Zeros of Functions with Radicals

We find the zeros of a function with radicals.

Introduction

In the previous section we found zeros for rational functions and ran into the problem of extraneous solutions. Radical functions are another instance where we have to check for those.

Callout. *Earlier we discussed that domains of the polynomial functions given by the form x^n, for n a positive, whole number. If n is odd the range is $(-\infty, \infty)$. That means for every number b, there is a number a with $a^n = b$. In this case, there is exactly one number a with this property, which is denoted by $a = \sqrt[n]{b}$. This means $\sqrt[n]{b}$ exists, when n is odd, for all real numbers b.*

If n is even the range of x^n is $[0, \infty)$. This means that for every nonnegative number b, there is a number a with $a^n = b$. In this case, there are exactly two numbers a with this property (unless $b = 0$). Those numbers have the same absolule value, but one is positive and the other is negative. The positive one is denoted by $\sqrt[n]{b}$. This means $\sqrt[n]{b}$ exists, when n is even, only for nonnegative real numbers b and the result $\sqrt[n]{b}$ is nonnegative.

Definition 1. *The* ***principal n^{th} root function*** *is given by $\sqrt[n]{x}$, for n a positive, whole number called the* ***index of the radical****. The value of $\sqrt[n]{x}$ has the property that $\left(\sqrt[n]{x}\right)^n = x$.*

For n odd, the domain of $\sqrt[n]{x}$ is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$. For n even, the domain of $\sqrt[n]{x}$ is $[0, \infty)$ and the range is $[0, \infty)$.

Zeros of Rational Functions

Example 1. *Let g be the function given by $g(x) = \sqrt{1 - x^2}$. Find the zeros of g.*

Learning outcomes:

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Explanation.

$$\begin{aligned}
g(x) &= 0 \\
\sqrt{1-x^2} &= 0 \\
\left(\sqrt{1-x^2}\right)^2 &= (0)^2 \\
1-x^2 &= 0 \\
(1+x)(1-x) &= 0
\end{aligned}$$

From these factors, either $x=-1$ or $x=1$. Checking these gives:

$$\begin{aligned}
g(1) &= \sqrt{1-(1)^2} = 0 \\
g(-1) &= \sqrt{1-(-1)^2} = 0
\end{aligned}$$

Both of these check out, so the zeroes are $x = \pm 1$.

Example 2. *Let f be the function given by $f(x) = \sqrt{3x+7} - x - 1$. Find the zeros of the function f.*

Explanation. *To solve the equation $\sqrt{3x+7} - x - 1 = 0$, we'll isolate the radical and square both sides.*

$$\begin{aligned}
\sqrt{3x+7} - x - 1 &= 0 \\
\sqrt{3x+7} &= x+1 \\
\left(\sqrt{3x+7}\right)^2 &= (x+1)^2 \\
3x+7 &= x^2+2x+1 \\
x^2 - x - 6 &= 0 \\
(x+2)(x-3) &= 0
\end{aligned}$$

Setting these factors each equal to zero gives us the possible solutions $x=-2$ and $x=3$. Let's check them.

$$\begin{aligned}
f(3) &= \sqrt{3(3)+7} - (3) - 1 \\
&= \sqrt{16} - 4 = 0
\end{aligned}$$

$$\begin{aligned}
f(-2) &= \sqrt{3(-2)+7} - (-2) - 1 \\
&= \sqrt{1} + 1 = 2
\end{aligned}$$

Notice that $f(-2)$ is not zero, so $x=-2$ is an extraneous solution.

The only zero is $x=3$.

Example 3. *Let s be the function given by $s(t) = t - \sqrt[3]{t^3 + 3t^2 - 6t - 8} + 1$. Find the zeros of the function s.*

Explanation. *As before, we'll isolate the radical. However, the radical here is a cube root so we have to raise each side to the third power, instead of squaring them.*

$$\begin{aligned}
s(t) &= 0 \\
t - \sqrt[3]{t^3 + 3t^2 - 6t - 8} + 1 &= 0 \\
t + 1 &= \sqrt[3]{t^3 + 3t^2 - 6t - 8} \\
(t+1)^3 &= \left(\sqrt[3]{t^3 + 3t^2 - 6t - 8}\right)^3 \\
t^3 + 3t^2 + 3t + 1 &= t^3 + 3t^2 - 6t - 8 \\
9t &= -9 \\
t &= -1
\end{aligned}$$

The only possible zero is $t = -1$. Let's check it.

$$\begin{aligned}
s(-1) &= (-1) - \sqrt[3]{(-1)^3 + 3(-1)^2 - 6(-1) - 8} + 1 \\
&= -\sqrt[3]{-1 + 3 + 6 - 8} = 0.
\end{aligned}$$

The function s has a single zero, at $t = -1$.

Example 4. *Let f be the function given by $f(x) = 3 + \sqrt{4-x} - \sqrt{2x+1}$. Find the zeros of f.*

Explanation. *In this case, there are multiple radicals and we can't isolate them both simultaneously. Instead, we'll isolate just one of them first.*

$$\begin{aligned}
f(x) &= 0 \\
3 + \sqrt{4-x} - \sqrt{2x+1} &= 0 \\
3 + \sqrt{4-x} &= \sqrt{2x+1} \\
\left(3 + \sqrt{4-x}\right)^2 &= \left(\sqrt{2x+1}\right)^2 \\
9 + 6\sqrt{4-x} + (4-x) &= 2x + 1 \\
6\sqrt{4-x} &= 3x - 12 \\
3\left(2\sqrt{4-x}\right) &= 3(x-4) \\
2\sqrt{4-x} &= x - 4 \\
\left(2\sqrt{4-x}\right)^2 &= (x-4)^2 \\
4(4-x) &= x^2 - 8x + 16 \\
16 - 4x &= x^2 - 8x + 16 \\
x^2 - 4x &= 0 \\
x(x-4) &= 0
\end{aligned}$$

Setting each of these factors equal to zero gives the two possible zeros $x = 0$ and $x = 4$. Let's check them.

$$\begin{aligned} f(0) &= 3 + \sqrt{4-(0)} - \sqrt{2(0)+1} \\ &= 3 + \sqrt{4} - \sqrt{1} = 4 \end{aligned}$$

$$\begin{aligned} f(4) &= 3 + \sqrt{4-(4)} - \sqrt{2(4)+1} \\ &= 3 + \sqrt{0} - \sqrt{9} = 0. \end{aligned}$$

The possible zero at $x = 0$ is an extraneous solution. The only zero of f is $x = 4$.