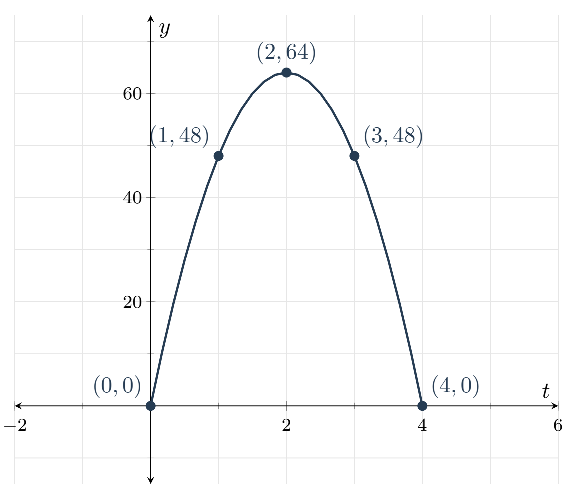
Parabolas

We explore polynomial functions.

Quadratic Graphs

Example 1. *Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of* 64 *feet per second. The height of the rocket can be modeled by the equation* $y = -16t^2 + 64t$*, where* t *is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.*

t	$-16t^2 + 64t$	$Point$
0	$-16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$-16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$-16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$-16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$-16(4)^2 + 64(4) = 0$	$(4, 0)$



A curve with the shape that we see in the above figure is called a **parabola**. Notice the symmetry in figure, how the y-values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

Learning outcomes:
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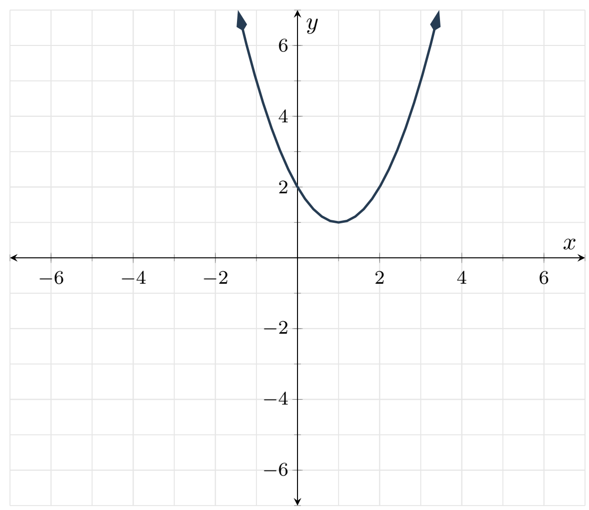
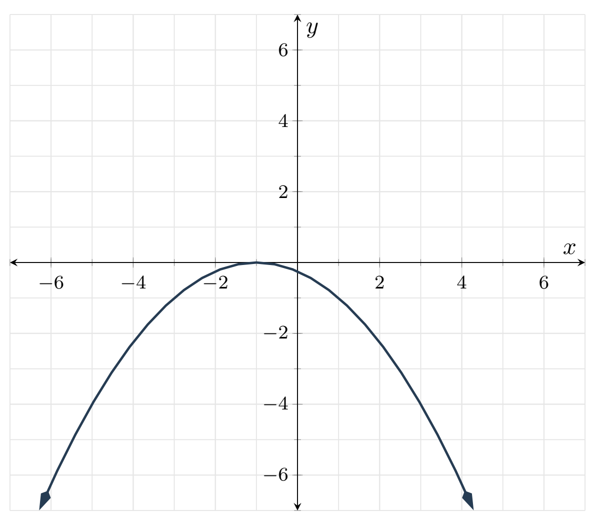
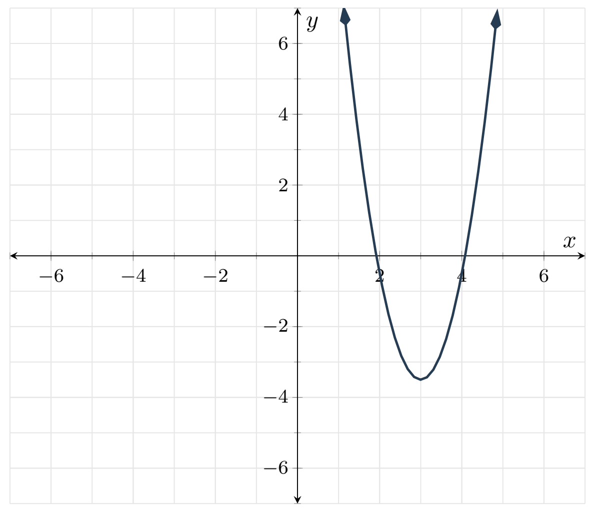
The first feature that we will talk about is the direction that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. That means that for large values of t, the at^2 term will be large and negative, and the resulting y-value will be low on the y-axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.







The graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola which opens upward or downward according to the sign of the leading coefficient a. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.

The **vertex** of a parabola is the highest or lowest point on the graph, depending uponon whether the graph opens downward or upward. In Example 1, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value. The **axis of symmetry** is a vertical line that passes through the vertex, cutting the parabola into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$". In Example 1, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the vertical axis is labeled y. In Example 1, the point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the vertical intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the horizontal axis is labeled x. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

It is possible for a quadratic graph to have zero, one, or two horizontal intercepts.

The figures below show an example of each.

Example 2. *Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.*

(a) *Find the vertex.*

(b) *Find the vertical intercept (i.e. the y-intercept).*

(c) *Find the horizontal or (i.e. the x-intercept(s)).*

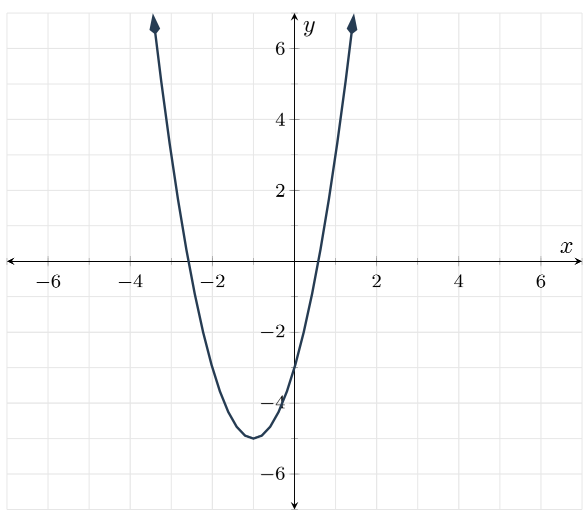
(d) *Find $f(-2)$.*

(e) *Solve $f(x) = 3$ using the graph.*

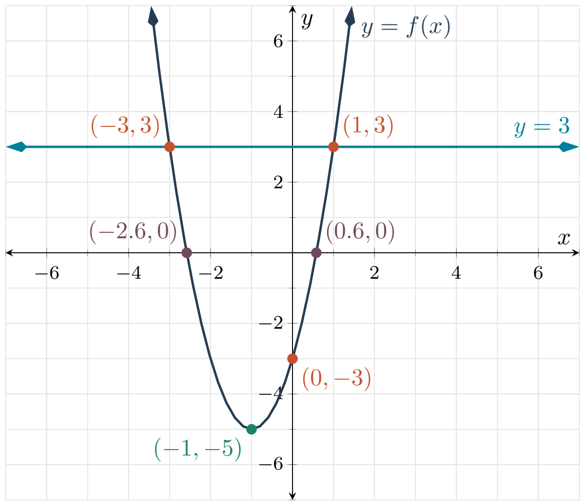
(f) *Solve $f(x) \leq 3$ using the graph.*

Explanation. *The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:*

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



(a) *The vertex is* $(-1, -5)$.

(b) *The vertical intercept is* $(0, -3)$.

(c) *The horizontal intercepts are approximately* $(-2.6, 0)$ *and* $(0.6, 0)$.

(d) *When* $x = -2$, $y = -3$, *so* $f(-2) = -3$.

(e) *The solutions to* $f(x) = 3$ *are the* x*-values where* $y = 3$. *We graph the horizontal line* $y = 3$ *and find the* x*-values where the graphs intersect. The solution set is* $\{-3, 1\}$.

(f) *The solutions are all of the* x*-values where the function's graph is below (or touching) the line* $y = 3$. *The interval is* $[-3, 1]$.

The Vertex Form of a Quadratic

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. we will learn another form called the vertex form.

