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Beyond volatility targeting

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Should the weight be $1/\sigma$ or $1/\sigma^2$?

In a past research note (*Understanding volatility targeting*, 4 October 2011) we have analysed the performance of simple asset allocation strategies that combine a passive index with cash. The weight on the index is inversely proportional to its volatility, in such a way that the volatility of the strategy itself is kept constant. In this document we study a more general set of strategies built so as to have a weight on the passive index equal to $1/\sigma^\gamma$. The goal of our analysis is to understand what determines the optimal choice of γ that maximises the Sharpe ratio in the long term. In particular, it would be interesting to establish under which conditions volatility targeting (γ=1) is optimal in this sense.

The role of alpha, index volatility and kurtosis

A simple theoretical model which allows for time varying volatility provides some interesting insights. We find that the optimal choice depends on three parameters. The higher the expected return of the passive index, the lower the optimal exponent γ. The higher the average volatility of the index, the higher the optimal solution. Finally, the optimal γ is an increasing function of the volatility of volatility.

In practice the optimum is close to $1/\sigma^2$

When we plug realistic parameter values into our formula we invariably find that the highest Sharpe ratio is obtained by dividing by a factor that is close to the variance, not by the volatility.

Is volatility targeting optimal?

Introduction

Volatility targeting is used to stabilise the volatility of a strategy…

Volatility targeting strategies (VTSs) have recently received considerable attention, partly due to the encouraging performance of low volatility strategies. A VTS can be thought of as simple asset allocation strategy that combines a passive index with cash. The weight on the index is inversely proportional to its volatility, so that the volatility of the strategy itself is kept constant at its target level.

…and tends to have positive impact on risk adjusted performance

We argued in a recent research note (*Understanding volatility targeting strategies*, 4 October 2011) that the Sharpe ratio of such a strategy should be higher than that of the passive index. A series of backtests, presented in the original note, suggest that this is indeed the case in the long term for all the stock market indices that we have considered.

Here we consider a generalisation of volatility targeting

Here we analyse a more general set of asset allocation strategies, in which the weight on the passive index can be proportional to any power of the index volatility, including the variance. Intuitively, choosing a weight that is proportional to a *high* power of σ would be effective in reducing volatility when the passive index is highly volatile. However, it would also result in highly leveraged positions when index volatility is particularly low.

The next sections formalise the intuition by deriving in a simple theoretical model the optimal balance between curbing volatility and stabilising weights.

The model

Cash and a risky asset (index) are available

Two assets are available, a passive index and cash. The latter earns a constant risk free return *r*. We consider a one factor stochastic volatility model of the returns to the index, which can be written as

$$\begin{aligned} y_t &= \mu + \sigma_t \varepsilon_t \quad \text{where} \qquad (1)\\ \sigma_t &\equiv \sigma_* e^{h_t/2}, \\ h_t &= \varphi h_{t-1} + \eta_t \end{aligned}$$

We derive the results in a standard stochastic volatility model

Here y_t is the index return, σ_t is its volatility,[1] ε_t is a random variable (i.i.d. standard normal) which represents *transitory* shocks to volatility. The process h_t represents the fluctuations of the log variance $\log \sigma_t^2$ around its long term mean $\log \sigma_*^2$. The innovation to h_t, which represents a *permanent* volatility shock, is i.i.d. and normally distributed: $\eta_t \sim N(0, q)$. σ_*^2, φ and q are constants. We assume $0 < \varphi < 1$ in order to impose stationarity.

The unconditional mean of *h* is zero, while its unconditional variance is given by

$$\sigma_h^2 \equiv Var(h_t) = \frac{q}{1-\varphi^2}.$$

[1] To be precise, conditional on σ_t the volatility of y_t is σ_t.

This model is the workhorse of the vast literature on stochastic volatility.

Choosing the weights

The weight on the risky index is proportional to volatility raised to the power of γ

Call w_t the weights on the index, so that the strategy returns are $w_t y_t + (1 - w_t) r$.

We use weights that are inversely proportional to the volatility raised to the power of γ:

$$w_t \equiv \frac{\bar{\sigma}}{\sigma_t^{\gamma}} \tag{2}$$

where $\bar{\sigma}$ is a constant target level.

It is important to bear in mind that we are implicitly assuming that volatility is known when the portfolio is rebalanced. We have shown in our original research note that, in practice, one can plug a volatility forecast from a simple model into (2) in order to keep volatility stable over time .

