The Famous Function $f(x) = 1/x$

Motivating Questions

- What is a possible explanation, in terms of functions, for the fact that one cannot divide by zero?
- Are sin, cos and tan really the only relevant trigonometric functions? Are there others? If so, how to understand them?

Introduction

We know that if a and b are two real numbers, then a/b makes sense, as long as b is not equal to zero. Let's look at what happens when we make divisions by numbers very close to zero, but not equal to zero. Take $a = 1$ for simplicity.

$$\frac{1}{0.1} = 10$$
$$\frac{1}{0.01} = 100$$
$$\frac{1}{0.001} = 1000$$
$$\frac{1}{0.0001} = 10000$$

This pattern makes us want to say that $1/0$ equals to $+\infty$ (whatever $+\infty$ means, at this point), but this doesn't work. To understand why, let's consider divisions by numbers very close to zero, but this time negative.

$$\frac{1}{-0.1} = -10$$
$$\frac{1}{-0.01} = -100$$
$$\frac{1}{-0.001} = -1000$$
$$\frac{1}{-0.0001} = -10000$$

Learning outcomes:
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The same reasoning as before would tempt us to say that $1/0$ equals $-\infty$. And this raises the question of whether ∞ or $-\infty$ is the better choice. While on an instinctive psychological level we could think that $+\infty$ is better than $-\infty$, there's really no way to decide — and this turns out to be related to the concept of *limit*, which you'll learn in Calculus.

Graph and asymptotics

To continue our discussion in a more precise way, let's consider the function f, defined for all real numbers *except for zero*, given by $f(x) = 1/x$. This is a very famous function, particularly useful as the building block for *rational functions*, which we'll discuss soon. Note that essentially what we have just done in the introduction was to consider the values

$$f(0.1), f(0.01), f(0.001) \text{ and } f(0.0001),$$

as well as

$$f(-0.1), f(-0.01), f(-0.001), \text{ and } f(-0.0001).$$

To get a good idea of the behavior a function has, our main strategy so far has been to just consider its graph. Naturally, plugging a handful of values won't cut it. Let's see what happens when we go to the other extreme and make divisions by very large numbers:

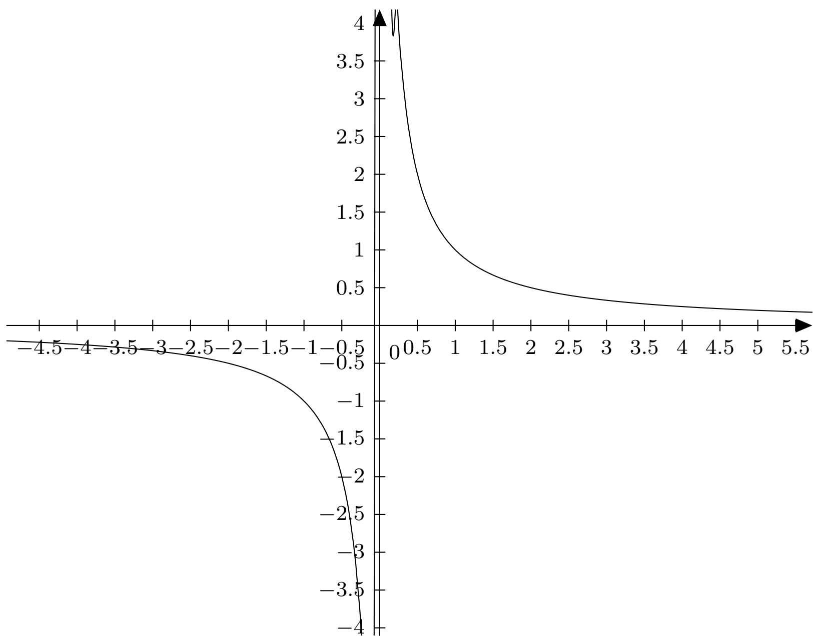
$$\begin{aligned} \frac{1}{10} &= 0.1 \\ \frac{1}{100} &= 0.01 \\ \frac{1}{1000} &= 0.001 \\ \frac{1}{10000} &= 0.0001 \end{aligned}$$

And from the negative side:

Algebraically, the explanation is simple: if one could make sense of $1/0$ and say that equals some number c, then this would give $1 = 0 \cdot c$, so $1 = 0$ — which is a complete collapse of the number system we have to deal with in our daily lives. But this doesn't give intuition for what is going on.

$$\frac{1}{-10} = -0.1$$
$$\frac{1}{-100} = -0.01$$
$$\frac{1}{-1000} = -0.001$$
$$\frac{1}{-10000} = -0.0001$$

Here's what the graph looks like.



[MAKE BETTER GRAPH, WILL FIX TIKZ AFTER DRAFTS ARE DONE]

Here's what we can immediately see from the graph, confirming our intuition from the several divisions previously done:

Asymptotics of $1/x$.

- If $x \to +\infty$, then $1/x \to 0$ (reads "when x tends to $+\infty$, $1/x$ tends to 0").
- If $x \to 0^+$, then $1/x \to +\infty$ (reads "when x tends to zero from the right, $1/x$ tends to $+\infty$").
- If $x \to 0^-$, then $1/x \to -\infty$ (reads "when x tends to zero from the left, $1/x$ tends to $-\infty$").

- If $x \to -\infty$, then $1/x \to 0$ (reads "when x tends to $-\infty$, $1/x$ tends to 0").

We say that the line $x = 0$ is a *vertical asymptote* for $f(x) = 1/x$, while the line $y = 0$ is a *horizontal asymptote*. We will discuss asymptotes of rational functions in general in the next unit. Next, as far as symmetries go, we can see that the graph is symmetric about the line $y = x$:

[ADD GRAPH AGAIN WITH ANTI-DIAGONAL INCLUDED]

This indicates that $f(x) = 1/x$ is an odd function. You can also see this algebraically via

$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x).$$

By the way, the graph of $f(x) = 1/x$ is called a *hyperbola*.

Application: Inverses of trigonometric functions

It turns out that applying f to some famous functions, such as the usual trigonometric functions sin, cos and tan, usually produces new interesting functions which can help to model several situations in a perhaps simpler way.

Definition The *cosecant*, *secant* and *cotangent* functions are defined, respectively, by

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \cot x = \frac{1}{\tan x}.$$

Exploration

a. For which values of x do we have that $\sin x = 0$? Draw the graph of sin.

b. For which values of x is csc undefined? Recall that one cannot divide by zero.

c. Repeat items (a) and (b) replacing sin and csc with cos and sec, respectively. What about tan and cot?

[THIS IS PROBABLY NOT VERY ADEQUATE. CHANGE LATER]

Warning: Do not confuse inverses of trigonometric functions, as discussed above, with inverse trigonometric functions such as arcsin, arccos and arctan (as in "inverse functions", as discussed in Section 3-2).

Summary

- The function $f(x) = 1/x$ is defined for all non-zero values of x. It is an odd function, and its asymptotics can be understood by its graph, called a *hyperbola*.
- We can compose $f(x) = 1/x$ with functions we frequently encounter, to produce new functions which may prove useful when modeling certain problems and real life situations. For instance, doing this to trigonometric functions, one obtains

  $$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \cot x = \frac{1}{\tan x}.$$

  They are called, respectively, the *cosecant*, *secant* and *cotangent* functions.