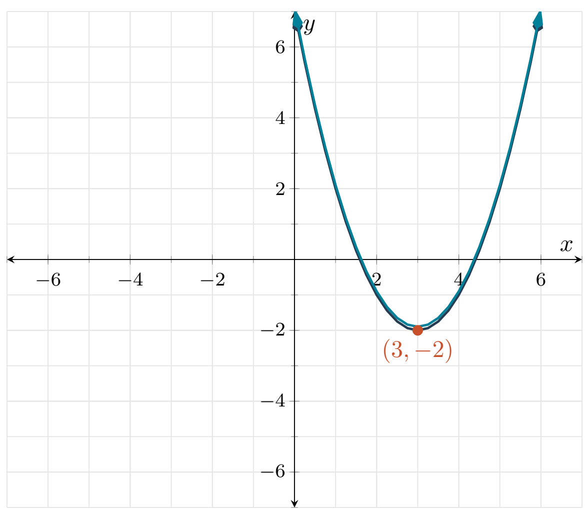
Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$\begin{aligned} g(x) &= (x-3)^2 - 2 \\ &= (x^2 - 6x + 9) - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

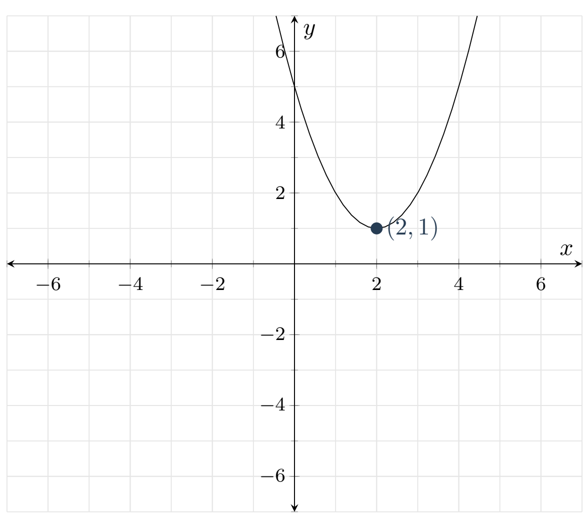
it is clear that f and g are the same function.

Graph of $f(x) = x^2 - 6x + 7$ and $g(x) = (x-3)^2 - 2$ the graphs of the two parabolas overlap each other completely

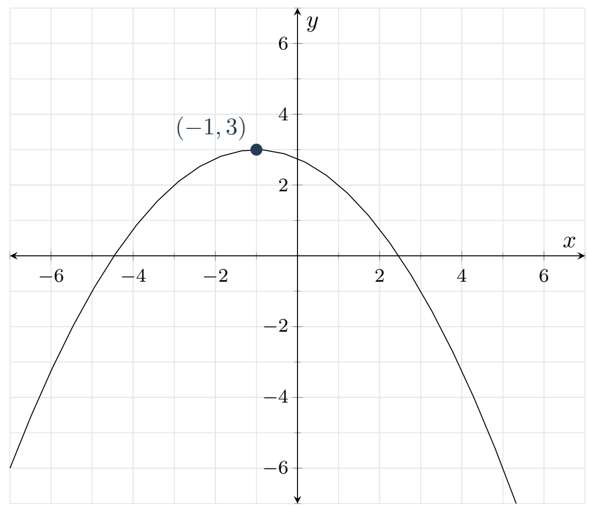


The formula given for g is said to be in vertex form because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x-3)^2 - 2$. The x-value is the solution to $(x-3) = 0$, and the y-value is the constant added at the end.

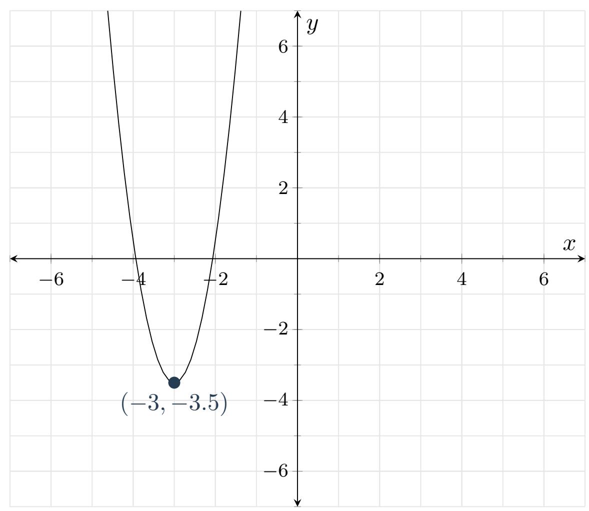
Example 3. *Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.*



$$r(x) = (x-2)^2 + 1$$



$$s(x) = -\frac{1}{4}(x+1)^2 + 3$$



$$t(x) = 4(x+3)^2 - 3.5$$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$r(2) = (2-2)^2 + 1 = 1$

The x-value is the solution to $(x-2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2, 1)$. Now we can define the vertex form of a quadratic function.

Callout. ***Vertex Form of a Quadratic Function*** *A quadratic function whose graph has vertex at the point* (h, k) *is given by*

$$f(x) = a(x - h)^2 + k$$