7.13 a) For this state the probabilities of an L_z measurement are

$$\mathsf{P}_{L_z=+2\hbar} = \left|\langle 2|\psi\rangle\right|^2 = \left|\langle 2|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = 0$$

$$\mathsf{P}_{L_z=+3\hbar} = \left|\langle 3|\psi\rangle\right|^2 = \left|\langle 1|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|\tfrac{\sqrt{3}}{2}\right|^2 = \tfrac{3}{4}$$

$$\mathsf{P}_{L_z=-2\hbar} = \left|\langle -2|\psi\rangle\right|^2 = \left|\langle -1|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|i\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

b) The energy $2\hbar^2/I$ corresponds to $|m| = 2$. For an energy measurement, the probabilities are

$$\mathsf{P}_{E_2} = \left|\langle 2|\psi\rangle\right|^2 + \left|\langle -2|\psi\rangle\right|^2 = \left|\langle 2|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 + \left|\langle -2|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|i\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$$\mathsf{P}_{E_3} = \left|\langle 3|\psi\rangle\right|^2 + \left|\langle -3|\psi\rangle\right|^2 = \left|\langle 3|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 + \left|\langle -3|\left(\tfrac{\sqrt{3}}{2}|3\rangle + i\tfrac{1}{2}|-2\rangle\right)\right|^2 = \left|\tfrac{\sqrt{3}}{2}\right|^2 = \tfrac{3}{4}$$

c) The expectation value of L_z is

$$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle = \sum_{m=-\infty}^{\infty} m\hbar\,\mathsf{P}_{L_z=m\hbar} = 3\hbar\mathsf{P}_{L_z=+3\hbar} + (-2\hbar)\mathsf{P}_{L_z=-2\hbar} = 3\hbar\tfrac{3}{4} + (-2\hbar)\tfrac{1}{4} = \tfrac{7}{4}\hbar$$

d) The expectation value of the energy is

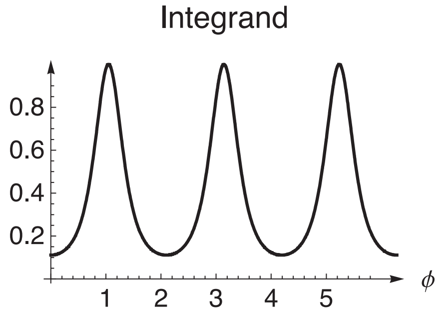
$$\langle E\rangle = \langle\psi|H|\psi\rangle = \sum_{m=0}^{\infty} E_m\,\mathsf{P}_{E_m} = \sum_{m=0}^{\infty}\frac{m^2\hbar^2}{2I}\mathsf{P}_{E_m} = \frac{2^2\hbar^2}{2I}\mathsf{P}_{E_2} + \frac{3^2\hbar^2}{2I}\mathsf{P}_{E_3} = \frac{\hbar^2}{2I}\left(4\tfrac{1}{4} + 9\tfrac{3}{4}\right) = \frac{31\hbar^2}{8I}$$

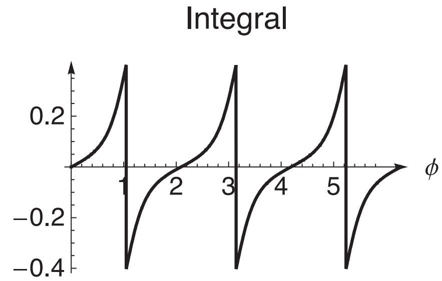
7.15 a) Normalize the wave function:

$$1 = \langle\psi|\psi\rangle = \int_0^{2\pi}\left|\frac{N}{2+\cos 3\phi}\right|^2 d\phi = |N|^2\int_0^{2\pi}\frac{1}{4+4\cos 3\phi + \cos^2 3\phi}\,d\phi$$

$$= |N|^2\left[\frac{4}{9\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{3\phi}{2}\right) - \frac{\sin 3\phi}{9(2+\cos 3\phi)}\right]_0^{2\pi}$$

The arctan function is problematic because of its limited range. Looking at the integrand and the resultant indefinite integral:





we see that we can do the integral from 0 to $2\pi/6$ and then multiply by 6 to avoid the arctan problem:

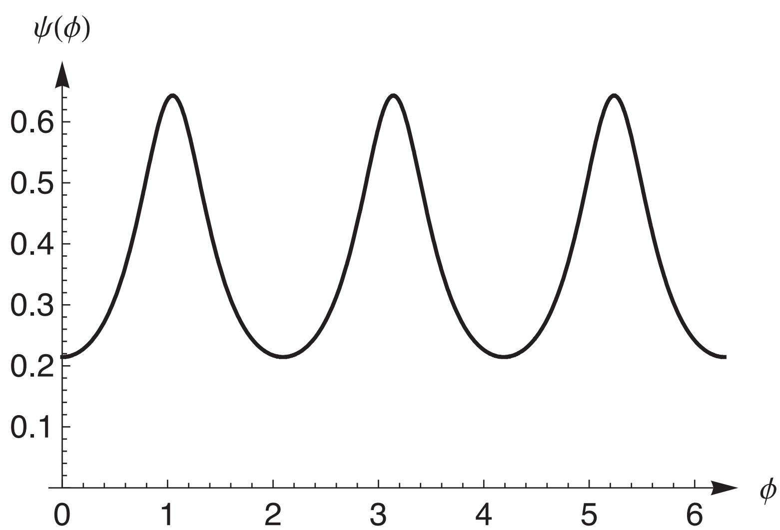
$$1 = 6|N|^2 \left[\frac{4}{9\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\frac{3\phi}{2} \right) - \frac{\sin 3\phi}{9(2+\cos 3\phi)} \right]_0^{2\pi/6}$$

$$= 6|N|^2 \left[\frac{4}{9\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\frac{\pi}{2} \right) \right] = 6|N|^2 \left[\frac{4}{9\sqrt{3}} \frac{\pi}{2} \right] = |N|^2 \left[\frac{4\pi}{3\sqrt{3}} \right]$$

The normalized wave function is

$$\psi(\phi) = \sqrt{\frac{3\sqrt{3}}{4\pi}} \frac{1}{2+\cos 3\phi}$$

b) Plot:



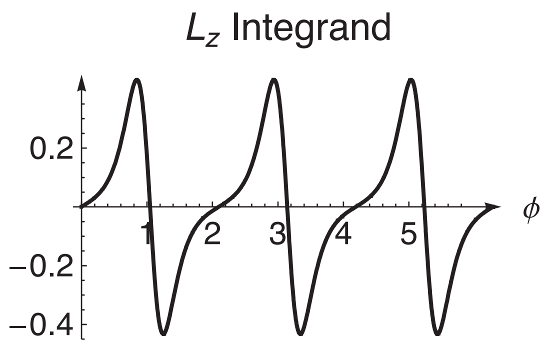
c) The expectation value of L_z is

$$\langle L_z \rangle = \langle \psi | L_z | \psi \rangle = \int_0^{2\pi} \psi^*(\phi)\ L_z\ \psi(\phi) d\phi = \int_0^{2\pi} \psi^*(\phi) \left(-i\hbar \frac{\partial}{\partial \phi} \right) \psi(\phi) d\phi$$

Insert the wave function to get

$$
\begin{aligned}
\langle L_z \rangle &= \frac{3\sqrt{3}}{4\pi}\int_0^{2\pi} \frac{1}{2+\cos 3\phi}\left(-i\hbar\frac{\partial}{\partial\phi}\right)\frac{1}{2+\cos 3\phi}\,d\phi \\
&= -i\hbar\frac{3\sqrt{3}}{4\pi}\int_0^{2\pi} \frac{1}{2+\cos 3\phi}\,\frac{3\sin 3\phi}{\left(2+\cos 3\phi\right)^2}\,d\phi \\
&= -i\hbar\frac{3\sqrt{3}}{4\pi}\int_0^{2\pi} \frac{3\sin 3\phi}{\left(2+\cos 3\phi\right)^3}\,d\phi
\end{aligned}
$$

Plot the integrand:



The integrand is clearly equally positive and negative, so the integral is zero. If it were not, then the expectation value would be imaginary, which is not physical.

$$\langle L_z \rangle = 0$$

7.28 The expansion of a general function in terms of Legendre polynomials is

$$f(z) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(z)$$

To find the expansion coefficients, use Fourier's trick, noting that the Legendre polynomials are not normalized to unity

$$
\begin{aligned}
\int_{-1}^{1} P_k(z) f(z)\,dz = \int_{-1}^{1} P_k(z)\sum_{\ell=0}^{\infty} c_\ell P_\ell(z)\,dz &= \sum_{\ell=0}^{\infty}\int_{-1}^{1} P_k(z) c_\ell P_\ell(z)\,dz \\
&= c_k \frac{2}{2k+1}
\end{aligned}
$$

