Precalculus with Review 2

January 7, 2022

Contents

Part 1

Functions Revisited

1.1 Review of Functions

Learning Objectives

- The objective of this section is to review major ideas from Math 1120 in preparation for Math 1121.

1.1.1 Famous Functions, Updated

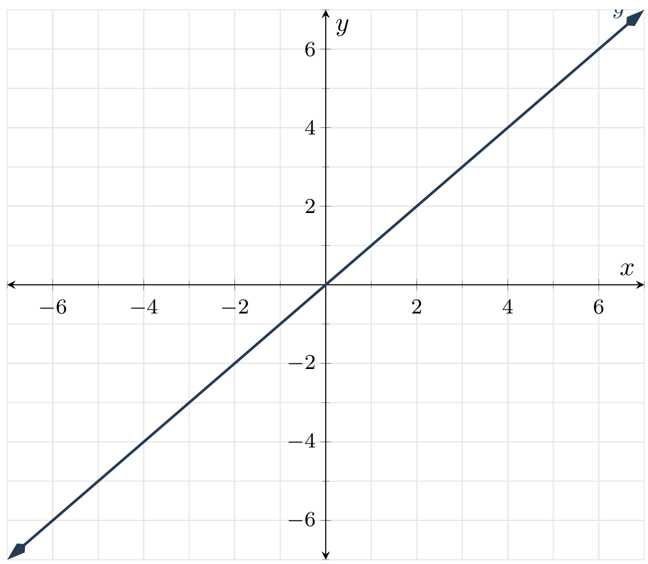
Now that we know about the domain and range, we can update our list of famous functions.

Linear Functions

Recall that the graph of a linear function is a line.

Example 1. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$	
x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: https://www.desmos.com/calculator/japnhapzvn

Properties of Linear Functions $y = mx + b$	
Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible?	Yes

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx+b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

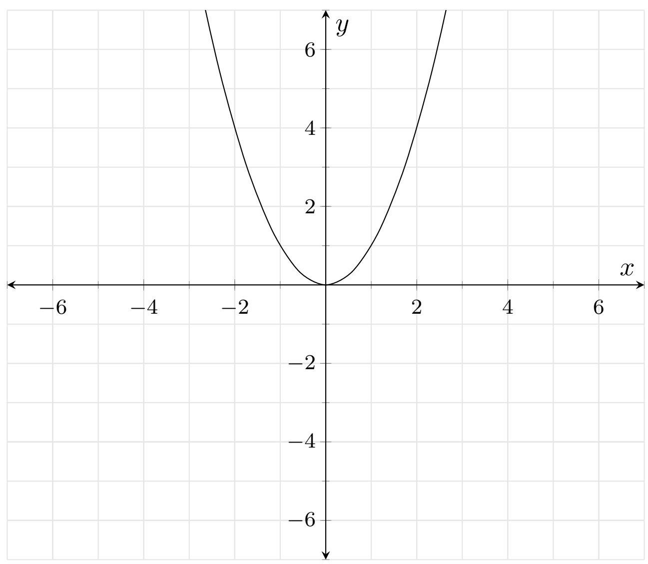
Domain and Range of Linear Functions $y = mx + b$	
Domain	$(-\infty, \infty)$
Range	If $m \neq 0$, $(-\infty, \infty)$; if $m = 0$, $\{b\}$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 2. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$	
x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Properties of Quadratic Functions $y = ax^2 + bx + c, a \neq 0$	
Periodic?	No
Odd?	No
Even?	If $b = 0$
One-to-one/invertible?	No

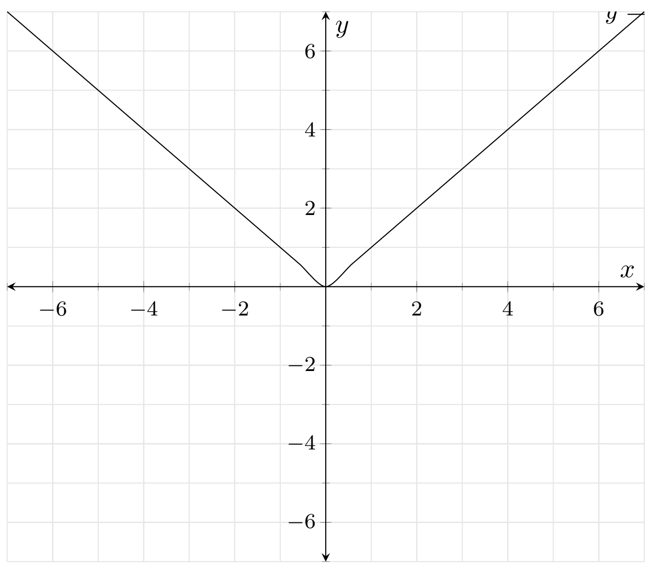
Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic functions have a vertex form $f(x) = d(x-h)^2+k$, where the vertex is at (h, k). If $d > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $d < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

Domain and Range of Quadratic Functions $y = d(x-h)^2 + k$	
Domain	$(-\infty, \infty)$
Range	If $d > 0$, $[k, \infty)$; if $d < 0$, $(\infty, k]$

Absolute Value

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = \lvert x \rvert$	
x	y
-2	2
-1	1
0	0
1	1
2	2

Properties of the Absolute Value Function $y = \lvert x \rvert$	
Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible?	No

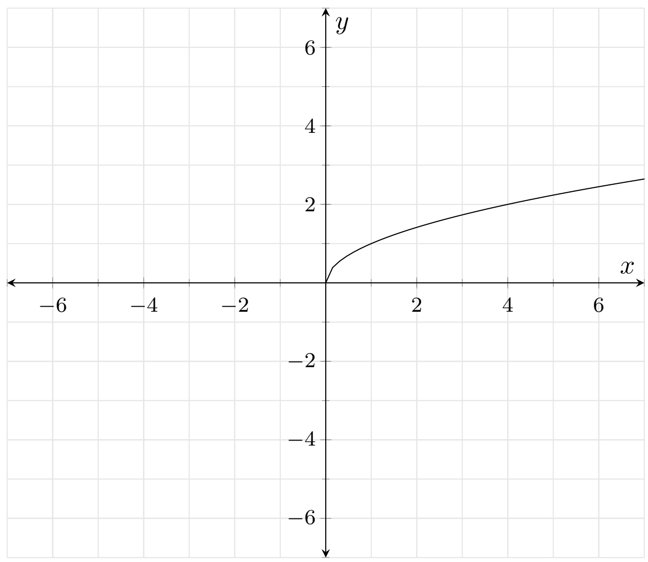
Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

Domain and Range of the Absolute Value Function $y = \lvert x \rvert$	
Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$	
x	y
0	0
1	1
4	2
9	3
25	5

Properties of the Square Root Function $y = \sqrt{x}$	
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function.

Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

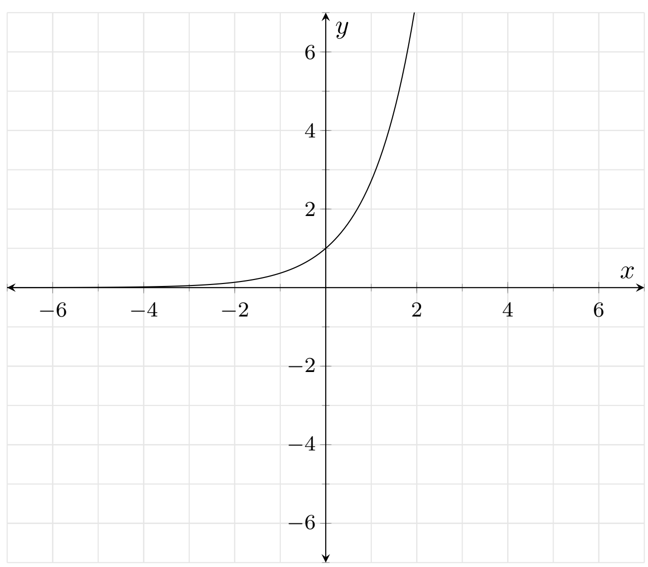
Domain and Range of the Square Root Function $y = \sqrt{x}$	
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$	
x	y
0	1
1	e
-1	$\frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/qsmvb7tiex`

Properties of the Exponential Functions $y = b^x$	
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

Domain and Range of the Exponential Functions $y = b^x$	
Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

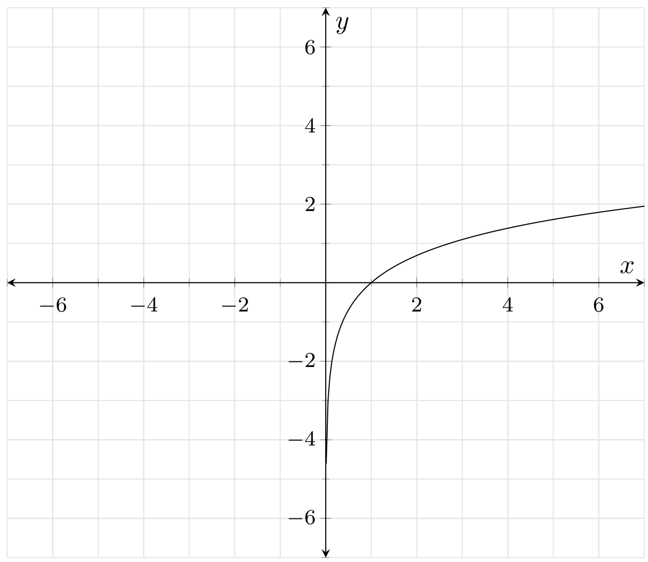
Logarithm

Another group of famous functions are logarithms.

Example 3. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = ln(x)$	
x	y
0	undefined
$\frac{1}{e}$	-1
1	0
e	1

You may notice that the table of values for $y = \ln(x)$ and $y = e^x$ are similiar. This is becase these two functions are interconnected. We will explore this more later in the course.

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/lxllnpdi6w`

Properties of the Logarithm Functions $y = \log_b(x)$	
Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible?	Yes

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

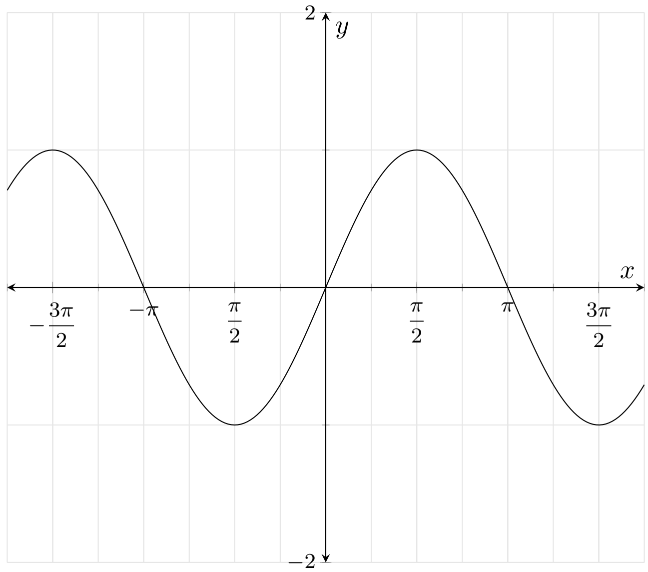
Domain and Range of the Logarithm Functions $y = \log_b(x)$	
Domain	$[0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Properties of the Sine Function $y = \sin(x)$	
Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible?	No

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$.

Domain and Range of the Sine Function $y = \sin(x)$	
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

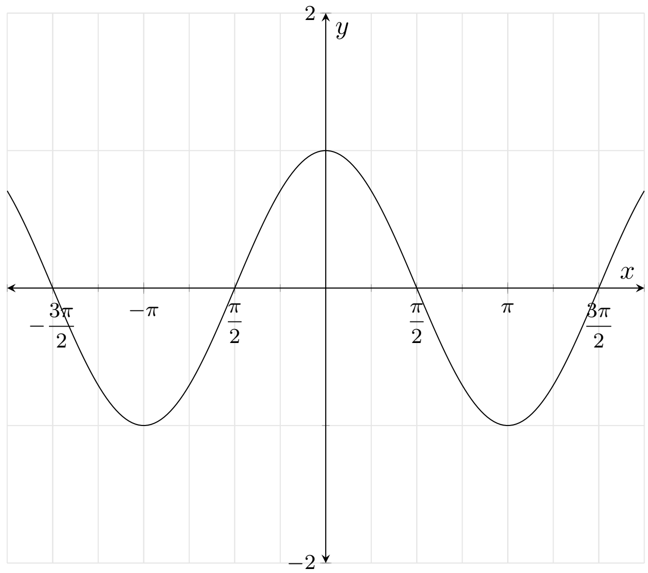
Desmos link: `https://www.desmos.com/calculator/vkxzcfv2aq`

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$	
x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in the following table

Properties of the Cosine Function $y = \cos(x)$	
Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible?	No

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Domain and Range of the Cosine Function $y = \cos(x)$	
Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

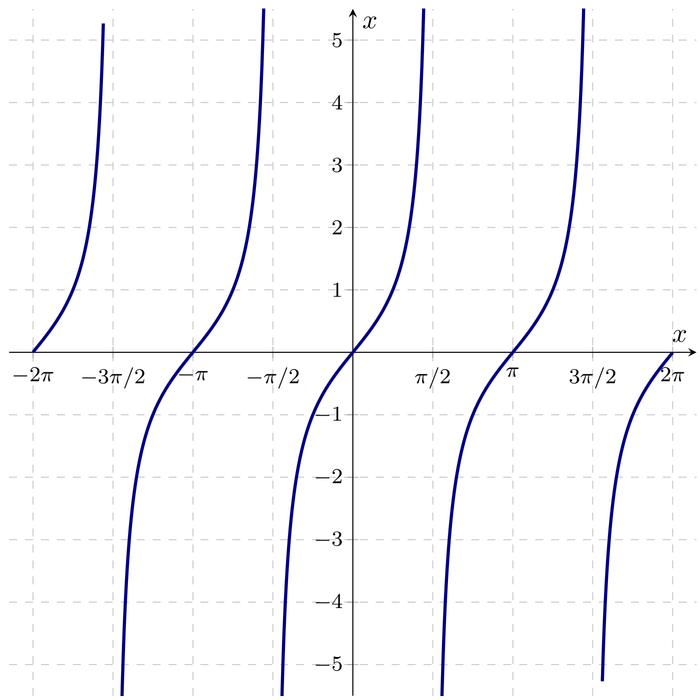
In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

Tangent

A function introduced in Section 4-1 is the tangent function,

$$y = \tan(x).$$



Important Values of $y = \tan(x)$	
x	y
$-\pi$	0
$-\dfrac{\pi}{2}$	undefined
0	0
$\dfrac{\pi}{2}$	undefined
π	0
$\dfrac{3\pi}{2}$	undefined
2π	0

As mentioned earlier, the tangent function is odd and periodic with period π.

Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in the following table.

Properties of the Tangent Function $y = \tan(x)$	
Periodic?	Yes, with period π
Odd?	Yes
Even?	No
One-to-one/invertible?	No

Note that the domain of the tangent function is all real numbers except for odd multiples of $\frac{\pi}{2}$, since tangent is undefined at those places. Furthermore, by looking at the graph, we can see that its range is $(-\infty, \infty)$.

Domain and Range of the Tangent Function $y = \tan(x)$	
Domain	$\cdots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots$
Range	$(-\infty, \infty)$

In general, we can consider $y = a\tan(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/1je3xt6hag`

1.2 Function Transformations

Learning Objectives

- Vertical and Horizontal Shifts
  - How to shift a function vertically
  - How to shift a function horizontally
  - Combining shifts and properties of quadratics (vertex, completing the square)
- Stretching Functions
  - Vertical stretch
  - Horizontal stretch
- Reflections of Functions
  - Reflections across the x-axis, the y-axis, the origin, $y = x$
  - Connect reflections to inverses, even, and odd functions

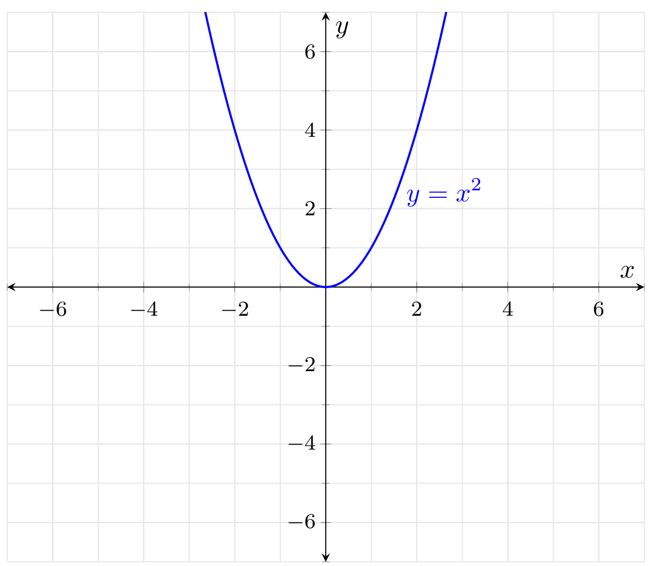
1.2.1 Vertical and Horizontal Shifts

Motivating Questions

- By adding a constant a to a function $f(x)$, what is the relation between the graphs of $y = f(x)$ and $y = f(x) + a$?
- By performing a "change of variable" $x \mapsto x - a$, what is the relation between the graphs of $y = f(x)$ and $y = f(x - a)$?
- How to use this new understanding to gain a deeper understanding of graphs of quadratic functions (i.e., parabolas)?

Introduction

Let's consider the two quadratic functions $f(x) = x^2$ and $g(x) = x^2 - 2x + 1$, defined for all real values of x. We know what their graphs look like:



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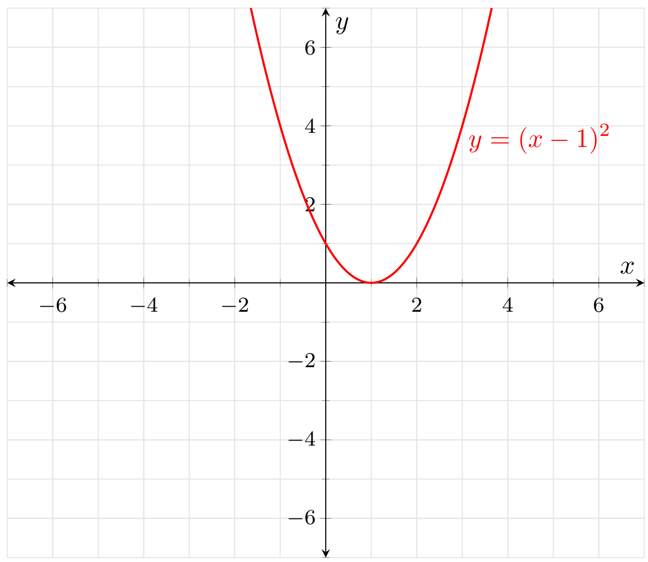
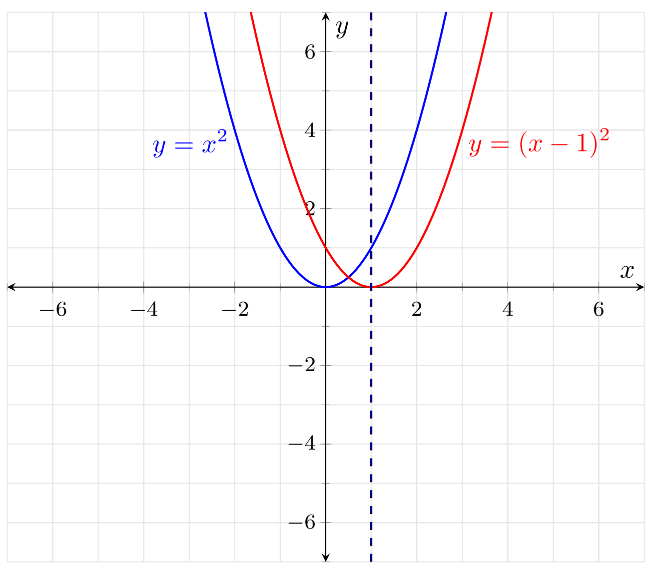


The graphs are very similar, down to the horizontal "width". In fact, drawing them together, we may see that they only differ by a horizontal translation:



Algebraically, one can see that this happens because

$$g(x) = x^2 - 2x + 1 = (x-1)^2 = f(x-1).$$

This hints at the following general fact: doing horizontal shifts to the graph of a function amounts to replacing x with "$x \pm$shift" inside $f(\cdot)$. In this unit, we'll

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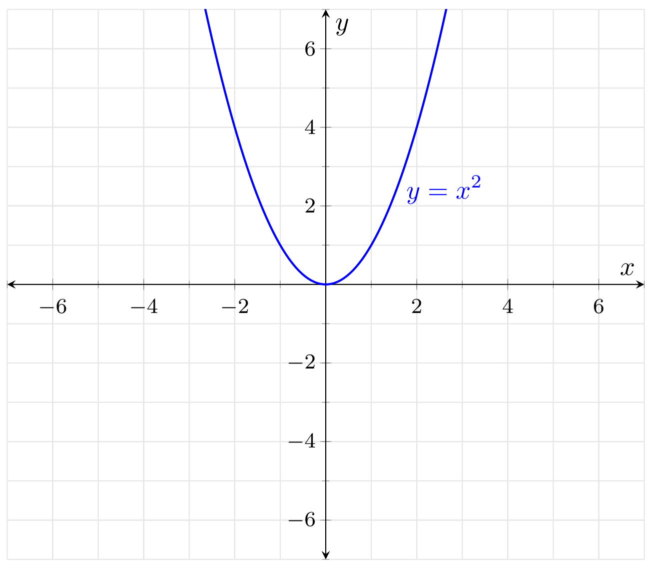
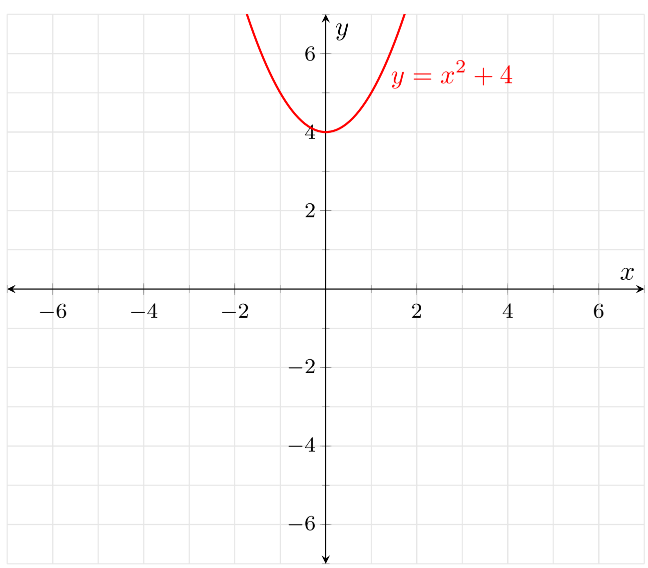
understand in more detail how to work with this, and also how to deal with vertical shifts, as opposed to horizontal shifts. Since vertical shifts are much easier to understand, that's where we'll begin.

Shifting a function vertically

Let's consider a very simple situation, where we have two functions $f(x) = x^2$ and $g(x) = x^2 + 4$. Graphing them, in order, we have that





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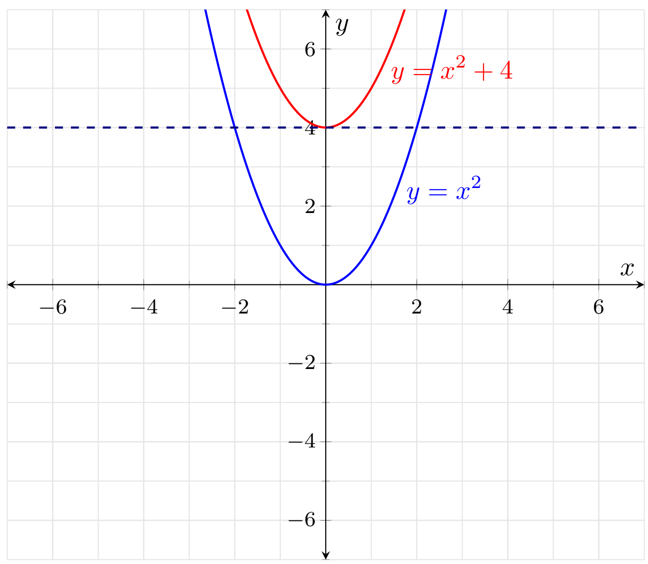
Clearly, $f(x)$ and $g(x)$ are directly related via $g(x) = f(x) + 4$, and seeing their graph together, we have that:



In other words, the graph of $y = g(x)$ was obtained from the graph of $y = f(x)$ by shifting it up exactly by 4 units. This is a very general phenomenon, that happens for any functions who differ by a constant.

Theorem (vertical shifts): Suppose f is a function and a is a positive number.

- To graph $y = f(x) + a$, shift the graph of $y = f(x)$ up a units, by adding a to the y-coordinates of the points on the graph of f.
- To graph $y = f(x) - a$, shift the graph of $y = f(x)$ down a units, by subtracting a from the y-coordinates of the points on the graph of f.

In the above setting, it is useful to call f the **parent function**.

Convention: Here, we'll always draw graphs of parent functions in blue, and graphs of the "child" functions in red. We'll indicate with a dashed line where the shift has happened.

Example 4. *For each of the following functions, find the parent function. How would the graph look like, in terms of the graph of the original function?*

a. $g(x) = 2e^{-x} - 1$.

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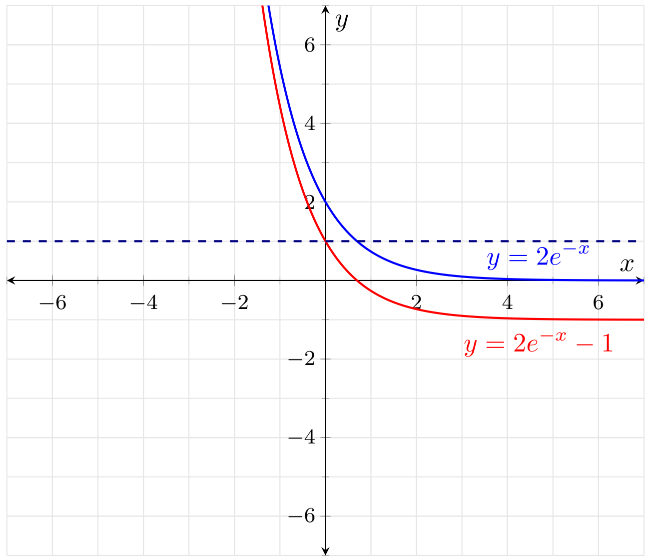
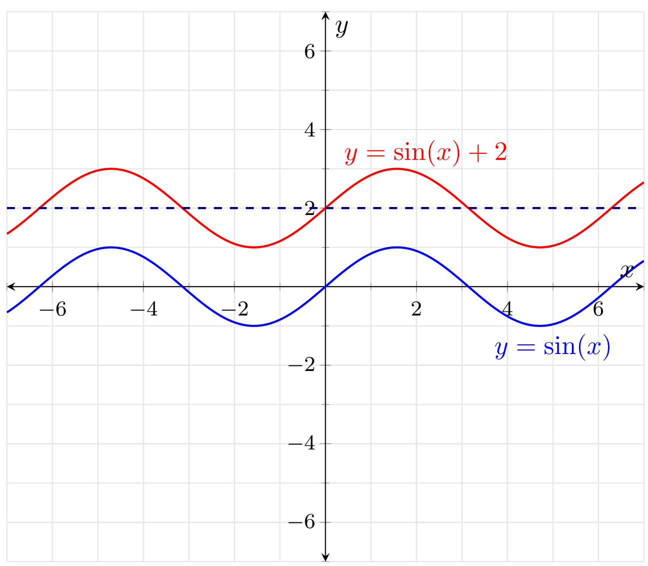
Explanation Setting $f(x) = 2e^{-x}$, we have that $g(x) = f(x) - 1$. So, to graph $y = g(x)$, we just need to consider the graph of $y = f(x)$ and shift it one unit down.



b. $g(x) = \sin(x) + 2$.

Explanation Here, we can just recognize that for $f(x) = \sin(x)$, we have $g(x) = f(x) + 2$. Thus, we just need to shift the graph of $y = \sin(x)$ up by 2 units.



