Newton Raphson Method to find root of any function

Algorithms | List of Mathematical Algorithms

Reading time: 35 minutes | Coding time: 10 minutes

Newton's Method, also known as **Newton-Raphson method**, named after Isaac Newton and Joseph Raphson, is a popular iterative method to find a good approximation for the root of a real-valued function *f(x) = 0.* It uses the idea that a continuous and differentiable function can be approximated by a straight line tangent to it.

Theory

For a continous and differentiable function *f(x)*, let x_0 be a good approximation for root *r* and let $r = x_0 + h$. The number h measures how far is x_0 is from true root.

Since $h = r - x_0$ is small we can use tangent line approximation as follow:

> $0 = f(r) = f(x_0 + h) \approx f(x_0) + hf'(x_0)$

And unless $hf'(x_0)$ is close to zero we have,

> $h \approx - f(x_0) / f'(x_0)$

So this gives us the root *r* as,

> $r = x_0 + h \approx x_0 - f(x_0) / f'(x_0)$

Now we put $x_1 = x_0 + h$, where x_1 is our new improved estimate, and we get the following equation

> $x_1 = x_0 - f(x_0) / f'(x_0)$

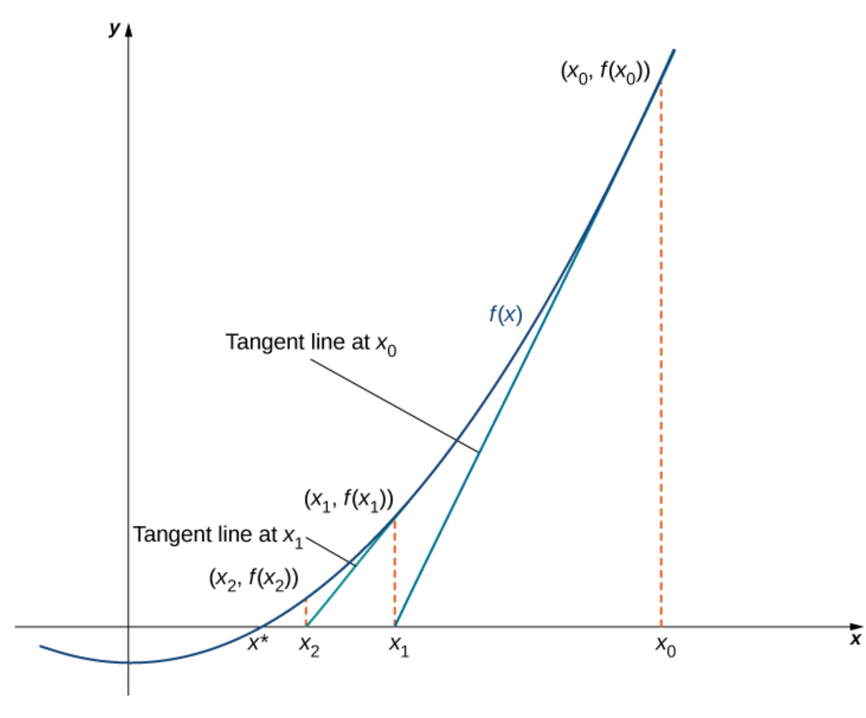
The next estimate x_2 is obtained from x_1 in exactly the same way as x_1 was obtained from x_0:

> $x_2 = x_1 - f(x_1) / f'(x_1)$

So we continue in this way, If x_n is the current estimate, then the next estimate x_{n+1} is given by :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We can see graphically how Newton's Method works as follow:



Newton Raphson Method.
Source: Openstax

We draw a tangent line to the graph of function f(x) at point $x = x_0$. This tangent will have the equation as $y= f'(x_0)(x - x_0) + f(x_0)$. Now, we find the x intercept of this tangent by putting y = 0. The new x that we find is the new estimate. This continues till we reach the root of the function.

Algorithm

For a given function f(x),the Newton Raphson Method algorithm works as follows:

```
1. Start

2. Define function as f(x)

3. Define derivative of function as g(x)

4. Input:
        a. Initial guess x0
        b. Tolerable Error e
        c. Maximum Iteration N

5. Do
        If g(x0) = 0
                Print "Mathematical Error"
                Stop
        End If

    x(i+1) = x(i) - f(x) / g(x)

        step = step + 1

        If step > N
                Print "Not Convergent"
                Stop
        End If

   While abs f(x1) > e

7. Print root as x(i+1)

8. Stop
```

Sample Problem

Now let's work with an example:

Find the root of function $f(x) = x^2 - 4x - 7$ taking initial guess as x = 5 using the Newton's Method to determine an approximation to the root that is accurate to at least within 10^{-9}.

