"Optimization Techniques"

It is a theory and subject related to finding optimal points (referring to maxima or minima).

It allow as to explore various methods or algorithms that can be applied to mathematical models to get the solution.

* Mathematical models allow us to relate the real-life problem with mathematical variables, tools and solution strategies.

- It allows us to choose from alternatives available in situation.
- The best mathematical model gives us the best (optimal) solution.

Optimization techniques in SCM (Supply Chain Management)

→ From the Practicle situation we have to develop mathematical model

& then we want to see the how the model does look like - linear, quadratic, non-linear, etc and we try to search solution strategies.

Components -
→ Decision variable
1) General model
2) Objective function
3) Constraints & Optimize
4) Restrictions

Practicle situation
→ alternatives
→ Constraints
→ linear
→ non linear

Formulation of Linear Programming Problem

* Linear Programming - mathematical concept that is used to find the optimal solution of linear function.

Objective function - max. / min.
Constraints
Decision variable (DV)
Restrictions on DV

} we have to find.

* General LPP may be stated as:
- degree of DV is one i.e $x_1^{①}, x_2^{①}$, etc.
- Max. / Min $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
- subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$

  $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$

  $\vdots$
- Non-negative restrictions $x_1, x_2 \ldots x_n \geq 0$

steps to formulating LLP -(4) steps

1) Identify decision variable & assign symbols (eg. $x, y, z, \ldots$ or $x_1, x_2 \ldots$)
DV are those quantities whose values we wish to determine.

2) Identify the set of constraints and express them in terms of inequalities involving the decision variables.

3) Identify the objective function and express in terms of DV.

4) Add the non-negative conditions.

example of LPP -
1) Max. / Min (Profit, cost, time)
2) Trim losses problem
3) Transportation problem (cost minimization)
4) Scheduling problems (manpower or machine)

Methods for Solving linear Programming Problem

→ Graphical Solution
→ Algebraic method / finding Basic Solution
→ Simplex method & its variants

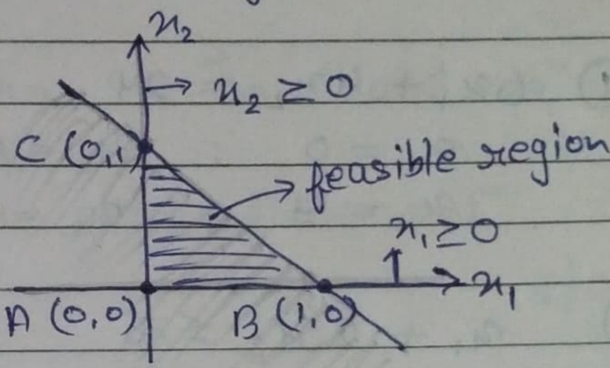
* <u>Graphical solution</u>
LPP which involve only two decision variable (x_1 & x_2) can be solved by it.

we may get
1 unique solution
2 Two solution / Infinite soln / Alternate solution
3 Unbounded solution
↳ LPP. that is, objective function value "z" is not bounded and region is unbounded)
4 Bounded LPP, with unbounded region
5 Infeasible solution (no solution)
↳ There is nothing common in constraint ~~and~~ or LPP.

Q. Maximize $\quad z = x_1 + x_2$
subject to $\quad x_1 + x_2 \le 1 \; ; \; x_1, x_2 \ge 0$
↳ define region

$\cancel{x_1}^{0} + x_2 = 1 \qquad x_2 = 1$
$x_1 + \cancel{x_2}^{0} = 1 \qquad x_1 = 1$

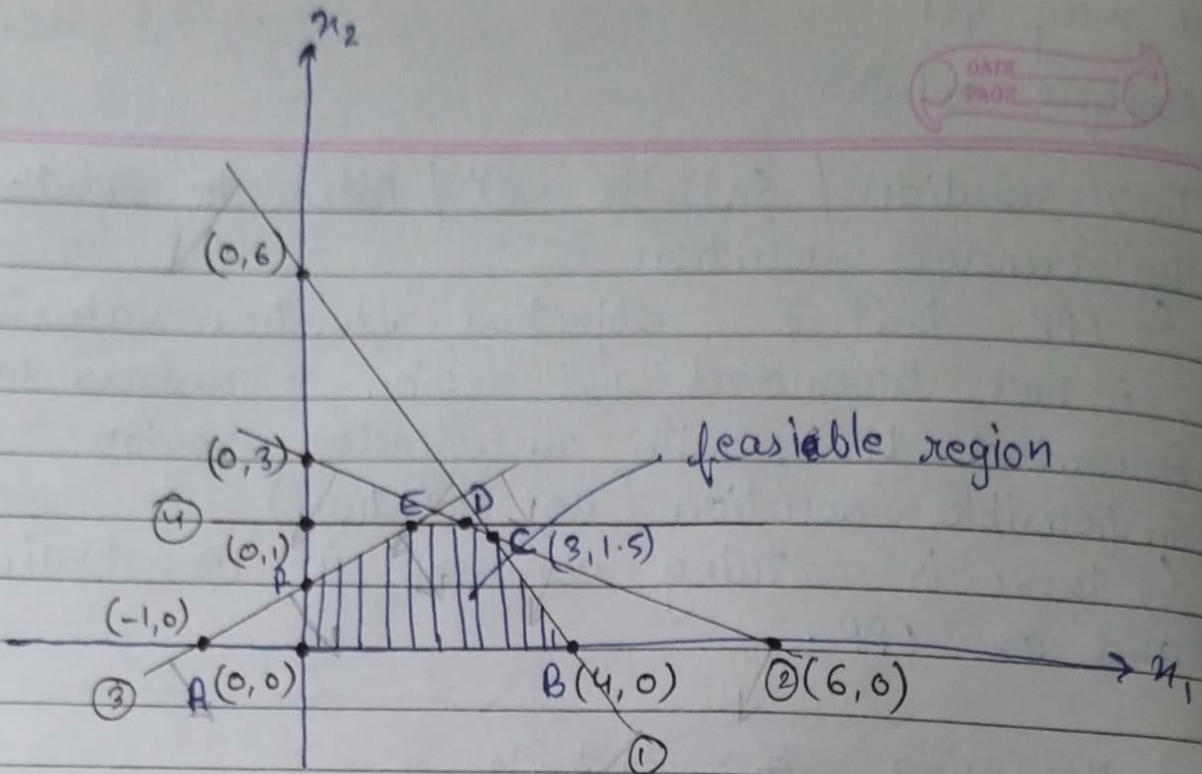
z at $A = 0 + 0 = 0$
z at $B = 1 + 0 = 1$ } max
z at $C = 0 + 1 = 1$



so there are 2 points which gives us the max. value i.e C & B. And also the line segment which join this points gives ~~the~~ optimal solution

$z_{max} = 1$ at (x_1, x_2)
↳ $(0,1) \; . \; (1,0) \; (1/2, 1/2)$

Q. Maximize $\quad z = 5x_1 + 4x_2$
subject to $\quad 6x_1 + 4x_2 \le 24$
$x_1 + 2x_2 \le 6$
$-x_1 + x_2 \le 1$
$x_2 \le 2$
$x_1, x_2 \ge 0$



① & ② → (3, 1.5)
② & ④ → (2, 2)
③ & ④ → (1, 2)

① $6x_1 + 4x_2 \overset{\le}{=} 24$

$x_1 = 0$; $x_2 = 6$ (0, 6)

$x_1 = 4$ $x_2 = 0$ (4, 0)

② $x_1 + 2x_2 \le 6$

$x_1 = 0$ $x_2 = 3$ (0, 3)

$x_1 = 6$ $x_2 = 0$ (6, 0)

③ $-x_1 + x_2 \le 1$ $0 + 0 \le 1$ ✓

$-(-2) + 0 \le 1$ ✗

$x_1 = 0$; $x_2 = 1$ (0, 1)

$x = -1$ $x_2 = 0$ (-1, 0)

now check z at all corner points.

A(0,0), B(4,0), C(3,1.5), D(2,2)

E(1,2), F(0,1)

z at A – 0, B – 20, C → 21 (max.), D → 18

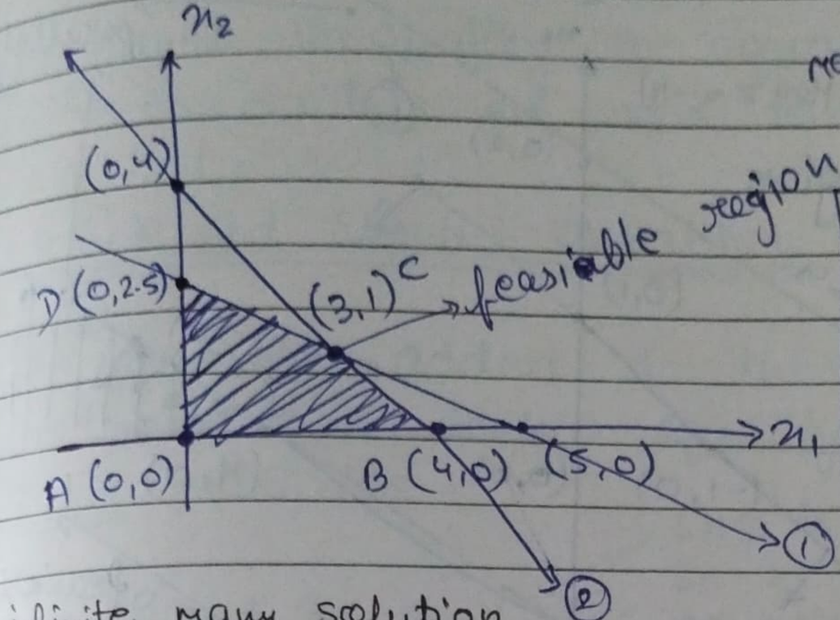
E → 13 F → 4 (unique solⁿ)

Q. Maximize $z = 2x_1 + 4x_2$

Subject to $x_1 + 2x_2 \le 5$

$x_1 + x_2 \le 4$

$x_1, x_2 \ge 0$



A (0,0) $z \to 0$

B (4,0) $z \to 8$

mom $\left\{ \begin{array}{l} C(3,1) \ z \to 10 \\ D(0,2.5) \ z \to 10 \end{array} \right.$

* optimal value

↳ C & D

i.e 10.

* optimal sol^n of $z = 10$

(x_1, x_2)

(3,1) or (0,2.5)

infinite many solution at line segment

NOTE → Objective function & Constraint eq^n are parallel or not.
if parallel then we get infinite sol^n on line segment.

6|0|...

↑

$z = 6x_1 + x_2 = C$

Q. Maximize $z = 6x_1 + x_2$

Subject to $2x_1 + x_2 \ge 3$

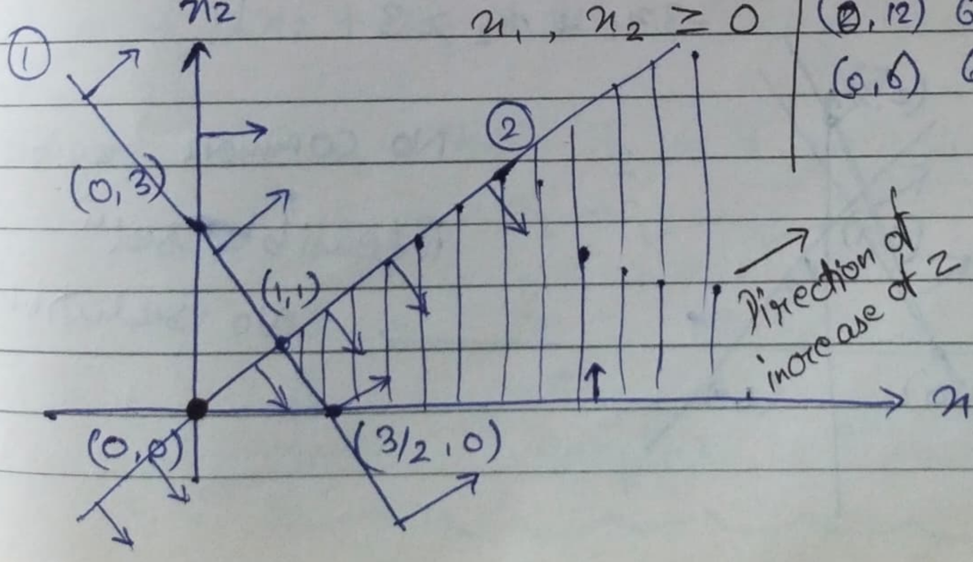
$x_1 - x_2 \ge 0$

$x_1, x_2 \ge 0$

(0,6) $6x_1 + x_2 = 6$

(0,12) $6x_1 + x_2 = 12$

(0,0) $6x_1 + x_2 = 0$



∴ unbounded region

∴ unbounded sol^n

$z \to \infty$

$(x_1, x_2) \to \infty$

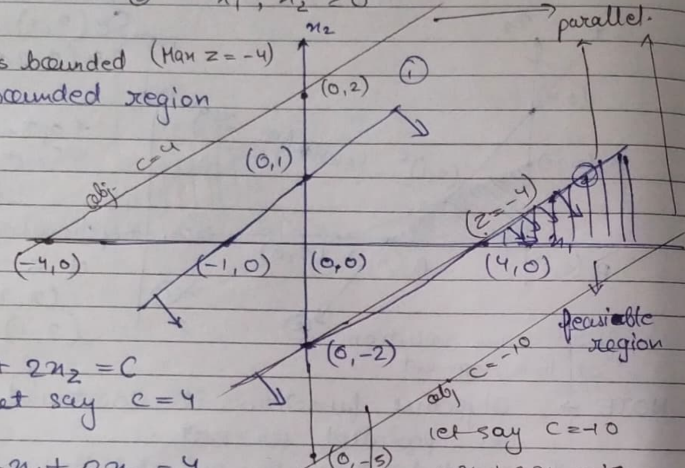
Q. $Z = 2x_2 - x_1$ (Maximize)

Subject to ① $x_1 - x_2 \geq -1$

② $-0.5x_1 + x_2 \leq -2$

NRC. → ③ $x_1, x_2 \geq 0$

- LPP is bounded (Max $Z = -4$)
- unbounded region



parallel.

feasible region

$-x_1 + 2x_2 = C$

let say $C = 4$

$-x_1 + 2x_2 = 4$

$x_1 = 0$; $x_2 = 2$

$x_1 = -4$; $x_2 = 0$

let say $C = -10$

$-x + 2x_2 = -10$

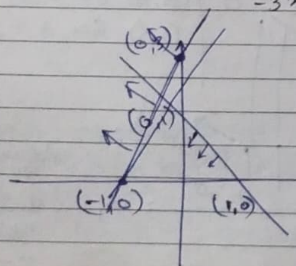
$x = 0$; $x_2 = -5$

$x_1 = +10$; $x_2 = 0$

Q. Maximize $Z = x_1 + x_2$

Subject to $x_1 + x_2 \leq 1$ | $x_1, x_2 \geq 0$

$-3x_1 + x_2 \geq 3$



No common region

infeasible solⁿ

→ No solution

* Algebraic Method / Finding Basic Solution.

