1 Averaging in Sensor Networks

The sensor networks under consideration, following the following averaging algorithm

$$x_i(k+1) = average(x_i(k), \{x_j \forall j \in \mathcal{N}(i)\}) \tag{1}$$

where $\mathcal{N}(i)$ is the set of nodes neighbouring (connected) to the i^{th} node in the following network of sensors.

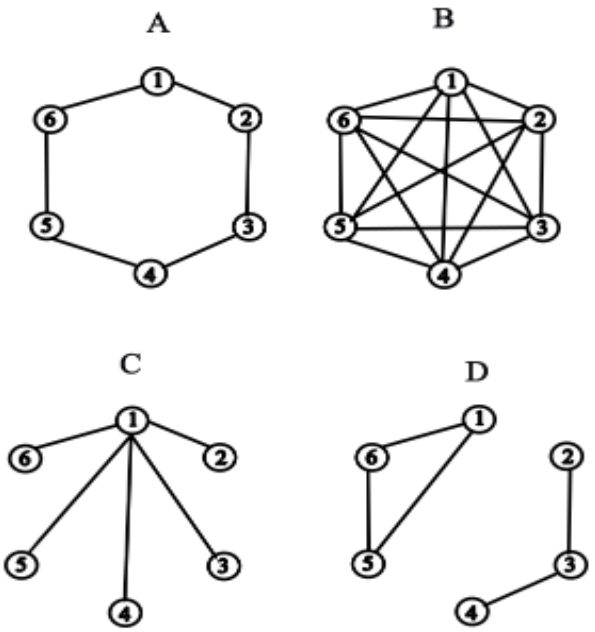


Figure 1: A : Cycle Graph, B : Complete Graph, C Star Graph, D : Disconnected Graph

1.1 Formulation

Formulating the network systems at hand using adjacency matrices **A** (Appendix A.1) for each of the graph configurations, we can model the evolution of the sensor readings as

$$\vec{x_i}(k+1) = \mathbf{A} * \vec{x_i}(k) \tag{2}$$

Since the algorithm is for averaging, the adjacency matrix **A** is row stochastic.

The initial conditions are random.

1.2 Convergence Conditions

In order to check for convergence at each node, we first define the conditions which correspond to convergence. Without loss of generality, we can define the *convergence_margin* as

$$convergence\_margin = 0.01 * min(x_i(0)) \tag{3}$$

Convergence is defined to be achieved if the maximum change of the sensor value among all nodes, is less than the convergence margin. The simulation is stopped once convergence is achieved.

