Homework 1
SM2001 : Data-Driven Methods in Engineering
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Problem 1

<u>SVD</u> Let the economy SVD be given as

$$\underset{\sim}{A}_{2\times3} = \underset{\sim}{U}_{2\times2} \cdot \hat{\underset{\sim}{\Sigma}}_{2\times2} \cdot \hat{\underset{\sim}{V}}^*_{2\times3}$$

$\underset{\sim}{U}, \underset{\sim}{V}$ are unitary $\quad \underset{\sim}{U}\cdot\underset{\sim}{U}^* = I, \; \underset{\sim}{V}^*\underset{\sim}{V} = I$

$$\underset{\sim}{A}_{2\times3} \cdot \underset{\sim}{A}^*_{3\times2} = (\underset{\sim}{U}_{2\times2} \cdot \hat{\underset{\sim}{\Sigma}}_{2\times2} \cdot \hat{\underset{\sim}{V}}^*_{2\times3}) \cdot (\underset{\sim}{U}_{2\times2} \cdot \hat{\underset{\sim}{\Sigma}}_{2\times2} \cdot \hat{\underset{\sim}{V}}^*_{2\times3})^*$$

$$= \underset{\sim}{U}_{2\times2} \cdot \hat{\underset{\sim}{\Sigma}}_{2\times2} \cdot \hat{\underset{\sim}{V}}^*_{2\times3} \cdot \hat{\underset{\sim}{V}}_{3\times2} \cdot \hat{\underset{\sim}{\Sigma}}_{2\times2} \cdot \underset{\sim}{U}^*_{2\times2} \quad = \underset{\sim}{U}_{2\times2} \cdot \hat{\underset{\sim}{\Sigma}}^2_{2\times2} \cdot \underset{\sim}{U}^*_{2\times2}$$

For the given $\underset{\sim}{A}$, $\quad \underset{\sim}{A} \cdot \underset{\sim}{A}^* = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \; u_1 = [1 \;\; 0], \; \lambda_1 = 1 \longrightarrow \sigma_1 = 1$

$\quad\;\; u_2 = [0 \;\; 1], \; \lambda_2 = 1 \longrightarrow \sigma_2 = 1$

$\therefore$ trivally $\quad \underset{\sim}{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\hat{\underset{\sim}{\Sigma}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \because u_1 = [1 \; 0], \; u_2 = [0 \;\; 1]$

$$\underset{\sim}{A} = \underset{\sim}{U} \, \hat{\underset{\sim}{\Sigma}} \cdot \hat{\underset{\sim}{V}}^* \xrightarrow{\underset{\sim}{U}^*} \underset{\sim}{U}^* \cdot \underset{\sim}{A} = \hat{\underset{\sim}{\Sigma}} \cdot \hat{\underset{\sim}{V}}^* \xrightarrow{(\;)^*} \underset{\sim}{A}^* \cdot \underset{\sim}{U} = \hat{\underset{\sim}{V}} \cdot \hat{\underset{\sim}{\Sigma}}$$

$\therefore$ we can obtain $\hat{\underset{\sim}{V}}$ as $\qquad \underset{\sim}{v}_i = \dfrac{A^* \cdot u_i}{\sigma_i}$

$$v_1 = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ economy SVD of $\underset{\sim}{A}$ is given by

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\underset{\sim}{A}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underset{\sim}{U}} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\hat{\underset{\sim}{\Sigma}}} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\hat{\underset{\sim}{V}}^*}$$

to compute full SVD, we find the complete basis of $\hat{\underset{\sim}{V}}$ and add zeroes to $\hat{\underset{\sim}{\Sigma}}$. $\quad \underset{\sim}{v}_1 = [0 \;\; 1 \;\; 0]^T, \; \underset{\sim}{v}_2 = [1 \;\; 0 \;\; 0]^T \quad$ trivially $\underset{\sim}{v}_3 = [0 \;\; 0 \;\; 1]^T$

Let $\hat{\underset{\sim}{v}}_3 = [1 \;\; 2 \;\; 3]^T \longleftarrow$ initial guess

$$\underset{\sim}{v}_3 = \hat{\underset{\sim}{v}}_3 - \frac{\langle \hat{v}_3, v_1 \rangle}{|v_1|^2} v_1 - \frac{\langle \hat{v}_3, v_2 \rangle}{|v_2|^2} \cdot v_2 \qquad \text{using gram-schmidt orthogonalization}$$

$$= [1 \;\; 2 \;\; 3]^T - \frac{2}{1}[0 \; 1 \; 0]^T - \frac{1}{1} \cdot [1 \; 0 \; 0]^T$$

$$\underset{\sim}{v}_3 = [0 \;\; 0 \;\; 3]^T \xrightarrow[\text{unitary basis}]{\text{normalize for}} \underset{\sim}{v}_3 = [0 \;\; 0 \;\; 1]^T$$

