Class 02, July 10, 2021

Submission - Class 01
- .py, .m, .tex, .pdf
- No late submission
- Due now

- "This is shown . . ."

Figure

1

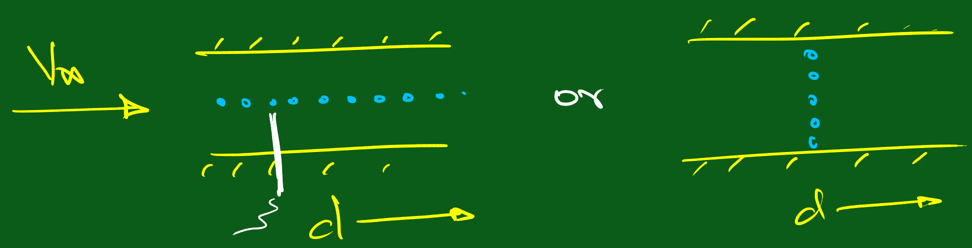
Figure 1

Fig. no
Table no
100 m/s

- x^2 x**2   x*x   x/4 → 0.25*x

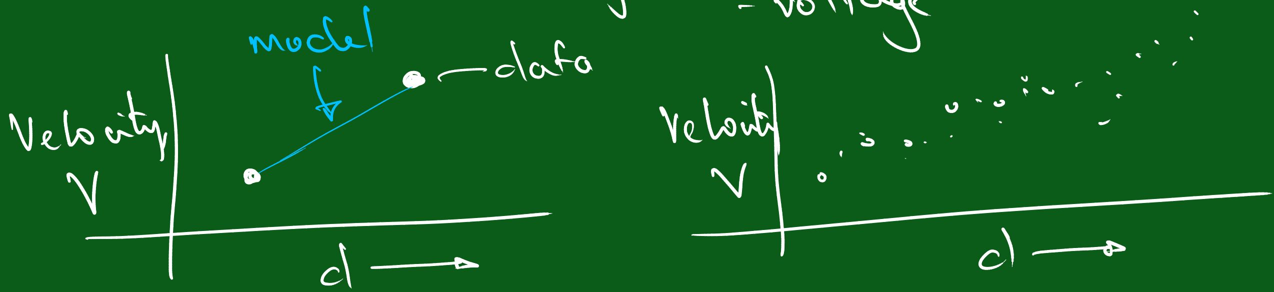
  x^3 = x*x*x

- Let us assume we are doing an experiment



How to find velocity?

- Hot-wire anemometer
  - → Thermal transducer
  - → Single point
  - → Velocity - resistance - voltage
- Pitot static tube
- Venturimeter
- PIV - ?



Eqn
↓
Tweak
↓
Least error

- linear eqn
- polynomial
- exponential

× Let us assume

$$V = \underbrace{m}\,d + \underbrace{c}$$

to be estimated

$$V_1 = m d_1 + c$$
$$V_2 = m d_2 + c$$
$$\vdots$$
$$V_N = m d_N + c$$

N observations

$$\underbrace{\begin{bmatrix} d_1 & 1 \\ d_2 & 1 \\ \vdots & \vdots \\ d_N & 1 \end{bmatrix}}_{N\times 2} \underbrace{\begin{Bmatrix} m \\ c \end{Bmatrix}}_{2\times 1} = \underbrace{\begin{Bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{Bmatrix}}_{N\times 1}$$

$$\begin{bmatrix} d_1 & 1 \\ d_2 & 1 \\ \vdots & \vdots \\ d_N & 1 \end{bmatrix} \begin{Bmatrix} m \\ c \end{Bmatrix} = \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix}$$

$$A \qquad x \quad = \bar{b}$$

* What does $Ax = \bar{b}$ mean in this context?

