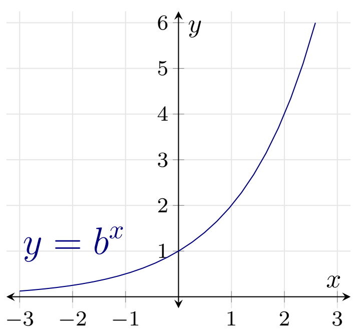
Zeros of Exponential Functions

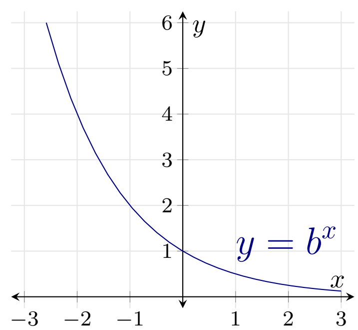
We examine the zeros of exponential and logarithmic functions.

Introduction

Zeros of Exponential Functions

As we saw previously, there are two varieties of elementary exponential functions: Increasing and Decreasing. The exponential function f given by $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Graphically, the two situations resemble the following.





These functions have domain $(-\infty, \infty)$ and range $(0, \infty)$. Notice that 0 is not in the range. That means the exponential function $f(x) = b^x$ has no zeros. The translated exponential functions, however, $g(x) = b^x + c$ will have a zero if c is negative.

Callout. *Remember that the natural logarithm,* $\ln(x)$*, is the inverse of the exponential function* e^x*.*

That means the composition $\ln(e^x) = x$ for all values of x. If we isolate the exponential on one side of our equation, we can use the logarithm to "undo" it.

Example 1. *Let f be the function given by $f(x) = 4e^x - 5$. Find the zeros of f.*

Learning outcomes:
Author(s): Bobby Ramsey

Explanation.

$$
\begin{aligned}
f(x) &= 0 \\
4e^x - 5 &= 0 \\
4e^x &= 5 \\
e^x &= \frac{5}{4} \\
\ln(e^x) &= \ln\left(\frac{5}{4}\right) \\
x &= \ln\left(\frac{5}{4}\right)
\end{aligned}
$$

This function has only a single zero, at $x = \ln\left(\frac{5}{4}\right)$.

The key to finding the zero in this example was being able to use the inverse function of e^x to bring down that variable. By examining the graphs of the exponentials above, you will notice that they pass the horizontal line test. That is, the exponential function $f(x) = b^x$ is a one-to-one function for any $b > 0$, $b \neq 1$. This means each of those exponential functions has an inverse, not just the base e exponential. These inverses are called logarithms.

Definition 1. *For a constant* $b > 0$, $b \neq 1$, *the* ***logarithm*** *with base* b, $\log_b(x)$, *is the inverse of the exponential function* b^x. *The domain of* $\log_b(x)$ *is* $(0, \infty)$ *and the range of* $\log_b(x)$ *is* $(-\infty, \infty)$.

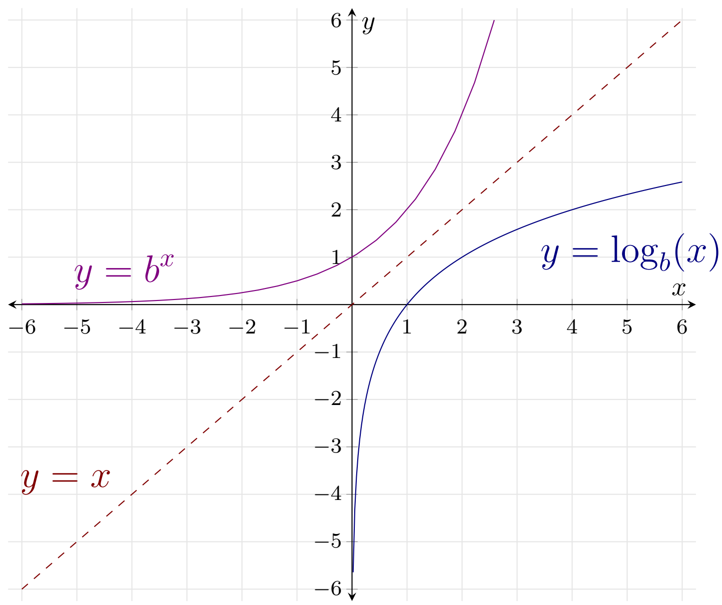
Remember that if f and f^{-1} are inverse functions, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

That the functions given by $\log_b(x)$ and b^x are inverses means:

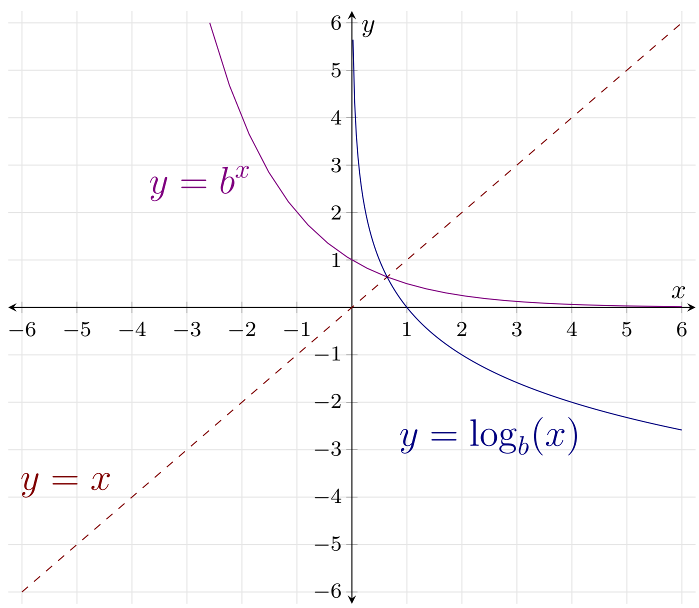
(a) $\log_b(b^x) = x$ for all x in $(-\infty, \infty)$

(b) $b^{\log_b(x)} = x$ for all x in $(0, \infty)$

The graphs of the exponentials b^x allow us to find the graphs of the corresponding logarithms by reflecting across the line $y = x$. For $b > 1$ we have this graph.



For $0 < b < 1$ we have this graph.



Here is a link to exponential functions and logarithms plotted on the same graph in Desmos. Move the slider for the base value of b and see how the two graphs respond. Desmos link: `https://www.desmos.com/calculator/q0aivjmasd`.

Example 2. *Let g be the function given by $g(x) = 2 \cdot 6^x - 5$. Find the zeros of* *the function g.*

Explanation. *Be careful with the order of operations here. Remember that $2 \cdot 6^x$ is not the same as 12^x.*

$$\begin{aligned} g(x) &= 0 \\ 2 \cdot 6^x - 5 &= 0 \\ 2 \cdot 6^x &= 5 \\ 6^x &= \frac{5}{2} \\ \log_6(6^x) &= \log_6\left(\frac{5}{2}\right) \\ x &= \log_6\left(\frac{5}{2}\right) \end{aligned}$$

The function g has a zero at $x = \log_6\left(\frac{5}{2}\right)$.

Example 3. *Let h be the function given by $h(t) = \left(\frac{1}{2}\right)^t + 3$. Find the zeros of h.*

Explanation.

$$\begin{aligned} h(t) &= 0 \\ \left(\frac{1}{2}\right)^t + 3 &= 0 \\ \left(\frac{1}{2}\right)^t &= -3 \end{aligned}$$

Our next step would be to take the logarithm, base $\frac{1}{2}$, of both sides to isolate the variable t, but that would mean taking the logarithm of -3. The domain of $\log_{1/2}(t)$ is $(0, \infty)$, so the logarithm of -3 does not exist. Said another way, the function $\left(\frac{1}{2}\right)^t$ has range $(0, \infty)$, so there is no value of t for which $\left(\frac{1}{2}\right)^t$ is -3.

This function has no zeros.

Notice that 0 is in the range of the logarithms. The fact that $b^0 = 1$ for all $b \neq 0$, means that for each logarithm, $\log_b(1) = 0$. Each logarithm $\log_b(x)$ has a zero at $x = 1$. If the function is modified, we can use the fact that $b^{\log_b(x)} = x$ for all x in $(-\infty, \infty)$ to find the zeros.

Example 4. *Let f be the function given by $f(x) = 3\log_5(x) + 7$. Find the zeros of f.*

Explanation.

$$\begin{aligned}
f(x) &= 0 \\
3\log_5(x) + 7 &= 0 \\
3\log_5(x) &= -7 \\
\log_5(x) &= -\frac{7}{3} \\
5^{\log_5(x)} &= 5^{-\frac{7}{3}} \\
x &= 5^{-\frac{7}{3}}
\end{aligned}$$

The function f has a zero at $x = 5^{-\frac{7}{3}}$.

Example 5. *Let k be the function given by $k(t) = \dfrac{2t\log_5(t)}{3e^t + 1}$. Find the zeros of f.*

Explanation. *We know that a fraction is zero precisely when the numerator is zero.*

$$\begin{aligned}
k(t) &= 0 \\
2t\log_5(t) &= 0
\end{aligned}$$

Setting these factors equal to zero we find either $2t = 0$, giving us the possible zero at $t = 0$, or $\log_5(t) = 0$, giving us the possible zero at $t = 1$. Let us check them.

$$\begin{aligned}
k(1) &= \frac{2(1)\log_5(1)}{3e^1 + 1} \\
&= \frac{2(0)}{3e + 1} = 0
\end{aligned}$$

However, $t = 0$ is not in the domain of k, since the $\log_5(t)$ factor would be undefined.

The function k has a zero at $x = 1$.