Now, the information required to perform the Newton Raphson Method is as follow:

- $f(x) = x^2 - 4x - 7$,
- Initial Guess x0 = 5,
- f´(x) = g(x) = 2x - 4,
- And tolerance $e = 10^{-9}$

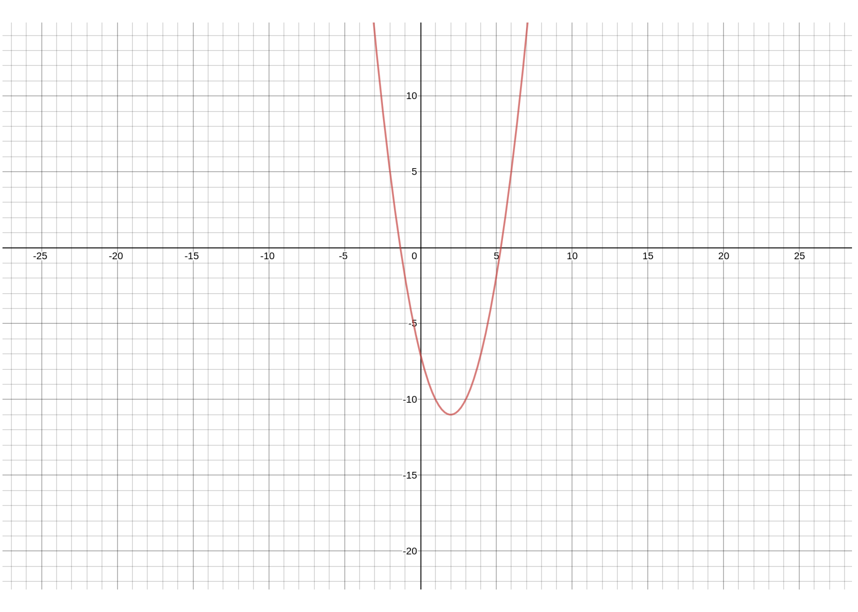


Figure: Plot of the function $f(x) = x^2 - 4x - 7$

Below we show the iterative process described in the algortihm above and show the values in each iteration:

Inputs

- $f(x) = x^2 - 4x - 7$,
- Initial Guess x0 = 5,
- And tolerance $e = 10^{-9}$

We will calculate the f´(x) :
f´(x) = g(x) = 2x - 4,

Iteration 1

x0 = 5
f(x0) = f(5) = -2

- Check if g(x0) is not 0
  g(x0) = g(5) = 2(5) - 4 = 6 which is not 0 ✔️
- We then proceed to calculate x1 :
  x1 = x0 - f(x0)/g(x0)
  x1 = 5 - (-2/6) = 5.333333333

Now we check the loop condition i.e. fabs(f(x1)) > e
f(x1) = f(5.333333333) = 0.1111111111
fabs(f(c)) = 0.1111111111 > e = 10^{-9} ✔️
The loop condition is true so we will perform the next iteration.

Iteration 2
x1 = 5.333333333

f(x1) = f(5.333333333) = 0.1111111111

- Check if g(x1) is not 0
  g(x1) = g(5.333333333) = 2(5.333333333) - 4 = 6.666666666 which is not 0 ✔️
- We then proceed to calculate x2 :
  x2 = x1 - f(x1)/g(x1)
  x1 = 5.333333333 - (0.1111111111/6) = 5.316666667

Now we check the loop condition i.e. fabs(f(x1)) > e
f(x2) = f(5.316666667) = 0.0002777777778
fabs(f(x2)) = 0.0002777777778 > e = 10^{-9} ✔️
The loop condition is true so we will perform the next iteration.

As you can see, it converges to a solution which depends on the tolerance and number of iteration the algorithm performs.

```
function(x) = x^2 - 4x - 7
Enter initial guess: 5

Enter precision of method: 0.000000001

iterations      x                function(x)
  1           5.333333333       0.1111111111
  2           5.316666667    0.0002777777778
  3           5.316624791    1.753601708e-09
  4            5.31662479                  0

Root = 5.31662479
f(x)=0
116 microseconds
```