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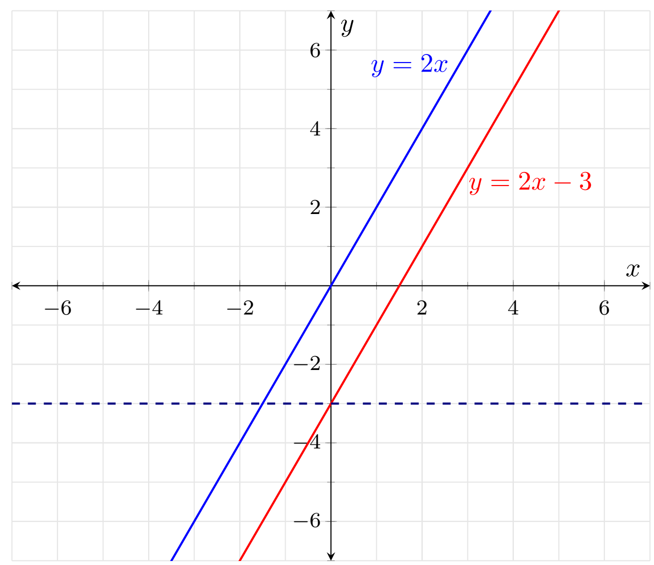
c. $g(x) = 2x - 3$

Explanation Granted, graphing a linear function poses little to no challenge, but understanding how things work in this setting might offer us some general insight on y-intercepts. If $f(x) = 2x$, then $g(x) = f(x) - 3$. Graphing $g(x) = 2x$ is easier than easy: just a line with slope 2 passing through the origin. And the shift down by 3 units comes last, as you would expect:



Shifting a function horizontally

Consider again the example given in the introduction, where we have $f(x) = x^2$ and $f(x-1) = x^2 - 2x + 1$. The first thing we would like to address is a source of frequent confusion when first learning this topic. Namely, we have replaced x with $x - 1$ in the formula for $f(x)$, but the graph of the modified function ended up shifted to the *right*, even though one might expect the shift to have happened to the *left*, due to the negative sign in the $x - 1$ factor!

Here is one safe way to think about it: imagine that you are standing on the x-axis and, say, at the origin of the cartesian plane, but that the graph of $y = f(x)$ is already drawn. Replacing x with $x - 1$ *does move* the x-axis to the left. But *you, the observer*, standing on the x-axis, sees the graph move to the right!

Alternatively, compare this with what happened with vertical shifts, but switching the roles of the x-axis and y-axis. More precisely, start with the graph of $y = f(x)$, then rotate it by 90° clockwise (this switches the axes). Replacing x with $x-1$ now brings the graph down by 1 unit. Finally, rotate everything back by 90° counterclockwise (this undoes the switching of the axes). The resulting

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graph is obtained from the original one by shifting it to the *right*, not left.

Theorem (horizontal shifts): Suppose f is a function and a is a positive number.

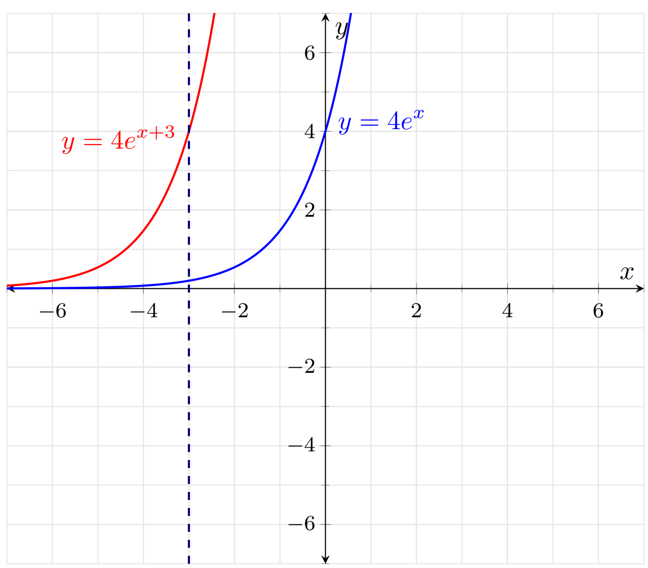
- To graph $y = f(x-a)$, shift the graph of $y = f(x)$ right a units, by adding a to the x-coordinates of the points on the graph of f.
- To graph $y = f(x+a)$, shift the graph of $y = f(x)$ left a units, by subtracting a from the x-coordinates of the points on the graph of f.

As before we'll continue to call f the **parent function**, whose graph will be drawn in blue, while the graphs of the "child" functions will be indicated in red.

Example 5. *For each of the following functions, find the parent function. How would the graph look like, in terms of the graph of the original function?*

a. $g(x) = 4e^{x+3}$.

Explanation Here, if we look at $f(x) = 4e^x$, then we have that $g(x) = 4e^{x+3} = f(x+3)$. So, to graph $y = g(x)$, we may just graph $y = f(x)$, and then shift it 3 units to the left.



b. $g(x) = \sin(x - (\pi/2))$.

Explanation Consider this time $f(x) = \sin x$. Then we have that $g(x) = f(x - (\pi/2))$, which means that to graph $y = g(x)$, we may take the graph of $y = \sin x$ and shift it to the right by $\pi/2$ units. We obtain:

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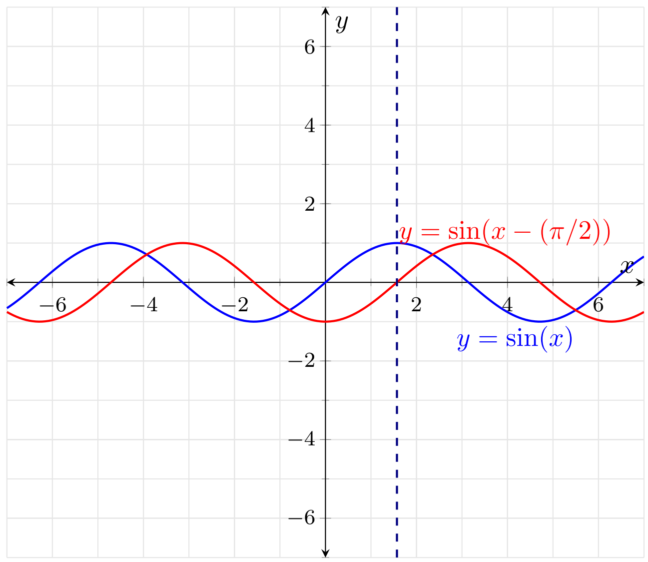

You might be thinking now that the graph of $y = \sin(x - (\pi/2))$ looks an awful lot like the graph of $y = -\cos x$. This is not a coincidence and indeed $\sin(x - (\pi/2)) = -\cos x$ is true for all values of x. We will explore such relations (and more) between trigonometric functions, in future units.

c. $g(x) = (x - 2)^2 - 5(x - 2) + 6$.

Explanation As you might be guessing by now, the parent function can be found by just seeking the shifted variable (in this case, $x - 2$), and replacing it with x. Meaning that if $f(x) = x^2 - 5x + 6$, then $g(x) = f(x - 2)$. Thus, to graph $y = g(x)$, we can just graph $y = f(x)$ and shift it 2 units to the right. Since we can write $f(x) = (x - 2)(x - 3)$, we know that f is a parabola which crosses the x-axis at $x = 2$ and $x = 3$, and it is concave up (we'll understand how to graph parabolas in a bit more of detail, finding also its vertex, on the next section). Hence:

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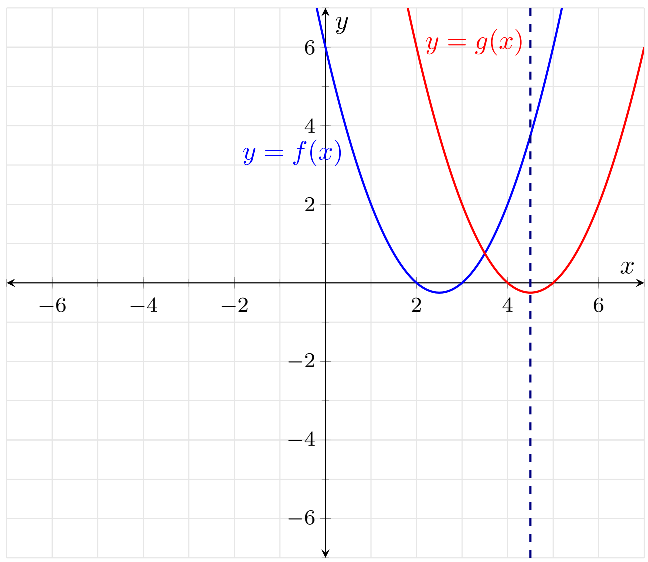


More on parabolas

Let's discuss what happens with parabolas and quadratic functions more precisely. Consider a generic $f(x) = ax^2 + bx + c$, with a, b and c real numbers, and assume that $a \neq 0$. We assume this because if a were zero, we would have a linear function instead of a quadratic one, which is not the focus here. The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

used to find the values of x for which $f(x) = 0$ or, in other words, the possible x-intercepts, is usually a source of grief for people studying quadratic functions for the first time. Let's try to remedy this by understanding where such formula comes from. The main strategy here is a little algebraic device called "completing the squares", which is also useful for finding the coordinates (h, k) of the vertex of the parabola given by the graph of $f(x)$.

Very briefly, the procedure consists in noting that $(x - h)^2 = x^2 - 2hx + h^2$ and paying close attention to the $-2hx$ term. Comparing this with the linear term you were given will tell you what h should be. Add and subtract whatever you need to in the quadratic function you were given, to produce $(x - h)^2$ (or, more generally, the multiple $a(x - h)^2$), and whatever constant term is left outside the factored $(\cdots)^2$ will be the desired k. We'll see several examples below.

Example 6. *For each of the following quadratic functions, rewrite it in the form $f(x) = a(x - h)^2 + k$, for suitable numbers h and k. Such point (h, k) is automatically the vertex of the parabola in question.*

a. $f(x) = x^2 - 4x + 7$.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Explanation Comparing $x^2 - 4x$ with $x^2 - 2hx$ suggests that $h = 2$ does the trick. Since $h^2 = 4$, we add and subtract 4 in the expression for the given $f(x)$, to get

$$f(x) = x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7 = (x - 2)^2 + 3.$$

Hence, the vertex of the parabola $y = x^2 - 4x + 7$ is the point $(2, 3)$.

b. $f(x) = 2x^2 + 6x + 12$.

Explanation This time, look at $2x^2 + 6x = 2(x^2 + 3x)$, and compare $x^2 + 3x$ with $x^2 - 2hx$. It seems like $h = -3/2$ is what we need. Note that $h^2 = 9/4$. Because of the 2 we had to factor out, we'll add and subtract $2 \cdot (9/4) = 9/2$ to complete the square. So

$$\begin{aligned} f(x) &= 2x^2 + 6x + 12 = 2x^2 + 6x + \frac{9}{2} - \frac{9}{2} + 12 \\ &= 2\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{2} + 12 = 2\left(x + \frac{3}{2}\right)^2 + \frac{15}{2} \\ &= 2\left(x - \left(-\frac{3}{2}\right)\right)^2 + \frac{15}{2}. \end{aligned}$$

Thus $(h, k) = (-3/2, 15/2)$ is the vertex of this parabola.

c. $f(x) = 3x^2 + 12x + 14$.

Explanation As before, we start focusing on $3x^2 + 12x = 3(x^2 + 4x)$. Compare $x^2 + 4x$ with $x^2 - 2hx$ to see that we need $h = -2$ here. Since $h^2 = 4$, let's add and subtract $3 \cdot 4 = 12$ from the original expression, to obtain

$$\begin{aligned} f(x) &= 3x^2 + 12x + 14 = (3x^2 + 12x + 12) - 12 + 14 \\ &= 3(x^2 + 4x + 4) + 2 = 3(x + 2)^2 + 2 \\ &= 3(x - (-2))^2 + 2. \end{aligned}$$

So, the vertex of this parabola has coordinates $(h, k) = (-2, 2)$.

Very generally, consider $f(x) = ax^2 + bx + c$, with a, b and c real numbers, with $a \neq 0$. Let's repeat everything we have done in the previous example, with a, b and c instead of concrete numbers. Here are the steps we can follow:

- First, look only at $ax^2 + bx = a(x^2 + bx/a)$.
- Compare $x^2 + bx/a$ with $x^2 - 2hx$. The h we need here is $h = -b/2a$. Note that $h^2 = b^2/4a^2$.
- Because of the a we had to factor out in the beginning, let's add and subtract $a \cdot (b^2/4a^2) = b^2/4a$ from the original expression.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

- Compute

$$\begin{aligned} ax^2 + bx + c &= \left(ax^2 + bx + \frac{b^2}{4a}\right) - \frac{b^2}{4a} + c \\ &= a\left(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}\right) + \frac{-b^2 + 4ac}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} \\ &= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}. \end{aligned}$$

- Hence, the vertex of the parabola described by $y = ax^2 + bx + c$ is given by

$$(h, k) = \left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right).$$

And with those computations in place, we are in fact very close to understanding where the quadratic formula came from. Solving $ax^2 + bx + c = 0$ is, by the above, the same as solving

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0.$$

Reorganize as

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a},$$

and divide by a to get

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Assuming that $b^2 - 4ac \geq 0$, we may take square roots on both sides:

$$\left|x + \frac{b}{2a}\right| = \frac{\sqrt{b^2 - 4ac}}{2|a|}.$$

Eliminating the absolute values, we have

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Now solve for x, by putting everything on the right side under a common denominator:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Mystery solved. For each choice of sign $\pm$, we get one x-intercept. Now, we also observe that the average of such solutions does give the x-coordinate of the vertex, as you might expect:

$$\frac{1}{2}\left(\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b+\sqrt{b^2-4ac}}{2a}\right)=\frac{1}{2}\left(\frac{-2b}{2a}\right)=-\frac{b}{2a}.$$

The y-coordinate of the vertex will be, naturally, $k = f(-b/2a)$. This can also be used as a shortcut to write quadratic functions in vertex-form.

Example 7. *Write $f(x) = x^2 - 8x + 15$ in vertex-form $f(x) = a(x-h)^2 + k$ without completing the squares explicitly.*

Explanation We can factor the quadratic as $f(x) = (x-3)(x-5)$. This means that the x-coordinate of the vertex is $h = (3+5)/2 = 4$, and so

$$k = f(4) = 4^2 - 8 \cdot 4 + 15 = -1.$$

Hence $x^2 - 8x + 15 = (x-4)^2 - 1$. As a quick sanity-check (particular to *this* example), note that factoring this last result as a difference of squares (because $1^2 = 1$) does give $(x-3)(x-5)$.

Summary

- Vertical shifts: given the graph of $y = f(x)$, we can draw the graph of $y = f(x) + a$, with $a > 0$, by shifting the graph of $y = f(x)$ up by a units. Similarly, the graph of $y = f(x) - a$ is obtained by shifting the original graph down by a units.
- Horizontal shitfs: given the graph of $y = f(x)$, we can draw the graph of $y = f(x-a)$, with $a > 0$, by shifting the graph of $y = f(x)$ by a units to the left. Similarly, the graph of $y = f(x+a)$ is obtained by shifting the original graph by a units to the right.
- For an arbitrary quadratic function $f(x) = ax^2 + bx + c$, we found formulas for the coordinates (h, k) of its vertex by completing the squares. We have also concluded that the x-coordinate h of the vertex is, in fact, the average of the x-intercepts of the parabola described as the graph of $y = f(x)$ and, in particular, we have seen how to deduce the famous quadratic formula with this general strategy.

1.2.2 Stretching Functions

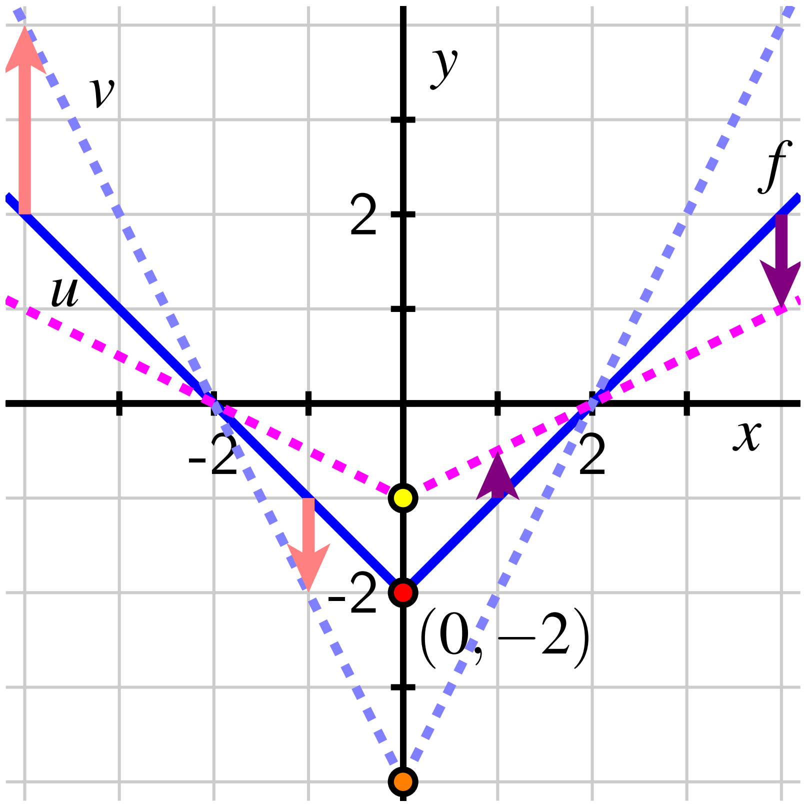
Vertical stretches and reflections

So far, we have seen the possible effects of adding a constant value to function output $f(x)+a$ and adding a constant value to function input $f(x+b)$. Each of these actions results in a translation of the function's graph (either vertically or horizontally), but otherwise leaving the graph the same. Next, we investigate the effects of multiplication the function's output by a constant.

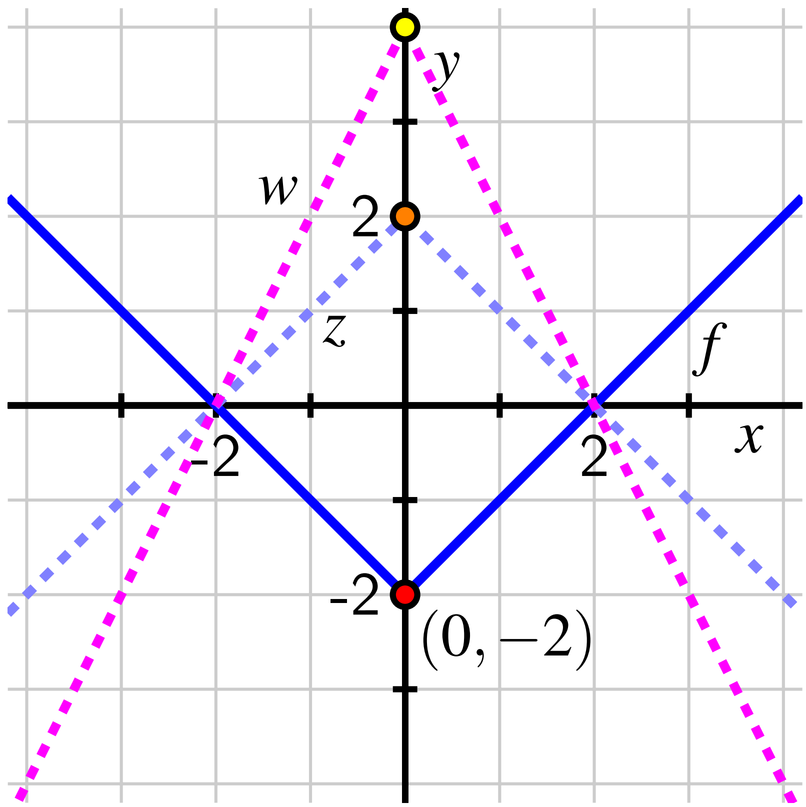
Example 8. *Given the parent function $y = f(x)$ pictured in below , what are the effects of the transformation $y = v(x) = cf(x)$ for various values of c?*

Explanation We first investigate the effects of $c = 2$ and $c = \frac{1}{2}$. For $v(x) = 2f(x)$, the algebraic impact of this transformation is that every output of f is multiplied by 2. This means that the only output that is unchanged is when $f(x) = 0$, while any other point on the graph of the original function f will be stretched vertically away from the x-axis by a factor of 2. We can see this in **image** where each point on the original dark blue graph is transformed to a corresponding point whose y-coordinate is twice as large, as partially indicated by the red arrows.

 Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)



Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)



In contrast, the transformation $u(x) = \frac{1}{2}f(x)$ is stretched vertically by a factor of $\frac{1}{2}$, which has the effect of compressing the graph of f towards the x-axis, as all function outputs of f are multiplied by $\frac{1}{2}$. For instance, the point $(0, -2)$ on the graph of f is transformed to the graph of $(0, -1)$ on the graph of u, and others are transformed as indicated by the purple arrows.

To consider the situation where $c < 0$, we first consider the simplest case where $c = -1$ in the transformation $z(x) = -f(x)$. Here the impact of the transformation is to multiply every output of the parent function f by -1; this takes any point of form (x, y) and transforms it to $(x, -y)$, which means we are reflecting each point on the original function's graph across the x-axis to generate the resulting function's graph. This is demonstrated in second graph where $y = z(x)$ is the reflection of $y = f(x)$ across the x-axis and will be discussed more in the next section.

Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)

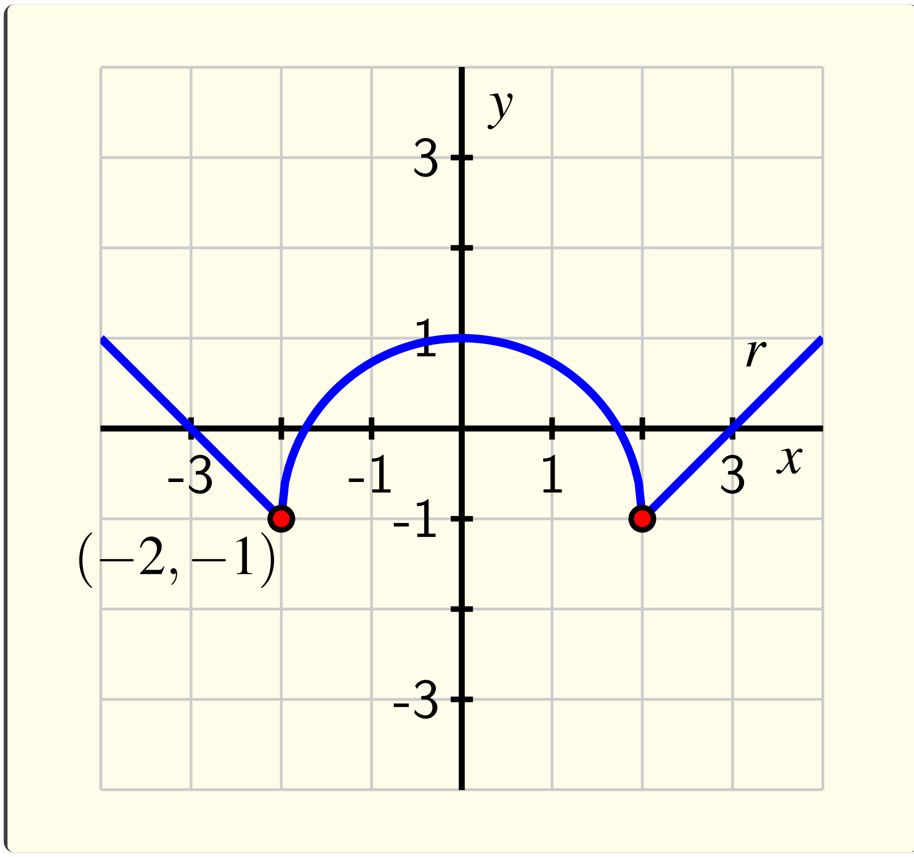
Finally, we also investigate the case where $c = -2$, which generates $y = w(x) = -2f(x)$. Here we can think of -2 as $-2 = 2(-1)$: the effect of multiplying by -1 first reflects the graph of f across the x-axis (resulting in w), and then multiplying by 2 stretches the graph of z vertically to result in w, as shown in second graph .

We summarize and generalize our observations from the graphs above as follows.

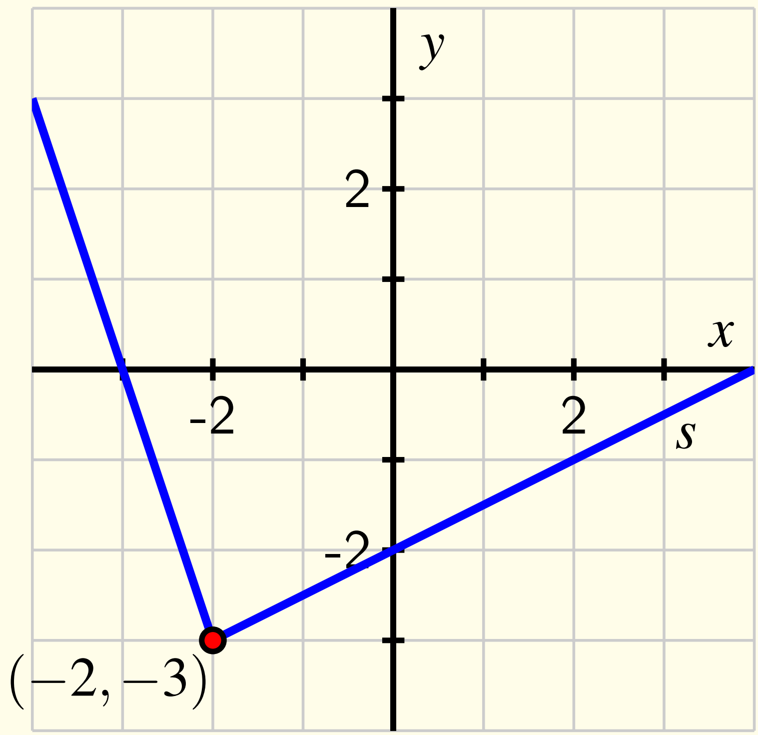
Given a function $y = f(x)$ and a real number $c > 0$, the transformed function $y = v(x) = cf(x)$ is a *vertical stretch* of the graph of f. Every point $(x, f(x))$ on the graph of f gets stretched vertically to the corresponding point $(x, cf(x))$ on the graph of v. If $0 < c < 1$, the graph of v is a compression of f toward the x-axis; if $c > 1$, the graph of v is a stretch of f away from the x-axis. Points where $f(x) = 0$ are unchanged by the transformation.
Given a function $y = f(x)$ and a real number $c < 0$, the transformed function $y = v(x) = cf(x)$ is a reflection of the graph of f across the x-axis followed by a vertical stretch by a factor of $|c|$.

Exploration
Consider the functions r and s given in the following graphs.

Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)



Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)

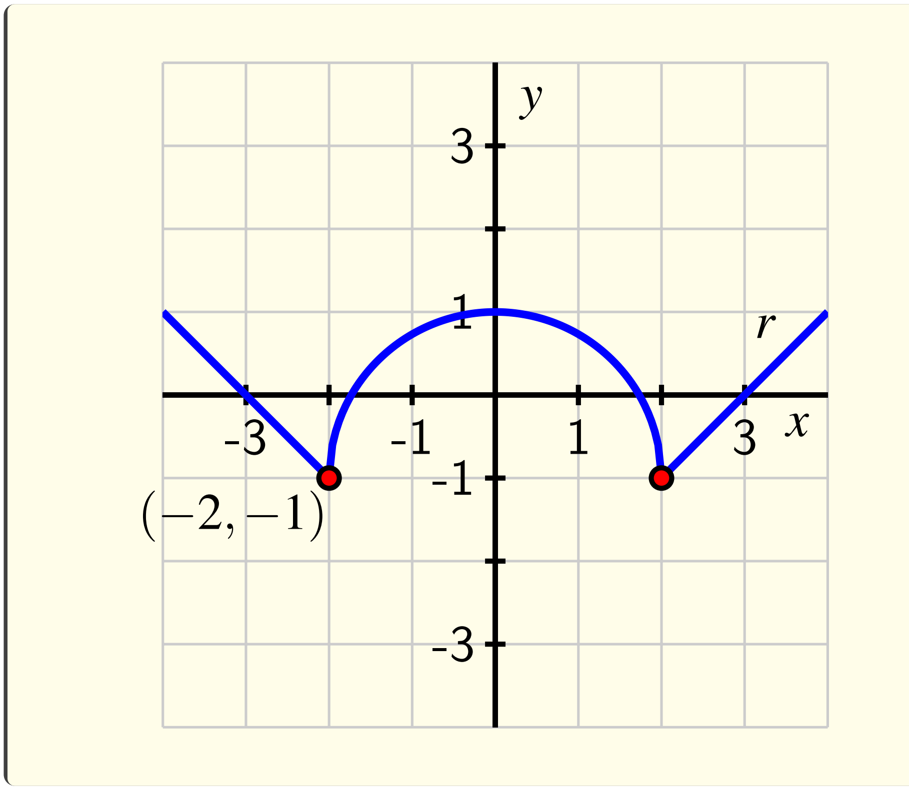


a. On the same axes as the plot of $y = r(x)$, sketch the following graphs: $y = g(x) = 3r(x)$ and $y = h(x) = \frac{1}{3}r(x)$. Be sure to label several points on each of r, g, and h with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in g and h from r.