1.3 Results and Observations

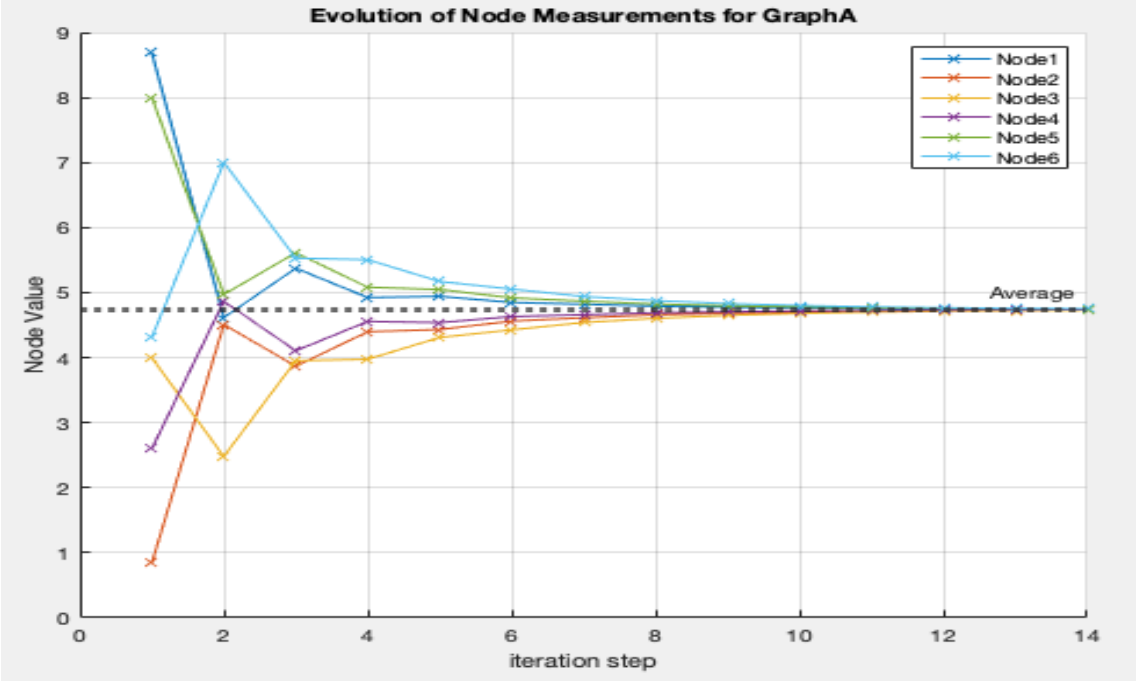


Figure 2: Evolution of Node States for Graph A

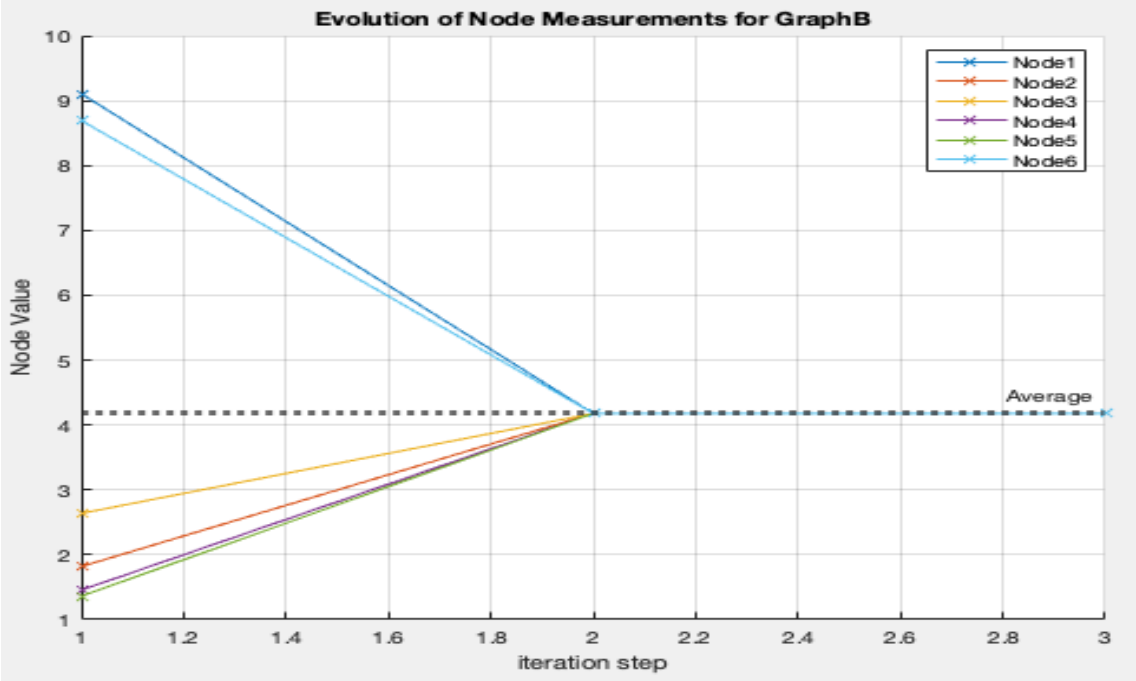


Figure 3: Evolution of Node States for Graph B

For graphs A and B, the sensor measurements converge to the average value of the initial measurement of the sensors.

Furthermore, it is seen that the convergence for the Complete Graph (B) happens in a single iteration. The Cycle Graph (A) on the other hand converges to the initial average but takes more number of iterations.

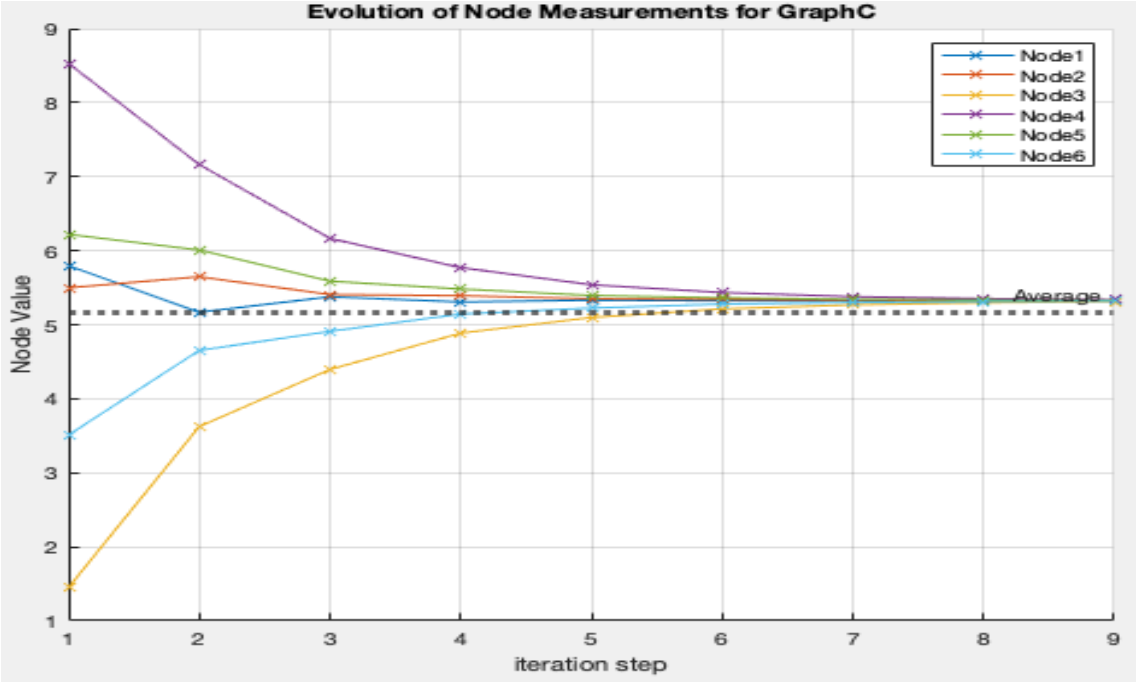


Figure 4: Evolution of Node States for Graph C

The measurements in graph C do not converge to the initial average value. There is a slight bias towards the initial measurement of Node 1 which is the central node and hence highest number of network connections. The other nodes are connected to Node 1 and hence the averaging of their measurement are influenced directly by Node 1 only and not by the other nodes directly.

