4.1 Consider the action of a single-particle state operator S_{1z} on a two-particle state vector. Let the two-particle state vector have spin up for particle 1 ($|+\rangle_1$) and spin down for particle 2 ($|-\rangle_2$). For a product state, we get

$$S_{1z}|\psi\rangle = S_{1z}|+\rangle_1|-\rangle_2 = \left(S_{1z}|+\rangle_1\right)|-\rangle_2 = \left(+\frac{\hbar}{2}|+\rangle_1\right)|-\rangle_2 = +\frac{\hbar}{2}|+\rangle_1|-\rangle_2 = +\frac{\hbar}{2}|\psi\rangle$$

For a sum state, we get

$$S_{1z}|\psi\rangle = S_{1z}\left(|+\rangle_1 + |-\rangle_2\right) = \left(S_{1z}|+\rangle_1 + S_{1z}|-\rangle_2\right) = \left(+\frac{\hbar}{2}|+\rangle_1 + S_{1z}|-\rangle_2\right) = ???$$

For a product state, the result makes sense, while for the sum state, the result is ambiguous because it is not clear how S_{1z} acts on $|-\rangle_2$, if at all.

4.2 (a) Act on $|\psi\rangle$ with S_{1z}

$$S_{1z}|\psi\rangle = \frac{1}{\sqrt{2}}\left(S_{1z}|+\rangle_1|-\rangle_2 - S_{1z}|-\rangle_1|+\rangle_2\right) = \frac{1}{\sqrt{2}}\left(\left(+\frac{\hbar}{2}\right)|+\rangle_1|-\rangle_2 - \left(-\frac{\hbar}{2}\right)|-\rangle_1|+\rangle_2\right)$$
$$= \frac{1}{\sqrt{2}}\frac{\hbar}{2}\left(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\right) \neq \lambda|\psi\rangle$$

The result is not proportional to $|\psi\rangle$, so $|\psi\rangle$ is not an eigenstate.

(b) Check normalization

$$\langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle+|_1\langle-|_2 - \langle-|_1\langle+|_2\right)\frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right)$$
$$= \frac{1}{2}\left(\langle+|_1\langle-|_2\,|+\rangle_1|-\rangle_2 - \langle-|_1\langle+|_2|+\rangle_1|-\rangle_2 - \langle+|_1\langle-|_2|-\rangle_1|+\rangle_2 + \langle-|_1\langle+|_2|-\rangle_1|+\rangle_2\right)$$
$$= \frac{1}{2}\left(\langle+|+\rangle_1\langle-|-\rangle_2 - \langle-|+\rangle_1\langle+|-\rangle_2 - \langle+|-\rangle_1\langle-|+\rangle_2 + \langle-|-\rangle_1\langle+|+\rangle_2\right)$$
$$= \frac{1}{2}\left(1\bullet1 - 0\bullet0 - 0\bullet0 + 1\bullet1\right)$$
$$= 1$$

So the state $|\psi\rangle$ is normalized.

4.3 The probability that observer *A* measures particle 1 to have spin up along an arbitrary direction is the sum of the joint probabilities of observer *A* measuring particle 1 to have spin up and observer *B* measuring particle 2 to have spin up and down. So first calculate the probability that observer *A* records a "+" along the general direction $\hat{\mathbf{n}}$ and that observer *B* records a "+" along the *z*-axis. We get

$$\mathcal{P}_{+\hat{\mathbf{n}},+z} = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2}\langle + | \right) |\psi\rangle \right|^2 = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2}\langle + | \right) \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2$$

$$= \frac{1}{2} \left| {}_{1:\hat{\mathbf{n}}}\langle + | + \rangle_1 \; {}_{2}\langle + | - \rangle_2 + {}_{1:\hat{\mathbf{n}}}\langle + | - \rangle_1 \; {}_{2}\langle + | + \rangle_2 \right|^2 = \frac{1}{2} \left| {}_{1:\hat{\mathbf{n}}}\langle + | - \rangle_1 \right|^2$$

$$= \frac{1}{2} \left| \left(\cos\frac{\theta}{2} \; {}_{1}\langle + | + e^{-i\phi} \sin\frac{\theta}{2} \; {}_{1}\langle - | \right) |-\rangle_1 \right|^2 = \frac{1}{2} \sin^2\frac{\theta}{2}$$