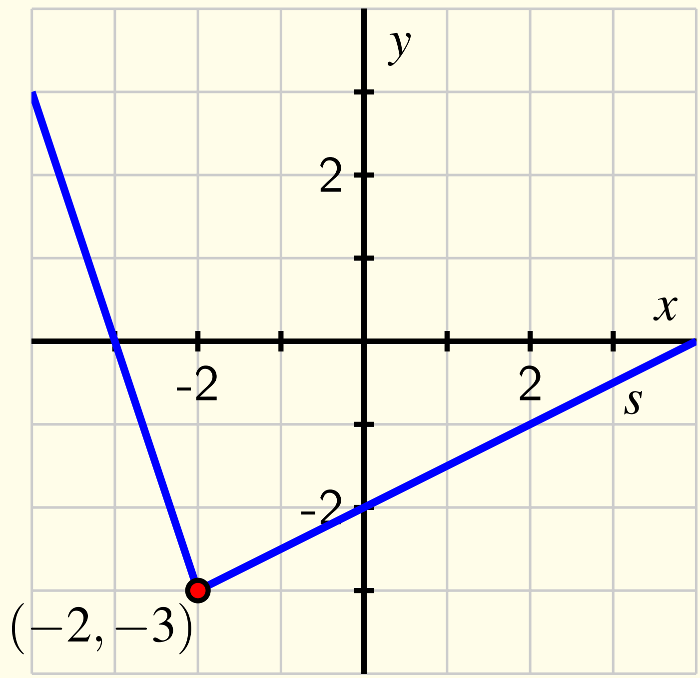
b. On the same axes as the plot of $y = s(x)$, sketch the following graphs: $y = k(x) = -s(x)$ and $y = j(x) = -\frac{1}{2}s(x)$. Be sure to label several points on each of r, g, and h with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in g and h from r.

c. On the additional copies of the two figures below, sketch the graphs of the following transformed functions: $y = m(x) = 2r(x+1) - 1$ (at left) and $y = n(x) = \frac{1}{2}s(x-2) + 2$. As above, be sure to label several points on each graph and indicate their correspondence to points on the original parent function.

Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)



Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)



d. Describe in words how the function $y = m(x) = 2r(x+1) - 1$ is the result of three elementary transformations of $y = r(x)$. Does the order in which these transformations occur matter? Why or why not?

Horizontal Stretches

Exploration Follow the Link to Desmos [a] .

(a) Make sure that the following graphs are enabled.

- $f(x) = \sqrt{4 - x^2}$
- $1.5f(x)$ or $1.5\sqrt{4 - x^2}$
- $2f(x)$ or $2\sqrt{4 - x^2}$
- $0.5f(x)$ or $0.5\sqrt{4 - x^2}$
- $0.25f(x)$ or $0.25\sqrt{4 - x^2}$

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

What effect do the 1.5, 2, 0.5 and 0.25 seem to have?

(b) Now disable the previous graphs and make sure that the following graphs are enabled.

- $f(x) = \sqrt{4 - x^2}$
- $f(1.5x)$ or $\sqrt{4 - (1.5x)^2}$
- $f(2x)$ or $r\sqrt{4 - (2x)^2}$
- $f(0.5x)$ or $\sqrt{4 - (0.5x)^2}$
- $f(0.25x)$ or $\sqrt{4 - (0.25x)^2}$

What effect do the 1.5, 2, 0.5 and 0.25 seem to have?

[a] Link: https://www.desmos.com/calculator/xjem27frqi

Let c be a positive real number then the following transformations result in stretches or shrinks of the graph $y = f(x)$

Horizontal Stretches or Shrinks

$$y = f\left(\frac{x}{c}\right)$$

The transformation is a stretch by factor c if $c > 1$.
The transformation is a shrink by factor c if $c < 1$.

Vertical Stretches or Shrinks

$$y = c \cdot f(x)$$

The transformation is a stretch by factor c if $c > 1$
The transformation is a shrink by factor c if $c < 1$

Example 9. *Let $f(x) = x^3 - 16x$. Find equations for the following transformations of $f(x)$.*

(a) *$g(x)$ is a vertical stretch of $f(x)$ by a factor of 3.*

(b) *$h(x)$ is a horizontal shrink of $f(x)$ by a factor of $\frac{1}{2}$.*

Explanation

(a) Transformation from $f(x)$ to $g(x)$

$$\begin{aligned} g(x) &= 3 \cdot f(x) \\ &= 3(x^3 - 16x) \\ &= 3x^3 - 48x \end{aligned}$$

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

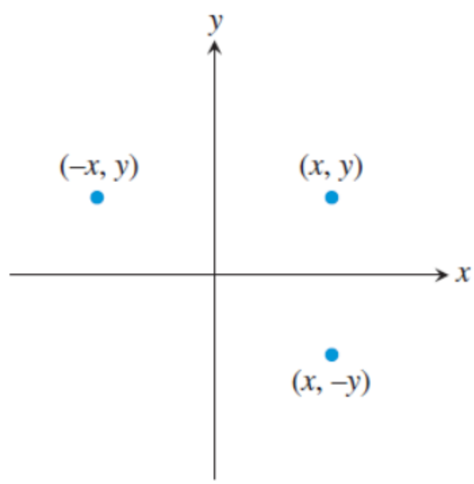
(b) Transformation from $f(x)$ to $h(x)$

$$\begin{aligned} h(x) &= f\left(\frac{x}{\frac{1}{2}}\right) \\ &= f(2x) \\ &= (2x)^3 - 16(2x) \\ &= 8x^3 - 32x \end{aligned}$$

1.2.3 Reflections of Functions

Reflections Across Axes

Points (x, y) and $(x, -y)$ are reflections of each other across the x-axis. Points (x, y) and $(-x, y)$ are reflections of each other across the y-axis. In general, two points that are symmetric with respect to a line are reflections of each other across that line.



The following transformations result in reflections of the graph of $y = f(x)$

- Reflection across the x-axis

$$y = -f(x)$$

- Reflection across the y-axis

$$y = f(-x)$$

- Reflection through the origin

$$y = -f(-x)$$

Example 10. *Find an equation for the reflection of* $f(x) = \dfrac{5x-9}{x^2+3}$ *across each axis.*
Explanation

Across the x-axis: $y = -f(x) = -\dfrac{5x-9}{x^2+3} = \dfrac{9-5x}{x^2+3}$

Across the y-axis: $y = f(-x) = \dfrac{5(-x)-9}{(-x)^2+3} = \dfrac{-5x-9}{x^2+3}$

Putting it Together

Transformations may be performed one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to the order.

Example 11. (a) *The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.*

- *a horizontal shift 2 units to the right*
- *a vertical stretch by a factor of 3*
- *a vertical translation 5 units up*

(b) *Apply the transformations above in the opposite order and find the equations of the graph that results.*

Explanation

(a) Applying the transformations in order we have

$y = x^2$	Original function
$y = (x-2)^2$	Horizontal shift
$y = 3(x-2)^2$	Vertical stretch
$y = 3(x-2)^2 + 5$	Vertical translation
$y = 3x^2 - 12x + 17$	Expanded form

(b) Applying the transformations in the opposite order we have

$y = x^2$	Original function
$y = x^2 + 5$	Vertical translation
$y = 3(x^2 + 5)$	Vertical stretch
$y = 3((x-2)^2 + 5)$	Horizontal translation
$y = 3x^2 - 12x + 27$	Expanded form

1.3 Solving Inequalities

Learning Objectives

- Solving Inequalities Graphically
  - Motivating solutions to inequalities
  - Definition of a solution to an inequality
  - Review finding zeros of equations
- Solving Inequalities without a Graph
  - Famous functions are continuous except at... IVT
  - Solving with a sign chart
  - Using signs of famous functions

1.3.1 Solving Inequalities Graphically

Motivating Questions

- What is a solution to an inequality?
- How can we use graphs to solve inequalities?

Introduction

Dabin and Melina are having a walking race. Dabin can walk 1 meter per second, but Melina can walk 2 meters per second. Since Melina is the faster walker, she gives Dabin a head start of 5 meters. At this point, we can ask a few questions about the race. Two questions we'll focus on are "When is Dabin in the lead?" and "When is Melina in the lead?" both of which can be answered by considering inequalities.

To start, let's define some relevant functions. The function D defined by $D(t) = 5 + t$ represents how far (in meters) Dabin has walked t seconds after the start of the race. Similarly, the function M defined by $M(t) = 2t$ represents how far Melina has walked t seconds after the start of the race. In our framework, asking when Dabin is in the lead is the same as asking for all t such that $D(t) > M(t)$. In the vocabulary that we'll use, we want to solve the inequality $D(t) > M(t)$.

Definition Say f and g are functions. A **solution** to the inequality $f(x) < g(x)$ is the set of x values where $f(x) < g(x)$. Similarly, a solution to the inequality $f(x) > g(x)$ is the set of x values where $f(x) > g(x)$.

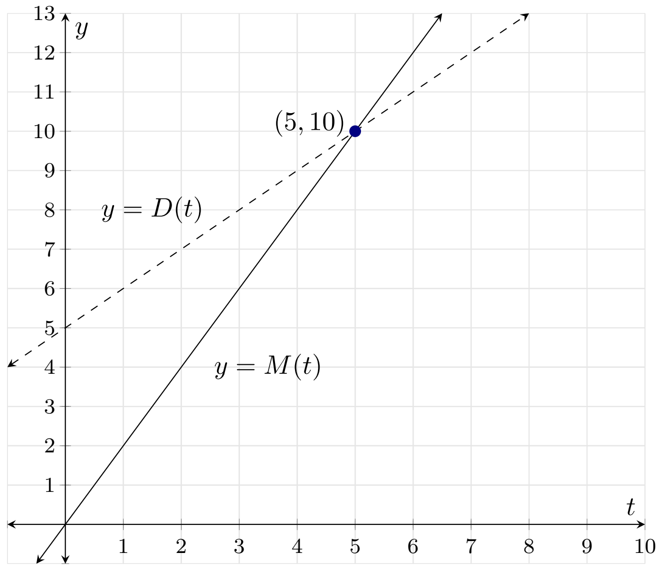
Note that we define a solution to an inequality as a set. We will often write the sets in interval notation.

Solving inequalities graphically

Example 12. *Let D be defined by $D(t) = 5+t$, and M be defined by $M(t) = 2t$. Find a solution to the inequality $D(t) < M(t)$.*

Explanation This example asks us to find the set of t values where $D(t) < M(t)$. One approach to inequalities of this form is to look at the graphs of the equations involved. The following figure shows the graphs of $y = D(t)$ and $y = M(t)$.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



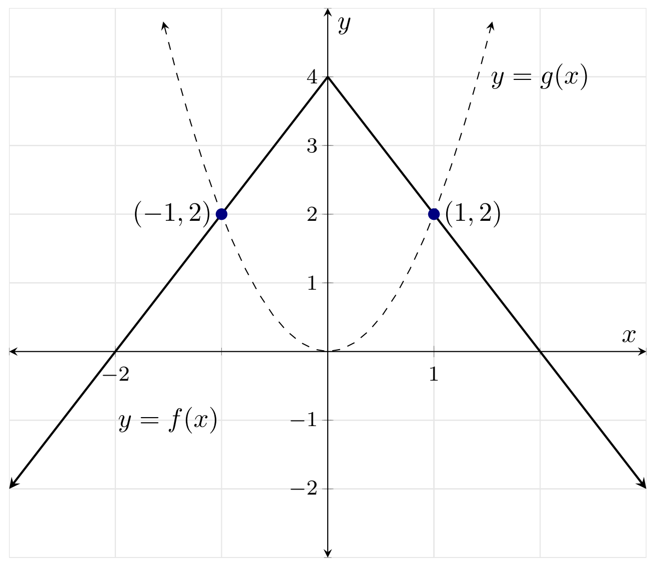
Because of the way we draw the graphs of functions, if $D(x) < M(x)$ for some x if and only if the graph of D lies below the graph of M at the point x. Using this information, we can see that if $t > 5$, then the graph of D lies below the graph of M. Therefore, the set of all t such that $t > 5$ is the solution to $D(t) < M(t)$. Writing this in interval notation, the solution is $(5, \infty)$.

Putting this in terms of the scenario described at the beginning of the section, Melinda is in the lead after 5 seconds.

> **Exploration** Find a solution to the inequality $D(t) < M(t)$.

Example 13. *Let f and g be functions whose graphs are shown below. Assume all important behavior of the functions is shown in the figure.*

Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



(a) *Solve the inequality* $f(x) < g(x)$.

(b) *Solve the inequality* $f(x) \geq g(x)$.

Explanation

(a) To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.

(b) To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. To solve the former equation we can look at the x-coordinates of the intersection points. This yields $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.

Now let's turn our attention to inequalities involving absolute values, which are often a source of confusion. The following theorem provides the complete story.

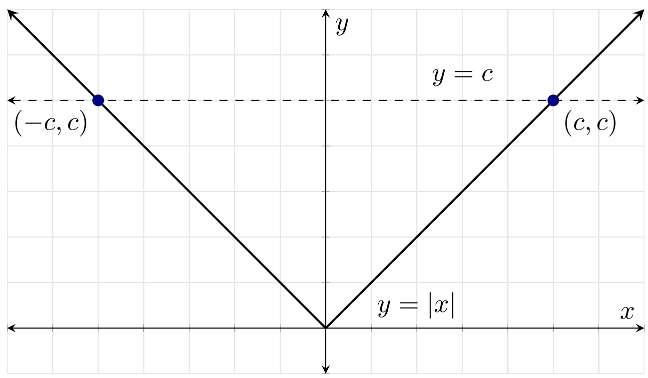
Theorem 1. *Let c be a real number.*

- *For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.*
- *For $c > 0$, $|x| \leq c$ is equivalent to $-c \leq x \leq c$.*
- *For $c \leq 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \leq c$ has no solution.*
- *For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.*

 Acknowledgements: Stitz Zeager Open Source Mathematics
(https://www.stitz-zeager.com/)

- *For $c \geq 0$, $|x| \geq c$ is equivalent to $x \leq -c$ or $x \geq c$.*
- *For $c < 0$, $|x| > c$ and $|x| \geq c$ are true for all real numbers.*

In light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which we know happens when $x = c$ or $x = -c$. Graphing, we get



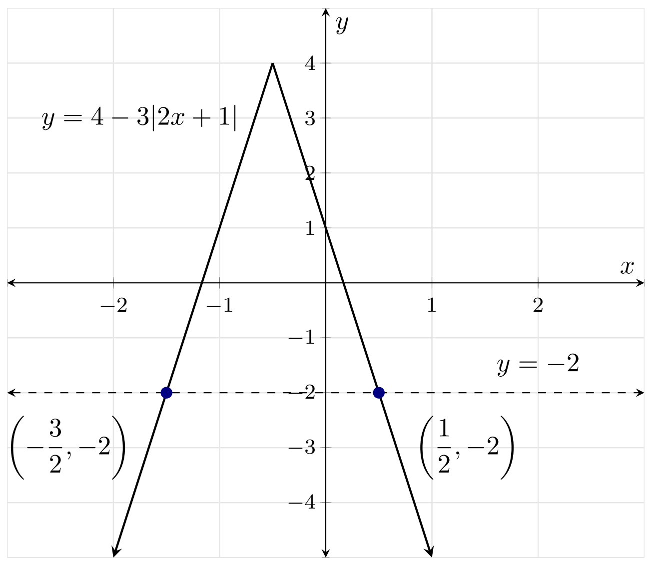
We see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in the theorem can be shown similarly. You can try changing the value of c using Desmos.

Desmos link: `https://www.desmos.com/calculator/dbpb01aybm`

Example 14. *Solve the inequality* $4 - 3|2x + 1| > -2$.

Explanation Let's start by graphing both sides of the inequality on the same axes.

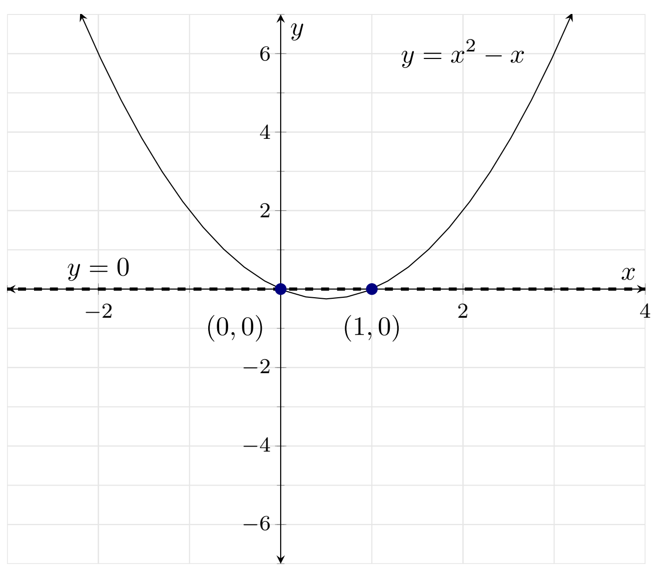
Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)



We see that the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$. Therefore, the solution in interval notation is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

Example 15. *Solve the inequality* $x^2 \leq x$.

Explanation We could start by graphing both sides of the inequality on the same graph, but here, we'll demonstrate another possible approach. Note that $x^2 \leq x$ is equivalent to the inequality $x^2 - x \leq 0$, by subtracting x from both sides. Now, let's graph $y = x^2 - x$ and $y = 0$ on the same axes.



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Notice that the two points of intersection are $(0, 0)$ and $(1, 0)$, so $x^2 - x = 0$ for $x = 0$ and $x = 1$. To find the solution to $x^2 - x < 0$, we can see that the graph of $y = x^2 - x$ lies below the graph of $y = 0$ between 0 and 1. Therefore, the solution is $[0, 1]$.

The above example illustrates a common technique. Rather than considering two functions f and g and asking when one is greater than, less than, or equal to the other, we can move one function to the other side, and consider the function $f - g$. Now, the problem becomes one of finding when the function $f - g$ is positive, negative, or zero.

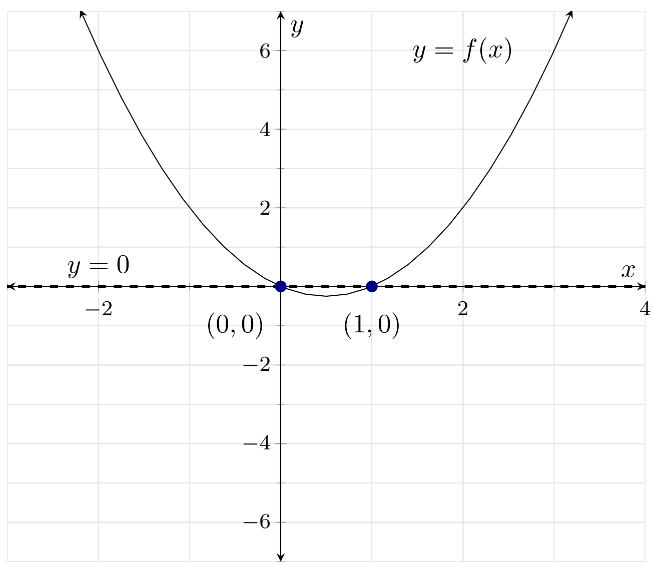
1.3.2 Solving Inequalities without a Graph

Motivating Questions

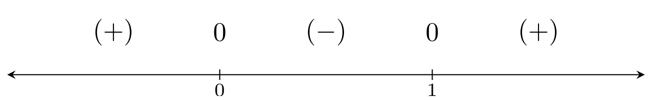
- How can we find solutions to inequalities without using a graph?
- How can we use the zeros of functions to solve inequalities?

Introduction and the Importance of Zeros

In the previous section, we constructed the following graph to solve the inequality $f(x) \leq 0$, where $f(x) = x^2 - x$.



We can see that the graph of f does dip below the x-axis between its two x-intercepts. The zeros of f are $x = 0$ and $x = 1$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. For every number in $(-\infty, 0)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, 0)$. Similarly, $f(x) < 0$ for all x in $(0, 1)$, and $f(x) > 0$ for all x in $(1, \infty)$. We can schematically represent this with the *sign diagram* below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the

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number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(0, 1)$. Adding in the zeros, the solution to $f(x) \leq 0$ is $[0, 1]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function.

Continuity and the Intermediate Value Theorem

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

This is a special case of a theorem called the Intermediate Value Theorem, or IVT for short. To talk about the IVT, we first need to discuss what it means for a function to be *continuous*.

> **Definition** *(Informal.)* We say a function f on an interval is **continuous** if the graph of f has no 'breaks' or 'holes' on that interval.

In further courses, you will learn a more formal definition of continuity, but for now, this will suffice.

Example 16. (a) *Linear and quadratic functions are continuous.*

(b) *In fact, all of our famous functions are continuous where they are defined.*

(c) *All polynomials are continuous.*

(d) *Rational functions are continuous where they are defined. In particular,* $\dfrac{1}{x}$ *is continuous on its domain,* $(-\infty, 0) \cup (0, \infty)$.

(e) *If f and g are continuous functions, so are $f + g$ and $f \cdot g$.*

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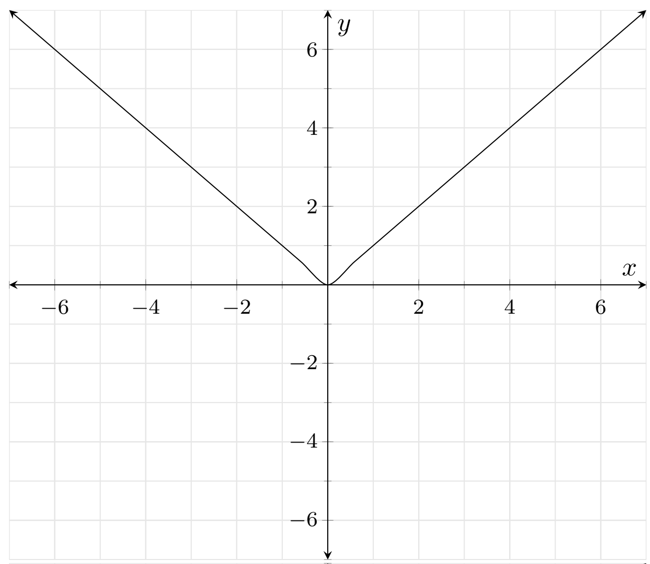

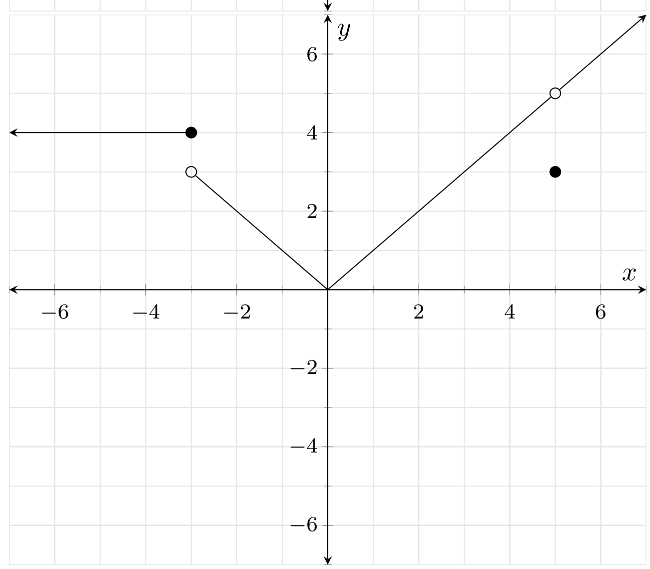

The function whose graph is shown on the left above is continuous, while the function whose graph is shown on the right above is not, since it has breaks in its graph.

One way to think about continuous functions is that they are the functions whose graphs you could draw on an infinite piece of paper without ever taking your pencil off the paper (except where they aren't defined). You will encounter and learn about continuous functions more in-depth in calculus, but for now, familiarity at this level will be enough.

Now that we know about continuous functions, we can state our version of the

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IVT.

Theorem 2 (Intermediate Value Theorem (Zero Version))**.** *Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.*

Reinterpreted, this means that the graph of a continuous function can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis.

Here's how we'll use the IVT to solve inequalities of the form $f(x) > 0$, where f is a continuous function. If a given interval does not contain a zero of f, then by the IVT either all the function values on the interval are positive or they're all negative. In this way, the IVT allows us to determine the sign of *all* of the function values on the interval by testing the function at just *one* value in the interval, which we're free to choose.

This gives us the following steps for solving an inequality involving a continuous function.

(a) Rewrite the inequality, if necessary, as a continuous function $f(x)$ on one side of the inequality and 0 on the other.

(b) Find the zeros of f and place them on the number line with the number 0 above them.

(c) Choose a real number, called a *test value*, in each of the intervals determined in step 2.

(d) Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.

(e) Choose the intervals which correspond to the correct sign to solve the inequality.

As you can see, the zeros of continuous functions are important, so in the examples that follow, we'll highlight the techniques we use to find zeros. It may also be useful to review methods for finding zeros that you've seen before.

Solving Inequalities Algebraically

Example 17. *Solve the inequality $3x^2 + x < 6x - 2$.*

Explanation To start, let's put the inequality in a nice form, with a continuous function on one side and 0 on the other. It doesn't matter which side the 0 is

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on, so we'll choose to rewrite the inequality as $3x^2 - 5x + 2 < 0$. Since quadratic functions are continuous, we can use the steps outlined in the previous section to solve the inequality.

First, we find the zeros of f, where $f(x) = 3x^2 - 5x + 2$. We could do this by using the quadratic formula, but let's factor. Factoring gives us $f(x) = (3x-2)(x-1)$. In order for $f(x) = 0$ to be true, we need $3x - 2 = 0$ or $x - 1 = 0$. This tells us that the zeros of f are $x = \dfrac{2}{3}$ and $x = 1$. This gives us a good start to our sign diagram:

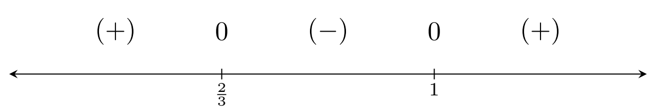
This sign diagram tells us that we have to check three intervals: $\left(-\infty, \dfrac{2}{3}\right)$, $\left(\dfrac{2}{3}, 1\right)$, and $(1, \infty)$. However, thanks to the IVT, we only need to check one test value per interval. Be careful not to choose $x = \dfrac{2}{3}$ or $x = 1$ as your test values!

For the interval $\left(-\infty, \dfrac{2}{3}\right)$, we choose $x = 0$ to be our test value and see that $f(0) = 3(0)^2 - 5(0) + 2 = 2$, which is positive.

For the interval $\left(\dfrac{2}{3}, 1\right)$, we choose $x = \dfrac{5}{6}$ to be our test value and see that $f(0) = 3\left(\dfrac{5}{6}\right)^2 - 5\left(\dfrac{5}{6}\right) + 2 = \dfrac{25}{12} - \dfrac{25}{6} + 2 = -\dfrac{1}{12}$, which is negative.

For the interval $(1, \infty)$, we choose $x = 2$ to be our test value and see that $f(2) = 3(2)^2 - 5(2) + 2 = 4$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was negative, we choose $\left(\dfrac{2}{3}, 1\right)$ as the solution to the inequality.

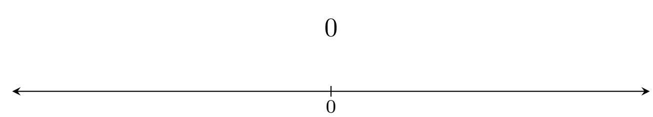
Example 18. *Solve the inequality $xe^x \geq -x$.*

Explanation Rewriting our inequality, we have $xe^x + x \geq 0$. Since linear and exponential functions are continuous and products and sums of continuous functions are continuous, we know that the function f defined by $f(x) = xe^x + x$ is a continuous function.

First, we find the zeros of f. We start by noticing that each term in $xe^x + x$ contains a factor of x, so we can factor that out and find $f(x) = x(e^x + 1)$. In order to solve the equation $f(x) = 0$, we need to solve $x = 0$ and $e^x + 1 = 0$. The first equation is already solved, and tells us that $x = 0$ is one zero of f. To solve the second equation, we calculate

$$\begin{aligned} e^x + 1 &= 0 \\ e^x &= -1, \end{aligned}$$

and note that the exponential function is never negative, so there are no solutions. Therefore, the only zero of f is $x = 0$. We now begin to construct the sign diagram.

