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Lecture 4 Notes

These notes correspond to Sections 1.5 and 1.6 in the text.

The Secant Method

One drawback of Newton's method is that it is necessary to evaluate $f'(x)$ at various points, which may not be practical for some choices of f . The *secant method* avoids this issue by using a finite difference to approximate the derivative. As a result, $f(x)$ is approximated by a *secant line* through two points on the graph of f , rather than a tangent line through one point on the graph.

Since a secant line is defined using two points on the graph of $f(x)$, as opposed to a tangent line that requires information at only one point on the graph, it is necessary to choose two initial iterates x_0 and x_1 . Then, as in Newton's method, the next iterate x_2 is then obtained by computing the x -value at which the secant line passing through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ has a y -coordinate of zero. This yields the equation

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1) + f(x_1) = 0$$

which has the solution

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

which can be rewritten as follows:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= x_1 \frac{f(x_1) - f(x_0)}{f(x_1) - f(x_0)} - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= \frac{x_1(f(x_1) - f(x_0)) - f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= \frac{x_1f(x_1) - x_1f(x_0) - x_1f(x_1) + x_0f(x_1)}{f(x_1) - f(x_0)} \\ &= \frac{x_0f(x_1) - x_1f(x_0)}{f(x_1) - f(x_0)}. \end{aligned}$$

This leads to the following algorithm.

Algorithm (Secant Method) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. The following algorithm computes an approximate solution x^* to the equation $f(x) = 0$.

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Choose two initial guesses  $x_0$  and  $x_1$ 
for  $k = 1, 2, 3, \dots$  do
    if  $f(x_k)$  is sufficiently small then
         $x^* = x_k$ 
        return  $x^*$ 
    end
     $x_{k+1} = \frac{x_{k-1}f(x_k) - x_kf(x_{k-1})}{f(x_k) - f(x_{k-1})}$ 
    if  $|x_{k+1} - x_k|$  is sufficiently small then
         $x^* = x_{k+1}$ 
        return  $x^*$ 
    end
end

```

Like Newton's method, it is necessary to choose the starting iterate x_0 to be reasonably close to the solution x^* . Convergence is not as rapid as that of Newton's Method, since the secant-line approximation of f is not as accurate as the tangent-line approximation employed by Newton's method.

Example We will use the Secant Method to solve the equation $f(x) = 0$, where $f(x) = x^2 - 2$. This method requires that we choose two initial iterates x_0 and x_1 , and then compute subsequent iterates using the formula

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \dots$$

We choose $x_0 = 1$ and $x_1 = 1.5$. Applying the above formula, we obtain

$$\begin{aligned} x_2 &= 1.4 \\ x_3 &= 1.41379310344828 \\ x_4 &= 1.41421568627451. \end{aligned}$$

As we can see, the iterates produced by the Secant Method are converging to the exact solution $x^* = \sqrt{2}$, but not as rapidly as those produced by Newton's Method. \square

We now prove that the Secant Method converges if x_0 is chosen sufficiently close to a solution x^* of $f(x) = 0$, if f is continuously differentiable near x^* and $f'(x^*) = \alpha \neq 0$. Without loss of generality, we assume $\alpha > 0$. Then, by the continuity of f' , there exists an interval $I_\delta = [x^* - \delta, x^* + \delta]$ such that

$$\frac{3\alpha}{4} \leq f'(x) \leq \frac{5\alpha}{4}, \quad x \in I_\delta.$$

It follows from the Mean Value Theorem that

$$x_{k+1} - x^* = x_k - x^* - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

$$\begin{aligned}
&= x_k - x^* - \frac{f'(\theta_k)(x_k - x^*)}{f'(\varphi_k)} \\
&= \left[1 - \frac{f'(\theta_k)}{f'(\varphi_k)}\right] (x_k - x^*),
\end{aligned}$$

where θ_k lies between x_k and x^* , and φ_k lies between x_k and x_{k-1} . Therefore, if x_{k-1} and x_k are in I_δ , then so are φ_k and θ_k , and x_{k+1} satisfies

$$|x_{k+1} - x^*| \leq \max \left\{ \left|1 - \frac{5\alpha/4}{3\alpha/4}\right|, \left|1 - \frac{3\alpha/4}{5\alpha/4}\right| \right\} |x_k - x^*| \leq \frac{2}{3} |x_k - x^*|.$$

We conclude that if $x_0, x_1 \in I_\delta$, then all subsequent iterates lie in I_δ , and the Secant Method converges at least linearly, with asymptotic rate constant $2/3$.

The order of convergence of the Secant Method can be determined using a result, which we will not prove here, stating that if $\{x_k\}_{k=0}^\infty$ is the sequence of iterates produced by the Secant Method for solving $f(x) = 0$, and if this sequence converges to a solution x^* , then for k sufficiently large,

$$|x_{k+1} - x^*| \approx S |x_k - x^*| |x_{k-1} - x^*|$$

for some constant S .

We assume that $\{x_k\}$ converges to x^* of order α . Then, dividing both sides of the above relation by $|x_k - x^*|^\alpha$, we obtain

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|^\alpha} \approx S |x_k - x^*|^{1-\alpha} |x_{k-1} - x^*|.$$

Because α is the rate of convergence, the left side must converge to a positive constant C as $k \rightarrow \infty$. It follows that the right side must converge to a positive constant as well, as must its reciprocal. In other words, there must exist positive constants C_1 and C_2

$$\frac{|x_k - x^*|}{|x_{k-1} - x^*|^\alpha} \rightarrow C_1, \quad \frac{|x_k - x^*|^{\alpha-1}}{|x_{k-1} - x^*|} \rightarrow C_2.$$

This can only be the case if there exists a nonzero constant β such that

$$\frac{|x_k - x^*|}{|x_{k-1} - x^*|^\alpha} = \left(\frac{|x_k - x^*|^{\alpha-1}}{|x_{k-1} - x^*|} \right)^\beta,$$

which implies that

$$1 = (\alpha - 1)\beta \quad \text{and} \quad \alpha = \beta.$$

Eliminating β , we obtain the equation

$$\alpha^2 - \alpha - 1 = 0,$$

which has the solutions

$$\alpha_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618, \quad \alpha_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618.$$

Since we must have $\alpha > 1$, the rate of convergence is 1.618.

The Bisection Method

Suppose that $f(x)$ is a continuous function that changes sign on the interval $[a, b]$. Then, by the Intermediate Value Theorem, $f(x) = 0$ for some $x \in [a, b]$. How can we find the solution, knowing that it lies in this interval?

The method of *bisection* attempts to reduce the size of the interval in which a solution is known to exist. Suppose that we evaluate $f(m)$, where $m = (a + b)/2$. If $f(m) = 0$, then we are done. Otherwise, f must change sign on the interval $[a, m]$ or $[m, b]$, since $f(a)$ and $f(b)$ have different signs. Therefore, we can cut the size of our search space in half, and continue this process until the interval of interest is sufficiently small, in which case we must be close to a solution. The following algorithm implements this approach.

Algorithm (Bisection) Let f be a continuous function on the interval $[a, b]$ that changes sign on (a, b) . The following algorithm computes an approximation p^* to a number p in (a, b) such that $f(p) = 0$.

