6.1 Use eV and nm as the primary units, so use $E_{rest} = mc^2$ where possible. For a nonrelativistic particle

$$\lambda_{deBroglie} = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{hc}{\sqrt{2mc^2 E}} \frac{hc}{\sqrt{2mc^2 \frac{1}{2} mv^2}} = \frac{hc}{mc^2 (v/c)}$$

a) 3 eV electron

$$\lambda_{dB} = \frac{hc}{\sqrt{2mc^2 E}} = \frac{1240\ eV\ nm}{\sqrt{2 \bullet 511\ keV \bullet 3eV}} = 0.71\ nm$$

b) 7 MeV proton

$$\lambda_{dB} = \frac{hc}{\sqrt{2mc^2 E}} = \frac{1240\ eV\ nm}{\sqrt{2 \bullet 938\ MeV \bullet 7\ MeV}} = 1.1 \times 10^{-5}\ nm$$

c) 200 m/s C_{60} Buckyball

$$\lambda_{dB} = \frac{hc}{mc^2 (v/c)} = \frac{1240\ eV\ nm}{(60 \bullet 12 \bullet 938\ MeV)(200/3 \times 10^8)} = 2.7 \times 10^{-3}\ nm$$

d) room temperature oxygen molecule O_2

$$v_{thermal} = \sqrt{\frac{2k_B T}{m}} = c\sqrt{\frac{2k_B T}{mc^2}} = c\sqrt{\frac{2 \bullet 8.61734 \times 10^{-5}\ eV/K \bullet 300\ K}{(2 \bullet 16 \bullet 938\ MeV)}} = 1.312 \times 10^{-6}\ c$$

$$\lambda_{dB} = \frac{hc}{mc^2 (v/c)} = \frac{1240\ eV\ nm}{(2 \bullet 16 \bullet 938\ MeV)(1.312 \times 10^{-6})} = 0.031\ nm$$

e) raindrop. Assume 2 mm diameter, terminal velocity 6 m/s

http://www.shorstmeyer.com/wxfaqs/float/rdtable.html

$$m_{raindrop} = \tfrac{4}{3}\pi r^3 \rho_{water} = \tfrac{4}{3}\pi (0.1\ cm)^3 (1\ g/cm^3) = 0.00419\ g$$

$$\lambda_{dB} = \frac{h}{mv} = \frac{6.626 \times 10^{-34}\ J\ s}{(0.00419 \times 10^{-3}\ kg)(6\ m/s)} = 2.64 \times 10^{-20}\ nm$$

f) person. Assume 100 kg, 1 m/s

$$\lambda_{dB} = \frac{h}{mv} = \frac{6.626 \times 10^{-34}\ J\ s}{(100\ kg)(1\ m/s)} = 6.63 \times 10^{-27}\ nm$$

The wavelength is typically important in interference and diffraction experiments. In interference experiments, the two path lengths must be stable to less than the wavelength to see fringes. In diffraction experiments, the diffraction angle is given by λ/a, where a is the size of the diffracting object. In the first four cases, quantum effects can be seen, in the last two, they would be exceedingly difficult.

6.2 (a) The normalization integral is

$$1=\langle\psi|\psi\rangle=\int_{-\infty}^{\infty}|\psi(x)|^2\,dx=\int_{-\infty}^{\infty}\left|Ae^{-x^2/a^2}\right|^2 dx=|A|^2\int_{-\infty}^{\infty}e^{-2x^2/a^2}\,dx=|A|^2\,a\sqrt{\frac{\pi}{2}}$$

giving the normalized wave function

$$\psi(x)=\left(\frac{2}{\pi a^2}\right)^{1/4}e^{-x^2/a^2}$$

(b) The expectation value of position is

$$\langle x\rangle=\langle\psi|x|\psi\rangle=\int_{-\infty}^{\infty}\psi^*(x)x\;\psi(x)dx=\int_{-\infty}^{\infty}x|\psi(x)|^2\,dx=\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty}xe^{-2x^2/a^2}\,dx=0$$

because the integrand is odd and the integration interval is even.
(c) To find the uncertainty, we first need the expectation value of the square of the position:

$$\left\langle x^2\right\rangle=\langle\psi|x^2|\psi\rangle=\int_{-\infty}^{\infty}\psi^*(x)x^2\;\psi(x)dx=\int_{-\infty}^{\infty}x^2|\psi(x)|^2\,dx=\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty}x^2e^{-2x^2/a^2}\,dx$$

$$=\left(\frac{2}{\pi a^2}\right)^{1/2}\frac{2}{\left(\sqrt{2}/a\right)^3}\int_0^{\infty}y^2e^{-y^2}\,dy=\left(\frac{2}{\pi a^2}\right)^{1/2}\frac{2a^3}{2\sqrt{2}}\frac{\sqrt{\pi}}{4}=\frac{a^2}{4}$$

The uncertainty of the position is

$$\Delta x=\sqrt{\left\langle x^2\right\rangle-\langle x\rangle^2}=\sqrt{\frac{a^2}{4}-0}=\frac{a}{2}$$

(d) The probability of finding the particle in the region between 0 and *a* is

$$\mathcal{P}_{0<x<a}=\int_0^a|\psi(x)|^2\,dx=\int_0^a\left|\left(\frac{2}{\pi a^2}\right)^{1/4}e^{-x^2/a^2}\right|^2dx=\left(\frac{2}{\pi a^2}\right)^{1/2}\int_0^a e^{-2x^2/a^2}\,dx$$

$$=\left(\frac{2}{\pi a^2}\right)^{1/2}\frac{1}{\left(\sqrt{2}/a\right)}\int_0^{\sqrt{2}}e^{-y^2}\,dy=\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{2}}e^{-y^2}\,dy$$

This integral can be looked up in a table and is often called the error function, defined by:

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

Thus we get

$$\mathcal{P}_{0<x<a}=\frac{1}{2}erf\left(\sqrt{2}\right)$$

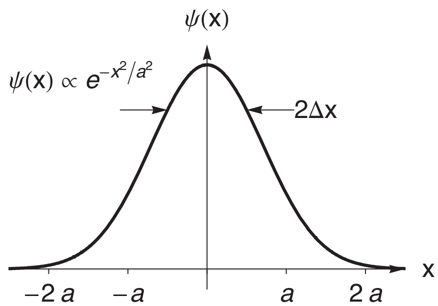
From the CRC handbook, we have

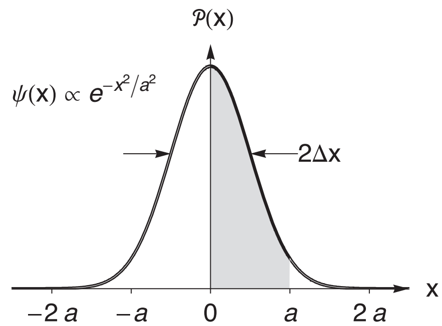
$$\begin{aligned}
erf(x) &= 2F\left(\sqrt{2}x\right)-1 \\
F(2) &= 0.9773 \\
erf\left(\sqrt{2}\right) &= 2(0.9773)-1 = 0.9546
\end{aligned}$$

yielding

$$\mathcal{P}_{0<x<a} = 0.4773$$

(e) The plots of the wave function and probability density:





(f) The expectation value of momentum is

$$\langle p\rangle = \langle\psi|p|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)dx = (-i\hbar)\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty}\left(\frac{-2x}{a^2}\right)e^{-2x^2/a^2}\,dx = 0$$

because the integrand is odd and the integration interval is even.

(g) To find the uncertainty, we first need the expectation value of the square of the momentum:

$$\begin{aligned}
\langle p^2\rangle &= \langle\psi|p^2|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)\left(-\hbar^2\frac{d^2}{dx^2}\right)\psi(x)dx \\
&= \left(-\hbar^2\right)\left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty}\left(\frac{4x^2}{a^4}-\frac{2}{a^2}\right)e^{-2x^2/a^2}\,dx \\
&= \left(-2\hbar^2\right)\left(\frac{2}{\pi a^2}\right)^{1/2}\left(\frac{4}{a^4\left(\sqrt{2}/a\right)^3}\int_0^{\infty}y^2e^{-y^2}\,dy-\frac{2}{a^2\left(\sqrt{2}/a\right)}\int_0^{\infty}e^{-y^2}\,dy\right) \\
&= \left(-2\hbar^2\right)\left(\frac{2}{\pi a^2}\right)^{1/2}\frac{1}{a}\left(\sqrt{2}\frac{\sqrt{\pi}}{4}-\sqrt{2}\frac{\sqrt{\pi}}{2}\right) = \frac{\hbar^2}{a^2}
\end{aligned}$$

The uncertainty of the momentum is

$$\Delta p = \sqrt{\langle p^2\rangle-\langle p\rangle^2} = \sqrt{\frac{\hbar^2}{a^2}-0} = \frac{\hbar}{a}$$

(h) The uncertainty principle is

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

In this case we get

$$\Delta p \Delta x = \frac{\hbar}{a}\frac{a}{2} = \frac{\hbar}{2}$$

so this is a minimum uncetainty state.

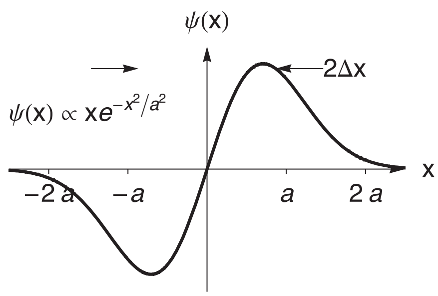
(ii) For the state $\psi(x) = Axe^{-x^2/a^2}$, similar calculations yield

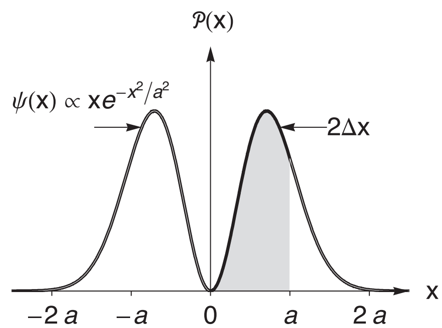
Normalized state: $$\psi(x) = 2\left(\frac{2}{\pi a^6}\right)^{1/4} xe^{-x^2/a^2}$$

Expectation value of x: $$\langle x \rangle = 0$$

Position uncertainty: $$\Delta x = a\sqrt{3}/2$$

Probability: $$\mathcal{P}_{0<x<a} = 0.3693$$





Expectation value of p: $$\langle p \rangle = 0$$

Momentum uncertainty: $$\Delta p = \hbar\sqrt{3}/a$$

Uncertainty product: $$\Delta p = 3\hbar/2 > \hbar/2$$

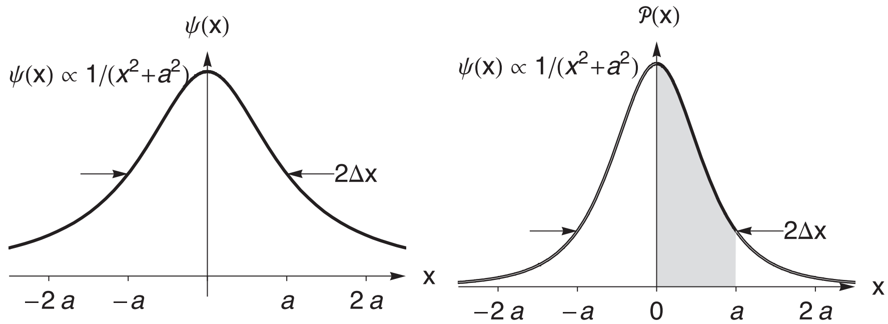
(iii) For the state $\psi(x) = A/\left(x^2 + a^2\right)$, we get

