"Integer Programming Problems" (IPP)

Optimize (Min / Max) $z = c_1 x_1 + \dots + c_n x_n$

S.t. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le, \ge, = b_1$

$a_{11}x_1 + \dots + a_{1n}x_n \le, \ge, = b_2$

$a_{m1}x_1 + \dots + a_{mn}x_n \le, \ge, = b_m$

$x_1, x_2, x_3, \dots x_n \ge 0$

& $x_1, x_2, x_3, \dots x_n$ are integer (some or all)

* Types of IPP

1. Pure IPP when $x_i \in \mathbb{Z} \; \forall i$.
2. Mixed IPP $x_i \in \mathbb{Z}$ for some i
3. zero-one IPP $x_i \in \mathbb{Z} \; \forall i$; $x_i = \{0, 1\}$

* Methods to solve IPP

1. Branch Bound tech. → IPP with 2 decision variable
2. Fractional Cut Method (Gomory's constraints) → (2 or more than 2 decision variables)

Branch & Bound Technique

two decision variable.

LPP : opt $z = C^T X$

$AX = , \ge , \le b$

$X \ge 0$

IPP : add x_1, x_2 are integer.

Step 1: Solve the associated LPP with graphical method.

If decision variables are integer then it is the optimal sol^n for IPP too, if not move to step 2.

Step 2: $x_1 \notin Z$, $x_2 \notin Z$ or x_1 & $x_2 \notin Z$

Do the branching with respect to any decision variables ~~are~~ ~~integers~~ ~~then~~ having non-integeral value.

$x_1 = 5/2$ $x_2 = 4$

Suppose $x_1 \notin Z$; $x_1 \in$ Rational no.

$5/2 \to 2.5$

$x \le 2$

$(x_1, x_2) \xrightarrow{z} x_1 \le 2$ $(0, 1, 2)$

$(x_1, x_2) \xrightarrow{\text{other possibility}} x_2 \ge 3$ $(3, 4)$

Step 3: Repeat step 2, until all DV are integers.

Q. Max $z = 7x_1 + 9x_2$

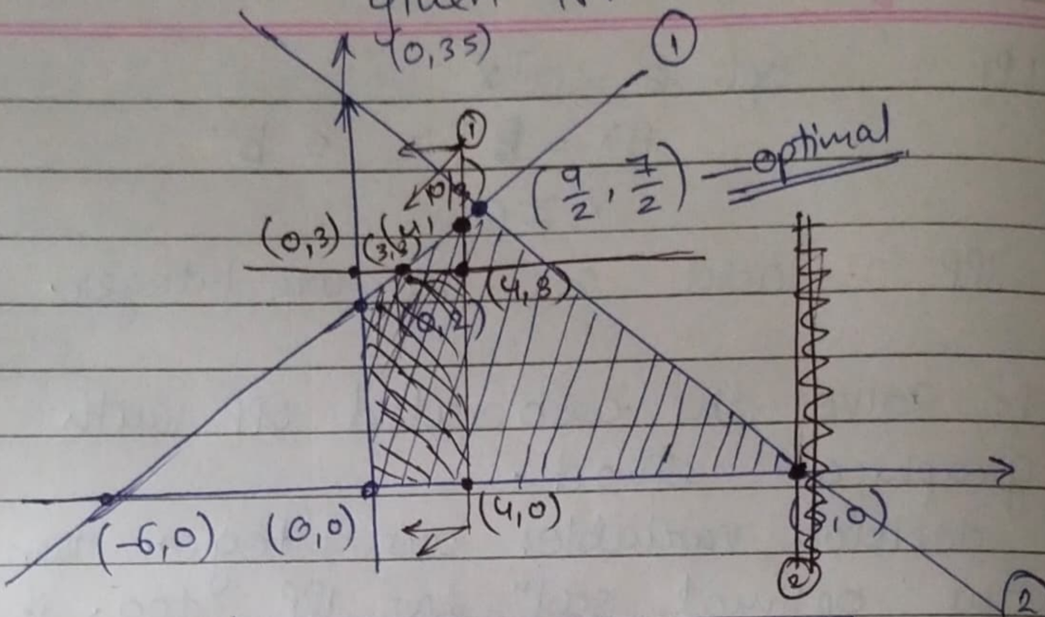
s.t. $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_1, x_2 \ge 0$ $x_1, x_2 \in Z$

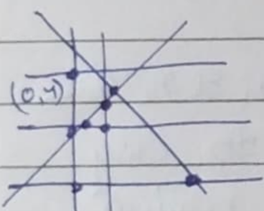
IPP.

