9.1. The product of a and $a^\dagger$ is

$$aa^\dagger = \frac{m\omega}{2\hbar}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}\left[\hat{p}\hat{x} - \hat{x}\hat{p}\right]\right)$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}\left[\hat{p},\hat{x}\right]\right) = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega}\left[\hat{x},\hat{p}\right]\right)$$

The commutator of $\hat{x}$ and $\hat{p}$ is $\left[\hat{x},\hat{p}\right] = i\hbar$, giving

$$aa^\dagger = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) + \frac{1}{2}$$

9.2. To show that the raising operator deserves its name, act with $a^\dagger$ on an energy eigenstate $|E\rangle$, and then consider what happens when the Hamiltonian *H* acts on $a^\dagger|E\rangle$:

$$H\left(a^\dagger|E\rangle\right) = Ha^\dagger|E\rangle$$

The commutator $\left[H,a^\dagger\right] = +\hbar\omega a^\dagger$ tells us that $Ha^\dagger = a^\dagger H + \hbar\omega a^\dagger$, so we get

$$\begin{aligned} H\left(a^\dagger|E\rangle\right) &= \left(a^\dagger H + \hbar\omega a^\dagger\right)|E\rangle \\ &= a^\dagger H|E\rangle + \hbar\omega a^\dagger|E\rangle \end{aligned}$$

Now use the energy eigenvalue equation $H|E\rangle = E|E\rangle$ to obtain

$$\begin{aligned} H\left(a^\dagger|E\rangle\right) &= a^\dagger E|E\rangle + \hbar\omega a^\dagger|E\rangle \\ &= \left(E + \hbar\omega\right)\left(a^\dagger|E\rangle\right) \end{aligned}$$

This tells us that when the new ket $a^\dagger|E\rangle$ is acted on by the Hamiltonian *H*, the result is the same ket $a^\dagger|E\rangle$ multiplied by the factor $\left(E + \hbar\omega\right)$, which means that the new ket $a^\dagger|E\rangle$ is also an eigenstate of *H*, but with an energy eigenvalue $\left(E + \hbar\omega\right)$ that is larger than the eigenvalue *E* of the original ket $|E\rangle$ by one quantum of energy. So $a^\dagger$ has earned its name as the "raising operator."

9.3. The unnormalized wave function is $e^{-m\omega x^2/2\hbar}$. Now normalize:

$$1 = \int_{-\infty}^{\infty}\left|\varphi_0(x)\right|^2 dx = \int_{-\infty}^{\infty}\left|Ae^{-m\omega x^2/2\hbar}\right|^2 dx = |A|^2\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}\, dx$$

From Eq. (F.22) we get the integral, giving

$$1 = |A|^2\, 2\frac{1}{2\sqrt{m\omega/\hbar}}\sqrt{\pi} = |A|^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

Choosing the normalization constant to be real and positive gives

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

9.4. The forbidden region is beyond x_0 (or - x_0) where

$$E = T + U = \tfrac{1}{2} m\omega^2 x_0^2 = \tfrac{1}{2}\hbar\omega$$
$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

The probability of finding the particle in the forbidden region is

$$\mathcal{P}_{|x|>x_0} = 2\int_{x_0}^{\infty} \left|\varphi_0(x)\right|^2 dx = 2\int_{x_0}^{\infty} \left|\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}\right|^2 dx = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{\sqrt{\hbar/m\omega}}^{\infty} e^{-m\omega x^2/\hbar}\, dx$$

This integral can be looked up in a table and is often called the error function, defined by:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$
$$erfc(z) = 1 - erf(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt$$

We just need to make ours look like it. Change variables to $t = \sqrt{m\omega/\hbar}\,x$:

$$\mathcal{P}_{|x|>x_0} = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-t^2}\, dt = erfc(1) = 1 - erf(1)$$

From the CRC handbook, we have

$$erfc(x) = 1 - erf(x) = 2\left(1 - F\left(\sqrt{2}x\right)\right)$$
$$F\left(\sqrt{2}\right) = 0.9213$$
$$erfc(1) = 2(1 - 0.9213) = 0.157$$
$$\mathcal{P}_{|x|>x_0} = 0.157$$

9.5. To determine the proper scaling factor, consider the norm of the state $a^\dagger|n\rangle$. The rules of Hermitian conjugation allows us to write the norm as

$$\left| a^\dagger|n\rangle\right|^2 = \left(\langle n|a\right)\left(a^\dagger|n\rangle\right) = \langle n|aa^\dagger|n\rangle$$

The product $a^\dagger a$ is the number operator *N*, so the commutator $\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a = 1$ gives us the product $aa^\dagger$: $aa^\dagger = N + 1$. Substituting above gives

$$\left| a^\dagger|n\rangle\right|^2 = \langle n|(N+1)|n\rangle = \langle n|(n+1)|n\rangle = (n+1)\langle n|n\rangle = n+1$$

where we have used the normalization ($\langle n|n\rangle = 1$) of the energy/number eigenstates $|n\rangle$. Let c be the proportionality factor between the state $a^\dagger|n\rangle$ and the eigenstate $|n+1\rangle$:

$$a^\dagger|n\rangle = c|n+1\rangle$$

Because $|n\rangle$ and $|n+1\rangle$ are normalized to unity, we can use that to find the constant c:

$$\left|a^\dagger|n\rangle\right|^2 = \left|c|n+1\rangle\right|^2$$
$$n+1 = |c|^2$$

By convention, we choose the proportionality constant to be real and positive (an overall phase is not measurable) and obtain

$$\boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

9.6 Assume that

$$a|n\rangle = |n-1\rangle$$
$$a^\dagger|n\rangle = |n+1\rangle$$

Find action of $aa^\dagger$ and $a^\dagger a$ using these assumptions:

$$aa^\dagger|n\rangle = a|n+1\rangle = |n+1-1\rangle = |n\rangle$$
$$a^\dagger a|n\rangle = a^\dagger|n-1\rangle = |n-1+1\rangle = |n\rangle$$

The commutator action is

$$\left[a,a^\dagger\right]|n\rangle = \left(aa^\dagger - a^\dagger a\right)|n\rangle = aa^\dagger|n\rangle - a^\dagger a|n\rangle = |n\rangle - |n\rangle = 0$$

Therefore a and $a^\dagger$ would commute:

$$\left[a,a^\dagger\right] = 0$$

9.7. The orthogonality relation gives

$$\langle 1|0\rangle = \int_{-\infty}^{\infty}\varphi_1^*(x)\varphi_0(x)\,dx = \int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{m\omega}{2\hbar}}(2x)\,e^{-m\omega x^2/2\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}\,dx$$
$$= \left(\frac{m\omega}{\hbar}\right)\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} xe^{-m\omega x^2/\hbar}\,dx$$

The integrand is odd and the region of integration is even with respect to the origin, so the integral is zero. Equivalently, we can do the indefinite integral to get

$$\langle 1|0\rangle = \left(\frac{m\omega}{\hbar}\right)\sqrt{\frac{1}{2\pi}}\left(\frac{\hbar}{2m\omega}\right)\left[e^{-m\omega x^2/\hbar}\right]_{-\infty}^{\infty} = \left(\frac{m\omega}{\hbar}\right)\sqrt{\frac{1}{2\pi}}\left(\frac{\hbar}{2m\omega}\right)[0-0] = 0$$

9.8. The classical oscillator motion is described by

$$x(t) = x_0 \sin(\omega t)$$

where x_0 is the amplitude of the sinusoidal oscillation. The probability $\mathcal{P}(x)dx$ that the particle is found to be between positions x and $x + dx$ is simply the ratio of the time the particle spends in that region to the total time. Since the motion is symmetric with respect to the origin we can limit our analysis to the half cycle from $-x_0$ to $+x_0$, for which the total time is half a period

$$\frac{T}{2} = \frac{\pi}{\omega}$$

The time that the particle spends in the infinitesimal region dx is given by the ratio of the length of that region to the velocity at that point

$$dt = \frac{dx}{v(x)}$$

To find the velocity, let's use conservation of mechanical energy, noting that the energy at the end points is purely potential energy:

$$\begin{aligned} &\tfrac{1}{2} m\omega^2 x_0^2 = \tfrac{1}{2} m\omega^2 x^2 + \tfrac{1}{2} mv^2 \\ &v^2 = \omega^2 \left(x_0^2 - x^2\right) \end{aligned}$$

