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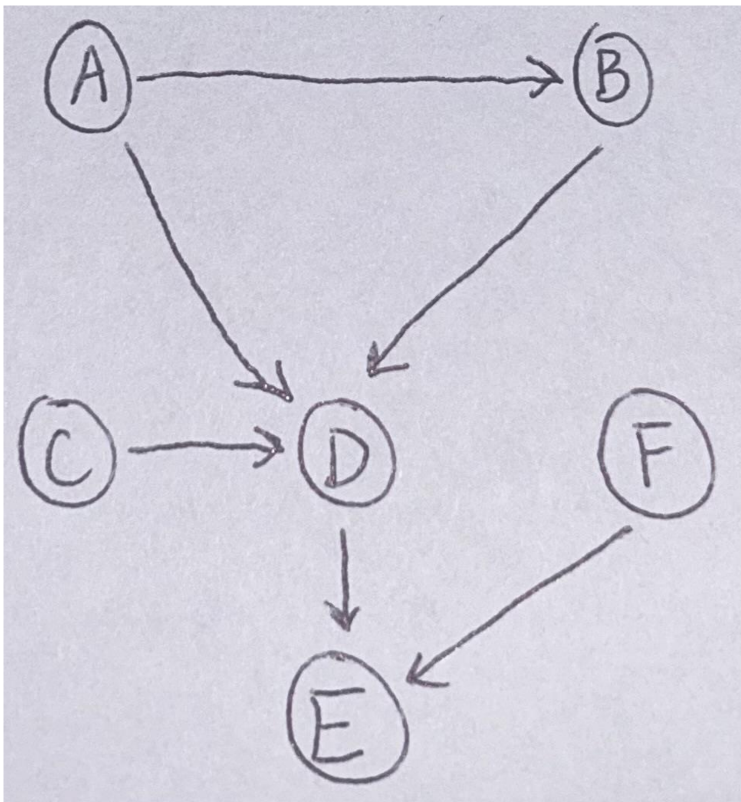
COMP-5600

Assignment 3: Probabilistic Graphical Models

1. $P(A, B, C, D, E, F) = P(A) \cdot P(C) \cdot P(E|A) \cdot P(B|A, C) \cdot P(D|B, C) \cdot P(F|D)$

2. 

3. (Part 1)

$$P(+u \mid +e) = \frac{P(+e \wedge +u)}{P(+e)}$$

$$P(+e \wedge +u) = \sum_{i,h,t} P(+e \mid +t \wedge +u) \cdot P(+t \mid i) \cdot P(+u \mid i \wedge h) \cdot P(i) \cdot P(h)$$

$$P(+e) = \sum_{i,h,t,u} P(+e \mid t \wedge e) \cdot P(t \mid i) \cdot P(u \mid i \wedge h) \cdot P(i) \cdot P(h)$$

sum:

$$f_1(+u \mid +e) \propto \sum_{i,h,t} P(+e \mid +t \wedge +u) \cdot P(t \mid i) \cdot P(+u \mid i \wedge h) \cdot P(i) \cdot P(h)$$

$$f_2(-u \mid +e) \propto \sum_{i,h,t} P(+e \mid t \wedge -u) \cdot P(-u \mid i \wedge h) \cdot P(i) \cdot P(h)$$

$$P(+u \mid +e) = \frac{f_1}{f_1 + f_2}$$

$$P(-u \mid +e) = \frac{f_2}{f_1 + f_2}$$

$$f_1(+u \mid +e) \propto \sum_{i,h,t} P(+e \mid t \wedge +u) \cdot P(t \mid i) \cdot P(+u \mid i \wedge h) \cdot P(i) \cdot P(h)$$

$$f_{1\,+i \wedge +h} = P(+i) \cdot P(+h) \cdot (P(+e \mid +t \wedge +u) \cdot P(+t \mid +i) \cdot P(+u \mid +i \wedge +h) + P(+e \mid -t \wedge +u) \cdot P(-t \mid +i) \cdot P(+u \mid +i \wedge +h))$$