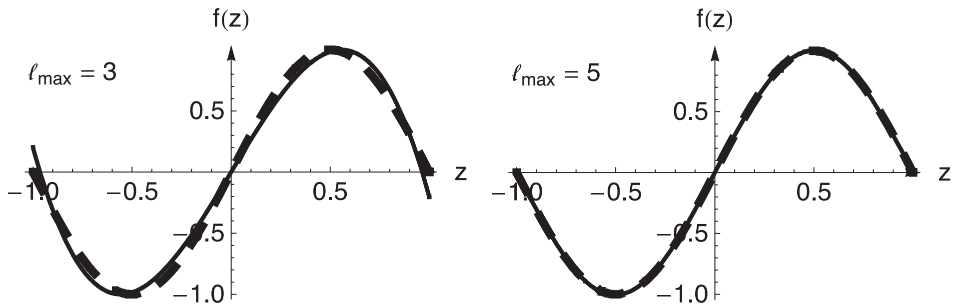
This gives the coefficients

$$c_\ell = \left(\ell + \tfrac{1}{2}\right)\int_{-1}^{1} P_\ell(z) f(z)\,dz$$

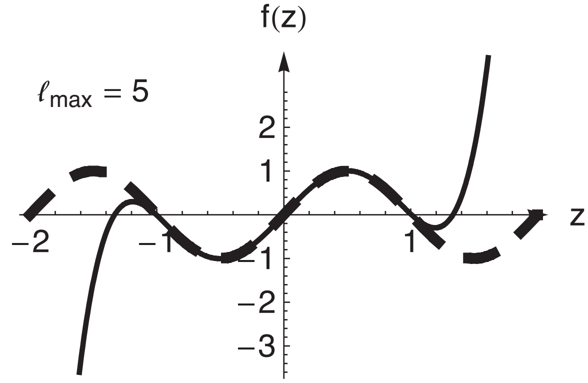
For the function $f(z) = \sin \pi z$, Matlab gives the coefficients

$$c_\ell = 0,\ 0.955,\ 0,\ -1.158,\ 0,\ 0.219,\ 0,\ -0.017,\ 0,\ 0.0007,\ \ldots$$

The 7th term is 1% of the 1st and 3rd terms (all even terms are zero), so a sum to $\ell = 5$ should provide a good approximation at the 1% level. The plots below confirm that.



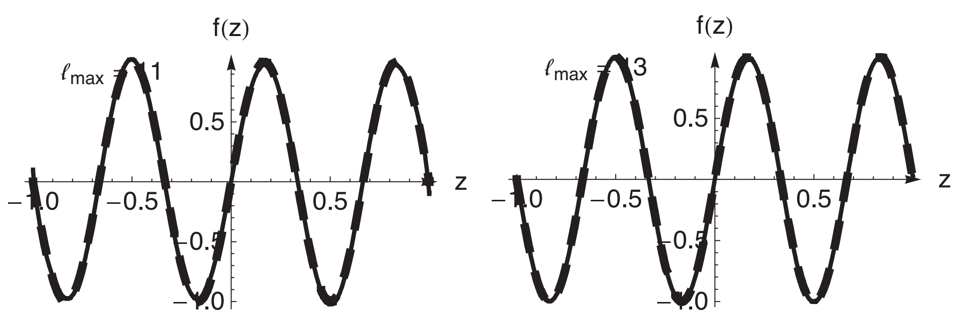
Over the range $-2 < z < 2$, the fit is terrible (see below) because the Legendre polynomials are designed to be used over the range $-1 < z < 1$.



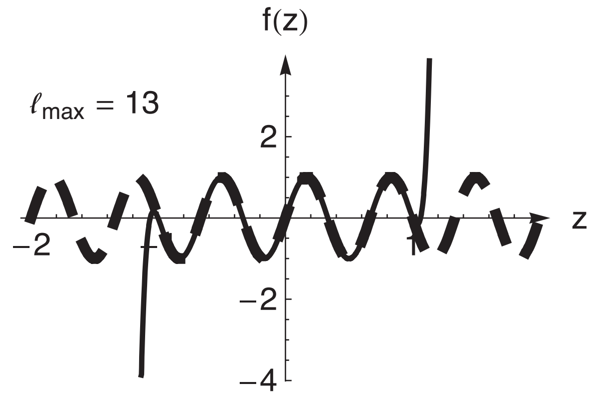
For the function $f(z) = \sin 3\pi z$, Matlab gives the coefficients

$$c_\ell = 0, 0.318, 0, 0.617, 0, -0.073, 0, -1.99, 0, 1.59, 0, -0.557, 0, 0.116, 0, -0.016, \ldots$$

The 15th term is 1% of the 7th and 9th terms (all even terms are zero), so a sum to $\ell = 13$ should provide a good approximation at the 1% level. The plots below confirm that.