Thus we arrive at

$$\mathcal{P}(x)dx = \frac{dt}{T/2} = \frac{dt}{\pi/\omega} = \frac{dx}{v(x)}\frac{\omega}{\pi} = \frac{dx}{\omega\sqrt{x_0^2 - x^2}}\frac{\omega}{\pi} = \frac{dx}{\pi\sqrt{x_0^2 - x^2}}$$

From which we can find the classical probability density

$$\mathcal{P}(x) = \frac{1}{\pi\sqrt{x_0^2 - x^2}}$$

You can check that is properly normalized such that

$$\int_{-x_0}^{x_0} \mathcal{P}(x)dx = 1$$

9.9. i) Write the wave function using the constant $\beta^2 = m\omega/\hbar$:

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2 x^2/2}$$

The expectation value of position is

$$\langle x \rangle = \int_{-\infty}^{\infty} \varphi_0^*(x)\ x\ \varphi_0(x)dx = \int_{-\infty}^{\infty} x\left|\varphi_0(x)\right|^2 dx = 0$$

by symmetry since $|\varphi_0(x)|^2$ is even. For momentum:

$$\langle p\rangle = \int_{-\infty}^{\infty}\varphi_0^*(x)\frac{\hbar}{i}\frac{d}{dx}\varphi_0(x)dx = \int_{-\infty}^{\infty}\left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2x^2/2}\frac{\hbar}{i}\frac{d}{dx}\left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2x^2/2}dx$$
$$=\left(\frac{\beta^2}{\pi}\right)^{1/2}\frac{\hbar}{i}\int_{-\infty}^{\infty}e^{-\beta^2x^2/2}\left(-\beta^2x\right)e^{-\beta^2x^2/2}dx = \left(\frac{\beta^2}{\pi}\right)^{1/2}\frac{\hbar}{i}\left(-\beta^2\right)\int_{-\infty}^{\infty}xe^{-\beta^2x^2}dx$$
$$=0$$

For the squares:

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty}\varphi_0^*(x)x^2\varphi_0(x)dx = \int_{-\infty}^{\infty}\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\alpha x^2/2}x^2\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\alpha x^2/2}dx$$
$$=\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty}x^2e^{-\alpha x^2}dx = \sqrt{\frac{\alpha}{\pi}}\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha} = \frac{\hbar}{2m\omega}$$

$$\left\langle p^2\right\rangle = \int_{-\infty}^{\infty}\varphi_0^*(x)\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\varphi_0(x)dx = -\hbar^2\sqrt{\frac{\beta^2}{\pi}}\int_{-\infty}^{\infty}e^{-\beta^2x^2/2}\left(\frac{d}{dx}\right)^2e^{-\beta^2x^2/2}dx$$
$$=-\hbar^2\sqrt{\frac{\beta^2}{\pi}}\int_{-\infty}^{\infty}\left(\beta^4x^2-\beta^2\right)e^{-\beta^2x^2}dx = -\hbar^2\sqrt{\frac{\beta^2}{\pi}}\left[\beta^4\frac{1}{2\beta^2}\sqrt{\frac{\pi}{\beta^2}}-\beta^2\sqrt{\frac{\pi}{\beta^2}}\right] = \frac{\beta^2\hbar^2}{2}$$
$$=\frac{m\omega\hbar}{2}$$

ii) Now do the same for all states but using the operators a and $a^\dagger$.

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip+m\omega x\right)$$
$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(ip+m\omega x\right)$$
$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$
$$p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$

$$\langle x\rangle = \langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a^\dagger + a|n\rangle$$
$$=\sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a^\dagger|n\rangle + \langle n|a|n\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|\sqrt{n+1}|n+1\rangle + \langle n|\sqrt{n}|n-1\rangle\right]$$
$$=\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\langle n|n+1\rangle + \sqrt{n}\langle n|n-1\rangle\right] = 0 \quad \text{since} \quad \langle n|m\rangle = \delta_{nm}$$

$$\langle p\rangle = \langle n|p|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a^\dagger - a|n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a^\dagger|n\rangle - \langle n|a|n\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|\sqrt{n+1}|n+1\rangle - \langle n|\sqrt{n}|n-1\rangle\right]$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\langle n|n+1\rangle - \sqrt{n}\langle n|n-1\rangle\right] = 0 \quad \text{since} \quad \langle n|m\rangle = \delta_{nm}$$

Note also that $\langle n|a^2|n\rangle = 0$ and $\langle n|\left(a^\dagger\right)^2|n\rangle = 0$ in a similar manner, so that

$$\left\langle x^2\right\rangle = \langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^\dagger + a\right)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^\dagger\right)^2 + a^\dagger a + aa^\dagger + a^2|n\rangle$$
$$= \frac{\hbar}{2m\omega}\langle n|a^\dagger a + aa^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|\sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1}|n\rangle$$
$$= \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)$$

$$\left\langle p^2\right\rangle = \langle n|p^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|\left(a^\dagger - a\right)^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|\left(a^\dagger\right)^2 - a^\dagger a - aa^\dagger + a^2|n\rangle$$
$$= \frac{\hbar m\omega}{2}\langle n|a^\dagger a + aa^\dagger|n\rangle = \frac{\hbar}{2m\omega}\langle n|\sqrt{n}\sqrt{n} + \sqrt{n+1}\sqrt{n+1}|n\rangle$$
$$= \frac{\hbar m\omega}{2}(2n+1) = \hbar m\omega\left(n+\tfrac{1}{2}\right)$$

All of these agree with part (i)

iii) The uncertainty principle is $\Delta x\Delta p \geq \hbar/2$ where

$$\Delta x = \sqrt{\left\langle\left(x - \langle x\rangle\right)^2\right\rangle} = \sqrt{\left\langle x^2\right\rangle - \langle x\rangle^2} = \sqrt{\left\langle x^2\right\rangle} \quad \text{since} \quad \langle x\rangle = 0$$

$$\Delta p = \sqrt{\left\langle\left(p - \langle p\rangle\right)^2\right\rangle} = \sqrt{\left\langle p^2\right\rangle - \langle p\rangle^2} = \sqrt{\left\langle p^2\right\rangle} \quad \text{since} \quad \langle p\rangle = 0$$

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)}$$

$$\Delta p = \sqrt{\hbar m\omega\left(n+\tfrac{1}{2}\right)}$$

$$\Delta x\Delta p = \sqrt{\frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)}\sqrt{\hbar m\omega\left(n+\tfrac{1}{2}\right)} = \left(n+\tfrac{1}{2}\right)\hbar \geq \frac{\hbar}{2}$$

So the uncertainty relation is obeyed.

