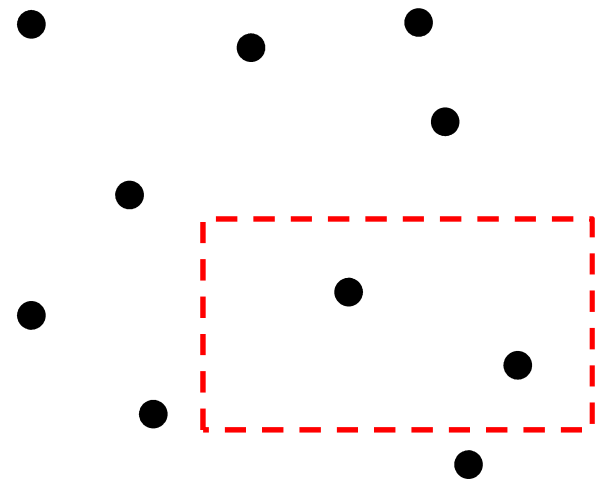


# 2D Range Query

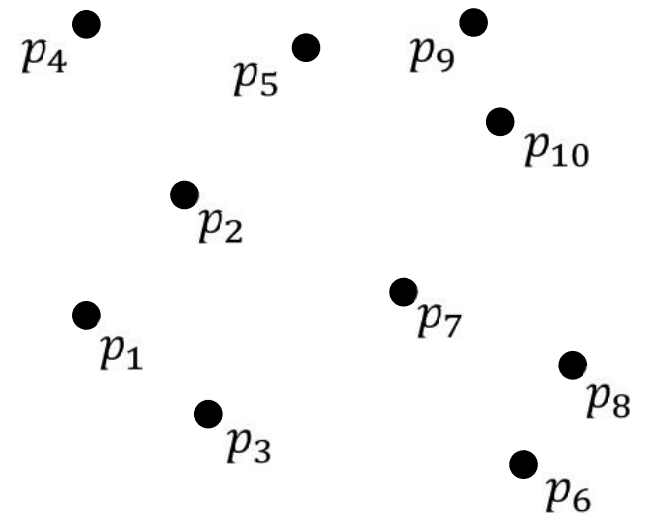
KD-tree and Range tree

# 2D Orthogonal Range Query

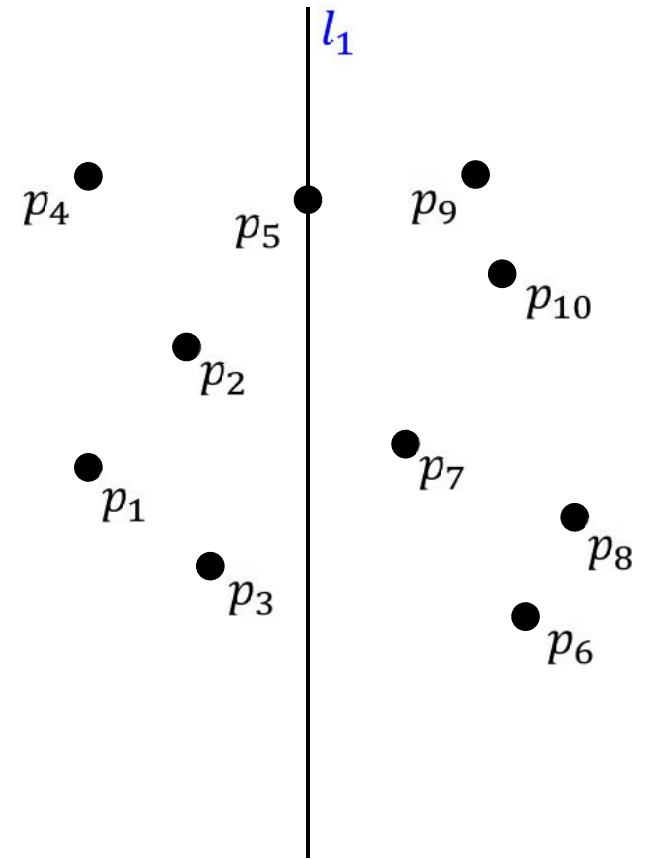
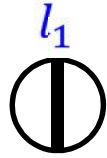
- Data: A set  $S$  of  $n$  points
- Query: Report/Count subset of  $S$  that lie in a rectangle range



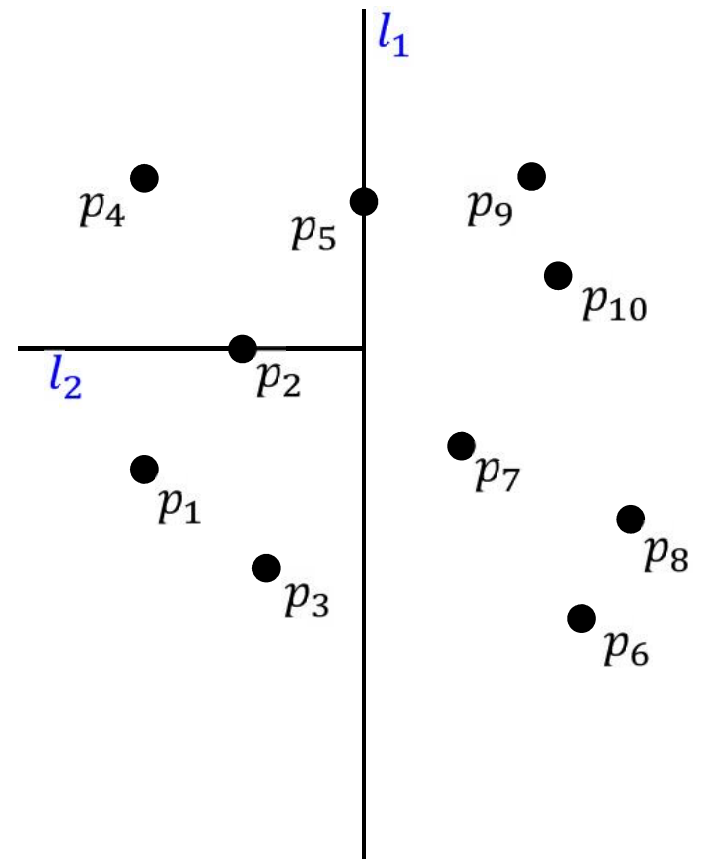
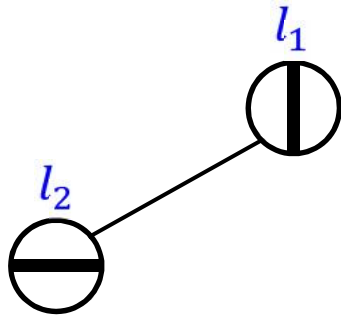
# K-d Tree



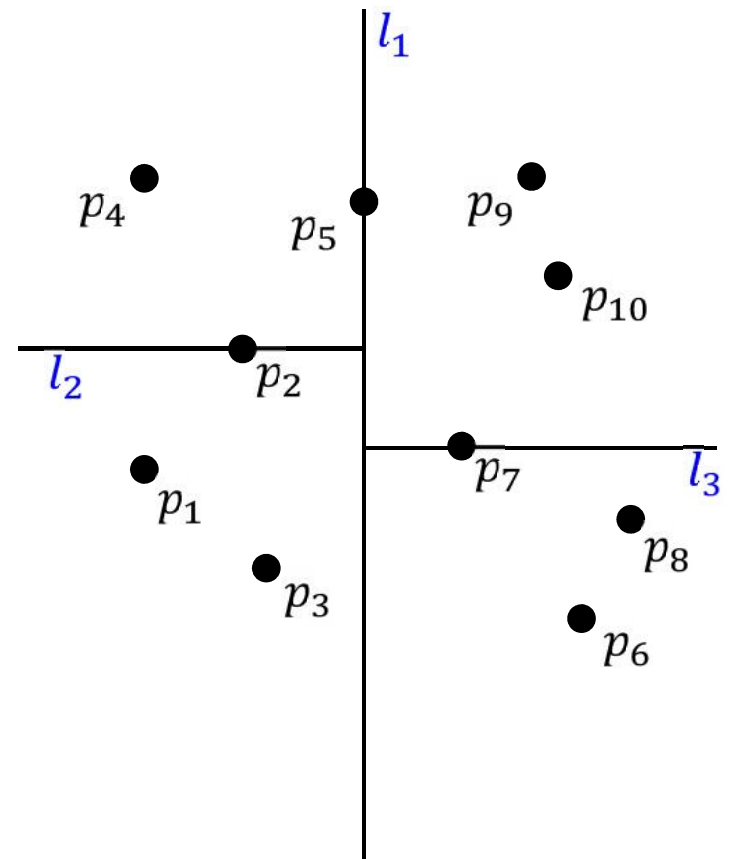
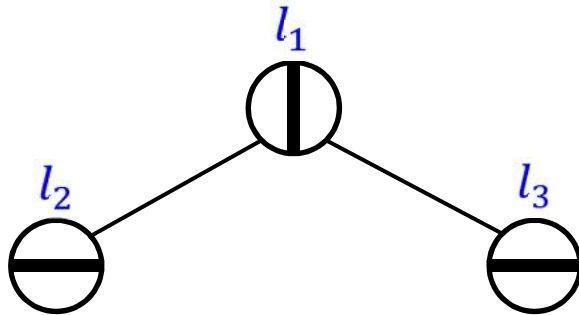
# K-d Tree



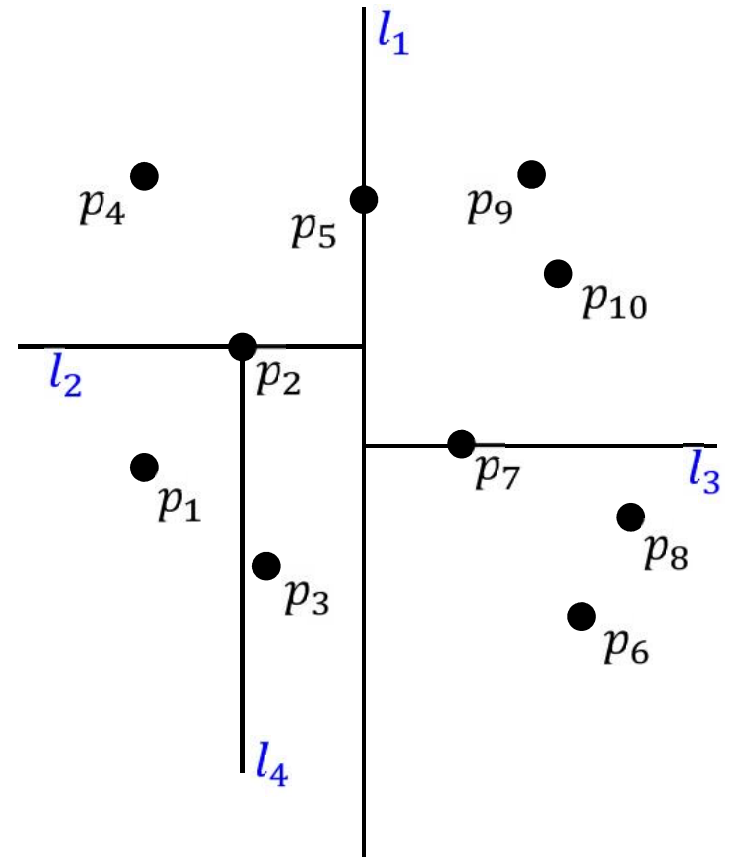
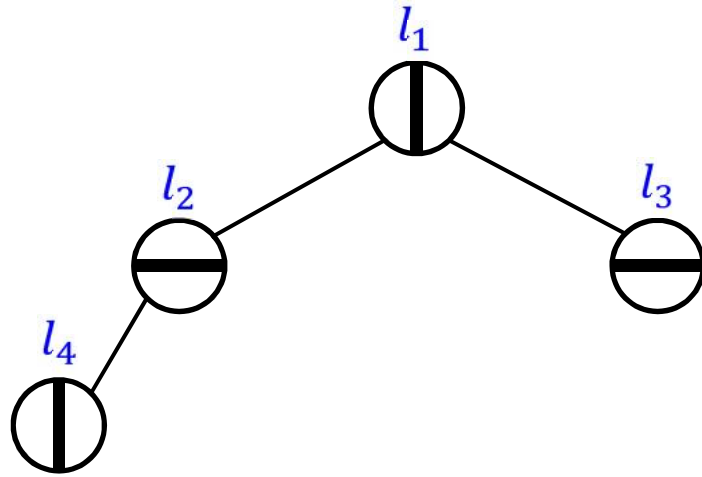
# K-d Tree