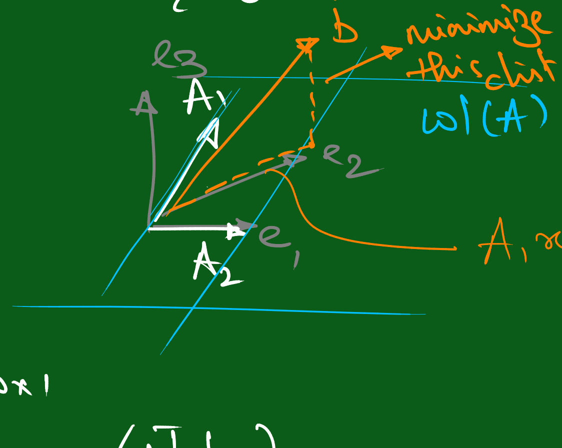
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{N1} & a_{N2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}$$

$$\underbrace{\begin{Bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{Bmatrix}}_{A_1} \underline{x_1} + \underbrace{\begin{Bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{Bmatrix}}_{A_2} \underline{x_2} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}$$

* $\bar{b}$ is a linear combination of the columns of A!
* Example

$$\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} x_1 + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} x_2 \approx \bar{b}$$

How can you be sure this can be satisfied?



$A_1 x_1 + A_2 x_2$

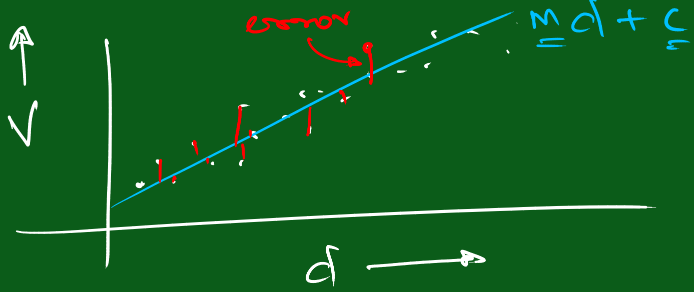
* $A_{N\times 2}\, x_{2\times 1} = b_{N\times 1}$

$$(A^T A)_{2\times 2}\, x_{2\times 1} = (A^T b)_{2\times 1}$$

How can you be sure this matrix is invertible?

$$x = (A^T A)^{-1} A^T b$$

solution for x!



m & c obtained such that it "best fits" the data

Linear Regression

$Ax = b$

find $\hat{x} = \arg\min_x \|b - Ax\|$

$Ax - b = \epsilon$ ← minimize this term

$\hat{x} = (A^T A)^{-1} A^T b$!

× $V = md^c \longrightarrow \log V = \underline{\log m} + \underline{c} \log d$

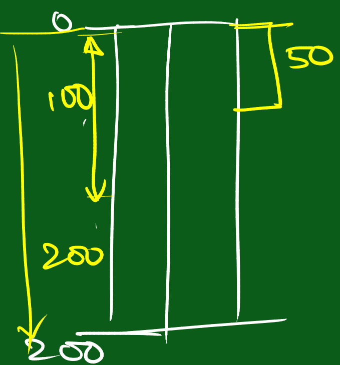
$V = md + c + kd^4 \longrightarrow$ valid?

× Task : data/scripts - class 02 - data 1.txt

2.txt

3.txt

200(x, y)



- Perform linear regression using $y = mx + c$
- Using first 50, 100, 200 points
-