9.10. The matrix elements of the ladder operators are given by

$$\langle m|a|n\rangle = \langle m|\sqrt{n}\,|n-1\rangle \qquad \langle m|a^\dagger|n\rangle = \langle m|\sqrt{n+1}\,|n+1\rangle$$
$$= \sqrt{n}\ \delta_{m,n-1} \qquad\qquad = \sqrt{n+1}\ \delta_{m,n+1}$$

In component notation this is

$$a_{mn} = \sqrt{n}\ \delta_{m,n-1} \qquad a^\dagger_{mn} = \sqrt{n+1}\ \delta_{m,n+1}$$

with the first index labeling the row and the second the column. For example, a few elements are

$$a_{00} = \sqrt{0}\delta_{0,0-1} = 0,\ a_{01} = \sqrt{1}\delta_{0,1-1} = 1,\ a_{02} = \sqrt{2}\delta_{0,2-1} = 0,\ a_{12} = \sqrt{2}\delta_{1,2-1} = \sqrt{2}$$
$$a^\dagger_{00} = \sqrt{0+1}\ \delta_{0,0+1} = 0,\ a^\dagger_{01} = \sqrt{1+1}\ \delta_{0,1+1} = 0,\ a^\dagger_{10} = \sqrt{0+1}\ \delta_{1,0+1} = 1$$

The explicit matrix representations are

$$a \doteq \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\ a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad a^\dagger \doteq \begin{pmatrix} a^\dagger_{00} & a^\dagger_{01} & a^\dagger_{02} & a^\dagger_{03} & \cdots \\ a^\dagger_{10} & a^\dagger_{11} & a^\dagger_{12} & a^\dagger_{13} & \cdots \\ a^\dagger_{20} & a^\dagger_{21} & a^\dagger_{22} & a^\dagger_{23} & \cdots \\ a^\dagger_{30} & a^\dagger_{31} & a^\dagger_{32} & a^\dagger_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using the equations above, we get the matrix representations

$$a \doteq \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad a^\dagger \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

9.11 a) Normalize:

$$|\psi(t=0)\rangle = A\left[|0\rangle + 2e^{i\pi/2}|1\rangle\right]$$
$$1 = \langle\psi|\psi\rangle = A^*\left(\langle 0| + 2e^{-i\pi/2}\langle 1|\right)A\left(|0\rangle + 2e^{i\pi/2}|1\rangle\right)$$
$$= |A|^2(1+4) = |A|^2\,5$$
$$\Rightarrow A = \tfrac{1}{\sqrt{5}}$$

b) Time evolution

$$\begin{aligned}\left|\psi(t)\right\rangle &= \tfrac{1}{\sqrt{5}}\left(e^{-iE_0t/\hbar}\left|0\right\rangle + 2e^{i\pi/2}e^{-iE_1t/\hbar}\left|1\right\rangle\right)\\ &= e^{-i\omega t/2}\,\tfrac{1}{\sqrt{5}}\left(\left|0\right\rangle + 2e^{i\pi/2}e^{-i\omega t}\left|1\right\rangle\right)\end{aligned}$$

c) Expectations values with ladder ops:

$$\begin{aligned}\left\langle x\right\rangle &= \left\langle\psi(t)\right|x\left|\psi(t)\right\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\left\langle\psi(t)\right|a^\dagger + a\left|\psi(t)\right\rangle\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\omega t/2}\,\tfrac{1}{\sqrt{5}}\left(\left\langle 0\right| + 2e^{-i\pi/2}e^{+i\omega t}\left\langle 1\right|\right)\left(a^\dagger + a\right)e^{-i\omega t/2}\,\tfrac{1}{\sqrt{5}}\left(\left|0\right\rangle + 2e^{i\pi/2}e^{-i\omega t}\left|1\right\rangle\right)\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{1}{5}\left[2e^{i\pi/2}e^{-i\omega t}\left\langle 0\right|a\left|1\right\rangle + 2e^{-i\pi/2}e^{+i\omega t}\left\langle 1\right|a^\dagger\left|0\right\rangle\right]\\ &= \sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{2}{5}\left[ie^{-i\omega t}\sqrt{1} - ie^{+i\omega t}\sqrt{1}\right] = \sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{4}{5}\sin\omega t\end{aligned}$$

Momentum expectation value:

$$\begin{aligned}\left\langle p\right\rangle &= \left\langle\psi(t)\right|p\left|\psi(t)\right\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\left\langle\psi(t)\right|a^\dagger - a\left|\psi(t)\right\rangle\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\omega t/2}\,\tfrac{1}{\sqrt{5}}\left(\left\langle 0\right| + 2e^{-i\pi/2}e^{+i\omega t}\left\langle 1\right|\right)\left(a^\dagger - a\right)e^{-i\omega t/2}\,\tfrac{1}{\sqrt{5}}\left(\left|0\right\rangle + 2e^{i\pi/2}e^{-i\omega t}\left|1\right\rangle\right)\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{1}{5}\left[-2e^{i\pi/2}e^{-i\omega t}\left\langle 0\right|a\left|1\right\rangle + 2e^{-i\pi/2}e^{+i\omega t}\left\langle 1\right|a^\dagger\left|0\right\rangle\right]\\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{2}{5}\left[-ie^{-i\omega t}\sqrt{1} - ie^{+i\omega t}\sqrt{1}\right] = \sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{4}{5}\cos\omega t\end{aligned}$$

Ehrenfest's theorem in this case is

$$\left\langle p\right\rangle = m\frac{d}{dt}\left\langle x\right\rangle = m\frac{d}{dt}\left[\sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{4}{5}\sin\omega t\right] = m\sqrt{\tfrac{\hbar}{2m\omega}}\,\tfrac{4}{5}\left[\omega\cos\omega t\right] = \sqrt{\tfrac{m\omega\hbar}{2}}\,\tfrac{4}{5}\cos\omega t$$

So it is satisfied.

9.12 The "fast" expansion leaves the *wave function* the same, but it's no longer the ground state of the new system. Our job then is to find the overlap between the original state and the ground state of the new system and square that to find the probability. The original wavefunction is the ground state of the potential with frequency ω_1:

$$\psi_i(x) = \left(\frac{m\omega_1}{\pi\hbar}\right)^{1/4} e^{-m\omega_1x^2/2\hbar}$$

a) The probability that the system starts in state $\psi_i(x)$ and is measured to be in the ground state $\varphi_0(x)$ of the new system is the projection squared:

$$\mathcal{P}_0 = \left|\left\langle\varphi_0\middle|\psi_i\right\rangle\right|^2 = \left|\int_{-\infty}^{\infty}\varphi_0^*(x)\psi_i(x)\,dx\right|^2$$

The ground state of the new system (potential with frequency ω_2) is

$$\varphi_0(x) = \left(\frac{m\omega_2}{\pi\hbar}\right)^{1/4} e^{-m\omega_2x^2/2\hbar}$$

The projection of $\psi_i(x)$ onto $\varphi_0(x)$ is

$$\langle\varphi_0|\psi_i\rangle = \int_{-\infty}^{\infty}\varphi_0^*(x)\psi_i(x)\,dx = \int_{-\infty}^{\infty}\left(\frac{m\omega_2}{\pi\hbar}\right)^{1/4} e^{-m\omega_2 x^2/2\hbar}\left(\frac{m\omega_1}{\pi\hbar}\right)^{1/4} e^{-m\omega_1 x^2/2\hbar}\,dx$$

$$= \left(\omega_1\omega_2\right)^{1/4}\left(\frac{m}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} e^{-m(\omega_1+\omega_2)x^2/2\hbar}\,dx = \left(\frac{2\left(\omega_1\omega_2\right)^{1/2}}{\omega_1+\omega_2}\right)^{1/2}$$

The probability is thus

$$\mathcal{P}_0 = \left|\langle\varphi_0|\psi_i\rangle\right|^2 = \frac{2\left(\omega_1\omega_2\right)^{1/2}}{\omega_1+\omega_2} = \frac{2\sqrt{\dfrac{\omega_1}{\omega_2}}}{1+\dfrac{\omega_1}{\omega_2}} = \frac{2\sqrt{\dfrac{\omega_2}{\omega_1}}}{1+\dfrac{\omega_2}{\omega_1}}$$