Sol. (1) Sol LPP associate with given IPP



$$Z = \left(\frac{9}{2}\right) + 9\left(\frac{7}{2}\right) = 63$$

$$x_1 = \frac{9}{2}, \quad x_2 = \frac{7}{2}$$



$x_1 = \frac{9}{2} = 4.5$

$x_1 \le 4$ $\qquad$ $x_1 \ge 5$

(1)

$Z = 58$

$x_1 = 4;\ x_2 = \frac{10}{3}$

(2)

$Z = 35$

$x_1 = 5;\ x_2 = 0$

$\downarrow$

$x_1, x_2 \in \mathbb{Z}$

$x_2 = \frac{10}{3} = 3.3$

$x_2 \le 3$ $\qquad$ $x_2 \ge 4$

(3)

$Z = 55$

$x_1 = 4;\ x_2 = 3$

No feasible Solution.

$\uparrow$

Optimal solution of IPP

$Z_{max} = 55$ at

$x_1 = 4, \quad x_2 = 3$

Remark 1. $Z_{max}(LPP) \geq Z_{max}(IPP)$

2. Consider a situation;

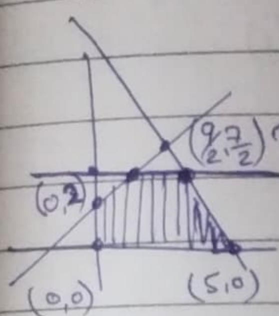
$\Longrightarrow$ feathomed of z.

stop opening branches when we already get optimal solution.

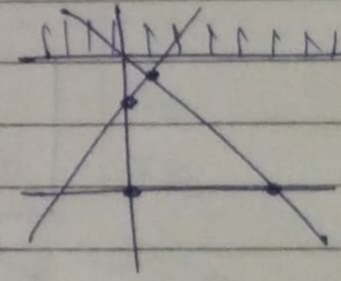
② Alternate of ① solⁿ.

$$\boxed{\begin{array}{c} Z = 63 \\ x = \frac{9}{2}\ ;\ x_2 = \frac{7}{2} \end{array}}$$

$x_2 = \frac{7}{2} = 3.5$



$x_2 \leq 3$; $x_2 \geq 4$



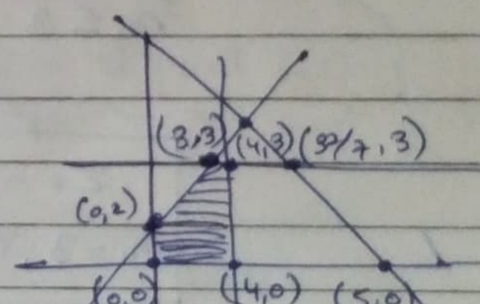
$$\boxed{\begin{array}{c} Z = 59 \\ x_1 = \frac{32}{7}\ ;\ x_2 = 3 \end{array}} \qquad \boxed{\begin{array}{c} \text{no feasible} \\ \text{Solution} \end{array}}$$

$x_1 \leq 4$; $x_1 \geq 5$

$$\boxed{\begin{array}{c} Z = 55 \\ x_1 = 4\ ;\ x_2 = 3 \end{array}} \qquad \boxed{\begin{array}{c} Z = 35 \\ x_1 = 5\ ;\ x_2 = 0 \end{array}}$$