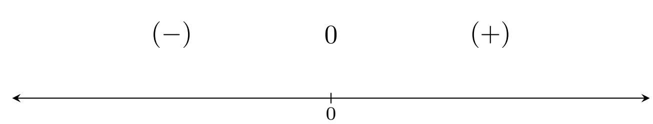


This sign diagram tells us that we have to check two intervals: $(-\infty, 0)$ and $(0, \infty)$. Again, thanks to the IVT, we only need to check one test value per interval.

For the interval $(-\infty, 0)$, we choose $x = -1$ to be our test value and see that $f(-1) = e^{-1} - 1$, which is negative. To see that $e^{-1} - 1$ is negative, notice that $e^{-1} = \dfrac{1}{e}$, and since $e > 1$, $\dfrac{1}{e} < 1$. Therefore, when we subtract 1 from e^{-1}, we obtain a negative number.

For the interval $(0, \infty)$, we choose $x = 1$ to be our test value and see that $f(1) = e^1 + 1$, which is positive.

We can now update our sign diagram to the following:



Since we wanted to find where f was non-negative, we choose $[0, \infty)$ as the solution to the inequality. Remember to include 0 in the interval, since zeros of f are also points where f is non-negative.

Example 19. *Solve the inequality* $2^{x^2 - 3x} \geq 16$.

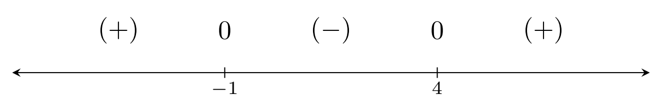
Explanation We set $r(x) = 2^{x^2-3x} - 16$ and solve the equivalent inequality $r(x) \geq 0$. The domain of r is all real numbers, so in order to construct our sign diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by taking logarithms, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. Therefore, the intervals in which we need to find test values are $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$.

For the interval $(-\infty, -1)$, we choose $x = -2$ to be our test value. We see that $r(-2) = 2^{4+6} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

For the interval $(-1, 4)$, we choose $x = 0$ to be our test value. We see that $r(0) = 2^0 - 16 = 2^0 - 2^4 < 0$, so r is negative on the interval.

For the interval $(4, \infty)$, we choose $x = 5$ to be our test value. We see that $r(5) = 2^{25-15} - 16 = 2^{10} - 2^4 > 0$, so r is positive on the interval.

We can now construct a sign diagram.



From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$ is on or above the x-axis.

Dealing with Difficult Denominators

Even after we feel comfortable with the procedure for solving inequalities involving continuous functions, you might still wonder about functions which aren't defined on all real numbers, such as rational functions or more generally, functions with denominators that could potentially evaluate to 0. The good news is that if f and g are continuous functions, the function $\dfrac{f}{g}$ is continuous wherever it is defined. Therefore, we can adapt our technique from before, but remembering that a change of sign *could* happen around a point where a function is undefined, so we need to add any places our functions are undefined to our sign diagram.

Example 20. *Solve the inequality* $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1.$

Explanation To solve the inequality, it may be tempting to begin by clearing denominators. The problem is that, depending on x, $(x - 1)$ may be positive (which doesn't affect the inequality) or $(x - 1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and begin to make a sign diagram.

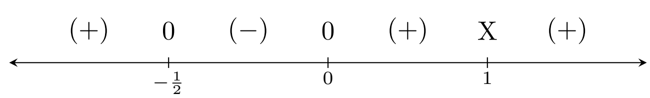
$$
\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1}-\frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right)-x(x-1)+1(2(x-1))}{2(x-1)} &\geq 0 && \text{get a common denominator} \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 && \text{expand}
\end{aligned}
$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The candidates for zeros of r are the solutions to $2x^3-x^2-x=0$, which we can find by factoring.

$$
\begin{aligned}
2x^3-x^2-x &= 0 \\
x(2x^2-x-1) &= 0 \\
x(2x+1)(x-1) &= 0.
\end{aligned}
$$

Therefore, the candidates for zeros of r are $x=0$, $x=-\frac{1}{2}$ and $x=1$. However, $x=1$ is not in the domain of r, since it is the solution to $2x-2=0$, which is the equation we get by setting the denominator equal to 0. However, x-values for which the function is undefined are also possible places where the sign of the function might change, so we should include them on the sign diagram. Since r is a rational function, it is continuous everywhere it is defined, so when constructing the sign diagram, we only need to consider the intervals between zeros or places where it is undefined. For us, these intervals will be $\left(-\infty,-\frac{1}{2}\right)$, $\left(-\frac{1}{2},0\right)$, $(0,1)$, and $(1,\infty)$.

Choosing test values in each test interval (we encourage you to check the calculation), we can construct the sign diagram below.



We used an X to denote that r is not defined at $x=1$.

We are interested in where $r(x)\geq 0$. We find $r(x)$ is positive on the intervals $\left(-\infty,-\frac{1}{2}\right)$, $(0,1)$ and $(1,\infty)$. We add to these intervals the zeros of r, $x=-\frac{1}{2}$, and $x=0$, to get our final solution: $\left(-\infty,-\frac{1}{2}\right]\cup[0,1)\cup(1,\infty)$.

 Acknowledgements: Stitz Zeager Open Source Mathematics
(https://www.stitz-zeager.com/)

Example 21. *Solve the inequality* $\dfrac{e^x}{e^x - 4} \leq 3$.

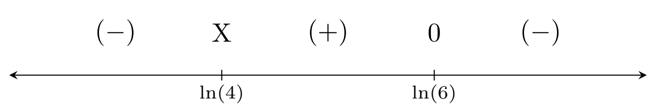
Explanation The first step we need to take to solve $\dfrac{e^x}{e^x - 4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned} \frac{e^x}{e^x - 4} &\leq 3 \\ \frac{e^x}{e^x - 4} - 3 &\leq 0 \\ \frac{e^x}{e^x - 4} - \frac{3\left(e^x - 4\right)}{e^x - 4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12 - 2e^x}{e^x - 4} &\leq 0 \end{aligned}$$

We set $r(x) = \dfrac{12 - 2e^x}{e^x - 4}$. We note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this by taking logarithms gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we set the numerator equal to zero and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[1] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. This indicates that we might want to use $\ln(3)$, $\ln(5)$, and $\ln(7)$ as our test values. While the prospect of determining the sign of $r\left(\ln(3)\right)$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r\left(\ln(3)\right) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly and construct the sign diagram.



From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

[1] This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Conclusion

We hope that the specific examples we've gone through illustrate a general principle when it comes to solving inequalities. First, we want to rewrite the inequality in a form where 0 is on one side and a nice-enough function is on the other side. Then, we use the fact that our functions are continuous on their domains to narrow down where possible sign changes can occur. From there, we can use test values to compute the sign of the function on intervals, and finish by putting our solution in interval notation.

1.4 Function Transformations Project

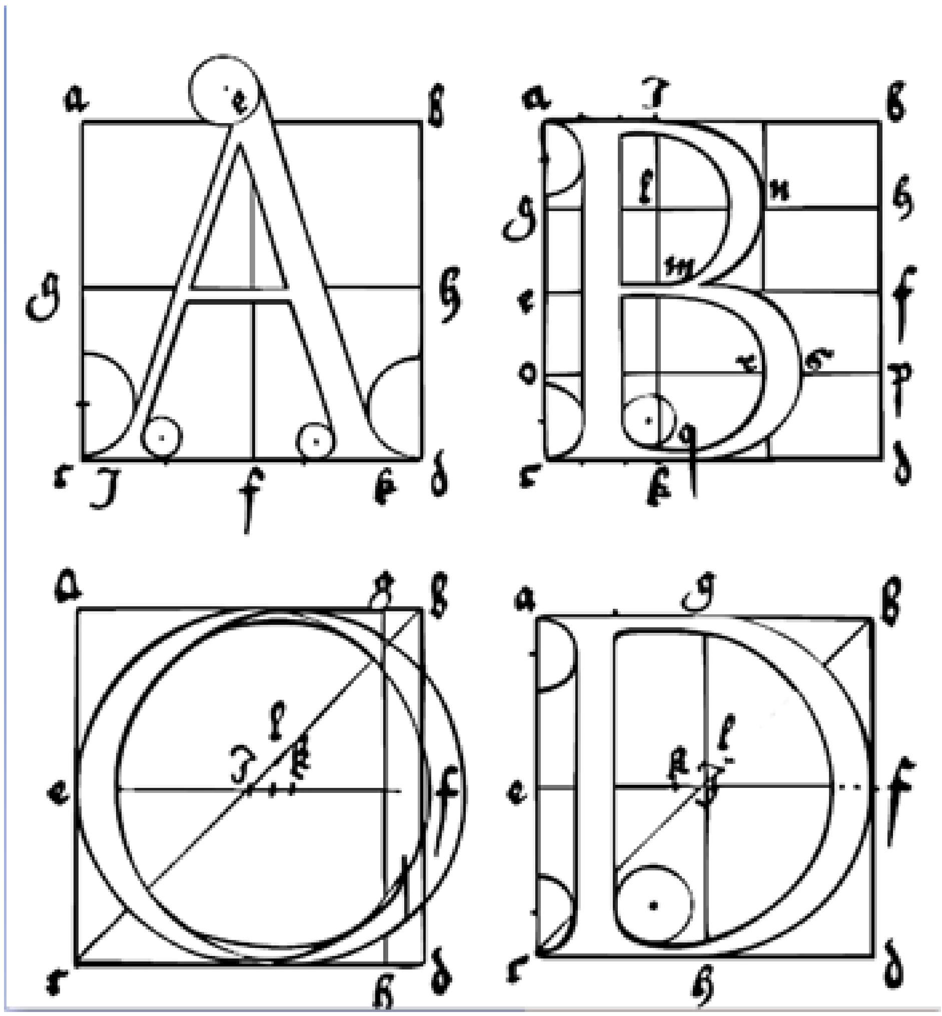
Learning Objectives

- Creating a Font
  - Definition of Function
  - Function Transformations
  - Domains and ranges, especially restricted domains
  - Interperting graphs

1.4.1 Creating a Font

Introduction

The graphs of linear, quadratic, exponential and power functions all have a characteristic shape. But the graphs of polynomials have a huge variety of different shapes.

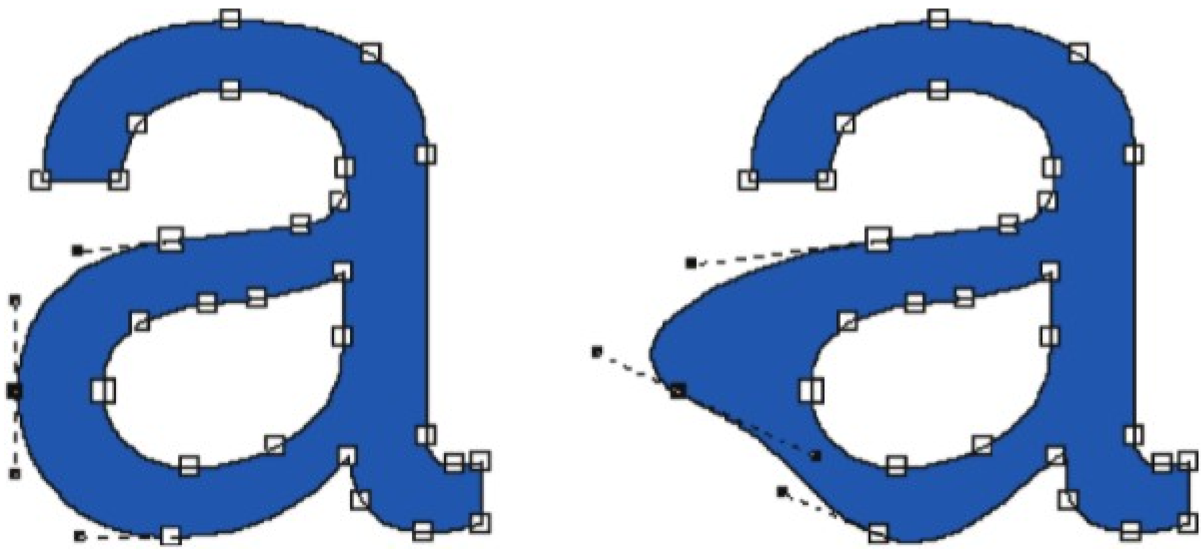


Ever since Gutenberg's invention of movable type in 1455, artists and printers have been interested in the design of pleasing and practical fonts. In 1525, Albrecht Durer published *On the Just Shaping of Letters*, which set forth a

 Acknowledgements: Yoshiwara Books (https://yoshiwarabooks.org/)

system of rules for the geometric construction of Roman capitals. The letters shown above are examples of Durer's font. Until the twentieth century, a ruler and compass were the only practical design tools, so straight lines and circular arcs were the only geometric objects that could be accurately reproduced.

With the advent of computers, complex curves and surfaces, such as the smooth contours of modern cars, can be defined precisely. In the 1960s the French automobile engineer Pierre Bézier developed a new design tool based on polynomials. **Bézier curves** are widely used today in all fields of design, from technical plans and blueprints to the most creative artistic projects. Many computer drawing programs and printer languages use quadratic and cubic Bézier curves.



Project

In this project, you will use the online graphing calculator Desmos to create your letter. Desmos can be accessed at http://desmos.com[2] .

Step 1: Choose **one letter from each category below** that you will create.

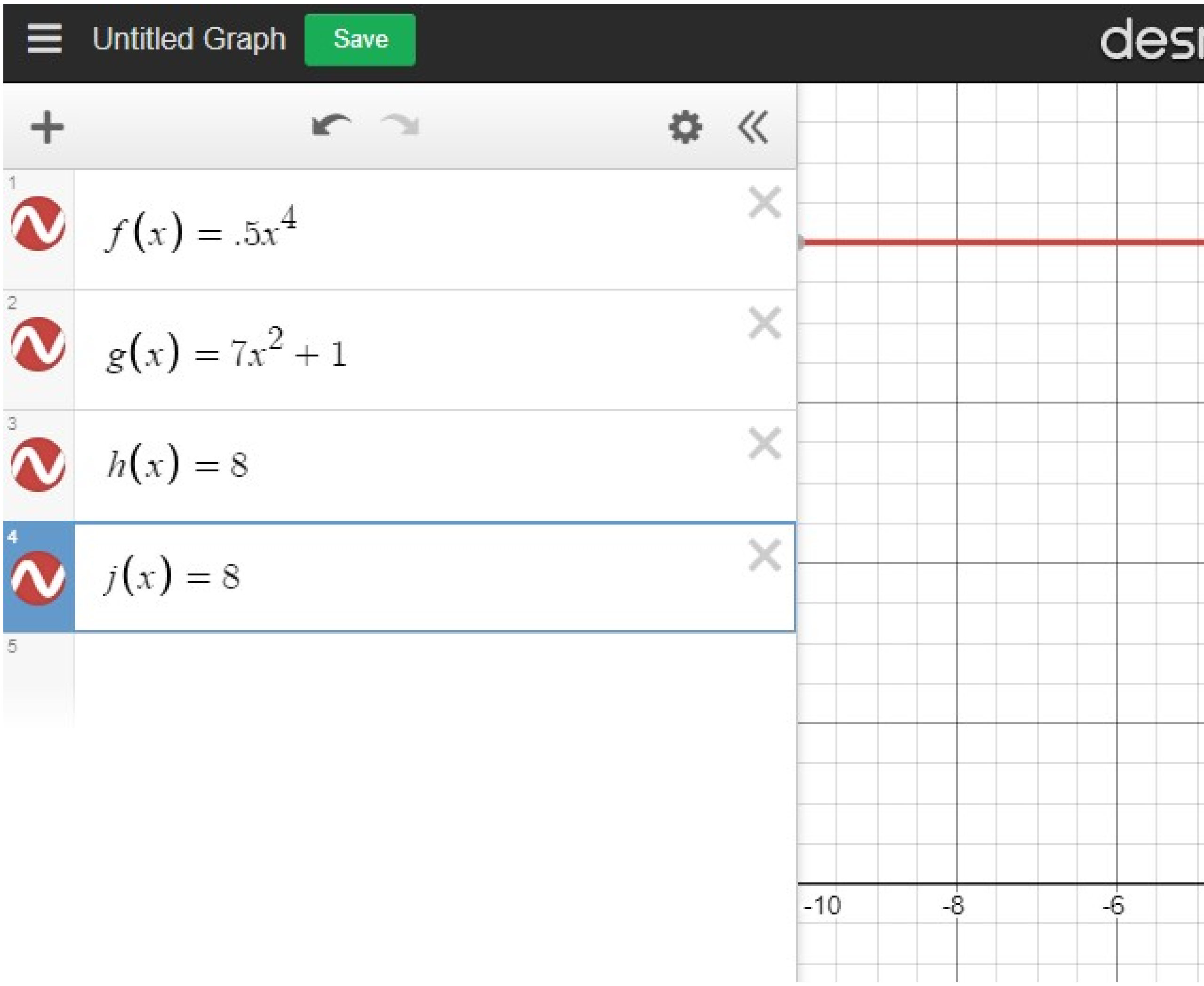
Category 1	X	Y	A	K	N	W
Category 2	J	R	h	S	U	Q
Category 3	G	m	g	f	e	a

[2]See http://desmos.com at http://desmos.com

 Acknowledgements: Yoshiwara Books (https://yoshiwarabooks.org/)

Step 2 Go to http://desmos.com[3] . You may wish to create an account and save your work, but you are not required to do so.

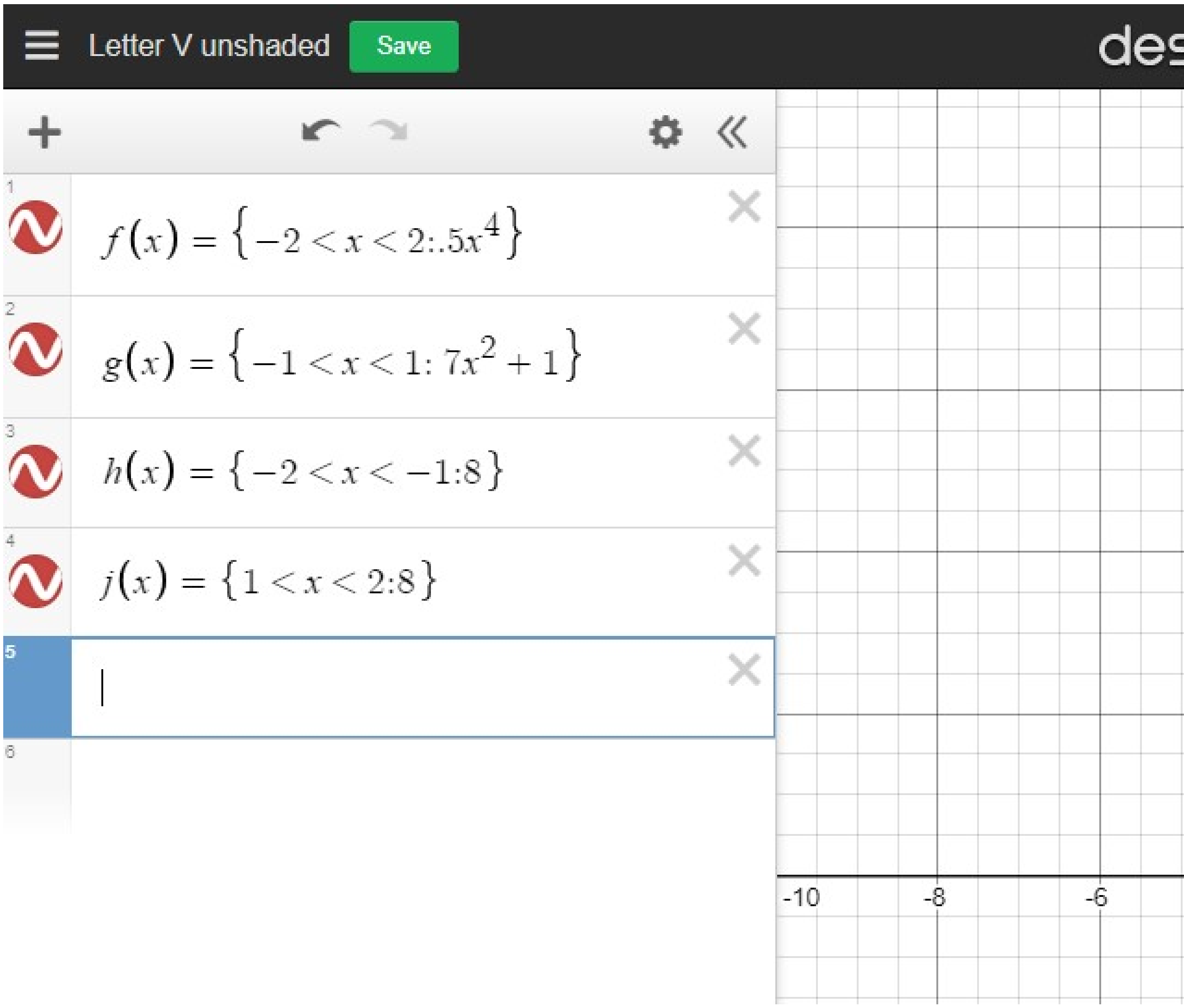
Step 2 Using our Famous Functions and function transformations, create an outline of your letter using functions. Your letter should sit on the x-axis and the top of the letter should be at $y = 8$ (except for lowercase letter such as e or a which should have a height of 4 units). Here is an example:



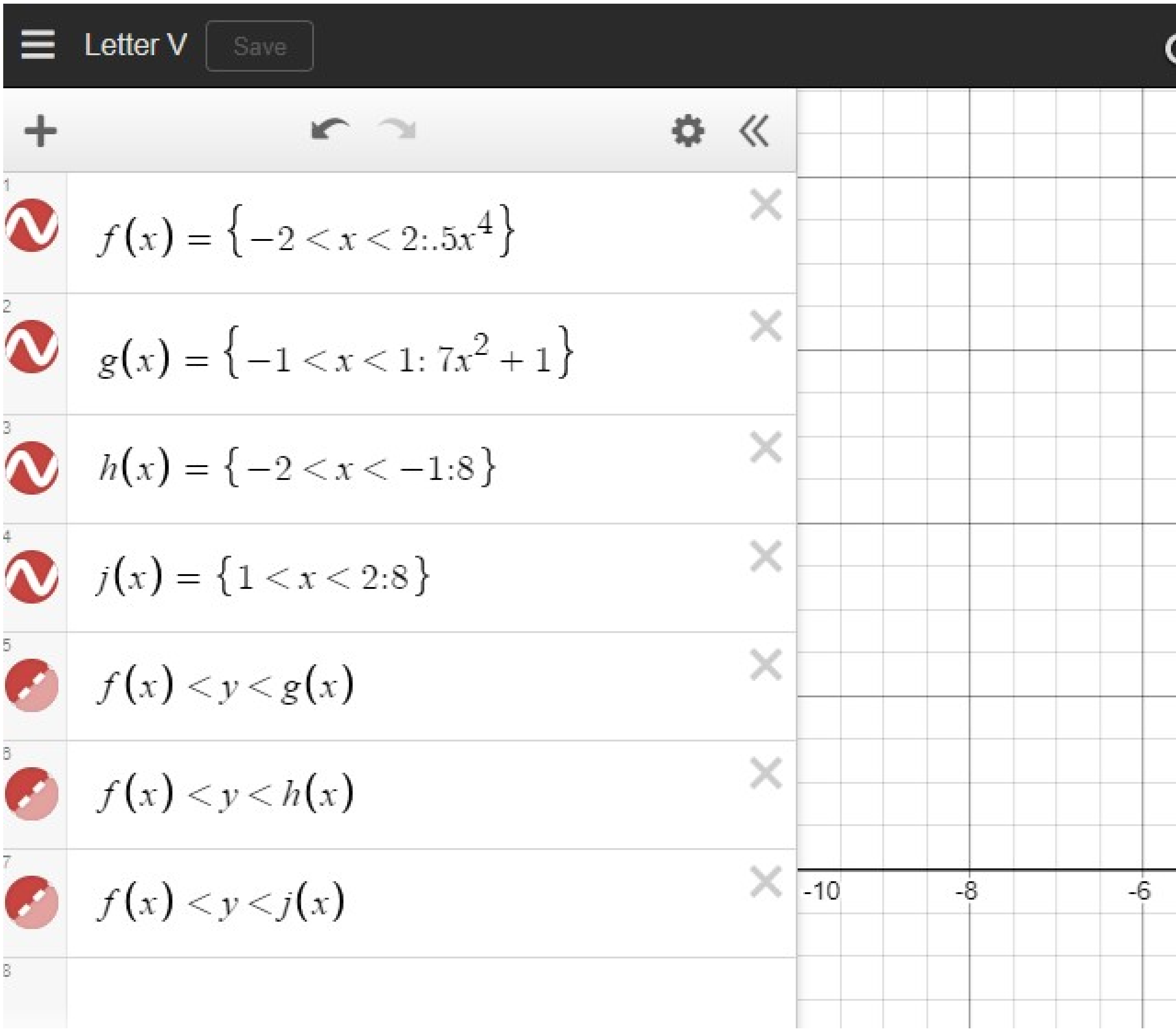
[3]See http://desmos.com at `http://desmos.com`

 Acknowledgements: Yoshiwara Books (https://yoshiwarabooks.org/)

Step 3 Restrict the domain of each of your functions so that extraneous parts of your functions which are not used to create your letter are not showing on your graph. Here is an example. Pay close attention to the formating to tell Desmos about the restricted domain.



Step 4 Use inequalities to tell Desmos how to shade in the inside of your letter. For example, if you write $f(x) < y < g(x)$, this tells Desmos to color in all the points of the form (x_0, y_0) where x_0 is in the domain of both $f(x)$ and $g(x)$ and y_0 is bigger than $f(x_0)$ and smaller than $g(x_0)$. Here is an example:



In order to get your picture to be all one color, you can long click on the colored circle next to the equation and pick your favorite color for each formula.

Step 5 Take a screenshot of your letter and the formulas that produced it just like the example above and save it.

Step 6 Repeat steps 2-5 with your other two chosen functions.

Step 7 Turn in your three screenshots.

Part 2

Origins of Trig

2.1 Right Triangle Trig

Learning Objectives

- Sine, Cosine, and Tangent
  - Similar triangles, and trig functions as ratios of triangle sides
  - Famous right triangles and deducing famous values
  - Find missing side
  - $\sin^2\theta + \cos^2\theta = 1$
- Secant, Cosecant, and Cotangent
- All From One, One From All
  - How to find values of all trig functions for an acute angle, given only one of such values.

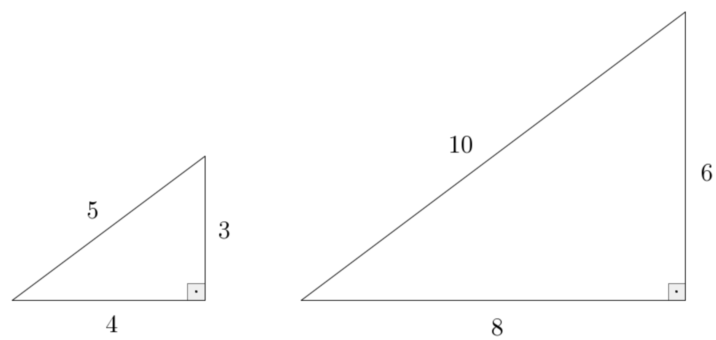
2.1.1 Sine, Cosine, and Tangent

Motivating Questions

- How to study, in a systematic way, ratios between the sides of a right triangle?
- What are the values of sine, cosine, and tangent, for the most frequent angles of $30°$, $45°$ and $60°$? And why?

Introduction

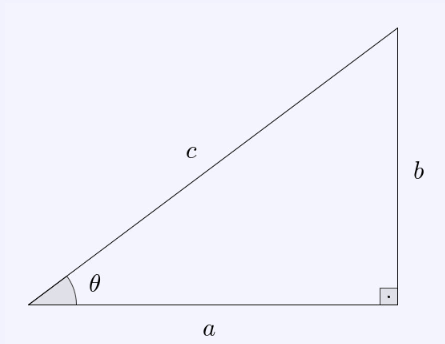
Recall that two triangles are called **similar** if one of them can be obtained by rescaling and moving around the other. Here's an example:



The dotted square symbol is a shorthand for "$90°$ degrees". Triangles which have a $90°$ angle are called **right triangles**, and will be the main focus of our discussion. What do similar triangles actually have in common? Certainly angles, but not necessarily the lengths of the sides. However, the **ratios** between any two sides of a triangle will remain the same, no matter how the triangle gets rescaled. Such ratios ultimately give us so much information about the given right triangle that they deserve special names: sine, cosine, and tangent.