Now calculate the probability that observer A records a "+" along the general direction $\hat{\mathbf{n}}$ and that observer B records a "-" along the z-axis. We get

$$\mathcal{P}_{+\hat{\mathbf{n}},-z} = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2}\langle - | \right) |\psi\rangle \right|^2 = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2}\langle - | \right) \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2$$

$$= \frac{1}{2} \left| \left(\cos\frac{\theta}{2} \; {}_{1}\langle + | + e^{-i\phi} \sin\frac{\theta}{2} \; {}_{1}\langle - | \right) |+\rangle_1 \right|^2 = \frac{1}{2} \cos^2\frac{\theta}{2}$$

Now sum the two probabilities:

$$\mathcal{P}_{+\hat{\mathbf{n}}A} = \mathcal{P}_{+\hat{\mathbf{n}},+z} + \mathcal{P}_{+\hat{\mathbf{n}},-z} = \frac{1}{2}\sin^2\frac{\theta}{2} + \frac{1}{2}\cos^2\frac{\theta}{2} = \frac{1}{2}$$

4.4 Calculate the probability that observer A records a "+" along the direction $\hat{\mathbf{n}}$ oriented at angles θ,ϕ and that observer B records a "-" along the same direction. For state $|\psi_a\rangle$, we get

$$\mathcal{P}_{+\hat{\mathbf{n}},-\hat{\mathbf{n}}} = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2:\hat{\mathbf{n}}}\langle - | \right) |\psi_a\rangle \right|^2 = \left| \left({}_{1:\hat{\mathbf{n}}}\langle + | \; {}_{2:\hat{\mathbf{n}}}\langle - | \right) \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2$$

$$= \frac{1}{2} \left| \left(\cos\frac{\theta}{2} \; {}_{1}\langle + | \; + \; e^{-i\phi} \sin\frac{\theta}{2} \; {}_{1}\langle - | \right) \left(\sin\frac{\theta}{2} \; {}_{2}\langle + | \; - \; e^{-i\phi} \cos\frac{\theta}{2} \; {}_{2}\langle - | \right) \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2$$

$$= \frac{1}{2} \left| -e^{-i\phi} \cos^2\frac{\theta}{2} - e^{-i\phi} \sin^2\frac{\theta}{2} \right|^2 = \frac{1}{2} \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \right) = \frac{1}{2}$$

Similarly, one gets $\mathcal{P}_{-\hat{\mathbf{n}},+\hat{\mathbf{n}}} = 1/2$, such that $\mathcal{P}_{same} = \mathcal{P}_{+\hat{\mathbf{n}},-\hat{\mathbf{n}}} + \mathcal{P}_{-\hat{\mathbf{n}},+\hat{\mathbf{n}}} = 1$, indicating perfect anticorrelation. For state $|\psi_b\rangle$, we get

$$\mathcal{P}_{+\hat{\mathbf{n}},-\hat{\mathbf{n}}}=\left|\left({}_{1:\hat{\mathbf{n}}}\langle+|\ {}_{2:\hat{\mathbf{n}}}\langle-|\right)|\psi_b\rangle\right|^2=\left|\left({}_{1:\hat{\mathbf{n}}}\langle+|\ {}_{2:\hat{\mathbf{n}}}\langle-|\right)\frac{1}{\sqrt{2}}\left(|+\rangle_{1x}|-\rangle_{2x}-|-\rangle_{1x}|+\rangle_{2x}\right)\right|^2$$

$$=\frac{1}{2}\left|\begin{array}{l}\left(\cos\frac{\theta}{2}\ {}_1\langle+|+e^{-i\phi}\sin\frac{\theta}{2}\ {}_1\langle-|\right)\left(\sin\frac{\theta}{2}\ {}_2\langle+|-e^{-i\phi}\cos\frac{\theta}{2}\ {}_2\langle-|\right)\\ \left\{\frac{1}{\sqrt{2}}\left(|+\rangle_1+|-\rangle_1\right)\frac{1}{\sqrt{2}}\left(|+\rangle_2-|-\rangle_2\right)-\frac{1}{\sqrt{2}}\left(|+\rangle_1-|-\rangle_1\right)\frac{1}{\sqrt{2}}\left(|+\rangle_2+|-\rangle_2\right)\right\}\end{array}\right|^2$$