```
for  $j = 1, 2, \dots$  do
     $p_j = (a + b)/2$ 
    if  $f(p_j) = 0$  or  $b - a$  is sufficiently small then
         $p^* = p_j$ 
        return  $p^*$ 
    end
    if  $f(a)f(p_j) < 0$  then
         $b = p_j$ 
    else
         $a = p_j$ 
    end
end
```

At the beginning, it is known that (a, b) contains a solution. During each iteration, this algorithm updates the interval (a, b) by checking whether f changes sign in the first half (a, p_j) , or in the second half (p_j, b) . Once the correct half is found, the interval (a, b) is set equal to that half. Therefore, at the beginning of *each* iteration, it is known that the current interval (a, b) contains a solution.

The test $f(a)f(p_j) < 0$ is used to determine whether f changes sign in the interval (a, p_j) or (p_j, b) . This test is more efficient than checking whether $f(a)$ is positive and $f(p_j)$ is negative, or vice versa, since we do not care which value is positive and which is negative. We only care whether they have different signs, and if they do, then their product must be negative.

In comparison to other methods, including some that we will discuss, bisection tends to converge rather slowly, but it is also guaranteed to converge. These qualities can be seen in the following result concerning the accuracy of bisection.

Theorem Let f be continuous on $[a, b]$, and assume that $f(a)f(b) < 0$. For each positive integer n , let p_n be the n th iterate that is produced by the bisection algorithm. Then the sequence $\{p_n\}_{n=1}^{\infty}$ converges to a number p in (a, b) such that $f(p) = 0$, and each iterate p_n satisfies

$$|p_n - p| \leq \frac{b - a}{2^n}.$$

It should be noted that because the n th iterate can lie anywhere within the interval (a, b) that is used during the n th iteration, it is possible that the error bound given by this theorem may be quite conservative.

Example We seek a solution of the equation $f(x) = 0$, where

$$f(x) = x^2 - x - 1.$$

Because $f(1) = -1$ and $f(2) = 1$, and f is continuous, we can use the Intermediate Value Theorem to conclude that $f(x) = 0$ has a solution in the interval $(1, 2)$, since $f(x)$ must assume every value between -1 and 1 in this interval.

We use the method of *bisection* to find a solution. First, we compute the midpoint of the interval, which is $(1 + 2)/2 = 1.5$. Since $f(1.5) = -0.25$, we see that $f(x)$ changes sign between $x = 1.5$ and $x = 2$, so we can apply the Intermediate Value Theorem again to conclude that $f(x) = 0$ has a solution in the interval $(1.5, 2)$.

Continuing this process, we compute the midpoint of the interval $(1.5, 2)$, which is $(1.5 + 2)/2 = 1.75$. Since $f(1.75) = 0.3125$, we see that $f(x)$ changes sign between $x = 1.5$ and $x = 1.75$, so we conclude that there is a solution in the interval $(1.5, 1.75)$. The following table shows the outcome of several more iterations of this procedure. Each row shows the current interval (a, b) in which we know that a solution exists, as well as the midpoint of the interval, given by $(a + b)/2$, and the value of f at the midpoint. Note that from iteration to iteration, only one of a or b changes, and the endpoint that changes is always set equal to the midpoint.

a	b	$m = (a + b)/2$	$f(m)$
1	2	1.5	-0.25
1.5	2	1.75	0.3125
1.5	1.75	1.625	0.015625
1.5	1.625	1.5625	-0.12109
1.5625	1.625	1.59375	-0.053711
1.59375	1.625	1.609375	-0.019287
1.609375	1.625	1.6171875	-0.0018921
1.6171875	1.625	1.62109325	0.0068512
1.6171875	1.62109325	1.619140625	0.0024757
1.6171875	1.619140625	1.6181640625	0.00029087

The correct solution, to ten decimal places, is 1.6180339887, which is the number known as the *golden ratio*. \square

For this method, it is easier to determine the order of convergence if we use a different measure of the error in each iterate x_k . Since each iterate is contained within an interval $[a_k, b_k]$ where $b_k - a_k = 2^{-k}(b - a)$, with $[a, b]$ being the original interval, it follows that we can bound the error $x_k - x^*$ by $e_k = b_k - a_k$. Using this measure, we can easily conclude that bisection converges linearly, with asymptotic error constant $1/2$.

Safeguarded Methods

It is natural to ask whether it is possible to combine the rapid convergence of methods such as Newton's method with "safe" methods such as bisection that are guaranteed to converge. This leads to the concept of *safeguarded methods*, which maintain an interval within which a solution is known to exist, as in bisection, but use a method such as Newton's method to find a solution within that interval. If an iterate falls outside this interval, the safe procedure is used to refine the interval before trying the rapid method.

An example of a safeguarded method is the *method of Regula Falsi*, which is also known as the *method of false position*. It is a modification of the secant method in which the two initial iterates x_0 and x_1 are chosen so that $f(x_0) \cdot f(x_1) < 0$, thus guaranteeing that a solution lies between x_0 and x_1 . This condition also guarantees that the next iterate x_2 will lie between x_0 and x_1 , as can be seen by applying the Intermediate Value Theorem to the secant line passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

It follows that if $f(x_2) \neq 0$, then a solution must lie between x_0 and x_2 , or between x_1 and x_2 . In the first scenario, we use the secant line passing through $(x_0, f(x_0))$ and $(x_2, f(x_2))$ to compute the next iterate x_3 . Otherwise, we use the secant line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Continuing in this fashion, we obtain a sequence of smaller and smaller intervals that are guaranteed to contain a solution, as in bisection, but interval is updated using a superlinearly convergent method, the secant method, rather than simply being bisected.

Algorithm (Method of Regula Falsi) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function that changes sign on the interval (a, b) . The following algorithm computes an approximate solution x^* to the equation $f(x) = 0$.