Again, the fit over the range $-2 < z < 2$ is terrible (see below).



See code we wrote in class….

7.29 a) For this state the probabilities of an L_z measurement are

$$\mathsf{P}_{L_z=+2\hbar} = \sum_{\ell=2}^{\infty}\left|\left\langle \ell 2 \middle| \psi \right\rangle\right|^2 = \sum_{\ell=2}^{\infty}\left|\left\langle \ell 2 \middle| \left(\tfrac{1}{\sqrt{2}}\left|1,-1\right\rangle + \tfrac{1}{\sqrt{3}}\left|10\right\rangle + i\tfrac{1}{\sqrt{6}}\left|00\right\rangle \right)\right\rangle\right|^2 = 0$$

$$\mathsf{P}_{L_z=-1\hbar} = \sum_{\ell=1}^{\infty}\left|\left\langle \ell,-1 \middle| \psi \right\rangle\right|^2 = \sum_{\ell=1}^{\infty}\left|\left\langle \ell,1 \middle| \left(\tfrac{1}{\sqrt{2}}\left|1,-1\right\rangle + \tfrac{1}{\sqrt{3}}\left|10\right\rangle + i\tfrac{1}{\sqrt{6}}\left|00\right\rangle \right)\right\rangle\right|^2 = \tfrac{1}{2}$$

$$\mathsf{P}_{L_z=0\hbar} = \sum_{\ell=0}^{\infty}\left|\left\langle \ell 0 \middle| \psi \right\rangle\right|^2 = \sum_{\ell=0}^{\infty}\left|\left\langle \ell 0 \middle| \left(\tfrac{1}{\sqrt{2}}\left|1,-1\right\rangle + \tfrac{1}{\sqrt{3}}\left|10\right\rangle + i\tfrac{1}{\sqrt{6}}\left|00\right\rangle \right)\right\rangle\right|^2 = \tfrac{1}{3} + \tfrac{1}{6} = \tfrac{1}{2}$$

b) The expectation value of L_z is

$$\left\langle L_z \right\rangle = \left\langle \psi \middle| L_z \middle| \psi \right\rangle = \sum_{m=-\infty}^{\infty} m\hbar\, \mathsf{P}_{L_z=m\hbar} = 0\hbar \mathsf{P}_{L_z=0\hbar} + \left(-1\hbar\right)\mathsf{P}_{L_z=-1\hbar} = 0\hbar\tfrac{1}{2} + \left(-1\hbar\right)\tfrac{1}{2} = -\tfrac{1}{2}\hbar$$

c) The expectation value of $\mathbf{L}^2$ is

$$\left\langle \mathbf{L}^2 \right\rangle = \left\langle \psi \middle| \mathbf{L}^2 \middle| \psi \right\rangle = \sum_{\ell m} c_{\ell m}^{*} \left\langle \ell m \right| \mathbf{L}^2 \sum_{\ell' m'} c_{\ell' m'} \left| \ell' m' \right\rangle = \sum_{\ell m} \left| c_{\ell m} \right|^2 \ell\left(\ell+1\right)\hbar^2$$
$$= \tfrac{1}{2}2\hbar^2 + \tfrac{1}{3}2\hbar^2 + \tfrac{1}{6}0\hbar^2 = \tfrac{5}{3}\hbar^2$$

d) The expectation value of the energy is

$$\left\langle E \right\rangle = \left\langle \psi \middle| H \middle| \psi \right\rangle = \sum_{\ell m} c_{\ell m}^{*} \left\langle \ell m \right| \frac{\mathbf{L}^2}{2I} \sum_{\ell' m'} c_{\ell' m'} \left| \ell' m' \right\rangle = \sum_{\ell m} \left| c_{\ell m} \right|^2 \ell\left(\ell+1\right)\frac{\hbar^2}{2I}$$
$$= \tfrac{1}{2}2\frac{\hbar^2}{2I} + \tfrac{1}{3}2\frac{\hbar^2}{2I} + \tfrac{1}{6}0\frac{\hbar^2}{2I} = \tfrac{5}{6}\frac{\hbar^2}{I}$$

e) The expectation value of the angular momentum component L_y is

$$\left\langle L_y \right\rangle = \left\langle \psi \middle| L_y \middle| \psi \right\rangle = \sum_{m=-\infty}^{\infty} m\hbar\, \mathsf{P}_{L_y=m\hbar}$$