First check whether this result makes sense. If $\omega_1 = \omega_2$ then nothing has been changed, and we see that $\mathcal{P}_0 = 1$ as it should.
b) If $\omega_2 = 1.7\omega_1$, then the probability is

$$\mathcal{P}_0 = \frac{2\sqrt{1.7}}{1+1.7} = 0.966$$

9.13 This new potential is "half" of the harmonic oscillator potential. Where the potentials are the same (x<0), the solutions should be the same. But for the new potential, the wave functions must be zero for x>0, where the potential energy is infinite. For the new wave functions to satisfy the continuity boundary condition, they must be zero at x=0. The odd numbered "full" potential wave functions GO TO ZERO at $x = 0$ and so will work for this new potential (at least the part of them for x<0). So the states of the new potential are the odd states of the "full" potential. Thus the energy levels are

$$E = \tfrac{3}{2}\hbar\omega,\ \tfrac{7}{2}\hbar\omega,\ \tfrac{11}{2}\hbar\omega,\ \tfrac{15}{2}\hbar\omega,\ .....$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \qquad for\ \ n = 1,3,5,7,...$$

$$E_m = \left(2m+\tfrac{3}{2}\right)\hbar\omega \qquad for\ \ m = 0,1,2,3,...$$

9.14 a) To solve this problem we want to write the state in terms of energy eigenstates. Rather than calculating the coefficients using spatial overlap integrals, let's try to write the state in a way that makes it obvious which energy states are included. First write out the harmonic oscillator functions to see how they relate to this state. To simplify notation, use the standard variable change $\xi = \sqrt{m\omega/\hbar}\;x = \beta x$:

$$\varphi_0(x)=\left(\tfrac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-m\omega x^2/2\hbar}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}e^{-\beta^2x^2/2}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}e^{-\xi^2/2}$$

$$\varphi_1(x)=\left(\tfrac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sqrt{\tfrac{m\omega}{\hbar}}\tfrac{1}{\sqrt{2}}2xe^{-m\omega x^2/2\hbar}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}\tfrac{1}{\sqrt{2}}2\beta xe^{-\beta^2x^2/2}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}\sqrt{2}\xi e^{-\xi^2/2}$$

$$\varphi_2(x)=\left(\tfrac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\tfrac{1}{\sqrt{8}}\left[4\tfrac{m\omega}{\hbar}x^2-2\right]e^{-m\omega x^2/2\hbar}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}\tfrac{1}{\sqrt{8}}\left[4\beta^2x^2-2\right]e^{-\beta^2x^2/2}=\left(\tfrac{\beta^2}{\pi}\right)^{\frac{1}{4}}\tfrac{1}{\sqrt{8}}\left[4\xi^2-2\right]e^{-\xi^2/2}$$

$$\Rightarrow\ e^{-\xi^2/2}=\varphi_0(x)\left(\tfrac{\pi}{\beta^2}\right)^{\frac{1}{4}};\ \ \xi e^{-\xi^2/2}=\tfrac{1}{\sqrt{2}}\varphi_1(x)\left(\tfrac{\pi}{\beta^2}\right)^{\frac{1}{4}};\ \ \left[2\xi^2-1\right]e^{-\xi^2/2}=\sqrt{2}\varphi_2(x)\left(\tfrac{\pi}{\beta^2}\right)^{\frac{1}{4}}$$

The initial wave function is thus

$$\begin{aligned}\psi(x,0)&=A\left(1-3\sqrt{\tfrac{m\omega}{\hbar}}x+2\tfrac{m\omega}{\hbar}x^2\right)e^{-m\omega x^2/2\hbar}=A\left(1-3\beta x+2\beta^2x^2\right)e^{-\beta^2x^2/2}\\&=A\left(1-3\xi+2\xi^2\right)e^{-\xi^2/2}=A\left(2(1)-3(\xi)+1\left(2\xi^2-1\right)\right)e^{-\xi^2/2}\\&=A\left(\tfrac{\pi}{\beta^2}\right)^{\frac{1}{4}}\left(2\varphi_0(x)-\tfrac{3}{\sqrt{2}}\varphi_1(x)+\sqrt{2}\varphi_2(x)\right)\end{aligned}$$

Switching to bra-ket notation, we have

$$|\psi(0)\rangle=C\left(2|0\rangle-\tfrac{3}{\sqrt{2}}|1\rangle+\sqrt{2}|2\rangle\right)$$

where C is the normalization constant. Normalize to find C

$$\begin{aligned}1&=\left|\langle\psi(0)|\psi(0)\rangle\right|=|C|^2\left(2\langle 0|-\tfrac{3}{\sqrt{2}}\langle 1|+\sqrt{2}\langle 2|\right)\left(2|0\rangle-\tfrac{3}{\sqrt{2}}|1\rangle+\sqrt{2}|2\rangle\right)\\&=|C|^2\left(4+\tfrac{9}{2}+2\right)=\tfrac{21}{2}|C|^2\qquad\Rightarrow C=\tfrac{\sqrt{2}}{\sqrt{21}}\end{aligned}$$

Now find the expectation value of the energy:

$$\begin{aligned}\langle E\rangle&=\sum_n E_n\mathcal{P}_{E_n}=\sum_n\left(n+\tfrac{1}{2}\right)\hbar\omega\mathcal{P}_{E_n}\\ \mathcal{P}_{E_n}&=\left|\langle n|\psi(0)\rangle\right|^2=\left|\langle n|\tfrac{\sqrt{2}}{\sqrt{21}}\left(2|0\rangle-\tfrac{3}{\sqrt{2}}|1\rangle+\sqrt{2}|2\rangle\right)\right|^2=\tfrac{2}{21}\left|2\langle n|0\rangle-\tfrac{3}{\sqrt{2}}\langle n|1\rangle+\sqrt{2}\langle n|2\rangle\right|^2\\&=\tfrac{2}{21}\left(4\delta_{n0}+\tfrac{9}{2}\delta_{n1}+2\delta_{n2}\right)\\ \langle E\rangle&=\sum_n\left(n+\tfrac{1}{2}\right)\hbar\omega\tfrac{2}{21}\left(4\delta_{n0}+\tfrac{9}{2}\delta_{n1}+2\delta_{n2}\right)=\tfrac{2}{21}\hbar\omega\left[4\tfrac{1}{2}+\tfrac{9}{2}\tfrac{3}{2}+2\tfrac{5}{2}\right]=\tfrac{2}{21}\hbar\omega\left[2+\tfrac{27}{4}+5\right]\\&=\tfrac{55}{42}\hbar\omega\cong 1.31\ \hbar\omega\end{aligned}$$

b) The wave function at the later time T can also be written as a superposition of states. We know how the original state evolves, so we can relate the two expressions:

$$\begin{aligned}
\psi(x,T) &= B\left(3-3i\sqrt{\tfrac{m\omega}{\hbar}}x-2\tfrac{m\omega}{\hbar}x^2\right)e^{-m\omega x^2/2\hbar} = B\left(3-3i\beta x-2\beta^2x^2\right)e^{-\beta^2x^2/2} \\
&= B\left(3-3\xi-2\xi^2\right)e^{-\xi^2/2} = B\left(2(1)-3i(\xi)-1\left(2\xi^2-1\right)\right)e^{-\xi^2/2} \\
&= B\left(\tfrac{\pi}{\beta^2}\right)^{\frac{1}{4}}\left(2\varphi_0(x)-\tfrac{3}{\sqrt{2}}i\varphi_1(x)-\sqrt{2}\varphi_2(x)\right)
\end{aligned}$$

or in Dirac notation

$$\left|\psi(T)\right\rangle = C\left(2|0\rangle-\tfrac{3}{\sqrt{2}}i|1\rangle-\sqrt{2}|2\rangle\right)$$

The expected Schrödinger time evolution is

$$\begin{aligned}
&\left|\psi(T)\right\rangle = Ce^{-i\omega t/2}\left(2|0\rangle-\tfrac{3}{\sqrt{2}}e^{-i\omega T}|1\rangle+\sqrt{2}e^{-i2\omega T}|2\rangle\right) \\
&\Rightarrow e^{-i\omega T}=i \quad and \quad e^{-i2\omega T}=-1 \\
&\Rightarrow \omega T=\frac{3\pi}{2}
\end{aligned}$$

$$\boxed{T=\frac{3\pi}{2\omega}}$$

9.15 The two energies are the n = 0, 1 states, so the measurement tells us that:

$$\mathcal{P}_{E_n} = \left|\left\langle n|\psi(t)\right\rangle\right|^2 = \tfrac{1}{2}\left(\delta_{n0}+\delta_{n1}\right)$$