$$=\frac{1}{2}\left|\begin{array}{l}\left(\cos\frac{\theta}{2}\ {}_1\langle+|+e^{-i\phi}\sin\frac{\theta}{2}\ {}_1\langle-|\right)\left(\sin\frac{\theta}{2}\ {}_2\langle+|-e^{-i\phi}\cos\frac{\theta}{2}\ {}_2\langle-|\right)\\ \left(-|+\rangle_1|-\rangle_2+|-\rangle_1|+\rangle_2\right)\end{array}\right|^2$$

$$=\frac{1}{8}\left|e^{-i\phi}\cos^2\frac{\theta}{2}+e^{-i\phi}\sin^2\frac{\theta}{2}\right|^2=\frac{1}{2}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)=\frac{1}{2}$$

Similarly, one gets $\mathcal{P}_{-\hat{\mathbf{n}},+\hat{\mathbf{n}}}=1/2$, such that $\mathcal{P}_{same}=\mathcal{P}_{+\hat{\mathbf{n}},-\hat{\mathbf{n}}}+\mathcal{P}_{-\hat{\mathbf{n}},+\hat{\mathbf{n}}}=1$, again indicating perfect anticorrelation.

4.5 Calculate the probability that observer A records a "+" along the direction $\hat{\mathbf{n}}_1$ oriented at an angle θ_1 with respect to the z-axis and that observer B records a "+" along the direction $\hat{\mathbf{n}}_2$ oriented at an angle θ_2 with respect to the z-axis. Let $\phi=0$ in both cases. We get

$$\mathcal{P}_{+\hat{\mathbf{n}}_1,+\hat{\mathbf{n}}_2}=\left|\left({}_{1:\hat{\mathbf{n}}_1}\langle+|\ {}_{2:\hat{\mathbf{n}}_2}\langle+|\right)|\psi\rangle\right|^2=\left|\left({}_{1:\hat{\mathbf{n}}_1}\langle+|\ {}_{2:\hat{\mathbf{n}}_2}\langle+|\right)\frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2-|-\rangle_1|+\rangle_2\right)\right|^2$$

$$=\frac{1}{2}\left|\left(\cos\frac{\theta_1}{2}\ {}_1\langle+|+\sin\frac{\theta_1}{2}\ {}_1\langle-|\right)\left(\cos\frac{\theta_2}{2}\ {}_2\langle+|+\sin\frac{\theta_2}{2}\ {}_2\langle-|\right)\left(|+\rangle_1|-\rangle_2-|-\rangle_1|+\rangle_2\right)\right|^2$$

$$=\frac{1}{2}\left|\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right|^2=\frac{1}{2}\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

Now calculate the probability that observer A records a "+" along the direction $\hat{\mathbf{n}}_1$ oriented at an angle θ_1 with respect to the z-axis and that observer B records a "-" along the direction $\hat{\mathbf{n}}_2$ oriented at an angle θ_2 with respect to the z-axis. Let $\phi=0$ in both cases. We get

$$\mathcal{P}_{+\hat{\mathbf{n}}_1,-\hat{\mathbf{n}}_2}=\left|\left({}_{1:\hat{\mathbf{n}}_1}\langle+|\ {}_{2:\hat{\mathbf{n}}_2}\langle-|\right)|\psi\rangle\right|^2=\left|\left({}_{1:\hat{\mathbf{n}}_1}\langle+|\ {}_{2:\hat{\mathbf{n}}_2}\langle-|\right)\frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2-|-\rangle_1|+\rangle_2\right)\right|^2$$

$$=\frac{1}{2}\left|\left(\cos\frac{\theta_1}{2}\ {}_1\langle+|+\sin\frac{\theta_1}{2}\ {}_1\langle-|\right)\left(\sin\frac{\theta_2}{2}\ {}_2\langle+|-\cos\frac{\theta_2}{2}\ {}_2\langle-|\right)\left(|+\rangle_1|-\rangle_2-|-\rangle_1|+\rangle_2\right)\right|^2$$

$$=\frac{1}{2}\left|-\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|^2=\frac{1}{2}\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