$$= 0.7 \cdot 0.6 \cdot (0.9 \cdot 0.8 \cdot 0.9 + 0.7 \cdot 0.2 \cdot 0.9) = \underline{0.32508}$$

$$f_{1\,+i \wedge -h} = P(+i) \cdot P(-h) \cdot (P(+e \mid +t \wedge +u) \cdot P(+t \mid +i) \cdot P(+u \mid +i \wedge -h) + P(+e \mid -t \wedge +u) \cdot P(-t \mid +i) \cdot P(+u \mid +i \wedge -h))$$

$$= 0.7 \cdot 0.4 \cdot (0.9 \cdot 0.8 \cdot 0.3 + 0.7 \cdot 0.2 \cdot 0.3) = \underline{0.07224}$$

$$f_{1\,-i \wedge +h} = P(-i) \cdot P(+h) \cdot (P(+e \mid +t \wedge +u) \cdot P(+t \mid -i) \cdot P(+u \mid -i \wedge +h) + P(+e \mid +t \wedge +u) \cdot P(-t \mid -i) \cdot P(+u \mid -i \wedge +h))$$

$$= 0.3 \cdot 0.6 \cdot (0.9 \cdot 0.5 \cdot 0.5 + 0.7 \cdot 0.5 + 0.5) = \underline{0.072}$$

$$f_{1\,-i \wedge -h} = P(-i) \cdot P(-h) \cdot (P(+e \mid +t \wedge +u) \cdot P(+t \mid -i) \cdot P(-u \mid -i \wedge -h) + P(+e \mid -t \wedge +u) \cdot P(-t \mid -i) \cdot P(+u \mid -i \wedge -h))$$

$$= 0.3 \cdot 0.4 \cdot (0.9 \cdot 0.5 \cdot 0.1 + 0.7 \cdot 0.5 \cdot 0.1) = \underline{0.0096}$$

$$f_1(+u \mid +e) \propto 0.32508 + 0.07224 + 0.072 + 0.0096 = \boxed{0.47892}$$

$$f_{2\,+i \wedge +h} = P(+i) \cdot P(+h) \cdot (P(+e \mid +t \wedge -u) \cdot P(+t \mid +i) \cdot P(-u \mid +i \wedge +h) + P(+e \mid -t \wedge -u) \cdot P(-t \mid +i) \cdot P(-u \mid +i \wedge +h))$$

$$= 0.7 \cdot 0.6 \cdot (0.5 \cdot 0.8 \cdot 0.1 + 0.3 \cdot 0.2 \cdot 0.1) = \underline{0.01932}$$

$$f_{2\,+i \wedge -h} = P(+i) \cdot P(-h) \cdot (P(+e \mid +t \wedge -u) \cdot P(+t \mid +i) \cdot P(-u \mid +i \wedge -h) + P(+e \mid -t \wedge -u) \cdot P(-t \mid +i) \cdot P(-u \mid +i \wedge -h))$$

$$= 0.7 \cdot 0.4 \cdot (0.5 \cdot 0.8 \cdot 0.7 + 0.3 \cdot 0.2 \cdot 0.7) = \underline{0.09016}$$

$$f_{2\,-i \wedge +h} = P(-i) \cdot P(+h) \cdot (P(+e \mid +t \wedge -u) \cdot P(+t \mid -i) \cdot P(-u \mid -i \wedge +h) + P(+e \mid -t \wedge -u) \cdot P(-t \mid -i) \cdot P(-u \mid -i \wedge +h))$$

$$= 0.3 \cdot 0.6 \cdot (0.5 \cdot 0.5 \cdot 0.9 + 0.3 \cdot 0.5 \cdot 0.9) = \underline{0.036}$$

$$f_{2\,-i \wedge -h} = P(-i) \cdot P(-h) \cdot (P(+e \mid +t \wedge -u) \cdot P(+t \mid -i) \cdot P(-u \mid -i \wedge -h) + P(+e \mid -t \wedge -u) \cdot P(-t \mid -i) \cdot P(-u \mid -i \wedge -h))$$

$$= 0.7 \cdot 0.6 \cdot (0.5 \cdot 0.8 \cdot 0.1 + 0.3 \cdot 0.2 \cdot 0.1) = \underline{0.0432}$$

$$f_2(+u \mid +e) \propto = 0.01932 + 0.09016 + 0.036 + 0.0432 = \boxed{0.18868}$$