Energy measurements are time independent, so we have no information on the time dependence of the amplitudes, but we know something about them from the Schrödinger time evolution. Thus the original state and the time-evolved state are:

$$\begin{aligned}
&\left|\psi(0)\right\rangle = c_0|0\rangle+c_1|1\rangle \quad\Rightarrow\quad |c_0|^2=|c_1|^2=\tfrac{1}{2} \\
&\left|\psi(0)\right\rangle = \tfrac{1}{\sqrt{2}}\left[e^{i\theta_0}|0\rangle+e^{i\theta_1}|1\rangle\right] \\
&\left|\psi(t)\right\rangle = e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left[e^{i\theta_0}|0\rangle+e^{i\theta_1}e^{-i\omega t}|1\rangle\right]
\end{aligned}$$

Now use this to find the expectation value of the position:

$$\begin{aligned}
\langle x\rangle &= \left\langle\psi(t)\right|x\left|\psi(t)\right\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\left\langle\psi(t)\right|a^\dagger+a\left|\psi(t)\right\rangle \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{-i\theta_0}\langle 0|+e^{-i\theta_1}\langle 1|e^{+i\omega t}\right)\left(a^\dagger+a\right)e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{i\theta_0}|0\rangle+e^{i\theta_1}|1\rangle e^{-i\omega t}\right) \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{2}\left[e^{-i\theta_1}e^{+i\omega t}e^{i\theta_0}\langle 1|a^\dagger|0\rangle+e^{-i\theta_0}e^{i\theta_1}e^{-i\omega t}\langle 0|a|1\rangle\right] \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{2}\left[e^{+i\omega t+i\theta_0-i\theta_1}+e^{-i\omega t-i\theta_0+i\theta_1}\right] = \sqrt{\tfrac{\hbar}{2m\omega}}\cos\left(\omega t+\theta_0-\theta_1\right)
\end{aligned}$$

Hence $\langle x\rangle = -\sqrt{\hbar/2m\omega}\sin\omega t$ implies that $\left(\theta_0-\theta_1\right)=\pi/2$. The overall phase is unknown but doesn't matter (cannot be measured). Now use this to find the expectation value of the momentum:

$$\begin{aligned}
\langle p\rangle &= \langle\psi(t)|p|\psi(t)\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\langle\psi(t)|a^\dagger - a|\psi(t)\rangle \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\frac{\omega t}{2}}\tfrac{i}{\sqrt{2}}\left(e^{-i\theta_0}\langle 0| + e^{-i\theta_1}\langle 1|e^{+i\omega t}\right)\left(a^\dagger - a\right)e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{i\theta_0}|0\rangle + e^{i\theta_1}|1\rangle e^{-i\omega t}\right) \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{2}\left(e^{-i\theta_1}e^{+i\omega t}e^{i\theta_0}\langle 1|a^\dagger|0\rangle - e^{-i\theta_0}e^{i\theta_1}e^{-i\omega t}\langle 0|a|1\rangle\right) \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{2}\left(e^{+i\omega t+i\theta_0-i\theta_1} - e^{-i\omega t-i\theta_0+i\theta_1}\right) = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\left(\omega t+\theta_0-\theta_1\right) = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\left(\omega t+\pi/2\right) \\
&= -\sqrt{\tfrac{m\omega\hbar}{2}}\cos\omega t
\end{aligned}$$

9.16 The two measured energies are the n = 1, 2 states, so the measurement tells us that:

$$\begin{aligned}
\mathcal{P}_{E_n} &= \left|\langle n|\psi(t)\rangle\right|^2 = |c_n|^2 \\
\mathcal{P}_{E_1} &= \left|\langle 1|\psi(t)\rangle\right|^2 = |c_1|^2 = 0.36 = \tfrac{9}{25} = \left(\tfrac{3}{5}\right)^2 \Rightarrow c_1 = \tfrac{3}{5}e^{i\theta_1} \\
\mathcal{P}_{E_2} &= \left|\langle 2|\psi(t)\rangle\right|^2 = |c_2|^2 = 0.64 = \tfrac{16}{25} = \left(\tfrac{4}{5}\right)^2 \Rightarrow c_2 = \tfrac{4}{5}e^{i\theta_2}
\end{aligned}$$

Thus we can construct the original state and the time evolved state:

$$\begin{aligned}
|\psi(0)\rangle &= \tfrac{1}{5}\left[3e^{i\theta_1}|1\rangle + 4e^{i\theta_2}|2\rangle\right] \\
|\psi(t)\rangle &= \tfrac{1}{5}\left[3e^{i\theta_1}e^{-iE_1t/\hbar}|1\rangle + 4e^{i\theta_2}e^{-iE_2t/\hbar}|2\rangle\right] = \tfrac{1}{5}\left[3e^{i\theta_1}e^{-i\frac{3\omega t}{2}}|1\rangle + 4e^{i\theta_2}e^{-i\frac{5\omega t}{2}}|2\rangle\right] \\
&= \tfrac{1}{5}e^{i\theta_1}e^{-i\frac{3\omega t}{2}}\left[3|1\rangle + 4e^{i(\theta_2-\theta_1)}e^{-i\omega t}|2\rangle\right]
\end{aligned}$$

Now use this to find the expectation value of the position:

$$\begin{aligned}
\langle x\rangle &= \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\langle\psi(t)|a^\dagger + a|\psi(t)\rangle \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{-i\theta_1}\langle 1| + 4e^{-i\theta_2}e^{+i\omega t}\langle 2|\right)\left(a^\dagger + a\right)e^{-i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{i\theta_1}|1\rangle + 4e^{i\theta_2}e^{-i\omega t}|2\rangle\right) \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{25}\left[12e^{-i\theta_2}e^{+i\omega t}e^{i\theta_1}\langle 2|a^\dagger|1\rangle + 12e^{-i\theta_1}e^{i\theta_2}e^{-i\omega t}\langle 1|a|2\rangle\right] \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{12}{25}\sqrt{2}\left[e^{+i\omega t+i\theta_1-i\theta_2} + e^{-i\omega t-i\theta_1+i\theta_2}\right] = \sqrt{\tfrac{\hbar}{m\omega}}\tfrac{24}{25}\cos\left(\omega t+\theta_1-\theta_2\right)
\end{aligned}$$

If $\langle x\rangle$ is a minimum at $t=0$, then $\cos(\theta_1-\theta_2)=-1$ and $(\theta_1-\theta_2)=\pi$. The overall phase is unknown but doesn't matter (cannot be measured). The time dependent wave function is

$$\begin{aligned}
\psi(x,t) &= e^{-i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{i\theta_1}\varphi_1(x) + 4e^{i\theta_2}e^{-i\omega t}\varphi_2(x)\right) = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) + 4e^{i(\theta_2-\theta_1)}e^{-i\omega t}\varphi_2(x)\right) \\
&= e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) + 4e^{-i\pi}e^{-i\omega t}\varphi_2(x)\right) = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) - 4e^{-i\omega t}\varphi_2(x)\right)
\end{aligned}$$