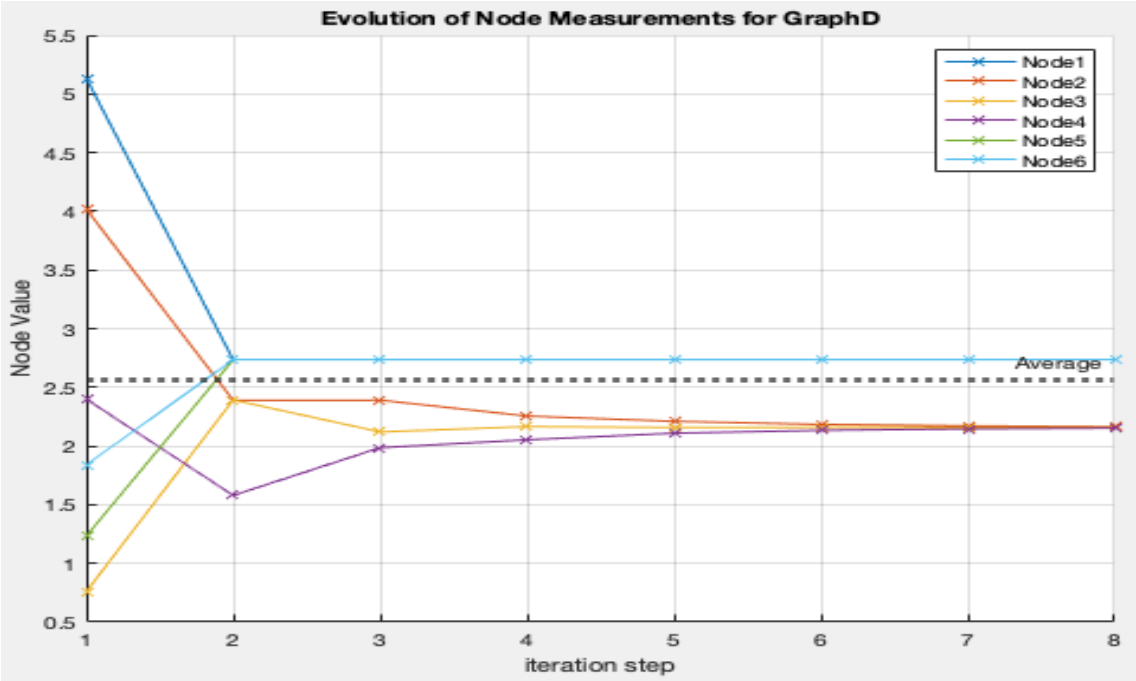


Figure 5: Evolution of Node States for Graph D

In Graph D, there are two seperate averaging dynamics which evolve.

1. The nodes 1, 5 and 6 form a Complete Graph (similar to Graph B) and hence the final measurement of these nodes converges to the average of the initial measurement of the nodes.

2. The nodes 2, 3 and 4 follow dynamics similar to the Star Graph. Node 3 is connected to both nodes 2 and 4. This causes a bias in averaging and hence the nodes converge to a measurement which is not the average of of their initial measurements but biased towards the measurement of Node 3.

2 Robots in Cyclic Pursuit

The n-bug system of robots under consideration are restricted to move on a circle under the following dynamics:

$$\theta_i(k+1) = mod(\theta_i(k) + u_i(k), s\pi) \tag{4}$$

2.1 Convergence Conditions

In order to check for convergence at each node, we first define the conditions which correspond to convergence. Without loss of generality, we can define the convergence margin. Convergence is defined to be achieved if the range of angular rates of all bots is less than the convergence margin. Beyond this point, all the bots can be considered to be moving at the same angular rate. The simulation is stopped once convergence is achieved.

There is also a limit on the iterations to stop the simulation if an equilibrium conditions not found.