Similarly, we get

$$\begin{aligned}
\mathcal{P}_{-\hat{\mathbf{n}}_1,-\hat{\mathbf{n}}_2} &= \left| \left({}_{1:\hat{\mathbf{n}}_1}\langle -| \; {}_{2:\hat{\mathbf{n}}_2}\langle -| \right) |\psi\rangle \right|^2 = \left| \left({}_{1:\hat{\mathbf{n}}_1}\langle -| \; {}_{2:\hat{\mathbf{n}}_2}\langle -| \right) \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2 \\
&= \frac{1}{2} \left| \left(\sin\frac{\theta_1}{2} \; {}_1\langle +| - \cos\frac{\theta_1}{2} \; {}_1\langle -| \right) \left(\sin\frac{\theta_2}{2} \; {}_2\langle +| - \cos\frac{\theta_2}{2} \; {}_2\langle -| \right) \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2 \\
&= \frac{1}{2} \left| -\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \right|^2 = \frac{1}{2}\sin^2\left(\frac{\theta_1 - \theta_2}{2} \right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_{-\hat{\mathbf{n}}_1,+\hat{\mathbf{n}}_2} &= \left| \left({}_{1:\hat{\mathbf{n}}_1}\langle -| \; {}_{2:\hat{\mathbf{n}}_2}\langle +| \right) |\psi\rangle \right|^2 = \left| \left({}_{1:\hat{\mathbf{n}}_1}\langle -| \; {}_{2:\hat{\mathbf{n}}_2}\langle +| \right) \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2 \\
&= \frac{1}{2} \left| \left(\sin\frac{\theta_1}{2} \; {}_1\langle +| - \cos\frac{\theta_1}{2} \; {}_1\langle -| \right) \left(\cos\frac{\theta_2}{2} \; {}_2\langle +| + \sin\frac{\theta_2}{2} \; {}_2\langle -| \right) \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right) \right|^2 \\
&= \frac{1}{2} \left| \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \right|^2 = \frac{1}{2}\cos^2\left(\frac{\theta_1 - \theta_2}{2} \right)
\end{aligned}$$

Hence the probability for same results is

$$\mathcal{P}_{same} = \mathcal{P}_{+\hat{\mathbf{n}}_1,+\hat{\mathbf{n}}_2} + \mathcal{P}_{-\hat{\mathbf{n}}_1,-\hat{\mathbf{n}}_2} = \sin^2\left(\frac{\theta_1 - \theta_2}{2} \right)$$

and the probability for opposite results is

$$\mathcal{P}_{same} = \mathcal{P}_{+\hat{\mathbf{n}}_1,-\hat{\mathbf{n}}_2} + \mathcal{P}_{-\hat{\mathbf{n}}_1,+\hat{\mathbf{n}}_2} = \cos^2\left(\frac{\theta_1 - \theta_2}{2} \right)$$

Now use these results to calculate the average probability that the results are the same and the probability that the results are opposite, considering all possible measurements. There are 9 different combinations of measurement directions for the pair of observers. The angle $\theta_1 - \theta_2$ is 0° in 1/3 of the measurements ($\hat{\mathbf{a}}\hat{\mathbf{a}}, \hat{\mathbf{b}}\hat{\mathbf{b}}, \hat{\mathbf{c}}\hat{\mathbf{c}}$) and 120° in 2/3 of the measurements ($\hat{\mathbf{a}}\hat{\mathbf{b}}, \hat{\mathbf{a}}\hat{\mathbf{c}}, \hat{\mathbf{b}}\hat{\mathbf{a}}, \hat{\mathbf{b}}\hat{\mathbf{c}}, \hat{\mathbf{c}}\hat{\mathbf{a}}, \hat{\mathbf{c}}\hat{\mathbf{b}}$), so the average probabilities are

$$\begin{aligned}
\mathcal{P}_{same} &= \frac{1}{3}\cdot\sin^2\frac{0°}{2} + \frac{2}{3}\cdot\sin^2\frac{120°}{2} = \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2} \\
\mathcal{P}_{opp} &= \frac{1}{3}\cdot\cos^2\frac{0°}{2} + \frac{2}{3}\cdot\cos^2\frac{120°}{2} = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{2}
\end{aligned}$$

These predictions of quantum mechanics are inconsistent with the inequalities derived from hidden variable theories. These probabilities have been experimentally measured to test whether local hidden variable theories are possible. The results agree with quantum mechanics and hence exclude the possibility of local hidden variable theories.