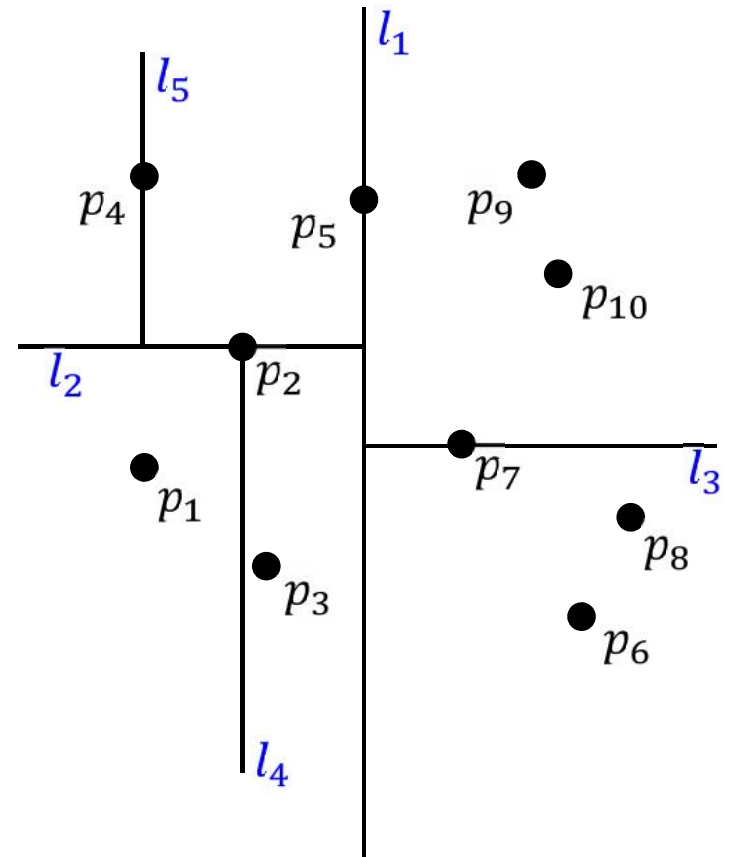
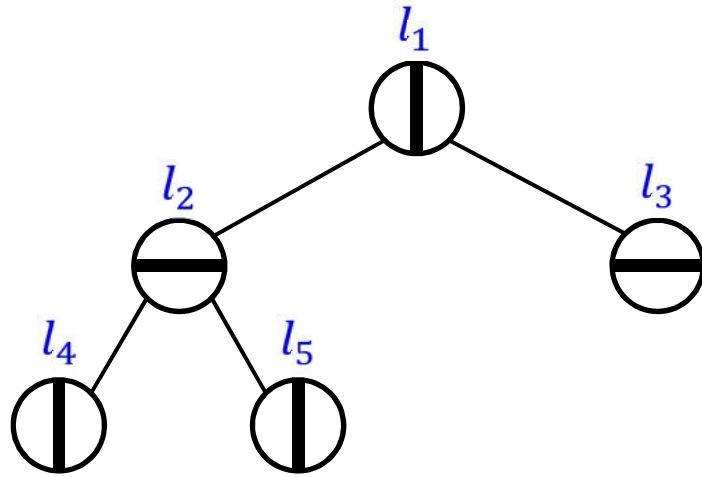
# K-d Tree



# K-d Tree

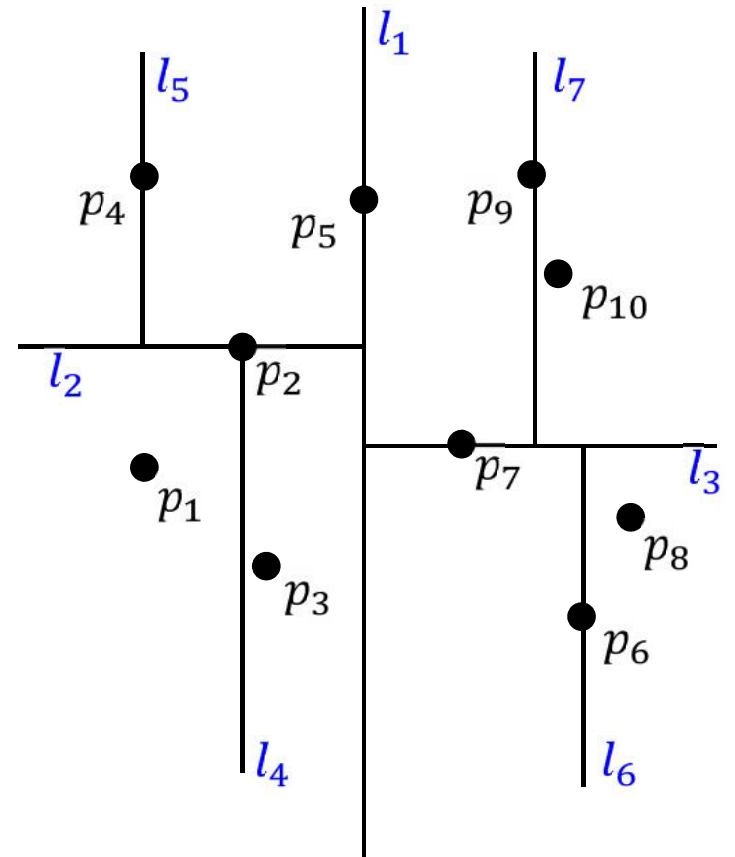
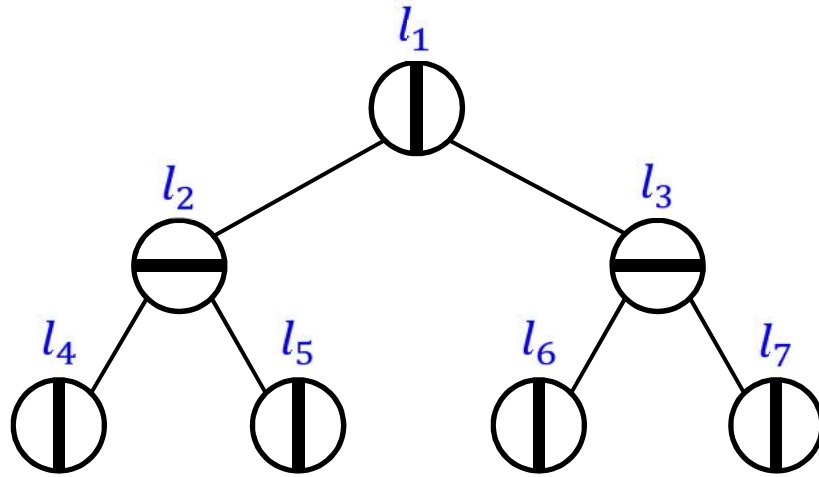


# K-d Tree

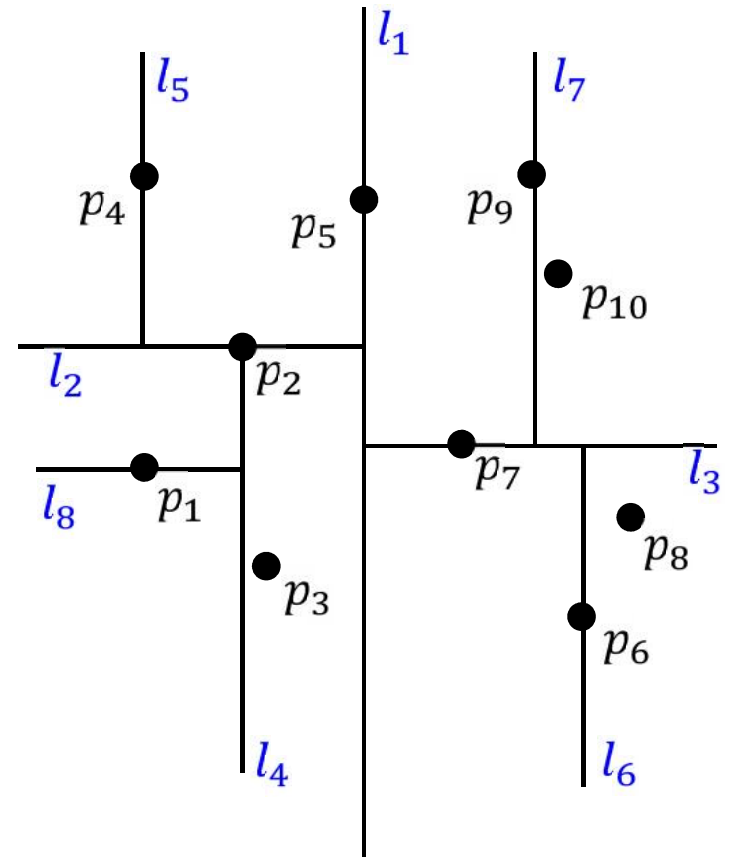
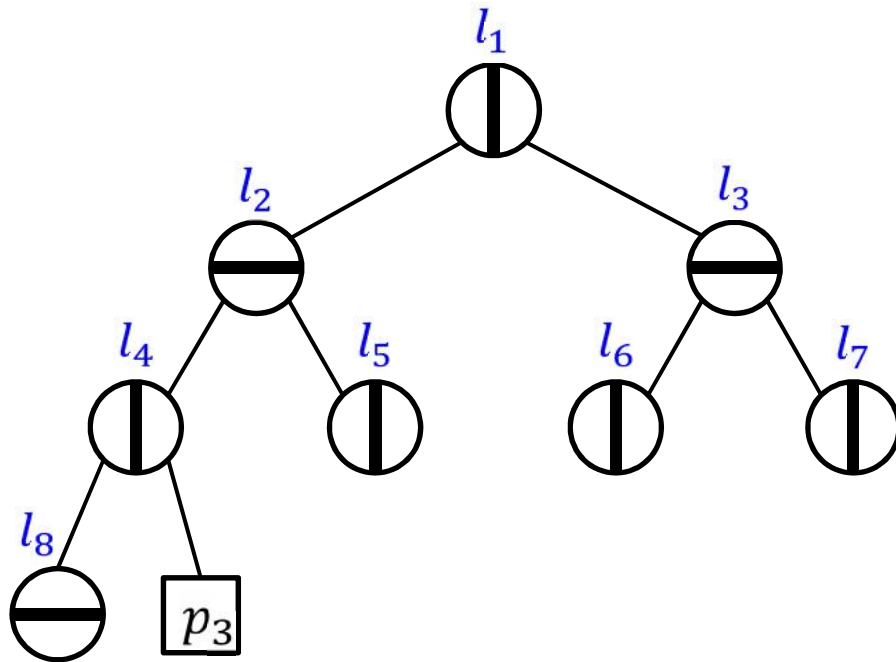




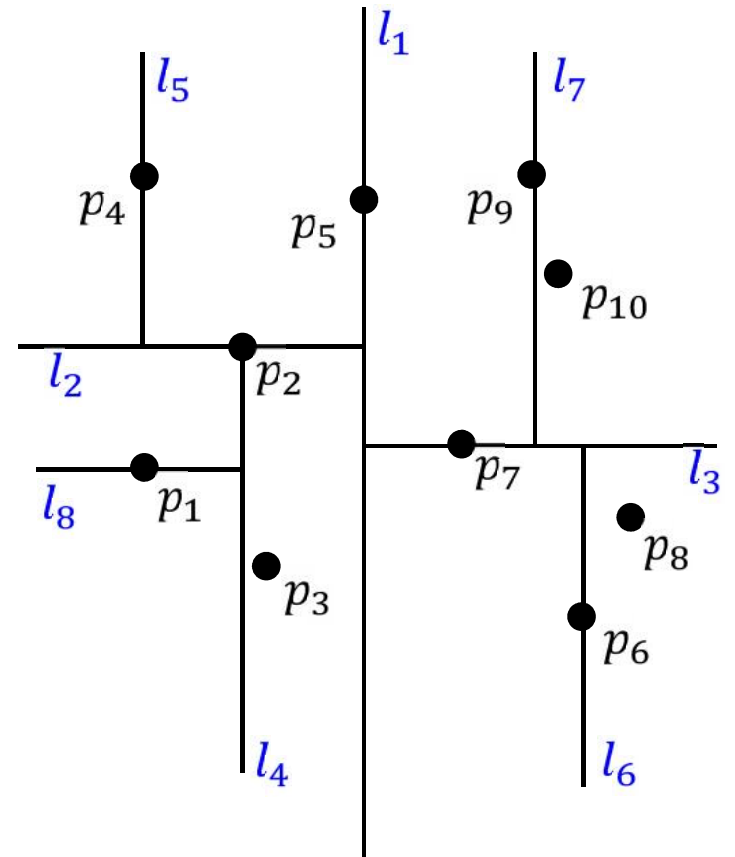
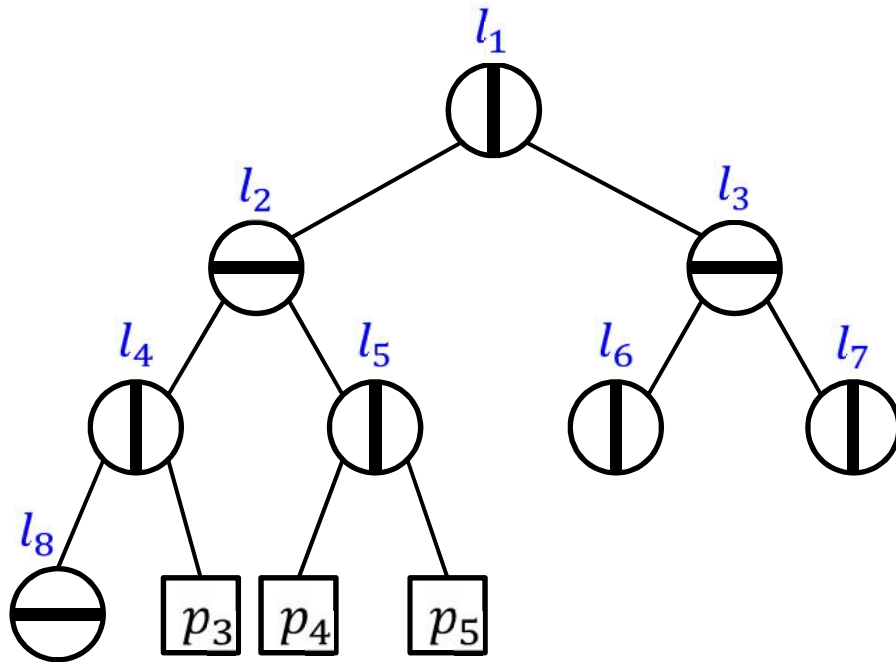
# K-d Tree