Data 1	m	c
50		
100		
200		

D2		

D3		

files to be submitted

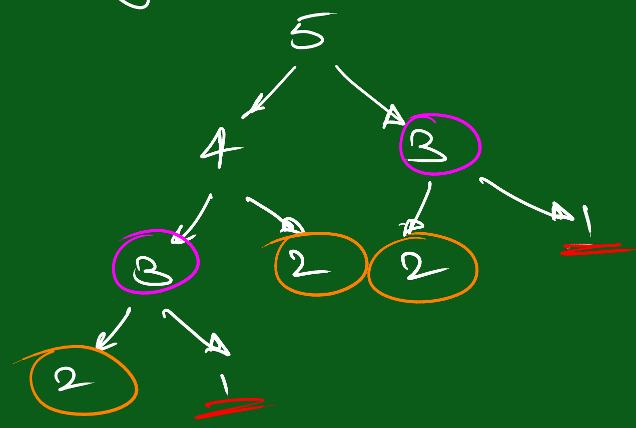
- .m
- .tex
- .pdf

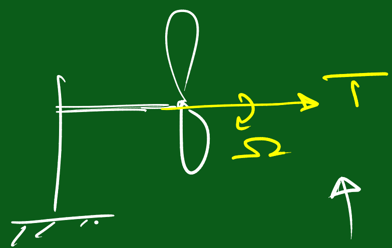
- Based on the behaviour of m & c, infer the type of data.
- Do you think a better model would work instead of $y = mx + c$? What? Why?
- Plot the data a scatter plot & a line for the model. Support previous inference
- Use octave — do not use inv(A) or A\b!
- Invert matrix using Gaussian-Elimination

Class 03, July 17 2021

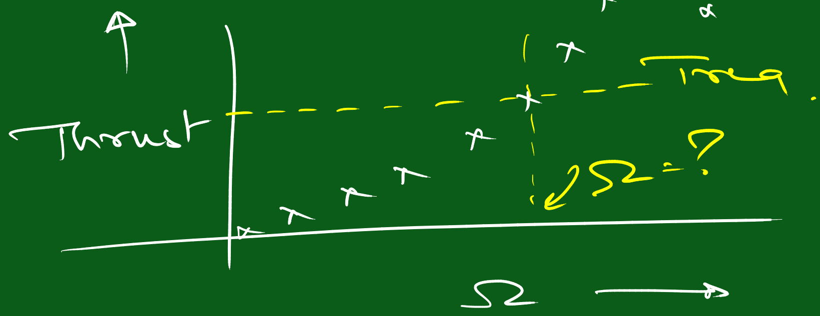
Submission_Class 02
- .m, .tex, .pdf
- Due now.

* Demo on Python
* Root Finding Algorithms





What is the Ω so that $T = T_{req}$?



$$T(\Omega) = T_{req}$$

$$F(\Omega) = T(\Omega) - T_{req} = 0$$

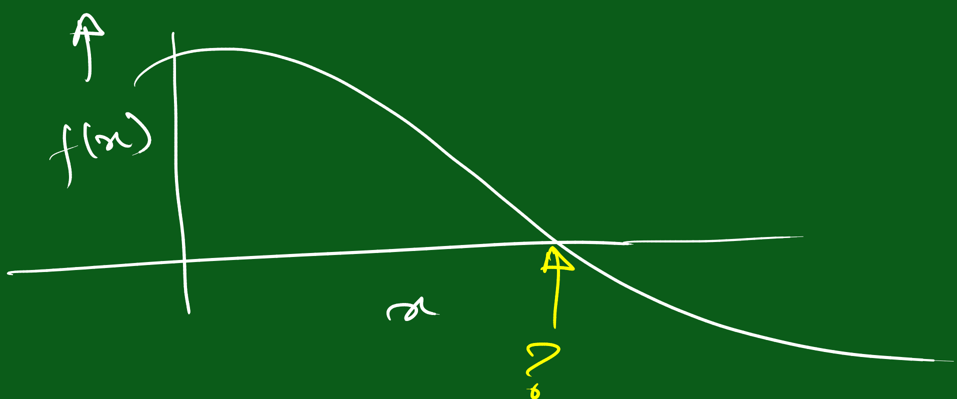
* In general we are interested in solving for