Definitions and examples

Definition (right triangle trig): Consider the following right triangle, with one angle θ (this is the lowercase greek letter "theta") indicated, and sides labeled a, b and c.



Then:

- The side labeled with a is called the **adjacent** side to θ.
- The side labeled with b is called the **opposite** side to θ.
- The side labeled with c is called the **hypotenuse** of the triangle.

With this in place, we define the **sine**, **cosine**, and **tangent** of θ, by

$$\sin\theta = \frac{b}{c}\left(= \frac{\text{opp.}}{\text{hyp.}}\right), \quad \cos\theta = \frac{a}{c}\left(= \frac{\text{adj.}}{\text{hyp.}}\right), \text{ and } \tan\theta = \frac{b}{a}\left(= \frac{\text{opp.}}{\text{adj.}}\right).$$

Remark The hypotenuse of a right triangle is always the side opposite to the right angle. Also note that $\tan\theta = \sin\theta/\cos\theta$. You might have seen the mnemonic "SOH CAH TOA" before: for example, "SOH" means "sine equals opposite over hypotenuse", and so on.

Example 22. *For each of the following triangles with given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute $\sin\theta$, $\cos\theta$ and $\tan\theta$.*

a. **Explanation** We have opp. $= 12$, adj. $= 5$ and hyp. $= 13$. This means that

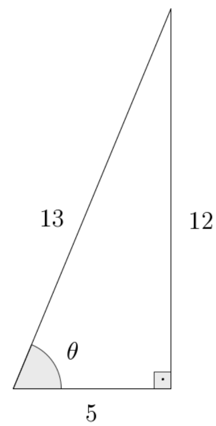
$$\sin\theta = \frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \text{and} \quad \tan\theta = \frac{12}{5}.$$

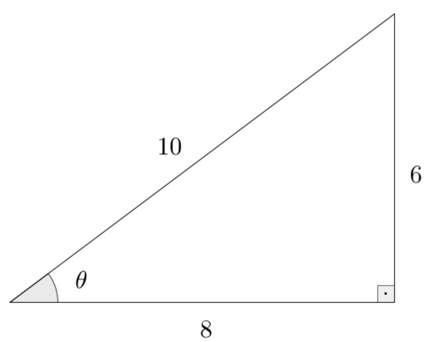
b. **Explanation** We have opp. $= 8$, adj. $= 6$ and hyp. $= 10$. This means that

$$\sin\theta = \frac{8}{10} = \frac{4}{5}, \quad \cos\theta = \frac{6}{10} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{8}{10} = \frac{4}{3}.$$

c. **Explanation** We have opp. $= 24$, adj. $= 18$ and hyp. $= 30$. This means that

$$\sin\theta = \frac{24}{30} = \frac{4}{5}, \quad \cos\theta = \frac{18}{30} = \frac{3}{5}, \quad \text{and} \quad \tan\theta = \frac{24}{18} = \frac{4}{3}.$$





Note that the values were the same values as in the previous item. This was expected, as the triangle there is similar to the triangle given here (the scaling factor is 3).

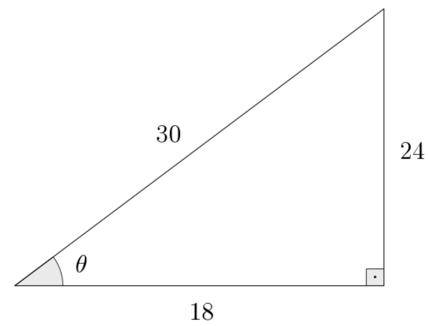
Remark Note that in all of the above examples, the values of $\sin\theta$ and $\cos\theta$ were always less than 1. This is always true, and a general consequence of the fact that the hypotenuse is always bigger than either of the other two sides.

Often, one has information about the angles, but not about all the sides. Knowing $\sin\theta$, $\cos\theta$ and $\tan\theta$ helps us find out missing sides of a given right triangle. For that, the following fact is extremely important:

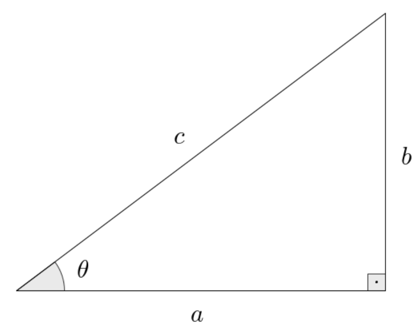
Theorem (Fundamental Identity): For any given angle θ, we have that

$$\sin^2\theta + \cos^2\theta = 1.$$

Here, $\sin^2\theta$ means $(\sin\theta)^2$, and similarly for $\cos^2\theta$.



Why is this true? Consider again a right triangle like below:



Then we know that $\sin\theta = b/c$ and $\cos\theta = a/c$. But the Pythagorean theorem also says that $a^2 + b^2 = c^2$. Putting all of this together, we have that

$$\sin^2\theta + \cos^2\theta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2}{c^2} + \frac{a^2}{b^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

as required.

Let's see how to apply this.

Example 23. *For each of the following triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

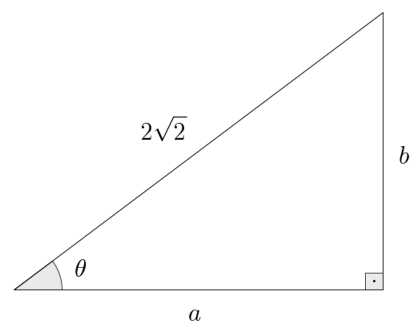
a. Given: $\sin\theta = \sqrt{2}/6$ *on*

Explanation From the given information, we know that

$$\frac{\sqrt{2}}{6} = \sin\theta = \frac{b}{2\sqrt{2}} \implies b = \frac{\sqrt{2}\times(2\sqrt{2})}{6} = \frac{2}{3}.$$

Now we use the Pythagorean theorem: the relation $a^2 + (2/3)^2 = (2\sqrt{2})^2$ gives us that

$$a^2 + \frac{4}{9} = 8 \implies a^2 = 8 - \frac{4}{9} = \frac{68}{9} \implies a = \frac{2\sqrt{17}}{3}.$$



Alternatively, to find the value of a, we can also use the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$ first — which then yields a. Here's how this goes:

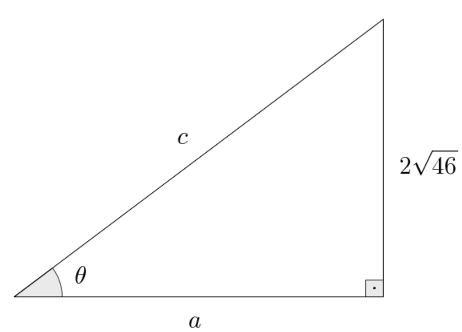
$$\left(\frac{\sqrt{2}}{6}\right)^2 + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{2}{36} \implies \cos^2\theta = \frac{34}{36},$$

and so $\cos\theta = \sqrt{34}/6$. Thus

$$\frac{\sqrt{34}}{6} = \cos\theta = \frac{a}{2\sqrt{2}} \implies a = \frac{2\sqrt{2}\times\sqrt{34}}{6} = \frac{2\sqrt{17}}{3},$$

as it should be. This is not something particular to this example: usually there is more than one strategy to solve this sort of problem. Which one is the best? You'll be the judge.

b. Given: $\cos\theta = \sqrt{3}/7$ *on*



Explanation Since this time we were given $\cos\theta$, but also the opposite side to θ, which does not appear on the expression for $\cos\theta$, we must rely more on the Pythagorean theorem instead. In any case, we know that

$$\frac{\sqrt{3}}{7} = \cos\theta = \frac{a}{c} \implies c = \frac{7a}{\sqrt{3}}.$$

Now, the Pythagorean relation reads $a^2 + (2\sqrt{46})^2 = (7a/\sqrt{3})^2$, and so:

$$a^2 + 184 = \frac{49a^2}{3} \implies 184 = \frac{49a^2}{3} - a^2$$

Continuing to manipulate this, we see that

$$184 = \frac{46a^2}{3} \implies a^2 = \frac{184 \times 3}{46} \implies a^2 = 12 \implies a = 2\sqrt{3}.$$

It remains to find the value of c. So we go back to the beginning and compute

$$c = \frac{7a}{\sqrt{3}} \implies c = \frac{7(2\sqrt{3})}{\sqrt{3}} \implies c = 14.$$

Values of trig functions for standard angles

We know that the sum of the inner angles of a triangle is always 180°. For right triangles, one of the angles is 90°, which means that the sum of the remaining two angles must also be 90°. Frequently we encounter triangles whose angles are 30°, 60° and 90°, and also triangles whose angles are 45°, 45° and 90°.

[figure]

These triangles have a special type of symmetry, which we'll exploit to find the values of sine, cosine, and tangent, for 30°, 45° and 60°. Finding the values of these trig functions for arbitrary angles, by hand, is a very difficult task. We will see later some trigonometric identities that may help us find such values for other angles but, in general, using a calculator (paying close attention to whether it is set to right "units") is the way to go.

For 30° **and** 60° Consider an equilateral triangle of side length ℓ. Equilateral means that all the sides have the same length. This implies that all the inner angles must be equal and, since they must add up to 180°, each of them equals 60°. But also draw a height h:

By the Pythagorean theorem, we know that

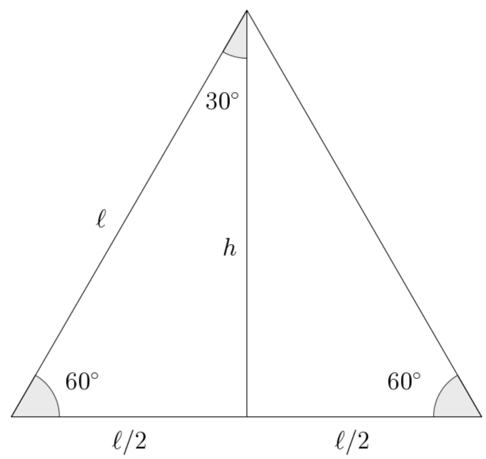
$$\ell^2 = \left(\frac{\ell}{2}\right)^2 + h^2,$$

and so we may compute:

$$\ell^2 = \frac{\ell^2}{4} + h^2 \implies \frac{3\ell^2}{4} = h^2 \implies h = \frac{\ell\sqrt{3}}{2}.$$

Now, relative to the 60° angle, we recognize

$$\text{opp.} = h = \frac{\ell\sqrt{3}}{2}, \quad \text{adj.} = \frac{\ell}{2}, \quad \text{and} \quad \text{hyp.} = \ell.$$



This means that

$$\sin(60^\circ) = \frac{h}{\ell} = \frac{\left(\frac{\ell\sqrt{3}}{2}\right)}{\ell} = \frac{\sqrt{3}}{2},$$

as well as

$$\cos(60^\circ) = \frac{\ell/2}{\ell} = \frac{1}{2} \quad \text{and} \quad \tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

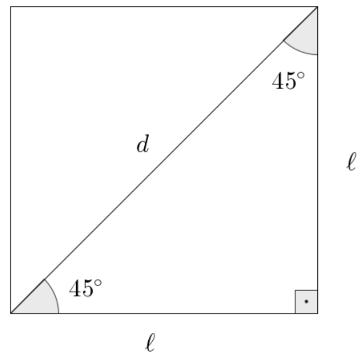
To find the values of $\sin(30^\circ)$, $\cos(30^\circ)$, and $\tan(30^\circ)$, we can use the same triangle, noting that the opposite side to 30° is the adjacent side to 60°, and that the adjacent side to 30° is the opposite side to 60°. Since the hypotenuse is always the side opposite to the right angle, we conclude that

$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}, \quad \cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

and, finally, that

$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Remark This is a general phenomenon: two acute angles are called **complementary** if they add up to 90°. In other words, the complementary angle to θ is always $90^\circ - \theta$, and $\sin\theta = \cos(90^\circ - \theta)$, as well as $\cos\theta = \sin(90^\circ - \theta)$. In particular, this justifies the name "cosine": it is the sine of the complement. We will discuss "coterminal angles" and "cofunctions" in more generality later.



For 45° Consider a square of side length ℓ, and draw a diagonal d.

By the Pythagorean theorem, $d^2 = \ell^2 + \ell^2 = 2\ell^2$ implies that $d = \ell\sqrt{2}$. Relative to either of the 45° angles, we have

$$\text{opp.} = \ell, \quad \text{adj.} = \ell, \quad \text{and} \quad \text{hyp.} = d = \ell\sqrt{2}.$$

Hence

$$\sin(45^\circ) = \cos(45^\circ) = \frac{\ell}{\ell\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \tan(45^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)} = 1.$$

Remark It is convenient to write $\sqrt{2}/2$ instead of $1/\sqrt{2}$ (similarly for $\sqrt{3}/3$ versus $1/\sqrt{3}$), even though the latter is mathematically acceptable, because it makes it easier to estimate. Namely, knowing that $\sqrt{2} \approx 1.414$, we know that $\sqrt{2}/2 \approx 0.707$, but when looking at $1/\sqrt{2}$, what does it mean to divide 1 by 1.414? This is the general reason why rationalizing fractions is useful.

Standard values

We can summarize what we have discovered here in a table. Besides our standard angles of 30°, 45°, and 60°, we can also consider 0° and 90° as extreme cases. Let's do a quick thought experiment to understand this: if a right triangle had an angle of 0°, this triangle would in fact collapse to a line segment, and we would have opp. $= 0$, while hyp. $=$ adj., suggesting we set $\sin(0^\circ) = 0$ and $\cos(0^\circ) = 1$. Since 0° and 90° are complementary, we're forced to set $\sin(90^\circ) = 1$ and $\cos(90^\circ) = 0$. But while

$$\tan(0^\circ) = \frac{\sin(0^\circ)}{\cos(0^\circ)} = \frac{0}{1} = 0,$$

computing $\tan(90°)$ does not make sense, as we would have a division by $\cos(90°) = 0$. We say that $\tan(90°)$ is **undefined**, or that it **does not exist** ("DNE" for short, as usual). So, we have:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

Those values should be committed to heart, but it's easier than what it seems. Here's how you can think about it:

- No need to memorize values for tangent: if you know $\sin\theta$ and $\cos\theta$, you can just compute $\tan\theta = \sin\theta/\cos\theta$.
- No need to memorize the values for cosine: recall that the cosine of an angle is the sine of the complement. So if you know values for sine, you're in business.
- How to memorize values for sine? The one thing you should remember here is that the values 0, $1/2$, $\sqrt{2}/2$, $\sqrt{3}/2$ and 1 will appear. What is their order? Simple: write them in increasing order, just like the angles from $0°$ to $90°$. So

  $$\sin(0°) = 0,\ \sin(30°) = \frac{1}{2},\ \sin(45°) = \frac{\sqrt{2}}{2},\ \sin(60°) = \frac{\sqrt{3}}{2},\ \sin(90°) = 1.$$

Summary

- We have defined sine, cosine, and tangent, as ratios between sides of a right triangle. For each angle θ, the fundamental identity $\sin^2\theta + \cos^2\theta = 1$ holds. It can be used together with the Pythagorean Theorem to get information about all sides of a given triangle, when some of them might be missing, provided you have some information about the angles.
- We have established the standard values of sine, cosine, and tangent, for the most frequent angles of $30°$, $45°$, and $60°$. Those values have been organized in a table. They are so frequent that knowing the values there by heart is useful, but exaggerated efforts into memorizing the table should not be wasted — understanding how the values are deduced pays off more in the long run.

2.1.2 Secant, Cosecant and Cotangent

Motivating Questions

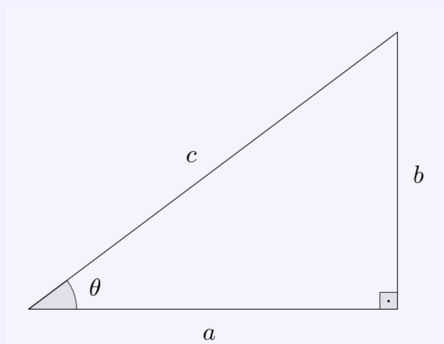
- How to use the reciprocal ratios of $\sin\theta$, $\cos\theta$, and $\tan\theta$, to also obtain information about a given right triangle?
- What are the values of such reciprocal rations for the standard angles of $30°$, $45°$ and $60°$?

Introduction

Briefly speaking, we have met three fundamental ratios between sides of a right triangle: sine, cosine, and tangent. But their reciprocals are also relevant ratios between the sides of the given triangle. Now, while such reciprocals ratios turn out to carry the same information as sine, cosine, and tangent, it is useful to know how to manipulate them as well. Later, when we study trigonometric functions as actual functions of a real parameter, discussing their graphs, symmetries, etc., more differences will become apparent.

Definitions and examples

Definition (right triangle trig – bis): Consider the following right triangle, with one angle θ indicated, and sides labeled a, b and c.



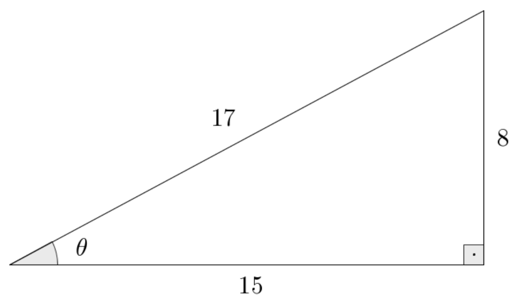
We define the **secant**, **cosecant**, and **cotangent** of θ, by

- $\sec\theta = \dfrac{c}{b} = \dfrac{1}{\cos\theta}\left(= \dfrac{\text{hyp.}}{\text{adj.}}\right)$;
- $\csc\theta = \dfrac{c}{a} = \dfrac{1}{\sin\theta}\left(= \dfrac{\text{hyp.}}{\text{opp.}}\right)$, and;

- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta} \left(= \dfrac{\text{adj.}}{\text{opp.}} \right)$.

Note that since for acute angles we always have $\sin\theta$ and $\cos\theta$ between 0 and 1, the reciprocals $\csc\theta$ and $\sec\theta$ will always be bigger than 1.

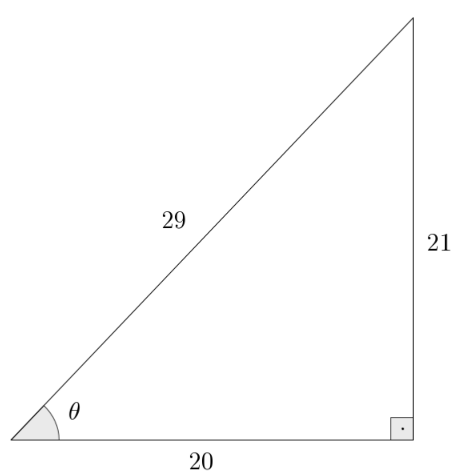
Example 24. *For each of the following triangles with a given angle θ, identify the adjacent (adj.), opposite (opp.) and hypotenuse (hyp.), and compute $\sec\theta$, $\csc\theta$, and $\cot\theta$.*



a. **Explanation** We have opp. = 8, adj. = 5 and hyp. = 17. This means that

$$\sec\theta = \frac{17}{15}, \quad \csc\theta = \frac{17}{8}, \quad \text{and} \quad \cot\theta = \frac{15}{8}.$$

Of course, you can find $\cos\theta$, $\sin\theta$, and $\tan\theta$ first, and then just flip all the fractions.



b. **Explanation** This time, we have opp. = 21, adj. = 20 and hyp. = 29.

This means that

$$\sec\theta = \frac{29}{20}, \quad \csc\theta = \frac{29}{21}, \quad \text{and} \quad \cot\theta = \frac{20}{21}.$$

Next, we had the fundamental identity $\sin^2\theta + \cos^2\theta = 1$. It turns out that with this, we may obtain two extra useful identities.

Theorem (Fundamental Identities – bis): For any given angle θ, we have that

$$1 + \cot^2\theta = \csc^2\theta \quad \text{and} \quad \tan^2\theta + 1 = \sec^2\theta,$$

where $\cot^2\theta$ means $(\cot\theta)^2$, and similarly for all other functions.

You should *not* think of those two extra identities as something more to be memorized. The only identity worth the trouble is $\sin^2\theta + \cos^2\theta = 1$. The following strategy is something you can quickly reproduce on a scrap paper if you need to recall these formulas, once you have understood the idea once:

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$:

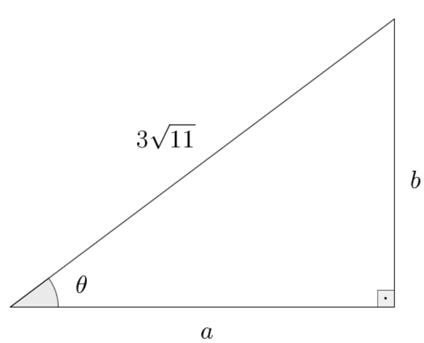
$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies 1 + \frac{\cos^2\theta}{\sin^2\theta} = \csc^2\theta \implies 1 + \cot^2\theta = \csc^2\theta.$$

- Divide both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \frac{\sin^2\theta}{\cos^2\theta} + 1 = \sec^2\theta \implies \tan^2\theta + 1 = \sec^2\theta.$$

Example 25. *For each of the given triangles, given the value of a trigonometric function at the indicated angle θ, find the lengths of the missing sides.*

a. *Given:* $\csc\theta = \sqrt{11}/2$ *on*



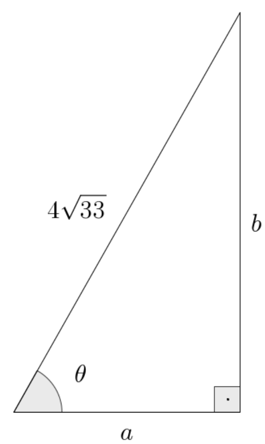
Explanation We start with

$$\frac{3\sqrt{11}}{a} = \csc\theta = \frac{\sqrt{11}}{2} \implies 6\sqrt{11} = a\sqrt{11} \implies a = 6.$$

It remains to find the value of b. This can be done with the Pythagorean Theorem, as follows: $a^2 + b^2 = c^2$ becomes

$$36 + b^2 = (3\sqrt{11})^2 \implies 36 + b^2 = 99 \implies b^2 = 63 \implies b = 3\sqrt{7}.$$

b. *Given:* $\cot\theta = 2\sqrt{2}/5$ *on*



Explanation Let's start again using the trigonometric function we were given:

$$\frac{a}{b} = \cot\theta = \frac{2\sqrt{2}}{5} \implies a = \frac{2b\sqrt{2}}{5}.$$

We cannot conclude anything else about a and b just from this, so we must resort to the Pythagorean Theorem again. The relation $a^2 + b^2 = c^2$ gives us that

$$\left(\frac{2b\sqrt{2}}{5}\right)^2 + b^2 = (4\sqrt{33})^2 \implies \frac{8b^2}{25} + b^2 = 528 \implies \frac{33b^2}{25} = 528.$$

Simplifying this, we have that

$$b^2 = 25 \times \frac{528}{33} = 25 \times 16 \implies b = 5 \times 4 \implies b = 20.$$

Now, we may go back and find a:

$$a = \frac{2b\sqrt{2}}{5} = \frac{40\sqrt{2}}{5} \implies b = 8\sqrt{2}.$$

Values of trig functions for standard angles – bis

Previously, we have obtained the following table of standard values for sine, cosine, and tangent:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

By simply inverting all of those values, we naturally obtain a similar table with standard values for secant, cosecant and cotangent. Of course, this is *not* another table you have to memorize, but we'll list it here for completeness:

	$0°$	$30°$	$45°$	$60°$	$90°$
$\csc\theta$	DNE	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	DNE
$\cot\theta$	DNE	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Note the undefined "extreme" values: $\csc(0°)$ is undefined, because that would be $1/\sin(0°)$, but $\sin(0°) = 0$ and we cannot divide by zero. Similarly for the other ones.

Summary

- We have defined secant, cosecant, and cotangent, as the reciprocal ratios of cosine, sine, and tangent. For each angle θ, we have the associated fundamental identities $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$, which can be easily deduced from the good old $\sin^2\theta + \cos^2\theta = 1$. Again, such identities can be used together with the Pythagorean Theorem to obtain information about sides of a right triangle.
- We have summarized (again in a table) the standard values of secant, cosecant, and cotangent, for the most frequent angles of $30°$, $45°$, and $60°$.

2.1.3 All From One, One From All

Motivating Questions

- Do all the trigonometric functions we have seen so far carry the same information?
- How to find all trigonometric functions, given a single one of them?

Introduction

We have encountered six trigonometric functions of an acute angle θ so far: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$. They all help up get information about right triangles having θ as one of the inner angles. But here is the thing: at this stage, they all carry the same information. All of these quantities are positive real numbers, and we have not only the Pythagorean Theorem, but also the fundamental relations

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Recall that while the first one is the most important one, the second two are immediate consequences of the first, by dividing it by $\sin^2\theta$ and $\cos^2\theta$, respectively. With all of this in place, once you have one of the six trigonometric values at θ, you can in fact find all of them. We'll explore this in this section, with several examples.

How to find all trigonometric functions, given one of them?

There are a few main facts one should keep in mind here.

- You know $\sin\theta$ if and only if you know $\csc\theta$.
- You know $\cos\theta$ if and only if you know $\sec\theta$.
- You know $\tan\theta$ if and only if you know $\cot\theta$.
- If you know $\sin\theta$ and $\cos\theta$, you know $\tan\theta$.

And there are two strategies: using just the trigonometric identities and proceeding algebraically (let's call this "strategy 1"), or drawing a suitable right triangle and thinking of opp., adj. and hyp. (let's call this "strategy 2"). We'll illustrate both of them with several examples, but in the end of the day, you may choose whichever strategy you'd like (unless specifically instructed otherwise).

Example 26. *Let θ be an acute angle. In all of the following problems, given the value of a certain trigonometric function at the value θ, find the remaining five.*

a. *Given:* $\sin\theta = 1/4$.

Explanation

- **Strategy 1:** Let's find $\cos\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \implies \frac{1}{16} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{1}{16},$$

so

$$\cos^2\theta = \frac{15}{16} \implies \cos\theta = \frac{\sqrt{15}}{4}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

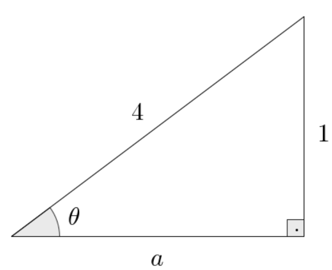
Flipping the fractions

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4} \quad \text{and} \quad \tan\theta = \frac{\sqrt{15}}{15},$$

respectively, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

- **Strategy 2:** We start drawing the following triangle: So, we find the



missing side a, using the Pythagorean relation $a^2 + b^2 = c^2$, which reads and gives:

$$a^2 + 1^2 = 4^2 \implies a^2 + 1 = 16 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Now we have all the sides, so finding all the ratios is immediate:

$$\sin\theta = \frac{1}{4}, \quad \cos\theta = \frac{\sqrt{15}}{4}, \quad \text{and} \quad \tan\theta = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}.$$

Flipping all the fractions, we obtain

$$\csc\theta = 4, \quad \sec\theta = \frac{4\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \sqrt{15}.$$

b. Given: $\cos\theta = 2/3$.

Explanation

- **Strategy 1:** Let's find $\sin\theta$ first, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\sin^2\theta + \left(\frac{2}{3}\right)^2 = 1 \implies \sin^2\theta + \frac{4}{9} = 1 \implies \sin^2\theta = 1 - \frac{4}{9},$$

so

$$\sin^2\theta = \frac{5}{9} \implies \sin\theta = \frac{\sqrt{5}}{3}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}.$$

Flipping the fractions

$$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3} \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{2},$$

respectively, we obtain

$$\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

- **Strategy 2:** This time, here's the triangle we'll use: We'll use the Pythagorean relation $a^2+b^2 = c^2$ to find the missing side b, as follows:

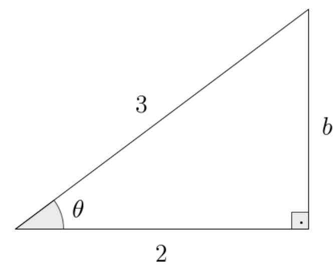
$$2^2 + b^2 = 3^2 \implies 4 + b^2 = 9 \implies b^2 = 5,$$

so $b = \sqrt{5}$. Again, with all the sides, we can read all the main ratios:

$$\sin\theta = \frac{\sqrt{5}}{3}, \quad \cos\theta = \frac{2}{3}, \quad \text{and} \quad \tan\theta = \frac{\sqrt{5}}{3}.$$

Taking reciprocals, we get the rest:

$$\csc\theta = \frac{3\sqrt{5}}{5} \quad \sec\theta = \frac{3}{2}, \quad \text{and} \quad \cot\theta = \frac{2\sqrt{5}}{5}.$$



c. *Given:* $\tan\theta = 5/4$.