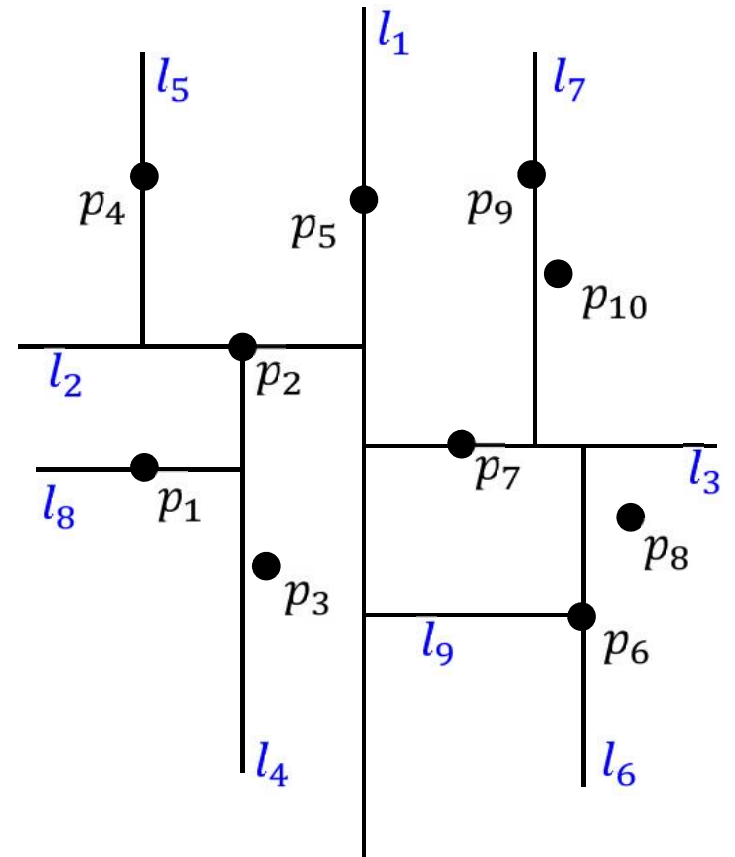
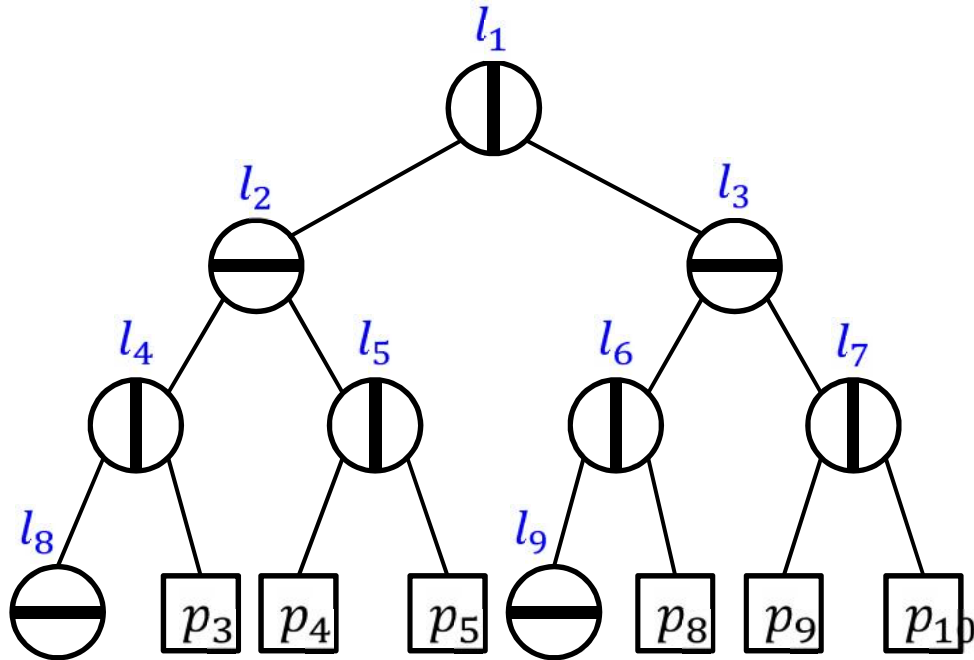
# K-d Tree



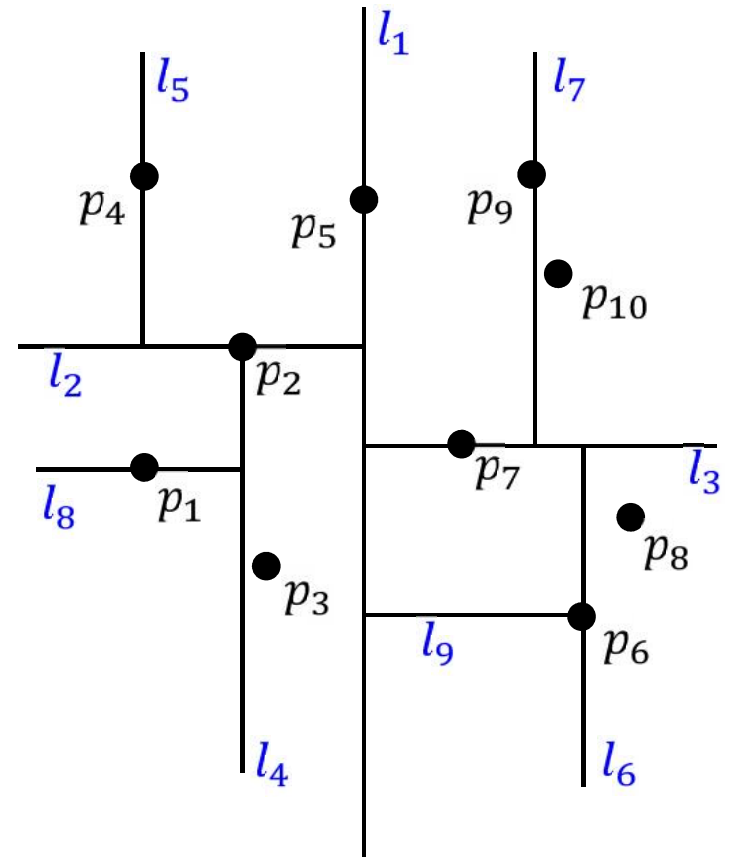
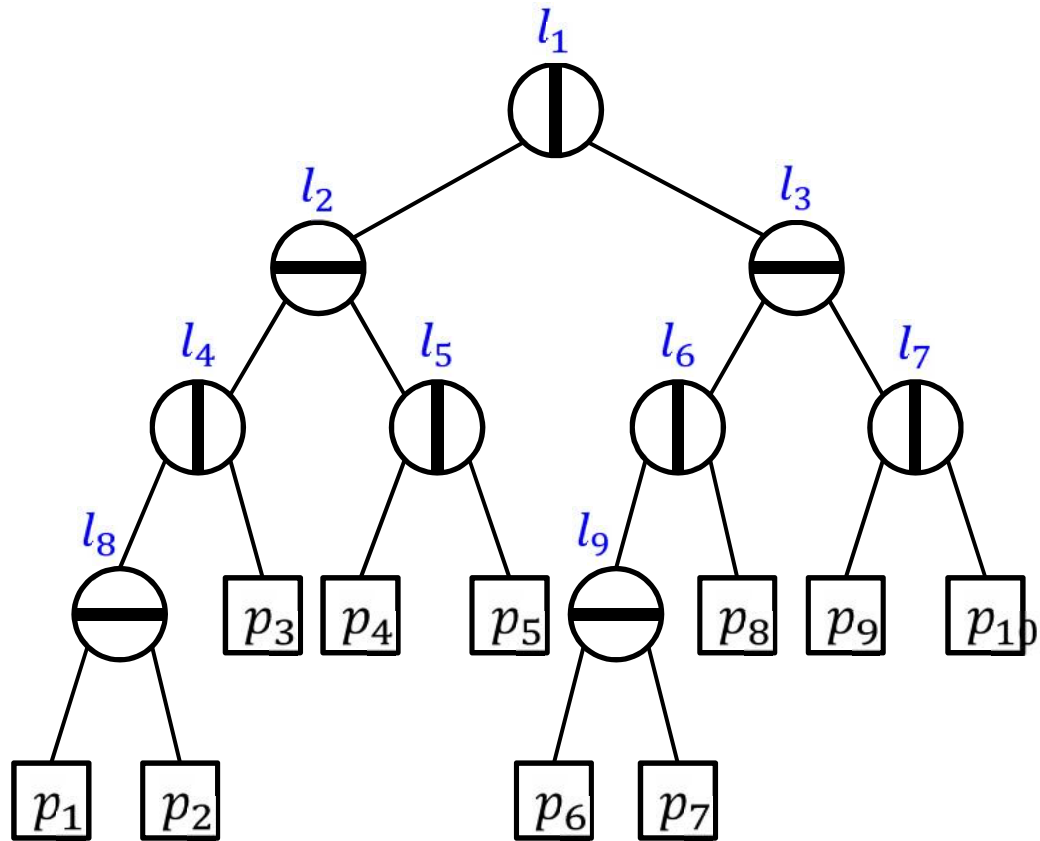
# K-d Tree



# K-d Tree

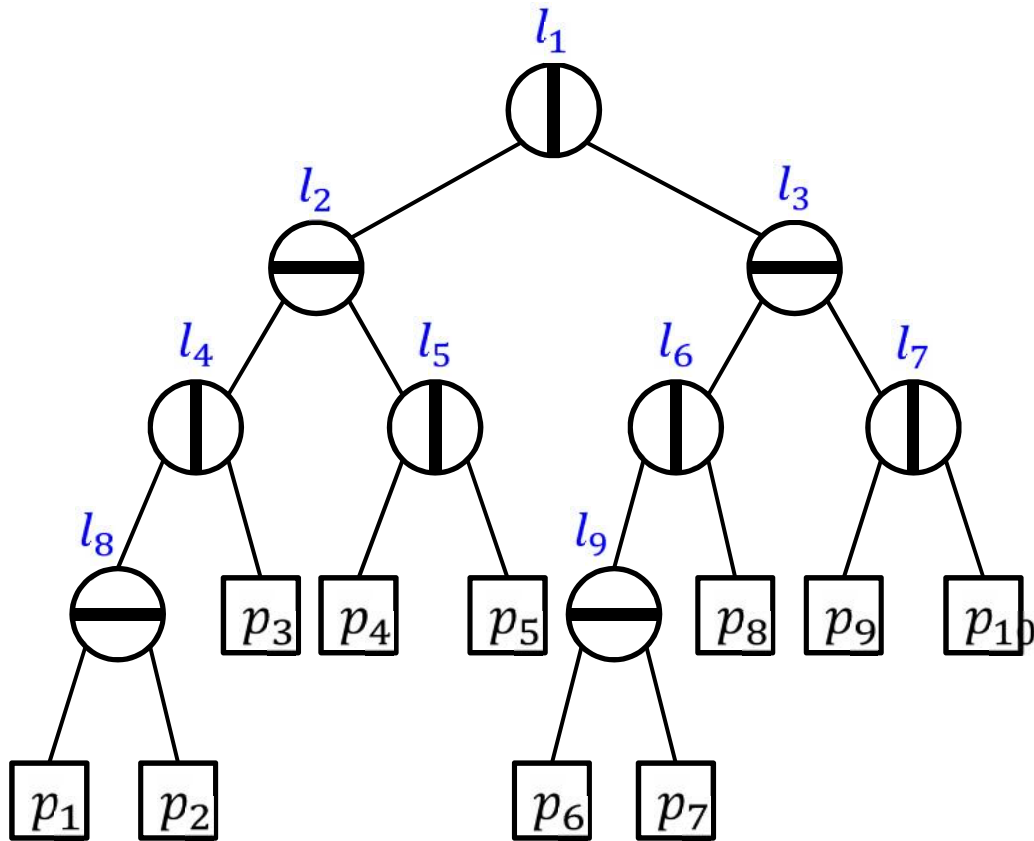


# K-d Tree



# K-d Tree

- Space:  $O(n)$



# K-d Tree

- If  $t$  is a node:
  - $t.val$ : cut value
  - $t.dir$ : cut direction
  - $t.left, t.right$ : child
- If  $t$  is a leaf:
  - $t.pt$ : point