$\therefore$ complete SVD of $\underset{\sim}{A}$

$$\boxed{\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\underset{\sim}{A}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underset{\sim}{U}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\underset{\sim}{\Sigma}} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underset{\sim}{V}^*}}$$

<u>Psuedo-inverse</u> $\quad \underset{\sim}{A} = \underset{\sim}{U} \cdot \hat{\underset{\sim}{\Sigma}} \cdot \hat{\underset{\sim}{V}}^* \xrightarrow[\text{inverse}]{\text{psuedo}} \underset{\sim}{A}^+ = \hat{\underset{\sim}{V}} \cdot \hat{\underset{\sim}{\Sigma}}^{-1} \cdot \underset{\sim}{U}^*$

$$\therefore \quad A^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \cdots \quad \boxed{A^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

Problem 2

$$\|\underset{\sim}{A}_{n\times m}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|A_{ij}|^2}$$

Let SVD of $\underset{\sim}{A}_{n\times m}$ be given as $\underset{\sim}{A}_{n\times m} = \underset{\sim}{U}_{n\times n} \cdot \underset{\sim}{\Sigma}_{n\times m} \cdot \underset{\sim}{V}^*_{m\times m}$

unitary matrices

$\because$ $\|\underset{\sim}{A}\|_F = \|\underset{\sim}{A}\underset{\sim}{B}\|_F = \|\underset{\sim}{C}\underset{\sim}{A}\|_F$ if $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are unitary matrices

$\therefore$ $\|\underset{\sim}{U}^*\underset{\sim}{A}\cdot\underset{\sim}{V}\| = \|A\|_F$ $\therefore$ $\|\underset{\sim}{\Sigma}\| = \|A\|_F$

The only non-zero elements of $\underset{\sim}{\Sigma}$ are the singular values

① if $m>n$, A has n singular values $\sigma_1, \sigma_2, \dots \sigma_n$

② if $n>m$, A has m singular values $\sigma_1, \sigma_2, \dots\dots \sigma_m$

$\therefore$ by definition of $\|\cdot\|_F$

$$\|\underset{\sim}{\Sigma}\|_F = \sqrt{\sum_{j=1}^{\min(m,n)}\sigma_j}$$

$\because$ $\|A\|_F = \|\underset{\sim}{\Sigma}\|_F$ $\boxed{\|A\|_F = \sqrt{\sum_{j=1}^{\min(m,n)}\sigma_j}}$ Hence Proved

Problem 3

For a real vector space V, an inner product on V is a function that assigns a real number $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$ to each pair of vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in V, satisfying the following properties, for any $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ in V, and $r \in \mathbb{R}$

① symmetry $\quad \langle u, v \rangle = \langle v, u \rangle$

② Linearity in the first arguement

$$\langle r \cdot u, v \rangle = r \langle u, v \rangle$$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

③ Positive-Definiteness

$$\langle u, u \rangle \geq 0$$

$$\langle u, u \rangle = 0 \text{ if and only if } u = 0$$

For given defined inner product $\quad \langle a, b \rangle = b^T M a$

which we can write as $\quad \langle a, b \rangle = b \cdot (Ma)$ $\quad \therefore$ real number ✓

dot product

① symmetry $\quad \langle a, b \rangle = b^T M a = b \cdot (Ma)$

$$= (Ma) \cdot b$$

$$= (Ma)^T b \quad = a^T M^T b$$

for $\langle a, b \rangle = \langle b, a \rangle \quad a^T M^T b = a^T M b$ ← from definition

from above

$\boxed{\therefore M^T = M \quad \text{M should be symmetric}}$

② $\quad \langle ra, b \rangle = b^T M (ra) = r \; b^T M a = r \langle a, b \rangle$ ✓

and

$$\langle a + b, c \rangle = c^T M (a + b) = c^T M a + c^T M b = \langle a, c \rangle + \langle b, c \rangle \checkmark$$