\# Solve System of equation, but in LPP we have system of inequalities. So steps

1. Convert inequality to equality.
   ↳ to do this we need to know

* Slack and Surplus Variables

A variable added to the '≤' type constraints to achieve equality is called Slack variable.

$x_1 + 3x_2 + x_3 \leq 4$

LHS is less than 4 so to convert it in equality we need to add something in LHS which should be non-negative ~~or greater~~ ~~than~~ let say S_1 is added.

$x_1 + 3x_2 + x_3 + S_1 = 4$

$\therefore S_1 \geq 0$

A variable subtracted from the '≥' type constraints to achieve equality is called surplus variable.

$x_1 + 3x_2 + x_3 \geq 4$

$x_1 + 3x_2 + x_3 - S_2 = 4$ $\therefore S_2 \geq 0$

≤	≥	=
Slack	Surplus	No change

Basic Solution (BS)

we can write any system of equation in matrix form. So

Consider a system "$AX = b$" $(A)_{m\times n}$ (Rank $= m$) of m equations with n variables $(n > m)$ and Put $(n-m)$ variables to zero and the resulting system $BX_B = b$ solution is called Basic solution.

example

Let we have → ① Maximize $Z = x_1 + x_2$

Subject to $x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

OR

$Z = x_1 + x_2 + 0\cdot S_1$

$x_1 + x_2 + S_1 = 1$

$x_1 \ge 0,\ x_2 \ge 0,\ S_1 \ge 0$

$x_1 + x_2 + S_1 = 1$

$$\begin{bmatrix} 1, & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ S_1 \end{bmatrix} = 1$$

$AX = b$

$(A)_{1\times 3}\, X = b$

$m = 1$ equations, $n = 3$ variables

$\therefore\ n - m = 2$

			Basic solution
case 1.	$x_1 = x_2 = 0$;	$S_1 = 1$	$(0\ 0\ 1)$
2.	$x_1 = S_1 = 0$;	$x_2 = 1$	$(0\ 1\ 0)$
3.	$x_2 = S_1 = 0$;	$x_1 = 1$	$(1\ 0\ 0)$

② Maximize $Z = 2x_1 + 3x_2$

subject to $2x_1 + x_2 \le 4$

$x_1 + 2x_2 \le 5$ & $x_1, x_2 \ge 0$

$Z = 2x_1 + 3x_2 + 0\cdot S_1$

$2x_1 + x_2 + S_1 = 4$

$x_1 + 2x_2 + S_2 = 5$

$x_1 \ge 0$, $x_2 \ge 0$, $S_1 \ge 0$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

total solⁿ ^nC_m

$(A)_{2\times 4}\, X = b$ → $B(X_B) = b$

m, n

Non-Basic variables — Basic variables

$4 - 2 = 2$

Case 1. $x_1 = x_2 = 0$; $S_1 = 4$ $S_2 = 5$

$(0\ \ 0\ \ 4\ \ 5)$

$2x_1 + x_2 + S_1 = 4$

$x_1 + 2x_2 + S_2 = 5$

$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Basic Matrix

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

all possible solⁿ.

cases	Basic variables	Non-Basic V	Solution (X_B)
1.	S_1, S_2	$x_1 = x_2 = 0$	$[0, 0, 4, 5]$
$S_1 = (-ve)$ × 2.	x_1, S_1	$x_2 = S_2 = 0$	$[5, 0, -6, 0]$
3.	x_1, S_2	$x_2 = 0 = S_1$	$[2, 0, 0, 3]$

4.	$x_2\ S_1$	$x_1 = S_2 = 0$	$[0, 5/2, 3/2, 0]$
$S_2 =$ (-ve) ~~5.~~	x_2, S_2	$x_1 = S_1 = 0$	$[0, 4, 0, -3]$
6.	x_1, x_2	$S_1 = S_2 = 0$	$[1, 2, 0, 0]$

all remain solⁿ are Basic solution / feasible solⁿ for objective function.
or BFS.

→ System of equation in Matrices form

$AX = b$ $\quad (A)_{m \times n}$

we reduce A by put (n−m) variables to 0. we get

$B X_B = b$ $\quad (B)_{m \times m}$

i.e $|B| \neq 0$

And if Basic solution satisfy the non-negative restriction then known as Basic feasible Solution else infeasible basic soution.

⟹ X → all decision variables (original & slack / surplus)

$AX = b$

↳ (X_B, X_N) — X_N → Non-basic variable; X_B → Basic variable

$B X_B = b$

↳ (X_B)

$B X_B = b$

$(B)_{m \times m} (X_B)_{m \times 1} = (b)_{m \times 1}$

$\therefore \boxed{X_B = B^{-1} b}$ & $\boxed{X_N = 0}$

We can use this equation when we have ~~more~~ ~~equation~~ large system of equations.

$\Rightarrow$ BFS ≤ BS ≤ ^nC_m

→ Degenerate and non-degenerate basic feasible solutions: A BFS with at least one basic variable assuming the value zero is called a degenerate BFS; else called non-degenerate BFS.

eg. $X = (\underbrace{x_1 \quad x_2}_{BV} \quad \underbrace{S_1 \quad S_2}_{NBV})$

$= (2, 3, 0, 0)$ ↳ (+ve) — BFS

$= (-2, 3, 0, 0)$ ↳ one (-ve) — infeasible BS

$= (0, 5, 0, 0)$ ↳ one zero. — degenerate BFS

→ Optimal BFS : BFS that optimize the objective function value.

Connection of Graphical & Algebraic method

(Connection of corner point & BFS)

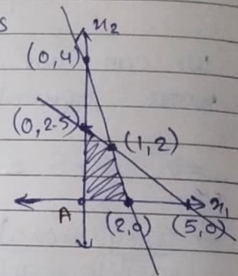
$\boxed{\text{Corner} \leq \text{BFS}}$ → equal when degenerate BFS else $\leq$.

Q. Maximize $Z = 2x_1 + 3x_2$

subject to $2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 5$

$x_1, x_2 \geq 0$



$\therefore \quad Z = 2x_1 + 3x_2 + S_1 \cdot 0 + S_2 \cdot 0$

$2x_1 + x_2 + S_1 = 4$ | $x_1 \geq 0$

$x_1 + 2x_2 + S_2 = 5$ | $x_2 \geq 0$

| $S_1 \geq 0$

| $S_2 \geq 0$

Non. BV $=0$	BV	BS	Associated corners	feasible	objective value
(x_1, x_2)	$(S_1 S_2)$	$(4,5)$	A	yes	0
× (x_1, S_1)	$(x_2 S_2)$	$(4,-3)$	F	NO	-
(x_1, S_2)	$(x_2 S_1)$	$(2.5, 1.5)$	B	yes	7.5
(x_2, S_1)	$(x_1 S_2)$	$(2,3)$	D	yes	4
× (x_2, S_2)	$(x_1 S_1)$	$(5,-6)$	E	NO	-
$(S_1 S_2)$	$(x_1 x_2)$	$(1,2)$	C	yes	8

$Z = 2x_1 + 3x_2 + 0.S_1 + 0.S_2$

In 1st case

$Z = 0 + 0 + 0 + 0 = 0$

3rd $\quad Z = 3(2.5) = 7.5$ ....

★ $Z = 8$ (Max.) optimal solution

$(x_1 = 1 \quad x_2 = 2)$

4 BFS	4 corners

$\boxed{\text{Vertices or corners} \leq \text{BFS} \leq \text{BS} \leq {}^nC_m}$

Q. Maximize $Z = 2x_1 + 3x_2 + 5x_3$

subject to $6x_1 + 7x_2 - 9x_3 \geq 4$

$x_1 + x_2 + 4x_3 = 10$

$x_1 \geq 0, \; x_3 \geq 0, \; x_2$ - unrestricted in sign

means we do not know $x_2 \geq 0$; $x_2 \leq 0$

$$\left.\begin{aligned} x_2 &= 7 - 0 \\ &= x_2^+ - x_2^- \\ x_2 &= 0 - 7 \\ &= x_2^+ - x_2^- \end{aligned}\right\} \; x_2 \text{ as two non negative decision variable.}$$

$\therefore \quad 6x_1 + 7x_2^+ - 7x_2^- - 9x_3 - S_1 = 4$

$x_1 + x_2^+ - x_2^- + 4x_3 = 10$

$$\begin{bmatrix} 6 & 7 & -7 & -9 & -1 \\ 1 & 1 & -1 & +4 & 0 \end{bmatrix}_{2\times 5} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ S_1 \end{bmatrix}_{1\times 5} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$^5C_2 = {}^nC_m \Rightarrow 10$ cases.

B.V. $= x_2^+, x_2^-$ — these two columns are Linearly dependent.

$$B = \begin{vmatrix} 7 & -7 \\ 1 & -1 \end{vmatrix} = 0 \rightarrow$$ this will not give feasible solution so not considered.

$AX = b \longrightarrow B(X_B) = b$

Standard form of LPP

An LPP is said to be standard form, if the following conditions are satisfied:

1) The objective function must be either maximization or minimization

2) All the constraints should ~~be~~ hold with an equality sign.

3) RHS of all constraints is non-negative.

4) All the variables involved in the problem are non-negative.

$$\boxed{\begin{array}{l} \text{Opt } Z = CX \quad (c_1x_1 + c_2x_2 + \cdots + c_nx_n) \\ AX \underbrace{\le, \ge, =}_{\text{slack/surplus}} b \\ X \ge 0 \end{array}}$$

$AX = b$

$b \ge 0, \quad X \ge 0$

Q Write the LPP in standard form

Max $Z = x_1 + x_2 - 2x_3$

Subject to
$$x_1 + x_2 + x_3 \le 15$$
$$2x_1 - x_2 + x_3 \le -10$$
$$x_1 + 2x_3 = 10$$
$$x_1, x_2 \ge 0; \ x_2 - \text{unristricted in sign.}$$

→ Maximize $Z = x_1 + x_2 - 2x_3$ (fine)

→ Decision Variables $x_1 \ge 0, \ x_2 \ge 0$ but x_3 unrestricted in sign.

$\therefore \ x_3 = x_3' - x_3'' \qquad x_3' \ge 0, \ x_3'' \ge 0$

$\therefore$ Max $Z = x_1 + x_2 - 2(x_3' - x_3'')$

subject to
$$x_1 + x_2 + x_3' - x_3'' \le 15$$
$$2x_1 - x_2 + x_3' + x_3' \le -10$$
$$x_1 + 2x_3' + 2x_3'' = 10$$

→ Constraints into equality & RHS should +ve.