Explanation

- **Strategy 1:** The only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$, so we might as well use it. It reads

$$\left(\frac{5}{4}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{25}{16} + 1 \implies \sec^2\theta = \frac{41}{16},$$

and so

$$\sec\theta = \frac{\sqrt{41}}{4} \implies \cos\theta = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41}.$$

With this, we could in principle find $\sin\theta$, by using $\sin^2\theta + \cos^2\theta = 1$ as usual. But there is a simpler way. Namely, we use that

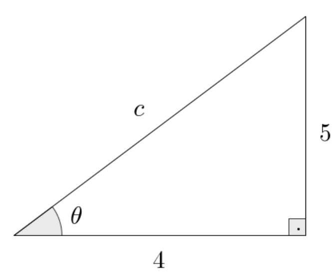
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{5}{4}\cdot\frac{4\sqrt{41}}{41} \implies \sin\theta = \frac{5\sqrt{41}}{41}.$$

Meaning that

$$\csc\theta = \frac{\sqrt{41}}{5} \quad \text{and} \quad \cot\theta = \frac{4}{5},$$

and we are done.

- **Strategy 2:** Now, we set a right triangle with legs 4 and 5, like below:



Since the hypotenuse c is missing, applying the Pythagorean Theorem is even easier:

$$c^2 = 4^2 + 5^2 = 16 + 25 = 41 \implies c = \sqrt{41}.$$

With this in place, we read from the triangle the main ratios as

$$\sin\theta = \frac{5\sqrt{41}}{41}, \quad \cos\theta = \frac{4\sqrt{41}}{41} \quad \text{and} \quad \tan\theta = \frac{5}{4}.$$

And taking reciprocals:

$$\csc\theta = \frac{\sqrt{41}}{5}, \quad \sec\theta = \frac{\sqrt{41}}{4} \quad \text{and} \quad \cot\theta = \frac{4}{5}.$$

d. Given: $\sec\theta = 7/3$.

Explanation

- **Strategy 1:** We immediately know that $\cos\theta = 3/7$, so let's find $\sin\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

$$\sin^2\theta + \left(\frac{3}{7}\right)^2 = 1 \implies \sin^2\theta + \frac{9}{49} = 1 \implies \sin^2\theta = 1 - \frac{9}{49},$$

so

$$\sin^2\theta = \frac{40}{49} \implies \sin\theta = \frac{2\sqrt{10}}{7}.$$

With this, we have that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{10}/7}{3/7} = \frac{2\sqrt{10}}{3}.$$

Flipping the remaining fractions

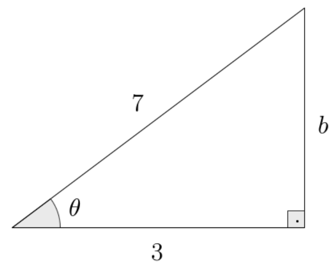
$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3},$$

respectively, we obtain

$$\csc\theta = \frac{7}{2\sqrt{10}} = \frac{7\sqrt{10}}{20} \quad \text{and} \quad \cot\theta = \frac{3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20}.$$

- **Strategy 2:** Let's draw a triangle with hypotenuse 7 and adjacent side to θ having length 3: Let's use, as usual, the Pythagorean relation $a^2 + b^2 = c^2$ to find b. It gives us that

$$3^2 + b^2 = 7^2 \implies 9 + b^2 = 49 \implies b^2 = 40 \implies b = 2\sqrt{10}.$$



So we have that

$$\sin\theta = \frac{2\sqrt{10}}{7}, \quad \cos\theta = \frac{3}{7}, \quad \text{and} \quad \tan\theta = \frac{2\sqrt{10}}{3}.$$

Taking reciprocals and rationalizing each of them, we also get

$$\csc\theta = \frac{7\sqrt{10}}{20}, \quad \sec\theta = \frac{7}{3}, \quad \text{and} \quad \cot\theta = \frac{3\sqrt{10}}{20}.$$

e. Given: $\csc\theta = 8/7$.

Explanation

- **Strategy 1:** We immediately know that $\sin\theta = 7/8$, so let's find $\cos\theta$ next, using the first fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We have

  $$\left(\frac{7}{8}\right)^2 + \cos^2\theta = 1 \implies \frac{49}{64} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{49}{64},$$

  so

  $$\cos^2\theta = \frac{15}{64} \implies \cos\theta = \frac{\sqrt{15}}{8}.$$

  With this, we have that

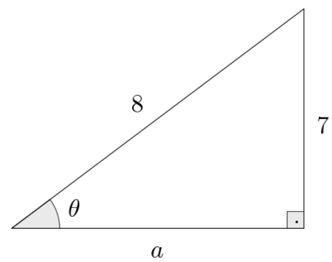
  $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{7/8}{\sqrt{15}/8} = \frac{7}{\sqrt{15}} = \frac{7\sqrt{15}}{15}.$$

  Flipping the remaining fractions

  $$\cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7}{\sqrt{15}},$$

  respectively, we obtain

  $$\sec\theta = \frac{8}{\sqrt{15}} = \frac{8\sqrt{15}}{15} \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}.$$



- **Strategy 2:** Consider the following triangle: Let's find a using the Pythagorean relation $a^2 + b^2 = c^2$, which becomes

$$a^2 + 7^2 = 8^2 \implies a^2 + 49 = 64 \implies a^2 = 15,$$

so that $a = \sqrt{15}$. Having all the sides of the triangle, we read that

$$\sin\theta = \frac{7}{8}, \quad \cos\theta = \frac{\sqrt{15}}{8}, \quad \text{and} \quad \tan\theta = \frac{7\sqrt{15}}{15}.$$

Taking reciprocals, we get

$$\csc\theta = \frac{8}{7}, \quad \sec\theta = \frac{8\sqrt{15}}{15}, \quad \text{and} \quad \cot\theta = \frac{\sqrt{15}}{7}$$

as well.

f. Given: $\cot\theta = 2/9$.

Explanation

- **Strategy 1:** We immediately know that $\tan\theta = 9/2$ and, again, the only fundamental identity we have involving $\tan\theta$ is $\tan^2\theta + 1 = \sec^2\theta$. It reads

$$\left(\frac{9}{2}\right)^2 + 1 = \sec^2\theta \implies \sec^2\theta = \frac{81}{4} + 1 \implies \sec^2\theta = \frac{85}{4},$$

and so

$$\sec\theta = \frac{\sqrt{85}}{2} \implies \cos\theta = \frac{2}{\sqrt{85}} = \frac{2\sqrt{85}}{85}.$$

Again, instead of using $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$, we can just argue that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \implies \sin\theta = \tan\theta\cos\theta = \frac{9}{2}\cdot\frac{2\sqrt{85}}{85} \implies \sin\theta = \frac{9\sqrt{85}}{85}.$$

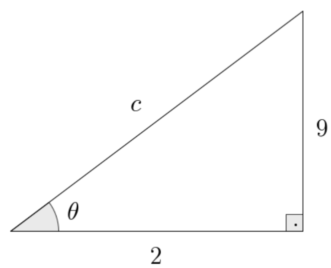
Meaning that

$$\csc\theta = \frac{\sqrt{85}}{9},$$

and so we are done.

As an aside, try to solve this problem again without immediately using that $\tan\theta = 9/2$, start only with $\cot\theta = 2/9$, use the identity $1 + \cot^2\theta = \csc^2\theta$ and go from there, it is instructive.

- **Strategy 2:** Again, let's set up a convenient triangle: Here, in par-



ticular, note that the scale and proportions in this picture are completely off. This is ok, since drawing triangles is just meant to help us organize what is the opposite side to θ, what is the adjacent side, and what is the hypotenuse. It doesn't matter how bad your picture looks, as long as the "positions" are correct. In any case, we immediately find c with

$$c^2 = a^2 + b^2 = 2^2 + 9^2 = 4 + 81 = 85,$$

so $c = \sqrt{85}$. Now we can read all the ratios and rationalize them to obtain:

$$\sin\theta = \frac{9\sqrt{85}}{85}, \quad \cos\theta = \frac{2\sqrt{85}}{85}, \quad \text{and} \quad \tan\theta = \frac{9}{2}.$$

Take reciprocals and rationalize whatever is needed to get

$$\csc\theta = \frac{\sqrt{85}}{9}, \quad \sec\theta = \frac{\sqrt{85}}{2}, \quad \text{and} \quad \cot\theta = \frac{2}{9}$$

as well.

Let's summarize the highlights of the strategy, from the algebraic perspective.

(a) If you're given sin or cos, use the fundamental identity to find the other one. Then find tan = sin / cos, and flip all the fractions to get csc, sec and tan.

(b) If you're given csc or sec, flip it to get sin or cos, and proceed as (a).

(c) If you're given tan, use $\tan^2\theta + 1 = \sec^2\theta$ to find $\sec\theta$. Once you have $\sec\theta$, you have $\cos\theta$. Then proceed as (a).

(d) If you're given cot, flip it to get tan, and proceed as (c).

Note that we're employing a mathematician's general philosophy here: take a problem and reduce it to something which you already know how to solve (namely, we're arguing that — morally — if you know how to solve the problem when you were given either sin or cos, then you in fact know how to solve it when given *any* of the six trigonometric functions). And also from the geometric perspective, the strategy is even easier to describe: recognize the trigonometric function you were given in terms of opp., adj. and hyp., then draw a right triangle with this information. You will be missing one side, which can be found with the Pythagorean Theorem. Once you have all sides, you can find all the ratios between sides.

Of course, the two above ways to go about this are not the only ones, but they're as good a recipe as any. In any case, you have room for creativity here. And even if one method seems easier than the other, it is useful to be comfortable with both, as this is already a good chance to start getting acquainted with trigonometry identities, which will be indispensable later.

We will see later how to define and deal with trigonometric functions for angles which are not necessarily acute. Then, everything we did here becomes slightly more subtle, as one must now pay attention to signs (for example, we'll have that $\cos(120°) = -1/2$). But the overall program of using the fundamental trigonometric identities and the relations between the main trigonometric functions (sin, cos, and tan) with their reciprocals (csc, sec, and cot) will always be useful.

Summary

- We have illustrated, with several examples, two ways to find all the values of the trigonometric functions at an acute angle θ, once we know one of the values. This can be done algebraically by exploring trigonometric identities, or geometrically by drawing the "correct" triangle and applying the Pythagorean Theorem to find the missing side – to then read all ratios directly from the triangle itself.

2.2 The Unit Circle

Learning Objectives

- The Unit Circle
  - Degrees and Radians
  - Reference Angles
  - The Definition of trigonometric functions in terms of the Unit Circle
  - Evaluating trigonometric functions at standard angles

2.3 Trig Identities

Learning Objectives

- Trig Identities
  - What expressions composed of trig functions are identically equal for all values of x?
- Trig Identity Applications
  - How can applying trig identities help us solve problems?

2.3.1 Trigonometric Identities

Pythagorean Identities and Conjugates

From the previous section, we have found some identities. We will now summarize what we have already found and begin to introduce new identities. These will help us to breakdown and simplify trigonometric equations that will hopefully make our lives easier.

The Pythagorean Identities:

(a) $\cos^2(\theta) + \sin^2(\theta) = 1$.

Common Alternate Forms:

- $1 - \sin^2(\theta) = \cos^2(\theta)$
- $1 - \cos^2(\theta) = \sin^2(\theta)$

(b) $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

Common Alternate Forms:

- $\sec^2(\theta) - \tan^2(\theta) = 1$
- $\sec^2(\theta) - 1 = \tan^2(\theta)$

(c) $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

Common Alternate Forms:

- $\csc^2(\theta) - \cot^2(\theta) = 1$
- $\csc^2(\theta) - 1 = \cot^2(\theta)$

Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;

if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$:
  $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$
- $1 - \sin(\theta)$ and $1 + \sin(\theta)$:
  $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$:
  $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$:
  $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$:
  $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$:
  $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Verifying Identities

Remark **Strategies for Verifying Identities**

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Example 27.

More Identities

In the previous sections, we saw the utility of the Pythagorean Identities. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond. Our first set of identities is the 'Even / Odd' identities.

Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

2.3.2 Trigonometric Identities

Pythagorean Identities and Conjugates

From the previous section, we have found some identities. We will now summarize what we have already found and begin to introduce new identities. These will help us to breakdown and simplify trigonometric equations that will hopefully make our lives easier.

The Pythagorean Identities:

(a) $\cos^2(\theta) + \sin^2(\theta) = 1$.

Common Alternate Forms:

- $1 - \sin^2(\theta) = \cos^2(\theta)$
- $1 - \cos^2(\theta) = \sin^2(\theta)$

(b) $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

Common Alternate Forms:

- $\sec^2(\theta) - \tan^2(\theta) = 1$
- $\sec^2(\theta) - 1 = \tan^2(\theta)$

(c) $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

Common Alternate Forms:

- $\csc^2(\theta) - \cot^2(\theta) = 1$
- $\csc^2(\theta) - 1 = \cot^2(\theta)$

Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;
  if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$;
  if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$;

if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Pythagorean Conjugates

- $1 - \cos(\theta)$ and $1 + \cos(\theta)$:
  $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$
- $1 - \sin(\theta)$ and $1 + \sin(\theta)$:
  $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta) - 1$ and $\sec(\theta) + 1$:
  $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$:
  $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $\csc(\theta) - 1$ and $\csc(\theta) + 1$:
  $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$:
  $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Verifying Identities

Remark **Strategies for Verifying Identities**

- Try working on the more complicated side of the identity.
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- Use the Pythagorean Identities to ‘exchange’ sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

Example 28.

More Identities

In the previous sections, we saw the utility of the Pythagorean Identities. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond. Our first set of identities is the 'Even / Odd' identities.

Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Part 3

Trigonometric Functions

3.1 The Unit Circle to the Function Graph

Learning Objectives

- –
  –
  –

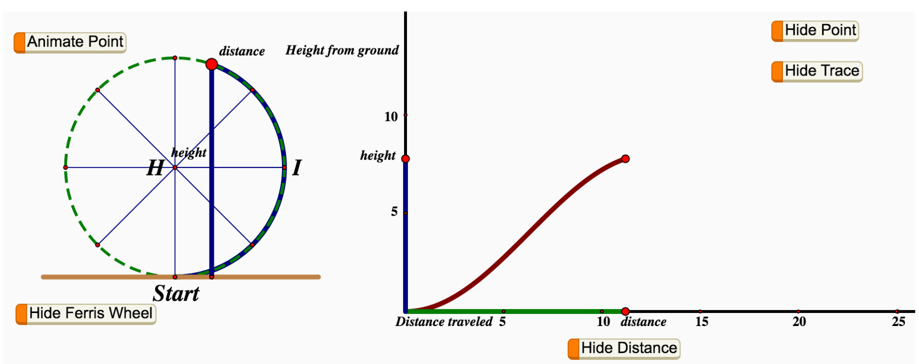
3.1.1 Traversing A Circle

Motivating Questions

- How does a point traversing a circle naturally generate a function?
- What are some important properties that characterize a function generated by a point traversing a circle?

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. You may recall from when we first studied periodic function that we considered the case of taking a ride on a ferris wheel. We considered your height, h, above the ground and how your height changed in tandem with the distance, d, that you have traveled around the wheel. We saw snapshot of this situation, which is available as a full animation at http://gvsu.edu/s/0Dt[4] .

A snapshot of the motion of a cab moving around a ferris wheel. Reprinted with permission from Illuminations by the National Council of Teachers of Mathematics. All rights reserved.



Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Exploration In the context of the ferris wheel pictured in above, assume that the height, h, of the moving point (the cab in which you are riding), and the distance, d, that the point has traveled around the circumference of the ferris wheel are both measured in meters. Further, assume that the circumference of the ferris wheel is 150 meters. In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

[4]See http://gvsu.edu/s/0Dt at `http://gvsu.edu/s/0Dt`

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

a. Recall that the circumference, C, of a circle is connected to the circle's radius, r, by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?

b. How high is the cab after it has traveled 1/4 of the circumference of the circle?

c. How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{150}{\pi} \approx 47.75$ meters?

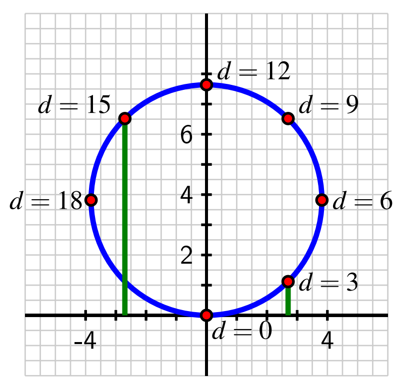
d. Can h be thought of as a function of d? Why or why not?

e. Can d be thought of as a function of h? Why or why not?

f. Why do you think the curve shown above has the shape that it does? Write several sentences to explain.

Circular Functions

The natural phenomenon of a point moving around a circle leads to interesting relationships. Let's consider a point traversing a circle of circumference 24 and examine how the point's height, h, changes as the distance traversed, d, changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen below where each noted point lies 3 additional units along the circle beyond the preceding one.



Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located 1/4 of the way along the circle) is at

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a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in the graph above as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can determine the other entries in the table below.

Data for height, h, as a function of distance traversed, d.

d	0	3	6	9	12	15	18	21	24
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

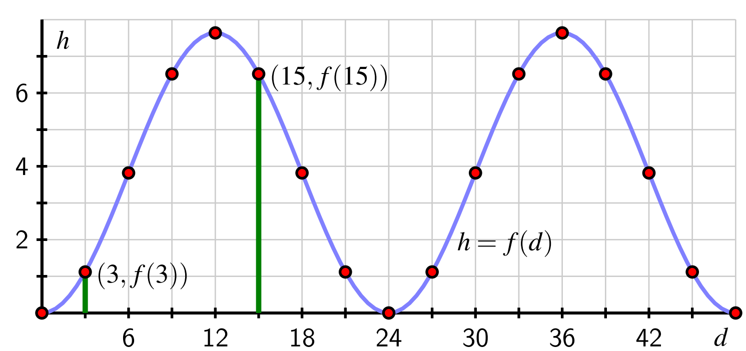
Moreover, if we now let the point continue traversing the circle, we observe that the d-values will increase accordingly, but the h-values will repeat according to the already-established pattern, resulting in the data in the table below.

Additional data for height, h, as a function of distance traversed, d.

d	24	27	30	33	36	39	42	45	48
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown below.

The height, h, of a point traversing a circle of radius 24 as a function of distance, d, traversed around the circle.



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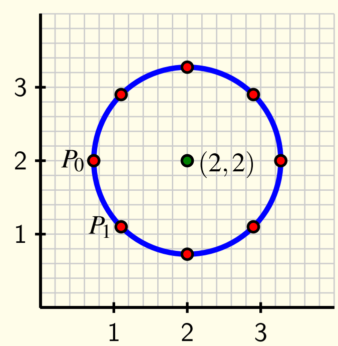
The function $h = f(d)$ we have been discussing is an example of what we will call a **circular function**. Indeed, it is apparent that if we:

- take any circle in the plane,
- choose a starting location for a point on the circle,
- let the point traverse the circle continuously,
- and track the height of the point as it traverses the circle,

the height of the point is a function of distance traversed and the resulting graph will have the same basic shape as the curve shown in the graph above. It also turns out that if we track the location of the x-coordinate of the point on the circle, the x-coordinate is also a function of distance traversed and its curve has a similar shape to the graph of the height of the point (the y-coordinate). Both of these functions are circular functions because they are generated by motion around a circle.

Exploration Consider the circle pictured below that is centered at the point $(2, 2)$ and that has circumference 8. Assume that we track the y-coordinate (that is, the height, h) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.



a. How far along the circle is the point P_1 from P_0? Why?

b. Label the subsequent points in the figure P_2, P_3, ... as we move counterclockwise around the circle. What are the exact coordinates of P_2? of P_4? Why?

c. Determine the coordinates of the remaining points on the circle (exactly where possible, otherwise approximately) and hence complete the entries in the table below that track the height, h, of the point traversing the circle as a function of distance traveled, d.

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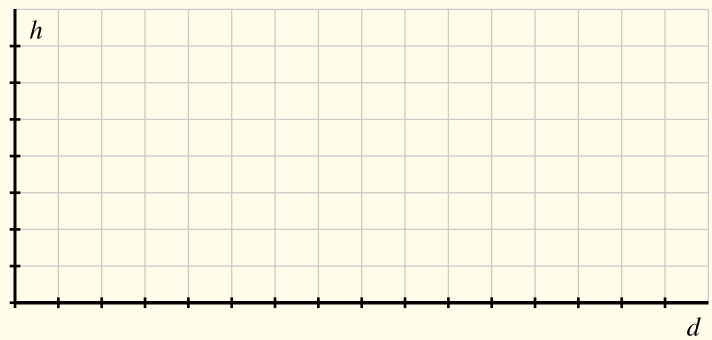
d. By plotting the points in the table and connecting them in an intuitive way, sketch a graph of h as a function of d on the axes provided over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.

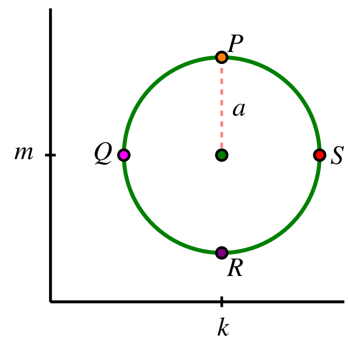


e. What is similar about your graph in comparison to the one in we created at the beginning of this section? What is different?

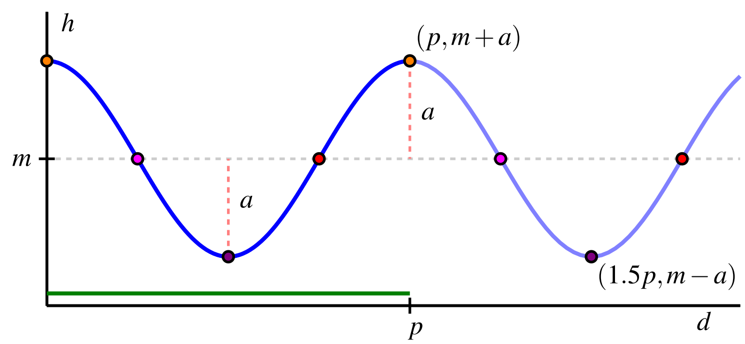
f. What will be the value of h when $d = 51$? How about when $d = 102$?

Properties of Circular Functions

Every circular function has several important features that are connected to the circle that defines the function. For the discussion that follows, we focus on circular functions that result from tracking the y-coordinate of a point traversing counterclockwise a circle of radius a centered at the point (k, m). Further, we will denote the circumference of the circle by the letter p.



We assume that the point traversing the circle starts at P in teh graph of the circle above. Its height is initially $y = m + a$, and then its height decreases to $y = m$ as we traverse to Q. Continuing, the point's height falls to $y = m - a$ at R, and then rises back to $y = m$ at S, and eventually back up to $y = m + a$ at the top of the circle. If we plot these heights continuously as a function of distance, d, traversed around the circle, we get the curve shown below.



This curve has several important features for which we introduce important terminology.

> **Definition** The **midline** of a circular function is the horizontal line $y = m$ for which half the curve lies above the line and half the curve lies below. If the circular function results from tracking the y-coordinate of a point traversing a circle, $y = m$ corresponds to the y-coordinate of the center of the circle. In addition, the **amplitude** of a circular function is the maximum deviation of the curve from the midline. Note particularly that the value of the amplitude, a, corresponds to the radius of the circle that generates the curve.

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Because we can traverse the circle in either direction and for as far as we wish, the domain of any circular function is the set of all real numbers. From our observations about the midline and amplitude, it follows that the range of a circular function with midline $y = m$ and amplitude a is the interval $[m - a, m + a]$.

This graph is an example of a periodic function. Recall the definition of a periodic function.

Definition Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x+k) = f(x)$ for every possible choice of x. The smallest value p for which $f(x+p) = f(x)$ for every choice of x is called the **period** of f.

For a circular function, the period is always the circumference of the circle that generates the curve. In the graph of the function above, we see how the curve has completed one full cycle of behavior every p units, regardless of where we start on the curve.

Circular functions arise as models for important phenomena in the world around us, such as in a *harmonic oscillator*. Consider a mass attached to a spring where the mass sits on a frictionless surface. After setting the mass in motion by stretching or compressing the spring, the mass will oscillate indefinitely back and forth, and its distance from a fixed point on the surface turns out to be given by a circular function.

The Average Rate of Change of a Circular Function

Just as there are important trends in the values of a circular function, there are also interesting patterns in the average rate of change of the function. These patterns are closely tied to the geometry of the circle.

For the next part of our discussion, we consider a circle of radius 1 centered at $(0, 0)$, and consider a point that travels a distance d counterclockwise around the circle with its starting point viewed as $(1, 0)$. We use this circle to generate the circular function $h = f(d)$ that tracks the height of the point at the moment the point has traversed d units around the circle from $(1, 0)$. Let's consider the average rate of change of f on several intervals that are connected to certain fractions of the circumference.

Remembering that h is a function of distance traversed along the circle, it follows that the average rate of change of h on any interval of distance between two points P and Q on the circle is given by

$$AV_{[P,Q]} = \frac{\text{change in height}}{\text{distance along the circle}},$$

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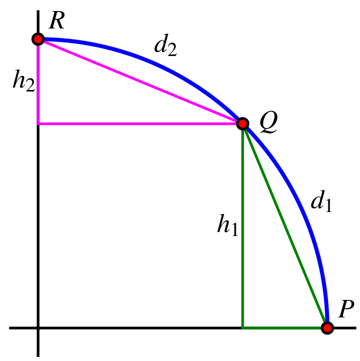
where both quantities are measured from point P to point Q.

First, we consider points P, Q, and R where Q results from traversing $1/8$ of the circumference from P, and R $1/8$ of the circumference from Q. In particular, we note that the distance d_1 along the circle from P to Q is the same as the distance d_2 along the circle from Q to R, and thus $d_1 = d_2$. At the same time, it is apparent from the geometry of the circle that the change in height h_1 from P to Q is greater than the change in height h_2 from Q to R, so $h_1 > h_2$. Thus, we can say that

$$AV_{[P,Q]} = \frac{h_1}{d_1} > \frac{h_2}{d_2} = AV_{[Q,R]}.$$



The differences in certain average rates of change appear to become more extreme if we consider shorter arcs along the circle. Next we consider traveling $1/20$ of the circumference along the circle. In the graph below, points P and Q lie $1/20$ of the circumference apart, as do R and S, so here $d_1 = d_5$. In this situation, it is the case that $h_1 > h_5$ for the same reasons as above, but we can say even more. From the green triangle, we see that $h_1 \approx d_1$ (while $h_1 < d_1$), so that $AV_{[P,Q]} = \frac{h_1}{d_1} \approx 1$. At the same time, in the magenta triangle in the figure we see that h_5 is very small, especially in comparison to d_5, and thus $AV_{[R,S]} = \frac{h_5}{d_5} \approx 0$. Hence, in this graph,

$$AV_{[P,Q]} \approx 1 \text{ and } AV_{[R,S]} \approx 0.$$

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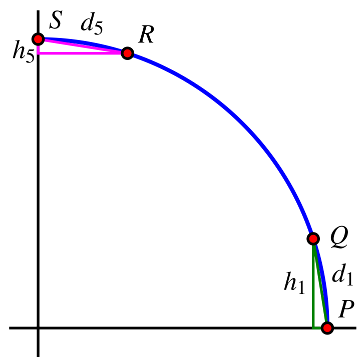
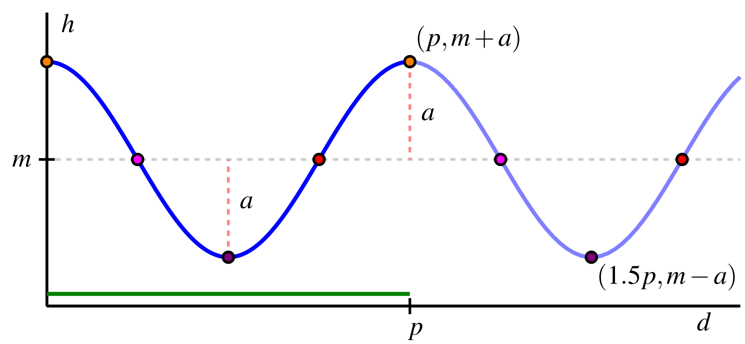


This information tells us that a circular function appears to change most rapidly for points near its midline and to change least rapidly for points near its highest and lowest values.