③ for $a \neq 0 \in \mathbb{R}^n$

$$\langle a, a \rangle = a^T M a > 0$$

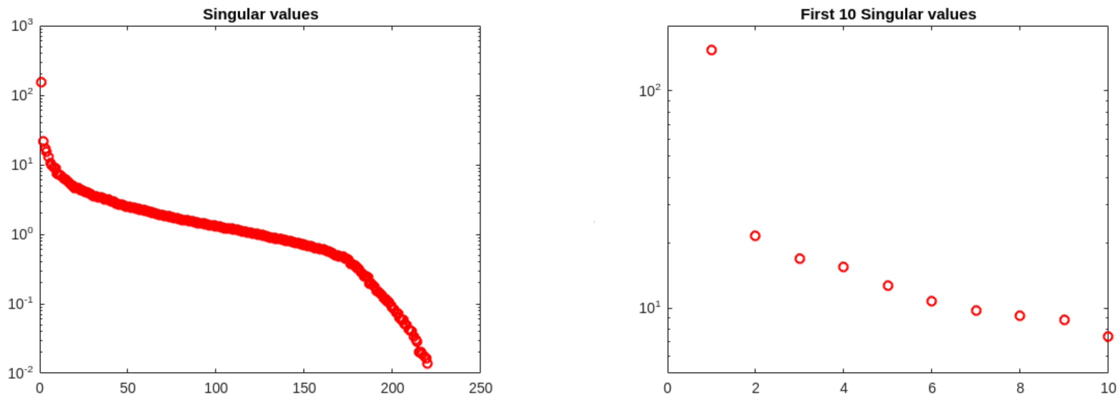
$\langle a, a \rangle = 0$ iff $a = 0$

$\boxed{\therefore M \text{ is positive definite}}$

$\therefore$ for $\langle x, y \rangle = y^T \cdot M \cdot x$ to be an inner product,

$\boxed{M \text{ must be symmetric positive definite}}$

Problem 4



(a) All Singular Values

(b) First 10 Singular Values

Figure 1: Singular Values Plot

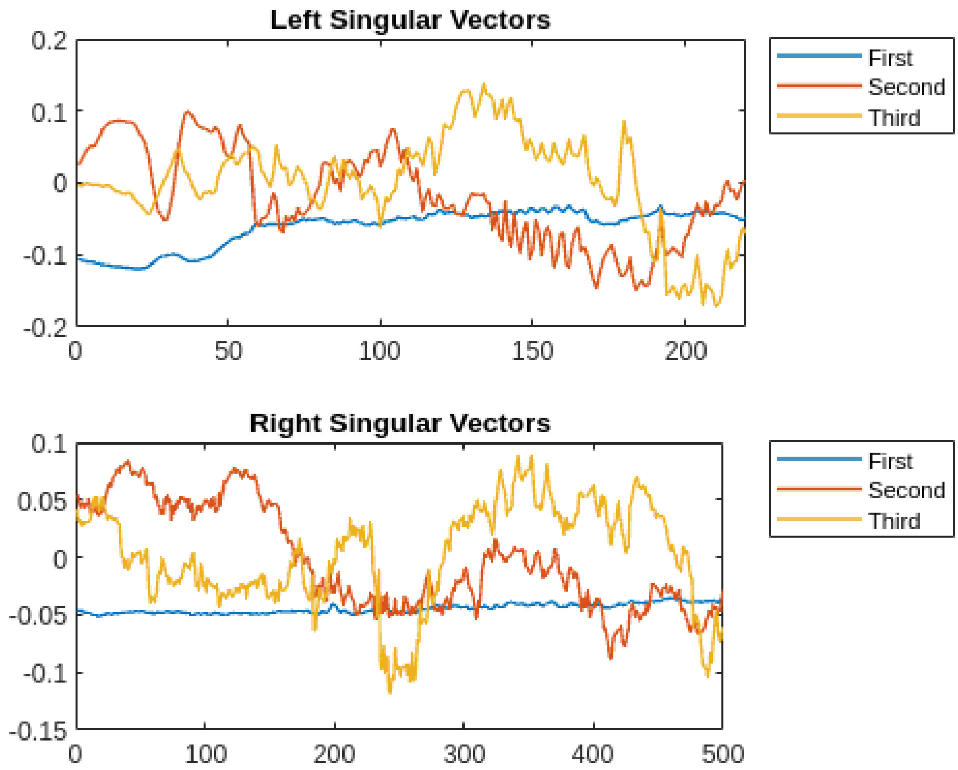


Figure 2: First 3 Singular Vectors

Command Window :

```
Square of Frobenius norm of projected data : 24476.487024
Sum of squares of singular values of projected space : 24476.487024
```

The square of Frobenius norm of the data projected on the first 3 singular vectors is equal to the sum of squares of the singular values of the projection space.