Normalized state: $$\psi(x) = \sqrt{\frac{2a^3}{\pi}}\frac{1}{x^2 + a^2}$$

Expectation value of x: $$\langle x \rangle = 0$$

Position uncertainty: $$\Delta x = a$$

Probability: $$\mathcal{P}_{0<x<a} = 0.4092$$



Expectation value of p: $\langle p \rangle = 0$

Momentum uncertainty: $\Delta p = \hbar / a\sqrt{2}$

Uncertainty product: $\Delta p = \hbar / \sqrt{2} > \hbar/2$

6.3 The momentum eigenstate before the shutter is

$$|p_0\rangle \doteq \varphi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i p_0 x/\hbar}.$$

a) After the shutter, the wave function is limited in spatial extent. The particles have velocity $v_0 = p_0/m$, so the length of the wavepacket after the shutter is $d = v_0\tau = p_0\tau/m$. Hence the new wave function is

$$\psi(x) = \begin{cases} 0 & , \quad x<0 \\ \frac{1}{\sqrt{d}} e^{i p_0 x/\hbar} & , \quad 0<x<d \\ 0 & , \quad x>d \end{cases}$$

where the new scale factor ensures normalization over this finite interval.
b) The momentum space wave function is given by the Fourier transform:

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar}\, dx$$

Performing the transform gives

$$\begin{aligned}\phi(p) &= \frac{1}{\sqrt{d}}\int_0^d \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0x/\hbar}e^{-ipx/\hbar}\,dx = \frac{1}{\sqrt{p_0\tau/m}\sqrt{2\pi\hbar}}\int_0^{p_0\tau/m} e^{i(p_0-p)x/\hbar}\,dx \\
&= \frac{1}{\sqrt{p_0\tau/m}\sqrt{2\pi\hbar}}\left[\frac{e^{i(p_0-p)x/\hbar}}{i(p_0-p)/\hbar}\right]_0^{p_0\tau/m} = \frac{1}{\sqrt{p_0\tau/m}\sqrt{2\pi\hbar}}\left[\frac{e^{i(p_0-p)p_0\tau/m\hbar}-1}{i(p_0-p)/\hbar}\right] \\
&= \frac{e^{i(p_0-p)p_0\tau/2m\hbar}}{\sqrt{p_0\tau/m}\sqrt{2\pi\hbar}}\left[\frac{e^{i(p_0-p)p_0\tau/2m\hbar}-e^{-i(p_0-p)p_0\tau/2m\hbar}}{i(p_0-p)/\hbar}\right] \\
&= \frac{2e^{i(p_0-p)p_0\tau/2m\hbar}}{\sqrt{p_0\tau/m}\sqrt{2\pi\hbar}}\left[\frac{\sin\left((p_0-p)p_0\tau/2m\hbar\right)}{(p_0-p)/\hbar}\right]\end{aligned}$$

The momentum probability distribution is

$$\begin{aligned}\mathcal{P}(p) = |\phi(p)|^2 &= \frac{4}{(2\pi\hbar)p_0\tau/m}\left[\frac{\sin\left((p_0-p)p_0\tau/2m\hbar\right)}{(p_0-p)/\hbar}\right]^2 \\
&= \frac{4\left(p_0^2\tau^2/4m^2\right)}{(2\pi\hbar)p_0\tau/m}\left[\frac{\sin\left((p_0-p)p_0\tau/2m\hbar\right)}{(p_0-p)p_0\tau/2m\hbar}\right]^2 \\
&= \frac{p_0\tau}{2\pi m\hbar}\operatorname{sinc}^2\left(\frac{(p_0-p)p_0\tau}{2m\hbar}\right)\end{aligned}$$

The sinc function is peaked at $p = p_0$, as expected.

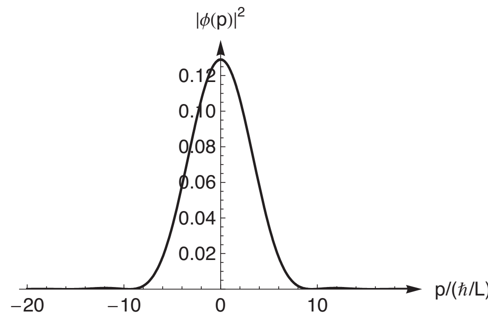
6.4 The energy eigenstates in a square well are

$$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}.$$

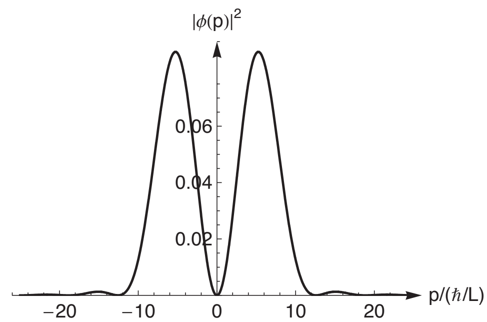
Using Eq. (6.31), the momentum space wave function is

$$\begin{aligned}\phi_n(p) = \langle p|\varphi_n\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi_n(x)e^{-ipx/\hbar}\,dx = \frac{1}{\sqrt{2\pi\hbar}}\int_0^L\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}e^{-ipx/\hbar}\,dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\frac{1}{2i}\int_0^L\left(e^{in\pi x/L}-e^{-in\pi x/L}\right)e^{-ipx/\hbar}\,dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\frac{1}{2i}\left[\frac{e^{in\pi-ipL/\hbar}-1}{in\pi/L-ip/\hbar}-\frac{e^{-in\pi-ipL/\hbar}-1}{-in\pi/L-ip/\hbar}\right] \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\frac{n\pi}{L}\left[\frac{(-1)^n e^{-ipL/\hbar}-1}{(p/\hbar)^2-(n\pi/L)^2}\right] \\
&= \frac{1}{\sqrt{\pi\hbar L}}\frac{n\pi\hbar^2}{L}\left[\frac{(-1)^n e^{-ipL/\hbar}-1}{(p)^2-(n\pi\hbar/L)^2}\right]\end{aligned}$$

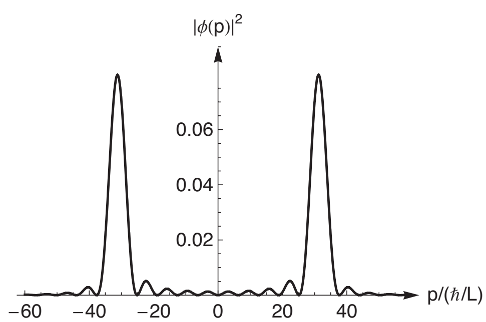
Plot for n = 1:



Plot for n = 2:



Plot for n = 10:



In each case, the momentum probability distribution is symmetric because there is no preferred direction for particle motion. The probability is concentrated near the values expected for a wave with wavelength $\lambda = 2L/n$ and momentum $p = h/\lambda = n\pi\hbar/L$.

6.5 For a free particle, the commutator of momentum and the Hamiltonian is

$$\left[\hat{p}, H\right] = \left[\hat{p}, \frac{\hat{p}^2}{2m}\right] = \left[\hat{p}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{p}\right].$$

In the position representation, this is represented by

$$
\begin{aligned}
\left[\hat{p},H\right] &\doteq \left(-i\hbar\frac{d}{dx}\right)\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 - \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2\left(-i\hbar\frac{d}{dx}\right)\\
&\doteq \frac{i\hbar^3}{2m}\frac{d^3}{dx^3} - \frac{i\hbar^3}{2m}\frac{d^3}{dx^3}\\
&\doteq 0
\end{aligned}
.
$$

In the abstract representation, the commutator is

$$\left[\hat{p},H\right] = \left[\hat{p},\frac{\hat{p}^2}{2m}\right] = \left[\hat{p}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{p}\right] = \frac{\hat{p}^3}{2m} - \frac{\hat{p}^3}{2m} = 0\ .$$

So the momentum and Hamiltonian of a free particle commute.

6.6 The commutator is defined by its action on a state vector $|\psi\rangle$ as:

$$\left[\hat{x},\hat{p}\right]|\psi\rangle = \hat{x}\hat{p}|\psi\rangle - \hat{p}\hat{x}|\psi\rangle\,.$$

In wave function language (the position representation), we express position as a multiplicative factor, $\hat{x}\doteq x$, momentum as a derivative operator $\hat{p}\doteq -i\hbar\, d/dx$, and the state vector is represented by $|\psi\rangle \doteq \psi(x)$. Then the commutator becomes

$$\left[\hat{x},\hat{p}\right]|\psi\rangle \doteq x\left(-i\hbar\frac{d}{dx}\right)\psi(x) - \left(-i\hbar\frac{d}{dx}\right)x\psi(x)$$

The second term requires the chain rule, yielding

$$
\begin{aligned}
\left[\hat{x},\hat{p}\right]|\psi\rangle &\doteq -i\hbar x\frac{d\psi(x)}{dx} + i\hbar x\frac{d\psi(x)}{dx} + i\hbar\psi(x)\\
&\doteq i\hbar\psi(x)
\end{aligned}
$$

This is very clearly non-zero, and tells us that the position and momentum operators do not commute.

In the momentum representation, we express position as a derivative operator $\hat{x}\doteq i\hbar\, d/dp$, momentum as a multiplicative factor $\hat{p}\doteq p$, and the state vector is represented by $|\psi\rangle \doteq \phi(p)$. Then the commutator becomes

$$\left[\hat{x},\hat{p}\right]|\psi\rangle \doteq \left(i\hbar\frac{d}{dp}\right)p\phi(p) - p\left(i\hbar\frac{d}{dp}\right)\phi(p)$$

The first term requires the chain rule, yielding

$$
\begin{aligned}
\left[\hat{x},\hat{p}\right]|\psi\rangle &\doteq i\hbar\phi(p) + pi\hbar\frac{d\phi(p)}{dp} - pi\hbar\frac{d\phi(p)}{dp}\\
&\doteq i\hbar\phi(p)
\end{aligned}
$$

Again, this is very clearly non-zero, and tells us that the position and momentum operators do not commute.