# K-d Tree

- If  $t$  is a node:
  - $t.val$ : cut value
  - $t.dir$ : cut direction
  - $t.left, t.right$ : child
- If  $t$  is a leaf:
  - $t.pt$ : point

$$T(n) = O(n) + 2T(n/2) \\ = O(n \log n)$$

$O(n)$

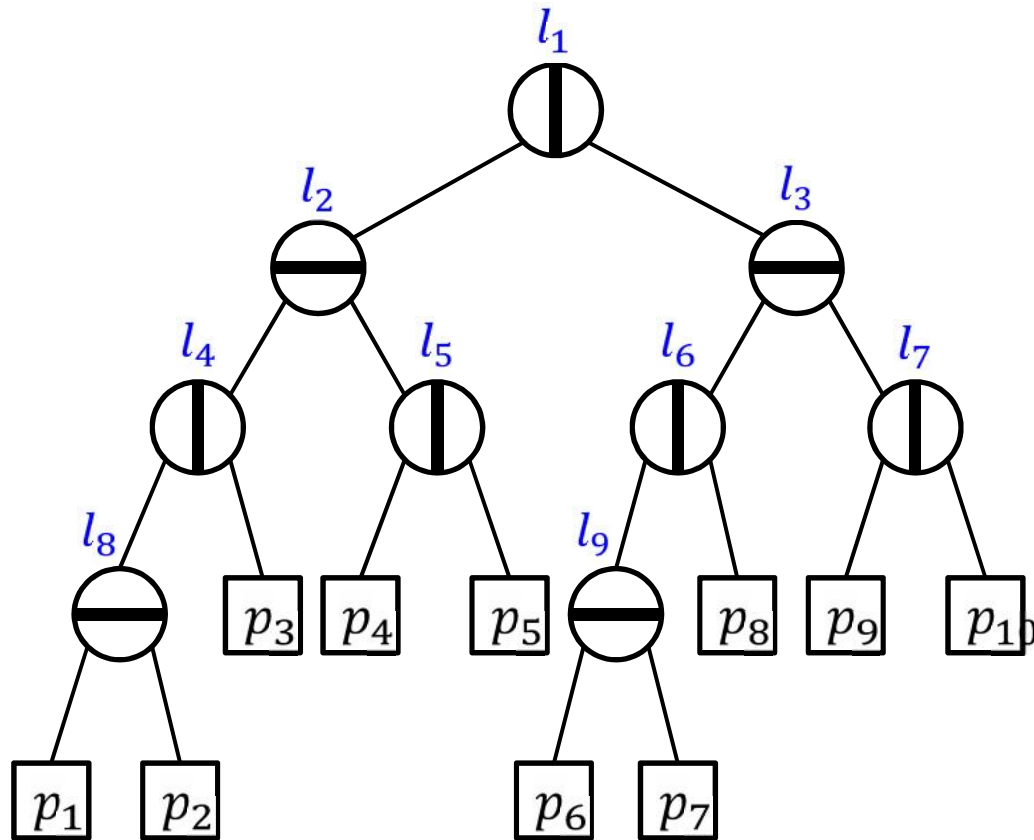
$2T(n/2)$

**BuildTree** ( $S, d$ ) //  $d$ : direction

1. If  $|S|=1$ , return leaf  $t$  where
  1.  $t.pt$  is the point of  $S$
2.  $x$  be median of  $d$ -th coordinates of all points in  $S$
3.  $L$  ( $R$ ) be subset of  $S$  whose  $d$ -th coordinates are no greater than (greater than)  $x$
4. Return node  $t$  where
  1.  $t.val = x$
  2.  $t.dir = d$
  3.  $t.left = \text{BuildTree}(L, 3-d)$
  4.  $t.right = \text{BuildTree}(R, 3-d)$

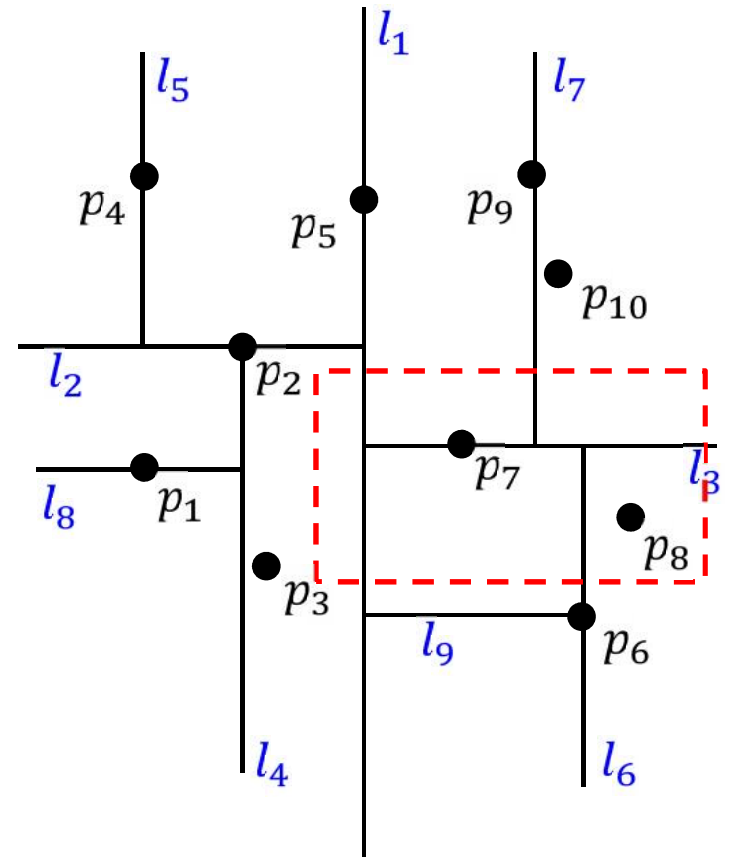


# K-d Tree

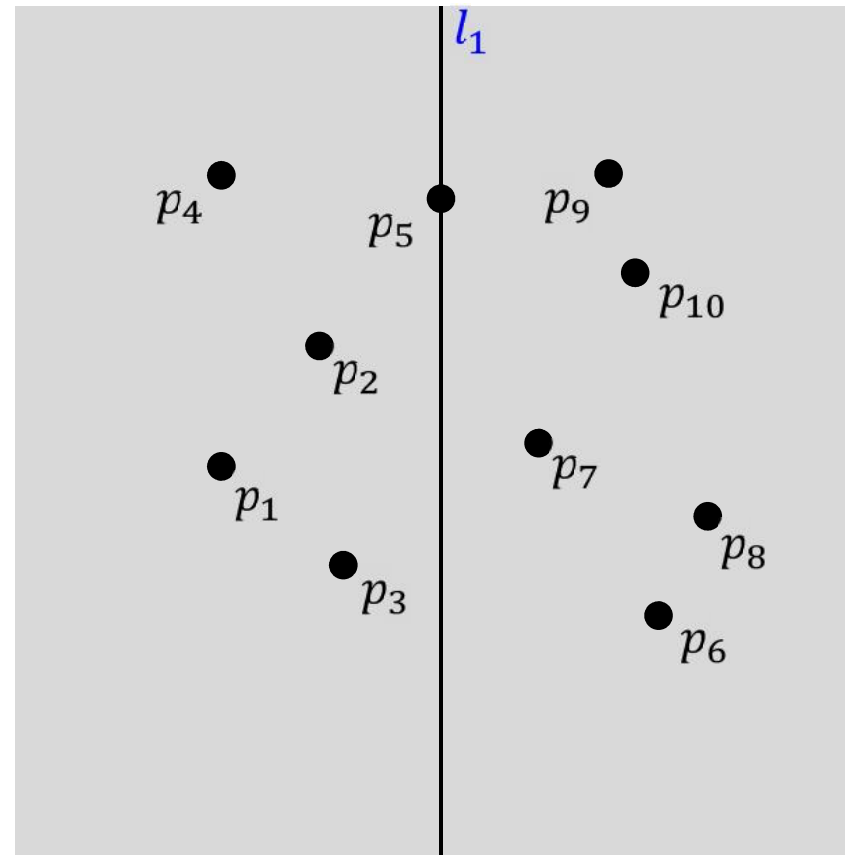
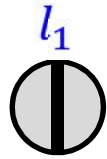


- Space:  $O(n)$
- Build time:  $O(n \log n)$

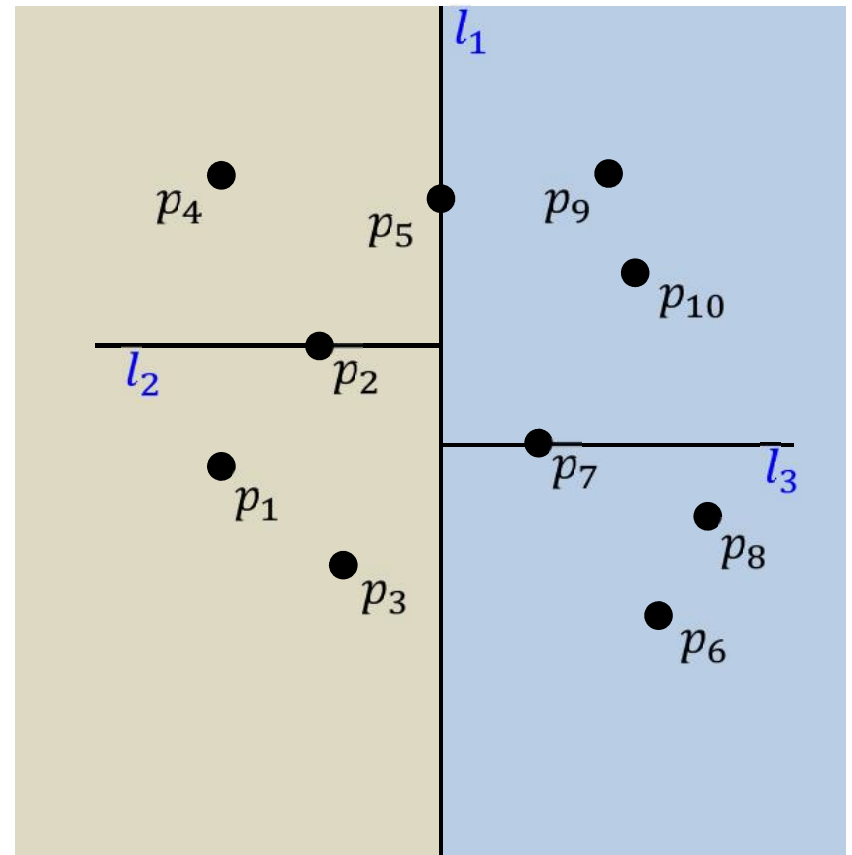
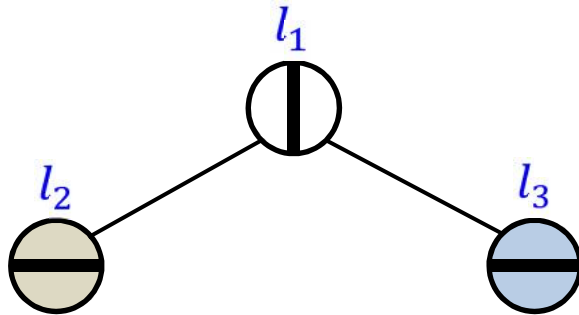
# Query a K-d tree



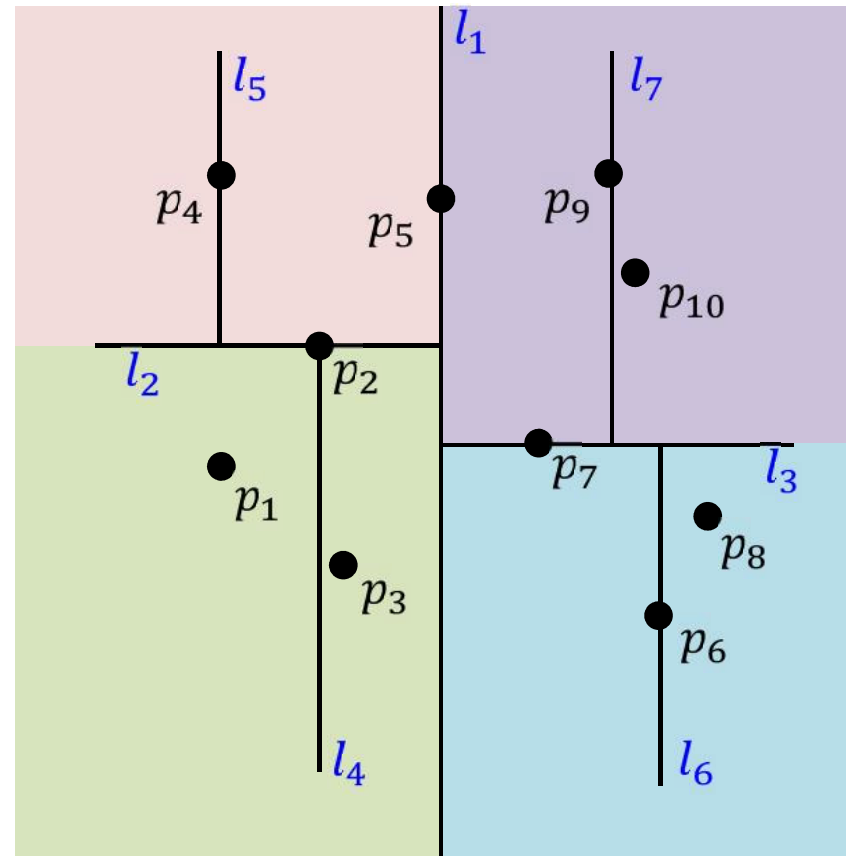
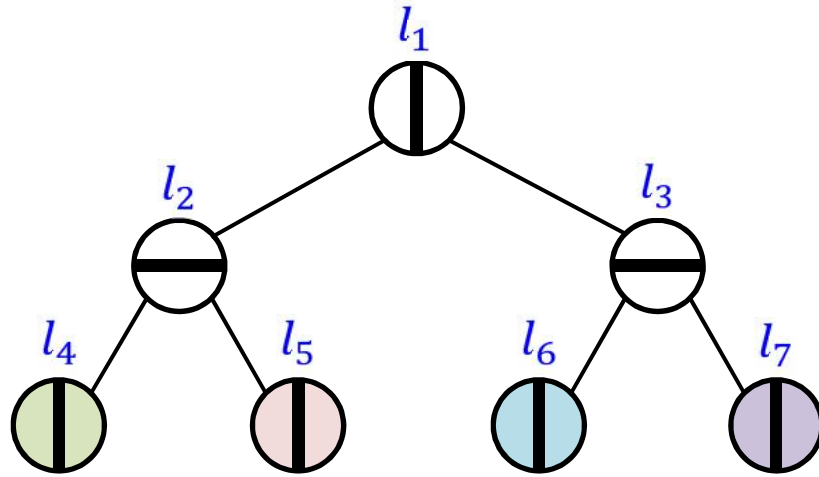
# Range of a node