```

repeat
   $c = \frac{af(b)-bf(a)}{f(b)-f(a)}$ 
  if  $f(c) = 0$  or  $b - a$  is sufficiently small then
     $x^* = c$ 
    return  $x^*$ 
end
```

```

if  $f(a) \cdot f(c) < 0$  then
     $b = c$ 
else
     $a = c$ 
end
end

```

Example We use the Method of Regula Falsi (False Position) to solve $f(x) = 0$ where $f(x) = x^2 - 2$. First, we must choose two initial guesses x_0 and x_1 such that $f(x)$ changes sign between x_0 and x_1 . Choosing $x_0 = 1$ and $x_1 = 1.5$, we see that $f(x_0) = f(1) = -1$ and $f(x_1) = f(1.5) = 0.25$, so these choices are suitable.

Next, we use the Secant Method to compute the next iterate x_2 by determining the point at which the secant line passing through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ intersects the line $y = 0$. We have

$$\begin{aligned}
 x_2 &= x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)} \\
 &= 1 - \frac{(-1)(1.5 - 1)}{0.25 - (-1)} \\
 &= 1 + \frac{1.5 - 1}{0.25 + 1} \\
 &= 1 + \frac{0.5}{1.25} \\
 &= 1.4.
 \end{aligned}$$

Computing $f(x_2)$, we obtain $f(1.4) = -0.04 < 0$. Since $f(x_2) < 0$ and $f(x_1) > 0$, we can use the Intermediate Value Theorem to conclude that a solution exists in the interval (x_2, x_1) . Therefore, we compute x_3 by determining where the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ intersects the line $y = 0$. Using the formula for the Secant Method, we obtain

$$\begin{aligned}
 x_3 &= x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)} \\
 &= 1.5 - \frac{(0.25)(1.4 - 1.5)}{-0.04 - 0.25} \\
 &= 1.41379.
 \end{aligned}$$

Since $f(x_3) < 0$ and $f(x_2) < 0$, we do not know that a solution exists in the interval (x_2, x_3) . However, we do know that a solution exists in the interval (x_3, x_1) , because $f(x_1) > 0$. Therefore, instead of proceeding as in the Secant Method and using the Secant line determined by x_2 and x_3 to compute x_4 , we use the secant line determined by x_1 and x_3 to compute x_4 . \square