1 Hamiltonian Formulation

The Hamiltonian $H(\theta)$ is a 4x4 arrowhead matrix defined as:

$$H(\theta) = \begin{pmatrix} E_0(\theta) & c & c & c \\ c & E_1(\theta) & 0 & 0 \\ c & 0 & E_2(\theta) & 0 \\ c & 0 & 0 & E_3(\theta) \end{pmatrix} \tag{1}$$

where $E_0(\theta) = \hbar\omega + \sum_{i=0}^{2} V_x^{(i)}(\theta)$ and $E_{i+1}(\theta) = E_0(\theta) + V_a^{(i)}(\theta) - V_x^{(i)}(\theta)$ for $i = 0, 1, 2$. Here ω is a frequency parameter, $c = 0.2$ is a fixed coupling constant, and V_x and V_a are potential terms that depend on the parameter vector $\boldsymbol{R}(\theta)$. These potential terms are defined as follows:

$$V_x^{(i)}(R_i) = a \cdot (R_i)^2 + b \cdot R_i + c \tag{2}$$

$$V_a^{(i)}(R_i) = a \cdot (R_i - x_{\text{shift}})^2 + c \tag{3}$$

2 R_θ generation

The R_θ vector traces a perfect circle orthogonal to the $x = y = z$ line using the `create_perfect_orthogonal_vectors` function from the Arrowhead/generalized package.

3 Berry Connection

The Berry connection $A(\theta)$ is calculated using:

$$A_n(\theta_j) = \langle n(\theta_j) | i\partial_\theta | n(\theta_j) \rangle \tag{4}$$

where $|n(\theta_j)\rangle$ are the eigenstates of $H(\theta_j)$.

4 Berry Phase

The Berry phase γ_n for state n is obtained by integrating the Berry connection:

$$\gamma_n = \int_0^{2\pi} A_n(\theta) d\theta \tag{5}$$

5 Verification

We verify the eigenvalue equation $H(\theta)|n(\theta)\rangle = E_n(\theta)|n(\theta)\rangle$ by comparing:

$$H(\theta)|n(\theta)\rangle \quad \text{vs} \quad E_n(\theta)|n(\theta)\rangle \tag{6}$$

6 Visualization

For each state n, we plot:

- Magnitude of $H(\theta)|n(\theta)\rangle$ and $E_n(\theta)|n(\theta)\rangle$
- Real and imaginary components separately
- All four vector components