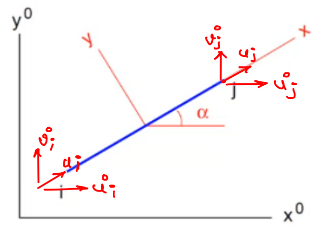
GCS (x^0, y^0) ; LCS (x, y) GCS : $(u_i^0, v_i^0, u_j^0, v_j^0)$ 4 DOF LCS : (u_i, u_j) 2 DOF



$$L = x_j - x_i$$
$$= \sqrt{(x_j^0 - x_i^0)^2 + (y_j^0 - y_i^0)^2}$$

$$\cos\alpha = \frac{1}{L}(x_j^0 - x_i^0)$$
$$\sin\alpha = \frac{1}{L}(y_j^0 - y_i^0)$$

Displacements:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u^0 \\ v^0 \end{Bmatrix}$$

Coordinates:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} x^0 \\ y^0 \end{Bmatrix}$$

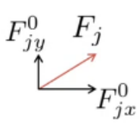
Thus, for a truss element (slender bar), GCS in terms of LCS can be expressed as :

$$u_i = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} u_i^0 \\ v_i^0 \end{Bmatrix} \qquad u_j = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} u_j^0 \\ v_j^0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} u_i^0 \\ v_i^0 \\ u_j^0 \\ v_j^0 \end{Bmatrix}$$

Similarly, we can decompose the forces in the GCS

$$F_{ix}^0 = F_i \cos\alpha \quad , \quad F_{iy}^0 = F_i \sin\alpha$$
$$F_{jx}^0 = F_j \cos\alpha \quad , \quad F_{jy}^0 = F_j \sin\alpha$$



$$\begin{Bmatrix} F_{ix}^0 \\ F_{iy}^0 \\ F_{jx}^0 \\ F_{jy}^0 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & \sin\alpha \end{bmatrix} \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

we observe that,

$$\boxed{\{u\} = [R]\{u^0\}}$$

and

$$\boxed{\{F^0\} = [R]^T\{F\}}$$

this can be seen due to the following reason

Element formulation in the LCS $\{F\} = [K]\{u\} \longrightarrow \{u^0\}, \{F^0\}$ in GCS

we know that $\{u\} = [R]\{u^0\}$

$$[K][R]\{u^0\} = \{F\} \xrightarrow{[R]^T \times} [R]^T[K][R]\{u^0\} = [R]^T\{F\}$$

$$\boxed{[R]^T[K][R]\{u^0\} = \{F^0\}}$$

Thus, $[K^0] = [R]^T[K][R]$

For a 2D truss element

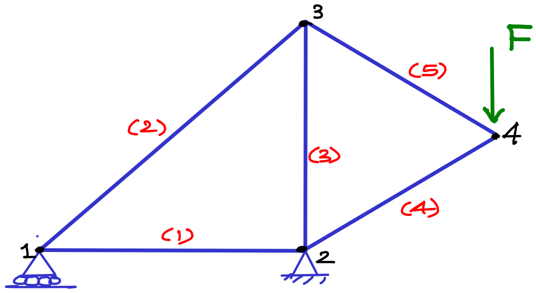
- 4 DOF $\therefore [K]_{4\times4}$
- symmetric
- all elements on main diagonal are positive

$$[K^0] = [R]^T[K][R] = \frac{AE}{L} \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

$$[K^0] = \frac{AE}{L} \begin{bmatrix} cc & cs & -cc & -cs \\ cs & ss & -cs & -ss \\ -cc & -cs & cc & cs \\ -cs & -ss & cs & ss \end{bmatrix}$$

example :

2. 2D truss frame subjected to a force F on a node



co-ordinates of nodes.

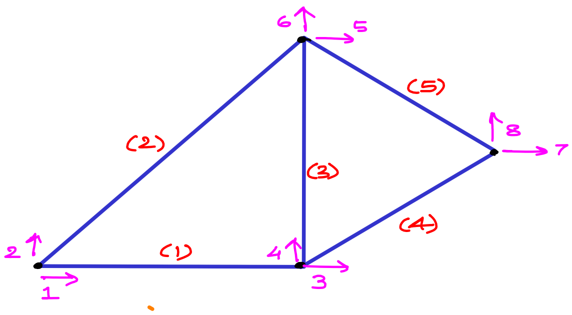
n	x	y
1	−15	0
2	0	0
3	0	20
4	24	10

c.s area of bars

(e)	A
1	3
2	6.25
3	4
4	4.394
5	4.394

$E = 21e4$ $F = 105$

Set up the Finite Element System of Equations



Since, the system has 4 nodes, it has 8 DOF $\therefore [K_{sys}]_{8\times8}$

element	degrees of freedom (d.o.f.)
1	1 2 3 4
2	1 2 5 6
3	3 4 5 6
4	3 4 7 8
5	5 6 7 8

we start by deriving the stiffness matrix for each element (in the global co-ordinate system)

never try to find the exact angle of any element, directly find cos and sin using co-ordintaes

$$[K^0] = \frac{AE}{L}\begin{bmatrix} cc & cs & -cc & -cs \\ cs & ss & -cs & -ss \\ -cc & -cs & cc & cs \\ -cs & -ss & cs & ss \end{bmatrix}$$

$$[K]^{(1)} = \frac{E}{10}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} u_1\ 1 \\ v_1\ 2 \\ u_2\ 3 \\ v_2\ 4 \end{matrix}$$

$$[K]^{(3)} = \frac{E}{10}\begin{array}{c} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \end{array} \begin{matrix} u_2\ 3 \\ v_2\ 4 \\ u_3\ 5 \\ v_3\ 6 \end{matrix}$$

