1.1. a)

$$|\psi_1\rangle = 3|+\rangle + 4|-\rangle$$

To normalize, introduce an overall complex multiplicative factor and solve for this factor by imposing the normalization condition:

$$\begin{aligned}
|\psi_1\rangle &= C\big(3|+\rangle + 4|-\rangle\big) \\
1 = \langle\psi_1|\psi_1\rangle &= \big\{C^*\big(3\langle+| + 4\langle-|\big)\big\}\big\{C\big(3|+\rangle + 4|-\rangle\big)\big\} \\
&= C^*C\big(9\langle+|+\rangle + 12\langle+|-\rangle + 12\langle-|+\rangle + 16\langle-|-\rangle\big) = C^*C(25) \\
|C|^2 &= \frac{1}{25}
\end{aligned}$$

Because an overall phase is physically meaningless, we choose C to be real and positive: $C = 1/5$. Hence the normalized input state is

$$|\psi_1\rangle = \tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle .$$

Likewise:

$$\begin{aligned}
|\psi_2\rangle &= C\big(|+\rangle + 2i|-\rangle\big) \\
1 &= \big\{C^*\big(\langle+| - 2i\langle-|\big)\big\}\big\{C\big(|+\rangle + 2i|-\rangle\big)\big\} = C^*C\big(\langle+|+\rangle + 4\langle-|-\rangle\big) = |C|^2(5) \\
|\psi_2\rangle &= \tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle
\end{aligned}$$

and

$$\begin{aligned}
|\psi_3\rangle &= C\big(3|+\rangle - e^{i\pi/3}|-\rangle\big) \\
1 &= \big\{C^*\big(3\langle+| - e^{-i\pi/3}\langle-|\big)\big\}\big\{C\big(3|+\rangle - e^{i\pi/3}|-\rangle\big)\big\} = C^*C\big(9\langle+|+\rangle + 1\langle-|-\rangle\big) = |C|^2(10) \\
|\psi_3\rangle &= \tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle
\end{aligned}$$

b) The probabilities for state 1 are

$$\mathcal{P}_{1,+} = \big|\langle+|\psi_1\rangle\big|^2 = \big|\langle+|\big(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\big)\big|^2 = \big|\tfrac{3}{5}\langle+|+\rangle + \tfrac{4}{5}\langle+|-\rangle\big|^2 = \big|\tfrac{3}{5}\big|^2 = \tfrac{9}{25}$$

$$\mathcal{P}_{1,-} = \big|\langle-|\psi_1\rangle\big|^2 = \big|\langle-|\big(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\big)\big|^2 = \big|\tfrac{3}{5}\langle-|+\rangle + \tfrac{4}{5}\langle-|-\rangle\big|^2 = \big|\tfrac{4}{5}\big|^2 = \tfrac{16}{25}$$

For the other axes, we get

$$\mathcal{P}_{1,+x} = \big|{}_x\langle+|\psi_1\rangle\big|^2 = \Big|\big(\tfrac{1}{\sqrt{2}}\langle+| + \tfrac{1}{\sqrt{2}}\langle-|\big)\big(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\big)\Big|^2 = \Big|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} + \tfrac{1}{\sqrt{2}}\tfrac{4}{5}\Big|^2 = \tfrac{49}{50}$$

$$\mathcal{P}_{1,-x} = \big|{}_x\langle-|\psi_1\rangle\big|^2 = \Big|\big(\tfrac{1}{\sqrt{2}}\langle+| - \tfrac{1}{\sqrt{2}}\langle-|\big)\big(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\big)\Big|^2 = \Big|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} - \tfrac{1}{\sqrt{2}}\tfrac{4}{5}\Big|^2 = \tfrac{1}{50}$$

$$\mathcal{P}_{1,+y} = \left|{}_y\langle + | \psi_1 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} - \tfrac{i}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{1,-y} = \left|{}_y\langle - | \psi_1 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} - \tfrac{i}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{1}{2}$$

The probabilities for state 2 are

$$\mathcal{P}_{2,+} = \left|\langle + | \psi_2 \rangle\right|^2 = \left|\langle +|\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{5}}\right|^2 = \tfrac{1}{5}$$

$$\mathcal{P}_{2,-} = \left|\langle - | \psi_2 \rangle\right|^2 = \left|\langle -|\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{2i}{\sqrt{5}}\right|^2 = \tfrac{4}{5}$$

$$\mathcal{P}_{2,+x} = \left|{}_x\langle + | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{10}} + \tfrac{2i}{\sqrt{10}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{2,-x} = \left|{}_x\langle - | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{10}} - \tfrac{2i}{\sqrt{10}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{2,+y} = \left|{}_y\langle + | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{10}} + \tfrac{2}{\sqrt{10}}\right|^2 = \tfrac{9}{10}$$

$$\mathcal{P}_{2,-y} = \left|{}_y\langle - | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle + \tfrac{2i}{\sqrt{5}}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{10}} - \tfrac{2}{\sqrt{10}}\right|^2 = \tfrac{1}{10}$$

The probabilities for state 3 are

$$\mathcal{P}_{3,+} = \left|\langle + | \psi_3 \rangle\right|^2 = \left|\langle +|\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 = \left|\tfrac{3}{\sqrt{10}}\right|^2 = \tfrac{9}{10}$$

$$\mathcal{P}_{3,-} = \left|\langle - | \psi_3 \rangle\right|^2 = \left|\langle -|\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 = \left|-\tfrac{1}{\sqrt{10}}e^{i\pi/3}\right|^2 = \tfrac{1}{10}$$

$$\begin{aligned}\mathcal{P}_{3,+x} &= \left|{}_x\langle + | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 \\ &= \left|\tfrac{3}{\sqrt{20}} - \tfrac{1}{\sqrt{20}}e^{i\pi/3}\right|^2 = \left(\tfrac{9}{20} + \tfrac{1}{20} - \tfrac{3}{20}2\cos\tfrac{\pi}{3}\right) = \tfrac{7}{20}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{3,-x} &= \left|{}_x\langle - | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 \\ &= \left|\tfrac{3}{\sqrt{20}} + \tfrac{1}{\sqrt{20}}e^{i\pi/3}\right|^2 = \left(\tfrac{9}{20} + \tfrac{1}{20} + \tfrac{3}{20}2\cos\tfrac{\pi}{3}\right) = \tfrac{13}{20}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{3,+y} &= \left|{}_y\langle + | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 \\ &= \left|\tfrac{3}{\sqrt{20}} + \tfrac{i}{\sqrt{20}}e^{i\pi/3}\right|^2 = \left(\tfrac{9}{20} + \tfrac{1}{20} - \tfrac{3}{20}2\sin\tfrac{\pi}{3}\right) = \tfrac{1}{20}\left(10 - 3\sqrt{3}\right) \cong 0.24\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{3,-y} &= \left|{}_y\langle - | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{3}{\sqrt{10}}|+\rangle - \tfrac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle\right)\right|^2 \\ &= \left|\tfrac{3}{\sqrt{20}} - \tfrac{i}{\sqrt{20}}e^{i\pi/3}\right|^2 = \left(\tfrac{9}{20} + \tfrac{1}{20} + \tfrac{3}{20}2\sin\tfrac{\pi}{3}\right) = \tfrac{1}{20}\left(10 + 3\sqrt{3}\right) \cong 0.76\end{aligned}$$

c) Matrix notation:

$$\left|\psi_1\right\rangle \doteq \tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\left|\psi_2\right\rangle \doteq \tfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\left|\psi_3\right\rangle \doteq \tfrac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix}$$

d) Probabilities in matrix notation

$$\mathcal{P}_{1,+} = \left|\langle +|\psi_1\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{3}{5}\right|^2 = \tfrac{9}{25}$$

$$\mathcal{P}_{1,-} = \left|\langle -|\psi_1\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}$$

$$\mathcal{P}_{1,+x} = \left|{}_x\langle +|\psi_1\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} + \tfrac{1}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{49}{50}$$

$$\mathcal{P}_{1,+x} = \left|{}_x\langle -|\psi_1\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} - \tfrac{1}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{1}{50}$$

$$\mathcal{P}_{1,+y} = \left|{}_y\langle +|\psi_1\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} - \tfrac{i}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{1,-y} = \left|{}_y\langle -|\psi_1\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{3}{5} + \tfrac{i}{\sqrt{2}}\tfrac{4}{5}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{2,+} = \left|\langle + | \psi_2 \rangle\right|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{1}{\sqrt{5}} \right|^2 = \tfrac{1}{5}$$

$$\mathcal{P}_{2,-} = \left|\langle - | \psi_2 \rangle\right|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{2i}{\sqrt{5}} \right|^2 = \tfrac{4}{5}$$

$$\mathcal{P}_{2,+x} = \left|{}_x\langle + | \psi_2 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{1}{\sqrt{10}} + \tfrac{2i}{\sqrt{10}} \right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{2,+x} = \left|{}_x\langle - | \psi_2 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{1}{\sqrt{10}} - \tfrac{2i}{\sqrt{10}} \right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{2,+y} = \left|{}_y\langle + | \psi_2 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{1}{\sqrt{10}} + \tfrac{2}{\sqrt{10}} \right|^2 = \tfrac{9}{10}$$

$$\mathcal{P}_{2,-y} = \left|{}_y\langle - | \psi_2 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \tfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \tfrac{1}{\sqrt{10}} - \tfrac{2}{\sqrt{10}} \right|^2 = \tfrac{1}{10}$$

$$\mathcal{P}_{3,+} = \left|\langle + | \psi_3 \rangle\right|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{3}{\sqrt{10}} \right|^2 = \tfrac{9}{10}$$

$$\mathcal{P}_{3,-} = \left|\langle - | \psi_3 \rangle\right|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{-e^{i\pi/3}}{\sqrt{10}} \right|^2 = \tfrac{1}{10}$$

$$\mathcal{P}_{3,+x} = \left|{}_x\langle + | \psi_3 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{3}{\sqrt{20}} - \tfrac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \tfrac{7}{20}$$

$$\mathcal{P}_{3,+x} = \left|{}_x\langle - | \psi_3 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{3}{\sqrt{20}} + \tfrac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \tfrac{13}{20}$$

$$\mathcal{P}_{3,+y} = \left|{}_y\langle + | \psi_3 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{3}{\sqrt{20}} + \tfrac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \tfrac{1}{20}\left(10 - 3\sqrt{3}\right) \cong 0.24$$

$$\mathcal{P}_{3,-y} = \left|{}_y\langle - | \psi_3 \rangle\right|^2 = \left| \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \tfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \tfrac{3}{\sqrt{20}} - \tfrac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \tfrac{1}{20}\left(10 + 3\sqrt{3}\right) \cong 0.76$$

1.2 a)
State 1

$$|\psi_1\rangle = \tfrac{1}{\sqrt{3}}|+\rangle + i\tfrac{\sqrt{2}}{\sqrt{3}}|-\rangle$$
$$|\phi_1\rangle = a|+\rangle + b|-\rangle$$
$$\langle\phi_1|\psi_1\rangle = 0 \quad \Rightarrow \quad \left(a^*\langle+| + b^*\langle-|\right)\left(\tfrac{1}{\sqrt{3}}|+\rangle + i\tfrac{\sqrt{2}}{\sqrt{3}}|-\rangle\right) = 0$$
$$a^*\tfrac{1}{\sqrt{3}} + ib^*\tfrac{\sqrt{2}}{\sqrt{3}} = 0 \quad \Rightarrow \quad a^* = -ib^*\sqrt{2}$$
$$|a|^2 + |b|^2 = 1 \quad \Rightarrow \quad |a|^2 + \tfrac{|a|^2}{2} = 1 \quad \Rightarrow \quad a = \tfrac{\sqrt{2}}{\sqrt{3}}$$
$$|\phi_1\rangle = \tfrac{\sqrt{2}}{\sqrt{3}}|+\rangle - i\tfrac{1}{\sqrt{3}}|-\rangle$$

State 2

$$|\psi_2\rangle = \tfrac{1}{\sqrt{5}}|+\rangle - \tfrac{2}{\sqrt{5}}|-\rangle$$
$$|\phi_2\rangle = a|+\rangle + b|-\rangle$$
$$\langle\phi_2|\psi_2\rangle = 0 \quad \Rightarrow \quad \left(a^*\langle+| + b^*\langle-|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle - \tfrac{2}{\sqrt{5}}|-\rangle\right) = 0$$
$$a^*\tfrac{1}{\sqrt{5}} - b^*\tfrac{2}{\sqrt{5}} = 0 \quad \Rightarrow \quad a^* = b^*2$$
$$|a|^2 + |b|^2 = 1 \quad \Rightarrow \quad |a|^2 + \tfrac{|a|^2}{4} = 1 \quad \Rightarrow \quad a = \tfrac{2}{\sqrt{5}}$$
$$|\phi_2\rangle = \tfrac{2}{\sqrt{5}}|+\rangle + \tfrac{1}{\sqrt{5}}|-\rangle$$

State 3

$$|\psi_3\rangle = \tfrac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\tfrac{1}{\sqrt{2}}|-\rangle$$
$$|\phi_3\rangle = a|+\rangle + b|-\rangle$$
$$\langle\phi_3|\psi_3\rangle = 0 \quad \Rightarrow \quad \left(a^*\langle+| + b^*\langle-|\right)\left(\tfrac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\tfrac{1}{\sqrt{2}}|-\rangle\right) = 0$$
$$a^*\tfrac{1}{\sqrt{2}} + e^{i\pi/4}b^*\tfrac{\sqrt{1}}{\sqrt{2}} = 0 \quad \Rightarrow \quad a^* = -e^{i\pi/4}b^* \quad \Rightarrow \quad b = -ae^{i\pi/4}$$
$$|a|^2 + |b|^2 = 1 \quad \Rightarrow \quad |a|^2 + |a|^2 = 1 \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}$$
$$|\phi_3\rangle = \tfrac{1}{\sqrt{2}}|+\rangle - e^{i\pi/4}\tfrac{1}{\sqrt{2}}|-\rangle$$