In Dirac notation, both calculations can be expressed as

$$[\hat{x},\hat{p}]|\psi\rangle = i\hbar|\psi\rangle$$

so the commutator as an operator is represented as

$$[\hat{x},\hat{p}] \doteq i\hbar$$

6.7 The Dirac normalization condition is

$$\langle p''|p'\rangle = \delta(p''-p').$$

The inner product of momentum eigenstates is

$$\begin{aligned}\langle p''|p'\rangle &= \int_{-\infty}^{\infty}\varphi_{p''}^{*}(x)\varphi_{p'}(x)\,dx = \int_{-\infty}^{\infty}A^{*}e^{-ip''x/\hbar}Ae^{ip'x/\hbar}\,dx \\ &= |A|^{2}\int_{-\infty}^{\infty}e^{i(p'-p'')x/\hbar}\,dx\end{aligned}.$$

To do this integral, recall the definition of a Fourier transform pair $f(x)$ and $a(k)$ from Eqs. (D.14) and (D.15):

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(k)e^{ikx}\,dk \qquad \Leftrightarrow \qquad a(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f(x)\,dx.$$

Now consider the inverse Fourier transform of $a(k)=\delta(k-k')$:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(k-k')e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}e^{ik'x}.$$

Hence, the Fourier transform is

$$a(k) = \delta(k-k') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\frac{1}{\sqrt{2\pi}}e^{ik'x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k'-k)x}\,dx.$$

Changing variables from wave vector to momentum gives

$$\delta\left(\frac{p}{\hbar}-\frac{p'}{\hbar}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(p'-p)x/\hbar}\,dx$$

and using the relation $\delta(ax)=\delta(x)/|a|$ results is

$$\delta(p''-p') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}e^{i(p'-p'')x/\hbar}\,dx.$$

Hence the inner product becomes

$$\langle p''|p'\rangle = |A|^{2}\,2\pi\hbar\,\delta(p''-p')$$

and the momentum eigenstates satisfy Dirac normalization if we set the normalization constant to

$$A = \frac{1}{\sqrt{2\pi\hbar}}\,.$$

The "normalized" momentum eigenstates are

$$\varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\,e^{ipx/\hbar}\,.$$

If we use the wave vector eigenstates:

$$\varphi_k(x) = Ae^{ikx}$$

the inner product is

$$\langle k''|k'\rangle = \int_{-\infty}^{\infty}\varphi_{k''}^{*}(x)\varphi_{k'}(x)\,dx = |A|^2\int_{-\infty}^{\infty}e^{i(k'-k'')x}\,dx = |A|^2\,2\pi\delta(k'-k'')$$

The Dirac normalization condition is now

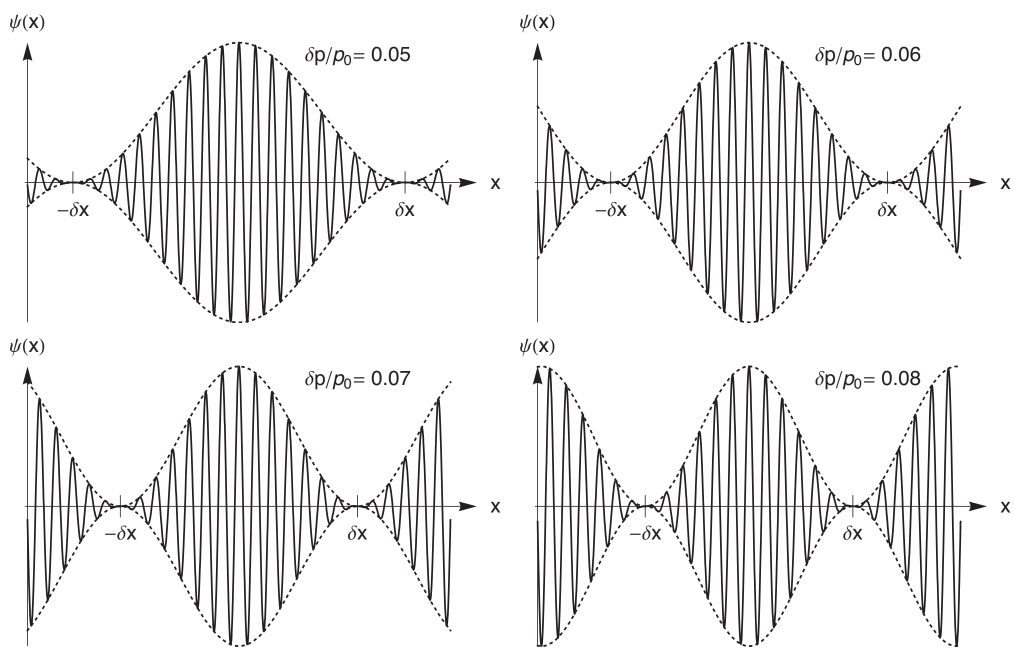
$$\langle k''|k'\rangle = \delta(k''-k').$$

and yields the normalized states

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}\,e^{ikx}\,.$$

The two different forms of the eigenstates have different normalization factors because the Dirac delta function has units of 1/argument. Momentum and wave vector differ by a factor of $\hbar$, so that difference is reflected in the normalization factors.

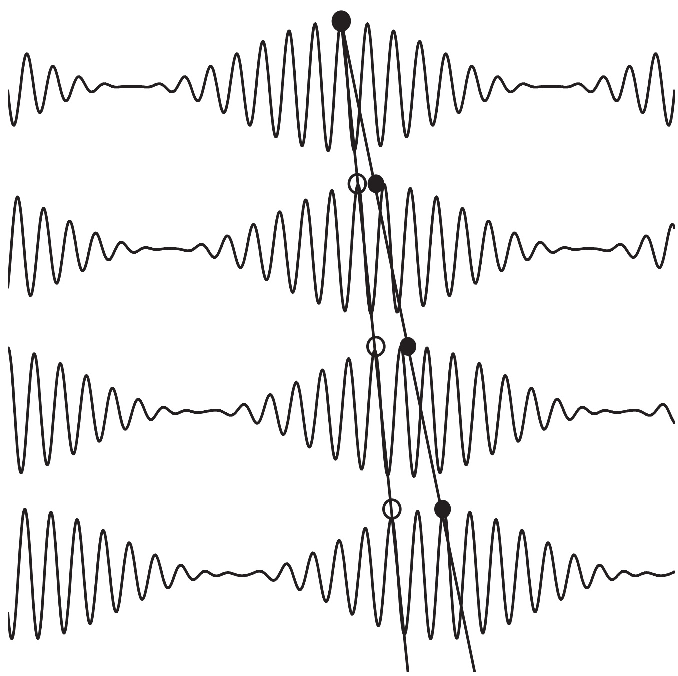
6.8 Here is a plot as the side mode spacing δp varies



As the spacing δp increases, the spatial extent δx, defined here as the location of the interference minima, decreases. To relate the momentum spread and the position spread, note that the spatial interference minima are located where the phases of the side mode waves are π out of phase with the central sinusoid. These phases are determined by the arguments of the $e^{ip_j x/\hbar}$ terms in Eq. (6.32). If we assume that the wave packet maximum, where the three waves are in phase, is at $x = 0$, then the destructive interference minimum on the right is at $x = \delta x$, as indicated above. To calculate δx, set the phase difference between the upper side mode ($p = p_0 + \delta p$) and the central mode ($p = p_0$) equal to π and solve

$$\begin{aligned}
&\frac{(p_0+\delta p)\delta x}{\hbar} - \frac{p_0 \delta x}{\hbar} = \pi \\
&\frac{\delta p \delta x}{\hbar} = \pi \\
&\delta p \delta x = \pi \hbar
\end{aligned}$$

The animation below shows that the velocity of the envelope—the group velocity—is twice the velocity of the wiggles within the envelope—the phase velocity.



6.9 The Gaussian integral is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\beta^2} \right)^{\frac{1}{4}} e^{-(p-p_0)^2/4\beta^2} e^{ipx/\hbar} e^{-ip^2 t/2m\hbar} \, dp$$

Collect terms in the exponent that are quadratic and linear in the momentum:

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{2\pi\beta^2}\right)^{\frac{1}{4}} e^{-p_0^2/4\beta^2}\int_{-\infty}^{\infty} e^{p\left(ix/\hbar+p_0/2\beta^2\right)-p^2\left(1/4\beta^2+it/2m\hbar\right)}\,dp$$

and use Eq. (F.23) to get

$$\begin{aligned}\psi(x,t)&=\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{2\pi\beta^2}\right)^{\frac{1}{4}} e^{-p_0^2/4\beta^2}\,\frac{\sqrt{\pi}}{\sqrt{1/4\beta^2+it/2m\hbar}}\,e^{\left(-x^2/\hbar^2+p_0^2/4\beta^4+ip_0x/\hbar\beta^2\right)/\left(1/\beta^2+i2t/m\hbar\right)}\\&=\frac{\sqrt{2\beta}}{\sqrt{\hbar\sqrt{2\pi}}}\,\frac{1}{\sqrt{1+i2t\beta^2/m\hbar}}\,e^{-p_0^2/4\beta^2}e^{\left(-x^2/\hbar^2+p_0^2/4\beta^4+ip_0x/\hbar\beta^2\right)/\left(1/\beta^2+i2t/m\hbar\right)}\\&=\frac{\sqrt{2\beta}}{\sqrt{\hbar\sqrt{2\pi}}}\,\frac{1}{\sqrt{1+i2t\beta^2/m\hbar}}\,e^{\left(ip_0x/\hbar-ip_0^2t/2m\hbar-x^2\beta^2/\hbar^2\right)/\left(1+i2t\beta^2/m\hbar\right)}\end{aligned}$$

Now add and subtract terms in the exponent to change form:

$$\begin{aligned}\psi(x,t)&=\frac{\sqrt{2\beta}}{\sqrt{\hbar\sqrt{2\pi}}}\,\frac{1}{\sqrt{1+i2t\beta^2/m\hbar}}\,e^{\left(ip_0x/\hbar-ip_0^2t/2m\hbar-x^2\beta^2/\hbar^2\right)/\left(1+i2t\beta^2/m\hbar\right)}\\&=\frac{\sqrt{2\beta}}{\sqrt{\hbar\sqrt{2\pi}}}\,\frac{1}{\sqrt{1+i2t\beta^2/m\hbar}}\,e^{\left[\left(ip_0x/\hbar-ip_0^2t/2m\hbar\right)\left(1+i2t\beta^2/m\hbar\right)-x^2\beta^2/\hbar^2+2xp_0t\beta^2/m\hbar^2-p_0^2t^2\beta^2/m^2\hbar^2\right]/\left(1+i2t\beta^2/m\hbar\right)}\\&=\frac{\sqrt{2\beta}}{\sqrt{\hbar\sqrt{2\pi}}}\,\frac{1}{\sqrt{1+i2t\beta^2/m\hbar}}\,e^{\left(ip_0x/\hbar-ip_0^2t/2m\hbar\right)}e^{-\left(x-p_0t/m\right)^2\beta^2/\hbar^2\left(1+i2t\beta^2/m\hbar\right)}\end{aligned}$$

Using the definitions

$$\gamma=1+\frac{it}{\tau}\,,\qquad \tau=\frac{m\hbar}{2\beta^2}\,,$$

the result is

$$\psi(x,t)=\frac{\sqrt{2\beta}}{\sqrt{\hbar\gamma\sqrt{2\pi}}}\,e^{ip_0\left(x-p_0t/2m\right)/\hbar}e^{-\left(x-p_0t/m\right)^2\beta^2/\hbar^2\gamma}$$

6.10 The expectation value of position is

$$\langle x\rangle=\int_{-\infty}^{\infty}x\ \mathcal{P}(x,t)\,dx=\int_{-\infty}^{\infty}x\left|\psi(x,t)\right|^2dx$$

Substituting the Gaussian wave packet gives

$$\begin{aligned}\langle x\rangle &= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} x\left|e^{ip_0\left(x-p_0t/2m\right)/\hbar}e^{-\left(x-p_0t/m\right)^2\beta^2/\hbar^2\gamma}\right|^2 dx \\ &= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} xe^{-\left(x-p_0t/m\right)^2\left(1/\gamma+1/\gamma^*\right)\beta^2/\hbar^2}\,dx = \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} xe^{-2\left(x-p_0t/m\right)^2\beta^2/\hbar^2\left|\gamma\right|^2}\,dx \\ &= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty}\left(z+p_0t/m\right)e^{-2z^2\beta^2/\hbar^2\left|\gamma\right|^2}\,dz = \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\left(0+\frac{p_0t}{m}\frac{\sqrt{\pi}\hbar\sqrt{\left|\gamma^2\right|}}{\sqrt{2}\beta}\right) \\ &= \frac{p_0}{m}t\end{aligned}$$

as in Eq. (6.56).

