Take Home

1. The implicit midpoint rule is

$$y_{n+1} = y_n + hf\left(t_n + \frac{1}{2}h, \frac{1}{2}\left(y_n + y_{n+1}\right)\right) \tag{1}$$

Define the error $e_n = y(t_n) - y_n$. By Taylor's theorem

$$y(t_{n+1}) = y\left(t_n + \frac{1}{2}h + \frac{1}{2}h\right) = y\left(t_n + \frac{1}{2}h\right) + \frac{1}{2}hy'\left(t_n + \frac{1}{2}h\right) + \mathcal{O}(h^2) \tag{2}$$

$$y(t_n) = y\left(t_n + \frac{1}{2}h - \frac{1}{2}h\right) = y\left(t_n + \frac{1}{2}h\right) - \frac{1}{2}hy'\left(t_n + \frac{1}{2}h\right) + \mathcal{O}(h^2) \tag{3}$$

Therefore

$$y(t_{n+1}) = y(t_n) + hy'\left(t_n + \frac{1}{2}h\right) + \mathcal{O}(h^2) \tag{4}$$

$$y\left(t_n + \frac{1}{2}h\right) = \frac{1}{2}\left(y(t_n) + y(t_{n+1})\right) + \mathcal{O}(h^2) \tag{5}$$

It follows that

$$\begin{aligned}
|e_{n+1}| &= |y(t_{n+1}) - y_{n+1}| \\
&= \left|y(t_n) + hy'\left(t_n + \frac{1}{2}h\right) - y_n - hf\left(t_n + \frac{1}{2}h, \frac{1}{2}\left(y_n + y_{n+1}\right)\right)\right| + \mathcal{O}(h^2) \\
&\le |e_n| + h\left|f\left(t_n + \frac{1}{2}h, y\left(t_n + \frac{1}{2}h\right)\right) - f\left(t_n + \frac{1}{2}h, \frac{1}{2}\left(y_n + y_{n+1}\right)\right)\right| + \mathcal{O}(h^2)
\end{aligned}$$

Use the Lipschitz condition for the convergence:

$$\begin{aligned}
&\left|f\left(t_n + \frac{1}{2}h, y\left(t_n + \frac{1}{2}h\right)\right) - f\left(t_n + \frac{1}{2}h, \frac{1}{2}\left(y_n + y_{n+1}\right)\right)\right| \le \lambda\left|y\left(t_n + \frac{1}{2}h\right) - \frac{1}{2}\left(y_n + y_{n+1}\right)\right| \\
&\le \frac{1}{2}\lambda\left|y(t_n) + y(t_{n+1}) - \frac{1}{2}\left(y_n + y_{n+1}\right)\right| + \mathcal{O}(h^2) \le \frac{1}{2}\lambda\left(|e_n| + |e_{n+1}|\right)
\end{aligned}$$

Consequently,

$$|e_{n+1}| \le |e_n| + \frac{1}{2}h\lambda\left(|e_n| + |e_{n+1}|\right) + \mathcal{O}(h^2) \tag{6}$$

$$|e_{n+1}| \le \gamma\,|e_n| + \mathcal{O}(h^2) \le \gamma\,|e_n| + Ah^2 \quad \text{where} \quad \gamma = \frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda} \tag{7}$$

for some A large enough. Since $|e_0| = |y(t_0) - y_0| = 0$, then by induction we have

$$\begin{aligned}
&|e_1| \le Ah^2, \quad |e_2| \le \gamma Ah^2 + Ah^2 \\
&|e_3| \le \left(\gamma^2 + \gamma + 1\right)Ah^2 \\
&\quad \cdots \\
&|e_n| \le \left(\gamma^{n-1} + \cdots + \gamma + 1\right)Ah^2 = \frac{\gamma^n - 1}{\gamma - 1}Ah^2
\end{aligned}$$

We now claim that

$$\frac{\gamma^n - 1}{\gamma - 1} A h^2 \to 0 \quad \text{as} \quad n \to \infty \quad \text{where} \quad h = \frac{T - t_0}{n}. \tag{8}$$

To see this, we recall for $x > 0$ that

$$1 + x \leq 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x \tag{9}$$

and for $0 < x < \frac{1}{2}$ that

$$\frac{1}{1-x} = 1 + \frac{x}{1-x} \leq 1 + 2x \leq e^{2x} \tag{10}$$

Since $h \to 0$ we may assume $h\lambda \leq 1$ and conclude $\gamma^n \leq \left(e^{h\lambda/2} e^{h\lambda}\right)^2 = e^{3(T-t_0)\lambda/2}$. Now

$$\gamma - 1 = \frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda} - 1 = \frac{h\lambda}{1 - \frac{1}{2}h\lambda} \quad \text{implies} \quad \frac{1}{\gamma - 1} = \frac{1 - \frac{1}{2}h\lambda}{h\lambda} \tag{11}$$

Therefore,

$$|e_n| \leq \frac{\gamma^n - 1}{\gamma - 1} A h^2 \leq \left(e^{3(T-t_0)\lambda/2} - 1\right) \frac{1 - \frac{1}{2}h\lambda}{h\lambda} A h^2 \leq \left(e^{3(T-t_0)\lambda/2} - 1\right) \frac{1}{\lambda} A h \to 0 \tag{12}$$

as $h \to 0$ and $n \to \infty$. This shows the implicit midpoint rule is convergent. $\square$

2. Set

$$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})) \tag{13}$$

Since $y'(t_i) = f(t_i, y(t_i))$, we can write equation (13) as

$$y(t_{i+1}) = y(t_i) + ahy'(t_i) + bhy'(t_{i-1}) + chy'(t_{i-2}) \tag{14}$$

Expanding both sides of (14) in Taylor series about t_i, we obtain

$$y(t_i) + hy'(t_i) + \frac{1}{2}h^2 y''(t_i) + \frac{1}{6}h^3 y'''(t_i) + O\left(h^4\right) \tag{15}$$

$$= y(t_i) + ahy'(t_i) + bh\left(y'(t_i) - hy''(t_i) + \frac{1}{2}h^2 y'''(t_i) + O\left(h^3\right)\right) \tag{16}$$

$$+ ch\left(y'(t_i) - 2hy''(t_i) + \frac{4}{2}h^2 y'''(t_i) + O\left(h^3\right)\right) \tag{17}$$

$$= y(t_i) + (a+b+c)hy'(t_i) + (-b-2c)h^2 y''(t_i) + \left(\frac{1}{2}b + 2c\right)h^3 y'''(t_i) + O\left(h^4\right) \tag{18}$$

We obtain

$$1 = a + b + c \tag{19}$$

$$\frac{1}{2} = -b - 2c \tag{20}$$

$$\frac{1}{6} = \frac{1}{2}b + 2c \tag{21}$$

which gives $a = \frac{23}{12}, b = -\frac{16}{12}, c = \frac{5}{12}$. Plugging these into (13), we obtain

$$y(t_{i+1}) = y(t_i) + \frac{h}{12}\left[23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2}))\right] + O\left(h^4\right) \tag{22}$$

or

$$w_{i+1} = w_i + \frac{h}{12}\left[23f\left(t_i, w_i\right) - 16f\left(t_{i-1}, w_{i-1}\right) + 5f\left(t_{i-2}, w_{i-2}\right)\right] \tag{23}$$

The order of the local truncation for the Adams-Bashforth three-step explicit method is $\tau(h) = O\left(h^3\right)$. $\square$