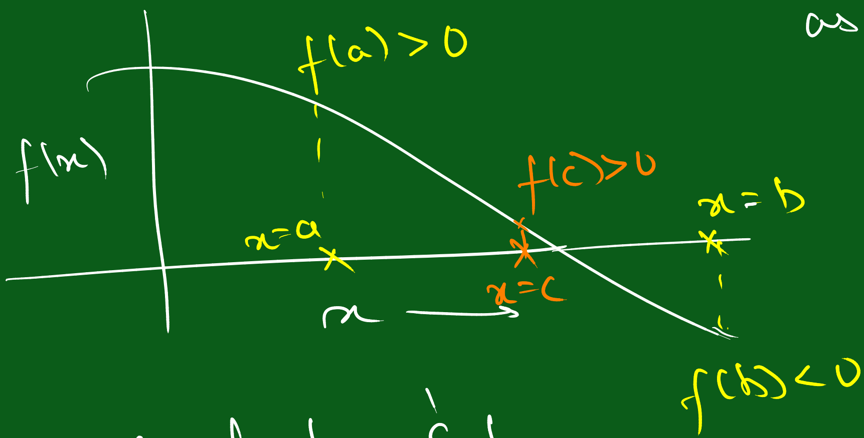
$$f(x) = 0$$

↳ scalar or vector

* Look at three methods — solving them numerically

1. Bisection method.





assume f(x) continuous

$x = a,\ f(a) \geq 0$

$x = b,\ f(b) < 0$

At least one root

$x = c,\ c \in (a, b)$

Step 1: find a & b

Step 2: $c = \frac{a+b}{2}$ → if $f(c) > 0$ then $a = c$
→ if $f(c) < 0$ then $b = c$

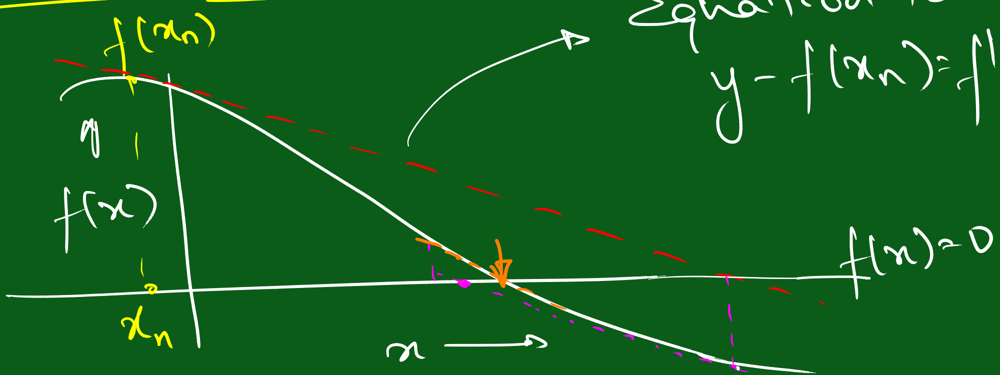
repeat till $f(c) = 0$

- limited precision
- we are okay with $|f(c)| < \epsilon$ ~ 10^{-6}

def bisector(f, a, b, $\boxed{\epsilon, N_{max}}$)