Max $Z = x_1 + x_2 - 2x_3' + 2x_3'' + s_1 \cdot 0 + s_2 \cdot 0$

Subject to

* multiply 2nd constraint by (-ve) sign.

$$x_1 + x_2 + x_3' - x_3'' \le 15$$
$$-2x_1 + x_2 - x_3' - x_3'' \ge 10$$
$$x_1 + 2x_3' + 2x_3'' = 10$$

* add slack & surplus variables

$$x_1 + x_2 + x_3' - x_3'' + S_1 = 15$$
$$-2x_1 + x_2 - x_3' + x_3'' - S_2 = 10$$
$$x_1 + 2x_3' - 2x_3'' = 10$$

$$x_1, x_2, x_3', x_3'', S_1, S_2, \geq 0$$

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let $x_2 \geq 2 \longrightarrow x_2 - 2 \geq 0$

Conversion

$$x_2' = x_2 - 2 \geq 0$$
$$x_2' \geq 0$$
$$x_2' = x_2 - 2$$
$$\boxed{x_2 = x_2' + 2}$$

Q. Max $z = x_1 + x_2$

s.t. $|4x_1 + x_2| \leq 6$

$7x_1 - x_2 \geq 5$

$x_1 \geq 0 \quad x_2 \geq 0$

OR

Max. $Z = x_1 + x_2$

$$-6 \leq 4x_1 + x_2 \leq 6$$
$$7x_1 - x_2 \geq 5$$

OR

Max $z = x_1 + x_2$

$$4x_1 + x_2 \geq -6$$
$$4x_1 + x_2 \leq 6$$
$$7x_1 - x_2 \geq 5$$

OR

$$-4x_1 - x_2 \leq 6$$
$$4x_1 + x_2 \leq 6$$
$$7x_1 - x_2 \geq 5$$

(remaining process is same)

Q. Max $z = x_1 + 2x_2 - x_3$

s.t. $x_1 + x_2 - x_3 \leq 5$

$$-x_1 + 2x_2 + 3x_3 \geq -4$$
$$2x_1 + 3x_2 - 4x_3 \geq 3$$
$$x_1 + x_2 + x_3 = 2$$

$x_1 \geq 0 \quad x_2 \geq b \; ; \quad x_3$ unrestricted in sign

$\hookrightarrow x_3' \geq 0 \quad x_3'' \geq 0$

$\hookrightarrow x_2 \geq b$

$$x_2 - b \geq 0$$

$\boxed{x_2 = x_2' + b}$; $x_2' \geq 0$

$$x_2' = x_2 - b$$

$\therefore$ Max $z = x_1 + 2(x_2' + b) - x_3' + x_3''$

remaining process is same.

Mathematics Behind Simplex Method

Matrices & Determinants

A matrix is a rectangular array of mn numbers, real or complex, arranged in m-rows & n-cols.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{12} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

eg. $A = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \end{bmatrix}_{2\times 3}$

$\Rightarrow m = n \rightarrow$ square matrix

$\Rightarrow 1\times n \rightarrow$ row matrix

$\Rightarrow m\times 1 \rightarrow$ col matrix

⇒ a_{ij} $\forall i,j$ is zero → null matrix

⇒ A square matrix ; principle diagonal = unity
all other elements are zero → Identity matrix

$[1]_{1\times1}$ $\begin{bmatrix}1&0\\0&1\end{bmatrix}_{2\times2}$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}_{4\times4}$ ⇒ $A=(a_{ij})_{m\times n}$

$B=(b_{ij})_{m\times n}$

$A=B$ iff $a_{ij}=b_{ij}$ $\forall i,j$

* Transpose of matrix

$A=\begin{bmatrix}2&3&5\\1&0&2\end{bmatrix} \longrightarrow A^T=\begin{bmatrix}2&1\\3&0\\5&2\end{bmatrix}$

* Operations → Addition

$A=(a_{ij})_{m\times n}$

$B=(b_{ij})_{m\times n}$

$A+B=(c_{ij})_{m\times n}$

$c_{ij}=a_{ij}+b_{ij}$

Multiplication

$A=(a_{ij})_{m\times n}$

$B=(b_{ij})_{m\times p}$

* col of A = row of B

$AB=(C_{ik})_{m\times p}$

eg. of multiplication

$A=\begin{bmatrix}4&5&6\\1&0&2\end{bmatrix}_{2\times3}$ $B=\begin{bmatrix}1&0\\3&2\\0&1\end{bmatrix}_{3\times2}$

$AB=\begin{bmatrix}4+15+0&0+10+6\\1+0+0&0+0+2\end{bmatrix}_{2\times2}=\begin{bmatrix}19&16\\1&2\end{bmatrix}$

Determinant : let A be a square matrix, then determinant of A / det A / |A| is defined as

$$|A| = \det A = \sum_{j=1}^{n}\left((-1)^{i+j}\det M_{ij}\right)a_{ij}$$

$A=a_{ij}$

M_{ij} = minor of a_{ij}

eg. $A=\begin{bmatrix}4&0&2\\1&0&3\\2&4&6\end{bmatrix}_{3\times3}$

$=4\left((-1)^2\cdot(-12)\right) - 0\left(\quad 1 \quad\right) + 2\left((-1)^4\cdot 4\right)$

$=\underline{\underline{56}}$

Inverse - A is a square matrix, if there exist a matrix B such that
$AB=BA=I$
B is inverse of A . ∴ $\boxed{A^{-1}=B}$
∴ $|A|\neq0$ → A is non singular
$|A|=0$ → A is singular matrix

Elementary transformation of a matrix
An elementary row or column transformation of a matrix is one of the following types

Type 1 : Interchange of any two rows/ columns

Type 2 : Multiply elements of a row / column by non - zero scalar.

Type 3 : Adding to any row / column, a scalar multiple of another row/col.

$$I \cdot A = A$$

$$\boxed{B \cdot A = I}$$

↳ B act as A^{-1}

ex. find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row operations.

* $A = I \cdot A$

↓

$I = B \cdot A$ → B act as A^{-1}

$$[A : I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Interchange $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$3 - 3\times1 = 0$

$1 - 3\times2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$ & $R_3 \to R_3 - 5R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$R_1 \to R_1 + R_3$ & $R_2 \to R_2 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$[I : B]$ $\boxed{B \text{ is } A^{-1}}$

\# Rank of Matrix - denoted by rank(A) or $\rho(A)$

Rank of matrix is the order of atleast "higher order non-vanishing minor".

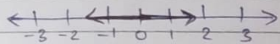
→ extract square matrix from given matrix $(A)_{2\times3}$ if possible. if determinant of any one is not equal to zero. so the rank of that matrix is two.

Linear Equation

Vectors : A vector in R^n is an ordered tuple of n real numbers.

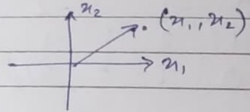
$a = (a_1, a_2, \ldots a_n)$

R^1 = real line



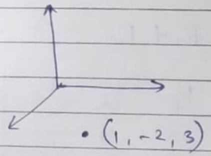
$a(a_1)$

R^2 = plane



$a = (a_1, a_2)$

R^3 = space



$a = (a_1, a_2, a_3)$

R^n n dimensional space (Euclidean space)

$a \in R^n \qquad a = (a_1, a_2, \ldots a_n) \quad a_i \in R$

Properties of vectors

- Equal vector ; $a = (a_1, a_2, \ldots a_n)$; $b = (b_1, b_2, \ldots b_n)$
  $a = b$ iff $a_i = b_i$
- Sum / diff. of vector
  $a \pm b = (a_1 \pm b_1, a_2 \pm b_2, \ldots a_n \pm b_n)$
- multiplication of vector by a scalar
  $\lambda a = \lambda(a_1, a_2, \ldots a_n)$
  $= (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$
  and $\lambda(a+b) = \lambda a + \lambda b$
  $\lambda(\mu a) = \lambda \mu a$

- Row and Column Vector : An n-vector $a = (a_1, a_2, a_3, \ldots, a_n)$ is called row vector. And $a^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is called column vector.

- Linear Combination of vector

$a_1 = (2, 3)$, $a_2 = (1, 5)$, $a_3 = (4, 0)$
$\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 1$

$\underline{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3} = (2(2,3) + 4(1,5) + 1(4,0))$

$= (4, 6) + (4, 20) + (4, 0)$
$= (12, 26) \in \mathbb{R}^2$
linear combination

Linear dependent / linear Independent Vectors

A set of vectors $a_1\ a_2 \ldots a_k \in \mathbb{R}^n$, for $\lambda_1, \lambda_2, \ldots \lambda_k$ scalars are linear combination - $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_k a_k$ equal to zero.

→ holds for some $\lambda_i \neq 0$ (or atleast one i) - LD
→ holds only if $\lambda_i = 0$ for all i - LI

eg. ① $a_1 = (1, 2)$ $\quad a_2 = (2, 4)$ → LD
$\lambda_1 = 2 \quad \lambda_2 = -1$

LC $\quad \lambda_1 a_1 + \lambda_2 a_2 = 0$
$2(1, 2) + (-1)(2, 4) = 0$
$(0, 0)$

② $a_1 = (1,0,0)$ $a_2 = (0,1,0)$

$a_3 = (0,0,1) \leftarrow (LI)$

$\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

$0(1,0,0) + 0(0,1,0) + 0(0,0,1) = 0$

$(0,0) \rightarrow$

$$\left.\begin{array}{ll} \mathbb{R}^1 & e_1 = (1) \\ \mathbb{R}^2 & e_1 = (1,0) \\ & e_2 = (0,1) \\ \mathbb{R}^3 & e_1 = (1,0,0) \\ & e_2 = (0,1,0) \\ & e_3 = (0,0,1) \end{array}\right\}$$ Linearly independent (LI) standard vectors.

Basic set - A L.I. set of vectors $a_1, a_2, \dots a_k$ of $\mathbb{R}^n$, having property that any vector of $\mathbb{R}^n$ can be expressed as linear combination of these vector, is called Basic set of $\mathbb{R}^n$.

eg. ① $a_1 = (0,1)$ $a_2 = (1,0)$

$$\begin{aligned}(4,2) &= \lambda_1 a_1 + \lambda_2 a_2 \\ &= 4(0,1) + 2(1,0) \\ &= (0,4) + (2,0) \\ &= (4,2)\end{aligned}$$

② $a_1 = (1,2)$ $a_2 = (5,6)$

check vectors are L.I.

ⓐ $\lambda_1(1,2) + \lambda_2(5,6) = (0,0)$

$(\lambda_1, 2\lambda_1) + (5\lambda_2, 6\lambda_2) = (0,0)$

by properties of vectors

$\lambda_1 + 5\lambda_1 = 0$	$2\lambda_2 + 6\lambda_2 = 0$
$6\lambda_1 = 0$	$8\lambda_2 = 0$
$\lambda_1 = 0$	$\lambda_2 = 0$

$\therefore$ given vectors are L.I.

ⓑ $\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} = 6 - 10 = -4$ $\hookrightarrow$ L.I.

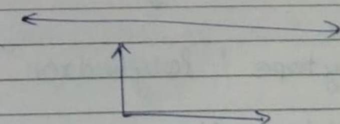
$\boxed{|A| \neq 0 \rightarrow \text{L.I.}}$

$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 = -4$

Hyperplanes -

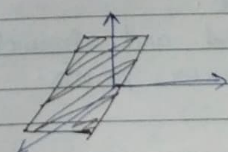
R^1 $ax = b$

R^2 $ax + by = c$



R^2 $ax + by + cz = d$

or $a_1x_1 + a_2x_2 + a_3x_3 = a$



(n-dimensional plane)

hyperplane

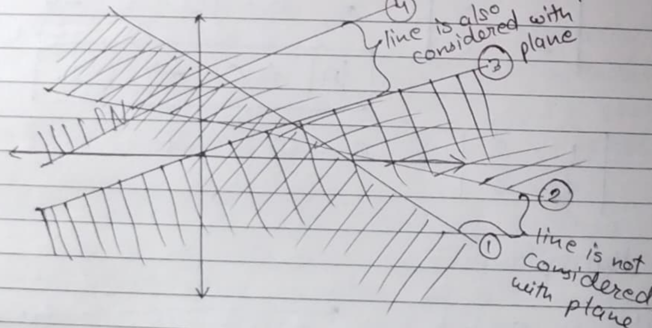
R^n $(a_1x_1 + a_2x_2 + \dots + a_nx_n) = d$

Note : In LPP, objective function is
$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$
same as hyperplane.

close and Open Half Spaces

In $\mathbb{R}^n$
① $a_1x_1 + \dots + a_nx_n < d$
② $a_1x_1 + \dots + a_nx_n > d$
} open half space

③ $a_1x_1 + \dots + a_nx_n \le d$
④ $a_1x_1 + \dots + a_nx_n \ge d$
} closed half space

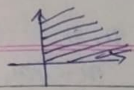


Polytope / Polyhedron

- The intersection of a finite no. of closed half spaces ; is called a polyhedron.
- If polyhedron is bounded then it is called a polytope.

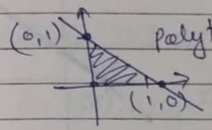
Polytope is closed and bounded having finite no. of corner points.

$S = \{(x_1, x_2) ; x_1 \ge 0 \text{ \& } x_2 \ge 0\}$



~~hyp~~ polyhedron

$S = \{(x_1, x_2) ; x_1 + x_2 \le 1 ; x_1 \ge 0 ; x_2 \ge 0$



polytope

eg.

Min $Z = 2x_1 + 5x_2 - x_3$ → hyperplane
s.t. $x_1 + x_2 + x_3 \le 1$
$2x_1 + x_2 \ge 5$
} Half ~~s~~ closed space or open
$x_1 \ge 0 ; x_2 \ge 0$

$\therefore$ $x_1 + x_2 + x_3 + S_1 = 1$
$2x_1 + x_2 - S_2 = 5$

find intersection & find polyhedron / polytope.