γ=0 gives a passive strategy, γ=1 is volatility targeting…

… and γ=2 corresponds to ***dividing by the variance***

Different choices of γ result in different volatility driven strategies. Setting $\gamma = 1$ corresponds to volatility targeting as defined in our original paper, i.e. it ensures that the conditional volatility of $w_t y_t$ is constant over time. If $\gamma = 0$ then the weight is constant, i.e. we hold the passive index and cash in fixed proportions. Finally, if γ is set equal to 2 then the weights are inversely proportional to the variance of the index.

The expectation of the excess return to the index is $\mu - r \equiv \mu_*$.

Choosing the target volatility level

To compute the moments of the VTS we need to set a target volatility level in (2), $\bar{\sigma}$. A natural choice is $\bar{\sigma} = \sigma_*^{\gamma} \exp\left(-\frac{\sigma_h^2}{8}\gamma^2\right)$ which implies that the expected weight on the index, $E(w_t)$, is equal to one for any value of γ. In other words, we are on average holding the index. The exponential in the formula can be thought of as a *convexity correction*, which is necessary because the model is nonlinear. In fact, Jensen's inequality implies that without the correction we would have

$$E(w_t) = E\left(\frac{\bar{\sigma}}{\sigma_t^{\gamma}}\right) = E\left(e^{-\gamma h_t/2}\right) > e^{-\gamma E(h_t)/2} = 1.$$

This is true regardless of the normality assumption, however the specific convexity correction used above depends on the assumption that h_t is Gaussian. In any case, the Sharpe ratio does not depend on the choice of $\bar{\sigma}$ because (2) is linear in $\bar{\sigma}$ and therefore the term disappears when we compute the ratio of mean to standard deviation.

The optimal strategy

Computing expected return...

The expected return of the strategy can be easily calculated:

$$E(w_t(y_t - r)) = \mu_*.$$

It does not depend on γ, which means that all the strategies considered in the analysis have the same expected return, which is also the expected return of the passive index.

...and variance

On noting that ε_t and h_s are independent for any *s,t* we can use the properties of the lognormal distribution to show that the variance of the returns to our strategy is given by[2]

$$Var(w_t(y_t - r)) = \mu_*^2\left[\exp\left(\frac{\sigma_h^2}{4}\gamma^2\right) - 1\right] + \sigma_*^2 \exp\left(\frac{\sigma_h^2}{4}\left(2(1-\gamma)^2 - \gamma^2\right)\right). \tag{3}$$

The variance accounts for the uncertainty on future weights

Expression (3) is made up of two terms. The former accounts for the impact of the variability of weights on the variance of returns. As we argued in the introduction, the higher γ is, the more volatile the resulting weights. In practice, however, this term is negligible for realistic parameter values.

The latter term in (3) represents the contribution to the variance of the product $w_t \sigma_t \varepsilon_t$. It is worth noting that this term is minimised by choosing $\gamma = 2$.

It is easy to show that the expressions found in our original work are special cases of (3) with $\gamma = 0$ (the passive index) and $\gamma = 1$ (volatility targeting).

Computing the Sharpe ratio...

We are now in a position to compute the (squared) Sharpe ratio:

$$SR(\gamma)^2 = E(w_t(y_t - r))^2 / Var(w_t(y_t - r)) .$$

...and optimising

Since the numerator is independent of γ, in order to maximise the Sharpe ratio we can look for the value of γ which minimises the denominator:

$$\min_{\gamma} Var(w_t(y_t - r)).$$

Note that the function is bounded from below (as it is nonnegative) and diverges as $\gamma \to \infty$ or $\gamma \to -\infty$. We can set the first derivative to zero and analyse the solutions. This yields the equation

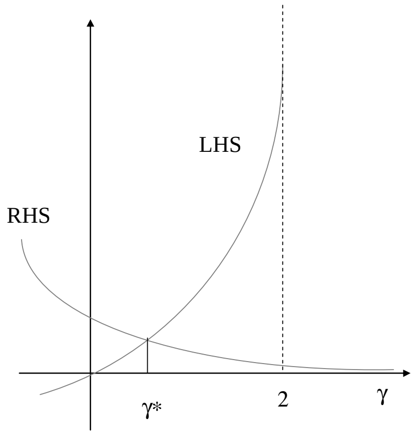
$$\frac{\gamma}{2-\gamma}\frac{\mu_*^2}{\sigma_*^2} = \exp\left(\sigma_h^2\left(\frac{1}{2} - \gamma\right)\right). \tag{4}$$

The value γ that solves (4) maximises the Sharpe ratio. It can be computed numerically given the parameter values. However, a graphical illustration can

[2] Details of the rather tedious algebra are available upon request.

help understanding how the expected return and volatility of the passive index (plus the volatility of its volatility) are related to the optimal strategy.

Figure 1: Solving for the optimal parameter gamma



Source: UBS quant.

