$$

Identifying the Bell states yields

$$\begin{aligned}\left|\psi_{ABC}\right\rangle = \frac{1}{2}\Big\{&\left|\beta_{00}\right\rangle_{AC}\left(\alpha_0|0\rangle_B+\alpha_1|1\rangle_B\right)\\ &+\left|\beta_{01}\right\rangle_{AC}\left(\alpha_1|0\rangle_B+\alpha_0|1\rangle_B\right)\\ &+\left|\beta_{10}\right\rangle_{AC}\left(\alpha_0|0\rangle_B-\alpha_1|1\rangle_B\right)\\ &+\left|\beta_{11}\right\rangle_{AC}\left(\alpha_1|0\rangle_B-\alpha_0|1\rangle_B\right)\Big\}\end{aligned}$$

16.17 (i) If Alice's Bell-state measurement indicates that the A and C qubits are in the state $\left|\beta_{00}\right\rangle_{AC}$, then Bob's qubit B is in the state

$$|\psi\rangle_B = \alpha_0|0\rangle_B + \alpha_1|1\rangle_B$$

This is exactly the secret state that Carol gave to Alice, so Bob does not need to apply a transformation, which is equivalent to the identity transformation.
(ii) If Alice's Bell-state measurement indicates that the A and C qubits are in the state $\left|\beta_{01}\right\rangle_{AC}$, then Bob's qubit B is in the state

$$|\psi\rangle_B = \alpha_1|0\rangle_B + \alpha_0|1\rangle_B$$

To transform this state to the secret state that Carol gave to Alice, Bob applies the U_{NOT} transformation. Let's check that Bob gets the desired result:

$$\begin{aligned}U_{NOT}|\psi\rangle_B &= U_{NOT}\left(\alpha_1|0\rangle_B + \alpha_0|1\rangle_B\right) \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}\\ &= \alpha_0|0\rangle_B + \alpha_1|1\rangle_B = \left|\psi_{\text{secret}}\right\rangle_B\end{aligned}$$

(iii) If Alice's Bell-state measurement indicates that the A and C qubits are in the state $\left|\beta_{10}\right\rangle_{AC}$, then Bob's qubit B is in the state

$$|\psi\rangle_B = \alpha_0|0\rangle_B - \alpha_1|1\rangle_B$$

To transform this state to the secret state that Carol gave to Alice, Bob applies the U_Z transformation. Let's check that Bob gets the desired result:

$$U_Z|\psi\rangle_B = U_Z\left(\alpha_0|0\rangle_B - \alpha_1|1\rangle_B\right) \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$
$$= \alpha_0|0\rangle_B + \alpha_1|1\rangle_B = |\psi_{\text{secret}}\rangle_B$$

(iv) If Alice's Bell-state measurement indicates that the *A* and *C* qubits are in the state $|\beta_{11}\rangle_{AC}$, then Bob's qubit *B* is in the state

$$|\psi\rangle_B = \alpha_1|0\rangle_B - \alpha_0|1\rangle_B$$

To transform this state to the secret state that Carol gave to Alice, Bob applies the $U_Z U_{NOT}$ transformation. Let's check that Bob gets the desired result:

$$U_Z U_{NOT}|\psi\rangle_B = U_Z U_{NOT}\left(\alpha_1|0\rangle_B - \alpha_0|1\rangle_B\right) \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ -\alpha_0 \end{pmatrix} = -\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$
$$= -\left(\alpha_0|0\rangle_B + \alpha_1|1\rangle_B\right) = -|\psi_{\text{secret}}\rangle_B$$

In this case, the state carries an overall phase shift (-1), but that is not physical so the state is the same.