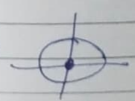
Convex Set

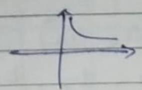
Set : A set is a collection of well defined objects called elements.

→ tabular form
$S = \{1, 2, 3, \dots\}$

→ set builder form
$S = \{x : x \in N\}$

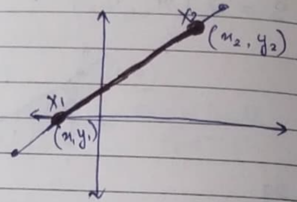
eg. ① $S = \{(x_1, x_2) ; x_1^2 + x_2^2 = r^2\}$
② $S = \{(x_1, x_2) ; x_1 x_2 \le 1 ; x_1, x_2 \ge 0\}$

①


②


Convex set - A set S is said to be convex if for any two points x_1 & $x_2 \in S$, their line joining $x = \lambda x_1 + (1-\lambda) x_2 \in S$, $0 \le \lambda \le 1$.

* line joining two points



$S = \mathbb{R}^2$; let $x_1, x_2 \in S = \mathbb{R}^2$

line joining x_1 & x_2 is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = \lambda \text{ (say)}$$

$$x = x_1 + \lambda(x_2 - x_1) \quad ; \quad y = y_1 + \lambda(y_2 - y_1)$$
$$x = x_1(1-\lambda) + \lambda x_2 \quad ; \quad y = y_1(1-\lambda) + \lambda y_2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = (1-\lambda)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$X = (1-\lambda) X_1 + \lambda X_2$$

$0 \le \lambda \le 1$

$$X = (1-\lambda) X_1 + \lambda X_2$$

let $\lambda = 0$; $X = X_1$

$\lambda = 1$; $X = X_2$

$0 \le \lambda \le 1$ $\hookrightarrow$ $X = (1-\lambda) X_1 + \lambda X_2$

Note : $S = \mathbb{R}^4$

$X_1 = (a_1, a_2, a_3, a_4)$

$X_2 = (b_1, b_2, b_3, b_4)$

$$X = (1-\lambda) X_1 + \lambda \cdot X_2$$
$$= (1-\lambda)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

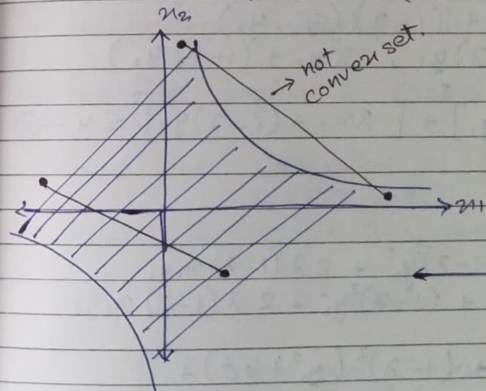
Two ways to check convex sets :

a) Graphically

b) Algebraically

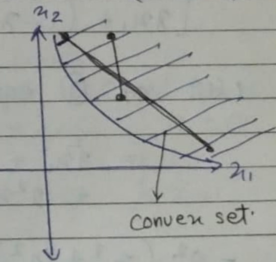
eg. ① $S = \{ x \in \mathbb{R}^2 : x_1 x_2 \le 4 \}$

where $x = (x_1, x_2)$

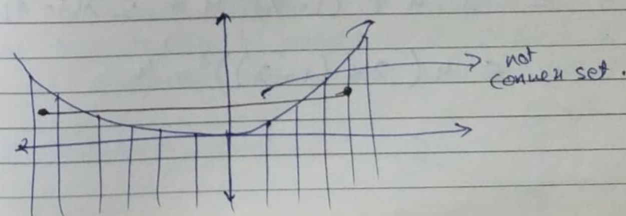


② $S = \{ x \in \mathbb{R}^2 : x_1 x_2 \ge 4 \}$

$x = (x_1, x_2)$



③ $S = \{ x \in \mathbb{R}^2 : x^2 \ge 4y \}$



Algebraic - Show that the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}$ is convex.

Take $x_1, x_2 \in S$

$$X_1 = (x_1, x_2) \; ; \; x_1^2 + x_2^2 \le 4 \quad \text{--- (1)}$$
$$X_2 = (y_1, y_2) \; ; \; y_1^2 + y_2^2 \le 4 \quad \text{--- (2)}$$

To prove

$$x = (1-\lambda) X_2 + \lambda X_1 \in S \; ; \qquad 0 \le \lambda \le 1$$

$$X = \lambda(x_1, x_2) + (1-\lambda)(y_1, y_2)$$
$$= (\lambda x_1 + (1-\lambda) y_1, \; \lambda x_2 + (1-\lambda) y_2)$$

$$[\lambda x_1 + (1-\lambda) y_1]^2 + [\lambda x_2 + (1-\lambda) y_2]^2 \le 4 \quad \text{--- (3)}$$

LHS of (3) eqn

$$= \lambda^2 x_1^2 + (1-\lambda)^2 y_1^2 + 2\lambda(1-\lambda) x_1 y_1 + \lambda^2 x_2^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda) x_2 y_2$$

$$= \lambda^2 (x_1^2 + x_2^2) + (1-\lambda)^2 (y_1^2 + y_2^2) + 2\lambda(1-\lambda)[x_1 y_1 + x_2 y_2]$$

$$\le \lambda^2 . 4 + (1-\lambda)^2 4 + 2\lambda(1-\lambda) . 4$$

$$\le 4(\lambda + (1-\lambda))^2 = 4$$

Convex Hull : let $S \subseteq \mathbb{R}^n$; then the smallest convex set containing the given set S is called the convex hull of S, denoted by Conv(S).

* Basic result on Convex set.

- → Union of two convex set is not convex.
- → The intersection of two convex set is a convex set.
- → The hyperplane is convex set.
- → The half spaces are convex set.
- → Polyhedron / Polytype is convex set.
- → Feasible region is convex set.

Convex linear Combination of n points

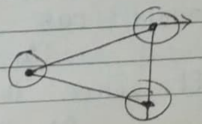
$X_1, X_2 \ldots\ldots X_n \in S \subseteq \mathbb{R}^n$, Then CLC of these n points is $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n$; $\lambda_i \ge 0 \; \forall i$.

$$X^* = \lambda_1 X_1 + \cdots + \lambda_n X_n \; ; \; \lambda_i \ge 0 \; ; \; \sum_{i=1}^{n} \lambda_i = 1$$

$$\boxed{X^* \in S} \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$$

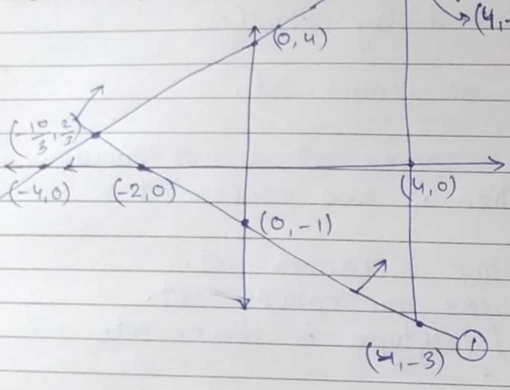
* Vertex or extreme Point

A point x which cannot be written as a CLC of two distinct points, is called a vertex or Extreme Point or corner point.



Q. All extreme point

$S = \{(x_1, x_2) : x_1 + 2x_2 \ge -2,\ -x_1 + x_2 \le 4,\ x_1 \le 4\}$
and represent the point (2, 3) as the convex linear combinations of extreme points S.



$$(2,3) = \lambda_1\left(\frac{-10}{3}, \frac{2}{3}\right) + \lambda_2(4,8) + \lambda_3(4,-3)$$

$$= \left(-\frac{10}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3,\ \frac{2}{3}\lambda_1 + 8\lambda_2 - 3\lambda_3\right)$$

$\therefore\ -\frac{10}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3 = 2$

$\frac{2}{3}\lambda_1 + 8\lambda_2 - 3\lambda_3 = 3$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

we can find unique solution for $\lambda_1, \lambda_2, \lambda_3$.
& hence it is possible to write (2,3) as CLC.

Simplex – Simplex is closed fig / line / point whose any point can be written in CLC where $\sum \lambda_i = 1$, $\lambda_i \ge 0$.

simplex in $\mathbb{R}^n$

Let $X_1, X_2, X_3, \ldots, X_n, X_{n+1}$ be $(n+1)$ points in $\mathbb{R}^n$. Then the convex set spanned by these $(n+1)$ points is called a n-simplex in $\mathbb{R}^n$.

eg. ① 0- simplex $\quad X_1 = \lambda_1 X_1 \quad \lambda_1 = 1,\ \lambda_1 \ge 0$

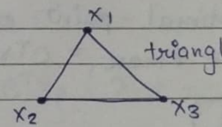
• point

② 1- Simplex $\quad X = \lambda_1 X_1 + \lambda_2 X_2 \quad \lambda_1 + \lambda = 1,\ \lambda_1, \lambda_2 \ge 0$

line segment

all points lie on line.

③ 2- Simplex

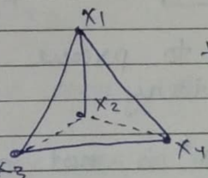


triangle $\quad X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

$\lambda_1, \lambda_2, \lambda_3 \ge 0$

④ 3- Simplex



tetrahedron

$X = \sum_{i=1}^{3} \lambda_i X_i$

$\sum_{i=1}^{3} \lambda_i = 1$

$\lambda_i \ge 0$

Fundamental theorem of LP

Statement : If the feasible region of an LPP is a convex polyhedron, then there exist an optimal solution to the LPP and atleast one corner point (or basic feasible solution) must be optimal.

Proof Existence & BFS / corner optimal

LPP : Max $z = C^T X$

st $AX = b$

$X \geq 0$

$C, X \in \mathbb{R}^n$

} Matrix notation of LPP.

feasible region $S = \{ X \mid AX = b, X \geq 0 \}$

S is convex polyhedron, it is non-empty closed bounded.

→ optimal solution must exist.

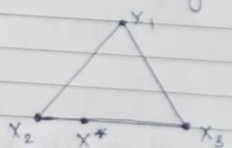
let X^* be optimal point of the given LPP

$\therefore \quad z_0 = z_{max} = C^T X^*$

$\boxed{X^* \in S}$

Now we have to prove X^* is corner point so on contrary.

let say X^* is not corner point



As S is polytope it has finite no. of extreme points (closed polyhedron) $X_1, X_2, \ldots, X_k$.

for $X^* \in S$

$$X^* = \sum_{i=1}^{k} \lambda_i X_i \; ; \; \sum_{i=1}^{k} \lambda_i = 1 \; ; \; \lambda_i \geq 0 \;\; \forall i$$

{ ✯ any point of S can be expressed as a convex linear combination of extreme points of S. }

$$C^T X^* = C^T \sum_{i=1}^{k} \lambda_i X_i$$

$$C^T X^* = C^T (\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_k X_k)$$

$$C^T X^* = \lambda_1 C^T X_1 + \cdots + \lambda_k C^T X_k$$

replace all values by max. value as we considere of as maximizer.

let max. value $C^T X_p$

$$C^T X^* \leq \lambda_1 C^T X_p + \cdots + \lambda_k C^T X_p$$

$$C^T X^* \leq C^T X_p \, (\cancel{\lambda_1 + \cdots + \lambda_k})^{\to 1}$$

$$C^T X^* \leq C^T X_p$$

$\therefore$ optimal solution occurs at corner point.

A bounded feasible region always gives us optimal solution.

- unique
- infinite

* Optimality & feasibility criteria for Simplex Method

Algebraic method

↳ total BFS $\leq {}^nC_m$

↳ z at BFS & select one.

simplex method

may the corner point is not lie in feasible region.

Theorem : Consider a LPP

$$\text{Max } z = C^T X$$
$$AX = b$$
$$X \geq 0.$$

Let X_B be BFS and If some $z_j - c_j < 0$ and for that j some $\lambda_{ij} > 0$, then there exists a new BFS $\hat{X}_B$ subject to

$$z(\hat{X}_B) \geq z(X_B)$$

Proof : Given LPP $\text{Max } Z = C^T X$
s.t. $AX = b$
$X \geq 0$

$(A)_{m \times n}$ $\quad \rho(A) = m$

$$A = (a_1, a_2, a_3, \ldots a_n)$$

select BFS we get

$$B X_B = b$$

$$(B)_{m \times m} = (b_1, b_2, \ldots b_m)$$

* optimality criteria

$z_j - c_j \geq 0$	Max
$z_j - c_j \leq 0$	Min

* feasibility $\quad \dfrac{X_{Br}}{\lambda_{rj}} = \text{Min} \left\{ \dfrac{X_{Bi}}{\lambda_{ij}} ; \lambda_{ij} > 0 \right\}$

Simplex Method & its Variants

For applying simplex method take LPP into standard form.

Simplex method is same as Basic solution or Algebraic method, only thing is we are not checking the value at all solution. we ~~are~~ will start from one of the basic feasible solution, and then based on the ~~next~~ optimality & feasibility criteria we will select the next BFS which will ensure that the value of z is improve & if criteria ensure that there is no improvement scope we will stop.