The expectation value of momentum is easier to calculate in momentum space:

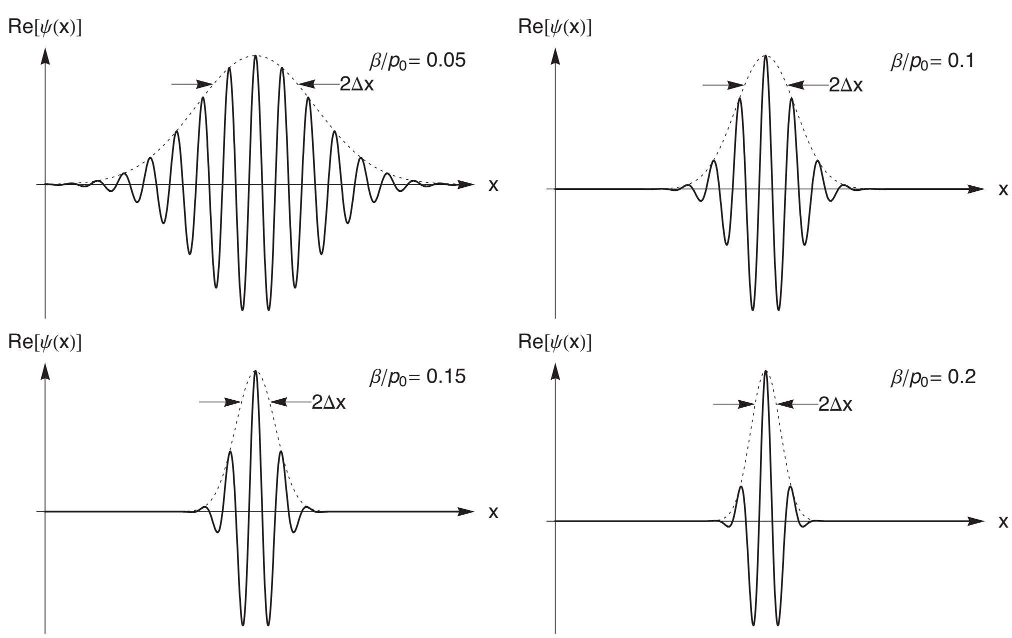
$$\langle p\rangle = \int_{-\infty}^{\infty} p\ \mathcal{P}(p,t)\,dp = \int_{-\infty}^{\infty} p\ \left|\phi(p,t)\right|^2 dp$$

Substituting the momentum distribution $\mathcal{P}(p) = \left|\phi(p)\right|^2 = e^{-(p-p_0)^2/2\beta^2}\Big/\beta\sqrt{2\pi}$ gives

$$\begin{aligned}\langle p\rangle &= \frac{1}{\beta\sqrt{2\pi}}\int_{-\infty}^{\infty} pe^{-(p-p_0)^2/2\beta^2}\,dp = \frac{1}{\beta\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(z+p_0\right)e^{-z^2/2\beta^2}\,dz \\ &= \frac{1}{\beta\sqrt{2\pi}}\left(0+p_0\sqrt{\pi}\sqrt{2}\beta\right) = p_0\end{aligned}$$

as in Eq. (6.59).

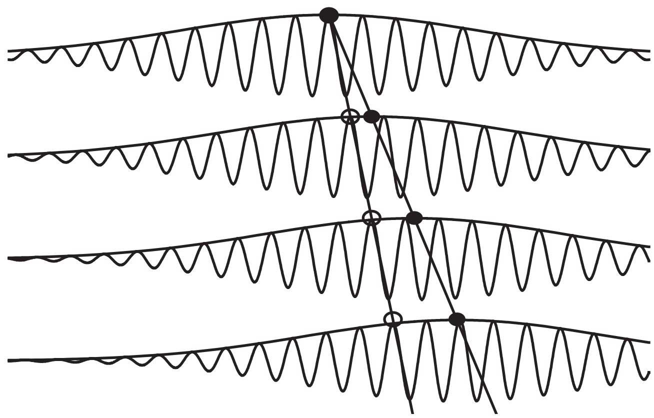
6.11 Here is a plot as the width β of the momentum distribution varies



As the width β increases, the spatial extent Δx, defined here as the standard deviation of the probability distribution, decreases. The relation between the momentum spread and the position spread is given by Eq. (6.70):

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1+\left(\frac{2\beta^2 t}{m\hbar}\right)^2}$$

The animation below shows that the velocity of the envelope—the group velocity—is twice the velocity of the wiggles within the envelope—the phase velocity.



For longer times, you can also see the spread of the wave packet.

6.12 A propagating Gaussian wave packet has the wave function

$$\psi(x,t) = \frac{\sqrt{2\beta}}{\sqrt{\hbar\gamma\sqrt{2\pi}}} e^{ip_0(x-p_0t/2m)/\hbar} e^{-(x-p_0t/m)^2\beta^2/\hbar^2\gamma}$$

where

$$\gamma = 1+\frac{it}{\tau}\,, \qquad \tau = \frac{m\hbar}{2\beta^2}\,.$$

The breadth of the wave function is characterized by the position uncertainty $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$. From Problem 6.10 or Eq. (6.56), we have $\langle x\rangle = p_0 t/m$. To find the uncertainty we need the expectation value

$$\left\langle x^2\right\rangle = \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} x^2 \left| e^{ip_0(x-p_0t/2m)/\hbar} e^{-(x-p_0t/m)^2\beta^2/\hbar^2\gamma}\right|^2 dx$$

$$= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} x^2 e^{-(x-p_0t/m)^2(1/\gamma+1/\gamma^*)\beta^2/\hbar^2}\,dx = \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty} x^2 e^{-2(x-p_0t/m)^2\beta^2/\hbar^2\left|\gamma^2\right|}\,dx$$

$$= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\int_{-\infty}^{\infty}\left(z+p_0t/m\right)^2 e^{-2z^2\beta^2/\hbar^2\left|\gamma^2\right|}\,dz$$

$$= \frac{2\beta}{\hbar\sqrt{2\pi\left|\gamma^2\right|}}\left(\frac{\sqrt{\pi}\hbar^3\left|\gamma^2\right|\sqrt{\left|\gamma^2\right|}}{4\sqrt{2}\beta^3} + \frac{p_0^2t^2}{m^2}\frac{\sqrt{\pi\left|\gamma^2\right|}\hbar}{\sqrt{2}\beta}\right) = \left(\frac{\hbar^2\left|\gamma^2\right|}{4\beta^2} + \frac{p_0^2t^2}{m^2}\right)$$

Hence, the position uncertainty is

$$\Delta x = \sqrt{\left\langle x^2\right\rangle - \left\langle x\right\rangle^2} = \sqrt{\left(\frac{\hbar^2\left|\gamma^2\right|}{4\beta^2} + \frac{p_0^2t^2}{m^2}\right) - \left(\frac{p_0t}{m}\right)^2} = \frac{\hbar\sqrt{\left|\gamma^2\right|}}{2\beta} = \frac{\hbar}{2\beta}\sqrt{1+\left(\frac{2\beta^2t}{m\hbar}\right)^2}\ .$$

This is a growing function of time, so the wave packet broadens in position space as it moves. In momentum space we have $\left\langle p\right\rangle = p_0$. The expectation value of p^2 is

$$\left\langle p^2\right\rangle = \frac{1}{\beta\sqrt{2\pi}}\int_{-\infty}^{\infty} p^2 e^{-(p-p_0)^2/2\beta^2}\,dp = \frac{1}{\beta\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(z+p_0\right)^2 e^{-z^2/2\beta^2}\,dz$$

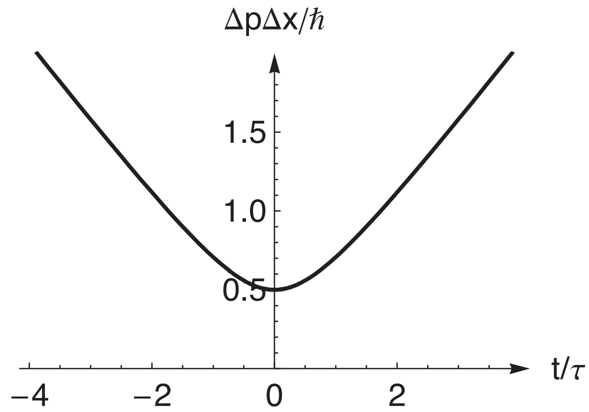
$$= \frac{1}{\beta\sqrt{2\pi}}\left(\frac{\sqrt{\pi}}{2}2\sqrt{2}\beta^3 + 0 + p_0^2\sqrt{\pi}\sqrt{2}\beta\right) = \beta^2 + p_0^2$$

$$\Delta p = \sqrt{\left\langle p^2\right\rangle - \left\langle p\right\rangle^2} = \sqrt{\beta^2 + p_0^2 - p_0^2} = \beta\,.$$

The momentum space width is constant.
The uncertainty product is

$$\Delta p\Delta x = \beta\frac{\hbar}{2\beta}\sqrt{1+\left(\frac{2\beta^2t}{m\hbar}\right)^2} = \frac{\hbar}{2}\sqrt{1+\left(\frac{2\beta^2t}{m\hbar}\right)^2} \geq \frac{\hbar}{2}\,,$$

so it is satisfied. The minimum uncertainty is at $t = 0$, at which time the wave packet has the minimum uncertainty allowed by quantum mechanics. For all other times the wave packet has a larger uncertainty.



6.13 For a harmonic wave of the form $e^{i(kx-\omega t)}$, the phase velocity is the ratio of the angular frequency to the wave vector

$$v_{phase} = \frac{\omega}{k}$$

Multiplying top and bottom by Planck's constant gets

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k}$$

Using Einstein's energy relation $E = \hbar\omega$ and de Broglie's relation $p = h/\lambda = \hbar k$ gives

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p}$$

Using the kinetic energy of a free particle gives

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m}$$

Using the classical relation between momentum and velocity of a particle $p = mv_{classical}$ results in

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{v_{classical}}{2}$$

For a wave that is a superposition of harmonic waves, the group velocity is the derivative of the dispersion relation:

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0}$$

Multiplying top and bottom by Planck's constant gets

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.\frac{d(\hbar\omega)}{d(\hbar k)}\right|_{k_0}$$

Using Einstein's energy relation $E = \hbar\omega$ and de Broglie's relation $p = h/\lambda = \hbar k$ gives

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.\frac{d(\hbar\omega)}{d(\hbar k)}\right|_{k_0} = \left.\frac{dE}{dp}\right|_{p_0}$$

Using the kinetic energy of a free particle gives

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.\frac{d(\hbar\omega)}{d(\hbar k)}\right|_{k_0} = \left.\frac{dE}{dp}\right|_{p_0} = \left.\frac{d\left(p^2/2m\right)}{dp}\right|_{p_0} = \frac{p_0}{m}$$

Using the classical relation between momentum and velocity of a particle $p = mv_{classical}$ results in

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0} = \left.\frac{d(\hbar\omega)}{d(\hbar k)}\right|_{k_0} = \left.\frac{dE}{dp}\right|_{p_0} = \left.\frac{d\left(p^2/2m\right)}{dp}\right|_{p_0} = \frac{p_0}{m} = v_{classical}$$

6.14 This wave function is not an eigenstate of momentum, which is clear if we write it as a sum of complex exponentials

$$\psi(x) = A\sin\left(p_0 x/\hbar\right) = \frac{A}{2i}\left(e^{ip_0x/\hbar} - e^{-ip_0x/\hbar}\right).$$