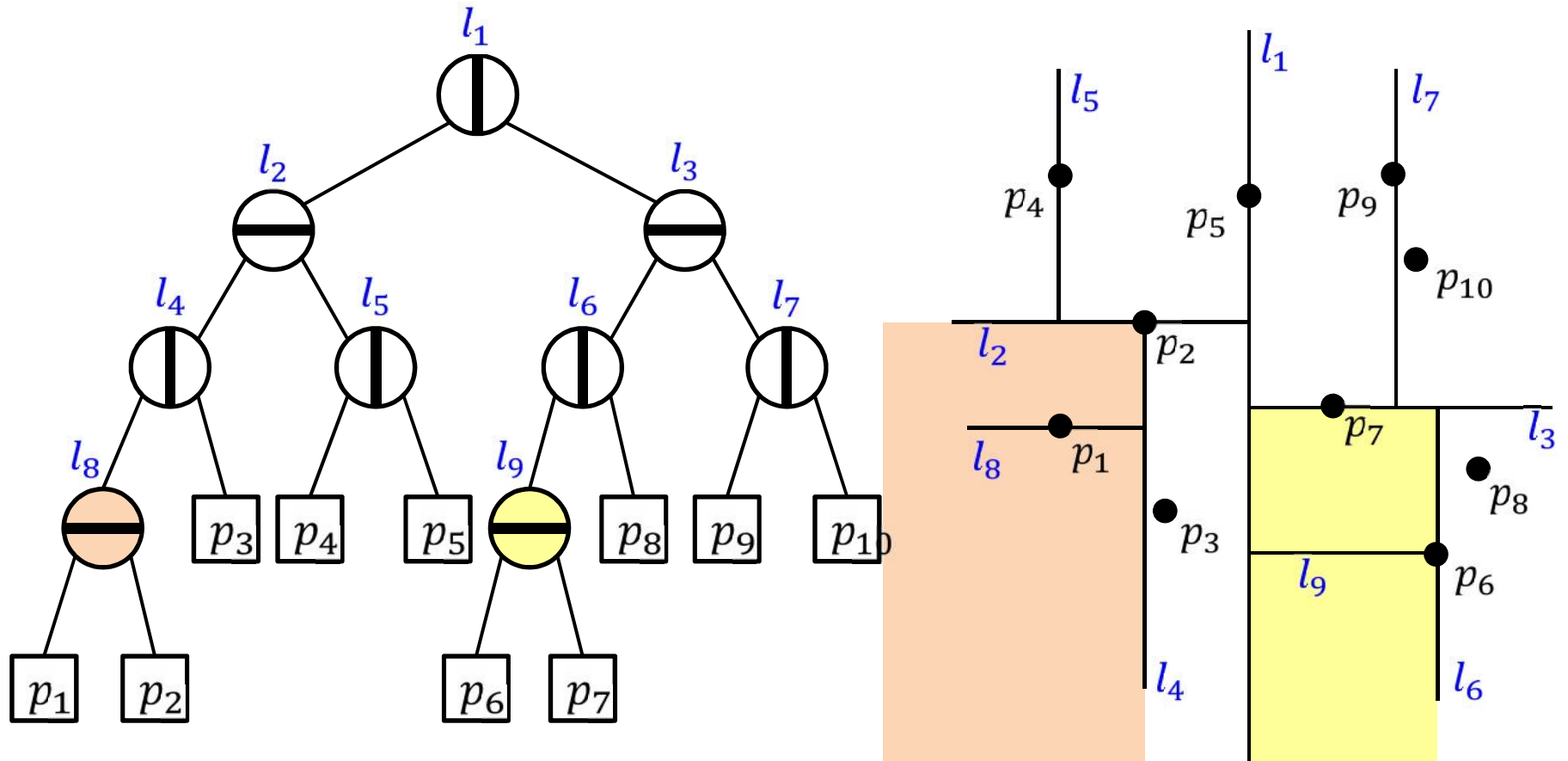
# Range of a node



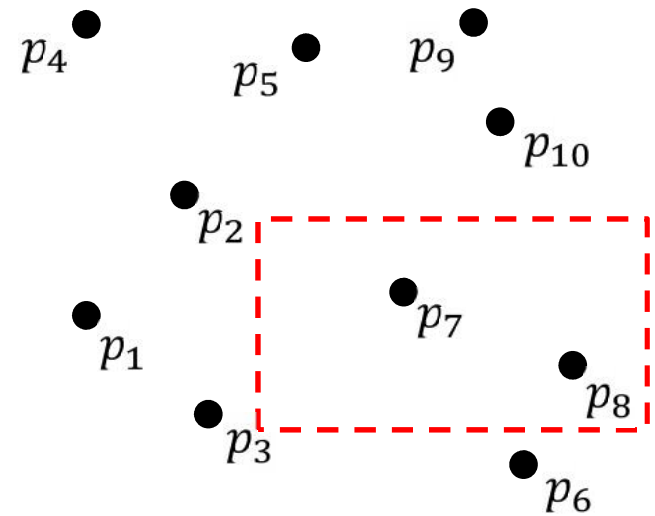
# Range of a node



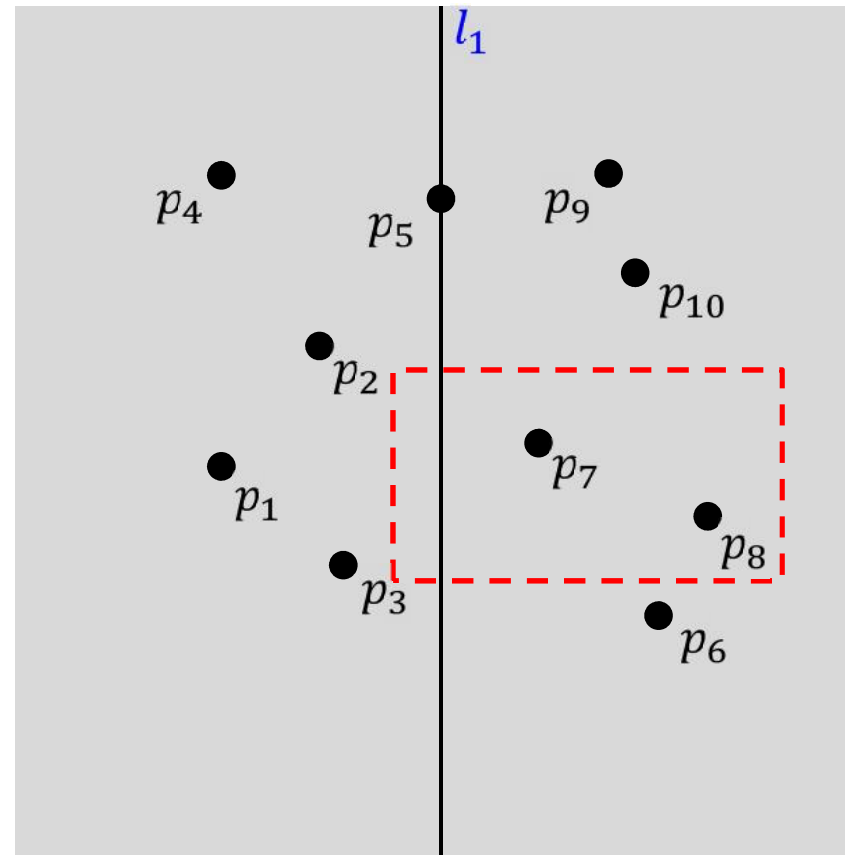
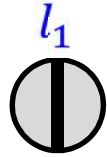
# Range of a node



# Query a K-d tree

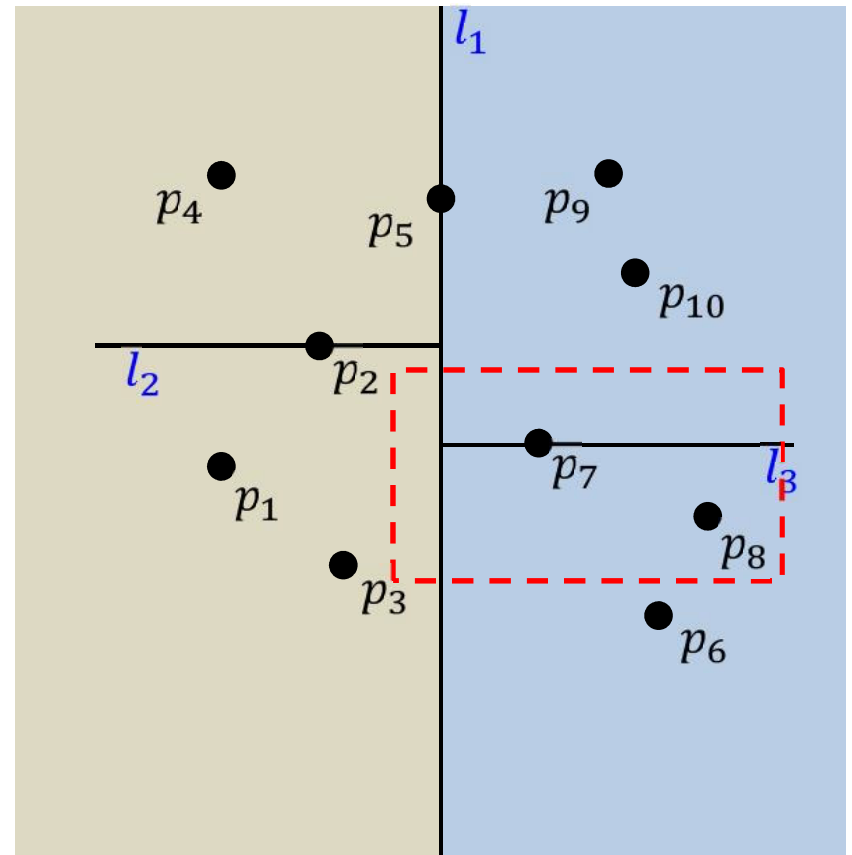
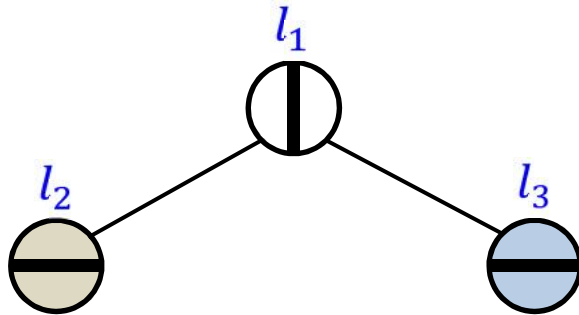


# Query a K-d tree

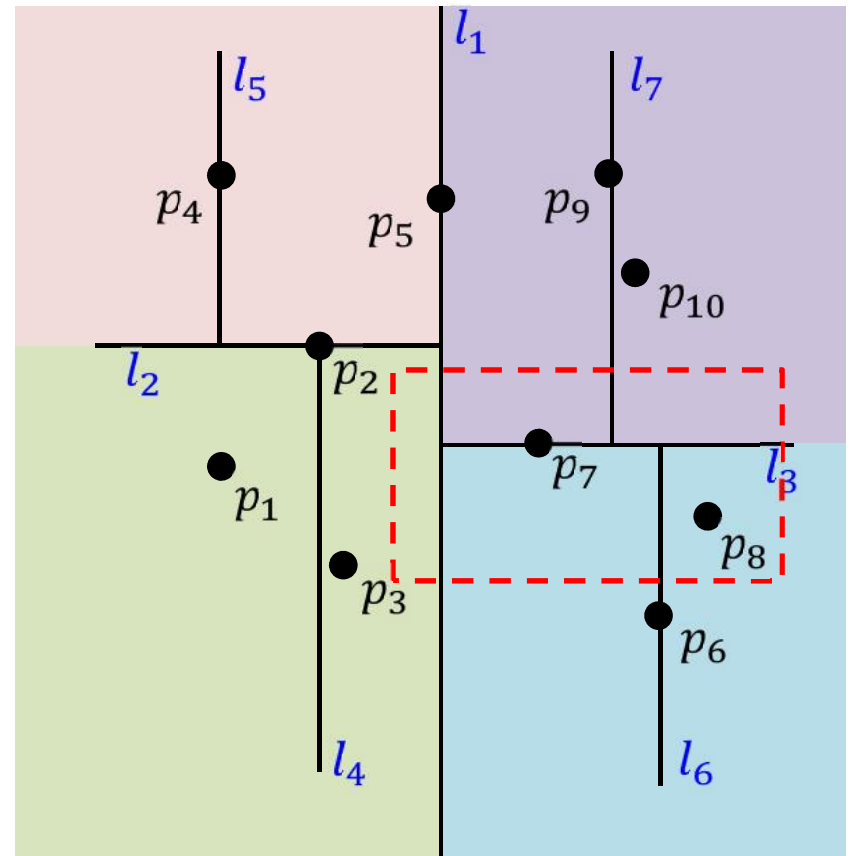
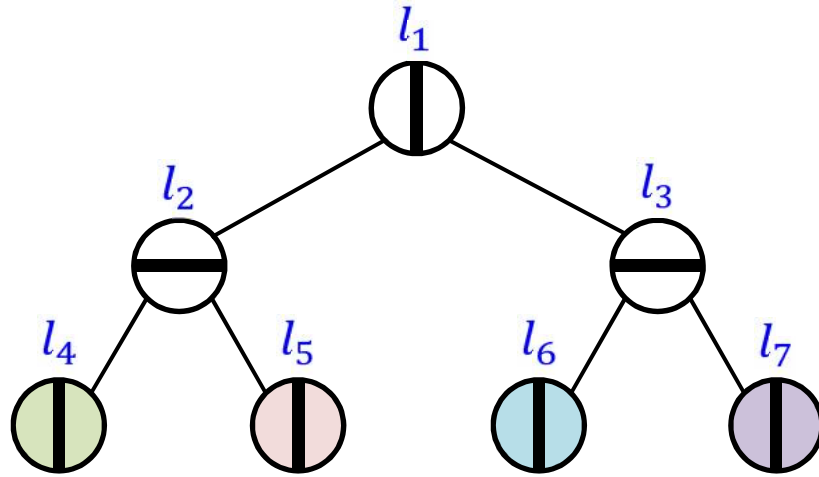