Simplex table

eg. $\quad \text{Max } z = 2x_1 + 3x_2$
s.t. $\quad x_1 - x_2 \leq 2$
$\quad -3x_1 + x_2 \leq 4$
$\quad x_1, x_2 \geq 0$

Standard LPP $\quad z = 2x_1 + 3x_2 + 0 \cdot S_1 + 0 \cdot S_2$
$\quad x_1 - x_2 + S_1 = 2$
$\quad -3x + x_2 + S_2 = 4$
$\quad x_1, x_2, S_1, S_2 \geq 0$

Scalars.

C_B	BV	x_1	x_2	S_1	S_2	X_B (solution)
	Z	$z_1 - c_1$	$z_2 - c_2$	$z_3 - c_3$	$z_4 - c_4$	Value of Z = 0
		$z_j - c_j$				
0	S_1	1	-1	1	0	$2 = x_{B1}$
					1	$4 = x_{B2}$

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix} \qquad AX = b$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \cancel{B}X_B = b$$

$$\boxed{X_B = b}$$

→ $\begin{bmatrix} 1 \\ -3 \end{bmatrix} a_1 = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} a_2 = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} a_3 = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} a_4 = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$z_j - C_j$: Net evaluation / Relative evaluation

$z_j - c_j = C_B$ (scalars) $- C_j$ objective function cost.

$z_1 - C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} - 2 = 0 - 2 = -2$

$z_2 - C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 3 = 0 - 3 = -3$

$z_3 - C_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 0$

$z_4 - C_4 = 0$

Simplex table layout

B.V	$x_1 \quad x_2 \; \cdots \; x_n$	X_B (solutions)
z	$z_j - c_j$	value of $z \geq 0$
S_1		x_{B1}
x_1		x_{B2}
x_2		x_{B3}

Steps to solve LPP with simplex method.

1. Write LPP in standard form.
2. Make 1st simplex table with initial BFS → DV
3. Check $z_j - c_j \geq 0 \;\forall j \longrightarrow$ optimal for (Max.)
   $z_j - c_j \leq 0 \;\forall j \rightarrow$ optimal for (Mini.)
4. If step 3 does not hold, find new BFS where

entering	leaving variables
[Max.] $z_j - c_j$ most ($-$ve) [Mini.] $z_j - c_j$ most ($+$ve)	$\dfrac{x_{Br}}{\lambda_{rj}} = \min\left\{\dfrac{x_{Bi}}{\lambda_{ij}} ; \lambda_{ij} > 0\right\}$

5. Make next simplex table using new BFS by apply row operation
6. Repeat step 3 onwards.

Q. Max $Z = 4x_1 + 3x_2$
s.t. $x_1 + x_2 \le 8$
$2x_1 + x_2 \le 10$
$x_1, x_2 \ge 0$

① standard form
$Z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2$

$x_1 + x_2 + S_1 = 8$
$2x_1 + x_2 + S_2 = 10$
$x_1, x_2, S_1, S_2 \ge 0$

entry

$z_j - c_j = C_B(\text{scalar}) - c_j$

B.V	x_1	x_2	S_1	S_2	X_B
Z	-4	-3	0	0	Value of Z = 0
0 / S_1	1	1	1	0	8
0 / S_2	2	1	0	1	10

Remove
pivot element

$AX = b$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times4} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

$BX_B = b$ $\quad (B)_{2\times2}$

① ($BV = S_1, S_2$) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$X_B = B^{-1} b$

$\boxed{X_B = b}$

$z_1 - c_1 = [0 \quad 0]\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4 = -4$

& similar for Others.

$$\therefore Z = \underbrace{4(x_1) + 3(x_2)}_{NBV} + 0(S_1) + 0(S_2)$$
$$= 0 + 0(8) + 0(10)$$
$$= 0$$

\# Optimality criteria $\quad z_j - c_j \ge 0 \;\; \forall j$ → DV
here $\quad -4 \ge 0$ ✗
$-3 \ge 0$ ✗
Does not satisfy
so we need to find entering variable and leaving variable

$$\hat{Z} = Z_0 - \theta_j(z_j - c_j)$$

next BFS, initial BFS, min. ratio

$$\frac{x_{Br}}{\lambda_{ri}} = \min\left\{\frac{X_{Bi}}{\lambda_{ij}} ; \lambda_{ij} > 0\right\}$$

$\lambda_{11} = 1 \qquad X_{B1} = 8$
$\lambda_{21} = 2 \qquad X_{B2} = 10$

$$\therefore \theta_j = \frac{X_{Br}}{\lambda_{ij}} = \min\left\{\frac{X_{Bi}}{\lambda_{ij}} ; \lambda_{ij} > 0\right\}$$
$$= \min\left\{\frac{8}{1}, \frac{10}{2}\right\}$$
$$\theta_j = \min\{8, 5\} = 5$$

Intersection of entry & leaving we get pivot element
we get ②.

② BV (s_1, x_1)

$$B = \begin{array}{cc} s_1 & x_1 \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$a_1 = \begin{bmatrix}1\\2\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}1\\2\end{bmatrix}$$
$$a_2 = \begin{bmatrix}1\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix}$$
$$a_3 = \begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix}$$
$$a_4 = \begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix}$$

→ have to find.

$$z_1 - c_1 = [0\ \ 4]\begin{bmatrix}0\\1\end{bmatrix} - 4 = 4 - 4 = 0$$

$$z_2 - c_2 = [0\ \ 4]\begin{bmatrix}1/2\\1/2\end{bmatrix} - 3 = (0+2) - 3 = -1$$

⋮ – and so on.

$$Z = CB \times X_B \Rightarrow [0\ \ 4]\begin{bmatrix}3\\5\end{bmatrix} = 20$$

	BV	CB	x_1	x_2	s_1	s_2	x_B
R_1	Z		-4	-3	0	0	0
R_2	s_1	0	1	1	1	0	8
R_3	s_2	0	$\boxed{2}$	1	0	1	10
$\hat{R}_1$	Z		0	-1	0	2	20
$\hat{R}_2$	s_1	0	0	$\boxed{1/2}$	1	$-1/2$	3
$\hat{R}_3$	x_1	4	1	$1/2$	0	$1/2$	5

$$\hat{R}_3 = \frac{R_3}{2} \ ; \quad \hat{R}_2 = R_2 - \frac{R_3}{2}$$

R_1	Z		1	0	2	+1	26
R_2	x_2		0	1	2	-1	6
R_3	x_1		1	0	-1	1	2

$R_2 \to 2\hat{R}_2$

$R_1 \to R_1 + 1(2\hat{R}_2)$

$R_3 \to R_3 - 1/2(\hat{R}_3)$

$z_j - c_j \geq 0$ ✓ so this is optimal solution

$Z_{max} = 26$ at $x_1 = 2$; $x_2 = 6$

Q. Mini $Z = x_1 - 3x_2 + 2x_3$

s.t. $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

$\hat{z} = z_0 - \theta_j(z_j - c_j)$

Method I

Max : Entering $z_j - c_j$ most (−ve)

$\hat{z} \geq z_0$

Min : Entering $z_j - c_j$ most (+ve)

$\hat{z} \leq z_0$

Method II

Convert min. problem to max problem.

Min $z = -$Max$(-z)$

Max $z = -$Min$(-z)$

$Z = x_1 - 3x_2 + 2x_3 + 0.S_1 + 0.S_2 + 0.S_3$

$3x_1 - x_2 + 2x_3 + S_1 = 7$

$-2x_1 + 4x_2 + S_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$

$s_1, S_2, S_3 \geq 0$

$x_1\ x_2\ x_3 \geq 0$

$$A = \begin{bmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$\boxed{x_B = b}$$

$$b = \begin{bmatrix}7\\12\\10\end{bmatrix}$$

(Method I) (Mini) → entering most (+ve) $(z_j - c_j)$

CB	BV	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Ratio
0	z	-1	3	-2	0	0	0	0	
0	s_1	3	-1	2	1	0	0	7	
← 0	s_2	-2	$\boxed{4}$	0	0	1	0	12	12/4
0	s_3	-4	3	8	0	0	1	10	10/3
R		1/2	0	-2	0	-3/4	0	-9	
←	s_1	$\boxed{5/2}$	0	2	1	1/4	0	10	
	x_2	-1/2	1	0	0	1/4	0	3	
	s_3	-5/2	0	8	0	-3/4	1	1	
	z	0	0	-12/5	-1/5	-8/10	0	-11	
	x_1	1	0	4/5	2/5	1/10	0	4	
	x_2	0	1	2/5	1/5	3/10	0	5	
	s_3	0	0	10	2/5	-1/2	1	11	

1) BV → $s_1\ x_2\ s_3$

$R_1 \rightarrow R_1 - 3(R_3/4)$

$R_2 \rightarrow R_2 + 1(R_3/4)$

$R_3 \rightarrow R_3/4$

$R_4 \rightarrow R_4 - 3(R_3/4)$

2) BV → $x_1\ x_2\ s_3$

$R_1 \rightarrow R_1 - 1/2\left(\frac{2}{5}R_2\right)$

$R_2 \rightarrow \frac{2}{5}R_2$ — pivot row.

$R_3 \rightarrow R_3 + 1/2\left(\frac{2}{5}R_2\right)$

$R_4 \rightarrow R_4 + 5/2\left(\frac{2}{5}R_2\right)$

all $z_j - c_j \le 0$

& optimal solution is

$z_{min} = -11$

at $x_1 = 4$

$x_2 = 5$

$x_3 = 0$

(Method II)

$$Min = -Max(-z)$$

$$Min\ z = x_1 - 3x_2 + 2x_3 + 0.s_1 + 0s_2 + 0s_3$$

$$\therefore -Max(-z) = x_1 - 3x_2 + 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\therefore Max(-z) = -x_1 + 3x_2 - 2x_3 - 0.s_1 - 0.s_2 - 0.s_3$$

In given constraints, there is no change.

$\therefore$ s.t $3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

$$A = \begin{bmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boxed{X_B = b} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} \ge 0$$

BFS.

Max. most (-ve) $z_j - c_j \geq 0$

CB	BV	x_1	x_2	x_3	S_1	S_2	S_3	X_B	
	$-z$	1	-3	$+2$	0	0	0	0	
0	s_1	3	-1	2	1	0	0	7	
← 0	s_2	-2	$\boxed{4}$	0	0	1	0	12	$12/4 = 3$
0	s_3	-4	3	8	0	0	1	10	$10/3 = 3.3$
	$-z$	$-1/2$	0	2	0	$3/4$	0	9	
←	s_1	$\boxed{5/2}$	0	2	1	$1/4$	0	10	$\frac{10}{5} \times 2$
	x_2	$-1/2$	1	0	0	$1/4$	0	3	
	s_3	$-5/2$	0	8	0	$-3/4$	1	1	
	$-z$	0	0	$12/5$	$1/5$	$8/10$	0	11	
	x_1	1	0	$4/5$	$2/5$	$1/10$	0	4	
	x_2	0	1	$2/5$	$1/5$	$3/10$	0	5	
	S_3	0	0	10	$2/5$	$-1/2$	1	11	

current all $z_j - c_j \geq 0$
so optimal solution

$\therefore -z = 11$

$\therefore \boxed{z = -11}$

at $x_1 = 4$
$x_2 = 5$
$x_3 = 0$

Simplex Method - Unbounded Solution

Theorem - let there exist a BFS to the given LPP. If for entering variable; $z_j - c_j$ most (-ve), the corresponding scalar $\lambda_{ij} \leq 0$. (Consider the case of maximization), then there does not exist any optimum value to LPP; we say it has unbounded solution.

→ $\lambda_{ij} \leq 0$ then unbounded solution.