: function — left — right — precision — max step count

2. Newtons method



Equation to tangent

$y - f(x_n) = f'(x_n)(x - x_n)$

- we want to find $y = 0$

$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$

unbounded ← $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

can be 0 ← $f'(x_n)$

also need derivative information

Start: x_0
x_1
$\vdots$
x_2
$\vdots$

$|f(x)| < \epsilon$

3. Secant Method →

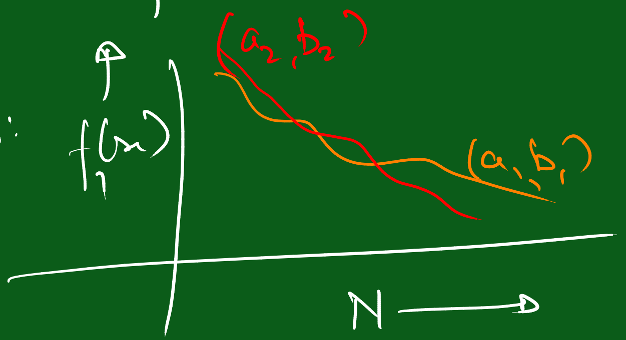
Everything in Python

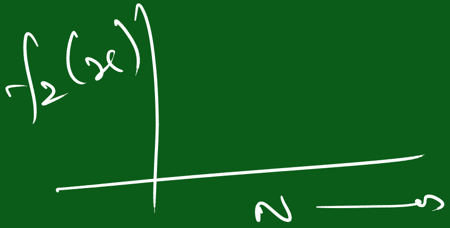
Report:
- Introduction (theory section with figs & equations) → must have a caption!
- Results & Analysis
- Conclusion
- References

Choose two unique starting sets

$$\begin{cases} f_1(x) = x^3 - 3x^2 - x + 9 \\ f_2(x) = e^x f_1(x) \end{cases}$$

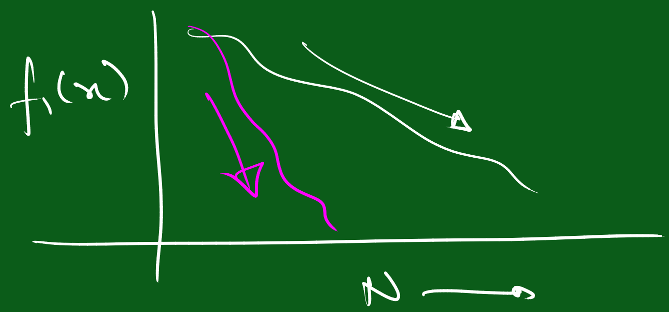
Bisection:
Newton
Secant





Newton → $f_3(x) = x^3 - 2x + 2$

Start at $x = 0$



Class 04: July 24, 2021

Submission: Class 03
- Due now
- .py, .tex, .pdf

multiple?

Quadrature

- → historically - determine area
- → Express solution in terms of integrals
- → numerical analysis: approximation of definite integral of a function

$$I = \int_a^b f(x)\,dx \approx \sum_i f(x_i) w_i$$

$\sum_i$: Summation over finite pieces

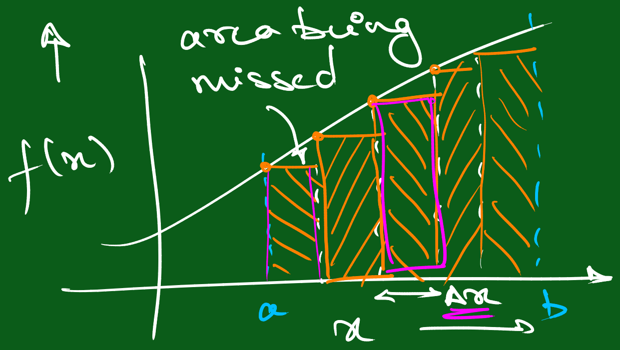
$f(x_i)$ → function value at x_i

w_i → weight given to the function at x_i

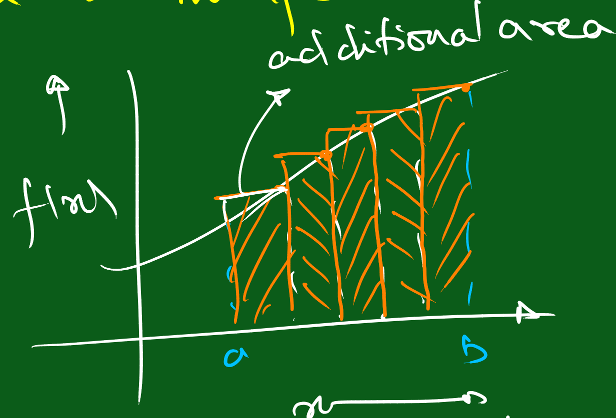
- Used in many places – forming basis function for integration
  - FEM
  - CFD schemes
  - Simple integration

