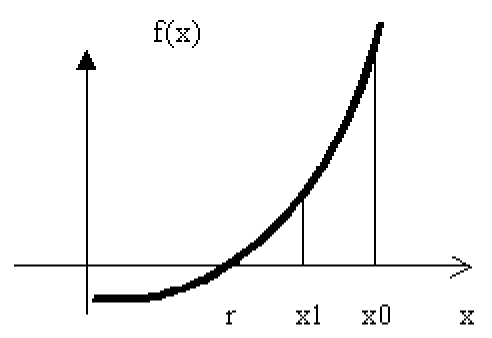
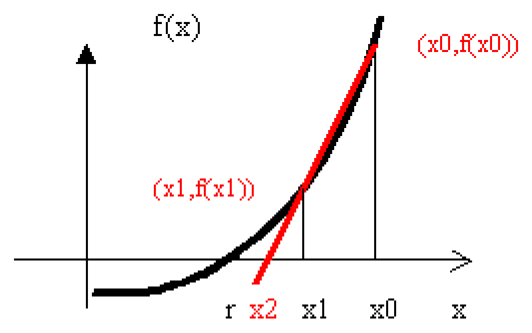
ROOT FINDING TECHNIQUES:
Secant method

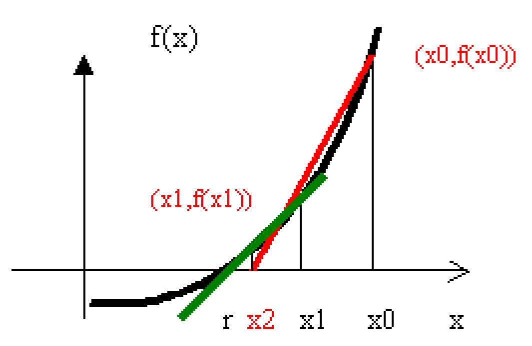
Consider a function f (x) which has the following graph:



Suppose that we want to locate the root **r** which lies near the points $\mathbf{x_0}$ and $\mathbf{x_1}$. The main idea in secant method is to approximate the curve with a straight line for x between the values of $\mathbf{x_0}$ and **r**. The straight line is assumed to be the secant which connects the two points (x_0, $f(x_0)$) and (x_1, $f(x_1)$).
Graphically we can represent this with the picture below, where the secant line and the new root estimate $\mathbf{x_2}$ are shown in red.



We can see that $\mathbf{x_2}$ is closer to the root **r** than either $\mathbf{x_1}$ or $\mathbf{x_0}$. If we were to repeat the process we can very easily see that $\mathbf{x_3}$ would be even closer to the root as it is shown below in green.



Analytically how can we obtain x_2 if we know x_0 and x_1? Let's try to write the equation of the red straight line. Its slope m is given by

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and its equation is:

$$y - f(x_0) = m\,(x - x_0)$$

The point x = x_2 corresponds to the point of the straight line where y = 0. Consequently we obtain:

$$-f(x_0) = m\,(x_2 - x_0)$$

If we solve this equation for x_2 we obtain the iteration formula:

$$x_2 = x_0 - \frac{f(x_0)}{m} = x_0 - \frac{f(x_0)}{\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

As an example of the method consider the following table which illustrates the steps needed to find the positive root of

$$x^2 - 2 = 0$$

We start with the initial guesses x0 = 2 and x1 = 1.8 .

x0	x1	f(x0)	f(x1)	m	x2	remarks
2	1.8	2	1.24	3.8	2 – 2 / 3.8 = 1.4737	We now go to next line by shifting x1 to x0 and x2 to x1 (follow the colors)
1.8	1.4737	1.24	0.1718	3.2737	1.8. – 1.24 / 3.2737 = 1.4212	We also shift f(x1) to f(x0) only one function evaluation per line
1.4737	1.4212	0.1718	0.0198	2.8952	1.4737-0.1718 / 2.8952 = 1.4144	Shift again. The root is approaching its exact value
1.4212	1.4144	0.0198	0.0005	2.8382	1.4212-0.0198/2.8382= 1.4142	At this point we reach the “exact” value considering 4 decimal digits

Note that the actual root of the equation is the square root of two, which is 1.414213562. If we round this value to 4 decimal places we have 1.4142. What is the error between this “exact” value and our approximation of the first, second, and third line of the table? Check to see how we dealt with this question in our study of Newton’s method and repeat the calculations for this example.

Here we want to illustrate another point of the iteration procedures. **How do we know when to stop since we normally don’t know the “exact” answer?**

Look at the table below where we list the values of “smallness of f(x)” and “closeness of x” and you can see that if you set some tolerances ahead of time, then you can have an if statement which will decide whether the iteration should stop. **WARNING: You must also have a bound on the maximum number of iterations, to prevent the divergent cases from running for ever.**

Line number (Indicated by n)	$\left\lvert \frac{f(x_n)}{f(0)} \right\rvert$	$\left\lvert \frac{x_{n+1} - x_n}{x_n} \right\rvert$
1	$\left\lvert \frac{1.24}{-2} \right\rvert = 0.62$	$\left\lvert \frac{1.4737 - 1.8}{1.8} \right\rvert = 0.18$
2		

	$\left\lvert\frac{0.1718}{-2}\right\rvert = 0.0859$	$\left\lvert\frac{1.4212-1.4737}{1.4737}\right\rvert = 0.0356$
3	$\left\lvert\frac{0.0198}{-2}\right\rvert = 0.0099$	$\left\lvert\frac{1.4144-1.4212}{1.4212}\right\rvert = 0.0048$
4	$\left\lvert\frac{0.0005}{-2}\right\rvert = 0.00025$	$\left\lvert\frac{1.4142-1.4144}{1.4144}\right\rvert = 0.0001$

PROS & CONS of the method:

1) When the method works, works fast. It is a bit slower than Newton's method but much faster than the bisection method
2) If we have the wrong starting values the method diverges (produces increasingly nonsensical results)
3) We only need to calculate the function itself at each step only once.

CHALLENGE

In our derivation of the iteration formula we used the slope of the secant and the point (x0, f(x0)) to write the equation of the secant line. What would happen if we used the point (x1, f(x1)) ? Explain.

TRY ON YOUR OWN

Complete the table below which, uses secant method to locate the root of the equation

$$f(x) = \tan(x) - x = 0$$

starting with x0= 4 and x1 = 3.8. Click on the line to see the solution.
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x0	x1	f(x0)	f (x1)	m	x2
4.4	4.3				