Normalize:

$$f_1(+u\,|+e) + f_2(+u\,|+e) = 0.47892 + 0.18868 = 0.6676$$

$\therefore$

$$P(+u\,|+e) = \frac{f_1(+u\,|+e)}{f_1(+u\,|+e) + f_2(-u\,|+e)} = \frac{0.47892}{0.6672} \approx 0.717$$

$$P(-u\,|+e) = \frac{f_1(-u\,|+e)}{f_1(+u\,|+e) + f_2(-u\,|+e)} = \frac{0.18868}{0.66676} \approx 0.283$$

$$\boxed{\text{Answer} = 0.717}$$

(Part 2)

(a) **T and U are independent.** False. T (good Test taker) and U (Understands material) are not independent because both are directly or indirectly dependent on the nodes I (Intelligent) and H (Hardworking). T is affected by I, and U is affected by both I and H.
(b) **T and U are conditionally independent given I, E, and H.** True. Given the variables I (Intelligent) and H (Hardworking), U (Understands material) is fully determined, and the relationship between T (good Test taker) and U is severed. Including E (high Exam score), which T and U jointly influence, does not change the conditional independence that arises by knowing I and H.
(c) **T and U are conditionally independent given I and H.** True. We have already established in part (b) that T and U are conditionally independent given I and H. Knowing I and H provides all the information needed to sever the dependency between T and U because the paths between them are "blocked" by the conditioning on their common causes.
(d) **E and H are conditionally independent given U.** False. E (high Exam score) is conditional on T (good Test taker) and U (Understands material), and since H (Hardworking) influences U, there exists an indirect path from E to H even when U is known, through the dependency of U on I (Intelligent), which also affects T. Therefore, knowing U does not block all paths between E and H due to this potential correlation through I.
(e) **E and H are conditionally independent given U, I, and T.** True. When we condition on both U (Understands material) and I (Intelligent), which influences U, and also on T

(good Test taker), we block all paths from E (high Exam score) to H (Hardworking), as the only path between them was U and I which are now known.
(f) **I and H are conditionally independent given E.** False. I (Intelligent) influences both T (good Test taker) and U (Understands material), which both influence E (high Exam score). H (Hardworking) influences U. Therefore, since both I and H affect U, which then influences E, I and H are not conditionally independent given E.
(g) **I and H are conditionally independent given T.** True. Given T (good Test taker), the link from I (Intelligent) to E (high Exam score) is "blocked," and since H (Hardworking) affects only U (Understands material), there is no active path between I and H given T. Therefore, I and H can be considered conditionally independent given T.
(h) **T and H are independent.** False. T (good Test taker) is influenced by I (Intelligent), and H (Hardworking) influences U (Understands material), which is also influenced by I. Since there is a common cause (I), there exists an indirect relationship between T and H.
(i) **T and H are conditionally independent given E.** False. Although E (high Exam score) depends on both T (good Test taker) and U (Understands material), conditioning on E does not "block" the path between T and H that goes through I (Intelligent) and U. That means the condition on E creates a collider (V-structure) at E, which, when conditioned on, actually opens up the path between T and H through U, given their common connection through I.
(j) **T and H are conditionally independent given E and U.** True. By conditioning on U (Understands material), we have blocked all paths from H (Hardworking) to T (good Test taker) since H only affects T through U, and now U is known. Additionally, whilst conditioning on E introduces an active path between all its parents due to being a collider, including U as a given variable blocks that path. Therefore, there is no active path left between T and H, making them conditionally independent given E and U.