b) Inner products

$$\langle\psi_1|\psi_1\rangle = \left(\tfrac{1}{\sqrt{3}}\langle+| - i\tfrac{\sqrt{2}}{\sqrt{3}}\langle-|\right)\left(\tfrac{1}{\sqrt{3}}|+\rangle + i\tfrac{\sqrt{2}}{\sqrt{3}}|-\rangle\right) = \tfrac{1}{3} + \tfrac{2}{3} = 1$$
$$\langle\psi_1|\psi_2\rangle = \left(\tfrac{1}{\sqrt{3}}\langle+| - i\tfrac{\sqrt{2}}{\sqrt{3}}\langle-|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle - \tfrac{2}{\sqrt{5}}|-\rangle\right) = \tfrac{1}{\sqrt{15}} + \tfrac{2\sqrt{2}i}{\sqrt{15}} = \tfrac{1}{\sqrt{15}}\left(1 + i2\sqrt{2}\right)$$
$$\langle\psi_1|\psi_3\rangle = \left(\tfrac{1}{\sqrt{3}}\langle+| - i\tfrac{\sqrt{2}}{\sqrt{3}}\langle-|\right)\left(\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{e^{i\pi/4}}{\sqrt{2}}|-\rangle\right) = \tfrac{1}{\sqrt{6}} - \tfrac{\sqrt{2}ie^{i\pi/4}}{\sqrt{6}} = \tfrac{1}{\sqrt{6}}(2 - i)$$
$$\langle\psi_2|\psi_1\rangle = \left(\tfrac{1}{\sqrt{5}}\langle+| - \tfrac{2}{\sqrt{5}}\langle-|\right)\left(\tfrac{1}{\sqrt{3}}|+\rangle + i\tfrac{\sqrt{2}}{\sqrt{3}}|-\rangle\right) = \tfrac{1}{\sqrt{15}} - \tfrac{2i}{\sqrt{15}} = \tfrac{1}{\sqrt{15}}\left(1 - i2\sqrt{2}\right)$$
$$\langle\psi_2|\psi_2\rangle = \left(\tfrac{1}{\sqrt{5}}\langle+| - \tfrac{2}{\sqrt{5}}\langle-|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle - \tfrac{2}{\sqrt{5}}|-\rangle\right) = \tfrac{1}{5} + \tfrac{4}{5} = 1$$
$$\langle\psi_2|\psi_3\rangle = \left(\tfrac{1}{\sqrt{5}}\langle+| - \tfrac{2}{\sqrt{5}}\langle-|\right)\left(\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{e^{i\pi/4}}{\sqrt{2}}|-\rangle\right) = \tfrac{1}{\sqrt{10}} - \tfrac{2e^{i\pi/4}}{\sqrt{10}} = \tfrac{1}{\sqrt{10}}\left(1 - \sqrt{2} - i\sqrt{2}\right)$$
$$\langle\psi_3|\psi_1\rangle = \left(\tfrac{1}{\sqrt{2}}\langle+| + \tfrac{e^{-i\pi/4}}{\sqrt{2}}\langle-|\right)\left(\tfrac{1}{\sqrt{3}}|+\rangle + i\tfrac{\sqrt{2}}{\sqrt{3}}|-\rangle\right) = \tfrac{1}{\sqrt{6}} + \tfrac{\sqrt{2}ie^{-i\pi/4}}{\sqrt{6}} = \tfrac{1}{\sqrt{6}}(2 + i)$$
$$\langle\psi_3|\psi_2\rangle = \left(\tfrac{1}{\sqrt{2}}\langle+| + \tfrac{e^{-i\pi/4}}{\sqrt{2}}\langle-|\right)\left(\tfrac{1}{\sqrt{5}}|+\rangle - \tfrac{2}{\sqrt{5}}|-\rangle\right) = \tfrac{1}{\sqrt{10}} - \tfrac{2e^{-i\pi/4}}{\sqrt{10}} = \tfrac{1}{\sqrt{10}}\left(1 - \sqrt{2} + i\sqrt{2}\right)$$
$$\langle\psi_3|\psi_3\rangle = \left(\tfrac{1}{\sqrt{2}}\langle+| + \tfrac{e^{-i\pi/4}}{\sqrt{2}}\langle-|\right)\left(\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{e^{i\pi/4}}{\sqrt{2}}|-\rangle\right) = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

1.3 Probability of measuring a_{n} in state $|\psi\rangle$ is

$$\mathcal{P}_{a_n} = \left|\langle a_n|\psi\rangle\right|^2$$

Probability of same measurement if state is changed to $e^{i\delta}|\psi\rangle$ is

$$\begin{aligned}\mathcal{P}_{a_n,NEW} &= \left|\langle a_n|e^{i\delta}\psi\rangle\right|^2 \\ &= \left|e^{i\delta}\langle a_n|\psi\rangle\right|^2 \\ &= \left|\langle a_n|\psi\rangle\right|^2\end{aligned}$$

So the probability is unchanged.

1.4

$$|+\rangle_x = a|+\rangle + b|-\rangle$$
$$|-\rangle_x = c|+\rangle + d|-\rangle$$

$$\mathcal{P}_{1,-x} = \left|{}_x\langle-|+\rangle\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,+x} = \left|{}_x\langle+|-\rangle\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,-x} = \left|{}_x\langle-|-\rangle\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{1,-x} = \left|{}_x\langle-|+\rangle\right|^2 = \left|\left\{c^*\langle+| + d^*\langle-|\right\}|+\rangle\right|^2 = \left|c^*\right|^2 = |c|^2 \quad\Rightarrow\quad |c|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,+x} = \left|{}_x\langle+|-\rangle\right|^2 = \left|\left\{a^*\langle+| + b^*\langle-|\right\}|-\rangle\right|^2 = \left|b^*\right|^2 = |b|^2 \quad\Rightarrow\quad |b|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,-x} = \left|{}_x\langle-|-\rangle\right|^2 = \left|\left\{c^*\langle+| + d^*\langle-|\right\}|-\rangle\right|^2 = \left|d^*\right|^2 = |d|^2 \quad\Rightarrow\quad |d|^2 = \tfrac{1}{2}$$

1.5 a) Possible results of a measurement of the spin component S_z are always $\pm\hbar/2$ for a spin-½ particle. Probabilities are

$$\mathcal{P}_{+\hbar/2} = \left|\langle +|\psi\rangle\right|^2 = \left|\langle +|\left(\tfrac{2}{\sqrt{13}}|+\rangle + i\tfrac{3}{\sqrt{13}}|-\rangle\right)\right|^2 = \left|\tfrac{2}{\sqrt{13}}\right|^2 = \tfrac{4}{13}$$

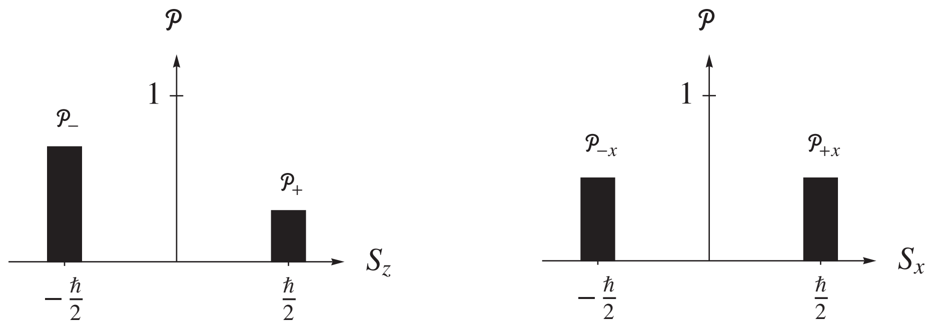
$$\mathcal{P}_{-\hbar/2} = \left|\langle -|\psi\rangle\right|^2 = \left|\langle -|\left(\tfrac{2}{\sqrt{13}}|+\rangle + i\tfrac{3}{\sqrt{13}}|-\rangle\right)\right|^2 = \left|\tfrac{3i}{\sqrt{13}}\right|^2 = \tfrac{9}{13}$$

b) Possible results of a measurement of the spin component S_x are always $\pm\hbar/2$ for a spin-½ particle. Probabilities are