By inspection, we see that this state is the superposition

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|p_0\rangle - |-p_0\rangle\right) \doteq \frac{1}{\sqrt{2}\sqrt{2\pi\hbar}}\left(e^{ip_0x/\hbar} - e^{-ip_0x/\hbar}\right),$$

so measurement of the momentum yields the results p_0 or $-p_0$. The expectation value of the momentum is

$$\begin{aligned}\langle p\rangle = \langle\psi|p|\psi\rangle &= \tfrac{1}{\sqrt{2}}\left(\langle p_0| - \langle -p_0|\right)p\,\tfrac{1}{\sqrt{2}}\left(|p_0\rangle - |-p_0\rangle\right)\\ &= \tfrac{1}{2}\left(\langle p_0|p|p_0\rangle - \langle p_0|p|-p_0\rangle - \langle -p_0|p|p_0\rangle + \langle -p_0|p|-p_0\rangle\right).\\ &= \tfrac{1}{2}\left(p_0 + (-p_0)\right) = 0\end{aligned}$$

The momentum probability distribution is obtained from the momentum space wave function. Using Eq. (6.31), we obtain

$$\begin{aligned}\phi(p) = \langle p|\psi\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\,dx = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2}\sqrt{2\pi\hbar}}\left(e^{ip_0x/\hbar} - e^{-ip_0x/\hbar}\right)e^{-ipx/\hbar}\,dx\\ &= \frac{1}{2\pi\hbar\sqrt{2}}\int_{-\infty}^{\infty}\left(e^{i(p_0-p)x/\hbar} - e^{-i(p_0+p)x/\hbar}\right)dx = \frac{1}{\sqrt{2}}\left(\delta(p-p_0) - \delta(p+p_0)\right)\end{aligned}$$

giving

$$\mathcal{P}(p) = |\phi(p)|^2 = \langle p|\psi\rangle = \tfrac{1}{2}\left(\delta(p-p_0) - \delta(p+p_0)\right)^2 = \tfrac{1}{2}\left(\delta(p-p_0) + \delta(p+p_0)\right)$$

To find the uncertainty, we need $\langle p^2 \rangle$:

$$\begin{aligned}\langle p^2 \rangle &= \langle \psi | p^2 | \psi \rangle = \tfrac{1}{\sqrt{2}}\big(\langle p_0| - \langle -p_0|\big) p^2 \tfrac{1}{\sqrt{2}}\big(|p_0\rangle - |-p_0\rangle\big) \\ &= \tfrac{1}{2}\big(\langle p_0|p^2|p_0\rangle - \langle p_0|p^2|-p_0\rangle - \langle -p_0|p^2|p_0\rangle + \langle -p_0|p^2|-p_0\rangle\big). \\ &= \tfrac{1}{2}\Big(p_0 + (-p_0)^2\Big) = p_0^2\end{aligned}$$

Hence the uncertainty is

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{p_0^2 - 0} = p_0 .$$

6.15 The position uncertainty is roughly the size of the box $\Delta x = a$. The uncertainty principle implies a minimum momentum

$$p_{\min} = \Delta p = \frac{\hbar}{2\Delta x} = \frac{\hbar}{2a}$$

and hence a minimum energy

$$E_{\min} = \frac{p_{\min}^2}{2m} = \frac{\hbar^2}{8ma^2}$$

For an electron in a 0.1 nm box, the energy is

$$E_{\min} = \frac{\hbar^2}{8ma^2} = \frac{(hc)^2}{32\pi^2 mc^2 a^2} = \frac{(1240\ eV\ nm)^2}{32\pi^2 \left(5.11\times 10^5\ eV\right)(0.1\ nm)^2} = 0.95\ eV$$

6.16 For a particle bound in a potential well $V(x)$, the energy is

$$E = \frac{p^2}{2m} + V(x)$$

For a particle confined to region Δx, the uncertainty principle implies a minimum momentum $p_{\min} = \Delta p = \hbar/2\Delta x$, and we estimate the ground state energy as

$$E_{gnd} = \frac{p_{\min}^2}{2m} + V(\Delta x) = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}k(\Delta x)^2$$

in the harmonic oscillator potential. Minimizing this with respect to Δx, gives

$$\frac{dE_{gnd}}{d(\Delta x)} = \frac{-2\hbar^2}{8m(\Delta x)^3} + k(\Delta x) = 0 \quad \Rightarrow \quad (\Delta x) = \left(\frac{\hbar}{2m\omega}\right)^{1/2},$$

where we have used $k = m\omega^2$, with ω being the resonant frequency of the oscillator. The resultant energy is

$$E_{gnd} = \frac{\hbar^2}{8m(\hbar/2m\omega)} + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right) = \frac{1}{2}\hbar\omega\,,$$

which is (miraculously) the correct energy.

6.17 For a particle bound in a potential well $V(x)$, the energy is

$$E = \frac{p^2}{2m} + V(x)$$

For a particle confined to region Δx, the uncertainty principle implies a minimum momentum $p_{\min} = \Delta p = \hbar/2\Delta x$, and we estimate the ground state energy as

$$E_{gnd} = \frac{p_{\min}^2}{2m} + V(\Delta x) = \frac{\hbar^2}{8m(\Delta x)^2} + a|\Delta x|$$

in the linear potential. Minimizing this with respect to Δx, gives

$$\frac{dE_{gnd}}{d(\Delta x)} = \frac{-2\hbar^2}{8m(\Delta x)^3} + a = 0 \quad \Rightarrow \quad (\Delta x) = \left(\frac{\hbar^2}{4ma}\right)^{1/3}.$$

The resultant energy is

$$E_{gnd} = \frac{\hbar^2}{8m(\hbar^2/4ma)^{2/3}} + a\left(\frac{\hbar^2}{4ma}\right)^{1/3} = \frac{3}{2}\left(\frac{\hbar^2 a^2}{4m}\right)^{1/3}.$$

6.18 For a particle bound in a potential well $V(x)$, the energy is

$$E = \frac{p^2}{2m} + V(x)$$

For a particle confined to region Δx, the uncertainty principle implies a minimum momentum $p_{\min} = \Delta p = \hbar/2\Delta x$, and we estimate the ground state energy as

$$E_{gnd} = \frac{p_{\min}^2}{2m} + V(\Delta x) = \frac{\hbar^2}{8m(\Delta x)^2} + b(\Delta x)^4$$

in this quartic potential. Minimizing this with respect to Δx, gives

$$\frac{dE_{gnd}}{d(\Delta x)} = \frac{-2\hbar^2}{8m(\Delta x)^3} + 3b(\Delta x)^3 = 0 \quad \Rightarrow \quad (\Delta x) = \left(\frac{\hbar^2}{12mb}\right)^{1/6}.$$

The resultant energy is

$$E_{gnd} = \frac{\hbar^2}{8m(\hbar^2/12mb)^{1/3}} + b\left(\frac{\hbar^2}{12mb}\right)^{2/3} = \frac{5}{2}\left(\frac{\hbar^4 b}{12^2 m^2}\right)^{1/3}.$$

6.19 For a particle bound in a potential well $V(x)$, the energy is

$$E = \frac{p^2}{2m} + V(x)$$

For a particle confined to region Δx, the uncertainty principle implies a minimum momentum $p_{\min} = \Delta p = \hbar/2\Delta x$, and we estimate the ground state energy as

$$E_{gnd} = \frac{p_{\min}^2}{2m} + V(\Delta x) = \frac{\hbar^2}{8m(\Delta x)^2} - \frac{e^2}{4\pi\varepsilon_0(\Delta x)}$$

in the Coulomb potential. Minimizing this with respect to Δx, gives

$$\frac{dE_{gnd}}{d(\Delta x)} = \frac{-2\hbar^2}{8m(\Delta x)^3} + \frac{e^2}{4\pi\varepsilon_0(\Delta x)^2} = 0 \quad \Rightarrow \quad (\Delta x) = \frac{\hbar^2\pi\varepsilon_0}{me^2}.$$

The resultant energy is

$$E_{gnd} = \frac{\hbar^2}{8m\left(\hbar^2\pi\varepsilon_0/me^2\right)^2} - \frac{e^2}{4\pi\varepsilon_0\left(\hbar^2\pi\varepsilon_0/me^2\right)} = -\frac{2me^4}{(4\pi\varepsilon_0)^2\hbar^2} = -2\alpha^2mc^2,$$

which is 4 times larger than the actual energy

6.20 For a particle in a square well from 0 -> L, the position expectation value is $\langle x\rangle = L/2$. The position uncertainty is

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}.$$

So first find $\langle x^2\rangle$:

$$\begin{aligned}
\langle x^2\rangle = \langle\varphi_1|x^2|\varphi_1\rangle &= \frac{2}{L}\int_0^L x^2\sin^2\frac{\pi x}{L}dx \\
&= \frac{2}{L}\left[\frac{x^3}{6} + \left(\frac{x^2}{4\pi/L} - \frac{1}{8(\pi/L)^3}\right)\sin\frac{2\pi x}{L} - \frac{x}{4(\pi/L)^2}\cos\frac{2\pi x}{L}\right]_0^L. \\
&= \frac{2}{L}\left[\frac{L^3}{6} - \frac{L}{4(\pi/L)^2}\right] = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)
\end{aligned}$$

The position uncertainty is

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) - \frac{L^2}{4}} = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2}}.$$

For a uniform probability density, we get

$$\langle x\rangle = \int_0^L x\mathcal{P}(x)\,dx = \int_0^L x\frac{1}{L}\,dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}$$

as expected, and

$$\langle x^2\rangle = \int_0^L x^2\mathcal{P}(x)\,dx = \int_0^L x^2\frac{1}{L}\,dx = \frac{1}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{L^2}{3}.$$

The position uncertainty is

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{L^2}{3} - \frac{L^2}{4}} = \frac{L}{\sqrt{12}},$$

which is larger than the ground state, as you might expect.