To find the probabilities of L_y measurements, use the L_y eigenstates in the calculations. Note that the $\ell=0$ single eigenstate $|00\rangle$ is the same in all bases ($|00\rangle = |00\rangle_x = |00\rangle_y$). The $\ell=1$ eigenstates are the same as the spin 1 system (see p. 60):

$$\begin{aligned}
|11\rangle_y &= \tfrac{1}{2}|11\rangle + i\tfrac{1}{\sqrt{2}}|10\rangle - \tfrac{1}{2}|1,-1\rangle \\
|10\rangle_y &= \tfrac{1}{\sqrt{2}}|11\rangle + \tfrac{1}{\sqrt{2}}|1,-1\rangle \\
|1,-1\rangle_y &= \tfrac{1}{2}|11\rangle - i\tfrac{1}{\sqrt{2}}|10\rangle - \tfrac{1}{2}|1,-1\rangle
\end{aligned}$$

Thus the probabilities are

$$\begin{aligned}
\mathsf{P}_{L_y=+1h} &= \sum_{\ell=1}^{\infty}\left|{}_y\langle \ell 1|\psi\rangle\right|^2 = \left|{}_y\langle 11|\psi\rangle\right|^2 = \\
&= \left|\left(\tfrac{1}{2}\langle 11| - i\tfrac{1}{\sqrt{2}}\langle 10| - \tfrac{1}{2}\langle 1,-1|\right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right|^2 = \left|-i\tfrac{1}{\sqrt{6}} - \tfrac{1}{2\sqrt{2}}\right|^2 = \tfrac{1}{6} + \tfrac{1}{8} = \tfrac{7}{24} \\
\mathsf{P}_{L_y=0h} &= \sum_{\ell=0}^{\infty}\left|{}_y\langle \ell 0|\psi\rangle\right|^2 = \left|{}_y\langle 00|\psi\rangle\right|^2 + \left|{}_y\langle 10|\psi\rangle\right|^2 = \\
&= \left|\langle 00|\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right|^2 + \left|\left(\tfrac{1}{\sqrt{2}}\langle 11| + \tfrac{1}{\sqrt{2}}\langle 1,-1|\right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right|^2 \\
&= \tfrac{1}{6} + \tfrac{1}{4} = \tfrac{10}{24} \\
\mathsf{P}_{L_y=-1h} &= \sum_{\ell=1}^{\infty}\left|{}_y\langle \ell,-1|\psi\rangle\right|^2 = \left|{}_y\langle 1,-1|\psi\rangle\right|^2 = \\
&= \left|\left(\tfrac{1}{2}\langle 11| + i\tfrac{1}{\sqrt{2}}\langle 10| - \tfrac{1}{2}\langle 1,-1|\right)\left(\tfrac{1}{\sqrt{2}}|1,-1\rangle + \tfrac{1}{\sqrt{3}}|10\rangle + i\tfrac{1}{\sqrt{6}}|00\rangle\right)\right|^2 = \tfrac{1}{6} + \tfrac{1}{8} = \tfrac{7}{24}
\end{aligned}$$

The three probabilities add to unity, as they must. The expectation value of the angular momentum component L_y is

$$\begin{aligned}
\langle L_y\rangle = \langle\psi|L_y|\psi\rangle &= \sum_{m=-\infty}^{\infty} m\hbar\,\mathsf{P}_{L_y=mh} = 1\hbar\mathsf{P}_{L_y=1h} + 0\hbar\mathsf{P}_{L_y=0h} + (-1\hbar)\mathsf{P}_{L_y=-1h} \\
&= 1\hbar\tfrac{7}{24} + 0\hbar\tfrac{10}{24} + (-1\hbar)\tfrac{7}{24} = 0\hbar
\end{aligned}$$