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{2}{\sqrt{13}}|+\rangle + i\tfrac{3}{\sqrt{13}}|-\rangle\right)\right|^2 = \left|\tfrac{2}{\sqrt{26}} + i\tfrac{3}{\sqrt{26}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{2}{\sqrt{13}}|+\rangle + i\tfrac{3}{\sqrt{13}}|-\rangle\right)\right|^2 = \left|\tfrac{2}{\sqrt{26}} - i\tfrac{3}{\sqrt{26}}\right|^2 = \tfrac{1}{2}$$

c) Histogram:



1.6 a) Possible results of a measurement of the spin component S_z are always $\pm\hbar/2$ for a spin-½ particle. Probabilities are

$$\mathcal{P}_{+\hbar/2} = \left|\langle +|\psi\rangle\right|^2 = \left|\langle +|\left(\tfrac{2}{\sqrt{13}}|+\rangle_x + i\tfrac{3}{\sqrt{13}}|-\rangle_x\right)\right|^2 = \left|\tfrac{2}{\sqrt{26}} + i\tfrac{3}{\sqrt{26}}\right|^2 = \tfrac{1}{2}$$

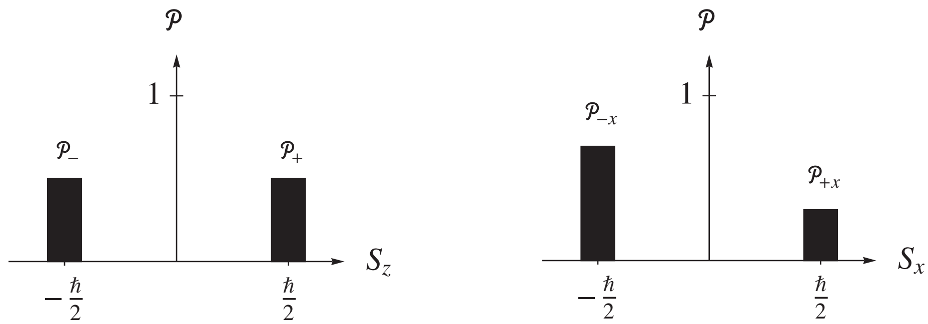
$$\mathcal{P}_{-\hbar/2} = \left|\langle -|\psi\rangle\right|^2 = \left|\langle -|\left(\tfrac{2}{\sqrt{13}}|+\rangle_x + i\tfrac{3}{\sqrt{13}}|-\rangle_x\right)\right|^2 = \left|\tfrac{2}{\sqrt{26}} - i\tfrac{3}{\sqrt{26}}\right|^2 = \tfrac{1}{2}$$

b) Possible results of a measurement of the spin component S_x are always $\pm\hbar/2$ for a spin-½ particle. Probabilities are

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi\rangle\right|^2 = \left|{}_x\langle +|\left(\tfrac{2}{\sqrt{13}}|+\rangle_x + i\tfrac{3}{\sqrt{13}}|-\rangle_x\right)\right|^2 = \left|\tfrac{2}{\sqrt{13}}\right|^2 = \tfrac{4}{13}$$

$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi\rangle\right|^2 = \left|{}_x\langle -|\left(\tfrac{2}{\sqrt{13}}|+\rangle_x + i\tfrac{3}{\sqrt{13}}|-\rangle_x\right)\right|^2 = \left|\tfrac{3i}{\sqrt{13}}\right|^2 = \tfrac{9}{13}$$

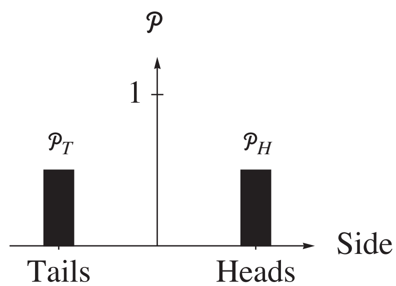
c) Histogram:



1.7 a) Heads or tails: H or T
b) Each result is equally likely so

$$\mathcal{P}_H = \tfrac{1}{2}$$
$$\mathcal{P}_T = \tfrac{1}{2}$$

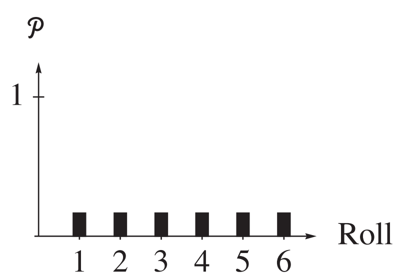
c) Histogram:



1.8 a) Six sides with 1, 2, 3, 4, 5, or 6 dots.
b) Each result is equally likely so

$$\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = \mathcal{P}_4 = \mathcal{P}_5 = \mathcal{P}_6 = \tfrac{1}{6}$$

c) Histogram:



1.9 a) 36 possible die combinations with 11 possible numerical results:

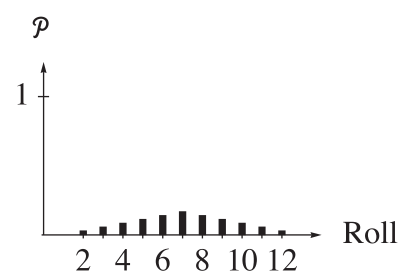
$$
\begin{aligned}
&2 = 1+1\\
&3 = 1+2, 2+1\\
&4 = 1+3, 2+2, 3+1\\
&5 = 1+4, 2+3, 3+2, 4+1\\
&6 = 1+5, 2+4, 3+3, 4+2, 5+1\\
&7 = 1+6, 2+5, 3+4, 4+3, 5+2, 6+1\\
&8 = 2+6, 3+5, 4+4, 5+3, 6+2\\
&9 = 3+6, 4+5, 5+4, 6+3\\
&10 = 4+6, 5+5, 6+4\\
&11 = 5+6, 6+5\\
&12 = 6+6
\end{aligned}
$$

b) Each possible die combination is equally likely, so the probabilities of the numerical results are the number of possible combinations divided by 36:

$$
\begin{aligned}
&\mathcal{P}_2 = \tfrac{1}{36},\ \mathcal{P}_3 = \tfrac{2}{36} = \tfrac{1}{18},\ \mathcal{P}_4 = \tfrac{3}{36} = \tfrac{1}{12},\ \mathcal{P}_5 = \tfrac{4}{36} = \tfrac{1}{9},\ \mathcal{P}_6 = \tfrac{5}{36},\ \mathcal{P}_7 = \tfrac{6}{36} = \tfrac{1}{6},\\
&\mathcal{P}_8 = \tfrac{5}{36},\ \mathcal{P}_9 = \tfrac{4}{36} = \tfrac{1}{9},\ \mathcal{P}_{10} = \tfrac{3}{36} = \tfrac{1}{12},\ \mathcal{P}_{11} = \tfrac{2}{36} = \tfrac{1}{18},\ \mathcal{P}_{12} = \tfrac{1}{36}
\end{aligned}
$$

Note that the sum of the probabilities is unity as it must be.

c) Histogram:



1.10 a) The probabilities for state 1 are

$$
\begin{aligned}
\mathcal{P}_{1,+} &= \left|\langle +|\psi_1\rangle\right|^2 = \left|\langle +|\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}\\
\mathcal{P}_{1,-} &= \left|\langle -|\psi_1\rangle\right|^2 = \left|\langle -|\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|i\tfrac{3}{5}\right|^2 = \tfrac{9}{25}\\
\mathcal{P}_{1,+x} &= \left|{}_x\langle +|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}\\
\mathcal{P}_{1,-x} &= \left|{}_x\langle -|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}\\
\mathcal{P}_{1,+y} &= \left|{}_y\langle +|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{49}{50}\\
\mathcal{P}_{1,-y} &= \left|{}_y\langle -|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\tfrac{4}{5}|+\rangle + i\tfrac{3}{5}|-\rangle\right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{50}
\end{aligned}
$$

The probabilities for state 2 are

$$\mathcal{P}_{2,+} = \left|\langle + | \psi_2 \rangle\right|^2 = \left|\langle + | \left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}$$
$$\mathcal{P}_{2,-} = \left|\langle - | \psi_2 \rangle\right|^2 = \left|\langle - | \left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-i\tfrac{3}{5}\right|^2 = \tfrac{9}{25}$$
$$\mathcal{P}_{2,+x} = \left|{}_x\langle + | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -| \right)\left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,-x} = \left|{}_x\langle - | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -| \right)\left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{2,+y} = \left|{}_y\langle + | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -| \right)\left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{50}$$
$$\mathcal{P}_{2,-y} = \left|{}_y\langle - | \psi_2 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -| \right)\left(\tfrac{4}{5} |+\rangle - i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{49}{50}$$

The probabilities for state 3 are

$$\mathcal{P}_{3,+} = \left|\langle + | \psi_3 \rangle\right|^2 = \left|\langle + | \left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-\tfrac{4}{5}\right|^2 = \tfrac{16}{25}$$
$$\mathcal{P}_{3,-} = \left|\langle - | \psi_3 \rangle\right|^2 = \left|\langle - | \left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|i\tfrac{3}{5}\right|^2 = \tfrac{9}{25}$$
$$\mathcal{P}_{3,+x} = \left|{}_x\langle + | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -| \right)\left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{3,-x} = \left|{}_x\langle - | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -| \right)\left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{i}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{3,+y} = \left|{}_y\langle + | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -| \right)\left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\tfrac{4}{5} + \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{1}{50}$$
$$\mathcal{P}_{3,-y} = \left|{}_y\langle - | \psi_3 \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -| \right)\left(-\tfrac{4}{5} |+\rangle + i\tfrac{3}{5} |-\rangle \right)\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\tfrac{4}{5} - \tfrac{1}{\sqrt{2}}\tfrac{3}{5}\right|^2 = \tfrac{49}{50}$$

b) States 2 and 3 differ only by an overall phase of $e^{i\pi} = -1$, so the measurement results are the same; the states are physically indistinguishable. States 1 and 2 have different relative phases between the coefficients, so they produce different results.

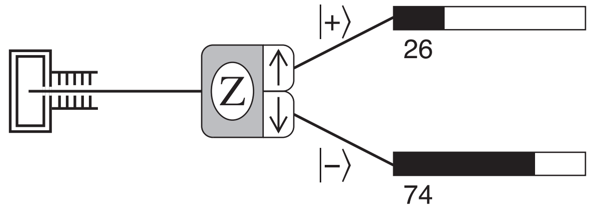
1.11 a) Possible results of a measurement of the spin component S_z are always $\pm\hbar/2$ for a spin-½ particle. Probabilities are

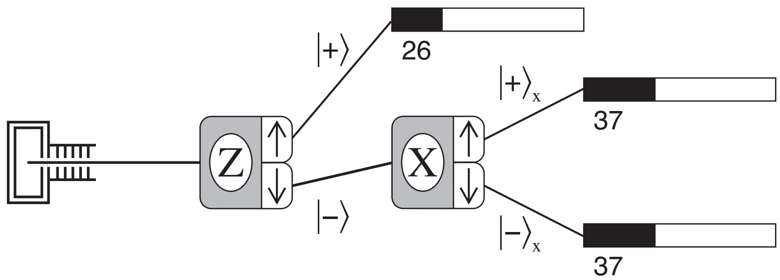
$$\mathcal{P}_{+\hbar/2} = \left|\langle + | \psi \rangle\right|^2 = \left|\langle + | \left(\tfrac{3}{\sqrt{34}} |+\rangle + i\tfrac{5}{\sqrt{34}} |-\rangle \right)\right|^2 = \left|\tfrac{3}{\sqrt{34}}\right|^2 = \tfrac{9}{34} \cong 0.26$$
$$\mathcal{P}_{-\hbar/2} = \left|\langle - | \psi \rangle\right|^2 = \left|\langle - | \left(\tfrac{3}{\sqrt{34}} |+\rangle + i\tfrac{5}{\sqrt{34}} |-\rangle \right)\right|^2 = \left|i\tfrac{5}{\sqrt{34}}\right|^2 = \tfrac{25}{34} \cong 0.74$$

b) After the measurement result of the spin component S_z is $-\hbar/2$, the system is in the $|-\rangle$ eigenstate corresponding to that result. The possible results of a measurement of the spin component S_x are always $\pm\hbar/2$ for a spin-½ particle. The probabilities are

$$\mathcal{P}_{+x} = \left|{}_x\langle + | \psi_{after} \rangle\right|^2 = \left|{}_x\langle + | - \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -| \right)|-\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{-x} = \left|{}_x\langle - | \psi_{after} \rangle\right|^2 = \left|{}_x\langle - | - \rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -| \right)|-\rangle\right|^2 = \left|-\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

c) Diagrams





1.12 For a system with three possible measurement results: a_1, a_2, and a_3, the three eigenstates are $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$

Orthogonality:

$$\langle a_1|a_2\rangle = 0$$
$$\langle a_1|a_3\rangle = 0$$
$$\langle a_2|a_3\rangle = 0$$

Normalization:

$$\langle a_1|a_1\rangle = 1$$
$$\langle a_2|a_2\rangle = 1$$
$$\langle a_3|a_3\rangle = 1$$

Completeness:

$$|\psi\rangle = c_1|a_1\rangle + c_2|a_2\rangle + c_3|a_3\rangle$$

1.13 a) For a system with three possible measurement results: a_1, a_2, and a_3, the three eigenstates $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ are

$$|a_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |a_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |a_3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

b) In matrix notation, the state is

$$|\psi\rangle \doteq \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

The state given is not normalized, so first we normalize it:

$$|\psi\rangle = C\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$1 = \langle\psi|\psi\rangle = C^*\begin{pmatrix} 1 & -2 & 5 \end{pmatrix}C\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = C^*C(1+4+25) = 1 \quad \Rightarrow \quad C = 1/\sqrt{30}$$

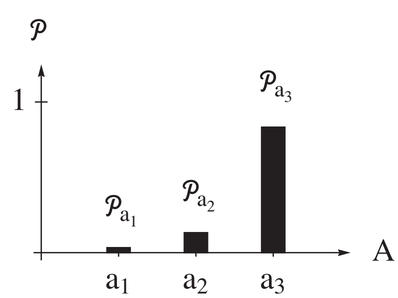
The probabilities are

$$\mathcal{P}_{a_1} = |\langle a_1|\psi\rangle|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{30}}\right|^2 = \tfrac{1}{30}$$

$$\mathcal{P}_{a_2} = |\langle a_2|\psi\rangle|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}\right|^2 = \left|-\tfrac{2}{\sqrt{30}}\right|^2 = \tfrac{4}{30}$$

$$\mathcal{P}_{a_3} = |\langle a_3|\psi\rangle|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}\right|^2 = \left|\tfrac{5}{\sqrt{30}}\right|^2 = \tfrac{25}{30}$$