Q. Max $z = 4x_1 + x_2$

s.t. $x_1 - x_2 \leq 1$
$-2x_1 + x_2 \leq 2$
$x_1, x_2 \geq 0$

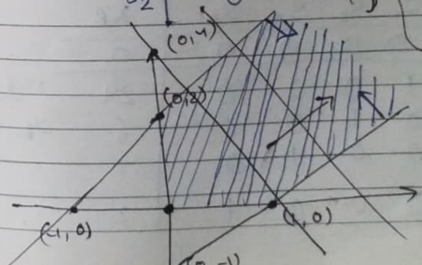
① $R_1 \to R_1 + 4R_2$
$R_2 \to R_2$
$R_3 \to R_3 + 2R_2$

standard form

Max $z = 4x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2$

s.t. $x_1 - x_2 + S_1 = 1$
$-2x_1 + x_2 + S_2 = 2$
$x_1, x_2, S_1, S_2 \geq 0$

BV	x_1	x_2	S_1	S_2	X_B
z	-4	-1	0	0	0
← S_1	$\boxed{1}$	-1	1	0	1
S_2	-2	1	0	1	2
z	0	-5	4	0	4
x_1	1	-1	1	0	1
S_2	0	-1	2	1	4



we have entering variable but do not have leaving V. that why current entity have unbounded solution

Q. Unique optimal solution in Simplex Method

$$\text{Max } z = 5x_1 + 4x_2$$

s.t.
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Standard form $\text{Max } z = 5x_1 + 4x_2 + 0\cdot s_1 + 0\cdot s_2 + 0\cdot s_3 + 0\cdot s_4$

$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$$

$$A = \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 \\ \end{array}$$
$$A = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 4}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4} \qquad x_B = \begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix}$$

most (-ve) ↓ x_1

BV	x_1	x_2	s_1	s_2	s_3	s_4	X_B	
	-5	-4	0	0	0	0	0	
← s_1	6	4	1	0	0	0	24	$24/6 \to 4$
s_2	1	2	0	1	0	0	6	$6/1$
s_3	-1	1	0	0	1	0	1	
s_4	0	1	0	0	0	1	2	

z	x_1	x_2 ↓	s_1	s_2	s_3	s_4	X_B
	0	-2/3	5/6	0	0	0	20
x_1	1	2/3	1/6	0	0	0	4
← s_2	0	4/3	-1/6	1	0	0	2
s_3	0	5/3	1/6	0	1	0	5
s_4	0	1	0	0	0	1	2
z	0	0	3/4	1/2	0	0	21
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

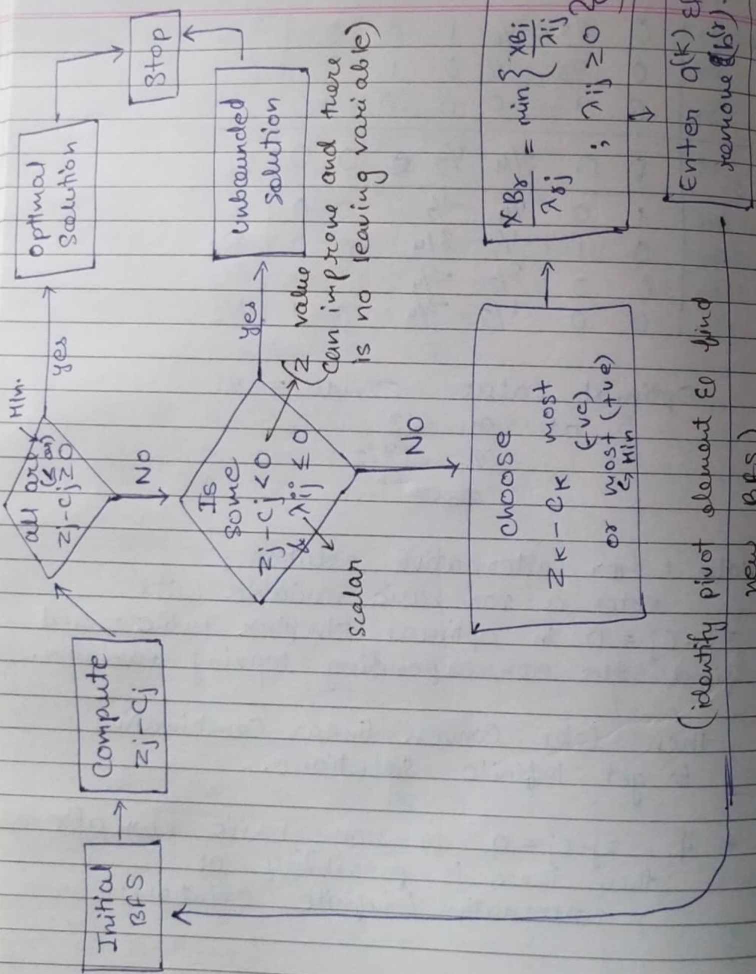
Optimal value $z_{max} = 21$
at $x_1 = 3$
$x_2 = 3/2$

Note : for alternative solution ; enter a non-basic variable with $z_j - c_j = 0$ in optimal simplex table ; and find the corresponding leaving variable.

then take convex linear combination to get infinite solutions.

* if $z_j - c_j = 0$ for non-basic variable then there is possibility of alternative / infinite solutions.

Summary of Simplex Method



Imp formulas for Simplex Method.

* B^{-1} is the inverse of Matrix of selected BV in that method.

1. $X_B = B^{-1} b$
2. Scalars $\lambda_j = B^{-1} a_j$
3. $z_j = C_B \times X_B \,/\, C_B B^{-1}$
4. $z_j = C_B (B^{-1} a_j)$
5. Note, for B.V "$z_j - c_j = 0$"
6. for NBV, "$z_j - c_j$" - calculate

Q. Max $z = 4x_1 + 3x_2$
s.t
$x_1 + x_2 \leq 8$
$2x_1 + x_2 \leq 10$
$x_1, x_2 \geq 0$

$z = 4x_1 + 3x_2 + 0 \cdot S_1 + 0 \cdot S_2$
$x_1 + x_2 + S_1 = 8$
$2x_1 + x_2 + S_2 = 10$
$x_1, x_2, S_1, S_2 \geq 0$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

C_B	BV	x_1	x_2	s_1	s_2	x_B
	z	−4	−3	0	0	0
0	S_1	1	1	1	0	8
0	S_2	2	1	0	1	10
	z	0		0		
	S_1	0		1		
	x_1	1		0		
		$B^{-1}a_1$	$B^{-1}a_2$	$B^{-1}a_3$	$B^{-1}a_4$	$B^{-1}b$

Transit to Big M or two Phase Method

When we can not find initial ready identity matrix to apply for simplex method and hence we have to transit to Big M or two Phase method.

In Big M or two Phase method we introduce artificial variable.

* Introduction & use of Artificial variable

LPP : While applying simplex method, we start with basic variables that gives initial BFS. Infact ; no computation is required to construct first simplex table.

eg. Min $z = 2x_1 + x_2$

s.t $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

$z = 2x_1 + x_2 + 0.S_1 + 0.S_2$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 - S_1 = 6$

$x_1 + 2x_2 + S_2 = 3$

$x_1 \; x_2 \; S_1 \; S_2 \geq 0$

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{3\times4}$$

$$\cancel{B = \begin{bmatrix} & & 0 \\ & & 0 \\ & & 1 \end{bmatrix}_{3\times3}}$$

BV = (__ , __ , S_2)

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - S_1 + a_2 = 6$

$x_1 + 2x_2 + S_2 = 3$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

BV = (a_1, a_2, S_2)

Big M method – It apply same mannar & have same feasible & optimal criteria as simplex method.

Introduce AI variable(s) for ready identity as sub-matrix of co-efficient matrix in LPP with high penalty /Cost (−M) in Maximization problem and penalty / cost (M) in Minimization problem.

$$C_a = \begin{cases} M & ; \text{ Minimization LPP problem} \\ -M & ; \text{ Maximization} \end{cases}$$

↳ cost of artificial variable.

Cases :

1. If no artificial variable remain in the basis (last simplex table) then the system is consistent.

2. If artificial variable remain in the last simplex table with positive value then the system is inconsistent / no solution or infeasible solution

3. If artificial variable remain in last simplex

table with zero value then the system is consistent. (That is the system is degenerate)
(artificial variable remain)

Q. Min $z = 2x_1 + 8x_2$

s.t. $5x_1 + 10x_2 = 150$

$x_1 \le 20$

$x_2 \ge 14$

$x_1, x_2 \ge 0$

standard LPP and introduce slack / surplus variable (s) with cost 0 & artificial variable(s) with cost M.

$$\text{Min } z = 2x_1 + 8x_2 + 0.s_1 + 0.s_2 + Ma_1 + Ma_2$$

$$5x_1 + 10x_2 + a_1 = 150$$

$$x_1 + s_1 = 20$$

$$x_2 - s_2 + a_2 = 14$$

$$a_1, a_2, x_1, x_2, s_1, s_2 \ge 0$$

$$A = \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & a_2 \end{array}$$
$$\begin{bmatrix} 5 & 10 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{array}{ccc} a_1 & s_1 & a_2 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad x_B = \begin{bmatrix} 150 \\ 20 \\ 14 \end{bmatrix}$$

$B.V = (a_1, s_1, a_2)$

C_B	BV	x_1	x_2 ↓	s_1	s_2	a_1	a_2	X_B	
	Z	5M−2	11M−8	0	−M	0	0	164M	
M	a_1	5	10	0	0	1	0	150	
0	s_1	1	0	1	0	0	0	20	
← M	a_2	0	1	0	−1	0	1	14	
	Z	5M−2	0	0	10M−8 ↓	0	−11M+8	112+10M	$R_1 \to R_1 - (11M-8)R_4$
← M	a_1	5	0	0	10	1	−10	10	$R_2 \to R_2 - 10R_4$
0	s_1	1	0	1	0	0	0	20	$R_3 \to R_3$
8	x_2	0	1	0	−1	0	1	14	$R_4 \to 1$.
	Z	2 ↓	0	0	0	X	X	120	
← 0	s_2	1/2	0	0	1	X	X	1	
0	s_1	1	0	1	0	X	X	20	
8	x_2	1/2	1	0	0	X	X	15	
	Z	0	0	0	−4	X	X	116	
2	x_1	1	0	0	2	X	X	2	
0	s_1	0	0	1	−2	X	X	18	
8	x_2	0	1	0	−1	X	X	14	

Optimal solution

$z_{min} = 116$.

at $x_1 = 2$

$x_2 = 14$

(artifical variable with non-zero value)

Q. Max $z = -x_1 + x_2$

s.t $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \ge 6$

$x_1, x \ge 0$

Max $z = -x_1 + x_2 + 0.s_1 + 0.s_2 - Ma_1$

$x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s = 6 + a_1$

$$A = \begin{array}{c} \begin{bmatrix} +1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & -1 & 1 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & a_1 \end{array} \end{array}$$

$$x_B = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} s_1 & a_1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

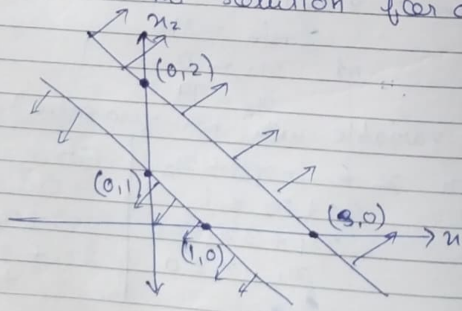
$(BV = s_1, a_1)$

C_B	BV	x_1	x_2 ↓	s_1	s_2	a_1	X_B	Ratio
	z	$-2M+1$	$-3M+1$	0	M	0	$-6M$	
← 0	s_1	1	1	1	0	0	1	
$-M$	a_1	2	3	0	-1	1	6	
	z	$M+2$	0	3	M	0	$-3M+1$	
	x_2	1	1	1	0	0	1	
	a_1	-1	0	-3	1	1	3	

$z_j - c_j \geq 0$ optimal

but artificial variable $a_1 = 3$ i.e. $(a_1 \neq 0)$

$\therefore$ No solution for above situation



Q. (Artificial Variable remain with zero value)

$$\text{Max } z = -2x_1 - x_2$$

s.t.
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

$$\text{Max } z = -2x_1 - x_2 + 0 \cdot s_1 + 0 \cdot s_2 - Ma_1 - Ma_2$$
$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 3$$

$$A = \begin{array}{c} \begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & a_2 \end{array} \end{array}$$

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} a_1 & a_2 & s_2 \end{array} \end{array}$$

$$x_B = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = b$$

C_B	BV	x_1 ↓	x_2	s_1	s_2	a_1	a_2	X_B	Range
	z	$-7M+2$	$-4M+1$	M	0	0	0	$-9M$	
← $-M$	a_1	3	1	0	0	1	0	3	3/3
$-M$	a_2	4	3	-1	0	0	1	6	6/4
0	s_2	1	2	0	1	0	0	3	3/1
	z	0	↓ $-5/3M+1/3$	M	$4M/3-2/3$	0	0	$-2M-2$	
	x_1	1	1/3	0	1/3	0	0	1	$1/\frac{1}{3}=3$
	a_2	0	5/3	-1	$-4/3$	1	0	2	6/5
←	s_2	0	5/3	0	$-1/3$	0	1	2	6/5
	z	0	0	M	$M-3/5$	0	$M-1/5$	$-12/5$	
	x_1	1	0	0	2/5	0	$-1/5$	3/5	
	a_2	0	0	-1	-1	1	-1	0	
	x_2	0	1	0	$-1/5$	0	3/5	6/5	

In above table, $z_j - c_j \geq 0$ ∴ optimality has arrived. artificial variable is in the final basis but it is a zero level.

∴ the system is feasible.

$z_{max} = \frac{-12}{5}$ at $x_1 = \frac{3}{5}$ $x_2 = \frac{6}{5}$

Two - Phase Method for solving LPP.

There are 2 phases

1) Phase 1 ensure feasibility.
2) Phase 2 we perform optimality.

Phase I : Introduce additional objective function in place of the given objective function, the new LPP formed is called Auxiliary LPP. The auxiliary LPP is

$$\boxed{\begin{array}{l} \text{Min } z^* = a_1 + a_2 + \dots + a_r \text{ (sum of all artificial variables)} \\ \text{s.t. constraints of original LPP } \& \ a_i \geq 0 \end{array}}$$

Cases :

or Max $z^* < 0$

1. Min $z^* > 0$, and AV remain in last simplex table. Then LPP has no solution.

2. Min $z^* = 0$, and no AV remain. Then LPP has feasible solution & go to Phase II.

3. Min $z^* = 0$, and AV remain ~~in~~ with zero level then LPP has feasible solution & go to phase II.

Phase II : Use the last table of Phase I and find new z-row with original objective function. Do simplex iteration if required to find optimal solution.

