APiE Assignment - Finite Volume for scalar equations (0.5EC)

Part (i) is compulsory and is required to pass the assignment. The other parts are all optional and are only required for the higher grades. It should be noted that part $(ii) - (iv)$ are independent i.e. it is possible to complete (i) and (iii) or (i) and (iv) only etc...

(i) Write a simple finite volume code to solve the one dimensional linear advection equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0.$$

Implement inflow on the left and outflow conditions on the right. Use the Lax-Friedrichs flux. Consider both (a) square wave initial conditions

$$\rho(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & 1 < x \leq 3 \\ 0 & 3 < x \leq 10 \end{cases}$$

and (b) triangular wave initial conditions

$$\rho(x) = \begin{cases} x & 0 \leq x \leq 1 \\ (2-x) & 1 < x \leq 2 \\ 0 & 2 < x \leq 10 \end{cases}$$

In the report please show the time evolution of the solution and explain how you implemented the boundary conditions.

(ii) Optional: Repeat question(i) for the one dimensional inviscid Burgers equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{2}\frac{\partial \rho^2}{\partial x} = 0.$$

(iii) Optional: Investigate the effect of adding a limiter, try minmod, superbee and Woodward. Comment on which gives the best results for each problem.

(iv) **Voluntary +1 points**: For the problem in (i) find the exact solution. For different grid resolutions consider the difference between the exact solution and your numerical solution. Hence comment on the error in your numerical method as a function of the grid size.

APiE Exercise - Finite Volume for systems of eqn's (0.5 EC)

Part (i) is compulsory and is required to pass the assignment. The other parts are all optional and are only required for the higher grades.

Euler equations in fluid dynamics describe the behaviour of inviscid flows. Ideal fluids that are assumed to have no viscocity are referred to as inviscid. Leonhard Euler actually derived these equations from the conservation of mass and momentum: [1] [2]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial t} = 0 \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + a^2 \rho)}{\partial t} = 0 \tag{2}$$

For simplicity, consider $a = 1$.

(i) Solve the equations on a unit domain $0 < x < 1$. Consider an inflow boundary condition on the left, given by $\rho = 0.5$ and $u = 1$, and an outflow condition on the right side. As initial conditions, take uniform density $\rho = 1$ and velocity $u = 1$. Plot and comment the evolution of the solution until it reaches a steady state. Comment on the steady state reached.

(ii) Consider now a wall at $x = 0.5$, separating the system into a uniform density and zero velocity region to the left, and no material at all to the right. (You may need to place a small density to the right; comment). Start your code at $t = 0$ with the wall already removed. Plot the evolution of the density profile with time. You should see a sharp front (shock) propagating into the empty space. What is the velocity of the shock?

(iii) Consider now solid walls boundary conditions at $x = 0$ and $x = 1$. Plot the evolution of the density.

(iv) Optional: Use the MUSCL scheme. Comment of the effect.

[1] This can be written as a system of equations, $\frac{\partial \vec{\omega}}{\partial t} + \frac{\partial \vec{f}(\vec{\omega})}{\partial x} = 0$, with $\vec{\omega} = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$ and $\vec{f} = \begin{pmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{pmatrix}$

[2] Note, the advection speeds for the Euler equation are $u \pm |a|$