Zeros of Rational Functions

We find the zeros of a rational function.

Introduction

Suppose Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

The time it takes Julia to drive the boat downstream is $\dfrac{12}{v+2}$ hours and upstream is $\dfrac{12}{v-2}$ hours. The function to model the whole trip's time is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours. The trip will take 8 hours, so we want $t(v)$ to equal 8, and we have:

$$\frac{12}{v-2} + \frac{12}{v+2} = 8.$$

To solve this equation algebraically, we would start by subtracting 8 from both sides to obtain:

$$\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0.$$

This has taken our equation involving rational functions, and converted it into the problem of determining the zeros of a single rational function. Namely, we are really just finding the zeros of $s(v) = \dfrac{12}{v-2} + \dfrac{12}{v+2} - 8$. (Notice that the function was changed by subtracting the 8, so we had to use a new name for it.)

In the same way, whenever we are asked to find the solution of a rational equation, it is equivalent to finding the zeros of a rational function instead.

Zeros of Rational Functions

Example 1. *Let us finish the calculation started in the Introduction. Find the zeros of* $s(v) = \dfrac{12}{v-2} + \dfrac{12}{v+2} - 8$.

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Explanation

We will begin by combining the left-hand side into a single fraction. Notice the fractions that appear have a common denominator of $(v-2)(v+2) = v^2 - 4$.

$$\begin{aligned}
s(v) &= \frac{12}{v-2} + \frac{12}{v+2} - 8 \\
&= \frac{12}{v-2} \cdot \left(\frac{v+2}{v+2}\right) + \frac{12}{v+2} \cdot \left(\frac{v-2}{v-2}\right) - 8\left(\frac{(v+2)(v-2)}{(v+2)(v-2)}\right) \\
&= \frac{12(v+2)}{(v-2)(v+2)} + \frac{12(v-2)}{(v+2)(v-2)} - \frac{8(v^2-4)}{(v+2)(v-2)} \\
&= \frac{12v+24}{(v+2)(v-2)} + \frac{12v-24}{(v+2)(v-2)} - \frac{8v^2-32}{(v+2)(v-2)} \\
&= \frac{(12v+24)+(12v-24)-(8v^2-32)}{(v+2)(v-2)} \\
&= \frac{-8v^2+(12v+12v)+(24-24+32)}{(v+2)(v-2)} \\
&= \frac{-8v^2+24v+32}{(v+2)(v-2)}.
\end{aligned}$$

That means $s(v) = 0$ is equivalent to the equation $\dfrac{-8v^2+24v+32}{(v+2)(v-2)} = 0$.

Since a fraction is zero if and only if the numerator is zero (and the denominator is nonzero), we need to look at $-8v^2 + 24v + 32 = 0$. We'll start by factoring, since we see a common factor of 8 in the coefficients. Actually, let's factor out -8 to clean up the sign of the leading term: $-8v^2+24v+32 = -8(v^2-3v-4)$. The quadratic factor $v^2 - 3v - 4$ can be factored to $(v-4)(v+1)$. That means:

$$\begin{aligned}
\frac{-8v^2+24v+32}{(v+2)(v-2)} &= 0 \\
-8v^2+24v+32 &= 0 \\
-8(v^2-3v-4) &= 0 \\
-8(v-4)(v+1) &= 0.
\end{aligned}$$

Setting each factor equal to 0 we see that either $-8 = 0$ (which is impossible), $v - 4 = 0$ (which gives a possible solution of $v = 4$), and $v + 1 = 0$ (which gives a possible solution of $v = -1$.

There are two POSSIBLE solutions, $v = -1$ and $v = 4$. The process of solving a rational equation like this can sometimes introduce extraneous solution. That is, a number that appears to be a solution, but doesn't actually satisfy the original equation.

Let's plug both of these possibilities back into our original formula for $s(v)$ to verify that they are actually solutions.

$$\begin{aligned} s(-1) &= \frac{12}{(-1)-2} + \frac{12}{(-1)+2} - 8 \\ &= \frac{12}{-3} + \frac{12}{1} - 8 \\ &= -4 + 12 - 8 = 0 \end{aligned}$$

$$\begin{aligned} s(4) &= \frac{12}{(4)-2} + \frac{12}{(4)+2} - 8 \\ &= \frac{12}{2} + \frac{12}{6} - 8 \\ &= 6 + 2 - 8 = 0. \end{aligned}$$

That means both $v = -1$ and $v = 4$ are solutions to the rational equation $\frac{12}{v-2} + \frac{12}{v+2} - 8 = 0$.

Let's remember where this example came from. In the example, v represented a speed, so it cannot be negative. The only solution is $v = 4$ miles per hour.

