Math 478/578 Midterm Spring '25, Take-home Part

1. (5 points)

   (a) Find the local truncation error of the implicit midpoint rule, which is given by Eq. (1.12) in the textbook by Iserles.

   (b) Prove the convergence of the midpoint rule method.

2. (5 points)

   Derive, in detail, the three-step Adams-Bashforth method.

3. (20 points, Computer Problem)

   Consider the following system of differential equations

$$y_1' = -2y_1 + y_2 + 2\sin t, \tag{1}$$
$$y_2' = -1.999y_1 + 0.999y_2 + 0.999(\sin t - \cos t). \tag{2}$$

   Write a program which solves the initial value problem (1) and (2) with the initial condition $(y_1(0), y_2(0)) = (2, 3.999)$ by the 2-step Adams-Bashforth method and the trapezoidal rule. Plot the solutions against the exact solutions on the same graph for each method on the time interval $[0, 5]$. What's the largest time step you can choose for each method to ensure the relative error at $t = 5$ is less than 0.01%? Also plot the error as function of time for each method for these time-step sizes. The exact solution to the problem is $y_1 = e^{-t} + e^{-0.001t} + \sin t$, $y_2 = e^{-t} + 1.999e^{-0.001t} + \cos t$.

   Computer programming requirement: make sure your programs are correct, accurate and efficient.