6.21 First normalize the wave function:

$$\begin{aligned}1 = \langle\psi|\psi\rangle &= \int_{-\infty}^{\infty}\psi^*(x)\psi(x)\,dx = \int_{-\infty}^{\infty}A^* e^{-ip_0x/\hbar}e^{-x^2/4\alpha^2}Ae^{ip_0x/\hbar}e^{-x^2/4\alpha^2}\,dx \\ &= |A|^2\int_{-\infty}^{\infty}e^{-x^2/2\alpha^2}\,dx = |A|^2\,\alpha\sqrt{2\pi}\end{aligned}.$$

The normalized wave function is

$$\psi(x) = \frac{1}{\sqrt{\alpha\sqrt{2\pi}}}e^{ip_0x/\hbar}e^{-x^2/4\alpha^2}.$$

(a) The expectation value of momentum calculated in the position representation is

$$\begin{aligned}\langle p\rangle = \langle\psi|p|\psi\rangle &= \int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi^*(x)\,dx \\ &= \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ip_0x/\hbar}e^{-x^2/4\alpha^2}\left(-i\hbar\frac{d}{dx}\right)e^{ip_0x/\hbar}e^{-x^2/4\alpha^2}\,dx \\ &= \frac{-i\hbar}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{ip_0}{\hbar} - \frac{x}{2\alpha^2}\right)e^{-x^2/2\alpha^2}\,dx = \frac{-i\hbar}{\alpha\sqrt{2\pi}}\left(\frac{ip_0}{\hbar}\alpha\sqrt{2\pi}\right) = p_0\end{aligned}.$$

This is expected from the form of the wave function.
(b) To calculate the expectation value in the momentum representation, we first need the momentum space wave function. Using Eq. (6.31), we obtain

$$\begin{aligned}\phi(p) = \langle p|\psi\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\,dx = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\alpha\sqrt{2\pi}}}e^{ip_0x/\hbar}e^{-x^2/4\alpha^2}e^{-ipx/\hbar}\,dx \\ &= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{\alpha\sqrt{2\pi}}}\int_{-\infty}^{\infty}e^{i(p_0-p)x/\hbar - x^2/4\alpha^2}\,dx\end{aligned}.$$

Use the integral in Eq. (F.23) to obtain

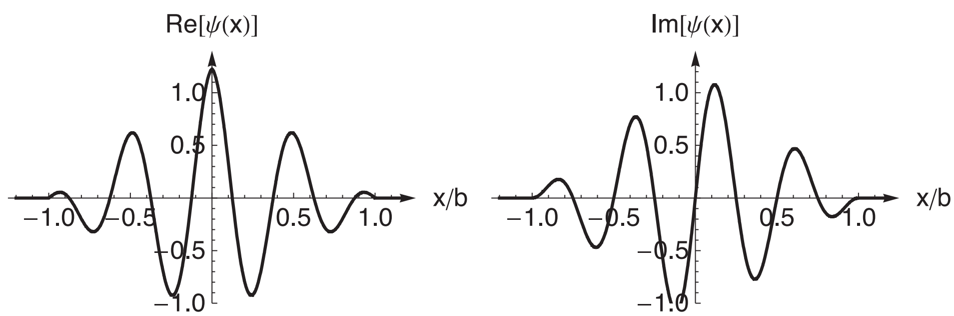
$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{\alpha\sqrt{2\pi}}}2\alpha\sqrt{\pi}e^{-\alpha^2(p-p_0)^2/\hbar^2} = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)e^{-\alpha^2(p-p_0)^2/\hbar^2}\ .$$

The expectation value of momentum in the momentum representation is

$$\langle p\rangle = \langle\psi|p|\psi\rangle = \int_{-\infty}^{\infty}\phi^*(p)\,p\ \phi(p)\,dp = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2}\int_{-\infty}^{\infty}p\ e^{-2\alpha^2(p-p_0)^2/\hbar^2}\ dp$$
$$= \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2}\frac{\hbar}{\alpha\sqrt{2}}\int_{-\infty}^{\infty}\left(p_0 + \frac{\hbar}{\alpha\sqrt{2}}y\right)e^{-y^2}\ dy = \left(\frac{2\alpha^2}{\pi\hbar^2}\right)^{1/2}\frac{\hbar}{\alpha\sqrt{2}}\left(p_0\sqrt{\pi}\right) = p_0 .$$

The same result, as expected.

6.22 (a) Plot the wave function, using $p_0 = 4\pi\hbar/b$, which gives a wavelength $\lambda = h/p_0 = b/2$:



(b) Normalize the wave function:

$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)\psi(x)\,dx = \int_{-b}^{b}A^*e^{-ip_0x/\hbar}\left(b-|x|\right)Ae^{ip_0x/\hbar}\left(b-|x|\right)dx$$
$$= |A|^2\int_{-b}^{b}\left(b^2 - 2b|x| + |x|^2\right)dx = 2|A|^2\int_0^b\left(b^2 - 2bx + x^2\right)dx$$
$$= 2|A|^2\left[xb^2 - bx^2 + x^3/3\right]_0^b = 2|A|^2\left[b^3/3\right] .$$

The normalized wave function is

$$\psi(x) = \sqrt{\frac{3}{2b^3}}e^{ip_0x/\hbar}\left(b-|x|\right).$$

(c) Using Eq. (6.31), the momentum space wave function is

$$\phi(p)=\langle p|\psi\rangle=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\,dx=\frac{1}{\sqrt{2\pi\hbar}}\int_{-b}^{b}\sqrt{\frac{3}{2b^3}}e^{ip_0x/\hbar}\left(b-|x|\right)e^{-ipx/\hbar}\,dx$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{3}{2b^3}}\left\{\int_{-b}^{0}e^{i(p_0-p)x/\hbar}(b+x)\,dx+\int_{0}^{b}e^{i(p_0-p)x/\hbar}(b-x)\,dx\right\}$$

$$=\sqrt{\frac{3}{4\pi\hbar b^3}}\left\{\left[\frac{e^{i(p_0-p)x/\hbar}\left(1+i(p_0-p)(b+x)/\hbar\right)}{(p_0-p)^2/\hbar^2}\right]_{-b}^{0}+\left[\frac{e^{i(p_0-p)x/\hbar}\left(-1+i(p_0-p)(b-x)/\hbar\right)}{(p_0-p)^2/\hbar^2}\right]_{0}^{b}\right\}$$

$$=\sqrt{\frac{3}{4\pi\hbar b^3}}\left\{\frac{\left(1+ib(p_0-p)/\hbar\right)-e^{-i(p_0-p)b/\hbar}}{(p_0-p)^2/\hbar^2}+\frac{-e^{i(p_0-p)b/\hbar}-\left(-1+ib(p_0-p)/\hbar\right)}{(p_0-p)^2/\hbar^2}\right\}$$

$$=\sqrt{\frac{3}{4\pi\hbar b^3}}\frac{\hbar^2\left\{2-e^{i(p_0-p)b/\hbar}-e^{-i(p_0-p)b/\hbar}\right\}}{(p_0-p)^2}$$

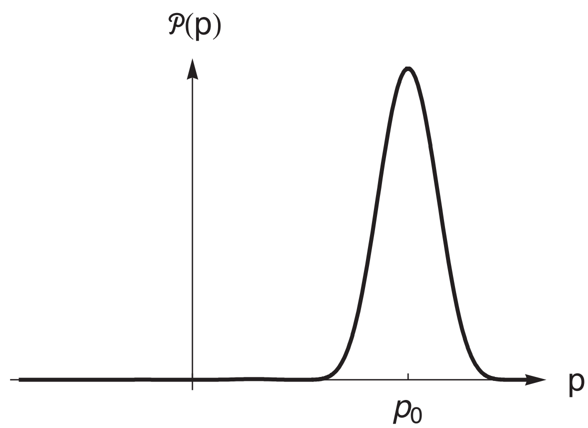
$$=\sqrt{\frac{3}{\pi\hbar b^3}}\frac{\hbar^2\left\{1-\cos\left((p_0-p)b/\hbar\right)\right\}}{(p_0-p)^2}$$

$$=\sqrt{\frac{3}{\pi\hbar b^3}}\frac{2\hbar^2\sin^2\left((p_0-p)b/2\hbar\right)}{(p_0-p)^2}$$

The momentum probability density is

$$\mathcal{P}(p)=|\phi(p)|^2=\frac{12\hbar^3}{\pi b^3}\frac{\sin^4\left((p_0-p)b/2\hbar\right)}{(p_0-p)^4}$$

Plot the momentum probability, using $p_0=4\pi\hbar/b$:



6.23 The position uncertainty is roughly the size of the nucleus $\Delta x=a=2\ fm$. This is 5 orders of magnitude smaller than the hydrogen atom, which implies a much larger energy, so we use the relativistic form $E=\sqrt{p^2c^2+m^2c^4}\simeq pc$. The uncertainty principle implies a minimum momentum

$$p_{\min} = \Delta p = \frac{\hbar}{2\Delta x} = \frac{\hbar}{2a}$$

and hence a minimum energy

$$E_{\min} = p_{\min} c = \frac{\hbar c}{2a}$$

For an electron in a 2 fm box, the energy is

$$E_{\min} = \frac{\hbar c}{2a} = \frac{hc}{4\pi a} = \frac{1240\ eV\ nm}{4\pi(2\ fm)} = 49\ MeV\ ,$$

which is definitely relativistic. The Coulomb potential energy for such a confined electron is

$$E_{Coulomb} = \frac{e^2}{4\pi\varepsilon_0 a} = \frac{e^2}{4\pi\varepsilon_0 \hbar c}\frac{hc}{2\pi a} = \alpha\frac{hc}{2\pi a} = \frac{1}{137}\frac{1240\ eV\ nm}{2\pi(2\ fm)} = 0.7\ MeV\ .$$

This potential energy well is not sufficient to confine this energetic electron, so electron confinement inside the nucleus by the Coulomb force is not sufficient. You would have to hypothesize some other force.

6.24 The boundary conditions for a wave incident on a well [Eq. (6.90)] are

$$\varphi_E(-a) : Ae^{-ik_1 a} + Be^{ik_1 a} = Ce^{-ik_2 a} + De^{ik_2 a}$$

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{x=-a} : ik_1 Ae^{-ik_1 a} - ik_1 Be^{ik_1 a} = ik_2 Ce^{-ik_2 a} - ik_2 De^{ik_2 a}$$

$$\varphi_E(a) : Ce^{ik_2 a} + De^{-ik_2 a} = Fe^{ik_1 a}$$

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{x=a} : ik_2 Ce^{ik_2 a} - ik_2 De^{-ik_2 a} = ik_1 Fe^{ik_1 a}$$

Solve the last two equations for C and D

$$\begin{aligned} Ce^{ik_2 a} + De^{-ik_2 a} &= Fe^{ik_1 a} \\ Ce^{ik_2 a} - De^{-ik_2 a} &= (k_1/k_2)Fe^{ik_1 a} \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2Ce^{ik_2 a} &= Fe^{ik_1 a}(1 + k_1/k_2) \\ 2De^{-ik_2 a} &= Fe^{ik_1 a}(1 - k_1/k_2) \end{aligned}$$

Similarly rearrange the first two equations:

$$\begin{aligned} 2Ae^{-ik_1 a} &= Ce^{-ik_2 a}(1 + k_2/k_1) + De^{ik_2 a}(1 - k_2/k_1) \\ 2Be^{ik_1 a} &= Ce^{-ik_2 a}(1 - k_2/k_1) + De^{ik_2 a}(1 + k_2/k_1) \end{aligned}$$

and substitute from above:

$$\begin{aligned} 4Ae^{-i2k_1 a} &= Fe^{-i2k_2 a}(1 + k_1/k_2)(1 + k_2/k_1) + Fe^{i2k_2 a}(1 - k_1/k_2)(1 - k_2/k_1) \\ 4B &= Fe^{-i2k_2 a}(1 + k_1/k_2)(1 - k_2/k_1) + Fe^{i2k_2 a}(1 - k_1/k_2)(1 + k_2/k_1) \end{aligned}$$

Simplify

$$4Ae^{-i2k_1a} = F\left[4\cos 2k_2a - \left(k_1/k_2 + k_2/k_1\right)2i\sin 2k_2a\right]$$
$$4B = F\left[-\left(k_1/k_2 - k_2/k_1\right)2i\sin 2k_2a\right]$$

and solve for the ratios *B*/*A* and *F*/*A*

$$\frac{F}{A} = \frac{e^{-i2k_1a}}{\cos 2k_2a - i\dfrac{k_1^2+k_2^2}{2k_1k_2}\sin 2k_2a}$$
$$\frac{B}{A} = \frac{ie^{-i2k_1a}\dfrac{k_2^2-k_1^2}{2k_1k_2}\sin 2k_2a}{\cos 2k_2a - i\dfrac{k_1^2+k_2^2}{2k_1k_2}\sin 2k_2a},$$

which agree with Eq. (6.91).