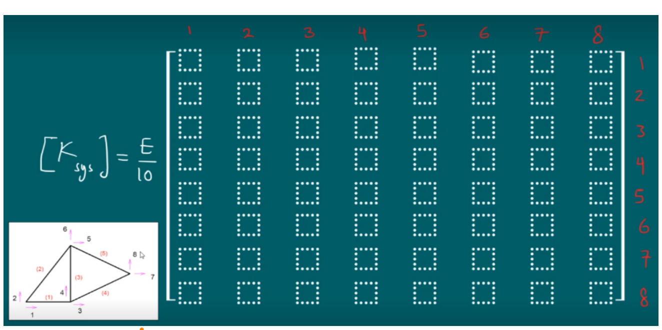
$$[K]^{(2)} = \frac{E}{10}\begin{array}{c} \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0.9 & 1.2 & -0.9 & -1.2 \\ 1.2 & 1.6 & -1.2 & -1.6 \\ -0.9 & -1.2 & 0.9 & 1.2 \\ -1.2 & -1.6 & 1.2 & 1.6 \end{bmatrix} \end{array} \begin{matrix} u_1\ 1 \\ v_1\ 2 \\ u_3\ 5 \\ v_3\ 6 \end{matrix}$$

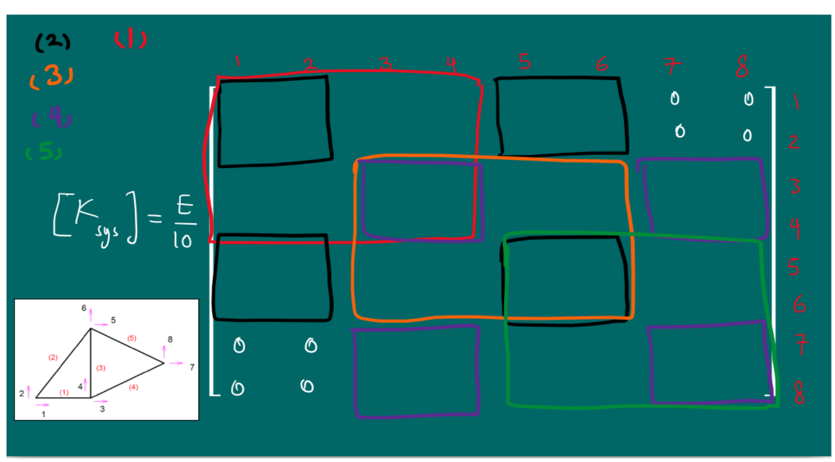
$$[K]^{(4)} = \frac{E}{10}\begin{array}{c} \begin{matrix} 3 & 4 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 1.44 & 0.6 & -1.44 & -0.6 \\ 0.6 & 0.25 & -0.6 & -0.25 \\ -1.44 & -0.6 & 1.44 & 0.6 \\ -0.6 & -0.25 & 0.6 & 0.25 \end{bmatrix} \end{array} \begin{matrix} u_2\ 3 \\ v_2\ 4 \\ u_4\ 7 \\ v_4\ 8 \end{matrix}$$

All element stiffness matrices have been scaled to E/10 to ease the formation of the system stiffness matrix

$$[K]^{(5)} = \frac{E}{10}\begin{array}{c} \begin{matrix} 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 1.44 & -0.6 & -1.44 & 0.6 \\ -0.6 & 0.25 & 0.6 & -0.25 \\ -1.44 & 0.6 & 1.44 & -0.6 \\ 0.6 & -0.25 & -0.6 & 0.25 \end{bmatrix} \end{array} \begin{matrix} u_3\ 5 \\ v_3\ 6 \\ u_4\ 7 \\ v_4\ 8 \end{matrix}$$

setting the system stiffness matrix



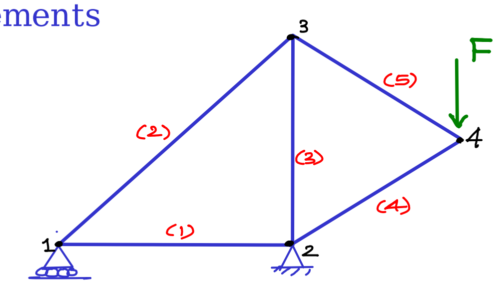


filling the element stiffness matrix values in the system stiffness matrix according to the corresponding degrees of freedom imposing boundary conditions on Force and Displacements

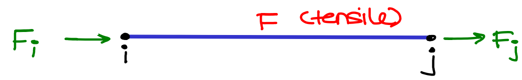
$$[K_{sys}]\{U\}=\{F\}$$

$$[K_{sys}]\begin{Bmatrix} u_1 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = R_{1y} \\ F_{2x} = R_{2x} \\ F_{2y} = R_{2y} \\ F_{3x} = 0 \\ F_{3y} = 0 \\ F_{4x} = 0 \\ F_{4y} = -F \end{Bmatrix}$$

Apply B.C.