Histogram:



c) In matrix notation, the state is

$$|\psi\rangle \doteq \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}$$

The state given is not normalized, so first we normalize it:

$$|\psi\rangle = C\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = C^*\begin{pmatrix} 2 & -3i & 0 \end{pmatrix} C\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} = C^*C(4+9+0) = 1 \quad \Rightarrow \quad C = 1/\sqrt{13}$$

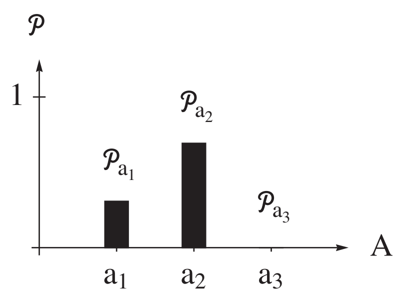
The probabilities are

$$\mathcal{P}_{a_1} = |\langle a_1|\psi\rangle|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\tfrac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 = \left|\tfrac{2}{\sqrt{13}}\right|^2 = \tfrac{4}{13}$$

$$\mathcal{P}_{a_2} = |\langle a_2|\psi\rangle|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 = \left|\tfrac{3i}{\sqrt{13}}\right|^2 = \tfrac{9}{13}$$

$$\mathcal{P}_{a_3} = |\langle a_3|\psi\rangle|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\tfrac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 = \left|\tfrac{0}{\sqrt{13}}\right|^2 = 0$$

Histogram:



1.14. There are four possible measurement results: $2\,eV$, $4\,eV$, $7\,eV$, and $9\,eV$. The probabilities are

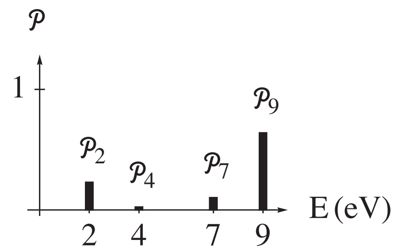
$$\mathcal{P}_{2\ eV} = |\langle 2\ eV|\psi\rangle|^2 = \left|\langle 2\ eV|\frac{1}{\sqrt{39}}\left\{3|2\ eV\rangle - i|4\ eV\rangle + 2e^{i\pi/7}|7\ eV\rangle + 5|9\ eV\rangle\right\}\right|^2 = \tfrac{9}{39}$$

$$\mathcal{P}_{4\ eV} = |\langle 4\ eV|\psi\rangle|^2 = \left|\langle 4\ eV|\frac{1}{\sqrt{39}}\left\{3|2\ eV\rangle - i|4\ eV\rangle + 2e^{i\pi/7}|7\ eV\rangle + 5|9\ eV\rangle\right\}\right|^2 = \tfrac{1}{39}$$

$$\mathcal{P}_{7\ eV} = \left|\langle 7\ eV|\psi\rangle\right|^2 = \left|\langle 2\ eV|\frac{1}{\sqrt{39}}\left\{3|2\ eV\rangle - i|4\ eV\rangle + 2e^{i\pi/7}|7\ eV\rangle + 5|9\ eV\rangle\right\}\right|^2 = \tfrac{4}{39}$$

$$\mathcal{P}_{9\ eV} = \left|\langle 9\ eV|\psi\rangle\right|^2 = \left|\langle 2\ eV|\frac{1}{\sqrt{39}}\left\{3|2\ eV\rangle - i|4\ eV\rangle + 2e^{i\pi/7}|7\ eV\rangle + 5|9\ eV\rangle\right\}\right|^2 = \tfrac{25}{39}$$

Histogram:



1.15 The probability is

$$\mathcal{P}_{\psi_f} = \left|\langle\psi_f|\psi_i\rangle\right|^2 = \left|\left(\tfrac{1-i}{\sqrt{3}}\langle a_1| + \tfrac{1}{\sqrt{6}}\langle a_2| + \tfrac{1}{\sqrt{6}}\langle a_3|\right)\left(\tfrac{i}{\sqrt{3}}|a_1\rangle + \sqrt{\tfrac{2}{3}}|a_2\rangle\right)\right|^2$$

$$= \left|\tfrac{i}{\sqrt{3}}\tfrac{1-i}{\sqrt{3}} + \sqrt{\tfrac{2}{3}}\tfrac{1}{\sqrt{6}}\right|^2 = \left|\tfrac{i}{3} + \tfrac{1}{3} + \tfrac{1}{3}\right|^2 = \tfrac{1}{9} + \tfrac{4}{9} = \tfrac{5}{9}$$

1.16 The measured probabilities are

$$\mathcal{P}_+ = \tfrac{1}{2} \quad \mathcal{P}_{+x} = \tfrac{3}{4} \quad \mathcal{P}_{+y} = 0.067$$
$$\mathcal{P}_- = \tfrac{1}{2} \quad \mathcal{P}_{-x} = \tfrac{1}{4} \quad \mathcal{P}_{-y} = 0.933$$

Write the input state as

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_+ = \left|\langle+|\psi\rangle\right|^2 = \left|\langle+|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |a|^2 = \tfrac{1}{2} \quad\Rightarrow\quad a = \tfrac{1}{\sqrt{2}}$$
$$\mathcal{P}_- = \left|\langle-|\psi\rangle\right|^2 = \left|\langle-|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |b|^2 = \tfrac{1}{2} \quad\Rightarrow\quad b = \tfrac{1}{\sqrt{2}}e^{i\phi}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\mathcal{P}_{+x} = \left|{}_x\langle+|\psi\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle+| + \langle-|\right\}\tfrac{1}{\sqrt{2}}\left\{|+\rangle + e^{i\phi}|-\rangle\right\}\right|^2 = \left|\tfrac{1}{2}\left\{1 + e^{i\phi}\right\}\right|^2$$
$$= \tfrac{1}{4}\left\{1 + e^{i\phi}\right\}\left\{1 + e^{-i\phi}\right\} = \tfrac{1}{4}\left\{1 + 1 + e^{i\phi} + e^{-i\phi}\right\} = \tfrac{1}{2}\left\{1 + \cos\phi\right\} = \tfrac{3}{4}$$
$$\cos\phi = \tfrac{1}{2} \quad\Rightarrow\quad \phi = \tfrac{\pi}{3} \quad or \quad \tfrac{5\pi}{3}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{+y} &= \left|{}_y\langle +|\psi\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle +| - i\langle -|\right\}\tfrac{1}{\sqrt{2}}\left\{|+\rangle + e^{i\phi}|-\rangle\right\}\right|^2 = \left|\tfrac{1}{2}\left\{1 - ie^{i\phi}\right\}\right|^2 \\
&= \tfrac{1}{4}\left\{1 - ie^{i\phi}\right\}\left\{1 + ie^{-i\phi}\right\} = \tfrac{1}{4}\left\{1 + 1 - ie^{i\phi} + ie^{-i\phi}\right\} = \tfrac{1}{2}\left\{1 + \sin\phi\right\} = 0.067 \\
\sin\phi &= -0.866 \quad \Rightarrow \quad \phi = \tfrac{4\pi}{3} \quad or \quad \tfrac{5\pi}{3} \quad \Rightarrow \quad \phi = \tfrac{5\pi}{3}
\end{aligned}$$