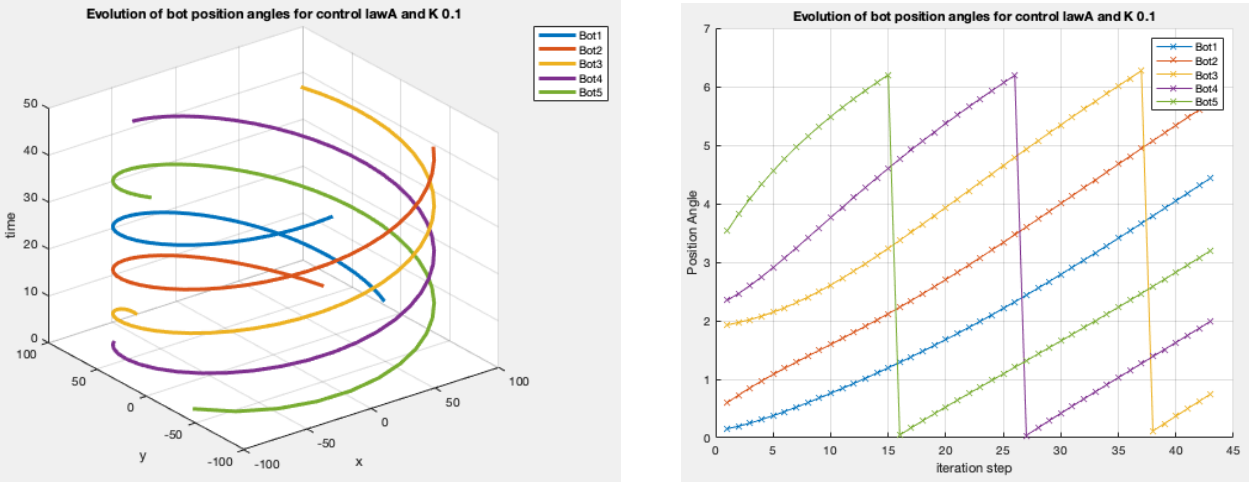
2.2 Part 1

The control law governing the system is defined as :

$$u_i(k) = Kdist_{cc}(\theta_i(k), \theta_{i+1}(k)) = Kmod(\theta_{i+1}(k) - \theta_i(k)) \tag{5}$$

where $dist_{cc}(\theta_i(k), \theta_{i+1}(k))$ is the counter- clockwise distance from θ_i to θ_{i+1}. K $\in$ [0,1] is the control gain

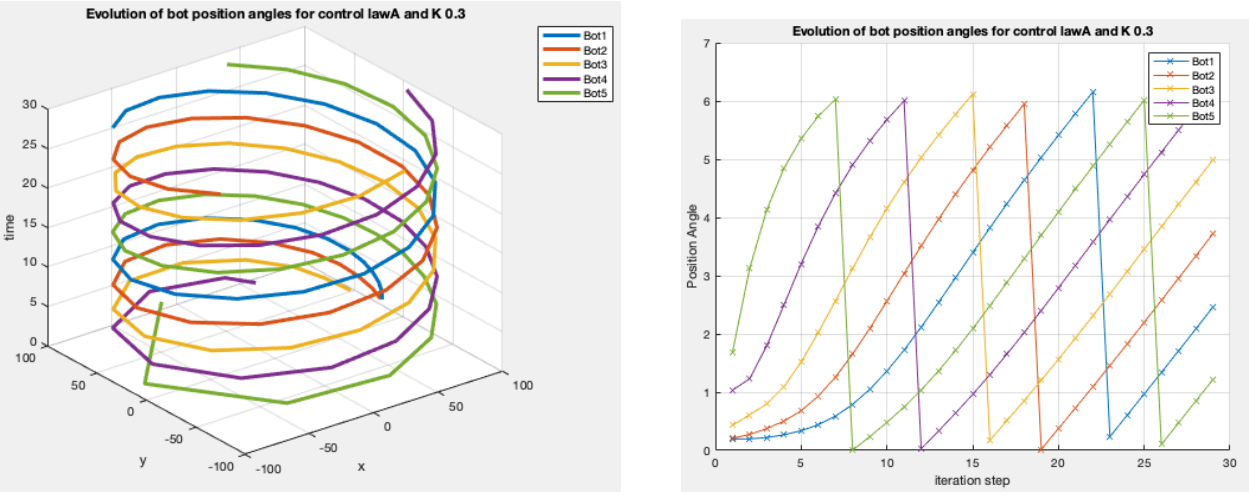
2.2.1 Results and Observations



(a) 3D representation

(b) 2D representation

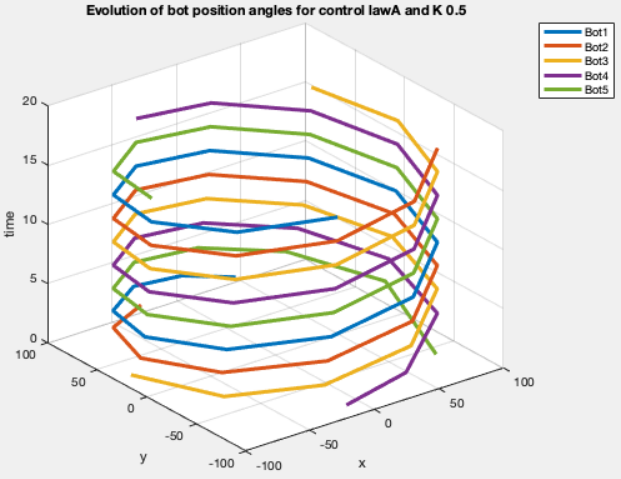
Figure 6: Evolution of states with Dynamics 1 and K = 0.1



