EXAMPLE 3.19

Calculate the pressure coefficient distribution around a circular cylinder using the source panel technique.

■ Solution

We choose to cover the body with eight panels of equal length, as shown in Figure 3.41. This choice is arbitrary; however, experience has shown that, for the case of a circular cylinder, the arrangement shown in Figure 3.41 provides sufficient accuracy. The panels are numbered from 1 to 8, and the control points are shown by the dots in the center of each panel.

Let us evaluate the integrals $I_{i,j}$ which appear in Equation (3.153). Consider Figure 3.42, which illustrates two arbitrary chosen panels. In Figure 3.42, (x_i, y_i) are the coordinates of the control point of the ith panel and (x_j, y_j) are the running coordinates over the entire jth panel. The coordinates of the boundary points for the ith panel are (X_i, Y_i) and (X_{i+1}, Y_{i+1}); similarly, the coordinates of the boundary points for the jth panel are (X_j, Y_j) and (X_{j+1}, Y_{j+1}). In this problem, $\mathbf{V}_\infty$ is in the x direction; hence, the angles between the x axis and the unit vectors $\mathbf{n}_i$ and $\boldsymbol{n}_j$ are β_i and β_j, respectively. Note that, in general, both β_i and β_j vary from 0 to 2π. Recall that the integral $I_{i,j}$ is

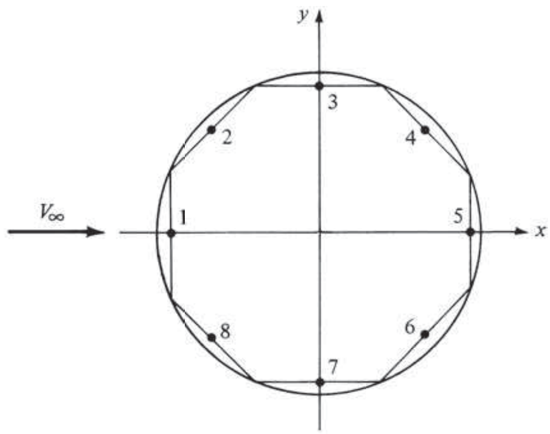

Figure 3.41 Source panel distribution around a circular cylinder.

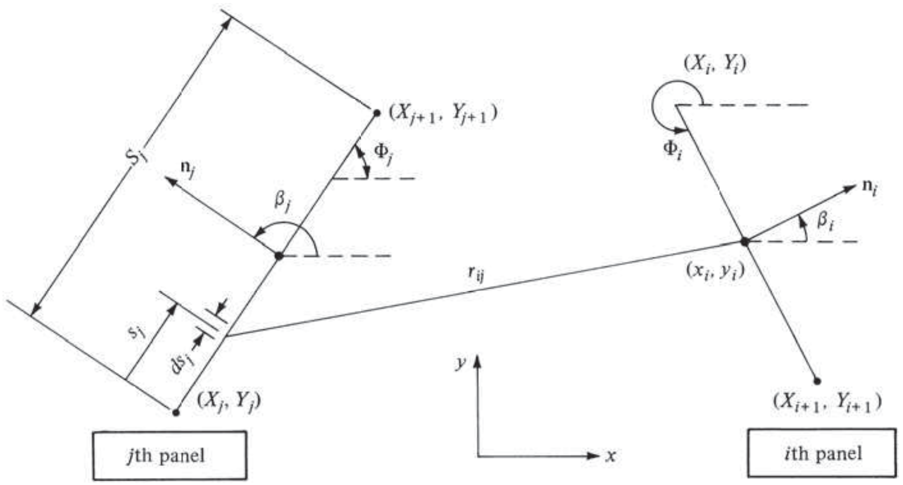

Figure 3.42 Geometry required for the evaluation of I_{ij}.

evaluated at the ith control point and the integral is taken over the complete jth panel:

$$I_{i,j} = \int_j \frac{\partial}{\partial n_i}(\ln r_{ij})\, ds_j \tag{3.158}$$