6.25 The transmission through a square well is unity when the well contains an integer number of half wavelengths [Eq. (6.97)]:

$$2k_2a = n\pi$$

In terms of energy this is

$$2\sqrt{2m\left(E+V_0\right)}a\Big/\hbar = n\pi$$

The width *a* is proportional to the integer *n*, so the minimum value of *a* occurs for the minimum value of *n*:

$$2\sqrt{2m\left(E+V_0\right)}a_{\min}\Big/\hbar = 1\pi$$
$$a_{\min} = \frac{\pi\hbar}{2\sqrt{2m\left(E+V_0\right)}} = \frac{hc}{4\sqrt{2mc^2\left(E+V_0\right)}}$$

Evaluating this gives

$$a_{\min} = \frac{1240\ eV\ nm}{4\sqrt{2\left(511\ keV\right)\left(32\ eV\right)}} = 0.054\ nm$$

The other possible energies for unity transmission are found with the other integers:

$$\frac{2m\left(E+V_0\right)}{\hbar^2} = \left(\frac{n\pi}{2a}\right)^2$$
$$E_n = \left(\frac{n\pi}{2a}\right)^2\frac{\hbar^2}{2m} - V_0 = n^2\frac{\pi^2\hbar^2}{8ma^2} - V_0.$$

Nothing that the lowest energy is 20 eV:

$$E_1 = \frac{\pi^2\hbar^2}{8ma^2} - V_0 = 20\ eV \quad \Rightarrow \quad \frac{\pi^2\hbar^2}{8ma^2} = E_1 + V_0 .$$

allows us to find the simple expression:

$$E_n = n^2\left(E_1 + V_0\right) - V_0 .$$

With values

$$\begin{aligned} E_n = n^2\left(E_1 + V_0\right) - V_0 &= E_1\ ,\ 4E_1 - 3V_0\ ,\ 9E_1 - 8V_0\ ,\ 16E_1 - 15V_0\ ,... \\ &= 20\ eV\ ,\ 116\ eV\ ,\ 276\ eV\ ,\ 500\ eV\ ,... \end{aligned} .$$

Every potential well has at least one bound state. To determine the number of bound states, find the parameter z_0 defined in Eq. (5.86):

$$z_0 = \sqrt{2mV_0a^2/\hbar^2}$$

Using *a* from above, we get

$$z_0 = \sqrt{\pi^2 V_0/4\left(E+V_0\right)} = \left(\pi/2\right)\sqrt{V_0/\left(E+V\right)_0} = \left(\pi/2\right)\sqrt{12eV/32eV} = 0.962$$

Recall from Chapter 5 that the well gains another bound state for each additional z_0 value of $\pi/2 = 1.57$. Hence this state has only one bound state.

6.26 The bound states of a finite well are determined by the parameter z_0 defined in Eq. (5.86):

$$z_0 = \sqrt{2mV_0a^2/\hbar^2} .$$

A well with 5 states has z_0 between 2π and 2.5π. This range of possibilities for z_0 corresponds to a range of possibilities for the well size 2*a*. The scattering resonances are given by Eq. (6.98)

$$E_n = n^2 \frac{\pi^2\hbar^2}{8ma^2} - V_0 .$$

In this case we do not know *n*, so we must treat the above as two equations with two unknowns (*n* and *a*), where we know only the possible range of z_0. Eliminating *a* first, we find

$$\frac{V_0}{4\left(E+V_0\right)} = \frac{z_0^2}{n^2\pi^2} ,$$

and solving for *n* gives

$$n = \sqrt{\frac{E+V_0}{V_0}}\frac{2z_0}{\pi} = \sqrt{\frac{11\ eV\ +\ 8\ eV}{8\ eV}}\left(\frac{z_0}{\pi/2}\right) = 1.54 \times \begin{cases} 4\ , & \text{minimum} \\ 5\ , & \text{maximum} \end{cases} .$$

So, the resonance must lie in the range

$$6.16 < n < 7.70 .$$

We conclude that the scattering resonance must have $n = 7$. Solving the resonance equation for the well half-width gives

$$a = \sqrt{\frac{n^2\pi^2\hbar^2}{8m(E_n + V_0)}} = \frac{nhc}{4\sqrt{2mc^2(E_n + V_0)}} = \frac{7(1240\ eV\ nm)}{4\sqrt{2(511\ keV)(19\ eV)}} = 0.49\ nm .$$

Thus the width of the well is $2a = 0.98\ nm$. The other possible energies for unity transmission are found with the other integers:

$$E_n = n^2 \frac{\pi^2\hbar^2}{8ma^2} - V_0 .$$

Nothing that the $n = 7$ resonance is 11 eV:

$$E_7 = 49\frac{\pi^2\hbar^2}{8ma^2} - V_0 = 11\ eV \quad \Rightarrow \quad \frac{\pi^2\hbar^2}{8ma^2} = \frac{1}{49}(E_7 + V_0).$$

Allows us to find the simple expression:

$$E_n = (n/7)^2 (E_1 + V_0) - V_0 .$$

Note that this result gives negative energies for $n < 5$. This is to be expected: the first scattering resonance has the same index as the highest bound state ($n = 5$ here). Starting with $n = 5$, the scattering resonances are

$$E_n = 1.69\ eV\ ,\ 5.96\ eV\ ,\ 11\ eV\ ,\ 16.82\ eV\ ,...\ .$$

6.27 The well parameter z_0 is

$$z_0 = \sqrt{\frac{2mV_0a^2}{\hbar^2}} = \sqrt{\frac{2(511keV)5eV(0.5nm/2)^2}{(1240eVnm/2\pi)^2}} = 2.864$$

which gives 2 bound states. The solutions to the transcendental equations are

$$z\tan(z) = \sqrt{z_0^2 - z^2} \quad \rightarrow \quad z = 1.155$$
$$-z\cot(z) = \sqrt{z_0^2 - z^2} \quad \rightarrow \quad z = 2.242$$

For this case, we put the zero of potential energy outside the well, so the energies of the bound states are

$$E = \frac{\hbar^2 z^2}{2ma^2} - 5eV = -4.185\ \text{eV}, \ \ -1.932\ \text{eV},$$

The scattering resonances are given by Eq. (6.98):

$$E_n = n^2 \frac{\pi^2\hbar^2}{8ma^2} - V_0 .$$

For this well, we get

$$E_n = n^2 \frac{(1240 eVnm)^2}{32(511keV)(0.5nm/2)^2} - 5eV = n^2 (1.504 eV) - 5eV .$$

Note that this result gives negative energies for $n < 2$. This is to be expected: the first scattering resonance has the same index as the highest bound state ($n = 2$ here). Starting with $n = 2$, the scattering resonances are

$$E_n = 1.018\ eV\ ,\ 8.541\ eV\ ,\ 19.07\ eV\ ,\ 32.61\ eV\ ,...\ .$$

6.28 The scattering resonances are given by Eq. (6.98), with the sign of the well/barrier changed:

$$E_n = n^2 \frac{\pi^2\hbar^2}{8ma^2} + V_0 .$$

For this well, we get

$$E_n = n^2 \frac{(1240 eVnm)^2}{32(511keV)(1nm/2)^2} + 5eV = n^2 (0.376 eV) + 5eV .$$

Starting with $n = 1$, the scattering resonances are

$$E_n = 5.376\ eV\ ,\ 6.505\ eV\ ,\ 8.385\ eV\ ,\ 11.02\ eV\ ,...\ .$$

6.29 a) When the energy of the incident particles is less than the height of the potential energy step, the wave function on the right side is a decaying exponential:

$$\varphi_E(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < 0 \\ Ce^{-qx}, & x > 0 \end{cases}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$
$$q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

The boundary conditions at the step are

$$\varphi(0): A + B = C$$
$$\left.\frac{d\varphi(x)}{dx}\right|_{x=0} : ikA - ikB = qC$$

Substitute the first equation into the second equation and solve for the ratio of the reflected amplitude to the incident amplitude

$$ikA - ikB = q(A+B)$$
$$ikA - qA = ikB + qB$$
$$\frac{B}{A} = \frac{ik+q}{ik-q}$$

The absolute square of this gives the reflection coefficient

$$R = \frac{|B|^2}{|A|^2} = \frac{ik+q}{ik-q} = \frac{k^2+q^2}{k^2+q^2} = 1$$

So 100% of the particles are reflected and there is no probability of transmission. There is some penetration of the wave function into the step, but the wave function decays to zero and never reaches infinity (where your detector is).

b) When the energy of the incident particles is greater than the height of the potential energy step, the wave function on the right side is a complex exponential:

$$\varphi_E(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < 0 \\ Ce^{ik_2x}, & x > 0 \end{cases}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$
$$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

The boundary conditions at the step are

$$\varphi(0): A + B = C$$
$$\left.\frac{d\varphi(x)}{dx}\right|_{x=0} : ik_1A - ik_1B = ik_2C$$

Substitute the first equation into the second equation and solve for the ratio of the reflected amplitude to the incident amplitude

$$ik_1A - ik_1B = ik_2(A+B)$$
$$ik_1A - ik_2A = ik_1B + ik_2B$$
$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

The absolute square of this gives the reflection coefficient

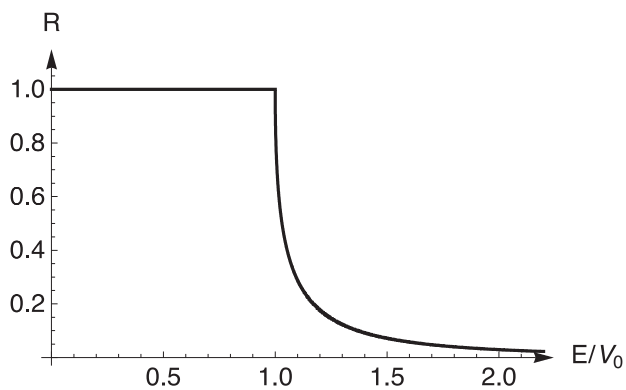
$$R = \frac{|B|^2}{|A|^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2$$

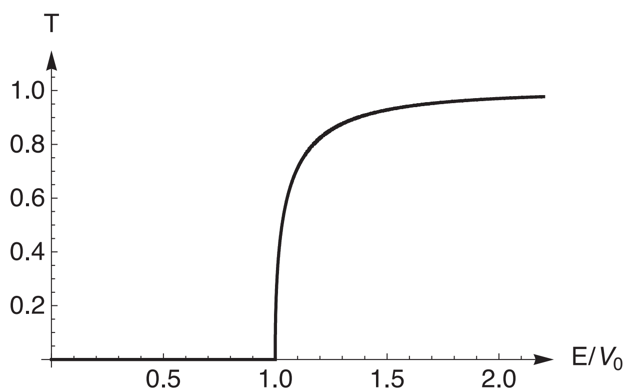
So less than 100% of the particles are reflected and there is some probability of transmission. To conserve particles, we require that $R + T = 1$, so the transmission is

$$T = 1 - R = 1 - \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{(k_1 + k_2)^2 - (k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{4k_1k_2}{(k_1 + k_2)^2} = \frac{4\sqrt{E}\sqrt{E - V_0}}{\left(\sqrt{E} + \sqrt{E - V_0}\right)^2}$$

Note that $T \neq |C|^2 / |A|^2$ in this problem because the wave speeds on the two sides are different.