Hence the input state is

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{5\pi}{3}}|-\rangle\right) = |+\rangle_{\hat{\mathbf{n}}\left(\theta=\frac{\pi}{2},\ \phi=\frac{5\pi}{3}\right)}$$

1.17 Follow the solution method given in the lab handout. (i) For unknown number 1, the measured probabilities are

$$\begin{aligned}
&\mathcal{P}_{+} = 1 \quad \mathcal{P}_{+x} = \tfrac{1}{2} \quad \mathcal{P}_{+y} = \tfrac{1}{2} \\
&\mathcal{P}_{-} = 0 \quad \mathcal{P}_{-x} = \tfrac{1}{2} \quad \mathcal{P}_{-y} = \tfrac{1}{2}
\end{aligned}$$

Write the unknown state as

$$|\psi_1\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{+} &= \left|\langle +|\psi_1\rangle\right|^2 = \left|\langle +|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |a|^2 = 1 \quad \Rightarrow \quad a = 1 \\
\mathcal{P}_{-} &= \left|\langle -|\psi_1\rangle\right|^2 = \left|\langle -|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |b|^2 = 0 \quad \Rightarrow \quad b = 0
\end{aligned}$$

Hence the unknown state is

$$|\psi_1\rangle = |+\rangle$$

which produces the probabilities

$$\begin{aligned}
\mathcal{P}_{+} &= \left|\langle +|\psi_1\rangle\right|^2 = \left|\langle +|+\rangle\right|^2 = 1 \\
\mathcal{P}_{-} &= \left|\langle -|\psi_1\rangle\right|^2 = \left|\langle 1|+\rangle\right|^2 = 0 \\
\mathcal{P}_{+x} &= \left|{}_x\langle +|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \tfrac{1}{2} \\
\mathcal{P}_{-x} &= \left|{}_x\langle -|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \tfrac{1}{2} \\
\mathcal{P}_{+y} &= \left|{}_y\langle +|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \tfrac{1}{2} \\
\mathcal{P}_{-y} &= \left|{}_y\langle -|\psi_1\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \tfrac{1}{2}
\end{aligned}$$

in agreement with the experiment.

(ii) For unknown number 2, the measured probabilities are

$$\begin{array}{lll} \mathcal{P}_{+} = \tfrac{1}{2} & \mathcal{P}_{+x} = \tfrac{1}{2} & \mathcal{P}_{+y} = 0 \\ \mathcal{P}_{-} = \tfrac{1}{2} & \mathcal{P}_{-x} = \tfrac{1}{2} & \mathcal{P}_{-y} = 1 \end{array}$$

Write the unknown state as

$$|\psi_2\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_{+} = \left|\langle +|\psi_2\rangle\right|^2 = \left|\langle +|\{a|+\rangle + b|-\rangle\}\right|^2 = |a|^2 = \tfrac{1}{2} \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}$$
$$\mathcal{P}_{-} = \left|\langle -|\psi_2\rangle\right|^2 = \left|\langle -|\{a|+\rangle + b|-\rangle\}\right|^2 = |b|^2 = \tfrac{1}{2} \quad \Rightarrow \quad b = \tfrac{1}{\sqrt{2}}e^{i\phi}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} \mathcal{P}_{+x} &= \left|{}_x\langle +|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\{\langle +| + \langle -|\}\tfrac{1}{\sqrt{2}}\{|+\rangle + e^{i\phi}|-\rangle\}\right|^2 = \left|\tfrac{1}{2}\{1 + e^{i\phi}\}\right|^2 \\ &= \tfrac{1}{4}\{1 + e^{i\phi}\}\{1 + e^{-i\phi}\} = \tfrac{1}{4}\{1 + 1 + e^{i\phi} + e^{-i\phi}\} = \tfrac{1}{2}\{1 + \cos\phi\} = \tfrac{1}{2} \end{aligned}$$
$$\cos\phi = 0 \quad \Rightarrow \quad \phi = \tfrac{\pi}{2} \ \ or \ \ \tfrac{3\pi}{2}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} \mathcal{P}_{+y} &= \left|{}_y\langle +|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\{\langle +| - i\langle -|\}\tfrac{1}{\sqrt{2}}\{|+\rangle + e^{i\phi}|-\rangle\}\right|^2 = \left|\tfrac{1}{2}\{1 - ie^{i\phi}\}\right|^2 \\ &= \tfrac{1}{4}\{1 - ie^{i\phi}\}\{1 + ie^{-i\phi}\} = \tfrac{1}{4}\{1 + 1 - ie^{i\phi} + ie^{-i\phi}\} = \tfrac{1}{2}\{1 + \sin\phi\} = 0 \end{aligned}$$
$$\sin\phi = -1 \quad \Rightarrow \quad \phi = \tfrac{3\pi}{2}$$

Hence the unknown state is

$$|\psi_2\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{3\pi}{2}}|-\rangle\right) = \tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right) = |-\rangle_y$$

which produces the probabilities

$$\begin{aligned} \mathcal{P}_{+} &= \left|\langle +|\psi_2\rangle\right|^2 = \left|\langle +|\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{-} &= \left|\langle -|\psi_2\rangle\right|^2 = \left|\langle -|\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{+x} &= \left|{}_x\langle +|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{-x} &= \left|{}_x\langle -|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| - \langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{+y} &= \left|{}_y\langle +|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| - i\langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = 0 \\ \mathcal{P}_{-y} &= \left|{}_y\langle -|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| + i\langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)\right|^2 = 1 \end{aligned}$$

in agreement with the experiment.
(iii) For unknown number 3, the measured probabilities are

$$\mathcal{P}_+ = \tfrac{1}{2} \quad \mathcal{P}_{+x} = \tfrac{1}{4} \quad \mathcal{P}_{+y} = 0.067$$
$$\mathcal{P}_- = \tfrac{1}{2} \quad \mathcal{P}_{-x} = \tfrac{3}{4} \quad \mathcal{P}_{-y} = 0.933$$

Write the unknown state as

$$|\psi_3\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_+ = \left|\langle +|\psi_3\rangle\right|^2 = \left|\langle +|\{a|+\rangle + b|-\rangle\}\right|^2 = |a|^2 = \tfrac{1}{2} \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}$$
$$\mathcal{P}_- = \left|\langle -|\psi_3\rangle\right|^2 = \left|\langle -|\{a|+\rangle + b|-\rangle\}\right|^2 = |b|^2 = \tfrac{1}{2} \quad \Rightarrow \quad b = \tfrac{1}{\sqrt{2}}e^{i\phi}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{+x} &= \left|{}_x\langle +|\psi_3\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\{\langle +| + \langle -|\}\tfrac{1}{\sqrt{2}}\{|+\rangle + e^{i\phi}|-\rangle\}\right|^2 = \left|\tfrac{1}{2}\{1 + e^{i\phi}\}\right|^2 \\
&= \tfrac{1}{4}\{1 + e^{i\phi}\}\{1 + e^{-i\phi}\} = \tfrac{1}{4}\{1 + 1 + e^{i\phi} + e^{-i\phi}\} = \tfrac{1}{2}\{1 + \cos\phi\} = \tfrac{1}{4} \\
\cos\phi &= -\tfrac{1}{2} \quad \Rightarrow \quad \phi = \tfrac{2\pi}{3} \quad or \quad \tfrac{4\pi}{3}
\end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{+y} &= \left|{}_y\langle +|\psi_3\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\{\langle +| - i\langle -|\}\tfrac{1}{\sqrt{2}}\{|+\rangle + e^{i\phi}|-\rangle\}\right|^2 = \left|\tfrac{1}{2}\{1 - ie^{i\phi}\}\right|^2 \\
&= \tfrac{1}{4}\{1 - ie^{i\phi}\}\{1 + ie^{-i\phi}\} = \tfrac{1}{4}\{1 + 1 - ie^{i\phi} + ie^{-i\phi}\} = \tfrac{1}{2}\{1 + \sin\phi\} = 0.067 \\
\sin\phi &= -0.866 \quad \Rightarrow \quad \phi = \tfrac{4\pi}{3} \quad or \quad \tfrac{5\pi}{3} \quad \Rightarrow \quad \phi = \tfrac{4\pi}{3}
\end{aligned}$$