In Dirac notation, the state is

$$|\psi(t)\rangle = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left[3|1\rangle - 4e^{-i\omega t}|2\rangle\right]$$

b) The expectation value of the momentum is:

$$\begin{aligned}\langle p\rangle &= \langle\psi(t)|p|\psi(t)\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\langle\psi(t)|a^\dagger - a|\psi(t)\rangle \\ &= i\sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\frac{3\omega t}{2}}e^{-i\theta_1}\tfrac{1}{5}\left(3\langle 1| - 4e^{+i\omega t}\langle 2|\right)\left(a^\dagger - a\right)e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3|1\rangle - 4e^{-i\omega t}|2\rangle\right) \\ &= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{25}\left(-12e^{+i\omega t}\langle 2|a^\dagger|1\rangle + 12e^{-i\omega t}\langle 1|a|2\rangle\right) = \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{12i}{25}\sqrt{2}\left(-e^{+i\omega t} + e^{-i\omega t}\right) \\ &= \sqrt{m\omega\hbar}\tfrac{24}{25}\sin\omega t\end{aligned}$$

c) The expectation value of the energy is

$$\begin{aligned}\langle E\rangle &= \sum_n E_n\mathcal{P}_{E_n} = \sum_n\left(n+\tfrac{1}{2}\right)\hbar\omega\mathcal{P}_{E_n} \\ \mathcal{P}_{E_n} &= \left|\langle n|\psi(t)\rangle\right|^2 = \left|\langle n|e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3|1\rangle - 4e^{-i\omega t}|2\rangle\right)\right|^2 = \tfrac{1}{25}\left|3\langle n|1\rangle - 4\langle n|2\rangle\right|^2 \\ &= \tfrac{1}{25}\left|9\delta_{n1} + 16\delta_{n2}\right| \\ \langle E\rangle &= \sum_n\left(n+\tfrac{1}{2}\right)\hbar\omega\tfrac{1}{25}\left|9\delta_{n1}+16\delta_{n2}\right| = \tfrac{1}{25}\hbar\omega\left[9\tfrac{3}{2}+16\tfrac{5}{2}\right] = \tfrac{1}{25}\hbar\omega\left[\tfrac{107}{2}\right] \\ &= \tfrac{107}{50}\hbar\omega = 2.14\hbar\omega\end{aligned}$$

9.17 a) Before we do any calculations we must normalize the state. Let's use Dirac notation to simplify the calculations.

$$\begin{aligned}|\psi(0)\rangle &= A\left[|0\rangle + 2|1\rangle + 2|2\rangle\right] \\ 1 = \langle\psi(0)|\psi(0)\rangle &= A^*\left[\langle 0| + 2\langle 1| + 2\langle 2|\right]A\left[|0\rangle + 2|1\rangle + 2|2\rangle\right] \\ &= |A|^2\left[\langle 0|0\rangle + 4\langle 1|1\rangle + 4\langle 2|2\rangle\right] = |A|^2\,9 \qquad \Rightarrow A = \tfrac{1}{3}\end{aligned}$$

We are free to choose A to be real and positive because an overall phase is not physical. Now let's find the time dependent state vector. The component states are energy eigenstates, so we simply multiply by the appropriate phase factors:

$$\begin{aligned}|\psi(0)\rangle &= \tfrac{1}{3}\left(|0\rangle + 2|1\rangle + 2|2\rangle\right) \\ |\psi(t)\rangle &= \tfrac{1}{3}\left(e^{-i\frac{E_0}{\hbar}t}|0\rangle + 2e^{-i\frac{E_1}{\hbar}t}|1\rangle + 2e^{-i\frac{E_2}{\hbar}t}|2\rangle\right) = \tfrac{1}{3}\left(e^{-i\frac{\omega}{2}t}|0\rangle + 2e^{-i\frac{3\omega}{2}t}|1\rangle + 2e^{-i\frac{5\omega}{2}t}|2\rangle\right) \\ &= \tfrac{1}{3}e^{-i\frac{\omega}{2}t}\left(|0\rangle + 2e^{-i\omega t}|1\rangle + 2e^{-i2\omega t}|2\rangle\right)\end{aligned}$$

Because there are three energy eigenstates in the state vector, there are three possible energies $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ that can be measured: $E_0 = \frac{1}{2}\hbar\omega,\ E_1 = \frac{3}{2}\hbar\omega,\ E_2 = \frac{5}{2}\hbar\omega$. The probabilities are

$$\mathcal{P}_{E_n} = \left|\langle n|\psi(t)\rangle\right|^2 = \left|\langle n|\tfrac{1}{3}e^{-i\frac{\omega}{2}t}\left(|0\rangle + 2e^{-i\omega t}|1\rangle + 2e^{-i2\omega t}|2\rangle\right)\right|^2$$

$$\boxed{\begin{aligned}\mathcal{P}_{E_0} &= \left(\tfrac{1}{3}\right)^2 = \tfrac{1}{9} = 0.11\overline{1}\\ \mathcal{P}_{E_1} &= \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9} = 0.44\overline{4}\\ \mathcal{P}_{E_2} &= \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9} = 0.44\overline{4}\end{aligned}}$$

The energy states are stationary states, so this result is time independent.

b) To find the expectation value of the momentum, use ladder operators rather than doing spatial integrals.

$$\begin{aligned}\langle p\rangle &= \langle\psi(t)|p|\psi(t)\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle\psi(t)|a^\dagger - a|\psi(t)\rangle\\ &= i\sqrt{\frac{m\omega\hbar}{2}}\,\tfrac{1}{3}e^{+i\frac{\omega}{2}t}\left[\langle 0| + 2e^{+i\omega t}\langle 1| + 2e^{+i2\omega t}\langle 2|\right]\left(a^\dagger - a\right)\tfrac{1}{3}e^{-i\frac{\omega}{2}t}\left[|0\rangle + 2e^{-i\omega t}|1\rangle + 2e^{-i2\omega t}|2\rangle\right]\\ &= i\sqrt{\frac{m\omega\hbar}{2}}\,\tfrac{1}{9}\left[-2e^{-i\omega t}\langle 0|a|1\rangle - 4e^{-i\omega t}\langle 1|a|2\rangle + 2e^{+i\omega t}\langle 1|a^\dagger|0\rangle + 4e^{+i\omega t}\langle 2|a^\dagger|1\rangle\right]\\ &= i\sqrt{\frac{m\omega\hbar}{2}}\,\tfrac{1}{9}\left[-2\sqrt{1}e^{-i\omega t} - 4\sqrt{2}e^{-i\omega t} + 2\sqrt{1}e^{+i\omega t} + 4\sqrt{2}e^{+i\omega t}\right]\\ &= i\sqrt{\frac{m\omega\hbar}{2}}\,\tfrac{1}{9}\left(2 + 4\sqrt{2}\right)\left(e^{+i\omega t} - e^{-i\omega t}\right)\end{aligned}$$

$$\boxed{\langle p\rangle = -\sqrt{\frac{m\omega\hbar}{2}}\,\tfrac{4}{9}\left(1 + 2\sqrt{2}\right)\sin\omega t}$$

c) Here again, Dirac notation is simpler. The energy states are stationary states, so this result is time independent.

$$\begin{aligned}\langle E\rangle &= \langle\psi|H|\psi\rangle = \tfrac{1}{3}\left[\langle 0| + 2\langle 1| + 2\langle 2|\right]H\,\tfrac{1}{3}\left[|0\rangle + 2|1\rangle + 2|2\rangle\right]\\ &= \tfrac{1}{9}\left[\langle 0|H|0\rangle + 4\langle 1|H|1\rangle + 4\langle 2|H|2\rangle\right] = \tfrac{1}{9}\left[\tfrac{1}{2}\hbar\omega + 4\tfrac{3}{2}\hbar\omega + 4\tfrac{5}{2}\hbar\omega\right] = \tfrac{33}{18}\hbar\omega\end{aligned}$$

$$\boxed{\langle E\rangle = \tfrac{11}{6}\hbar\omega \cong 1.83\ \hbar\omega}$$

The same result is obtained with the probability weighting method:

$$\langle E\rangle = \sum_n E_n\mathcal{P}_{E_n} = \tfrac{1}{2}\hbar\omega\tfrac{1}{9} + \tfrac{3}{2}\hbar\omega\tfrac{4}{9} + \tfrac{5}{2}\hbar\omega\tfrac{4}{9} = \tfrac{33}{18}\hbar\omega = \tfrac{11}{6}\hbar\omega$$

d) Now find the standard deviation of the energy.