$$I = \int_a^b f(x)\,dx$$

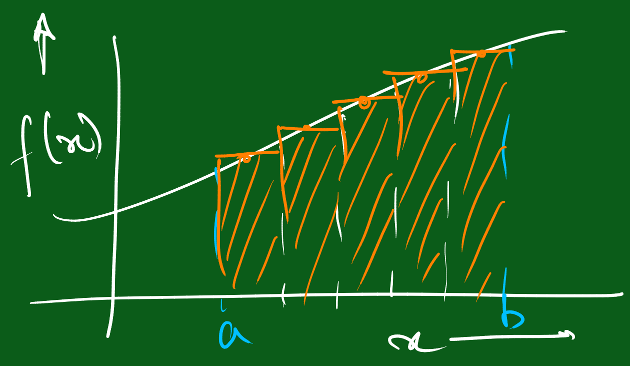
- Rectangular rule
  - left endpoint
  - right endpoint
  - midpoint
- Trapezoidal rule



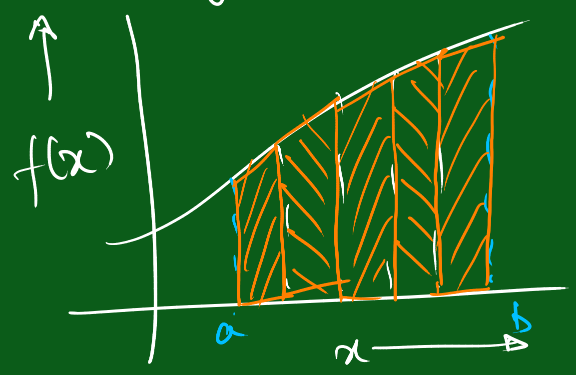
left-endpoint



right-endpoint

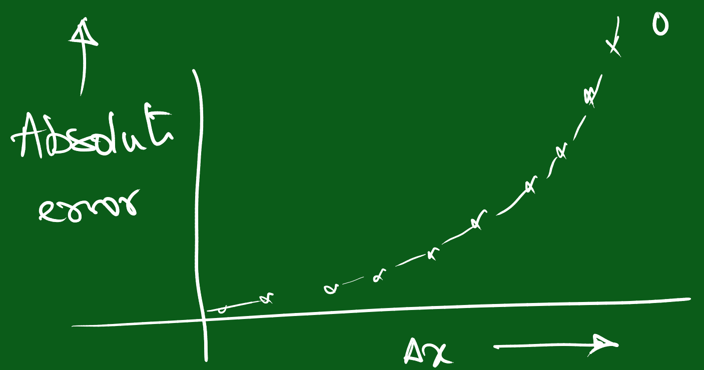


mid-point

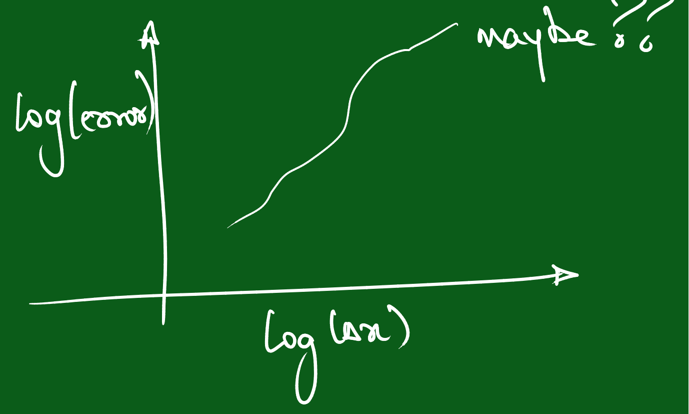


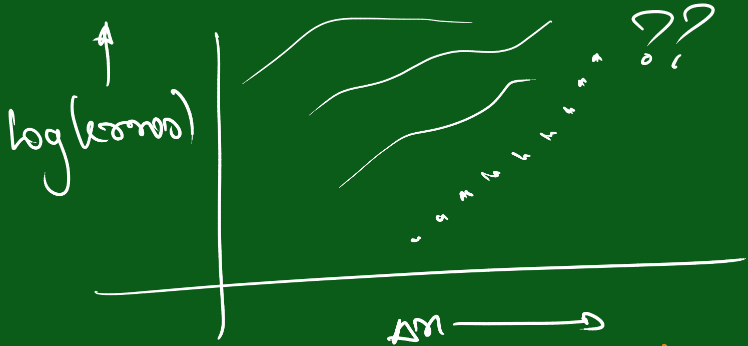
trapezoid rule

- Let us assume $I = \int_0^{\pi/2} \sin(x)\,dx$ ← Exact value known



or



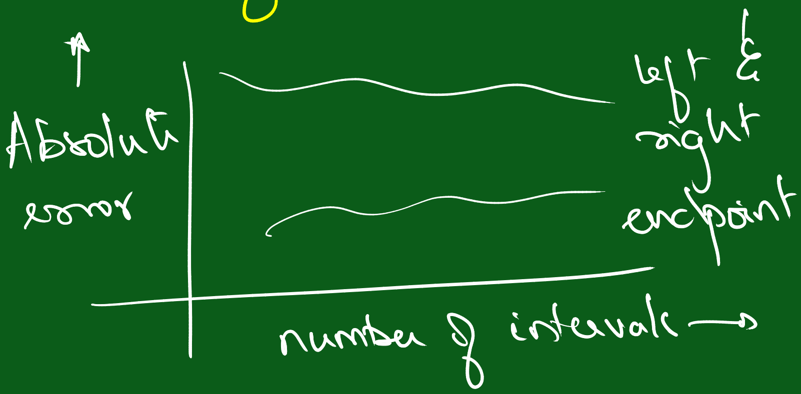


→ Fit an appropriate curve using linear regression

→ Identify the "order of accuracy" of each method.

what does it mean?

→ Comment on the accuracy of the four methods investigated



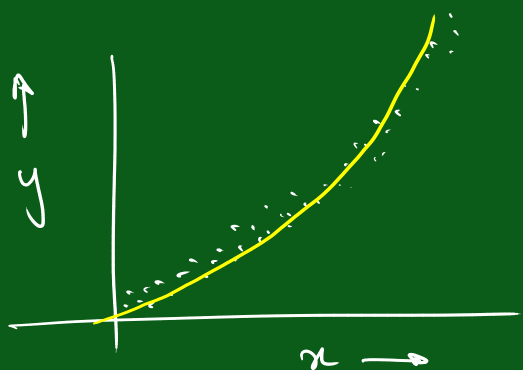
→ Conclusion?
lines up with any known mathematical result?

- Python, try to use functions

Class 05: July 31, 2021

Submission: Class 04
- Due now
- .py, .pdf, .tex
- Last class

QR Decomposition & Linear least squares



Finding a "least squares" solution

$$y = \underbrace{\beta_0 + \beta_1 x + \beta_2 x^2}_{?}$$

N observations

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2$$
$$\vdots$$
$$y_N = \beta_0 + \beta_1 x_N + \beta_2 x_N^2$$

$$\xrightarrow{\text{rewrite}} \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{Bmatrix}}_{x} = \underbrace{\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix}}_{b}$$

$$Ax = b$$

- Premultiply with A^T → Class 02 → we know this is also a least squares

$$A^T A x = A^T b$$

- Let us look at QR decomposition

$$A := \underbrace{[a_1 \; a_2 \; \cdots \; a_n]}_{\text{n linearly independant cols.}} \qquad \sum_{j=1}^{n} \alpha_j a_j = 0 \iff \alpha_j = 0$$