Newton Raphson method performed on the function f(x) = x^2 - 4x - 7

C++ Implementation

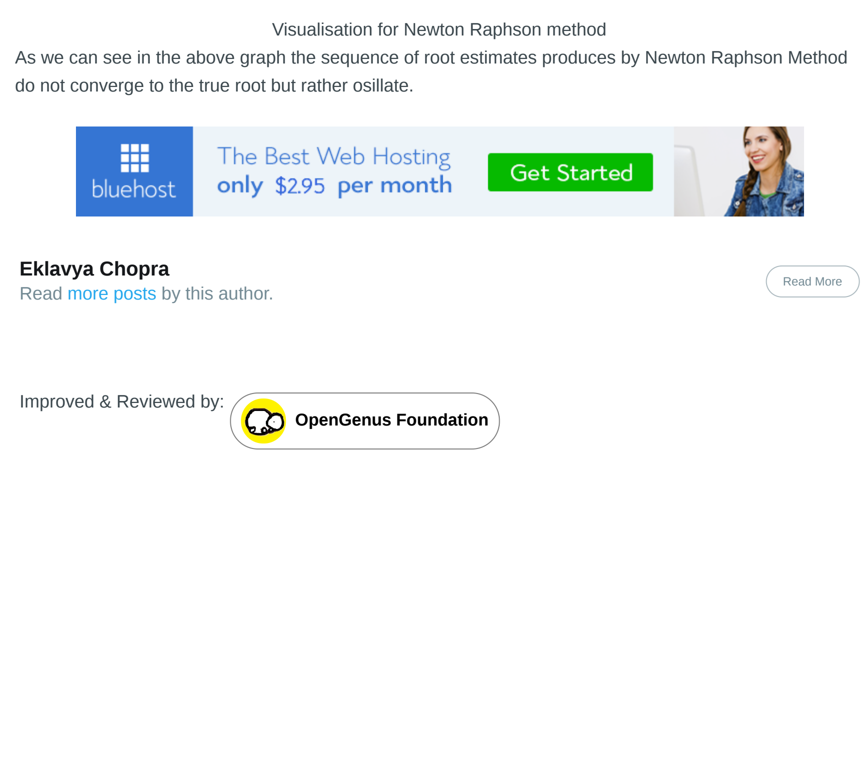
```cpp
#include <iostream>
#include <math.h>
#include<iomanip>
#include<chrono>
using namespace std::chrono;
using namespace std;
int N= 1000;   //  max iterations

static double function(double x);
double derivFunc(double x);
void newtonRaphson( double x, double precision);

int main() {
  double x0;
  double c;
  double precision;

  cout << "function(x) = x^2 - 4x - 7 "<<endl;


  cout << "Enter initial guess: ";
  cin >> x0;


  cout << "\nEnter precision of method: ";
  cin >> precision;


  newtonRaphson(x0,precision);

  return 0;
}

static double function(double x)    // f(x)
{
  return  pow(x,2) - 4*x -7;
}


double derivFunc(double x)   // f'(x) = g(x)
{
    return  2*x - 4 ;
}

void newtonRaphson(double x, double precision)
{

int iter=0;

cout<<setw(3)<<"\niterations"<<setw(8)<<"x"<<setw(30)<<"function(x)"<<endl;

auto start = high_resolution_clock::now();


  if(derivFunc(x)== 0 )
  {
     cout<<"Error";
     return;

  }
  double h = function(x) / derivFunc(x);

  do
  {

    h = function(x)/derivFunc(x);

    // x(i+1) = x(i) - f(x) / f'(x)
        x = x - h;

    iter++;
    cout<<setprecision(10)<<setw(3)<<iter<<setw(25)<<x<<setw(20)<<function(x)<<endl;

    if (iter > N)
    {
      cout<<" Not Convergent";
      break;
    }

  }while (fabs(function(x))>=precision);

  auto stop = high_resolution_clock::now();
  auto duration = duration_cast<microseconds>(stop - start);


  cout<<"\nRoot = "<<x<<endl;
  cout<<"f(x)="<<function(x)<<endl;
  cout << duration.count()<<" microseconds"<< endl;
}
```

Limitations

Newton Raphson Method is one of the **fastest method** which converges to root quickly (as it has quadratic convergence)
but there are also some drawbacks when this algorithm is used.

- We need to calculate the derivative for the function we are performing the iterations for which adds in more computation than simple methods like Bisection or Regula Falsi method.
- Since it is dependent on initial guess it may encounter a point where derivative may become zero and then not continue.
- Newton's method may not work if there are points of inflection, local maxima or minima around initial guess or the root.

Let us see an examples below demonstrating the limitation



Visualisation for Newton Raphson method

As we can see in the above graph the sequence of root estimates produces by Newton Raphson Method do not converge to the true root but rather osillate.

Eklavya Chopra
Read more posts by this author.
Read More

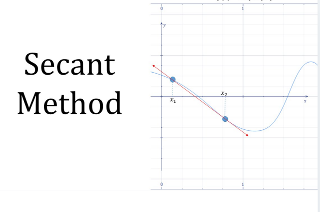
Improved & Reviewed by: OpenGenus Foundation

— OpenGenus IQ: Computing Expertise & Legacy —

Algorithms

- Generate all sub-strings of a string
- Shortest Subarray with at least K as sum
- Time and Space complexity of Radix Sort

See all 473 posts →



ALGORITHMS

Secant Method to find root of any function

Secant Method is a numerical method for solving an equation in one unknown. It avoids this issue of Newton's method by using a finite difference to approximate the derivative.

EKLAVYA CHOPRA

Regula-Falsi Method

ALGORITHMS

Regula Falsi Method for finding root of a polynomial

Regula Falsi method or the method of false position is a numerical method for solving an equation in one unknown. It is quite similar to bisection method algorithm and is one of the oldest approaches.

EKLAVYA CHOPRA