Problem 5

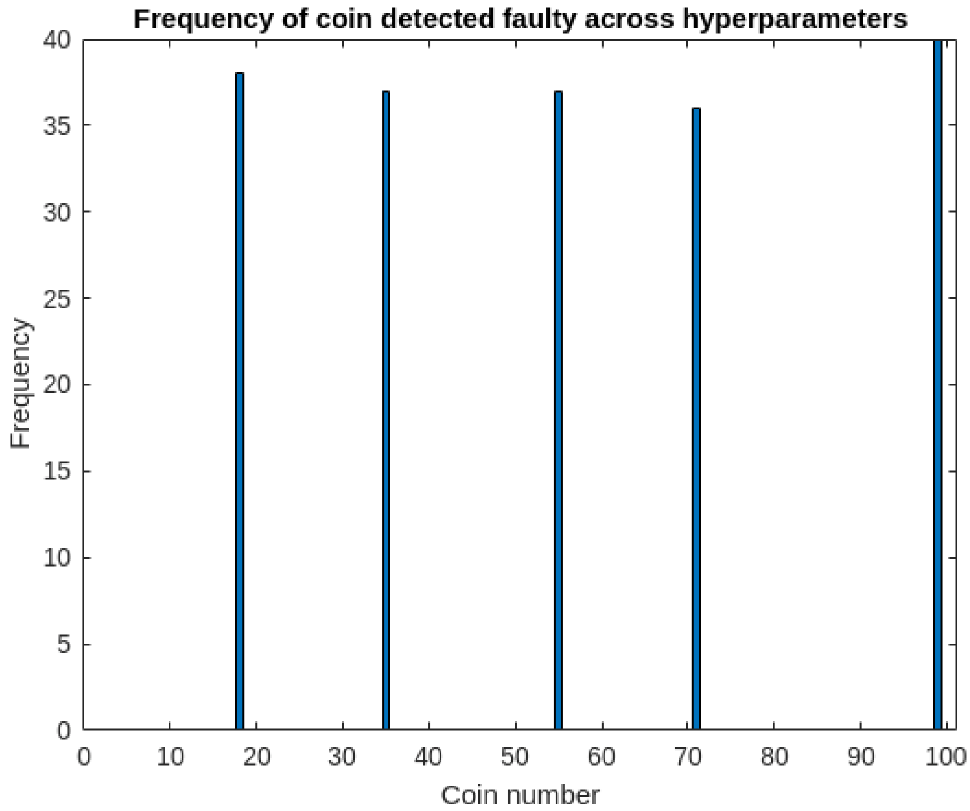


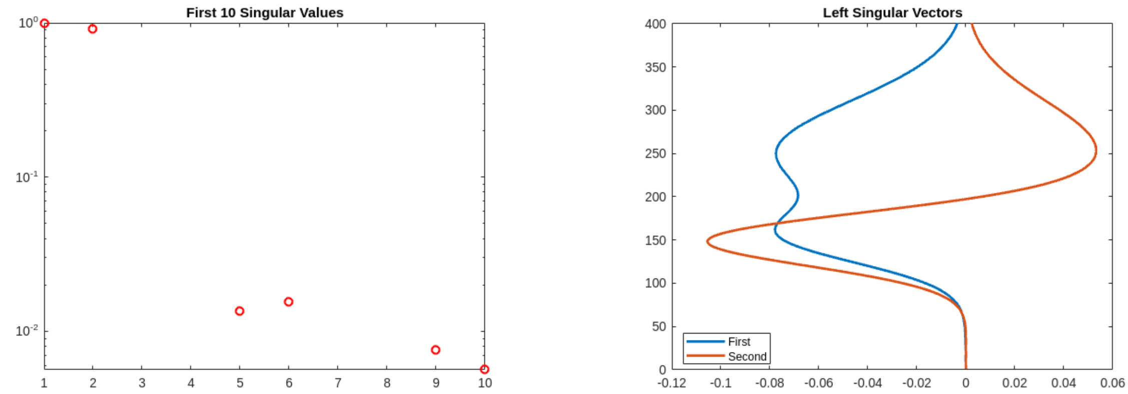
Figure 3: First 3 Singular Vectors

Command Window :

```
Indices of non-zero values, hence the faulty coins :
18
35
55
71
99
```

$\boxed{\text{Thus, 5 coins are faulty, which are coins numbered : 18, 35, 55, 71, 99}}$

Problem 6
Part (a)



(a) First 10 Singular Values

(b) First 2 Left Singular Vectors

Figure 4: Results from Singular Value Decomposition of the Data

The data is constructed by the summation of 2 signals, which are essentially oscillations at a certain frequency with a growth and a decay. Hence, the first two singular values being much larger than the rest suggests that the data can be expressed in a much compressed sense if the basis is changed such that it includes the two modes which are used to construct the data. In making such a change of basis, a large amount of information of the data can be stored using just the first two singular modes, which comes from the fact that the data was originally constructed using a combination of just 2 modes.