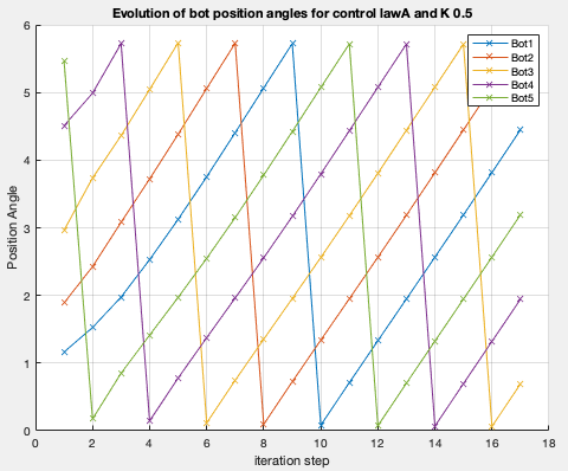
(a) 3D representation

(b) 2D representation

Figure 7: Evolution of states with Dynamics 1 and K = 0.3

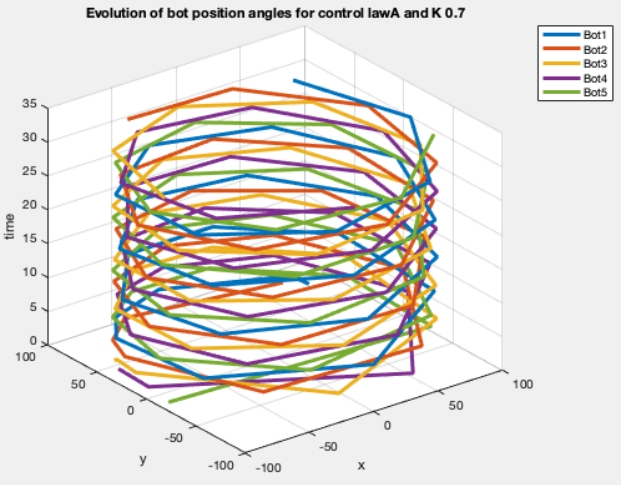


(a) 3D representation

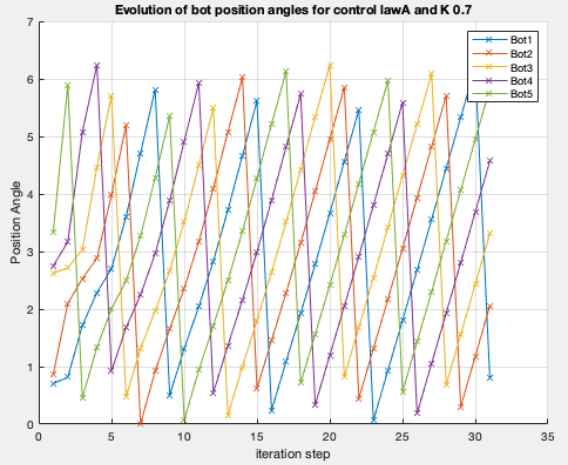


(b) 2D representation

Figure 8: Evolution of states with Dynamics 1 and K = 0.5



(a) 3D representation



(b) 2D representation

Figure 9: Evolution of states with Dynamics 1 and K = 0.7

It is seen that the equilibrium state is dynamic i.e. the equilibrium state consists of all the bots moving with the same angular velocity along the circle.

It is also seen that equilibrium is achieved for all test cases. After multiple test cases, it can be seen that for K $\in$ [0,1], equilibrium will always be achieved.

As the value of K is increased, keeping the convergence margin constant, The number of revolution the bots take before attaining equilibrium also increases.

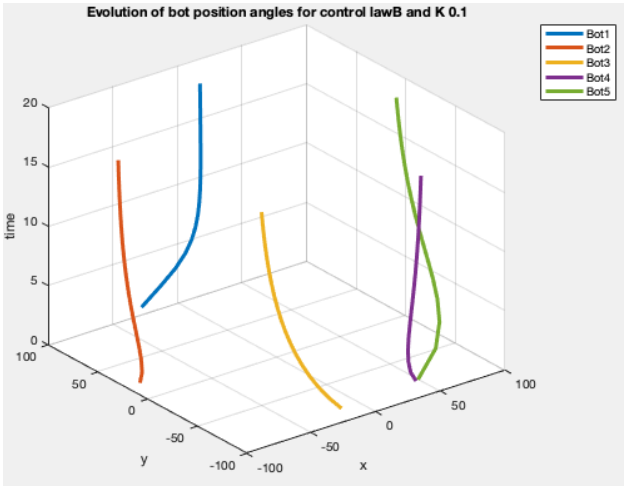
2.3 Part 2

The control law governing the system is defined as :

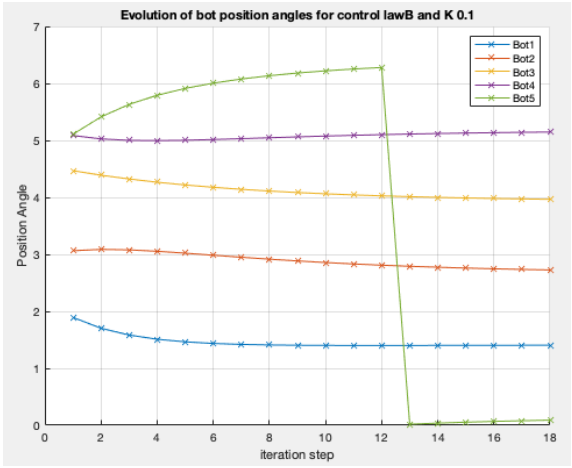
$$u_i(k) = Kdist_{cc}(\theta_i(k), \theta_{i+1}(k)) - Kdist_c(\theta_i(k), \theta_{i-1}(k)) \tag{6}$$

where $dist_c(\theta_i(k), \theta_{i-1}(k))$ is the counter- clockwise distance from θ_i to θ_{i+1}. K $\in$ [0,1] is the control gain