We can study the average rate of change not only on the circle itself, but also on a generic circular function graph, and thus make conclusions about where the function is increasing, decreasing, concave up, and concave down.



Summary

- When a point traverses a circle, a corresponding function can be generated by tracking the height of the point as it moves around the circle, where height is viewed as a function of distance traveled around the circle. We call such a function a *circular function*.
- Circular functions have several standard features. The function has a *midline* that is the line for which half the points on the curve lie above the line and half the points on the curve lie below. A circular function's *amplitude* is the maximum deviation of the

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function value from the midline; the amplitude corresponds to the radius of the circle that generates the function. Circular functions also repeat themselves, and we call the smallest value of p for which $f(x+p) = f(x)$ for all x the period of the function. The period of a circular function corresponds to the circumference of the circle that generates the function.

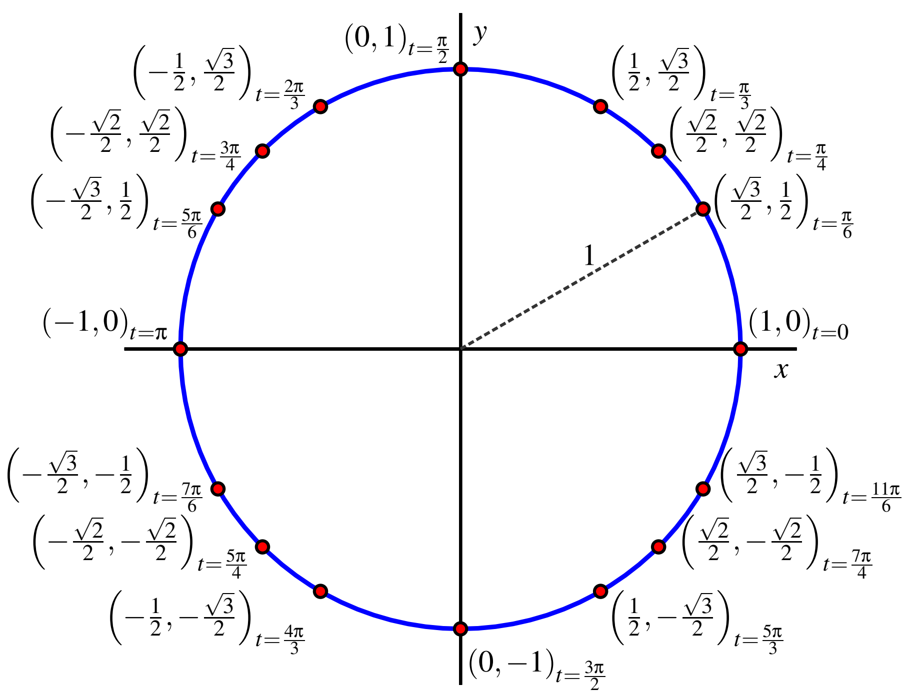
- Non-constant linear functions are either always increasing or always decreasing; quadratic functions are either always concave up or always concave down. Circular functions are sometimes increasing and sometimes decreasing, plus sometimes concave up and sometimes concave down. These behaviors are closely tied to the geometry of the circle.

3.1.2 The Sine and Cosine Functions

Motivating Questions

- What are the sine and cosine functions and how do they arise from a point traversing the unit circle?
- What important properties do the sine and cosine functions share?

In the last section, we saw how tracking the height of a point that is traversing a circle generates a periodic function. Previously, we also identified a collection of special points on the unit circle.



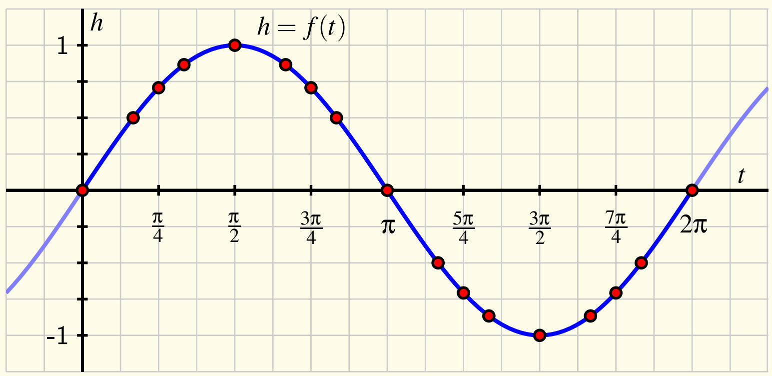
You can also use the *Desmos* file a

Desmos link: `https://www.desmos.com/calculator/jgddn7tzxg`

Acknowledgements: Active Prelude to Calculus
(https://activecalculus.org/prelude)

Exploration

If we consider the unit circle, start at $t = 0$, and traverse the circle counterclockwise, we may view the height, h, of the traversing point as a function of the angle, t, in radians. From there, we can plot the resulting (t, h) ordered pairs and connect them to generate the circular function pictured below.



a. What is the exact value of $h\left(\frac{\pi}{4}\right)$? of $h\left(\frac{\pi}{3}\right)$?

b. Complete the following table with the exact values of h that correspond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

c. What is the exact value of $h\left(\frac{11\pi}{4}\right)$? of $h\left(\frac{14\pi}{3}\right)$?

d. Give four different values of t for which $h(t) = -\frac{\sqrt{3}}{2}$.

The Definition of the Sine Function

The circular function that tracks the height of a point on the unit circle traversing counterclockwise from $(1, 0)$ as a function of the corresponding central angle

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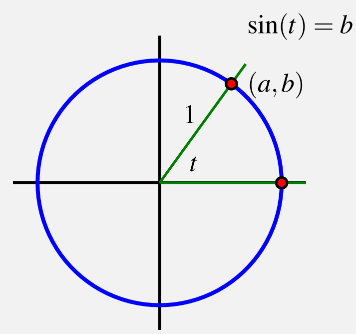
(in radians) is one of the most important functions in mathematics. As such, we give the function a name: the **sine** function.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1,0)$ and (a,b), we define the **sine of** t, denoted $\sin(t)$, by the rule

$$\sin(t) = b.$$

Because of the correspondence between radian angle measure and distance traversed on the unit circle, we can also think of $\sin(t)$ as identifying the y-coordinate of the point after it has traveled t units counterclockwise along the circle from $(1,0)$. Note particularly that we can consider the sine of negative inputs: for instance, $\sin(-\frac{\pi}{2}) = -1$.

Based on our earlier work with the unit circle, we know many different exact values of the sine function, and summarize these in in the table below:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Moreover, if we now plot these points in the usual way, we get the familiar circular wave function that comes from tracking the height of a point traversing a circle. We often call this graph the **sine wave**.

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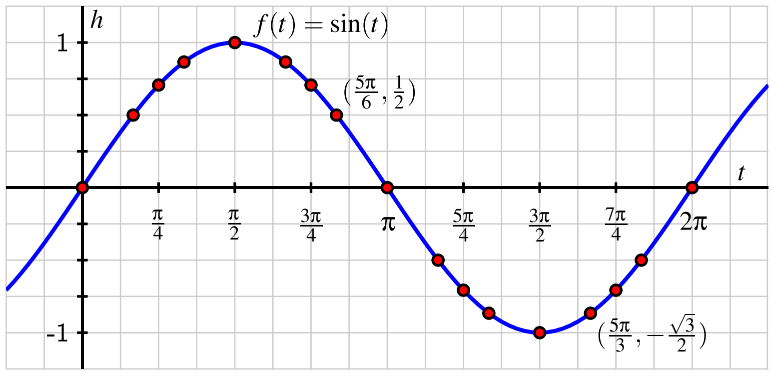


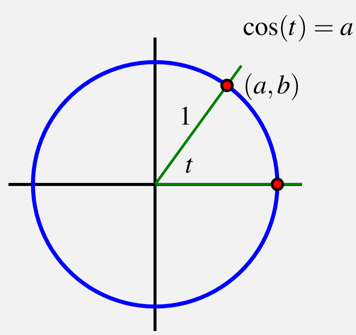
At [5] https://www.desmos.com/calculator/f9foqx24ct you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the sine graph.

Desmos link: `https://www.desmos.com/calculator/f9foqx24ct`

The Definition of the Cosine Function

Given any central angle of radian measure t in the unit circle with one side passing through the point $(1, 0)$, the angle generates a unique point (a, b) that lies on the circle. Just as we can view the y-coordinate as a function of t, the x-coordinate is likewise a function of t. We therefore make the following definition.

Definition



Given a central angle in the unit circle that measures t radians and that intersects the circle at both $(1, 0)$ and (a, b), we define the **cosine of** t,

[5]Link: `https://www.desmos.com/calculator/f9foqx24ct`

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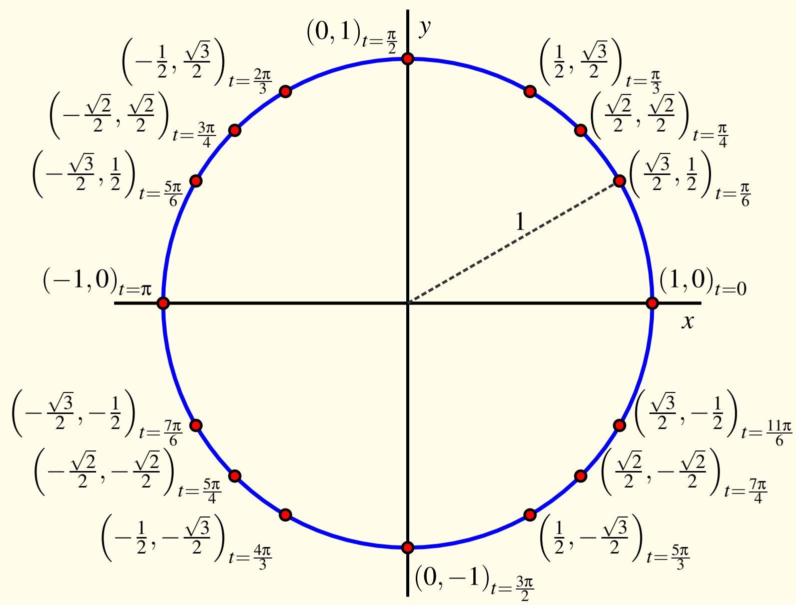
(https://activecalculus.org/prelude)

denoted $\cos(t)$, by the rule

$$\cos(t) = a.$$

Again because of the correspondence between the radian measure of an angle and arc length along the unit circle, we can view the value of $\cos(t)$ as tracking the x-coordinate of a point traversing the unit circle clockwise a distance of t units along the circle from $(1, 0)$. We now use the data and information we have developed about the unit circle to build a table of values of $\cos(t)$ as well as a graph of the curve it generates.

Exploration Let $k = g(t)$ be the function that tracks the x-coordinate of a point traversing the unit circle counterclockwise from $(1, 0)$. That is, $g(t) = \cos(t)$. Use the information we know about the unit circle to respond to the following questions.



a. What is the exact value of $\cos(\dfrac{\pi}{6})$? of $\cos(\dfrac{5\pi}{6})$? $\cos(-\dfrac{\pi}{3})$?

b. Complete the following table with the exact values of k that cor-

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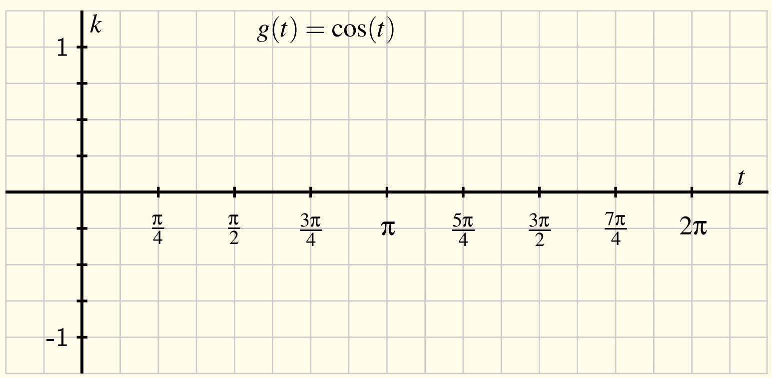
respond to the stated inputs.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
h									

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
h									

c. On the axes provided, sketch an accurate graph of $k = \cos(t)$. Label the exact location of several key points on the curve.



d. What is the exact value of $\cos(\frac{11\pi}{4})$? of $\cos(\frac{14\pi}{3})$?

e. Give four different values of t for which $\cos(t) = -\frac{\sqrt{3}}{2}$.

f. How is the graph of $k = \cos(t)$ different from the graph of $h = \sin(t)$? How are the graphs similar?

As we work with the sine and cosine functions, it's always helpful to remember their definitions in terms of the unit circle and the motion of a point traversing the circle. At [6] https://www.desmos.com/calculator/9s1ms0nlyf you can explore and investigate a helpful Desmos animation that shows how this motion around the circle generates the cosine graph.

Desmos link: https://www.desmos.com/calculator/9s1ms0nlyf

[6] Link: https://www.desmos.com/calculator/9s1ms0nlyf

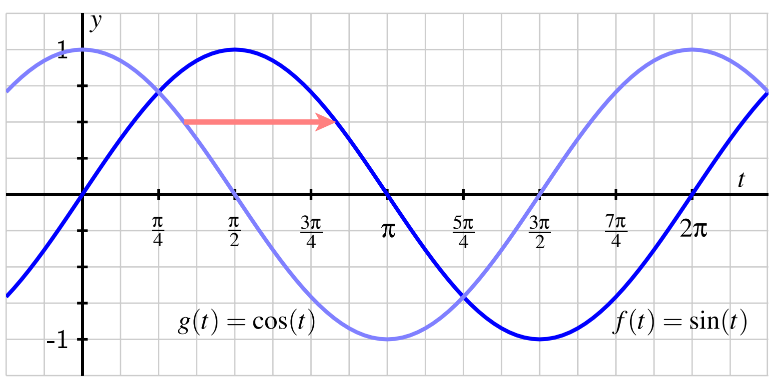
Properties of Sine and Cosine

Because the sine function results from tracking the y-coordinate of a point traversing the unit circle and the cosine function from the x-coordinate, the two functions have several shared properties of circular functions.

For both $f(t) = \sin(t)$ and $g(t) = \cos(t)$,

- the domain of the function is all real numbers;
- the range of the function is $[-1, 1]$;
- the midline of the function is $y = 0$;
- the amplitude of the function is $a = 1$;
- the period of the function is $p = 2\pi$.

It is also insightful to juxtapose the sine and cosine functions' graphs on the same coordinate axes. When we do, as seen in Figure **??**, we see that the curves can be viewed as horizontal translations of one another.



In particular, since the sine graph can be viewed as the cosine graph shifted $\frac{\pi}{2}$ units to the right, it follows that for any value of t,

$$\sin(t) = \cos(t - \frac{\pi}{2}).$$

Similarly, since the cosine graph can be viewed as the sine graph shifted left,

$$\cos(t) = \sin(t + \frac{\pi}{2}).$$

Because each of the two preceding equations hold for every value of t, they are often referred to as *identities*.

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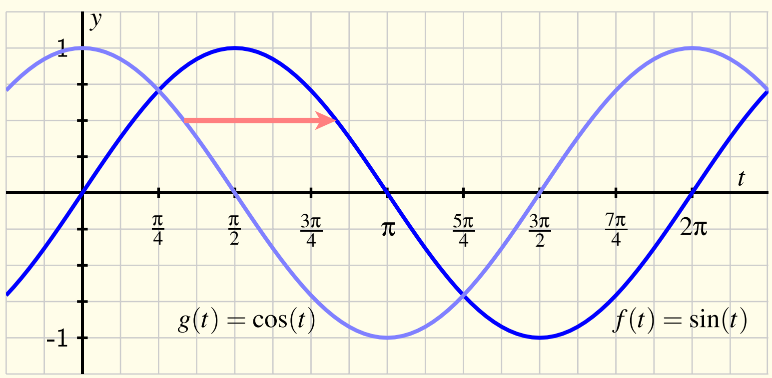
In light of the definitions of the sine and cosine functions, we can now view any point (x, y) on the unit circle as being of the form $(\cos(t), \sin(t))$, where t is the measure of the angle whose vertices are $(1, 0)$, $(0, 0)$, and (x, y). Note particularly that since $x^2 + y^2 = 1$, it is also true that $\cos^2(t) + \sin^2(t) = 1$. We call this fact the Fundamental Trigonometric Identity.

For any real number t,

$$\cos^2(t) + \sin^2(t) = 1.$$

There are additional trends and patterns in the two functions' graphs that we explore further in the following activity.

Exploration Use the figure below to assist in answering the following questions.



a. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing.

b. Give an example of the largest interval you can find on which $f(t) = \sin(t)$ is decreasing and concave down.

c. Give an example of the largest interval you can find on which $g(t) = \cos(t)$ is increasing.

d. Give an example of the largest interval you can find on which $g(t) = \sin(t)$ is increasing and concave up.

e. Without doing any computation, on which interval is the average rate of change of $g(t) = \cos(t)$ greater: $[\pi, \pi + 0.1]$ or $\left[\frac{3\pi}{2}, \frac{3\pi}{2} + 0.1\right]$? Why?

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

f. In general, how would you characterize the locations on the sine and cosine graphs where the functions are increasing or decreasingly most rapidly?

g. For which quadrants of the x-y plane is $\cos(t)$ negative for an angle in that quadrant?

Using Computing Technology

We have established that we know the exact value of $\sin(t)$ and $\cos(t)$ for any of the t-values labeled on the unit circle, as well as for any such $t \pm 2j\pi$, where j is a whole number, due to the periodicity of the functions. But what if we want to know $\sin(1.35)$ or $\cos(\frac{\pi}{5})$ or values for other inputs not in the table?

Any standard computing device a scientific calculator, *Desmos*, *Geogebra*, *WolframAlpha*, etc. has the ability to evaluate the sine and cosine functions at any input we desire. Because the input is viewed as an angle, each computing device has the option to consider the angle in radians or degrees. *It is always essential that you are sure which type of input your device is expecting.* Our computational device of choice is *Desmos*. In *Desmos*, you can change the input type between radians and degrees by clicking the wrench icon in the upper right and choosing the desired units. Radians is the default, and radians is what we will primarily use in both this class and calculus.

It take substantial and sophisticated mathematics to enable a computational device to evaluate the sine and cosine functions at any value we want; the algorithms involve an idea from calculus known as an infinite series. While your computational device is powerful, it's both helpful and important to understand the meaning of these values on the unit circle and to remember the special points for which we know the outputs of the sine and cosine functions exactly.

Exploration
Answer the following questions exactly wherever possible. If you estimate a value, do so to at least 5 decimal places of accuracy.

a. The x coordinate of the point on the unit circle that lies in the third quadrant and whose y-coordinate is $y = -\frac{3}{4}$.

b. The y-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = 2$ radians.

c. The x-coordinate of the point on the unit circle generated by a central angle in standard position that measures $t = -3.05$ radians.

d. The value of $\cos(t)$ where t is an angle in Quadrant II that satisfies

Acknowledgements: Active Prelude to Calculus (https://activecalculus.org/prelude)

$\sin(t) = \frac{1}{2}$.

e. The value of $\sin(t)$ where t is an angle in Quadrant III for which $\cos(t) = -0.7$.

f. The average rate of change of $f(t) = \sin(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

g. The average rate of change of $g(t) = \cos(t)$ on the intervals $[0.1, 0.2]$ and $[0.8, 0.9]$.

Summary

- The sine and cosine functions result from tracking the y- and x-coordinates of a point traversing the unit circle counterclockwise from $(1, 0)$. The value of $\sin(t)$ is the y-coordinate of a point that has traversed t units along the circle from $(1, 0)$ (or equivalently that corresponds to an angle of t radians), while the value of $\cos(t)$ is the x-coordinate of the same point.

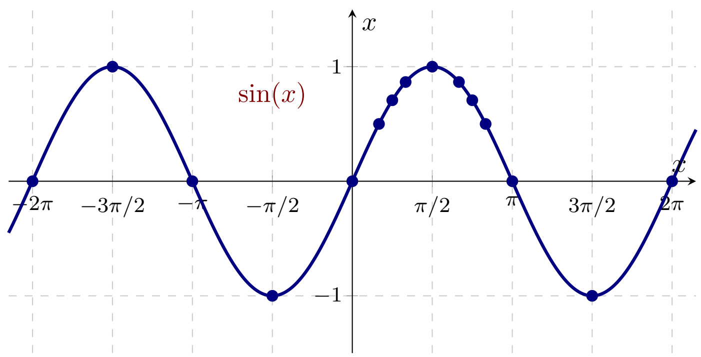
- The sine and cosine functions are both periodic functions that share the same domain (the set of all real numbers), range (the interval $[-1, 1]$), midline ($y = 0$), amplitude ($a = 1$), and period ($P = 2\pi$). In addition, the sine function is horizontal shift of the cosine function by $\frac{\pi}{2}$ units to the right, so $\sin(t) = \cos(t - \frac{\pi}{2})$ for any value of t.

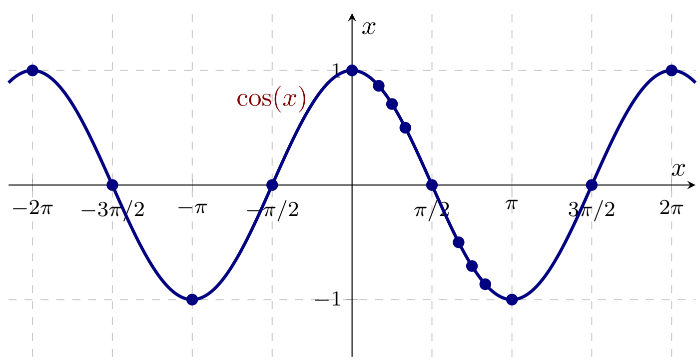
- If t corresponds to one of the special angles that we know on the unit circle, we can compute the values of $\sin(t)$ and $\cos(t)$ exactly. For other values of t, we can use a computational device to estimate the value of either function at a given input; when we do so, we must take care to know whether we are computing in terms of radians or degrees.

3.1.3 Creating a New Function: Tangent

Introduction

We are now going to determine the graph of the tangent function by analyzing what we now know about the sine and cosine functions. As a reminder, here is a graph of those functions with some important points marked. Specifically the points at all multiples of $\frac{\pi}{2}$ have been marked, as well as at the standard points $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$.



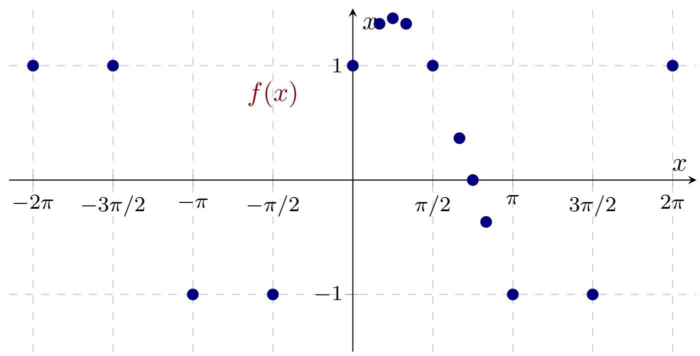


Graph of $f(x) = \sin(x) + \cos(x)$

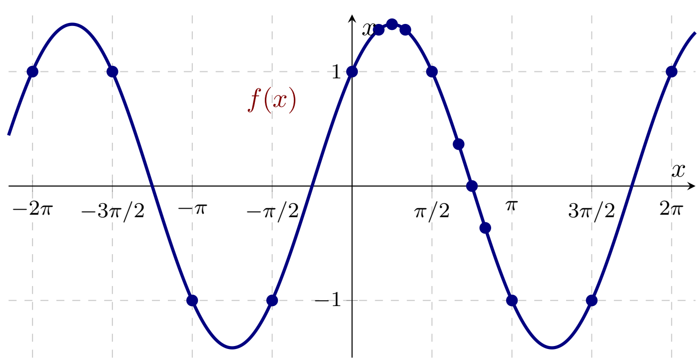
Before considering the function $\tan(x) = \frac{\sin(x)}{\cos(x)}$, lets consider the (possibly) more straightforward $f(x) = \sin(x) + \cos(x)$. Let's practice with this function creating a graph of a sum of two known functions.

What is the value of $f(0)$? We know $f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$.

We can easily calculate the values of f at all of the important points marked in the graphs above. Let us plot the points of f corresponding to them.



Those extra points plotted between $x = 0$ and $x = \pi$ show us the behavior of this function f. Notice that between $x = 0$ and $x = \frac{\pi}{2}$, the graph increases to a peak, then decreases in a very sinusoidal manner. If we continue this with standard values and "connect the dots", we end up with the following graph.



We've ended up with another periodic function that looks like a stretched and shifted version of sine or cosine.

Now, let's try again using division instead of addition.

Determining the Graph of Tangent

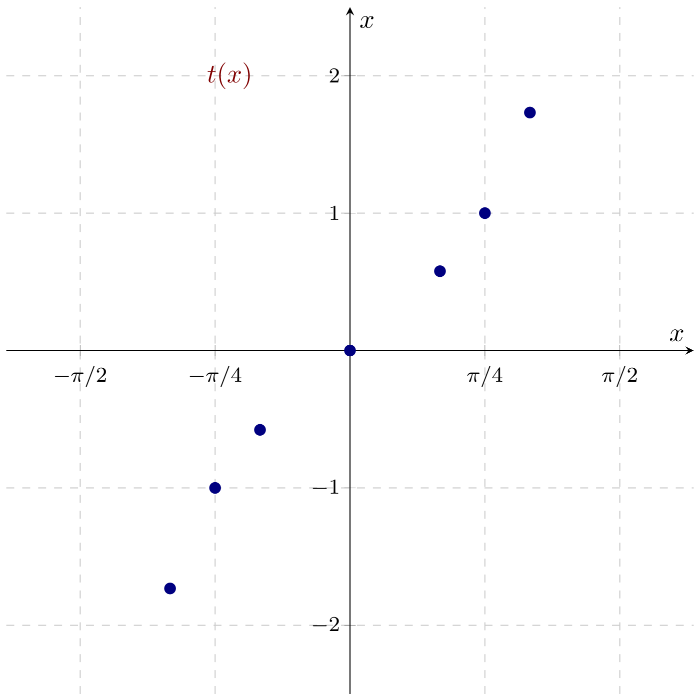
Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Notice that this function is undefined at all x-values with $\cos(x) = 0$. That means the function $\tan(x)$ is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

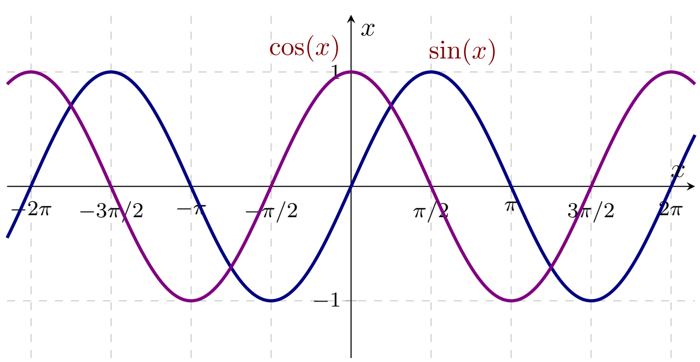
We can calculate values of $\tan(x)$ for other inputs. $\tan(0) = \dfrac{\sin(0)}{\cos(0)} = \dfrac{0}{1} = 0$. The following table lists some of the other values arising from this division. Notice that in this table we have chosen to rationalize the denominators of the fractions that have appeared. That is, we have written $\dfrac{1}{\sqrt{2}}$ as $\dfrac{\sqrt{2}}{2}$, by multiplying the fraction by 1 written in the form $\dfrac{\sqrt{2}}{\sqrt{2}}$. Similarly $\dfrac{1}{\sqrt{3}}$ is written as $\dfrac{\sqrt{3}}{3}$.

x	$\sin(x)$	$\cos(x)$	$t(x) = \dfrac{\sin(x)}{\cos(x)}$
$-\dfrac{\pi}{3}$	$-\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$-\sqrt{3}$
$-\dfrac{\pi}{4}$	$-\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	-1
$-\dfrac{\pi}{6}$	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{3}}{3}$
0	0	1	0
$\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{3}}{3}$
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1
$\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$

If we plot these points, we find the following graph.