Part (b)

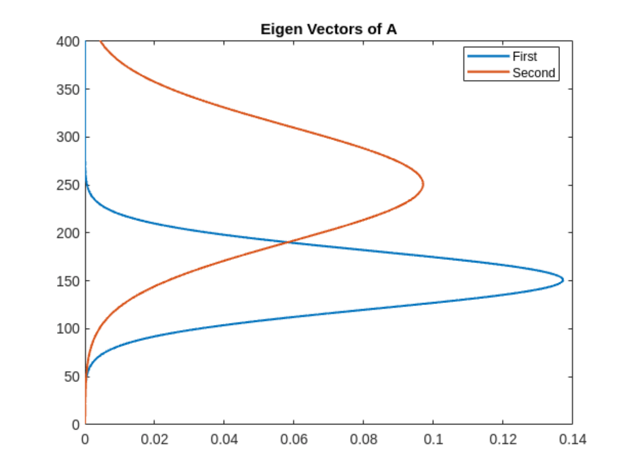


Figure 5: First 2 Eigen Vectors of matrix **A**

Comparison
It can be seen that while each of the first 2 left singular vectors obtained from SVD contain a mixture of frequencies, the eigen vectors of A, consists of only a single frequency each. This highlights the fact that in DMD, each mode corresponds to a single frequency, making it suitable for systems with well-defined oscillations. In contrast, SVD modes can represent a combination of frequencies, capturing more complex and overlapping dynamic behaviors in the system.

Part (c)

The eigen values of $\tilde{\mathbf{A}}$, which are the same as the eigen values of $\mathbf{A}$, can be used to compute the frequencies of the Dynamic Mode Decomposition (DMD) modes. The DMD frequencies i.e continuous-time eigenvalues λ_c, are calculated from the discrete-time eigenvalues λ_d as

$$\lambda_c = log(\lambda_d)/dt \tag{1}$$

These continuous-time eigen values are calculated and displayed.

Command Window :

```
0.0000 + 1.3000i
0.0000 + 4.1000i
```

These correspond to the frequencies, $omega1$ and $omega2$ in the code used to construct the original data.

Part (d)

To determine the best two spatial locations to measure the system in order to reconstruct the data, we use QR pivoting. The QR pivoting is to be done for the tailored basis. The given data is generated to depict a physical system which is essentially a linear combination of oscillations at a certain frequency with a growth and a decay, and hence the DMD modes would be the tailored basis for the data.

Since we want the best **two** spatial locations to measure the system, we select the first two positions of the pivots to place the sensors. These spatial positions are displayed.

Command Window :

```
152 243
```

Thus the spatial positions to place two sensors in order to best capture the data for optimal reconstruction is 152 and 243.

APPENDIX A

MATLAB Codes

Problem 4

```
clear all
close all
clc

X = readmatrix('HW1Q4.csv');

% compute economy SVD
[Ue, Se, Ve] = svd(X,'econ');

% plot singular values
figure(1)
semilogy(diag(Se),'ro','Linewidth',1.5)
title('Singular values')
%xlabel('')
%ylabel('Sine and Cosine Values')
figure(2)
semilogy(diag(Se),'ro','Linewidth',1.5)
xlim([0 10])
ylim([5 200])
title('First 10 Singular values')

%% reduce rank to 3 i.e projection of data in first 3 singular vectors
r = 3;
Ur=Ue(:,1:r);
Sr=Se(1:r,1:r);
Vr=Ve(:,1:r);

% plot first three left and right singular vectors
figure(3)
subplot(2,1,1), plot(real(Ur),'Linewidth',1)
xlim([0 length(Ur)])
title('Left Singular Vectors')
legend({'First','Second','Third'},'Location','bestoutside')

subplot(2,1,2), plot(real(Vr),'Linewidth',1)
xlim([0 length(Vr)])
title('Right Singular Vectors')
legend({'First','Second','Third'},'Location','bestoutside')

% reconstruct projected data to calculate Frobenius norm
X_r = Ur*Sr*Vr';
frobenius_norm = norm(X_r,"fro");
fprintf('Square of Frobenius norm of projected data : %f \n',
    frobenius_norm*frobenius_norm)
sum_singular_values = sum(diag(Sr*Sr));
fprintf('Sum of squares of singular values of projected space : %f \n
    \n', sum_singular_values)
```