$$\Delta E = \sqrt{\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle} = \sqrt{\left\langle E^2 - 2E\langle E \rangle + \langle E \rangle^2 \right\rangle} = \sqrt{\left\langle E^2 \right\rangle - \langle E \rangle^2}$$

$$\left\langle E^2 \right\rangle = \langle \psi | H^2 | \psi \rangle = \tfrac{1}{3}\left[\langle 0| + 2\langle 1| + 2\langle 2|\right] H^2 \tfrac{1}{3}\left[|0\rangle + 2|1\rangle + 2|2\rangle\right]$$

$$\left\langle E^2 \right\rangle = \tfrac{1}{9}\left[\langle 0|H^2|0\rangle + 4\langle 1|H^2|1\rangle + 4\langle 2|H^2|2\rangle\right] = \tfrac{1}{9}\left[\left(\tfrac{1}{2}\hbar\omega\right)^2 + 4\left(\tfrac{3}{2}\hbar\omega\right)^2 + 4\left(\tfrac{5}{2}\hbar\omega\right)^2\right]$$

$$\left\langle E^2 \right\rangle = \tfrac{137}{36}\hbar^2\omega^2$$

$$\Delta E = \sqrt{\tfrac{137}{36}\hbar^2\omega^2 - \left(\tfrac{33}{18}\right)^2 \hbar^2\omega^2} = \hbar\omega\sqrt{\tfrac{144}{18^2}}$$

$$\boxed{\Delta E = \tfrac{2}{3}\hbar\omega}$$

9.18. The differential equation tells us that we want a function whose derivative is equal to the function itself times a constant and *p:*

$$\frac{d}{dp}\phi_0(p) = -\frac{1}{m\omega\hbar}p\phi_0(p)$$

We know that the derivative of the exponential function e^p is itself, so to get the extra factor of p we need a p^2 in the exponent. To get the multiplicative factor correct, the function must be $e^{-p^2/2m\omega\hbar}$. Now normalize:

$$1 = \int_{-\infty}^{\infty} |\phi_0(p)|^2 \, dp = \int_{-\infty}^{\infty} \left|Ae^{-p^2/2m\omega\hbar}\right|^2 dp = |A|^2 \int_{-\infty}^{\infty} e^{-p^2/m\omega\hbar} \, dx$$

From Eq. (F.22) we get the integral, giving

$$1 = |A|^2 \, 2\frac{\sqrt{m\omega\hbar}}{2}\sqrt{\pi} = |A|^2 \sqrt{\pi m\omega\hbar}$$

Choosing the normalization constant to be real and positive gives

$$\phi_0(p) = \left(\frac{1}{\pi m\omega\hbar}\right)^{\frac{1}{4}} e^{-p^2/2m\omega\hbar}$$

9.19. The Fourier transform of the position-space wave function gives the momentum-space wave function:

$$\phi_0(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi_0(x) e^{-ipx/\hbar} \, dx$$

Use the parametrization $\beta = \sqrt{m\omega/\hbar}$ to write:

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar} = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}} e^{-\beta^2 x^2/2}$$

Now integrate using Eq. (F.23):

$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx}\,dx = \frac{\sqrt{\pi}}{a}e^{b^2/4a^2}$$

$$\phi_0(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}} e^{-\beta^2x^2/2}e^{ipx/\hbar}\,dx = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\beta^2x^2/2+ipx/\hbar}\,dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\sqrt{\pi}}{\sqrt{\beta^2/2}}e^{(ip/\hbar)^2/4(\beta^2/2)} = \frac{1}{\sqrt{\hbar}}\left(\frac{1}{\pi\beta^2}\right)^{\frac{1}{4}} e^{-p^2/2\hbar^2\beta^2}$$

Substituting $\beta = \sqrt{m\omega/\hbar}$ gives

$$\phi_0(p) = \left(\frac{1}{\pi m\omega\hbar}\right)^{\frac{1}{4}} e^{-p^2/2m\omega\hbar}$$

9.20. The initial wave function is

$$\psi(x,0) = \varphi_0(x-x_0) = \left(\beta^2/\pi\right)^{\frac{1}{4}} e^{-\beta^2(x-x_0)^2/2}.$$

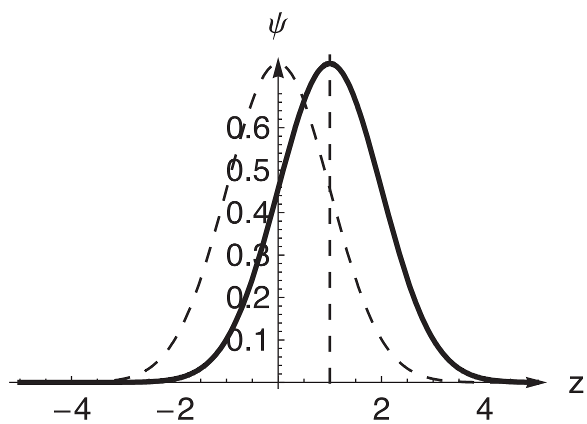
Use dimensionless variables to make plotting easier. Let $z = \beta x = x\sqrt{m\omega/\hbar}$. For the case of $x_0 = \sqrt{\hbar/m\omega}$, we have the dimensionless parameter $z_0 = \beta x_0 = \sqrt{\hbar/m\omega}\sqrt{m\omega/\hbar} = 1$, and the wave function is

$$\psi(z,0) = (1/\pi)^{\frac{1}{4}} e^{-(z-1)^2/2}.$$

a) Using the Mathematica command (with the function φ[n,z] defined as $\varphi_n(z)$:

```
Plot[{φ[0,z],φ[0,z-1]},{z,-5,5}]
```

gives the plot:



b) The overlap integrals

$$c_n = \langle n|\psi\rangle = \int_{-\infty}^{\infty}\varphi_n^*(z)\psi(z)\,dz$$

give the expansion coefficients for the energy state superposition:

$$\psi(z) = \sum_{n=0}^{\infty} c_n \varphi_n(z)$$

The Mathematica command

```
cn = Table[N[Integrate[φ[0,z-1]*φ[n,z],{z,-Infinity,Infinity}]],{n,0,9}]
```

gives the table of coefficients:

{0.7788, 0.5507, 0.2753, 0.1124, 0.03974, 0.01257,
0.00363, 0.00097, 0.00024, 0.000057}

The expected values for a coherent state are $c_n = \left(\alpha^n / \sqrt{n!}\right) e^{-\alpha^2/2}$ with $\alpha = x_0\sqrt{m\omega/2\hbar}$. For the case $x_0 = \sqrt{\hbar/m\omega}$, $\alpha = \sqrt{1/2}$ and the expected coefficients are $c_n = \left(1/\sqrt{2^n n!}\right) e^{-1/4}$, which are

{0.7788, 0.5507, 0.2753, 0.1124, 0.03974, 0.01257,
0.00363, 0.00097, 0.00024, 0.000057}

as expected. Check that 10 coefficients are sufficient to properly represent the wave function by summing the probabilities. For all states, we know that the sum is unity

$$\sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} \mathcal{P}_n = 1$$

For these 10 states, the Mathematica command to sum the probabilities is

```
NumberForm[Sum[cn[[i]]^2,{i,1,Length[cn]}],{20,20}]
```

and gives the result:

0.99999999982903300000

So we are missing only 2 parts in 10^{10} of the probability of this state by using these 10 states.

c) To find the expectation value of the energy perform the sum

$$\langle E \rangle = \sum_{n=0}^{9} \mathcal{P}_n E_n$$

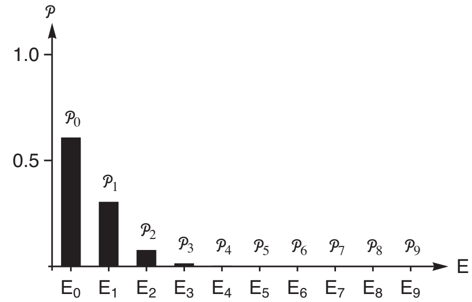
In Mathematica, this is

```
NumberForm[Sum[cn[[i]]^2*(i-1/2),i,1,Length[cn]],12,12]
```

giving the result:

0.999999998197

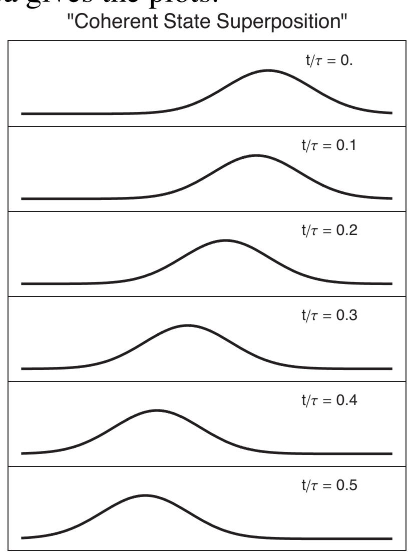
in units of $\hbar\omega$. Thus $\langle E \rangle = \hbar\omega$. This looks reasonable given the histogram (and also agrees with Problem 5.22):



d) The time dependent wave function is

$$\psi(z,t)=\sum_{n=0}^{\infty}c_n e^{-iE_nt/\hbar}\varphi_n(z)=e^{-i\omega t/2}\sum_{n=0}^{\infty}c_n e^{-in\omega t}\varphi_n(z)$$

Animation of this in Mathematica gives the plots:

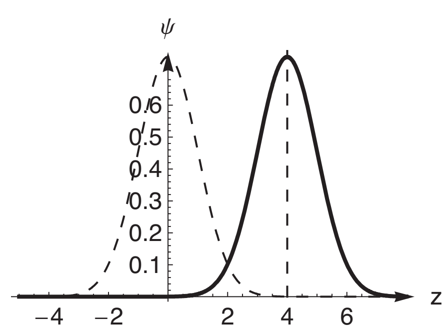


The Gaussian wave function moves back and forth within the harmonic well without changing its shape.

e) For the case of $x_0=4\sqrt{\hbar/m\omega}$, we have the dimensionless parameter $z_0=\beta x_0=\sqrt{m\omega/\hbar}\,4\sqrt{\hbar/m\omega}=4$, and the wave function is

$$\psi(z,0)=(1/\pi)^{\frac{1}{4}}e^{-(z-4)^2/2}.$$

The wave function plot:



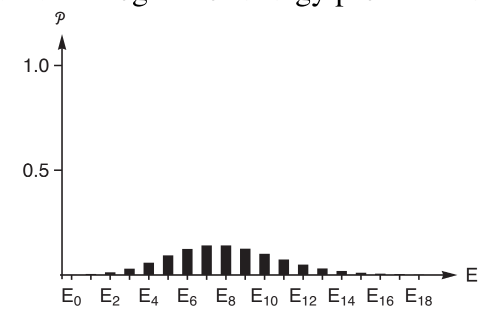
For the case $x_0 = 4\sqrt{\hbar/m\omega}$, $\alpha = 2\sqrt{2}$ and the expected coefficients are:

{0.0183156, 0.0518044, 0.103609, 0.169193, 0.239275, 0.302661, 0.349483, 0.373613, 0.373613, 0.352246, 0.315058, 0.268682, 0.219378, 0.172094, 0.130091, 0.0950052, 0.0671788, 0.0460843, 0.0307228, 0.0199356}

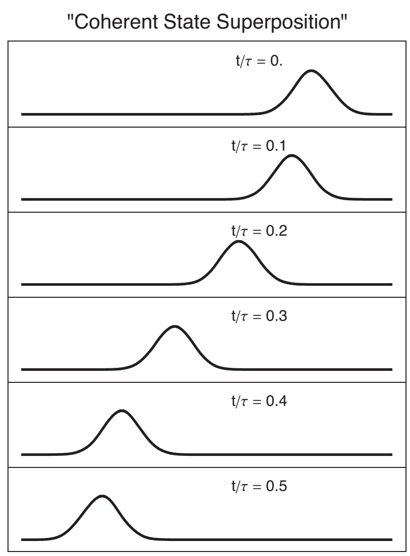
This time, 20 coefficients gets us 0.99975 of the total probability.
The energy expectation value is

$$\langle E \rangle = 8.5\hbar\omega$$

in agreement with Problem 5.22. Histogram of energy probabilities:



Animation:



Again, the wave function moves back and forth within the harmonic well without changing its shape. It is displaced farther from the origin, but has the same spatial width (the scale has changed).

9.21. The Morse potential is

$$V_M(R) = D_e\left(e^{-2\alpha(R-R_0)} - 2e^{-\alpha(R-R_0)}\right)$$

where the constant α is $\alpha = \omega\sqrt{\mu/2D_e}$. Expand this in a power series for small values of $\alpha(R-R_0)$:

$$
\begin{aligned}
V_M(R) &= D_e\left(e^{-2\alpha(R-R_0)} - 2e^{-\alpha(R-R_0)}\right) \\
&\cong D_e\left[\begin{array}{l}\left(1-2\alpha(R-R_0)+\frac{1}{2!}\left(2\alpha(R-R_0)\right)^2-\frac{1}{3!}\left(2\alpha(R-R_0)\right)^3\right) \\ -2\left(1-\alpha(R-R_0)+\frac{1}{2!}\left(\alpha(R-R_0)\right)^2-\frac{1}{3!}\left(\alpha(R-R_0)\right)^3\right)\end{array}\right] \\
&\cong D_e\left[\begin{array}{l}1-2\alpha(R-R_0)+2\alpha^2(R-R_0)^2-\frac{4}{3}\alpha^3(R-R_0)^3 \\ -2+2\alpha(R-R_0)-\alpha^2(R-R_0)^2+\frac{1}{3}\alpha^3(R-R_0)^3\end{array}\right] \\
&\cong D_e\left[-1+\alpha^2(R-R_0)^2-\alpha^3(R-R_0)^3\right]
\end{aligned}
$$

Hence the harmonic approximation (neglect cubic term) gives the potential

$$V_M(R) \cong -D_e + \alpha^2 D_e (R-R_0)^2$$

with a minimum at the bond length. The cubic correction term is

$$V_{cubic}(R) \cong -\alpha^3 D_e (R-R_0)^3$$

9.22. As we did for the harmonic oscillator, we use the energy basis as the default basis for a matrix representation. An operator is always diagonal in its own basis, and eigenvectors are unit vectors in their own basis. So the Hamiltonian is diagonal in the energy basis and the energy eigenstates are unit vectors in the energy basis. The diagonal elements of the Hamiltonian are the energy eigenvalues, so by inspection of our energy equation $E_n = n^3\hbar\omega$, the Hamiltonian is

$$H \doteq \begin{pmatrix} \hbar\omega & 0 & 0 & 0 & \cdots \\ 0 & 8\hbar\omega & 0 & 0 & \cdots \\ 0 & 0 & 27\hbar\omega & 0 & \cdots \\ 0 & 0 & 0 & 64\hbar\omega & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the rows and column are ordered $n = 1,2,3,\ldots$. In this matrix representation, the energy basis states are the unit vectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},\ |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix},\ |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix},\ \cdots$$

To find the matrix for *A*, find the individual matrix elements using the defining equation for *A*:

$$A|n\rangle = 3n^2|n+2\rangle$$

Take the projection onto an arbitrary state:

$$A_{mn} = \langle m|A|n\rangle = \langle m|3n^2|n+2\rangle = 3n^2\langle m|n+2\rangle = 3n^2\delta_{m,n+2}$$

Thus, for example, $A_{11} = 0$, $A_{13} = 0$, $A_{31} = 3$, $A_{42} = 12$, with the resultant matrix being

$$A \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 12 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 27 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
