Homework 2 - Questions

1. Determine whether the natural cubic spline that interpolates the table

x	0	1	2	3
y	1	1	0	10

is or is not the function

$$f(x) = \begin{cases} 1 + x - x^3 & x \in [0, 1] \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & x \in [1, 2] \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3 & x \in [2, 3]. \end{cases}$$

Answer. We calculate the second derivatives as follows.

$$\begin{aligned} f_1'(x) &= 1 - 3x^2 & f_1''(x) &= -6x \\ f_2'(x) &= 16 - 30x + 12x^2 & f_2''(x) &= 24x - 30 \\ f_3'(x) &= -68 + 54x - 9x^2 & f_3''(x) &= -18x + 54 \end{aligned}$$

Requirement 1: $f_1''(0) = f_3''(3) = 0$, so the natural boundary condition holds.

Requirement 2: $f_1''(1) = f_2''(1) = -6$ and $f_2''(2) = f_3''(2) = 18$.

Requirement 3: $f_1'(1) = f_2'(1) = -2$ and $f_2'(2) = f_3'(2) = 4$.

Requirement 4: $f_1(1) = f_2(1) = 1$ and $f_2(2) = f_3(2) = 0$.

Requirement 5: $f(x)$ is a cubic polynomial on each subinterval.

All 5 requirements are satisfied. Hence, $f(x)$ is a natural cubic spline. $\square$

2. Find the natural cubic spline function whose knots are $-1, 0$, and 1 and that takes the values $S(-1) = 13, S(0) = 7, S(1) = 9$.

Answer. Suppose the natural cubic spline function has the form

$$S(x) = \begin{cases} S_1(x), & x \in [-1, 0] \\ S_2(x), & x \in [0, 1], \end{cases}$$

where $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$, and $S_i'(x) = 3a_i x^2 + 2b_i x + c_i$, and $S_i''(x) = 6a_i x + 2b_i$ for $i = 1, 2$.

We have 8 unknowns: a_i, b_i, c_i, d_i for $i = 1, 2$. We write the equations that express the desired continuity and interpolation properties of the spline. First, the interpolation requires:

$$\begin{aligned} S_1(-1) = 13 \quad &\Longrightarrow \quad -a_1 + b_1 - c_1 + d_1 = 13 \\ S_1(0) = 7 \quad &\Longrightarrow \quad d_1 = 7 \\ S_2(0) = 7 \quad &\Longrightarrow \quad d_2 = 7 \\ S_2(1) = 9 \quad &\Longrightarrow \quad a_2 + b_2 + c_2 + d_2 = 9 \end{aligned}$$

From the continuity requirement of the derivatives at $x = 0$, we obtain

$$S_1'(0) = S_2'(0) \quad \Longrightarrow \quad c_1 = c_2$$

$$S_1''(0) = S_2''(0) \implies b_1 = b_2.$$

Finally, the "natural" end conditions give us

$$\begin{aligned} S_1''(-1) = 0 &\implies -6a_1 + 2b_1 = 0 \\ S_2''(1) = 0 &\implies 6a_2 + 2b_2 = 0 \end{aligned}$$

As a result, we obtain 8 linear equations. We solve them for 8 unknowns and obtain $a_1 = a_2 = \frac{4}{3}, \quad b_1 = b_2 = 4,$ $c_1 = c_2 = \frac{-10}{3}, \quad d_1 = d_2 = 7.$ □

3. Show how to use Richardson extrapolation if $L = \phi(h) + a_1 h + a_3 h^3 + a_5 h^5 + ...$

Answer. We compute the function L for the step size $\frac{h}{2}$:

$$L = \phi\left(\frac{h}{2}\right) + a_1 \frac{h}{2} + a_3 \left(\frac{h}{2}\right)^3 + ...$$

Then, we multiply this equation by 2 and subtract the given equation in the question:

$$2L - L = 2\phi\left(\frac{h}{2}\right) - \phi(h) - \frac{3}{4} a_3 h^3 - \frac{15}{16} a_5 h^5 + ...$$

We obtain the first extrapolation as

$$L_1 = 2\phi\left(\frac{h}{2}\right) - \phi(h) + O(h^3).$$

For the n-th extrapolation step, we take 2^{2n+1} as the multiplier and obtain the n-th extrapolation function:

$$L_n = \frac{2^{2n+1} L_{n-1}(\frac{h}{2}) - L_{n-1}(h)}{2^{2n+1} - 1}.$$

Each step cancels the next error term of odd degree. We can recursively eliminate the terms of higher odd degrees. □

4. Using Taylor series expansions, derive the error term for the formula $f''(x) \approx \frac{1}{h^2}[f(x) - 2f(x+h) + f(x+2h)]$.

Answer. By using Taylor series expansions, we derive

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(\xi_1) \tag{1}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(\xi_2) \tag{2}$$

If we add up equations (1) and (2), we obtain

$$f(x-h) - 2f(x) + f(x+h) = h^2 f''(x) + \frac{h^4}{12} f^{(4)}(\xi)$$

for some $\xi \in (x-h, x+h)$. This yields

$$f''(x) = \frac{1}{h^2}[f(x-h) - 2f(x) + f(x+h)] - \frac{h^2}{12} f^{(4)}(x) + O(h^4).$$

Taking the position $x+h$ as the center instead of x, we obtain

$$\frac{1}{h^2}[f(x) - 2f(x+h) + f(x+2h)] = f''(x+h) + \frac{h^2}{12} f^{(4)}(x+h) + O(h^4). \tag{3}$$

By the Taylor expansion of the second derivative $g(x) = f''(x)$, the deviation from x to $x+h$ expands as

$$\begin{aligned} f''(x+h) = g(x+h) &= g(x) + g'(x)h + \frac{h^2}{2}g''(x) + O(h^3) \\ &= f''(x) + hf'''(x) + \frac{h^2}{2}f^{(4)}(x) + O(h^2). \end{aligned} \tag{4}$$

Inserting (3) into (4) gives us

$$\frac{1}{h^2}[f(x) - 2f(x+h) + f(x+2h)] = f''(x) + hf'''(x) + \frac{7h^2}{12}f^{(4)}(x) + O(h^3),$$

and so we find the error term as

$$-hf'''(x) - \frac{7h^2}{12}f^{(4)}(x) + O(h^3).$$

□

5. **Coding Project**

   Using the US population Census data, do the following:

   (a) Determine the interpolation polynomial for these data.

   (b) Determine a cubic spline for these data.

   (c) Using both results, to answer the questions: what is the estimated US population on Jan. 1st, 2005? Which estimate do you think makes more sense?