The factors that determine the optimal solution

The two sides of the equation are depicted in Figure 1. The optimal value of γ, found at the point where LHS=RHS, always lies between 0 (the passive strategy) and 2 (dividing by the variance). A simple comparative statics exercise shows that:

1. The passive index is never optimal because the solution can never occur at $\gamma = 0$
2. The steeper the trend μ_*, the lower the optimal γ (increasing μ_* moves the curve LHS upwards for $\gamma > 0$)
3. The higher the long term volatility σ_*, the higher the optimal γ (increasing σ_* moves the curve LHS downwards for $\gamma > 0$)
4. The more volatile volatility is (i.e. the higher σ_h^2), the higher the optimal γ (increasing σ_h^2 moves the curve RHS upwards).

Weights close to 1/σ² are optimal in most cases

We plug realistic parameter values into the formula

Alternatively, the same result can be expressed (after dividing by $\exp\left(\sigma_h^2 / 2\right)$ and rearranging) as

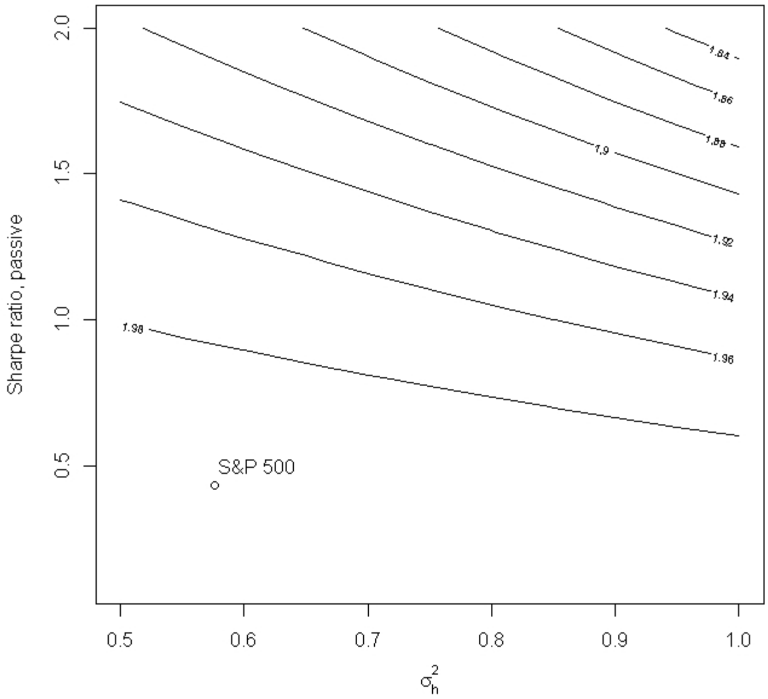
$$SR^{*2} = n\frac{\gamma e^{-\sigma_h^2 \gamma}}{2-\gamma}$$

where SR* is the annualised Sharpe ratio of the passive index and *n* the number of subperiods in a year (e.g. *n*=52 for a weekly model).

Because the expression depends only the passive Sharpe ratio and the volatility of volatility parameter σ_h, it is possible to illustrate the optimal choice of γ in a two-dimensional plot (Figure 2 for daily data, Figure 3 for weekly).

The parameter σ_h^2 is typically estimated to be between 0.5 and one for equity indices. If, for example, it were equal to 0.6 then assuming a median volatility of 20% would imply that the first and third quartiles of the stationary distribution of volatility are 15.4% and 26%.

Figure 2: Optimal choice of γ, daily data



Source: UBS quant. Sharpe ratios are annualised. The long term estimates for S&P 500, used in our original note *Understanding volatility targeting*, are 0.43 for the Sharpe ratio and 0.576 for the stationary variance of *h* (the point shown in the figure). The corresponding optimal choice of γ from solving (4) is 1.99.

