From Systems to Solutions

Motivating Questions.

- *What is a system of equations?*
- *What is a solution to a system?*
- *How can we solve systems of equations using graphs?*

Introduction

We have already seen many techniques for solving equations. Until now, however, we have only solved equations of the form $f(x) = 0$ for the variable x. In this section, we will consider equations with more than one variable and discuss how to solve them.

Consider a peculiar grocery store where the prices of all the items for sale are not listed, and you only find out the total cost of your purchase. Say you buy 6 mangos and 3 bananas, and your total cost is 9 dollars. Assume all the mangos cost the same amount and all the bananas cost the same amount. Without making any more purchases, is it possible find out how much a mango and a banana cost on their own?

Let's create an equation to describe this situation. Let x be a variable representing the cost of a mango, and let y be a variable representing the cost of a banana. Then, the equation

$$6x + 3y = 9$$

represents that buying 6 mangos at a cost of x dollars and 3 bananas at a cost of y dollars yields a total cost of 9 dollars.

You might have noticed that plugging $x = 1$ and $y = 1$ into the equation gives us a true statement, so you might conclude that mangos and bananas both cost 1 dollar. However, notice that plugging $x = 1.20$ and $y = 0.60$ into the equation also gives us a true statement, so it's also possible that mangos cost \$1.20 and bananas cost \$0.60. Even more worrying is that $x = 0$ and $y = 3$ also gives us a solution to the equation: is this store peculiar enough to be giving away mangos for free and charging \$3 per banana?

Examining the equation we set up can give us more insight. Let's rearrange the

Learning outcomes:

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equation to solve for y in terms of x:

$$\begin{aligned} 6x + 3y &= 9 \\ 3y &= 9 - 6x \\ y &= 3 - 2x. \end{aligned}$$

Now it becomes clearer what's going on. Whatever x is, we can find a value of y that satisfies our original equation. No matter the cost of a single mango, there's a way to price the bananas so that our equation is true! This means it's impossible to find the price of a single mango or a single banana with the information you've been given. We need more data!

Systems of linear equations

In order to collect more information, you go back to the store and buy 2 mangos and 2 banana for a total cost of \$3.20. This can be modeled by the equation

$$2x + 2y = 3.2.$$

Keep in mind that this x and y are the same x and y from before, so in order to find the cost of a mango and a banana, we must find x and y that satisfy both equations

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}$$

at the same time. This coupling of two (or more) linear equations is called a *system of linear equations.*

Definition 1. *A **linear equation of two variables** is an equation of the form*

$$a_1 x + a_2 y = c,$$

where a_1, a_2, and c are real numbers and at least one of a_1 and a_2 is nonzero.

*A **system of linear equations of two variables** is a collection of two or more linear equations of two variables.*

*We say a **solution** to a system of linear equations of two variables is a point (x, y) satisfying all equations in the system.*

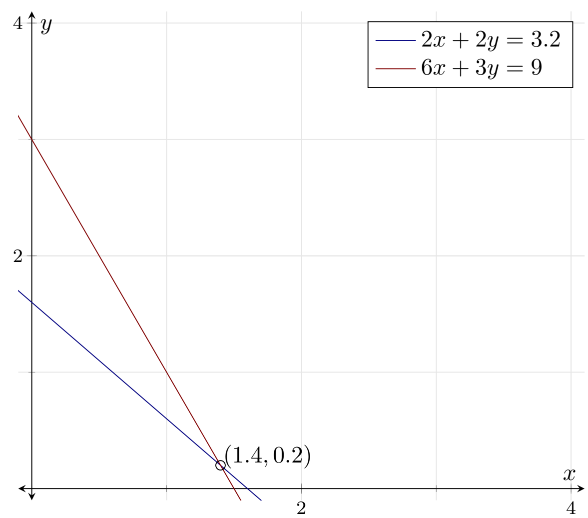
The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. Note that we can still have systems of non-linear equations, but they can be much more difficult to solve.

Finding solutions graphically

Let's return to our example from earlier and try to find a solution to the system of linear equation:

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}.$$

We want to find x and y satisfying both equations in the system. If x and y satisfy $6x + 3y = 9$, then the point (x, y) lies on the graph of $6x + 3y = 9$. Similarly, if x and y satisfy $2x+2y = 3.2$, then the point (x, y) lies on the graph of $2x + 2y = 3.2$. Therefore, to find any solutions, we can look at the graphs of $6x + 3y = 9$ and $2x + 2y = 3.2$, and see if there are any points that lie at the intersection of the two graphs:



By inspecting the graph, we see that these two lines intersect only at $(1.4, 0.2)$, so the only solution to the system is $x = 1.4$ and $y = 0.2$.

In context, this means that mangos cost \$1.40 each and bananas cost \$0.20 each. Note that in order to have exactly one solution to our system of linear equations in two variables, we needed the system to have two equations.

Note that not every system of linear equations will have one solution. If the graphs of the two equations are parallel, they will never intersect, so there won't be any solutions. Additionally, if the two equations are represented by the same graph, there will be infinitely many intersection points, and therefore, infinitely many solutions.

Next, we will see some methods for solving systems of equations algebraically.