Hence the unknown state is

$$|\psi_3\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right) = |+\rangle_{\hat{\mathbf{n}}\left(\theta=\frac{\pi}{2},\ \phi=\frac{4\pi}{3}\right)}$$

which produces the probabilities

$$\mathcal{P}_+ = \left|\langle +|\psi_3\rangle\right|^2 = \left|\langle +|\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_- = \left|\langle -|\psi_3\rangle\right|^2 = \left|\langle -|\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi_3\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}\left(1 + \cos\tfrac{4\pi}{3}\right) = \tfrac{1}{4}$$

$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi_3\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| - \langle -|\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}\left(1 - \cos\tfrac{4\pi}{3}\right) = \tfrac{3}{4}$$

$$\mathcal{P}_{+y} = \left|{}_y\langle + | \psi_3 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle + | - i\langle - |\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}\left(1+\sin\tfrac{4\pi}{3}\right) = \tfrac{1}{2}\left(1-\tfrac{\sqrt{3}}{2}\right) = 0.067$$

$$\mathcal{P}_{-y} = \left|{}_y\langle - | \psi_3 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle + | + i\langle - |\right)\tfrac{1}{\sqrt{2}}\left(|+\rangle + e^{i\frac{4\pi}{3}}|-\rangle\right)\right|^2 = \tfrac{1}{2}\left(1-\sin\tfrac{4\pi}{3}\right) = \tfrac{1}{2}\left(1+\tfrac{\sqrt{3}}{2}\right) = 0.933$$

in agreement with the experiment.
(iv) For unknown number 4, the measured probabilities are

$$\mathcal{P}_{+} = \tfrac{1}{4} \quad \mathcal{P}_{+x} = \tfrac{7}{8} \quad \mathcal{P}_{+y} = 0.283$$
$$\mathcal{P}_{-} = \tfrac{3}{4} \quad \mathcal{P}_{-x} = \tfrac{1}{8} \quad \mathcal{P}_{-y} = 0.717$$

Write the unknown state as

$$|\psi_4\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_{+} = \left|\langle + | \psi_4 \rangle\right|^2 = \left|\langle + |\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |a|^2 = \tfrac{1}{4} \quad \Rightarrow \quad a = \tfrac{1}{2}$$
$$\mathcal{P}_{-} = \left|\langle - | \psi_4 \rangle\right|^2 = \left|\langle - |\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = |b|^2 = \tfrac{3}{4} \quad \Rightarrow \quad b = \tfrac{\sqrt{3}}{2}e^{i\phi}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\mathcal{P}_{+x} = \left|{}_x\langle + | \psi_4 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle + | + \langle - |\right\}\tfrac{1}{2}\left\{|+\rangle + \sqrt{3}e^{i\phi}|-\rangle\right\}\right|^2 = \left|\tfrac{1}{2\sqrt{2}}\left\{1+\sqrt{3}e^{i\phi}\right\}\right|^2$$
$$= \tfrac{1}{8}\left\{1+\sqrt{3}e^{i\phi}\right\}\left\{1+\sqrt{3}e^{-i\phi}\right\} = \tfrac{1}{8}\left\{1+3+\sqrt{3}e^{i\phi}+\sqrt{3}e^{-i\phi}\right\} = \tfrac{1}{4}\left\{2+\sqrt{3}\cos\phi\right\} = \tfrac{7}{8}$$
$$\cos\phi = \tfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \phi = \tfrac{\pi}{6} \quad or \quad \tfrac{11\pi}{6}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\mathcal{P}_{+y} = \left|{}_y\langle + | \psi_4 \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle + | - i\langle - |\right\}\tfrac{1}{2}\left\{|+\rangle + \sqrt{3}e^{i\phi}|-\rangle\right\}\right|^2 = \left|\tfrac{1}{2\sqrt{2}}\left\{1-i\sqrt{3}e^{i\phi}\right\}\right|^2$$
$$= \tfrac{1}{8}\left\{1-i\sqrt{3}e^{i\phi}\right\}\left\{1+i\sqrt{3}e^{-i\phi}\right\} = \tfrac{1}{8}\left\{1+3-i\sqrt{3}e^{i\phi}+i\sqrt{3}e^{-i\phi}\right\} = \tfrac{1}{4}\left\{2+\sqrt{3}\sin\phi\right\} = 0.283$$
$$\sin\phi = -0.50 \quad \Rightarrow \quad \phi = \tfrac{7\pi}{6} \quad or \quad \tfrac{11\pi}{6} \quad \Rightarrow \quad \phi = \tfrac{11\pi}{6}$$

Hence the unknown state is

$$|\psi_4\rangle = \tfrac{1}{2}|+\rangle + \tfrac{\sqrt{3}}{2}e^{i\frac{11\pi}{6}}|-\rangle = \cos\tfrac{\pi}{3}|+\rangle + \sin\tfrac{\pi}{3}e^{i\frac{11\pi}{6}}|-\rangle = |+\rangle_{\hat{\mathbf{n}}\left(\theta=\frac{2\pi}{3},\ \phi=\frac{11\pi}{6}\right)}$$

which produces the probabilities

$$\mathcal{P}_{+} = \left|\langle +|\psi_4\rangle\right|^2 = \left|\langle +|\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{1}{4}$$

$$\mathcal{P}_{-} = \left|\langle -|\psi_4\rangle\right|^2 = \left|\langle -|\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{3}{4}$$

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi_4\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{1}{4}\left(2 + \sqrt{3}\cos\tfrac{11\pi}{6}\right) = \tfrac{7}{8}$$

$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi_4\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| - \langle -|\right)\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{1}{4}\left(2 - \sqrt{3}\cos\tfrac{11\pi}{6}\right) = \tfrac{1}{8}$$

$$\mathcal{P}_{+y} = \left|{}_y\langle +|\psi_4\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| - i\langle -|\right)\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{1}{4}\left(2 + \sqrt{3}\sin\tfrac{11\pi}{6}\right) = \tfrac{1}{4}\left(2 - \tfrac{\sqrt{3}}{2}\right) = 0.283$$

$$\mathcal{P}_{-y} = \left|{}_y\langle -|\psi_4\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle +| + i\langle -|\right)\tfrac{1}{2}\left(|+\rangle + \sqrt{3}e^{i\frac{11\pi}{6}}|-\rangle\right)\right|^2 = \tfrac{1}{4}\left(2 - \sqrt{3}\sin\tfrac{11\pi}{6}\right) = \tfrac{1}{4}\left(2 + \tfrac{\sqrt{3}}{2}\right) = 0.717$$

in agreement with the experiment.