Since

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

then

$$\begin{aligned}\frac{\partial}{\partial n_i}(\ln r_{ij}) &= \frac{1}{r_{ij}}\frac{\partial r_{ij}}{\partial n_i}\\ &= \frac{1}{r_{ij}}\frac{1}{2}[(x_i - x_j)^2 + (y_i - y_j)^2]^{-1/2}\\ &\quad\times\left[2(x_i - x_j)\frac{dx_i}{dn_i} + 2(y_i - y_j)\frac{dy_i}{dn_i}\right]\end{aligned}$$

or

$$\frac{\partial}{\partial n_i}(\ln r_{ij}) = \frac{(x_i - x_j)\cos\beta_i + (y_i - y_j)\sin\beta_i}{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{3.159}$$

Note in Figure 3.42 that Φ_i and Φ_j are angles measured in the counterclockwise direction from the x axis to the bottom of each panel. From this geometry,

$$\beta_i = \Phi_i + \frac{\pi}{2}$$

Hence,

$$\sin\beta_i = \cos\Phi_i \tag{3.160a}$$

$$\cos\beta_i = -\sin\Phi_i \tag{3.160b}$$

Also, from the geometry of Figure 3.38, we have

$$x_j = X_j + s_j\cos\Phi_j \tag{3.161a}$$

and

$$y_j = Y_j + s_j\sin\Phi_j \tag{3.161b}$$

Substituting Equations (3.159) to (3.161) into (3.158), we obtain

$$I_{i,j} = \int_0^{S_j} \frac{Cs_j + D}{s_j^2 + 2As_j + B} ds_j \tag{3.162}$$

where

$$A = -(x_i - X_j)\cos\Phi_j - (y_i - Y_j)\sin\Phi_j$$
$$B = (x_i - X_j)^2 + (y_i - Y_j)^2$$
$$C = \sin(\Phi_i - \Phi_j)$$
$$D = (y_i - Y_j)\cos\Phi_i - (x_i - X_j)\sin\Phi_i$$
$$S_j = \sqrt{(X_{j+1} - X_j)^2 + (Y_{j+1} - Y_j)^2}$$

Letting $\quad E = \sqrt{B - A^2} = (x_i - X_j)\sin\Phi_j - (y_i - Y_j)\cos\Phi_j$

we obtain an expression for Equation (3.162) from any standard table of integrals:

$$I_{i,j} = \frac{C}{2}\ln\left(\frac{S_j^2 + 2AS_j + B}{B}\right) + \frac{D - AC}{E}\left(\tan^{-1}\frac{S_j + A}{E} - \tan^{-1}\frac{A}{E}\right) \tag{3.163}$$

Equation (3.163) is a general expression for two arbitrarily oriented panels; it is not restricted to the case of a circular cylinder.

We now apply Equation (3.163) to the circular cylinder shown in Figure 3.41. For purposes of illustration, let us choose panel 4 as the ith panel and panel 2 as the jth panel; that is, let us calculate $I_{4,2}$. From the geometry of Figure 3.41, assuming a unit radius for the cylinder, we see that

$$X_j = -0.9239 \quad X_{j+1} = -0.3827 \quad Y_j = 0.3827$$
$$Y_{j+1} = 0.9239 \quad \Phi_i = 315^\circ \quad \Phi_j = 45^\circ$$
$$x_i = 0.6533 \quad y_i = 0.6533$$

Hence, substituting these numbers into the above formulas, we obtain

$$A = -1.3065 \quad B = 2.5607 \quad C = -1 \quad D = 1.3065$$
$$S_j = 0.7654 \quad E = 0.9239$$

Inserting the above values into Equation (3.163), we obtain

$$I_{4,2} = 0.4018$$

Return to Figures 3.41 and 3.42. If we now choose panel 1 as the jth panel, keeping panel 4 as the ith panel, we obtain, by means of a similar calculation, $I_{4,1} = 0.4074$. Similarly, $I_{4,3} = 0.3528$, $I_{4,5} = 0.3528$, $I_{4,6} = 0.4018$, $I_{4,7} = 0.4074$, and $I_{4,8} = 0.4084$.