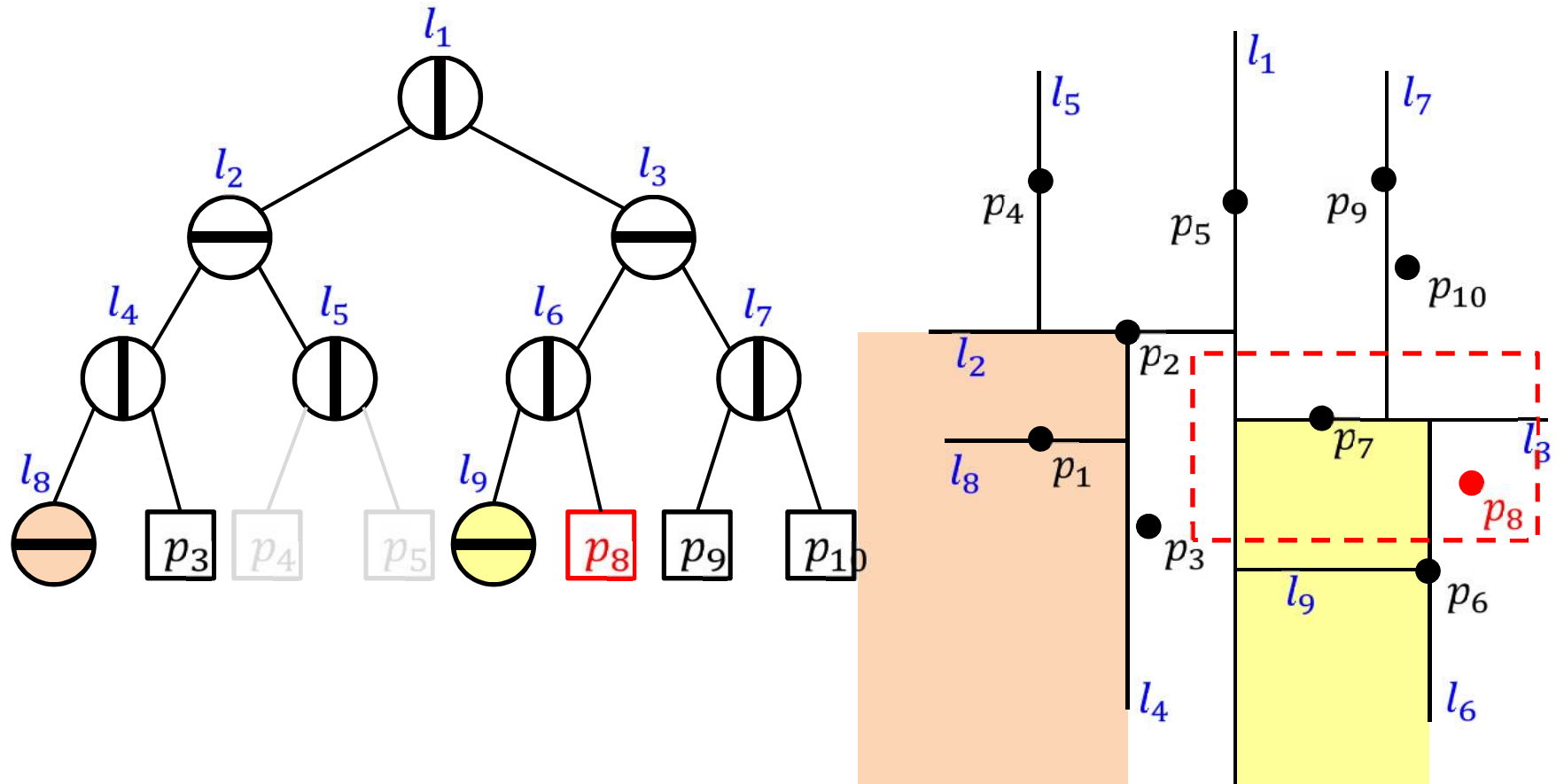
# Query a K-d tree



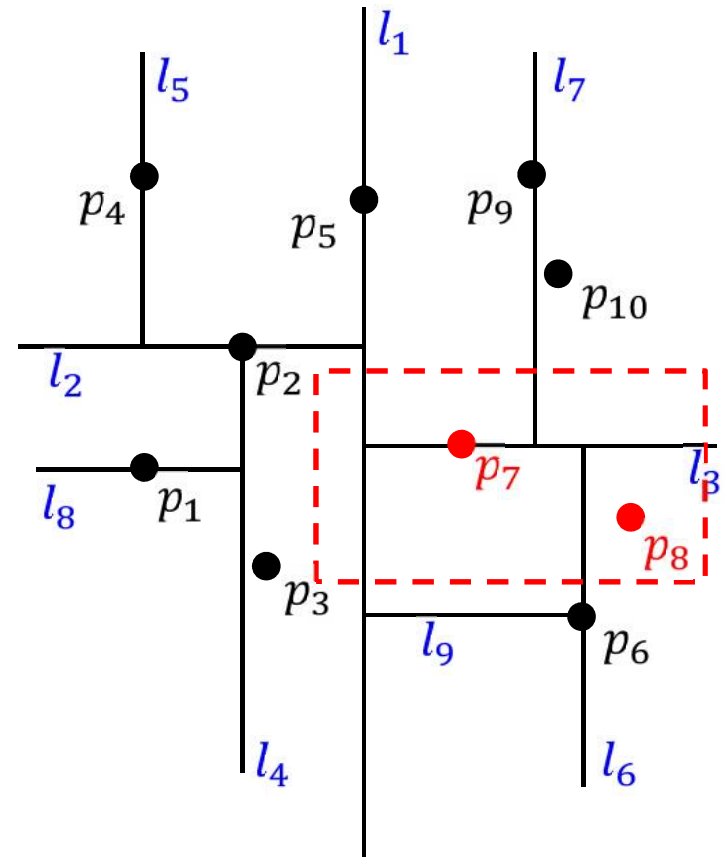
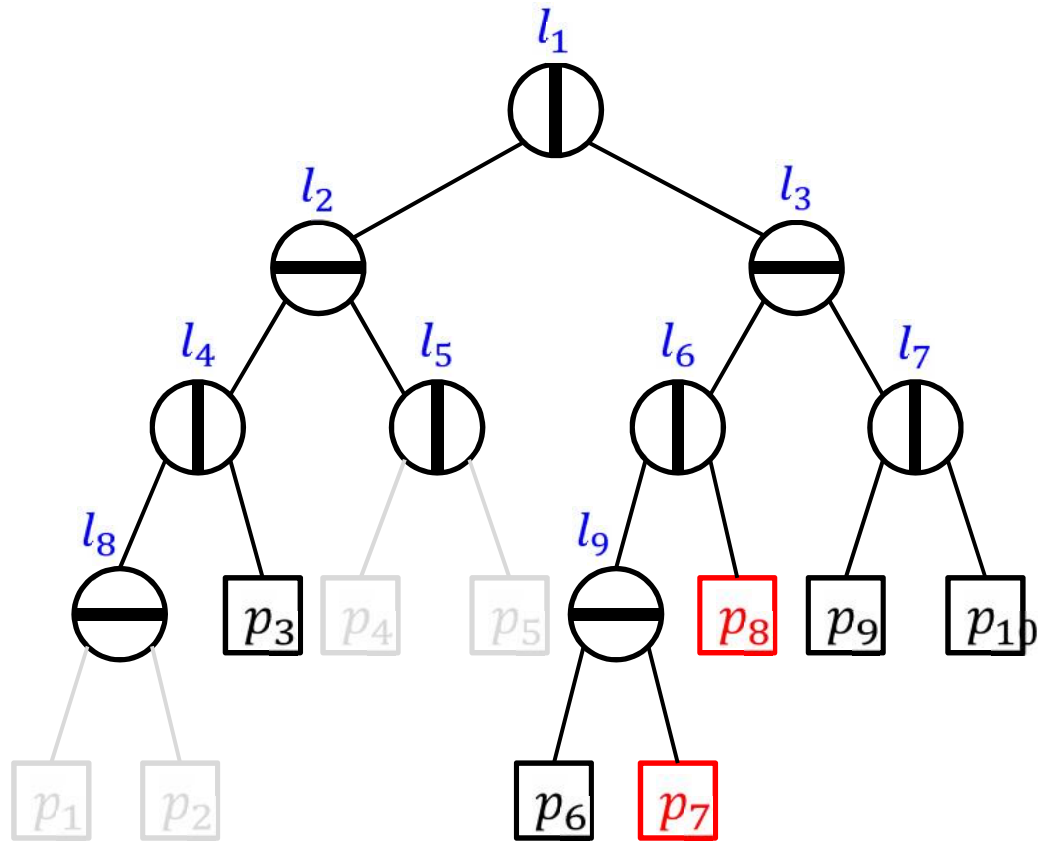
# Query a K-d tree



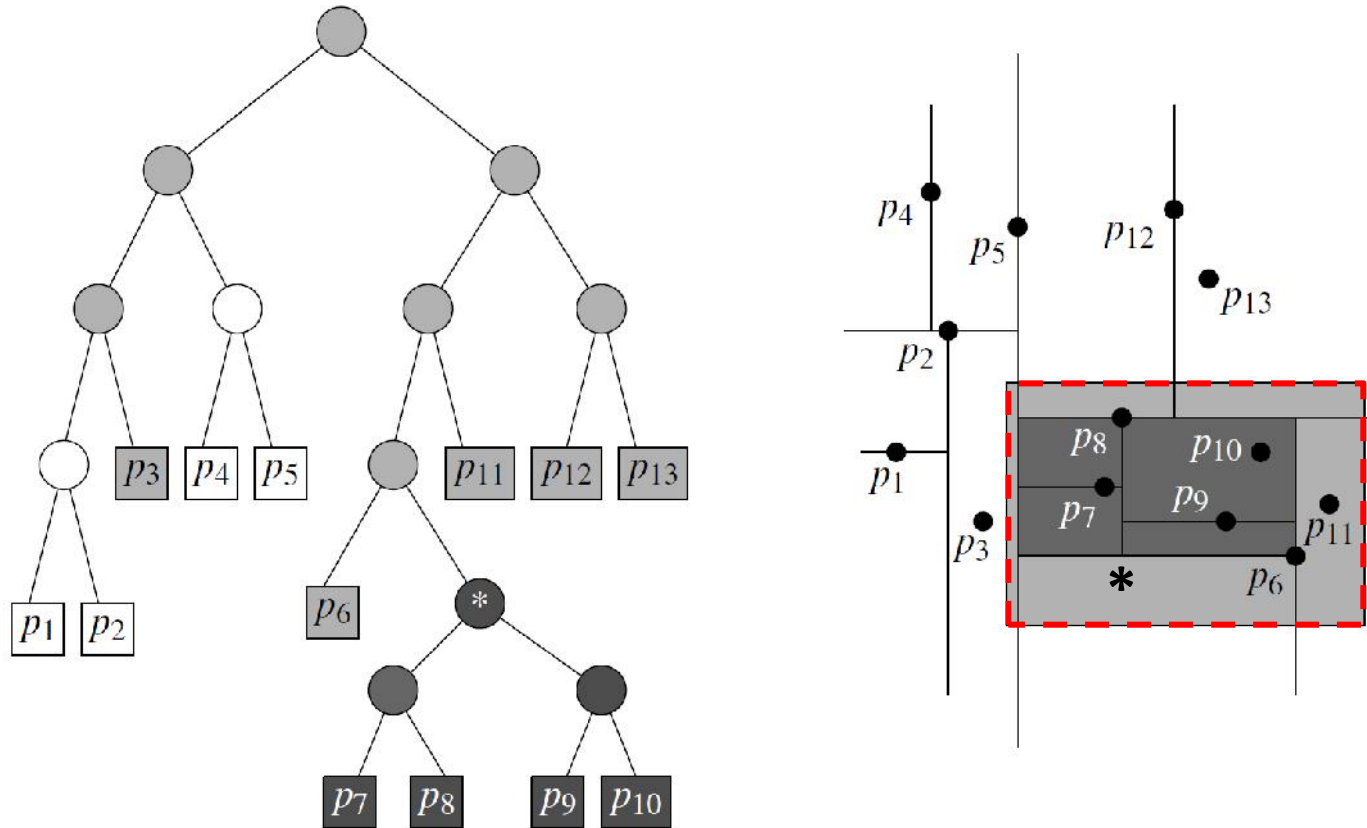
# Query a K-d tree



# Query a K-d tree



# Query a K-d tree



Range of \* lies within the query range.  
The whole subtree rooted at \* is reported.