2.3.1 Results and Observations

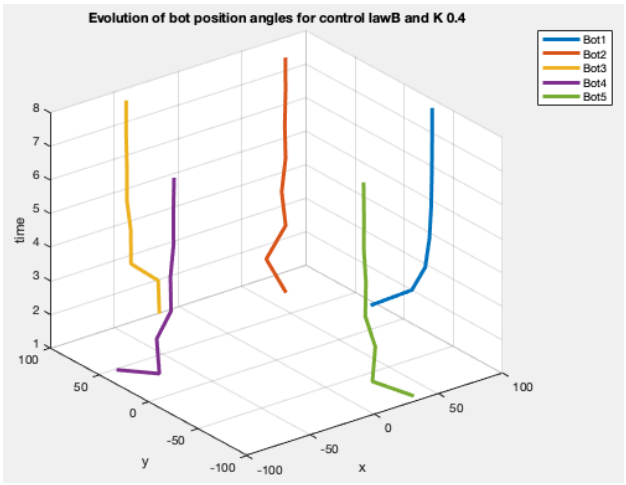


(a) 3D representation

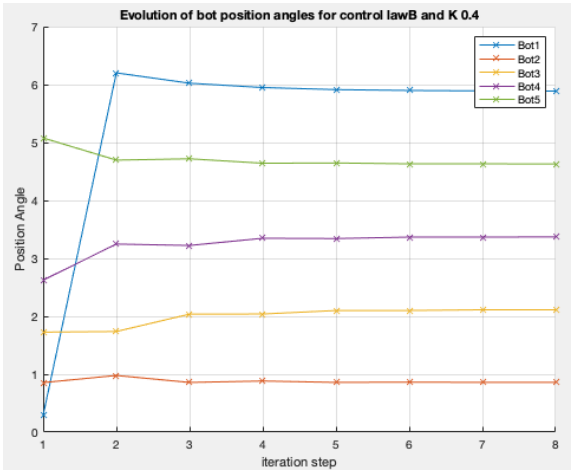


(b) 2D representation

Figure 10: Evolution of states with Dynamics 2 and K = 0.1

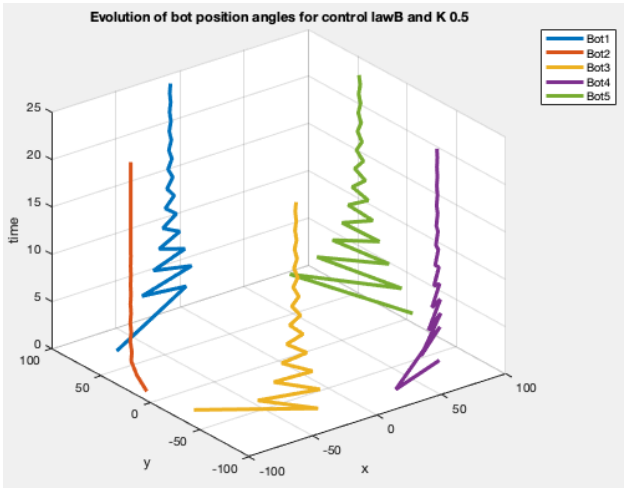


(a) 3D representation

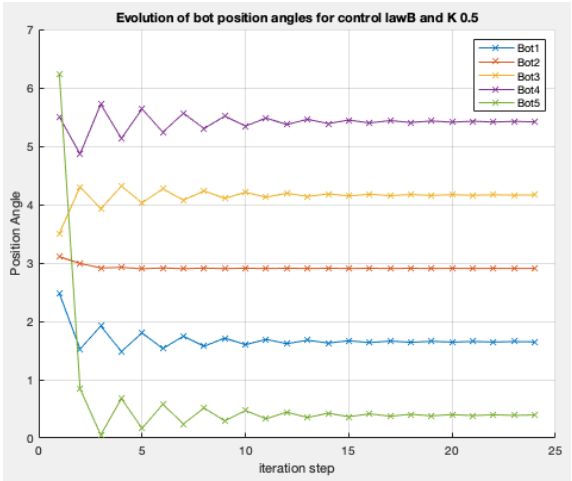


(b) 2D representation

Figure 11: Evolution of states with Dynamics 2 and K = 0.4

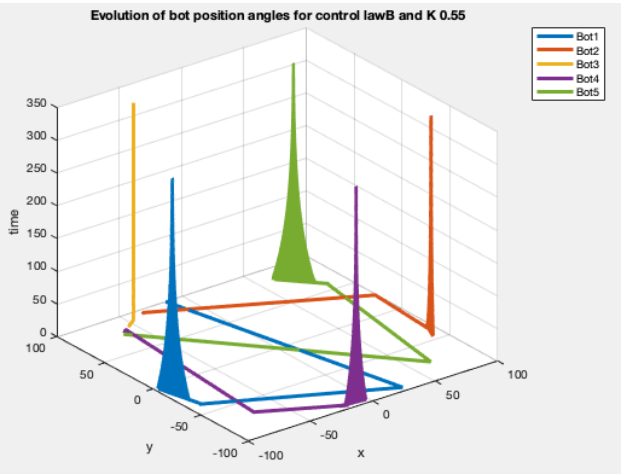


(a) 3D representation

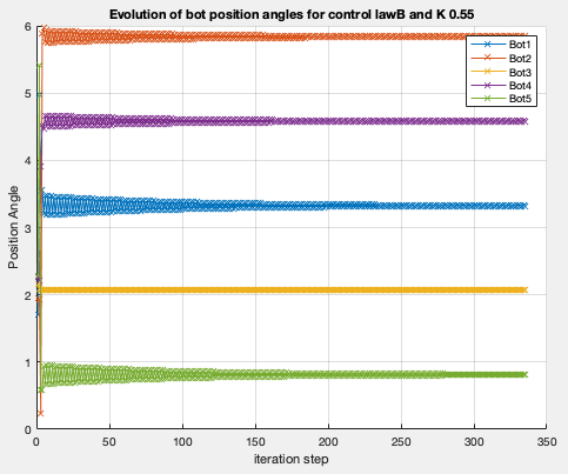


(b) 2D representation

Figure 12: Evolution of states with Dynamics 2 and K = 0.5



(a) 3D representation



(b) 2D representation

Figure 13: Evolution of states with Dynamics 2 and K = 0.55

It is seen that the equilibrium is static i.e. the equilibrium state consists of all bots at a position which does not vary with time.

It is seen that equilibrium is not achieved for all K $\in$ [0,1]. Beyond a certain control gain, no equilibrium point is found.

3 Kuramoto Oscillator Networks

The Kuramoto coupled-oscillator dynamics are given by :

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} sin(\theta_i - \theta_j) \tag{7}$$

where $\theta_i \in [0, 2\pi)$ is the phase angle, $\omega_i \in \mathcal{R}^+$ is the natural frequency of the ith oscillator. A = [a$_{ij}$] denote a real symmetric adjacency matrix (Appendix A.2) .