Return to Equation (3.153), which is evaluated for the ith panel in Figures 3.40 and 3.42. Written for panel 4, Equation (3.153) becomes (after multiplying each term by 2 and noting that $\beta_i = 45°$ for panel 4)

$$\begin{aligned} 0.4074\lambda_1 + 0.4018\lambda_2 + 0.3528\lambda_3 + \pi\lambda_4 + 0.3528\lambda_5 \\ + 0.4018\lambda_6 + 0.4074\lambda_7 + 0.4084\lambda_8 = -0.7071\, 2\pi V_\infty \end{aligned} \tag{3.164}$$

Equation (3.164) is a linear algebraic equation in terms of the eight unknowns, $\lambda_1, \lambda_2, \ldots, \lambda_8$. If we now evaluate Equation (3.153) for each of the seven other panels, we obtain a total of eight equations, including Equation (3.164), which can be solved simultaneously for the eight unknown λ's. The results are

$$\begin{array}{lll} \lambda_1/2\pi V_\infty = 0.3765 & \lambda_2/2\pi V_\infty = 0.2662 & \lambda_3/2\pi V_\infty = 0 \\ \lambda_4/2\pi V_\infty = -0.2662 & \lambda_5/2\pi V_\infty = -0.3765 & \lambda_6/2\pi V_\infty = -0.2662 \\ \lambda_7/2\pi V_\infty = 0 & \lambda_8/2\pi V_\infty = 0.2662 & \end{array}$$

Note the symmetrical distribution of the λ's, which is to be expected for the nonlifting circular cylinder. Also, as a check on the above solution, return to Equation (3.157). Since each panel in Figure 3.41 has the same length, Equation (3.157) can be written simply as

$$\sum_{j=1}^{n} \lambda_j = 0$$

Substituting the values for the λ's obtained into Equation (3.157), we see that the equation is identically satisfied.

The velocity at the control point of the ith panel can be obtained from Equation (3.156). In that equation, the integral over the jth panel is a geometric quantity that is evaluated in a similar manner as before. The result is

$$\begin{aligned} \int_j \frac{\partial}{\partial s}(\ln r_{ij})\,ds_j = \frac{D - AC}{2E} \ln \frac{S_j^2 + 2AS_j + B}{B} \\ - C\left(\tan^{-1}\frac{S_j + A}{E} - \tan^{-1}\frac{A}{E}\right) \end{aligned} \tag{3.165}$$

With the integrals in Equation (3.156) evaluated by Equation (3.165), and with the values for $\lambda_1, \lambda_2, \ldots, \lambda_8$ obtained above inserted into Equation (3.156), we obtain the velocities $V_1, V_2, \ldots, V_8$. In turn, the pressure coefficients $C_{p,1}, C_{p,2}, \ldots, C_{p,8}$ are obtained directly from

$$C_{p,i} = 1 - \left(\frac{V_i}{V_\infty}\right)^2$$

Results for the pressure coefficients obtained from this calculation are compared with the exact analytical result, Equation (3.101) in Figure 3.43. Amazingly enough, in spite of the relatively crude paneling shown in Figure 3.41, the numerical pressure coefficient results are excellent.

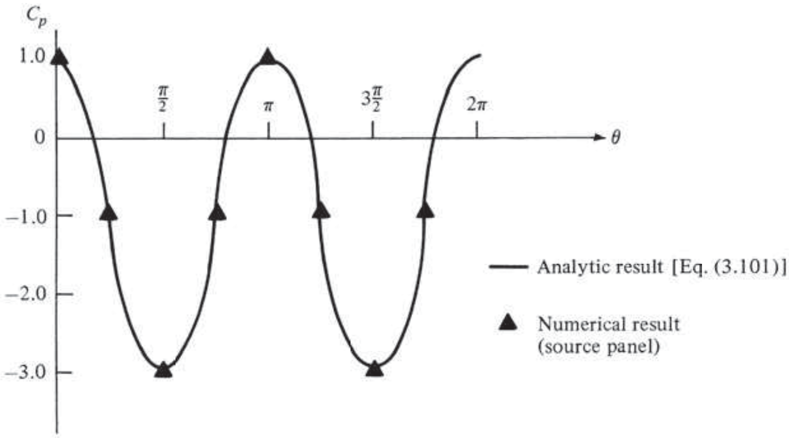

Figure 3.43 Pressure distribution over a circular cylinder; comparison of the source panel results and theory.