# Query a K-d tree

- If t is a node:
  - t.val: cut value
  - t.dir: cut direction
  - t.left, t.right: child
  - t.range: range
- If t is a leaf:
  - t.pt: point

```
Query (t, r)      //r: query range
1.  If t is a leaf
    1.  If t.pt is inside r, return t.pt
    2.  Else return NULL
2.  If t.range is inside r
    1.  ReportTree (t)
3.  Else if t.range intersects r
    1.  Return Query (t.left, r)
        Query (t.right, r)
```

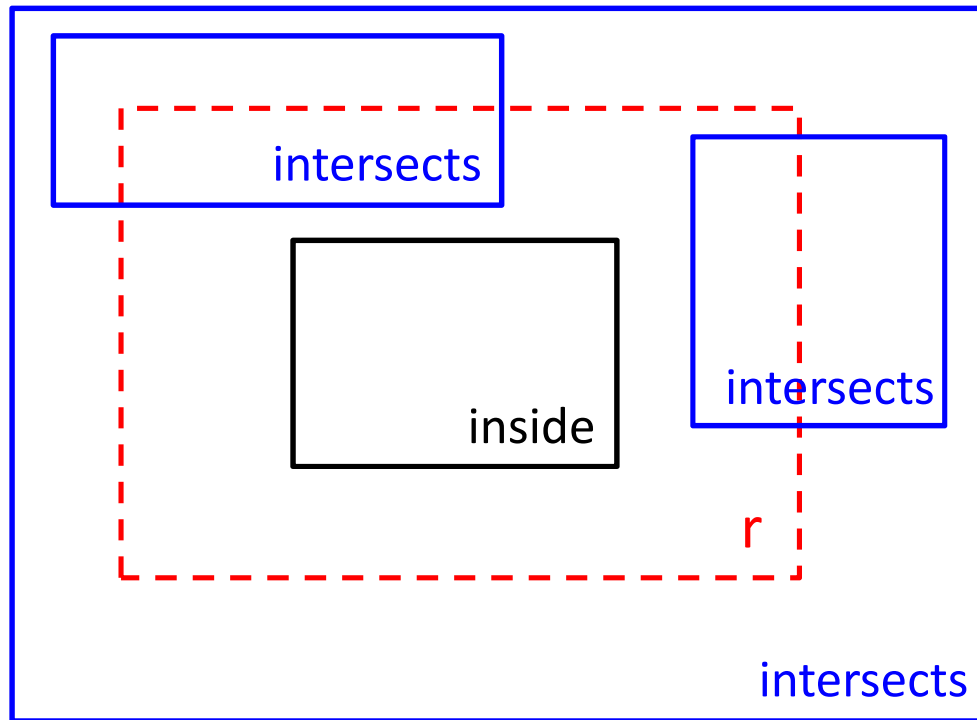
# Query a K-d tree

- Complexity:
  - Total time for (1-2):  $O(k)$
  - # calls to **Query**: ???

```
Query (t, r)      //r: query range
1.  If t is a leaf
    1.  If t.pt is inside r, return t.pt
    2.  Else return NULL
2.  If t.range is inside r
    1.  ReportTree (t)
3.  Else if t.range intersects r
    1.  Return Query (t.left, r)
        Query (t.right, r)
```

# Query a K-d tree

- Query( $t$ ,  $r$ ) is called if  $t$ 's parent's range intersects (but does not lie inside)  $r$



Such range must  
contain a border  
edge of  $r$



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

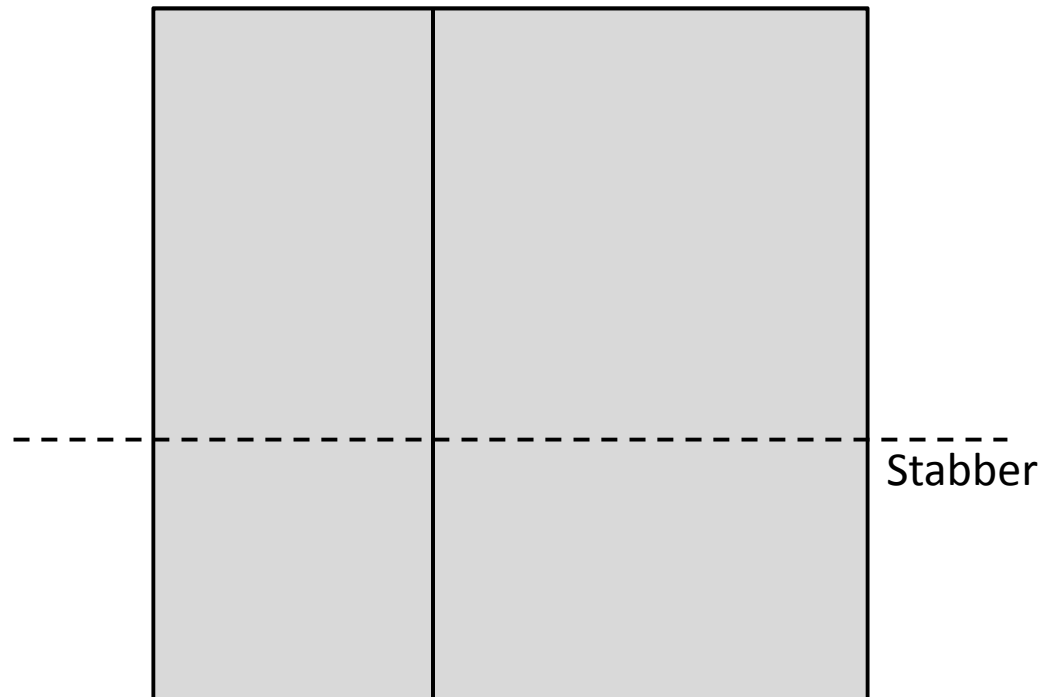
Tree level	# stabbed nodes
1	1



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

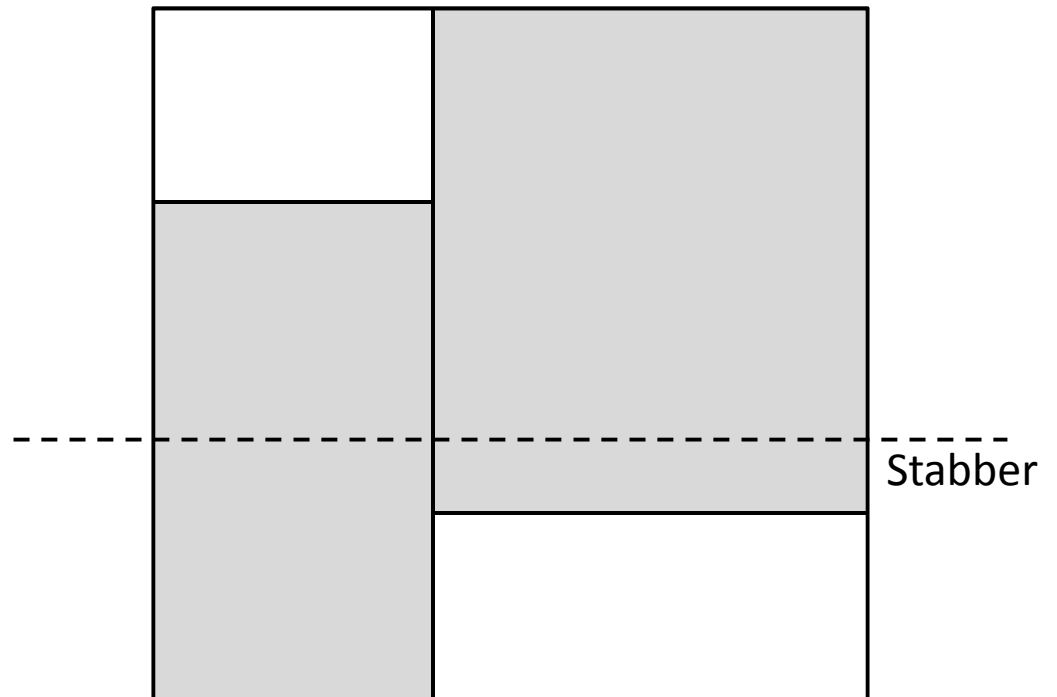
Tree level	# stabbed nodes
1	1
2	2



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

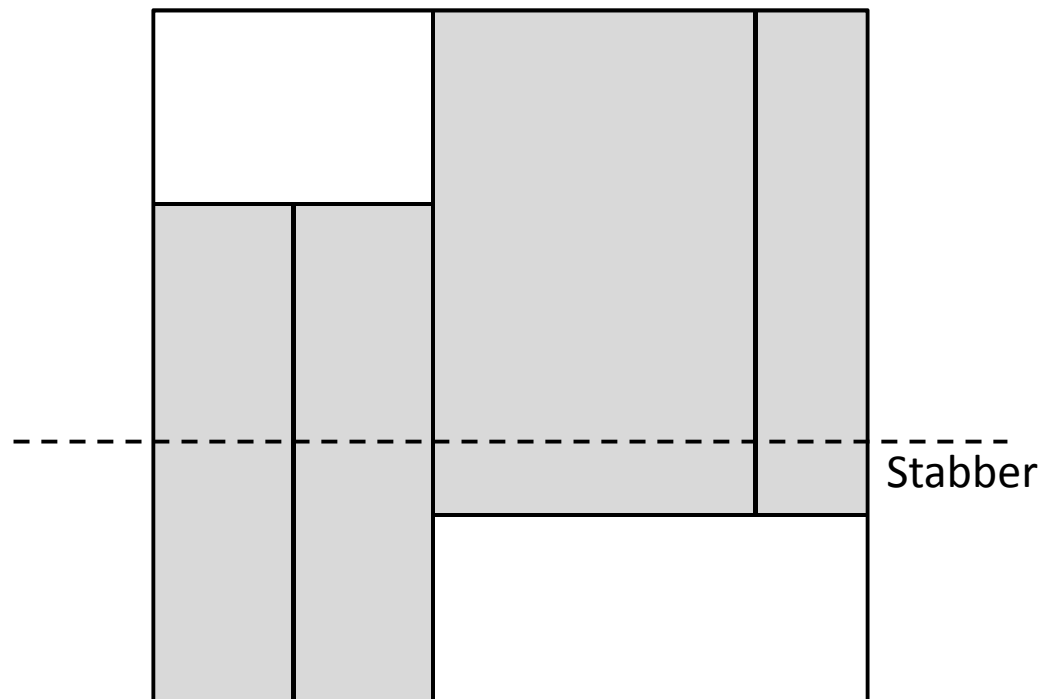
Tree level	# stabbed nodes
1	1
2	2
3	2



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

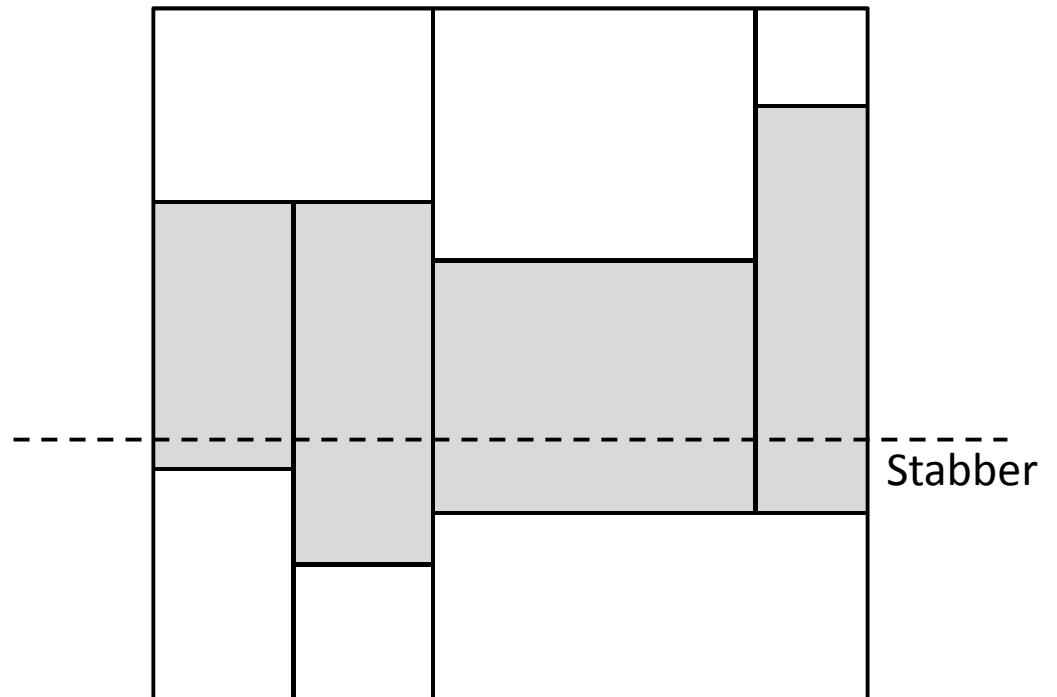
Tree level	# stabbed nodes
1	1
2	2
3	2
4	4



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?

Tree level	# stabbed nodes
1	1
2	2
3	2
4	4
5	4
$k$	$2^{\lfloor k/2 \rfloor}$



# Query a K-d tree

- How many nodes of a k-d tree can be “stabbed” by a line (i.e., the line passes through the node’s range)?
  - The root is stabbed.
  - All stabbed nodes form a binary tree of depth  $\frac{1}{2} \log n$ .
  - Hence total number is  $O(2^{\frac{1}{2} \log n} = \sqrt{n})$ .