Q. (infeasible)

Max $z = -x_1 + x_2$

s.t. $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

Max $z = -x_1 + x_2 + 0.s_1 - 0.s_2$

$x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 = 6$

all value ≥ 0

$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$

Auxiliary LPP

Min $z^* = a_1$ // (Max $z^* = -a$)

s.t $x_1 + x_2 + s_1 = 1$

$2x_1 + 3x_2 - s_2 + a_1 = 6$

all values ≥ 0

BV = (s_1, a_1)

	BV	x_1	x_2	s_1	s_2	a_1	X_B
	z	2	3	0	-1	0	6
← 0	s_1	1	1	1	0	0	1
1	a_1	2	3	0	-1	1	6
	z	-1	0	-3	-1	0	3
0	x_2	1	1	1	0	0	1
1	a_1	-1	0	-3	-1	1	3

$z^* = 6 > 0$

$z_j - c_j \leq 0$ ✓

$\boxed{a_1 = 3}$ $z^* = 3 > 0$

∴ Given LPP is infeasible.

Q. (no AV & feasible solution)

$\text{Min } z = 2x_1 + 8x_2$

s.t. $5x_1 + 10x_2 = 150$

$x_1 \leq 20$

$x_2 \geq 14$; $x_1, x_2 \geq 0$

Standard LPP by introducting slack, surplus variable with cost 0 & AV with cost ~~zero~~ one.

Phase I : $\text{Min } z^* = a_1 + a_2$

s.t. $5x_1 + 10x_2 = 150$

$x_1 + S_1 = 20$

$x_2 - S_1 = 14$

all value ≥ 0

$$A = \begin{bmatrix} 5 & 10 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$\therefore$ $5x_1 + 10x_2 + a_1 = 150$

$x_1 + S_1 = 20$

$x_2 - S_1 + a_2 = 14$

$$\therefore A = \begin{array}{cccccc} x_1 & x_2 & S_1 & S_2 & a_1 & a_2 \end{array}$$
$$\begin{bmatrix} 5 & 10 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{array}{ccc} a_1 & S_1 & a_2 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_B = b = \begin{bmatrix} 150 \\ 20 \\ 14 \end{bmatrix}$$

C_B	BV	x_1	x_2	S_1	S_2	a_1	a_2	X_B
	z	5	11	0	-1	0	0	164
1	a_1	5	10	0	0	1	0	150
0	S_1	1	0	1	0	0	0	20
← 1	a_2	0	1	0	-1	0	1	14
	z	5	0	0	10	0	-10	10
←	a_1	5	0	0	[illegible]	1	-[illegible]	80
	S_1	1	0	1	0	0	0	20
	x_2	0	1	0	-1	0	1	14
	z	0	0	0	0	1	-1	0
	S_2	1/2	0	0	1	1/10	-1	1
	S_1	1	0	1	0	0	0	20
	x_2	1/2	1	0	0	1/10	0	15

$[z_j - c_j \leq 0]$ ✗

$\boxed{z_j - c_j \leq 0}$ ✓

no. AV remain & so go to Phase II.

Phase II :

		cost → 2	8	0	0	
	BV	x_1	x_2	S_1	S_2	X_B
	z	2	0	0	0	120
← 0	S_2	1/2	0	0	1	1
0	S_1	1	0	1	0	20
8	x_2	1/2	1	0	0	15
	z	0	0	0	-4	116
	x_1	1	0	0	+2	2
	S_1	0	0	1	-2	18
	x_2	0	1	0	-1	4

$[z_j - c_j \geq 0]$ ✗

$\boxed{z_j - c_j \geq 0}$ ✓

Optimaility arrived

$z_{min} = 116$ at (x_1, x_2).

$x_1 = 2$

$x_2 = 4$

Q.

$$\text{Max } z = -2x_1 - x_2$$

s.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 - x_3 = 6$; $x_i \geq 0$

$x_1 + 2x_2 + x_4 = 3$

$$\text{Max } z = -2x_1 - x_2 + 0\cdot x_3 + 0\cdot x_4$$

Phase I : $\text{Min } z^* = a_1 + a_2$

s.t. $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - x_3 + a_2 = 6$

$x_1 + 2x_2 + x_4 = 3$

all value ≥ 0

$$A = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & a_1 & a_2 \end{array}$$
$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{array}{ccc} a_1 & a_2 & x_4 \end{array}$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X_B = b = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Original cost ②

BV	x_1	x_2	x_3	x_4	a_1	a_2	X_B
z	0	0	1	1	2	0	0
a_1	3	1	0	0	$\boxed{1}$	0	3
a_2	4	3	-1	0	0	1	6
x_4	1	2	0	1	0	0	3
z							

		-2	-1	0	0	1	1	
C_B	BV	x_1	x_2	x_3	x_4	a_1	a_2	X_B
	z	5	8	-1	0	1	1	9
1	a_1	3	1	0	0	1	0	3
1	a_2	4	3	-1	0	0	1	6
0	x_4	1	2	0	1	0	0	3.

Last table of Phase I

to remove a_2.

BV	x_1	x_2	x_3	x_4	a_1	a_2	X_B
z	0	0	1	1	2	0	0
x_1	1	0	0	$-1/5$	$2/5$	0	$3/5$
a_2	0	0	-1	-1	-1	1	0
x_2	0	1	0	$3/5$	$-1/5$	0	$6/5$

$z^* = 0$, $a = 0$ } move to Phase II

$R_1 \to R_1 + \frac{1}{5}(R_2)$

$R_2 \to -R_2$

$R_3 \to R_3 - \left(\frac{3}{5}(-R_2)\right)$

Phase II :

C_B	BV	x_1	x_2	x_3	x_4	X_B
	z	0	0	$+3/5$	0	$-12/5$
-2	x_1	1	0	$1/5$	0	$3/5$
0	x_4	0	0	1	1	0
-1	x_2	0	1	$-3/5$	0	$6/5$

$z_j - c_j \geq 0$ ✓

optimal solution

$z_{max} = -12/5$ at $(3/5, 6/5)$

at $x_1 = 3/5$

$x_2 = 6/5$

Unbounded solution
Big M & two Phase method

* Big M

$\text{Max } z = 3x_1 + 5x_2$

s.t. $x_1 - 2x_2 \leq 6$

$x_1 \le 10$

$x_2 \ge 1$; $x_1, x_2 \ge 0$

Standard LPP

$\text{Max } z = 3x_1 + 5x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 - Ma_1$

$x_1 - 2x_2 + S_1 = 6$

$x_1 + S_2 = 10$

$x_2 - S_3 + a_1 = 1$

all values ≥ 0

C_B	BV	x_1	x_2	S_1	S_2	S_3	a_1	X_B	
	$z \rightarrow$	-3	$-M-5$	0	0	M	0	$-M$	
0	S_1	1	-2	1	0	0	0	6	R_1
0	S_2	1	0	0	1	0	0	10	R_2
$\leftarrow -M$	a_1	0	$\boxed{1}$	0	0	-1	1	1	R_3
	z	-3	0	0	0	-5	5	5	
0	S_1	1	0	1	0	-2	2	8	
0	S_2	1	0	0	1	0	0	10	
5	x_2	0	1	0	0	-1	1	1	

we have entering variable but not have leaving variable

∴ unbounded solution.

* Two Phase

Auxiliary $\text{Min } z^* = a_1$

s.t. $x_1 - 2x_2 + S_1 = 6$

$x_1 + S_2 = 10$

$x_2 - S_2 + a_1 = 1$

all values ≥ 0

C_B	BV	x_1	x_2	S_1	S_2	S_3	a_1	X_B
	z	0	1	0	0	-1	0	1
0	S_1	1	-2	1	0	0	0	6
0	S_2	1	0	0	1	0	0	10
$\leftarrow 1$	a_1	0	$\boxed{1}$	0	0	-1	1	1

z	0	0	0	0	0	-1	0
S_1	1	0	1	0	-2	2	8
S_2	1	0	0	1	0	0	10
x_2	0	1	0	0	-1	1	1

$z^* = 0$ move to phase II

$a = 0$

$z_j - c_j \le 0$ ✗

	BV					~~a_1~~	X_B
	z	-3	0	0	0	-5	5
0	S_1	1	0	1	0	-2	8
0	S_2	1	0	0	1	0	10
5	x_2	0	1	0	0	-1	1

entering variable is available but leaving variable is not.

∴ unbounded solution.

"Dual of a Primal"

(Dual obtained from Primal)

LPP (Primal) ⟶ Dual (LPP)

$A \to A^T$

$(A^T)^T = A$

* Write Dual of a Primal

Primal (LPP) ⟶ Dual (LPP)

↓

Dual (Dual)

→ Write Primal into standard form.

↪ Standard Primal problem | Dual Problem

Standard Primal problem	Dual Problem
* Max $z = c_1x_1 + \dots + c_nx_n$	→ Min $z^* = b_1y_1 + \dots + b_my_m$
s.t. $AX = b$	s.t. $A^Ty \ge c^T$
$x \ge 0$	y – unrestricted of sign.
$A = (a_{ij})_{m\times n}$	$y = (y_1 \dots y_m)^T$
$b = (b_1 \dots b_m)^T$	
* Min $z = C^TX$	→ Max $z^* = b^Ty$
s.t. $AX = b$	s.t. $A^Ty \le c$
$X \ge 0$	y – unrestricted in sign

Remark :

a) for every Primal variable, there is a dual variable & constraints.

b) ~~for every~~ The coefficient of dual variables in the constraints are same as the coefficient of primal variables except that they are transposed.

c) no. PLPP constraints = no. of PLPP variables

d) The objective coeff. of PLPP variable become the RHS constrants of Dual constraints;

Standard Primal Objective → opt $z = C^TX$, $AX = b$; $X \ge 0$

	Objective	Dual constraints	variables
Maxi	Min	$\ge$	Unrestricted
Mini	Max	$\le$	Unrestricted

Q. Write dual of given Primal

Max $z = 5x_1 + 12x_2 + 4x_3$

s.t. $x_1 + 2x_2 + x_3 \le 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \ge 0$

Standard ↪ Max $z = 5x_1 + 12x_2 + 4x_3 + 0\cdot S_1$

s.t. $x_1 + 2x_2 + x_3 + S_1 = 10$ — y_1

$2x_1 - x_2 + 3x_3 = 8$ — y_2

Dual Min $z^* = 10y_1 + 8y_2$

s.t. ⟶

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} \qquad \boxed{A^T y \geq C}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 12 \\ 4 \\ 0 \end{bmatrix}$$

s.t.
$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 \geq 0$$

$y_2 \to$ unrestricted in sign

Q. Min $z = 15x_1 + 12x_2$

s.t. $x_1 + 2x_2 \geq 3$
$2x_1 - 4x_2 \leq 5$
$x_1, x_2 \geq 0$

standard → Min $z = 15x_1 + 12x_2 - 0 \cdot S_1 + 0 \cdot S_2$

s.t. $x_1 + 2x_2 - S_1 = 3$
$2x_1 - 4x_2 + S_2 = 5$

all value ≥ 0

Dual

↳ Max $z^* = 3y_1 + 5y_2$

s.t. $y_1 + 2y_2 \leq 15$
$2y_1 - 4y_2 \leq 12$
$-y_1 + 0 \cdot y_2 \leq 0$

$0 \cdot y_1 + y_2 \leq 0$

$\to y_1$ & $y_2 \to$ US.

$\therefore y_1 \geq 0$ & $y_2 \leq 0$

Q. Max $z = 5x_1 + 6x_2$

s.t. $x_1 + 2x_2 = 5$
$-x_1 + 5x_2 \geq 3$
$4x_1 + 7x_2 \leq 8$

x_1 - US ; $x_2 \geq 0$

↳ $x_1' - x_2'' \longrightarrow x_1' \geq 0 \quad x_2'' \geq 0$

standard Max $z = 5(x_1' - x_2'') + 6x_2$

s.t. $x_1' - x_2'' + 2x_2 = 5$
$-(x_1' - x_2'') + 5x_2 - S_1 = 3$
$4x_1 + 7x_2 + S_2 = 8$

all value ≥ 0.

Dual

↳ Min $z^* = 5y_1 + 3y_2 + 8y_3$

s.t. $y_1 - y_2 + 4y_3 \geq 5$
$-y_1 + y_2 - 4y_3 \geq -5$
$2y_1 + 5y_2 + 7y_3 \geq 6$
$-y_2 \geq 0 \Rightarrow y_2 \leq 0$
$+y_3 \geq 0$
y_1 - US.

Rules for Primal - Dual

Max	↔	Min
Constraints: $\geq$ $\leq$ $=$		Variables: $\leq$ $\geq$ Un-sign
Variables: ≥ 0 ≤ 0 Un-sing		Constraints: $\geq$ $\leq$

Weak Duality theorem

Consider Primal (LPP) Max $z = C^T X$

s.t. $AX = b$; $X \ge 0$

and Dual (LPP) Min $z^* = b^T W$

s.t. $A^T W \ge C$; $W \ge 0$

Let x be the feasible solⁿ of Primal & w be the feasible solⁿ of Dual then

$$\boxed{C^T X \le b^T W}$$

Remark - Primal - Min, Dual - Max then $\boxed{C^T X \ge b^T W}$

Find Dual variable value

Q. Find the optimal solution of Primal LPP

Max $z = 2x_1 + x_2$

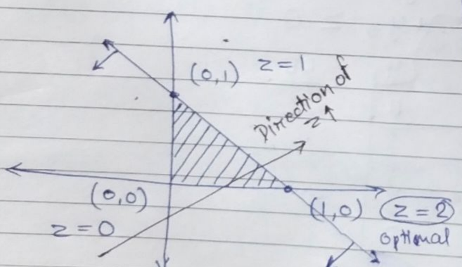
s.t. $x_1 + x_2 \le 1$

$x_1 \ge 0$

$x_2 \ge 2$

Write dual and find its optimal solution.