Problem 5

```
clear all
close all
clc

b = readmatrix('hw1Q5b.dat');
C = readmatrix('hw1Q5C.dat');

%% non-sparse solution using pseudo inverse
xL2 = pinv(C)*b;

%% sparse solution using LASSO regularization
xL1 = lassoglm(C,b);

%% Method 1 of analyszing LASSO solutions
% Find the indices of non-zero values
nonZeroIndices = find(xL1(:,1));

% Display the non-zero indices
disp('Indices of non-zero values, hence the faulty coins:');
disp(nonZeroIndices);

%% Method 2 of analyszing LASSO solutions
% Convert non-zero values to 1 using logical indexing
xL1(xL1 ~= 0) = 1;

% Calculate the sum of rows
rowSums = sum(xL1, 2);

% Create a histogram
figure(1)
bar(rowSums);
title('Frequency of coin detected faulty across hyperparameters')
xlabel('Coin number')
ylabel('Frequency')
```

Problem 6

```
clear all
close all
clc

dy  = 0.01;
y = (-2:dy:2)'; %spatial coordinate

dt = 0.1;
Nt = 101;
tend = dt*(Nt-1);
t = 0:dt:tend; %time

% define function
amp1 = 1;
y01 = 0.5;
sigmay1  = 0.6;

amp2 = 1.2;
y02 = -0.5;
sigmay2  = 0.3;
omega1 = 1.3;
omega2 = 4.1;

v1 = amp1*exp(-(y-y01).^2/(2*sigmay1^2));
v2 = amp2*exp(-(y-y02).^2/(2*sigmay2^2));
X = v1*exp(1i*omega1*t) + v2*exp(1i*omega2*t);


%% Part (a) : SVD analysis
% compute SVD
[U, S, V] = svd(X);

% plot singular values
semilogy(diag(S)/sum(diag(S)),'o')

% plot first two left singular vectors along the spatial coordinate y
figure(1)
plot(real(U(:,1:2)), 1:length(U(:,1:2)),'Linewidth',2)
ylim([0 length(U)])
title('Left Singular Vectors')
legend({'First','Second'},'Location','best')


%% Part (b) : DMD analysis
%% DMD
X1=X(:,1:end-1);
X2=X(:,2:end);

[U_X1,S_X1,V_X1]=svd(X1,'econ');

% Truncate to rank r
r = 10;  % Adjust as needed

Ur = U_X1(:, 1:r);
Sr = S_X1(1:r, 1:r);
Vr = V_X1(:, 1:r);



```

```
% Compute the approximate DMD matrix A
A_tilda = Ur'*X2*Vr/Sr;

% Compute the eigenvalues and eigenvectors of A
[eig_vec, eig_val] = eig(A_tilda);

% plot eigen values (same for A and A_tilda)
figure(2)
semilogy(real(diag(eig_val)),'ro','Linewidth',1.5)
title(['First ', num2str(r), ' Singular Values'])

% Compute eigen vectors of A
A = Ur * A_tilda * Ur';
[eig_vec_A, eig_val_A] = eig(A);
figure(3)
plot(real(eig_vec_A(:,1:2)), 1:length(eig_vec_A(:,1:2)),'Linewidth'
    ,1.5)
ylim([0 length(eig_vec_A)])
title('Eigen Vectors of A')
legend({'First','Second'},'Location','best')


%% Part (c) : Analysing eigenvalues of A
% Compute the DMD modes and frequencies
DMD_modes = X2 * Vr / Sr * eig_vec;
DMD_frequencies = log(diag(eig_val)) / dt;  % dt is the time step

% display frequencies corresponding to non-zero eigen values
%disp(DMD_frequencies(1:2,:));


%% Part (d) : best two spatial locations using QR pivoting method
% the tailored basis would be DMD modes
% Perform QR decomposition with pivoting
[Q, R, pivot] = qr(DMD_modes', 'vector');

% Q is the orthogonal matrix
% R is the upper triangular matrix,
% P is a permutation vector that indicates the column permutation
r = 2;          % we want best r spatial locations for sensors
sensors = pivot(1:r);
```