# Query a K-d tree

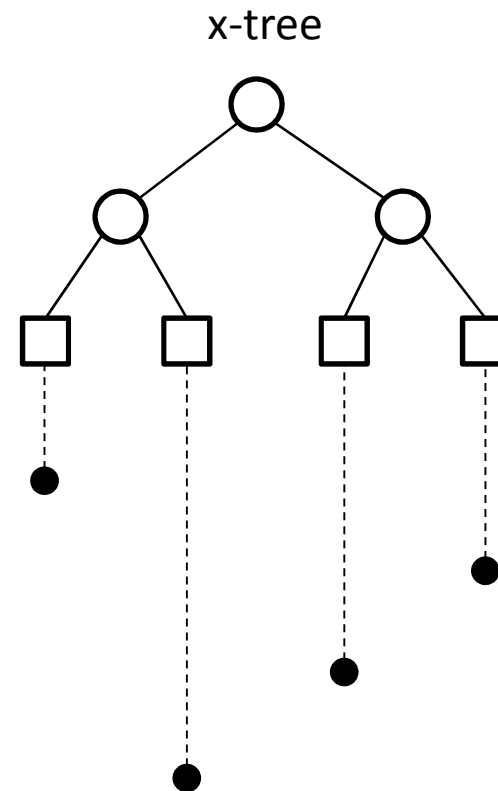
- Complexity:
  - Total time for (1-2):  $O(k)$
  - # calls to **Query**:  $O(\sqrt{n})$
  - Overall:
    - Report:  $O(\sqrt{n} + k)$
    - Count:  $O(\sqrt{n})$

```
Query (t, r)      //r: query range
1.  If t is a leaf
    1.  If t.pt is inside r, return t.pt
    2.  Else return NULL
2.  If t.range is inside r
    1.  ReportTree (t)
3.  Else if t.range intersects r
    1.  Return Query (t.left, r)
        Query (t.right, r)
```



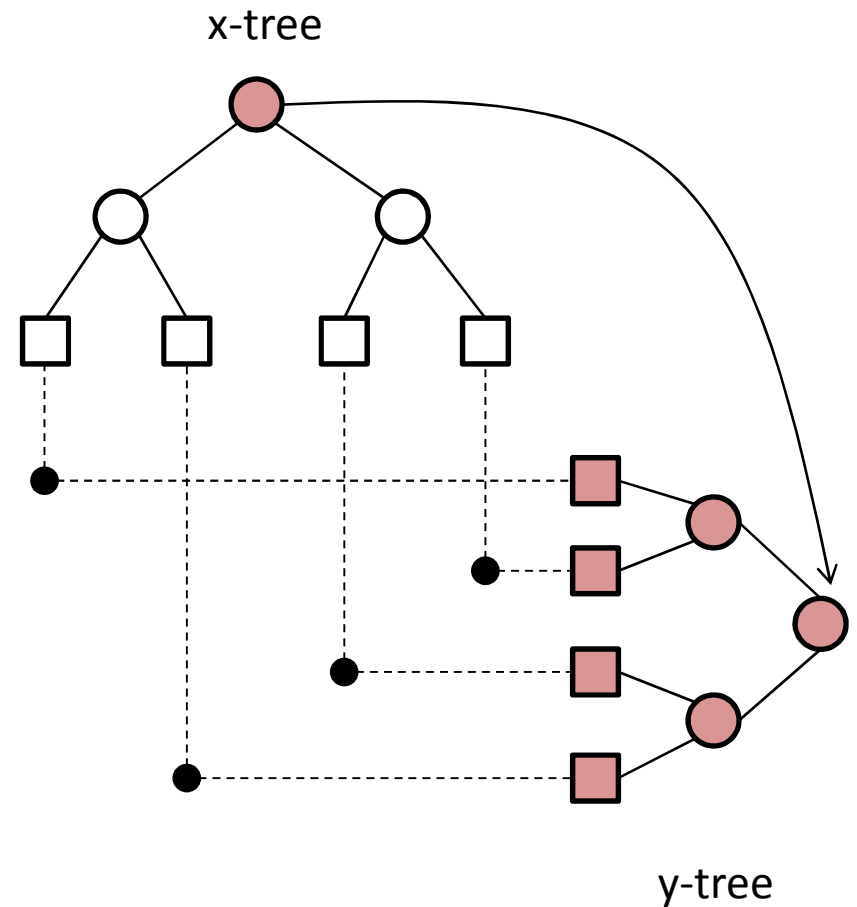
# Range Tree

- One binary tree in  $X$  (x-tree)



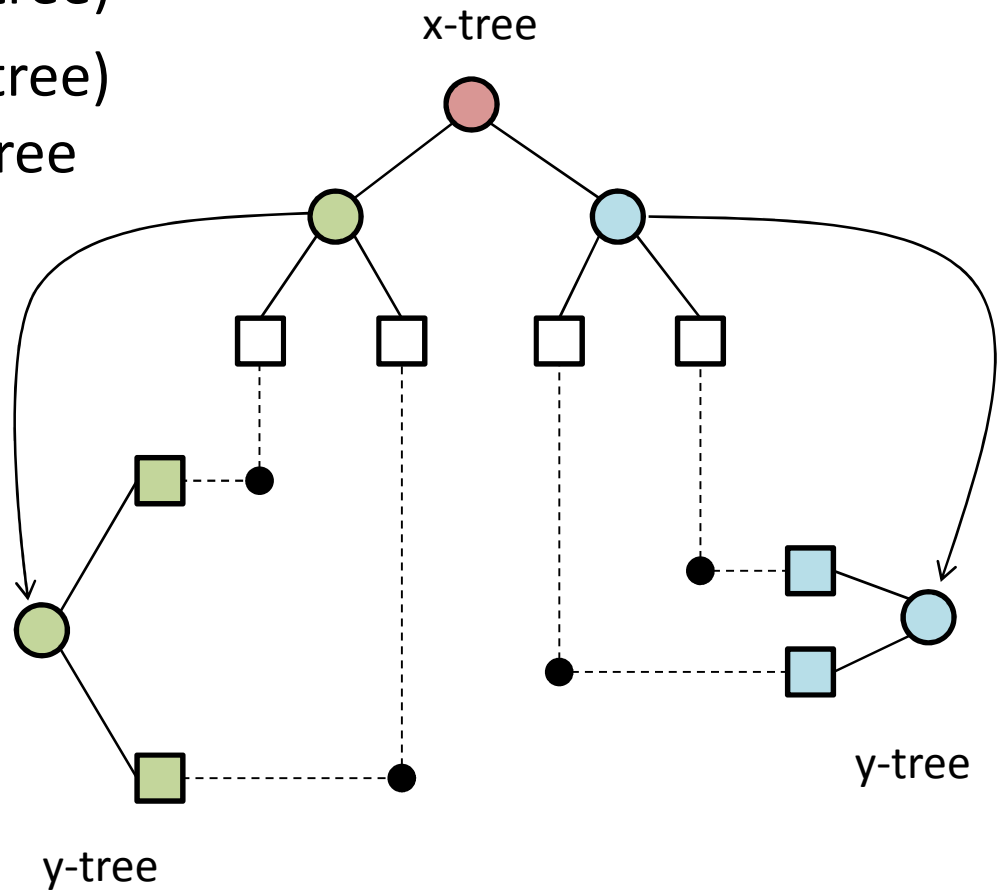
# Range Tree

- One binary tree in X (x-tree)
- One binary tree in Y (y-tree)  
for each node in the x-tree



# Range Tree

- One binary tree in X (x-tree)
- One binary tree in Y (y-tree)  
for each node in the x-tree

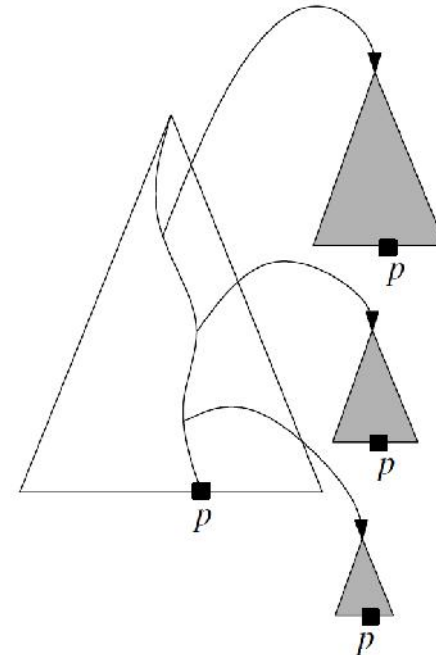


# Range Tree

- Space complexity:
  - Size of each tree (x- or y-) is linear to # of leaves
  - Let  $T_i$  be # of trees of which  $p_i$  is a leaf, total space is

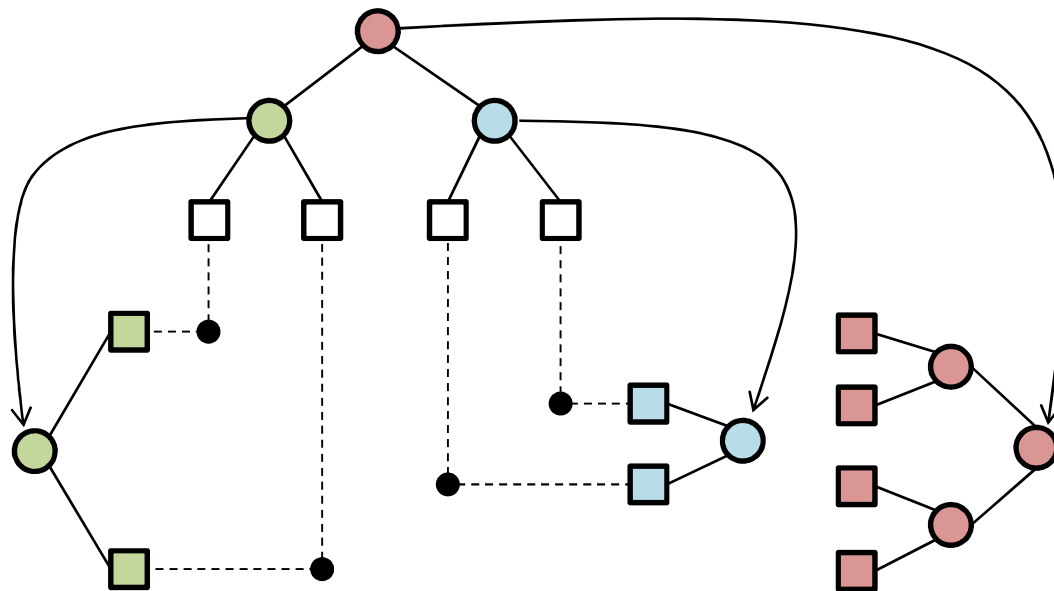
$$O\left(\sum_{i=1}^n T_i\right)$$

- $T_i = O(\log n)$
- Total space is  $O(n \log n)$



# Range Tree

How to build it?



# Range Tree

- If  $t$  is a node of x-tree:
  - $t.val$ : cut value
  - $t.left, t.right$ : child
  - $t.ytree$ : y-tree
- If  $t$  is a leaf of x-tree:
  - $t.pt$ : point
  - $t.ytree$ : a y-tree with a single point

$$T(n) = O(n) + 2T(n/2) \\ = O(n \log n)$$

$O(n)$

$2T(n/2)$

$O(n)$

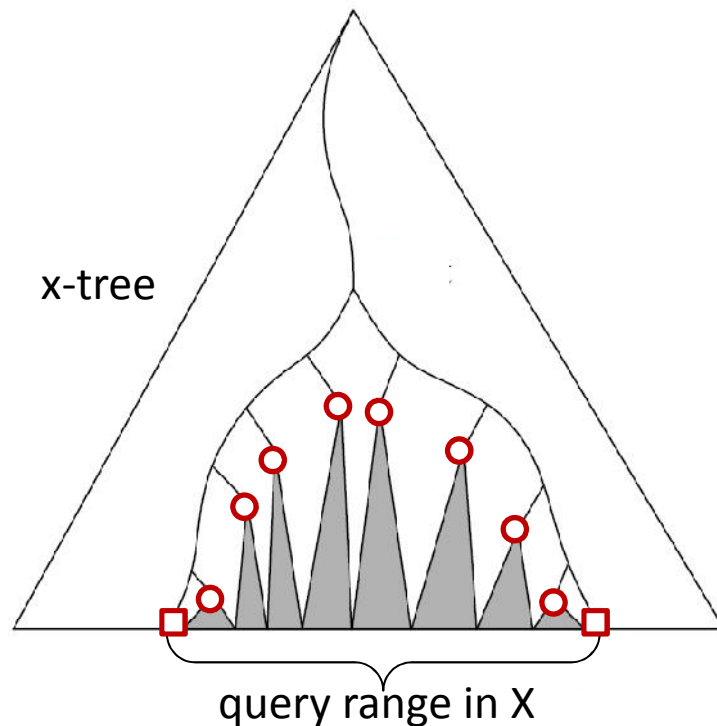
**BuildXTree** ( $S$ )    //  $S$ : point set

1. If  $|S|=1$ , return leaf  $t$  where
  1.  $t.pt$  and  $t.ytree$  are the point of  $S$
2.  $x$  be median of  $X$  coordinates of all points in  $S$
3.  $L$  ( $R$ ) be subset of  $S$  whose  $X$  coordinates are no greater than (greater than)  $x$
4. Return node  $t$  where
  1.  $t.val = x$
  2.  $t.left = \text{BuildXTree}(L)$
  3.  $t.right = \text{BuildXTree}(R)$
  4.  $t.ytree = \text{MergeYTree}(t.left.ytree, t.right.ytree)$

# Range Tree

- Space complexity:  $O(n \log n)$
- Building time:  $O(n \log n)$

# Query a range Tree



Complexity of QueryY():  $O(\log n_t + k_t)$

# Query() calls:  $O(\log n)$

Total complexity:  $O(\log^2 n + k)$

**Query** (t, rX, rY)

//rX, rY: query range in X and Y

1. If t is a leaf
  1. If t.pt is inside {rX,rY}, return t.pt
  2. Else return NULL
2. If t.range is inside rX
  - 1. **QueryY** (t.ytree, rY)
3. Else if t.range intersects rX
  1. Return **Query** (t.left, rX, rY)  
**Query** (t.right, rX, rY)

1D range query

Can be improved to  $O(\log n + k)$   
(using *fractional cascading*, see book/note)