To better understand the results at equilibrium, define the contribution to $\dot{\theta}$ from the phase difference as seperate parameter

$$\hat{\theta} = \sum_{j=1}^{n} a_{ij} sin(\theta_i - \theta_j) \tag{8}$$

The variation of $\hat{\theta}$ is insightful in analyzing the equilibrium of the coupled oscillator system.

3.1 Part 1

The coupled-oscillator with adjacency matrix $\mathbf{A}_1$,

$$\omega_i = i \qquad i \in 1, 2, 3, 4, 5 \tag{9}$$

and arbitrary initial conditions.

3.1.1 Results and Observations

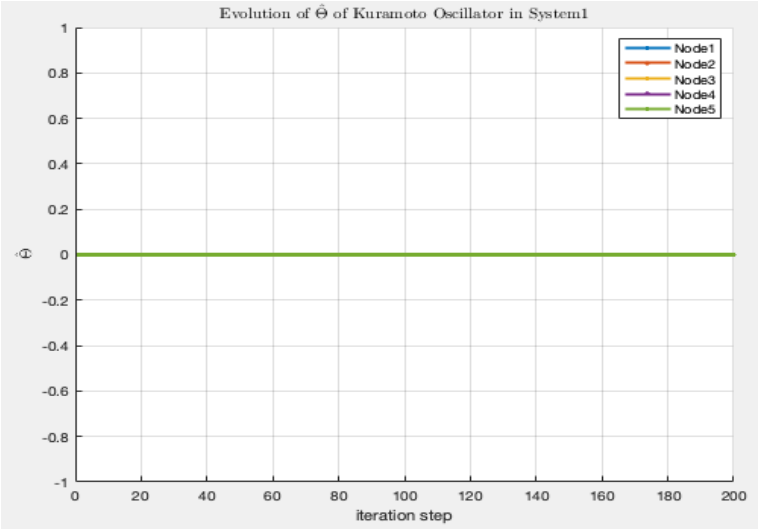


Figure 14: Evolution of Node States for Graph 1

It is seen that at equilibrium, the phase difference contribution converges to a constant value equal to zero (while the sum of phase differences might not necessarily add up to zero).

Thus, in the equilibrium condition for this configuration is from the beginning, and all the nodes oscillate at their natural frequency. All nodes can oscillate independently at their natural frequency because there is no coupling between the nodes (as per their adjacency matrix).

3.2 Part 2

The coupled-oscillator with adjacency matrix $\mathbf{A}_2$,

$$\omega_i = 5 \qquad i \in 1, 2, 3, 4, 5 \tag{10}$$

and initial conditions $\{0, \frac{2\pi}{5} + \frac{1}{2}, \frac{4\pi}{5}, \frac{6\pi}{5} + \frac{1}{10}, \frac{8\pi}{5}\}$