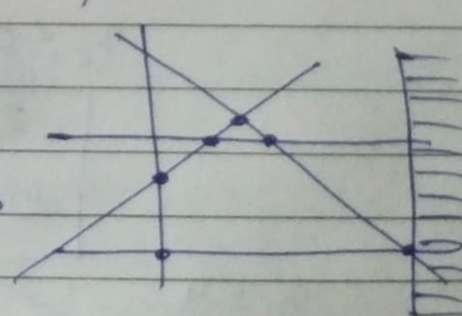
$\uparrow$

Optimal solution



$Z_{max} = 55$

at $x_1 = 4$; $x_2 = 3$



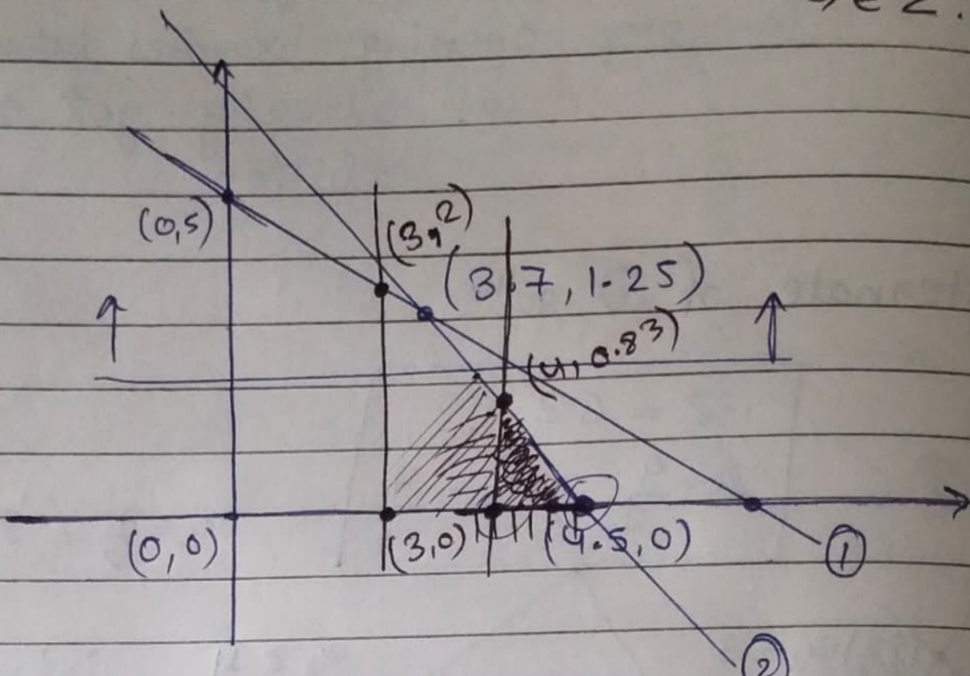
eg. $\text{Min } z = -5x_1 - 4x_2$

s.t. $x_1 + x_2 \le 5$

$10x_1 + 6x_2 \le 45$

$x_1, x_2 \ge 0$

$\hookrightarrow \in \mathbb{Z}$.



$z = -23.75$
$x_1 = 3.7 ; x_2 = 1.25$

$x_1 \le 3$ | $x_1 \ge 4$

$z = -23$
$x_1 = 3 ; x_2 = 2$

$z = -23.33$
$x_1 = 4 ; x_2 = 0.83$

$x_2 \le 0$ | $x_2 \ge 1$

$z = -22.5$
$x_1 = 4.5 ; x_2 = 0$

No feasible solution

$x_1 \le 4$ | $x_1 \ge 5$

$z = -20$
$x_1 = 4 ; x_2 = 0$

$\hookrightarrow$ optimal.

No feasible solution

* Gomoary's Cut Method (Procedure)

Solve IPP

→ Solve LPP ~~unit~~ by Simplex method

→ check Opt. soln $\in \mathbb{Z}$ or not.

→ if not, add a constraint

↳ Gomoary's constraint

→ Use Dual Simplex method.

C_B	BV	x_1	x_2	x_3	x_4	X_B
	Z	$Z_{11}-C_{11}$	$Z_{12}-C_{12}$	$Z_{13}-C_{13}$	$Z_{14}-C_{14}$	Z_0
	x_1	y_{11}	y_{12}	y_{13}	y_{14}	X_{B_1}
	x_2	y_{21}	y_{22}	y_{23}	y_{24}	X_{B_2}