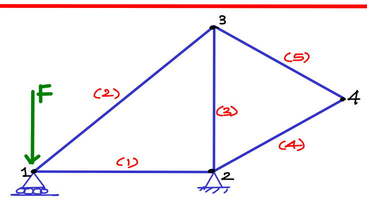


After finding these forces, we can find the force in each element by isolating the element and applying the truss element formulation in the LCS



$F_i = -F$

$F_j = F$



someone asked, what if we have a situation as depicted.

The system is over-constrained. Force is applied in y, and there is no possible displacement in y. This is what we should be avoiding when designing a system.

now we reduce the finite element system of equations

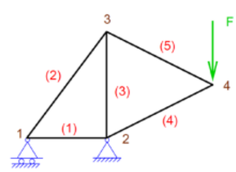
only upto 3 DOF in exam

$$\frac{E}{10}\begin{bmatrix} 2.9 & 1.2 & -2 & 0 & -0.9 & -1.2 & 0 & 0 \\ 1.2 & 1.6 & 0 & 0 & -1.2 & -1.6 & 0 & 0 \\ -2 & 0 & 3.44 & 0.6 & 0 & 0 & -1.44 & -0.6 \\ 0 & 0 & 0.6 & 2.25 & 0 & -2 & -0.6 & -0.25 \\ -0.9 & -1.2 & 0 & 0 & 2.34 & 0.6 & -1.44 & 0.6 \\ -1.2 & -1.6 & 0 & -2 & 0.6 & 3.85 & 0.6 & -0.25 \\ 0 & 0 & -1.44 & -0.6 & -1.44 & 0.6 & 2.88 & 0 \\ 0 & 0 & -0.6 & -0.25 & 0.6 & -0.25 & 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{y_1} \\ R_{x_2} \\ R_{y_2} \\ 0 \\ 0 \\ 0 \\ -105 \end{Bmatrix}$$

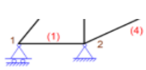
$$\{U\} = \begin{Bmatrix} 0.003 \\ 0 \\ 0 \\ 0 \\ 0.01667 \\ -0.00525 \\ 0.00943 \\ -0.03253 \end{Bmatrix}$$

Reduce the system

$$\frac{E}{10}\begin{bmatrix} 2.9 & -0.9 & -1.2 & 0 & 0 \\ -0.9 & 2.34 & 0.6 & -1.44 & 0.6 \\ -1.2 & 0.6 & 3.85 & 0.6 & -0.25 \\ 0 & -1.44 & 0.6 & 2.88 & 0 \\ 0 & 0.6 & -0.25 & 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -105 \end{Bmatrix}$$



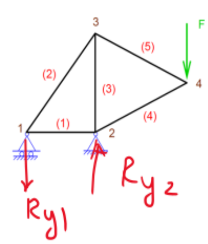
from the reduced FE system of equations, we can find all the displacements.
From the displacements, we can find out the Reaction Forces



$$\frac{E}{10}\begin{bmatrix} 2.9 & 1.2 & -2 & 0 & -0.9 & -1.2 & 0 & 0 \\ 1.2 & 1.6 & 0 & 0 & -1.2 & -1.6 & 0 & 0 \\ -2 & 0 & 3.44 & 0.6 & 0 & 0 & -1.44 & -0.6 \\ 0 & 0 & 0.6 & 2.25 & 0 & -2 & -0.6 & -0.25 \\ -0.9 & -1.2 & 0 & 0 & 2.34 & 0.6 & -1.44 & 0.6 \\ -1.2 & -1.6 & 0 & -2 & 0.6 & 3.85 & 0.6 & -0.25 \\ 0 & 0 & -1.44 & -0.6 & -1.44 & 0.6 & 2.88 & 0 \\ 0 & 0 & -0.6 & -0.25 & 0.6 & -0.25 & 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{y_1} \\ R_{x_2} \\ R_{y_2} \\ 0 \\ 0 \\ 0 \\ -105 \end{Bmatrix}$$

$\{U\} = \checkmark$

$$\{F\} = \begin{Bmatrix} 0 \\ -168 \\ 0 \\ 273 \\ 0 \\ 0 \\ 0 \\ -105 \end{Bmatrix}$$



Reaction Forces

$$R_{y_1} = 1.2\,u_1 - 1.2\,u_3 - 1.6\,v_3$$

$$R_{x_2} = -2\,u_1 - 1.44\,u_4 - 0.6\,v_4$$

$$R_{y_2} = -2\,v_3 - 0.6\,u_4 - 0.25\,v_4$$

Validate!

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum M = 0$$