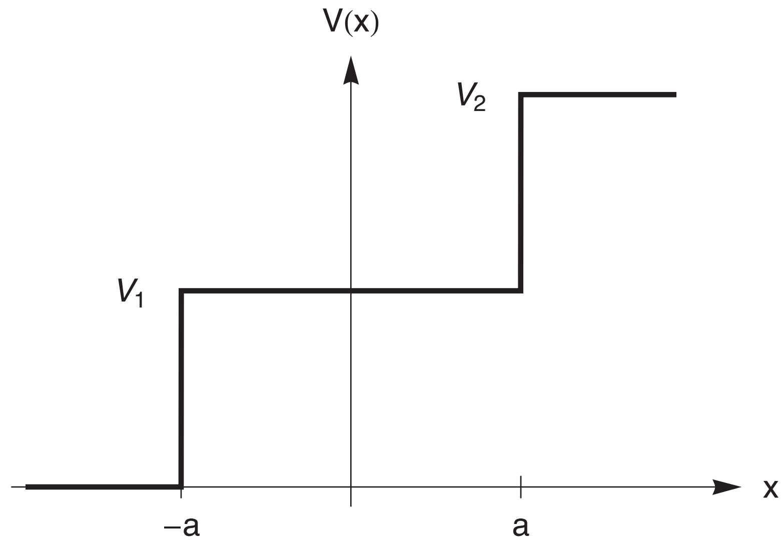
c) Plots:





The reflection is unity until the energy exceeds the step height, after which the reflection decreases monotonically. There are no resonances in this case because there are not two surfaces to generate multiple waves that can interfere (see Problem 6.30).

6.30 Consider the step potential shown below:



We put the potential steps at $x=a$ and $x=-a$ to make the algebra similar to the well problems done earlier. From Problem 6.29, we know that we need the incident energy to be above the final step height to get any transmission, so we assume that. The wave functions in the three regions are [see Eq. (6.89)]

$$\varphi_E(x)=\begin{cases} Ae^{ik_0x}+Be^{-ik_0x}, & x<-a \\ Ce^{ik_1x}+De^{-ik_1x}, & -a<x<a \\ Fe^{ik_2x}, & x>a \end{cases}$$

The wave vectors in the three regions are

$$k_0=\sqrt{\frac{2mE}{\hbar^2}}\ ,\quad k_1=\sqrt{\frac{2m(E-V_1)}{\hbar^2}}\ ,\quad k_2=\sqrt{\frac{2m(E-V_2)}{\hbar^2}}$$

The boundary conditions for a wave incident on the steps [see Eq. (6.90)] are

$$\varphi_E(-a): Ae^{-ik_0a}+Be^{ik_0a}=Ce^{-ik_1a}+De^{ik_1a}$$

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{x=-a}: ik_0Ae^{-ik_0a}-ik_0Be^{ik_0a}=ik_1Ce^{-ik_1a}-ik_1De^{ik_1a}$$

$$\varphi_E(a): Ce^{ik_1a}+De^{-ik_1a}=Fe^{ik_2a}$$

$$\left.\frac{d\varphi_E(x)}{dx}\right|_{x=a}: ik_1Ce^{ik_1a}-ik_1De^{-ik_1a}=ik_2Fe^{ik_2a}$$

Solve the last two equations for C and D

$$\begin{aligned} Ce^{ik_1a}+De^{-ik_1a}&=Fe^{ik_2a} \\ Ce^{ik_1a}-De^{-ik_1a}&=(k_2/k_1)Fe^{ik_2a} \end{aligned} \quad\Rightarrow\quad \begin{aligned} 2Ce^{ik_1a}&=Fe^{ik_2a}(1+k_2/k_1) \\ 2De^{-ik_1a}&=Fe^{ik_2a}(1-k_2/k_1) \end{aligned}$$

Similarly rearrange the first two equations:

$$2Ae^{-ik_0a} = Ce^{-ik_1a}\left(1+k_1/k_0\right) + De^{ik_1a}\left(1-k_1/k_0\right)$$
$$2Be^{ik_0a} = Ce^{-ik_1a}\left(1-k_1/k_0\right) + De^{ik_1a}\left(1+k_1/k_0\right)$$

and substitute from above:

$$4Ae^{-ik_0a}e^{-ik_2a} = Fe^{-i2k_1a}\left(1+k_2/k_1\right)\left(1+k_1/k_0\right) + Fe^{i2k_1a}\left(1-k_2/k_1\right)\left(1-k_1/k_0\right)$$
$$4Be^{ik_0a}e^{-ik_2a} = Fe^{-i2k_1a}\left(1+k_2/k_1\right)\left(1-k_1/k_0\right) + Fe^{i2k_1a}\left(1-k_2/k_1\right)\left(1+k_1/k_0\right)$$

Simplify

$$4Ae^{-i(k_0+k_2)a} = F\left[\left(1+k_2/k_0\right)2\cos 2k_1a - \left(k_2/k_1+k_1/k_0\right)2i\sin 2k_1a\right]$$
$$4Be^{i(k_0-k_2)a} = F\left[\left(1-k_2/k_0\right)2\cos 2k_1a - \left(k_2/k_1-k_1/k_0\right)2i\sin 2k_1a\right]$$

and solve for the ratios B/A and F/A

$$\frac{F}{A} = e^{-i(k_0+k_2)a}\frac{2}{\left(1+k_2/k_0\right)\cos 2k_1a - \left(k_2/k_1+k_1/k_0\right)i\sin 2k_1a}$$
$$\frac{B}{A} = e^{i2k_0a}\frac{\left(1-k_2/k_0\right)\cos 2k_1a - \left(k_2/k_1-k_1/k_0\right)i\sin 2k_1a}{\left(1+k_2/k_0\right)\cos 2k_1a - \left(k_2/k_1+k_1/k_0\right)i\sin 2k_1a}.$$

The absolute square of B/A gives the reflection coefficient

$$R = \frac{|B|^2}{|A|^2} = \frac{\left(1-k_2/k_0\right)^2\cos^2 2k_1a + \left(k_2/k_1-k_1/k_0\right)^2\sin^2 2k_1a}{\left(1+k_2/k_0\right)^2\cos^2 2k_1a + \left(k_2/k_1+k_1/k_0\right)^2\sin^2 2k_1a}$$

The reflection is zero and hence the transmission is unity when

$$2k_1a = (2n-1)\pi/2\ , \qquad n = 1,2,3,...$$

and

$$\left(k_2/k_1 - k_1/k_0\right) = 0 \quad \Rightarrow \quad k_1 = \sqrt{k_0k_2}\ ,$$

which in terms of energies is

$$E - V_1 = \sqrt{E\left(E-V_2\right)}$$
$$V_1 = E - \sqrt{E\left(E-V_2\right)} = E\left(1-\sqrt{1-V_2/E}\right)$$

For this choice of parameters, the waves reflected multiple times interfere destructively as in an optical anti-reflection (AR) coating.

6.31 The transmission probability is [Eq. (6.105)]

$$T = \frac{1}{1+\dfrac{V_0^2}{4E\left(V_0-E\right)}\sinh^2\left(\dfrac{2a}{\hbar}\sqrt{2m\left(V_0-E\right)}\right)}$$

For this barrier we get

$$T=\frac{1}{1+\frac{(10eV)^2}{4(5eV)(10eV-5eV)}\sinh^2\left(\frac{2(1nm/2)}{(1240eVnm/2\pi)}\sqrt{2(511keV)(10eV-5eV)}\right)}$$

$$=\frac{1}{1+1\sinh^2(11.454)}=4.5\times10^{-10}$$

6.32 The boundary conditions for a wave incident on a barrier [Eq. (6.104)] are

$$\varphi(-a): Ae^{-ika}+Be^{ika}=Ce^{-qa}+De^{qa}$$

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=-a}: ikAe^{-ika}-ikBe^{ika}=qCe^{-qa}-qDe^{qa}$$

$$\varphi(a): Ce^{qa}+De^{-qa}=Fe^{ika}$$

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=a}: qCe^{qa}-qDe^{-qa}=ikFe^{ika}$$

Solve the last two equations for *C* and *D*

$$\begin{aligned}Ce^{qa}+De^{-qa}&=Fe^{ika}\\ Ce^{qa}-De^{-qa}&=i(k/q)Fe^{ika}\end{aligned}\quad\Rightarrow\quad\begin{aligned}2Ce^{qa}&=Fe^{ika}(1+i\,k/q)\\ 2De^{-qa}&=Fe^{ika}(1-i\,k/q)\end{aligned}$$

Solve for the ratio of the growing exponential wave coefficient (*C*) to the decaying term coefficient (*D*):

$$\frac{C}{D}=e^{-2qa}\frac{(1+i\,k/q)}{(1-i\,k/q)}$$

and find the ratio of the absolute values:

$$\frac{|C|}{|D|}=e^{-2qa}\sqrt{\frac{(1+i\,k/q)(1-i\,k/q)}{(1-i\,k/q)(1+i\,k/q)}}=e^{-2qa}$$

Inside the barrier the growing part of the wave function is Ce^{qx} and the decaying part of the wave function is De^{-qx}. The ratio of their amplitudes is

$$\frac{\text{growing}}{\text{decaying}}=\frac{|C|e^{qx}}{|D|e^{-qx}}=\frac{|D|e^{-2qa}e^{qx}}{|D|e^{-qx}}=e^{-2q(a-x)}\le 1$$

Only at the far right side of the barrier are the two waves of equal magnitude.

6.33 The tunneling probability for a barrier of width d is (Eq. (6.105) with $d = 2a$)

$$T = \frac{1}{1 + \frac{\left(k^2+q^2\right)^2}{4k^2q^2}\sinh^2\left(qd\right)}$$

Expand the hyperbolic sine to get

$$T = \frac{1}{1 + \frac{\left(k^2+q^2\right)^2}{16k^2q^2}\left(e^{qd} - e^{-qd}\right)^2}$$

For $qd \gg 1$, the exponential term $e^{qd} \gg 1$ dominates, giving

$$T \cong \frac{1}{1 + \frac{\left(k^2+q^2\right)^2}{16k^2q^2}e^{2qd}} \cong \frac{16k^2q^2}{\left(k^2+q^2\right)^2}e^{-2qd}.$$

6.34 The barrier height is equal to the work function (5 eV), and the barrier width is $d = 2a = 1nm$ for the nominal 1 nA current. The bias voltage is minimal so the electron energy is close to zero. The decay constant in the well is

$$q = \frac{\sqrt{2m\left(V_0 - E\right)}}{\hbar} = \frac{\sqrt{2\left(511keV\right)\left(5eV - 0eV\right)}}{\left(1240eVnm/2\pi\right)} = 11.454$$

The tunneling current as a function of barrier width is [Eq. (6.108)]

$$I\left(d\right) = I_0 e^{-2qd}$$

which gives

$$I_0 = I\left(d\right)e^{+2qd} = I\left(1nm\right)e^{+2\left(11.454nm^{-1}\right)1nm} = 1nA\left(8.9\times10^9\right) = 8.9A$$

The currents at other separations are thus

$$I\left(d\right) = 8.9Ae^{-2qd} = 8.9Ae^{-2\left(11.454nm^{-1}\right)d}$$

$$I\left(0.8nm\right) = 8.9Ae^{-2\left(11.454nm^{-1}\right)0.8nm} = 98\ nA$$

$$I\left(1.2nm\right) = 8.9Ae^{-2\left(11.454nm^{-1}\right)1.2nm} = 10.2\ pA$$

$$I\left(2nm\right) = 8.9Ae^{-2\left(11.454nm^{-1}\right)2nm} = 1.1\times10^{-19}\ A$$