$X_{B_2}, X_{B_1} \notin \mathbb{Z}$

fractional part ≥ 0

$$X_{B1} = I_1 + F_1$$
$$X_{B2} = I_2 + F_2$$

choose the largest fractional value

$= \text{Max}\{F_1, F_2\}$ let say F_2 (largest)

$$y_{21}x_1 + y_{22}x_2 + y_{23}x_3 + y_{24}x_4 = X_{B_2}$$

$$(I_{21}+F_{21})x_1 + (I_{22}+F_{22})x_2 + (I_{23}+F_{23})x_3 + (I_{24}+F_{24})x_4 = I_2 + F_2$$

separate fractional part)

$$F_{21}x_1 + F_{22}x_2 + F_{23}x_3 + F_{24}x_4 = F_2$$

constraint equation $\boxed{F_2 - F_{21}x_1 - F_{22}x_2 - F_{23}x_3 - F_{24}x_4 \leq 0}$

Q. Max $z = x_1 + 2x_2$

s.t. $x_1 + x_2 \leq 7$

$2x_1 \leq 11$ $\qquad x_1, x_2 \geq 0$

$2x_2 \leq 7$ $\qquad \hookrightarrow \in Z$

Using Simplex method find optimal soln of associated LPP.

$$\text{Max } z = x_1 + 2x_2 + 0\cdot S_1 + 0\cdot S_2 + 0\cdot S_3$$

s.t. $x_1 + x_2 + S_1 = 7$

$2x_1 + S_2 = 11$

$2x_2 + S_3 = 7$

all variables ≥ 0

$R_3 \to \frac{R_3}{2}$

C_B	BV	x_1	x_2	S_1	S_2	S_3	X_B
	z	-1	-2	0	0	0	0
0	S_1	1	1	1	0	0	7
0	S_2	2	0	0	1	0	11
← 0	S_3	0	2	0	0	1	7
	z	-1	0	0	0	1	7
←	S_1	$\boxed{1}$	0	1	0	$-1/2$	$7/2$
	S_2	2	0	0	1	0	11
	x_2	0	1	0	0	$1/2$	$7/2$
	z	0	0	1	0	$1/2$	$21/2$
	x_1	1	0	1	0	$-1/2$	$7/2$
	S_2	0	0	-2	1	1	4
	x_2	0	1	0	0	$1/2$	$7/2$

Optimal $\quad z_{max} = 21/2$

$x_1 = \frac{7}{2} \quad x_2 = \frac{7}{2} \quad x_3 = 0$

$\hookrightarrow \notin Z$

$$x_1 = \frac{7}{2} = 3 + \frac{1}{2} = I + f$$
$$x_2 = \frac{7}{2} = 3 + \frac{1}{2} = I + f$$
$\} \geq 0$

→ 3^{rd} constraints − $0 \cdot x_1 + \cancel{1 \cdot x_2} + 0 \cdot S_1 + 0 \cdot S_2 + \frac{1}{2} \cdot S_3 = \frac{7}{2}$

not considered.

$$\frac{1}{2} S_3 = \cancel{3} + \frac{1}{2}$$

not considered

$$\frac{1}{2} - \frac{1}{2} S_3 \leq 0$$ → Gomoary's constraints.

$$\boxed{-\frac{1}{2} S_3 \leq -\frac{1}{2}}$$

$$\boxed{-\frac{1}{2} S_3 + x_{G_1} = -\frac{1}{2}}$$

→ optimal

BV	x_1	x_2	S_1	S_2	S_3	x_{G_1}	X_B
Z	0	0	1	[illegible]	1/2	0	21/2
x_1	1	0	1	0	−1/2	0	7/2
S_2	0	0	−2	1	1	0	4
x_2	0	1	0	0	1/2	0	7/2
x_{G_1}	0	0	0	0	−1/2	1	−1/2 →

(Dual simplex method)

BV	x_1	x_2	S_1	S_2	S_3	x_{G_1}	X_B
Z	0	0	1	0	0	1	10
x_1	1	0	1	0	0	−1	4
S_2	0	0	−2	1	0	2	3
x_2	0	1	0	0	0	1	3
S_3	0	0	0	0	1	−2	1

Optimal $Z_{max} = 10$ $x_1 = 4$

$x_2 = 3$

Q. Max $z = x_1 + 4x_2$

s.t. $2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ & $\in Z$.

Sol.

Simplex method on LPP associated with IPP.