Example 2. *Let f be the function given by $f(x) = \frac{1}{x-4} + x - \frac{x-3}{x-4}$. Find the zeros of the rational function f.*

Explanation

We are being asked to solve the equation

$$\frac{1}{x-4} + x - \frac{x-3}{x-4} = 0$$

Notice that the only denominators appearing in the fractions are $x - 4$, so the common denominator is $x - 4$. We start by combining these terms into a single fraction with that denominator.

$$\begin{aligned}
f(x) &= \frac{1}{x-4} + x - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + x \cdot \left(\frac{x-4}{x-4}\right) - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + \frac{x(x-4)}{x-4} - \frac{x-3}{x-4} \\
&= \frac{1}{x-4} + \frac{x^2-4x}{x-4} - \frac{x-3}{x-4} \\
&= \frac{(1)+(x^2-4x)-(x-3)}{x-4} \\
&= \frac{x^2+(-4x-x)+(1+3)}{x-4} \\
&= \frac{x^2-5x+4}{x-4}
\end{aligned}$$

Setting the numerator equal to zero and factoring gives the following.

$$\begin{aligned}
f(x) &= 0 \\
\frac{x^2-5x+4}{x-4} &= 0 \\
x^2-5x+4 &= 0 \\
(x-4)(x-1) &= 0.
\end{aligned}$$

By setting each of these factors equal to 0 we see that either $x-4=0$ (which gives a possible solution of $x=4$) and $x-1=0$ (which gives a possible solution of $x=1$). The two possible solutions are $x=4$ and $x=1$. Let's check them.

$$\begin{aligned}
f(1) &= \frac{1}{(1)-4} + (1) - \frac{(1)-3}{(1)-4} \\
&= \frac{1}{-3} + 1 - \frac{-2}{-3} \\
&= -\frac{1}{3} + 1 - \frac{2}{3} = 0.
\end{aligned}$$

However, $x=4$ is not in the domain of f, since it makes the denominators of the first and third terms zero. That is, $x=4$ is an extraneous solution.

The only solution is $x=1$.

Example 3. *Let g be the function given by* $g(p) = \dfrac{3}{p-2} + \dfrac{5}{p+2} - \dfrac{12}{p^2-4}$. *Find the zeros of the rational function g.*

Explanation

Since $p^2 - 4 = (p+2)(p-2)$, the least common denominator between these three fractions is $(p+2)(p-2) = p^2 - 4$. As before, we start by combining into a single fraction with that denominator.

$$\begin{aligned}
g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\
&= \frac{3}{p-2} \cdot \left(\frac{p+2}{p+2}\right) + \frac{5}{p+2} \cdot \left(\frac{p-2}{p-2}\right) - \frac{12}{p^2-4} \\
&= \frac{3(p+2)}{(p-2)(p+2)} + \frac{5(p-2)}{(p+2)(p-2)} - \frac{12}{p^2-4} \\
&= \frac{3p+6}{(p+2)(p-2)} + \frac{5p-10}{(p+2)(p-2)} - \frac{12}{(p+2)(p-2)} \\
&= \frac{(3p+6)+(5p-10)-(12)}{(p+2)(p-2)} \\
&= \frac{(3p+5p)+(6-10-12)}{(p+2)(p-2)} \\
&= \frac{8p-16}{(p+2)(p-2)}
\end{aligned}$$

Setting the numerator equal to zero gives the following.

$$\begin{aligned}
8p - 16 &= 0 \\
8p &= 16 \\
p &= \frac{16}{8} = 2.
\end{aligned}$$

There is one possible solution, at $p = 2$.

However, $p = 2$ is not in the domain of g, since it makes the denominators of the first and third terms zero. That is, $p = 2$ is an extraneous solution.

The function g does not have any zeros.

Let's look at this last example a bit more. We combined the three terms of $g(p)$

into a single fraction, but that fraction was not in its reduced form.

$$\begin{aligned} g(p) &= \frac{3}{p-2} + \frac{5}{p+2} - \frac{12}{p^2-4} \\ &= \frac{8p-16}{(p+2)(p-2)} \\ &= \frac{8(p-2)}{(p+2)(p-2)} \\ &= \frac{8\cancel{(p-2)}}{(p+2)\cancel{(p-2)}} \\ &= \frac{8}{(p+2)} \quad \text{, for } p \neq 2. \end{aligned}$$

Why was $p = -2$ not a zero of the function? Because it was also a zero of the denominator. (Notice the common factor of $p - 2$ in both the numerator and denominator.) Rewriting the fraction in lowest terms, we see that the numerator is never zero, since it's a constant 8.

From this example, it may seem that reducing the rational function to lowest terms will always help you bypass the extraneous solutions. That is not the case.

Example 4. *Let r be the function given by $r(t) = \dfrac{(t+3)(t^2-2t+1)}{t^2-1}$. Find the zeros of r.*

Explanation

Since we are already given the formula for r as a single fraction, let us simplify.

$$\begin{aligned} r(t) &= \frac{(t+3)(t^2-2t+1)}{t^2-1} \\ &= \frac{(t+3)(t-1)^2}{(t+1)(t-1)} \\ &= \frac{(t+3)(t-1)(t-1)}{(t+1)(t-1)} \\ &= \frac{(t+3)(t-1)\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} \\ &= \frac{(t+3)(t-1)}{(t+1)} \end{aligned}$$

The fraction will be zero when the numerator is zero. Setting each of the factors of the numerator equal to zero gives $t + 3 = 0$ (which gives a possible solution of $t = -3$), and $t - 1 = 0$ (which gives a possible solution of $t = 1$).

When we cancelled out the common factor of $t - 1$ from the numerator and the denominator, we changed the function without mentioning it. This new fraction $\dfrac{(t+3)(t-1)}{(t+1)}$ has $t = 1$ in its domain, but it is not in the domain of r. To be thorough, after that cancellation we should have written

$$r(t) = \frac{(t+3)(t-1)}{(t+1)}\text{, for } t \neq 1$$

to indicate that we're still using the original domain of r.

The function r has a single zero, at $x = -3$.