The optimal γ is typically close to 2

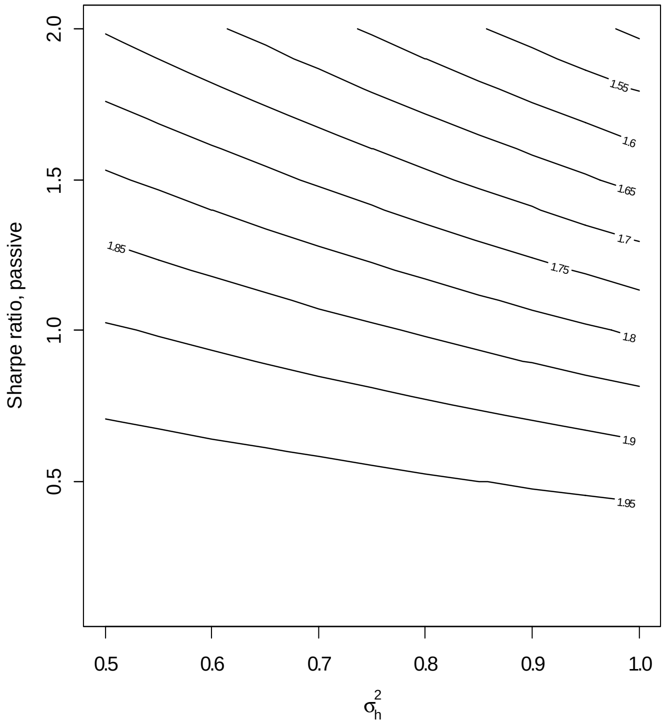
The optimal solution seems to be affected more by the Sharpe ratio than by the fourth moment of the distribution. What is striking is that, for any realistic value of the parameters, γ is not far from 2. This can be explained by the fact that the first term in (3) is typically negligible while the second term,

$$\sigma_*^2 \exp\left(\frac{\sigma_h^2}{4}\left(2(1-\gamma)^2-\gamma^2\right)\right), \tag{5}$$

is minimised at $\gamma = 2$. The two terms involving γ in (5) have a straightforward interpretation: The former is related to the variance of the term $\sigma_t^{1-\gamma}\varepsilon_t$ (i.e. the unpredictable component of return $\sigma_t\varepsilon_t$ divided by σ_t^{γ}) while the latter stems from the convexity correction. Setting $\gamma = 1$ (i.e. volatility targeting) minimises the first term as the volatility is kept constant at the target, but it also results in a larger value of the second term compared to the case where $\gamma = 2$. Dividing by the variance increases convexity and this has a positive effect on the Sharpe ratio.

Assuming weekly data (Figure 3) does not change the result significantly.

Figure 3: Optimal choice of γ, weekly data



Source: UBS quant. Sharpe ratios are annualised.

The improvement in Sharpe ratio compared to the passive strategy

Expression (5) can be used to gauge the improvement in Sharpe ratio that is obtained by selecting the optimal γ. The variance is approximately equal to $\sigma_*^2 \exp\left(\frac{\sigma_h^2}{2}\right)$ for the passive index, $\sigma_*^2 \exp\left(-\frac{\sigma_h^2}{4}\right)$ for volatility targeting and $\sigma_*^2 \exp\left(-\frac{\sigma_h^2}{2}\right)$ for $\gamma = 2$. Hence, assuming that σ_h^2 is between 0.5 and one we conclude that volatility targeting improves the Sharpe ratio compared to the passive index by a factor of 13 to 28%, while dividing by the variance results in an improvement between 20 and 45%.

Conclusion

The strategy that maximises risk adjusted performance is typically obtained for γ=2

We considered a set of simple asset allocation rules that can be viewed as a generalisation of VTSs. In particular, we allow the weight on the passive index to be inversely proportional to volatility raised to the power of γ, a free parameter. We then asked what is the optimal choice of γ, i.e. the one which results in the maximum Sharpe ratio. The answer is that, for realistic parameter values, the exponent should always be slightly below 2. In other words, weights that are roughly proportional to the variance (not to the standard deviation, i.e. the volatility) of the index seem to be optimal.

The factors that determine the optimal solution

The analysis also gives some guidance as to which factors determine the optimal weighting scheme. We find that the solution always lies between zero (holding the passive index) and 2 (dividing by the variance). It will be closer to 2 when the expected return to the passive index is low, when the average volatility of the index is high or when the uncertainty surrounding future volatility is high.

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UBS 12-Month Rating	Rating Category	Coverage[1]	IB Services[2]
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Neutral	**Hold/Neutral**	40%	35%
Sell	**Sell**	9%	15%
UBS Short-Term Rating	**Rating Category**	**Coverage[3]**	**IB Services[4]**
Buy	**Buy**	less than 1%	25%
Sell	**Sell**	less than 1%	17%

1:Percentage of companies under coverage globally within the 12-month rating category.
2:Percentage of companies within the 12-month rating category for which investment banking (IB) services were provided within the past 12 months.
3:Percentage of companies under coverage globally within the Short-Term rating category.
4:Percentage of companies within the Short-Term rating category for which investment banking (IB) services were provided within the past 12 months.

Source: UBS. Rating allocations are as of 31 March 2012.

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UBS 12-Month Rating	Definition
Buy	FSR is > 6% above the MRA.
Neutral	FSR is between -6% and 6% of the MRA.
Sell	FSR is > 6% below the MRA.
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