3.2.1 Results and Observations

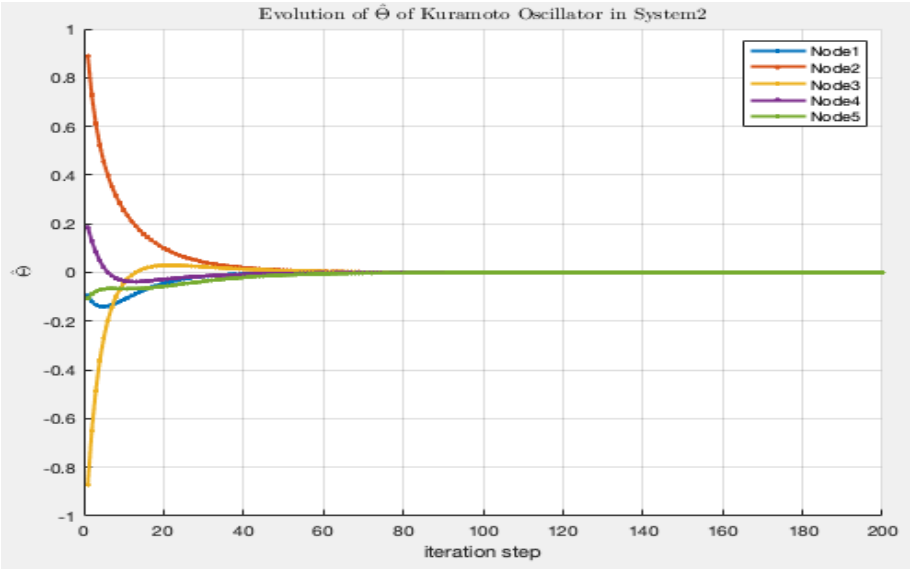


Figure 15: Evolution of Node States for Graph 2

It is seen again, that at equilibrium, the phase difference contribution converges to a constant value equal to zero.

Thus, in the equilibrium condition for this configuration is from the beginning, and all the nodes oscillate at their natural frequency. All nodes can oscillate at their natural frequency because their natural frequencies are same and hence in equilibrium there is no phase-difference coupling between the nodes.

3.3 Part 3

The coupled-oscillator with adjacency matrix $\mathbf{A}_3$,

$$\omega = \{1, 1, 1, 2.5, 2.5\} \tag{11}$$

and arbitrary initial conditions

3.3.1 Results and Observations

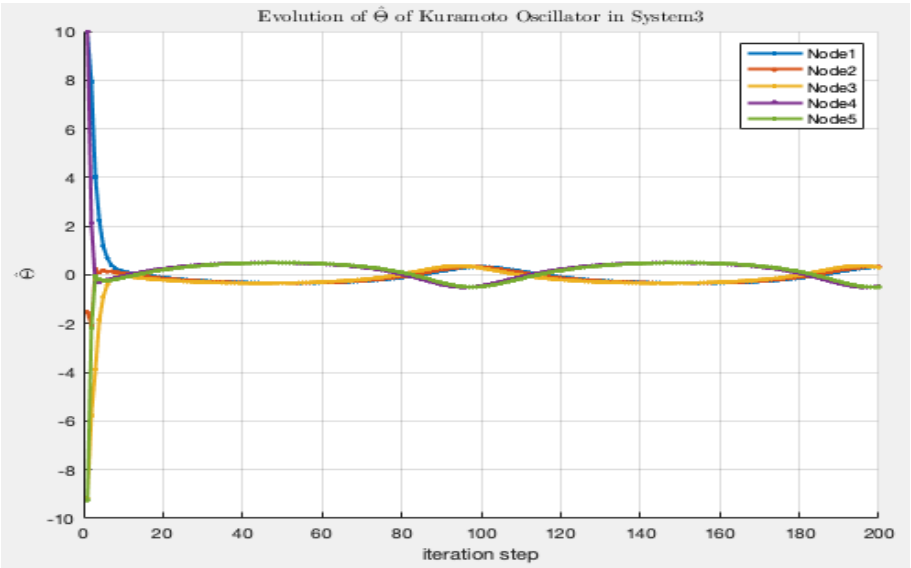


Figure 16: Evolution of Node States for Graph 3

It is seen that at equilibrium, the phase difference contribution oscillates about zero. The phase-contribution oscillation is in phase for nodes which have the same natural frequency, thus $\hat{\theta}$ for nodes 1,2 and 3 are in phase, while that for nodes 4 and 5 is in phase.

The phase-difference contribution for the nodes oscillates at the same frequency which is equal to the first resonant frequency of the natural frequencies i.e frequency of oscillation of phase-difference contribution is LCM(1, 2.5) = 5. Since the time-step for each iteration is 0.05 seconds, (as mentioned in the code) and the frequency is 100 steps, we can verify that the frequency of oscillation of phase-difference contribution is 5.

A Adjacency Matrices

A.1 Sensor Networks

A.1.1 Graph A

$$\mathbf{A}_A = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \tag{12}$$

A.1.2 Graph B

$$\mathbf{A}_B = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \tag{13}$$

A.1.3 Graph C

$$\mathbf{A}_C = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{14}$$

A.1.4 Graph D

$$\mathbf{A}_D = \begin{bmatrix} 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \tag{15}$$

A.2 Oscillator Networks

A.2.1 Graph 1

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

A.2.2 Graph 2

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 3 & 0 & 0 & 3 & 0 \end{bmatrix} \tag{17}$$

A.2.3 Graph 3

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 10 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 0 \\ 0 & 10 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 10 & 0 \end{bmatrix} \tag{18}$$