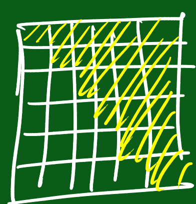
$$A = QR \rightarrow \text{upper triangular matrix}$$

Q ↙ orthogonal matrix

$$Q := [q_1 \; q_2 \; \cdots \; q_n] \qquad q_i^T q_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

n columns of Q form an orthonormal basis for the column span of A

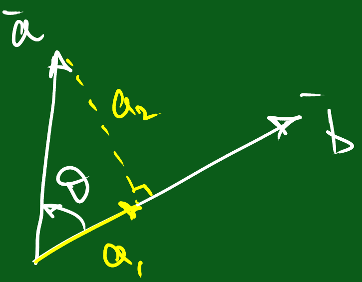
$$A^T A x = A^T b \qquad A = QR$$
$$R^T \underbrace{\boxed{Q^T Q}}_{I} R x = R^T Q^T b \qquad A^T = R^T Q^T$$
$$(R^T)^{-1} R^T R x = (R^T)^{-1} R^T Q^T b$$
$$\boxed{Rx = Q^T b}$$

 $x = \{Q^T b\}$

Back-substitution
- start from the last row and work up to the first row.

- We obtain Q & R from A using Gram-Schmidt process

What do we understand by projection?



$$\bar{a}_1 = a_1 \hat{b}$$

$$a_1 = \|\bar{a}\| \cos\theta = \bar{a} \cdot \hat{b}$$

$$\bar{a}_1 = (\bar{a} \cdot \hat{b})\hat{b}$$

$$= \left(\bar{a} \cdot \frac{\bar{b}}{\|\bar{b}\|}\right) \frac{\bar{b}}{\|\bar{b}\|}$$

$$A : [a_1 \; a_2 \; \cdots \; a_n]$$

$$Q := [q_1 \; q_2 \; \cdots \; q_n]$$

$$\text{proj}_b \, a = \frac{(\bar{a} \cdot \bar{b})\bar{b}}{(\bar{b} \cdot \bar{b})} = \frac{\langle a, b \rangle \bar{b}}{\langle b, b \rangle}$$

Inner product $\langle v, w \rangle = v^T w$

- let $u_1 = a_1$ $\qquad q_1 = \frac{u_1}{\|u_1\|}$

two nested loops??

$$u_2 = a_2 - \text{proj}_{u_1} a_2 \qquad q_2 = \frac{u_2}{\|u_2\|}$$

$\vdots$

$$u_k = a_k - \sum_{j=1}^{k-1} \text{proj}_{u_j} a_k \qquad q_k = \frac{u_k}{\|u_k\|}$$

$\rightarrow$ q_2 is orthonormal to q_1. we have "taken out" the projected component

- Rewrite a_i based on the q_i basis

$$a_1 = \langle q_1, a_1 \rangle q_1$$

$$a_2 = \langle q_1, a_2 \rangle q_1 + \langle q_2, a_2 \rangle q_2$$

$$A = QR$$

upper triangular matrix

$$[a_1 \; a_2 \; \cdots \; a_n] = [q_1 \; q_2 \; \cdots \; q_n] \begin{bmatrix} \langle q_1, a_1 \rangle & \langle q_1, a_2 \rangle & \cdots \\ 0 & \langle q_2, a_2 \rangle & \cdots \\ 0 & 0 & \ddots \end{bmatrix}$$

- Given a data set : find $\beta_0, \beta_1, \beta_2$ using QR decomposition
  - use Gram-Schmidt
  - use back substitution
  - no canned routines!
  - compare against the strategy used in Class 02
    - comments on accuracy and time taken
- There is an extra term
- Is $\beta_0 + \beta_1 x + \beta_2 x^2 + \boxed{\beta_4 x^4}$ is a better fit? Why or why not?
- Python & submit .py, .txt & .pdf.