Sol. Primal



Dual LPP

Min $w = y_1$

$\left.\begin{array}{l} y_1 \ge 2 \\ y_1 \ge 1 \\ y \ge 0 \end{array}\right\} y_1 \ge 2$

$\boxed{w = 2}$

Both give same optimal solution.

To find Dual variable value ; we use $\boxed{y^T = C_B B^{-1}}$

Q. Write dual of given LPP (Primal)

Max $z = 4x_1 + 3x_2$

s.t. $x_1 + x_2 \le 8$

$2x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$

Solve Primal, & find optimal solution of dual.

Simplex Method - Max $z = 4x_1 + 3x_2 + 0.S_1 + 0.S_2$

s.t. $x_1 + x_2 + S_1 = 8$

$2x_1 + x_2 + S_2 = 10$

all values ≥ 0

$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $x_B = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$

		4 ↓	3	0	0	
	BV	x_1	x_2	S_1	S_2	x_B
	z	-4	-3	0	0	0
0	S_1	1	1	1	0	8
← 0	S_2	2	1 ↓	0	1	10
	z	0	-3	0	0	8
← 0	S_1	0	1	1	0	8
4	x_1	1	0	-1	1	2
	z	0	0	2	1	26
3	x_2	0	1	2	-1	6
4	x_1	1	0	-1	1	2

(B^{-1})

$z_j - c_j \ge 0$

$R_2 - R_1 \to R_2$

$2 - 1 \to 1$

$R_2 - 1 \to R_2$

$Z_{max} = 26$ at $(x_1 = 2,\ x_2 = 6)$

* Dual (LPP)

$$\text{Min } W = 8w_1 + 10w_2$$
$$\text{s.t.} \quad w_1 + 2w_2 \geq 4$$
$$w_1 + w_2 \geq 3$$
$$w_1, w_2 \geq 0$$

$$w^T = C_B B^{-1}$$

$$\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$w_1 = 2 \qquad w_2 = 1$

$$W = 8(2) + 10(1)$$
$$= 16 + 10 = \underline{\underline{26}} \text{ optimal solution}$$

Method 1 : $w^T = C_B B^{-1}$

Method 2 : Read directly from simplex table of Primal

Method 3 : Complementary slackness conditions. (CSC)

(2). ~~surplus~~ ~~$S_1 \to w_3 \to (-S)$~~

Q. LPP (Primal) $\text{Max } z = 4x_1 + 10x_2$

$$\text{s.t.} \quad 2x_1 + x_2 \leq 50$$
$$2x_1 + 5x_2 \leq 100 \qquad ; x_1, x_2 \geq 0$$
$$2x_1 + 3x_2 \leq 90$$

(i) solve by simplex method

(ii) find optimal solution of dual (LPP)

(iii) solve (ii) by using CSC.

Sol.

BV	x_1	x_2	S_1	S_2	S_3	X_B
z	0	0	0	2	0	200
x_1	1	0	5/8	-1/8	0	150/8
x_2	0	1	-1/4	1/4	0	25/2
S_3	0	0	-1/2	-1/2	1	15

Optimal solution

$Z_{max} = 200$ at

$$\left.\begin{aligned} x_1 &= \tfrac{150}{8} \\ x_2 &= 25/2 \\ S_3 &= 15 \end{aligned}\right\} BV \qquad \left.\begin{aligned} S_1 &= 0 \\ S_2 &= 0 \end{aligned}\right\} NBV$$

Dual (LPP)

$$\text{Min } W = 50y_1 + 100y_2 + 90y_3$$
$$\text{s.t.} \quad 2y_1 + 2y_2 + 2y_3 \geq 4$$
$$y_1 + 5y_2 + 3y_3 \geq 10$$
$$y_1, y_2, y_3 \geq 0$$

$$2y_1 + 2y_2 + 2y_3 - S_1' = 4$$
$$y_1 + 5y_2 + 3y_3 - S_2' = 10$$

all values ≥ 0

optimal sol^n Dual $\to (y_1\ y_2\ y_3\ S_1'\ S_2') = (y_1, y_2, 0, 0)$

Primal $\to (x_1\ x_2\ S_1\ S_2\ S_3) \to$ vanish

$\therefore\ 2y_1 + 2y_2 = 4$

$y_1 + 5y_2 = 10$

$\hookrightarrow y_1 = 0 \quad y_2 = 2$

Dual $\rightarrow (y_1, y_2, S_1', S_2'')$

$(0, 2, 0, 0)$

$\text{Min } w = 50(0) + 100(2)$

$= \underline{200}$

Q. Primal (LPP)

$\text{Max } z = 2x_1 + x_2$

s.t. $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1, x_2 \ge 0$

Dual (LPP)

$\text{Min } Y = 10y_1 + 6y_2 + 2y_3 + y_4$

s.t.

$y_1 + y_2 + y_3 + y_4 \ge 2$

$2y_1 + y_2 - y_3 - 2y_4 \ge 1$

$y_1, y_2, y_3, y_4 \ge 0$

Sol Dual $\quad \text{Min } Y = 10y_1 + 6y_2 + 2y_3 + y_4$

s.t.

$y_1 + y_2 + y_3 + y_4 - S_1 = 2$

$2y_1 + y_2 - y_3 - 2y_4 - S_2 = 1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & -2 & 0 & -1 \end{bmatrix}$$

$(B \ne I)$ $\quad \therefore$ introduce AV a_1, a_2

$\therefore\ y_1 + y_2 + y_3 + y_4 - S_1 + a_1 = 2$

$2y_1 + y_2 - y_3 - 2y_4 - S_2 + a_2 = 1$

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -1 & -2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Initial Basic Solution table

		10	6	2	4	0	0	0	0	0
	BV	y_1	y_2	y_3	y_4	S_1	S_2	a_1	a_2	X_B
	Y	10-3M	6-2M	2	4-M	-M	-M	M	M	-3M
M	a_1	1	1	1	+1	-1	0	1	0	2
M	a_2	2	1	-1	-2	0	-1	0	1	1

final table

BV	y_1	y_2	y_3	y_4	S_1	S_2	a_1	a_2	X_B
Y	-2	0	0	-1	-4	-2	x	x	10
y_3	-1/2	0	1	3/2	-1/2	1/2	x	x	1/2
y_2	3/2	1	0	-1/2	-1/2	-1/2	x	x	3/2

$y_3\ y_2 \Rightarrow$ BV

Optimal $Y_{min} = 10$ at

$(y_1, y_2, y_3, y_4, S_1, S_2) = (0, 3/2, 1/2, 0, 0)$

Primal sol

$\hookrightarrow (x_1, x_2, S_1, S_2, S_3, S_4)$

$= (x_1, x_2, S_1, 0, 0, S_4)$

$\therefore\ \text{Max } Z = 2x_1 + x_2$

st. $x_1 + 2x_2 + S_1 = 10$

$x_1 + x_2 = 6$ — (I)

$x_1 - x_2 = 2$ — (II)

$x_1 - 2x_2 + S_4 = 1$

by ① & ⑪ $x_1 = 4$ $x_2 = 2$

↳ $s_1 = 2$ $s_4 = 1$

∴ optimal solⁿ of Primal

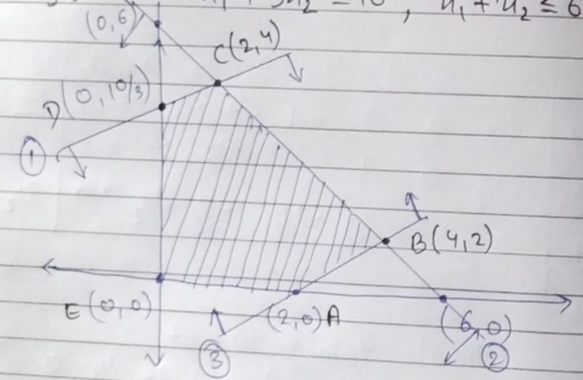
z_{max} at $(4, 2, 0, 0, 0, 1)$

$z_{max} = 2(4) + 2 = \underline{\underline{10}}$

Q. Solve the given LPP by graphical method & then apply CSC to find optimal solution of dual.

Min $Z = -x_1 + 2x_2$; $x_1, x_2 \geq 0$

s.t. $-x_1 + 3x_2 \leq 10$; $x_1 + x_2 \leq 6$; $x_1 - x_2 \leq 2$



Optimal solⁿ $z_{min} = -2$ at $A(2,0)$

∴ $-x + 3x_2 + S_1 = 10$

$x_1 + x_2 + S_2 = 6$

$x_1 - x_2 + S_3 = 2$

$\left(\begin{array}{l} x_1 = 2 \\ x_2 = 0 \end{array} \right)$

BV

$(x_1 \ x_2 \ S_1 \ S_2 \ S_3) = (2, 0, 12, 4, 0)$

NBV

Dual Max $y = -10y_1 - 6y_2 - 2y_3$

s.t. $y_1 - y_2 - y_3 \leq 1$

$-3y_1 - y_2 + y_3 \leq 2$; $y_1, y_2, y_3 \geq 0$

Dual optimal solⁿ $(\cancel{y_1} \ \cancel{y_2} \ y_3 \ \cancel{s_1} \ S_2)$

BV

from Primal → $(x_1 \ x_2 \ \cancel{S_1} \ S_2 \ S_3)$

NBV

$(y_1 \ y_2 \ y_3 \ s_1 \ S_2) = (0, 0, y_3, 0, S_2)$

↳ substitute it in syst of equatⁿ of Dual to get y_3 &

∴ $(0, 0, 1, 0, 1)$

Max $y = \underline{\underline{-2}}$.

Dual Simplex Method

In Dual Simplex Method, every table is optimal and we try to achieve feasibility.

When in the current table optimality & feasibility both achieve then this is the stopping criteria.

Q. use dual simplex method

Max $z = -2x_1 - x_2$

s.t. $2x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Sol^n Max $z = -2x_1 - x_2 + 0\cdot S_1 + 0\cdot S_2$

$2x_1 - x_2 - x_3 - S_1 = 3$

$x_1 - x_2 + x_3 - S_2 = 2$

all values ≥ 0.

$(B.V = S_1 \; S_2)$ $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

multiply by (-1) $\quad B^{-1} = -I$

$\therefore$ Max $z = -2x_1 - x_2$

s.t. $-2x_1 + x_2 + x_3 + S_1 = -3$

$-x_1 + x_2 - x_3 + S_2 = -2$

all values ≥ 0
variables

optimal

CB	BV	x_1	x_2	x_3	S_1	S_2	X_B
	z	2	1	0	0	0	0
	S_1	$\boxed{-2}$	1	1	1	0	$-3 \rightarrow$
	S_2	-1	1	-1	0	1	-2

feasibility is disturbed

we 1st remove X_{Bi} with most (-ve) value. & enter the variable which maintain optimality.

$\longrightarrow \min \left\{ \left| \frac{z_j - c_j}{\lambda_{ij}} \right| ; \lambda_{ij} < 0 \right\}$

BV	x_1	x_2	x_3	S_1	S_2	X_B
z	0	2	1	1	0	-3
x_1	1	$-1/2$	$-1/2$	$-1/2$	0	$3/2$
S_2	0	$1/2$	$\boxed{-3/2}$	$-1/2$	1	$-1/2 \rightarrow$
z	0	$5/3$	0	$2/3$	$2/3$	$-10/3$
x_1	1	$-2/3$	0	$-2/3$	$-1/3$	$5/3$
x_3	0	$-1/3$	1	$1/3$	$-2/3$	$1/3$

feasible

$R_1 \rightarrow R_1 - 2\left(\frac{R_2}{-2}\right)$

$R_2 \rightarrow R_2 / -2$

$R_3 \rightarrow R_3 + \left(\frac{R_2}{-2}\right)$

optimal sol^n $z_{max} = -\frac{10}{3}$

at $\left(x_1 = \frac{5}{3}, x_2 = 0, x_3 = \frac{1}{3}\right)$

Q. use dual simplex method to solve

Max $-x_1$

st. $x_1 - x_2 \geq 3$ $\quad x_1, x_2 \geq 0$

$-x_1 + x_2 \geq 4$

$\rightarrow$ infeasible solution.

BV	x_1	x_2	S_1	S_2	X_B
z	1	0	0	0	0
S_1	-1	1	1	0	-3
S_2	1	$\boxed{-1}$	0	1	$-4 \rightarrow$
z	1	0	0	0	
S_1	0	0	1	1	$-7 \rightarrow$
x_2	-1	1	0	-1	4

current table is optimal but not feasible because we do not find an entering variable.