Optimal simplex table is

C_B	BV	x_1	x_2	s_1	s_2	X_B
	z	1	0	1	0	7
4	x_2	$1/2$	1	$1/4$	0	$7/4$
→ 0	s_2	$7/2$	0	$-3/4$	1	$39/4$
		$\hookrightarrow 3+\frac{1}{2}$		$\hookrightarrow -1+\frac{1}{4}$		

Optimal $Z_{max} = 7$

at $x_1 = \frac{7}{4}$; $x_2 = \frac{39}{4} \notin Z$

#. $x_1 = \frac{7}{4} = 1 + \underset{f_1}{\frac{3}{4}}$ $\qquad x_2 = 9 + \underset{f_2}{\frac{3}{4}}$

$= \text{Max}[f_1, f_2] = \frac{3}{4}\ (s_2)$

Construct Gomoary's constraint for $\frac{3}{4}$

$X_{G_1} = -f_2 + f_{21} \times x_1 + f_{22} x_2 + f_{23} s_1 + f_{24} s_2$

$= -\frac{3}{4} + \frac{1}{2} x_1 + 0 + \frac{1}{4} s_1 + 0$

$X_{G_1} = -\frac{3}{4} + \frac{1}{2} x_1 + \frac{1}{4} s_1$

$$-\frac{3}{4} + \frac{1}{2}x_1 + \frac{1}{4}s_1 - x_{G_1} = 0$$

$-\frac{3}{4} \times \frac{2}{7}$ $\quad -\frac{3}{14}$ $\quad -3/4 \times 2$ $\quad -\frac{1}{2} \times (-2)$ $\quad -\frac{1}{4}(-2)$

$$\boxed{-\frac{1}{2}x_1 - \frac{1}{4}S_2 + X_{G_1} = -\frac{3}{4}} \quad (-1)$$

add in table

CB	BV	x_1	x_2	S_1	S_2	X_{G_1}	X_B
	Z	1	0	1	0	0	7
	x_2	1/2	1	1/4	0	0	7/4
	S_2	7/2	0	-3/4	1	0	39/4
	X_{G_1}	-1/2	0	-1/4	-1/4	1	-3/4

infeasible → remove.

BV	x_1	x_2	S_1	S_2	X_{G_1}	X_B
z	0	0	1/2	0	2	11/2
x_2	0	1	0	0	1	1
S_2	0	0	-5/2	1	7	9/2
x_1	1	0	1/2	0	-2	3/2

optimal $Z_{max} = 11/2$

at $x_1 = \frac{3}{2}$ $\quad S_2 = 9/2$ $\rightarrow \notin Z$

$x_2 = 1$

\# $x_1 = \frac{3}{2} = 1 + \frac{1}{2}$

$S_2 = \frac{9}{2} = 4 + \frac{1}{2}$ $\qquad \max\left[\frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2}$ $(x_1$

$$X_{G_2} = \frac{1}{2} + \left(0 \cdot x_1 + 0 \cdot x_2 + \frac{1}{2}S_1 + 0 \cdot S_2 + 0 \cdot X_{G_1}\right)$$

$$X_{G_2} = \frac{1}{2} + \frac{1}{2}S_1$$

$$\boxed{-\frac{1}{2} = -\frac{1}{2}S_1 + X_{G_2}}$$

C_B	BV	x_1	x_2	s_1	s_2	x_{G_1}	x_{G_2}	X_B
	z	0	0	1/2	0	2	0	11/2
4	x_2	0	1	0	0	1	0	1
0	s_2	0	0	-5/2	1	7	0	9/2
1	x_1	1	0	1/2	0	-2	0	3/2
0	x_{G_2}	0	0	-1/2	0	0	1	-1/2 →

$-\frac{1}{2}(-2) \to 1$

C_B	BV	x_1	x_2	S_1	S_2	X_{G_1}	X_{G_2}	X_B
	z	0	0	0	0	2	1	5
4	x_2	0	1	0	0	1	0	1
0	S_2	0	0	0	1	7	-5	7
1	x_1	1	0	0	0	-2	1	1
0	S_1	0	0	1	0	0	-2	1

optimal $Z_{max} = 5$

at $x_1 = 1$ $x_2 = 1$