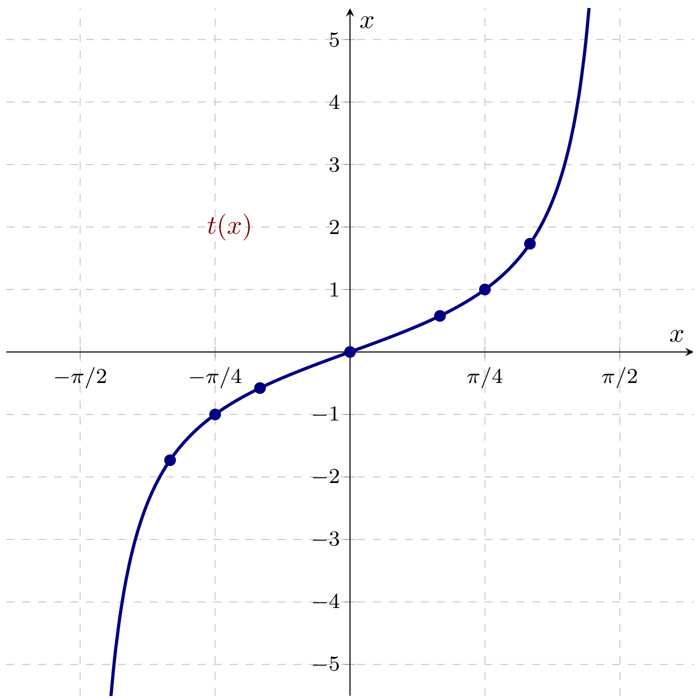
Let's think about what happens if x is a number really close to $\frac{\pi}{2}$, but just a little bit smaller than $\frac{\pi}{2}$. Notice from the graphs that the value of $\sin(x)$ will be a positive number that is really close to 1 and the value of $\cos(x)$ will be really close to 0 but still positive.



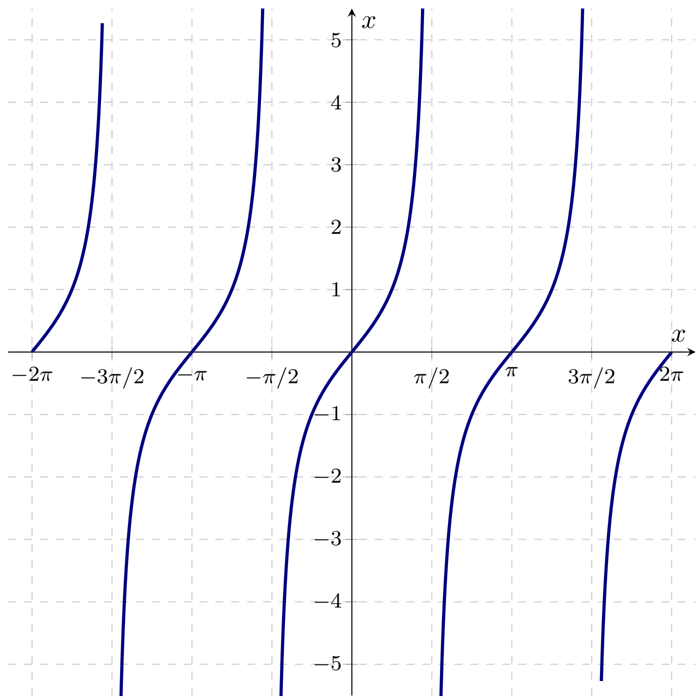
What happens if we take 1 and divide it by a small positive number? Let's look at a table of values to see.

z	$\frac{1}{z}$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{10}$	10
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Notice that as the numbers $z = 1,\ 1/2,\ 1/3,\ \ldots$ got smaller and smaller, the values of $\frac{1}{z} = 1,\ 2,\ 3,\ \ldots$ got larger and larger? That is the same thing we are noticing in the graph of the function t we are building above. For values of x really close to $\pi/2$, but still less than $\pi/2$, the value of $t(x)$ is basically 1 divided by a very small positive number. This table of values tells us that the smaller that denominator gets, the larger the fraction becomes. Adding this behavior to the graph of t gives the following.



By repeating similar calculations for other standard inputs, we arrive at the following graph.



As you can see from the graph, $\tan(x)$ is an odd, periodic function with period π (not 2π like sine and cosine).

3.1.4 Graphs of Secant, Cosecant, and Cotangent

Coming Soon

3.2 Trigonometric Functions

Learning Objectives

- –
  –
  –

3.2.1 Trig Functions as Functions

Trig Functions as Functions

3.2.2 Trig Functions as Functions

Trig Functions as Functions

3.2.3 Trig Functions as Functions

Trig Functions as Functions

3.3 Some Applications of Trigonometry

Learning Objectives

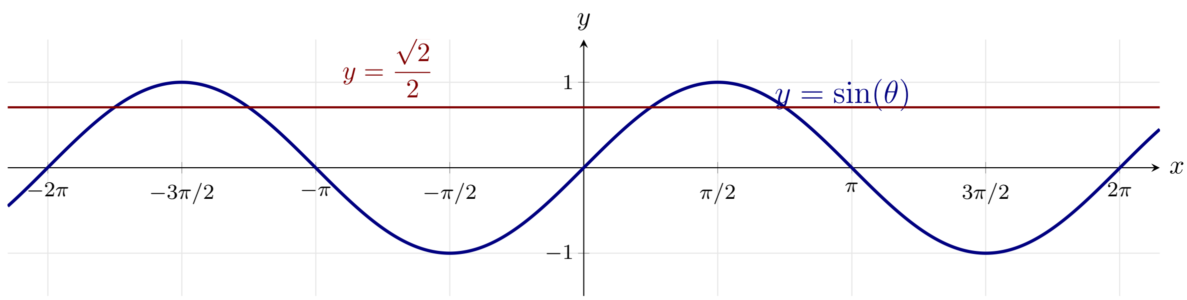
- Solving Trigonometric Equations
  - Solving elementary trigonometric equations.
  - Solving trigonometric equations using identities.
  - FInding solutions on restricted domains.
- Applications of Trigonometric Functions
  - Applications involving trigonometric functions

3.3.1 Solving Trigonometric Equations

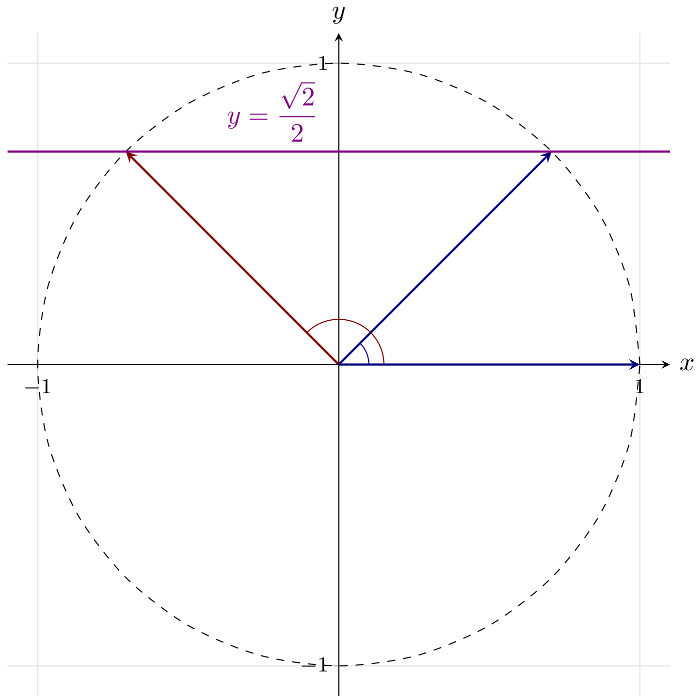
Introduction

Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. Something like $\sin(\theta) = \frac{\sqrt{2}}{2}$. We know that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, meaning that $\theta = \frac{\pi}{4}$ is a solution of this equation, but is that the only solution or *are there more*?

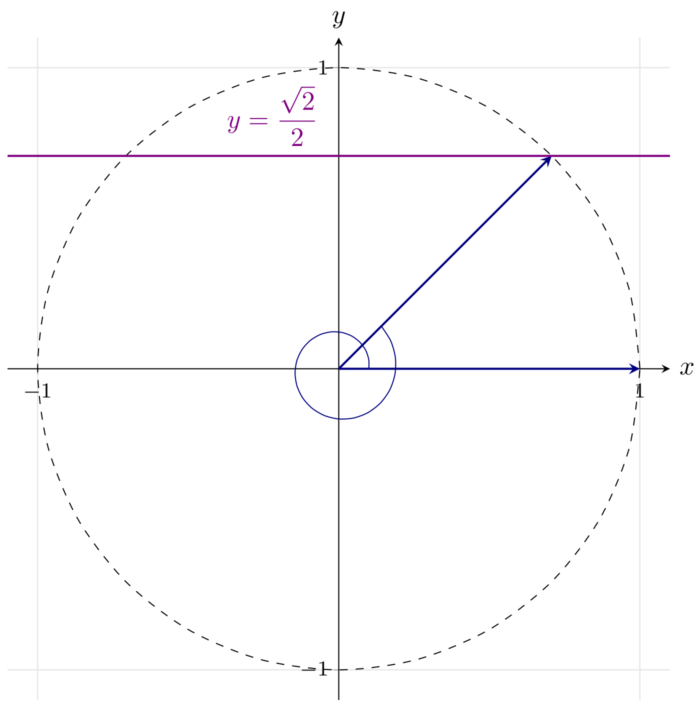
Let's look at the graph of the sine function.



Notice that the graph of $\sin(\theta)$ and the graph of the constant function $y = \frac{\sqrt{2}}{2}$ intersect many times, not just once. In fact, since sine is a periodic function, these graphs intersect infinitely many times. Each of these intersections represents a single solution of the equation $\sin(\theta) = \frac{\sqrt{2}}{2}$. We need a process to identify and write down each of these solutions. Let's start by looking at the unit circle. Remember that sine values correspond to the y-coordinate of points on the unit circle. This equation is asking us to find all the points on the unit circle with a y-coordinate of $\frac{\sqrt{2}}{2}$.



You see that there are two locations on the unit circle with y-coordinate equal to $\frac{\sqrt{2}}{2}$, one in the first quadrant and another in the second. As we mentioned earlier, the first quadrant angle is $\theta = \frac{\pi}{4}$. The angle in the second quadrant has reference angle $\frac{\pi}{4}$, which means the angle is $\frac{3\pi}{4}$. Those are the only two points on the circle with that y-coordinate, but remember that there are many other angles which are coterminal with those. For instance:



The only solutions are the angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and *all the angles coterminal with them*. Since the sine function has period 2π, that means any other solution has to be an integer multiple of 2π away from one of these first two solutions. Putting that together, our solutions are:

$$\theta = \frac{\pi}{4} + 2\pi k,\ \frac{3\pi}{4} + 2\pi k,\ k \text{ any integer.}$$

The steps we've followed are summarized in the following.

To solve a triginometric equation:

(a) Find the reference angle of the solutions. Typically the standard values will help identify this.

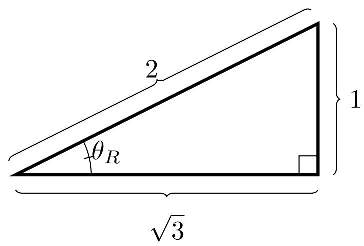
(b) Find all solutions on a single period of the function. Use the graph, the unit circle, and the reference angle to identify these.

(c) Find all solutions. Use the period of the function to find all requested solutions.

Example 29. *Solve the equation:*

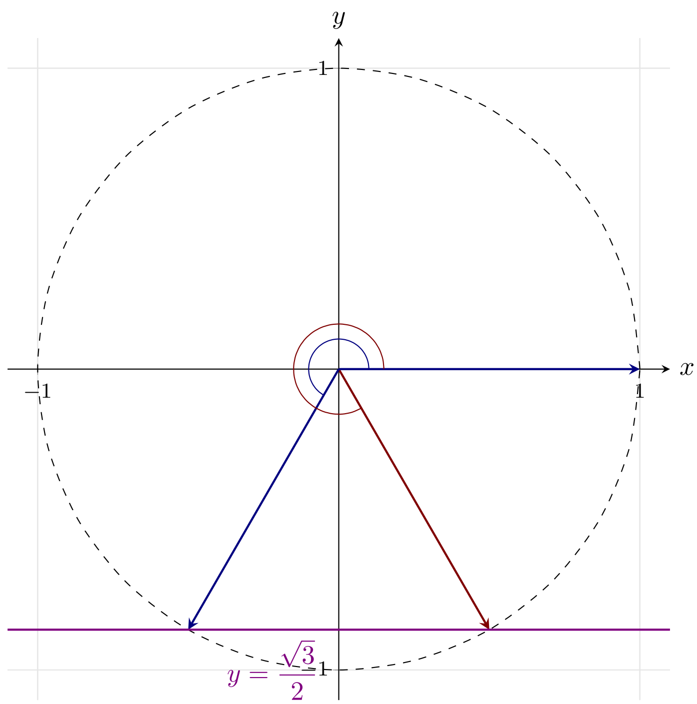
$$\sin(\theta) = -\frac{1}{2}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\sin(\theta_R) = \frac{1}{2}$ and the negative sign will be used to indicate the quadrant of the angle.



[2in]

From the picture we see $\theta_R = \frac{\pi}{6}$. Let's look at the unit circle.



In one period $[0, 2\pi)$, there are two angles that have reference angle $\frac{\pi}{6}$ and have negative sine value. One is in quadrant 3, and one in quadrant 4. That means the solutions in the interval $[0, 2\pi)$ are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

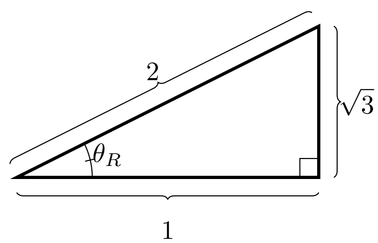
To find all solutions, we have to add all multiples of 2π to these. The solutions are then

$$\theta = \frac{5\pi}{6} + 2\pi k,\ \frac{11\pi}{6} + 2\pi k,\ k \text{ any integer.}$$

Example 30. *Solve the equation:*

$$\tan(\theta) = -\sqrt{3}.$$

Explanation We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\tan(\theta_R) = \sqrt{3}$ and the negative sign will be used to indicate the quadrant of the angle. Since tangent is opposite over adjacent, we have the following triangle.



[2in]

From the picture we see $\theta_R = \dfrac{\pi}{3}$. Let's look at the unit circle.

The tangent function has one period $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. In the interval $\left(-\dfrac{\pi}{2}, 0\right)$ (which is Quadrant IV), tangent is negative while in $\left(0, \dfrac{\pi}{2}\right)$ (which is Quadrant I), tangent is positive. For $\tan(\theta)$ to be negative in this interval, we need θ to be in $\left(-\dfrac{\pi}{2}, 0\right)$. The only angle in that interval with reference angle $\dfrac{\pi}{3}$ is $\theta = -\dfrac{\pi}{3}$. This is the only solution on this period.

> Remember that the tangent function has period π, unlike sine and cosine which have period 2π. On the period $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, tangent is one-to-one, so there is exactly one angle which gives the desired output value. Sine and cosine are not one-to-one across a full period.

To find all solutions, we have to add all multiples of π to this. The solutions are then

$$\theta = \frac{2\pi}{3} + \pi k,\ k \text{ any integer.}$$

Let's try one a bit more complicated.

Example 31. *Solve the equation:*

$$\cos(\theta)\left(\cos(\theta) + 1\right) = \sin^2(\theta).$$

Explanation We'll start by simplifying a bit.

$$\begin{aligned}
\cos(\theta)\left(\cos(\theta)+1\right) &= \sin^2(\theta)\\
\cos^2(\theta)+\cos(\theta) &= \sin^2(\theta)\\
\cos^2(\theta)-\sin^2(\theta)+\cos(\theta) &= 0\\
\cos^2(\theta)-\left(1-\cos^2\theta\right)+\cos(\theta) &= 0\\
2\cos^2(\theta)+\cos(\theta)-1 &= 0.
\end{aligned}$$

Notice that this equation is quadratic in $\cos(\theta)$. We can factor it like we try to do to solve any other quadratic equation:

$$\left(\cos(\theta)+1\right)\left(2\cos(\theta)-1\right)=0.$$

On the interval $[0,2\pi)$, $\cos(\theta)=-1$ has only one solution, $\theta=\pi$. For $\cos(\theta)=\frac{1}{2}$, we see that the reference angle $\theta_R=\frac{\pi}{3}$. Since cosine is positive in Quadrants I and IV, we find solutions $\theta=\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

All solutions are:

$$\theta=\pi+2\pi k,\ \frac{\pi}{3}+2\pi k,\ \frac{5\pi}{3}+2\pi k,\ k \text{ any integer.}$$

3.3.2 Applications of Trigonometry

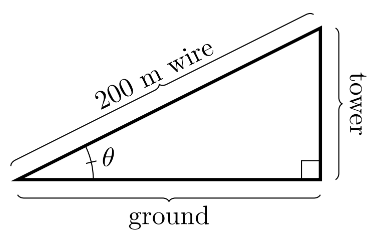
Applications of Trigonometry

In the previous sections, you have been learning about trigonometric functions in the abstract. In this section, we wish to apply them.

Example 32. *A wire 200 meters long is attached to the top of a tower. When pulled taut, it makes a* 60° *angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?*

Explanation

In application problems, we are often given data about angles measured in degrees. It is up to us to translate this into radians for our calculations. Let's call the angle we're given as θ, so $\theta = \frac{\pi}{3}$ radians. Let's draw a diagram of this situation.



[2in]

Based on this image, the height of the tower is the opposite the angle we know, and the distance along the ground is adjacent. That means the tower height is related to the angle by $\sin(\theta) = \frac{\text{tower}}{200}$ and the distance across the ground is given by $\cos(\theta) = \frac{\text{ground}}{200}$.

Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, the tower height can be computed by:

$$\begin{aligned}
\sin\left(\frac{\pi}{3}\right) &= \frac{\text{tower}}{200} \\
\text{tower} &= 200\sin\left(\frac{\pi}{3}\right) \\
&= 200\left(\frac{\sqrt{3}}{2}\right) \\
&= 100\sqrt{3}.
\end{aligned}$$

The tower has a height of $100\sqrt{3}$ m, which is approximately 173.21 m.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

The EXACT VALUE of the height is $100\sqrt{3}$ m. Saying that this is approximately 173.21 m is provided to give us an indication of scale. Our actual answer is the exact value, not this approximation.

We'll follow a similar calculation to find the distance from the base of the tower to the wire along the ground.

$$\begin{aligned}
\cos\left(\frac{\pi}{3}\right) &= \frac{\text{ground}}{200} \\
\text{ground} &= 200\cos\left(\frac{\pi}{3}\right) \\
&= 200\left(\frac{1}{2}\right) \\
&= 100.
\end{aligned}$$

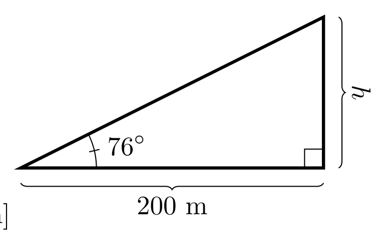
The wire hits the ground 100 m from the base of the tower.

Example 33. *A camera is setup 200 m from the base of a building, pointed at the top of the building. If the angle-of-elevation is measured as* $76°$*, find the height of the building.*

Explanation

The angle-of-elevation means the angle between the camera's line-of-sight and horizontal. Since the camera is setup 200 meters from the building, this gives us a right triangle where we know the base angle and the length of the adjacent side.

If we call the height of the building as h, then we have a triangle



[2in]

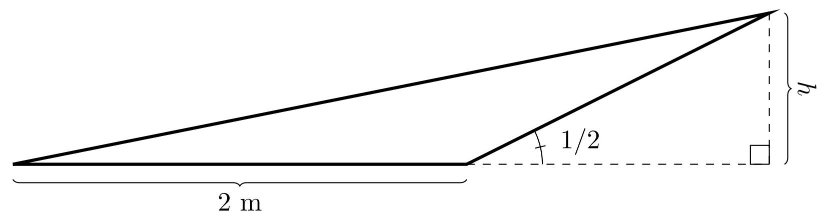
That means $\tan(76°) = \frac{h}{200}$, so $h = 200\tan(76°)$. The exact height is $200\tan(76°)$ meters. This is approximately 802.16 meters.

Example 34. *A 4 meter long piece of wire is going to be bent at its midpoint. The right side of the wire is bent up through an angle of* $\frac{1}{2}$*. The two ends of the wire are joined by a piece of string, creating an obtuse triangle. What is the area of the resulting triangle?*

Explanation

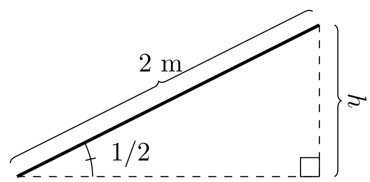
Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

That the wire is bent at its midpoint, means the resulting triangle will have two sides of length 2 m. Call the height of the triangle h.



[4in]

Let us focus on the dotted triangle on the right side



[2in]

Notice that the height of this right triangle is the same as the height of the obtuse triangle above. Since we know the hypotenuse and base angle of this right triangle, we can find the height (the opposite side) using sine.

$$\begin{aligned} \sin\left(\frac{1}{2}\right) &= \frac{h}{2} \\ 2\sin\left(\frac{1}{2}\right) &= h. \end{aligned}$$

Be careful here! That angle $\frac{1}{2}$ is not in degrees, it's in radians. (You can tell, because there is no "degrees symbol".) If you are going to approximate this value, make sure you are using radians.

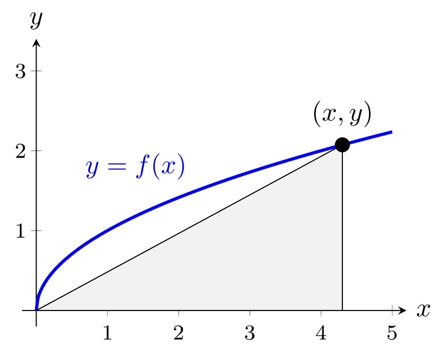
Now the area of the whole triangle is:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2)\left(2\sin\left(\frac{1}{2}\right)\right) \\ &= 2\sin\left(\frac{1}{2}\right). \end{aligned}$$

The exact value of the area is $2\sin\left(\frac{1}{2}\right)\ m^2$. Using a calculator, this is approximately $0.959\,m^2$. (Remember that m^2 is the abbreviation for "square meters".)

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Example 35. *A right triangle is constructed by taking a point (x, y) on the graph of the function $f(x) = \sqrt{x}$, drawing a line vertically downward to the x-axis, then connecting both of those points to the origin as in the picture below.*



For one particular point (x, y), the acute angle between the hypotenuse of the triangle and the positive x-axis is found to measure $\frac{\pi}{6}$ radians. Find the coordinates of the point (x, y).

Explanation

Since the hypotenuse of the right triangle runs from the origin, which has coordinates $(0, 0)$, to the point (x, y), the horizontal side of the triangle has length x and vertical side has length y. We know that the value of tangent is given by the ratio of the opposite side length divided by the adjacent side length. Using the given angle of $\frac{\pi}{6}$, this means $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{y}{x}$. That is, $x = y\sqrt{3}$

We also know that the point lies on the graph of $f(x) = \sqrt{x}$, which means $y = \sqrt{x}$.

This gives us a nonlinear system of two equations:

$$\begin{cases} x = y\sqrt{3} \\ y = \sqrt{x} \end{cases}$$

Squaring the bottom equation yields $y^2 = x$. When substituting the top equation into the bottom equation, we arrive at:.

$$\begin{aligned} y^2 &= y\sqrt{3} \\ y^2 - y\sqrt{3} &= 0 \\ y(y - \sqrt{3}) &= 0 \end{aligned}$$

so either $y = 0$ or $y = \sqrt{3}$.

The value $y = 0$ corresponds to the point $(0, 0)$ on the graph, which does not yield any angle. This is an extraneous solution, which is discarded.

 Acknowledgements: Stitz Zeager Open Source Mathematics (https://www.stitz-zeager.com/)

Substituting the value $y = \sqrt{3}$ into $x = y\sqrt{3}$ gives $x = (\sqrt{3})\sqrt{3} = 3$. The point is $\left(3, \sqrt{3}\right)$.

Part 4

Inverse Functions In Depth

4.1 Review of Inverse Functions

Learning Objectives

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4.1.1 Review of Inverse Functions

In Section 3-2-2, we briefly introduced the concept of *inverse functions*. Recall that for a one-to-one function f, we can define the inverse function f^{-1}. If we think of f as a process that takes some input x and produces some output $f(x)$, then providing $f(x)$ as an input to f^{-1} produces the original input x, and vice versa. Symbolically, we wrote that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

We learned several important principles, which we summarize below.

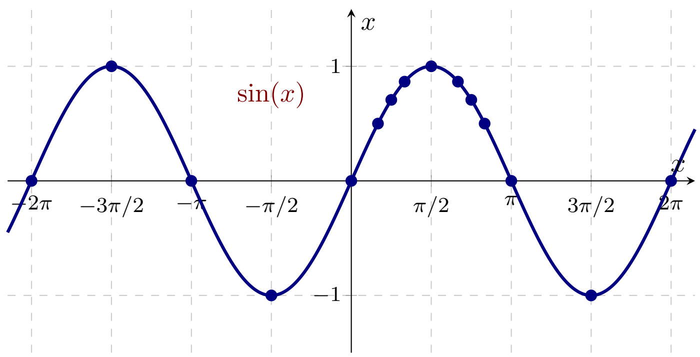
- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- A function f has an inverse function if and only if f is a *one-to-one* function.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.
- If $(a, f(a))$ is a point on the graph of f, then $(f(a), a)$ is a point on the graph of f^{-1}.
- The graph of f^{-1} is the graph of f reflected across the line $y = x$.
- The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- If f^{-1} is the inverse of f, then f is the inverse of f^{-1}.

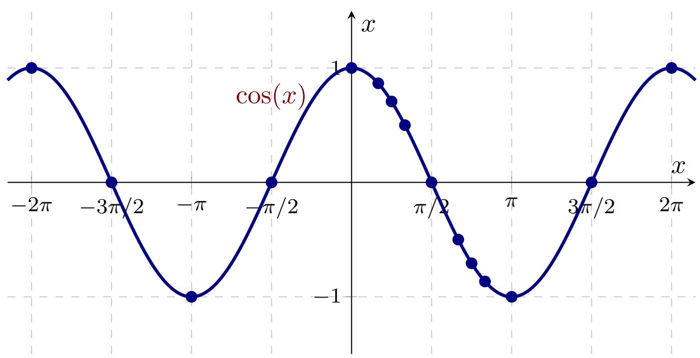
In this section, we'll explore inverse functions more in-depth.

4.1.2 Creating a New Function: Tangent

Introduction

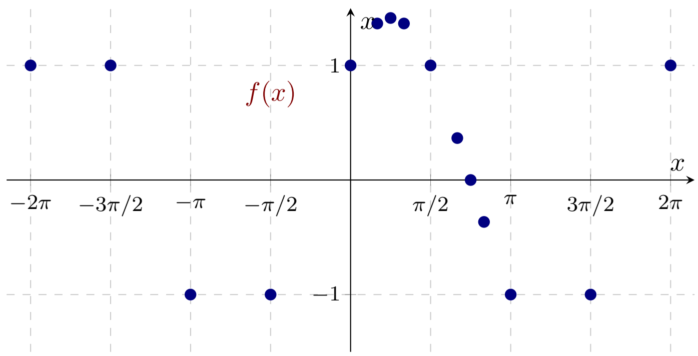
Let us return to a couple of famous functions we've briefly met with before: sine and cosine. As a reminder, here is a graph of those functions with some important points marked. Specifically the points at all multiples of $\frac{\pi}{2}$ have been marked, as well as at the standard points $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$.



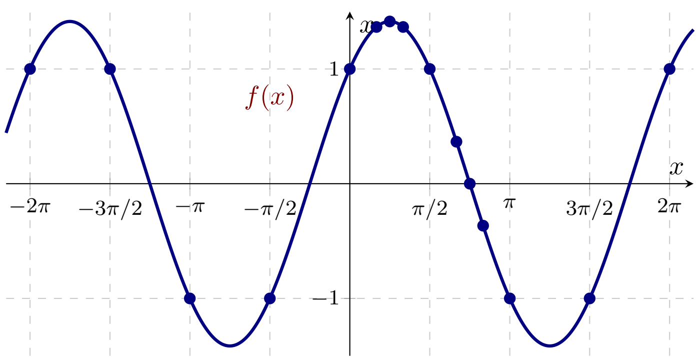


Call $f(x) = \sin(x) + \cos(x)$. What is the value of $f(0)$? We know $f(0) = \sin(0) + \cos(0) = 0 + 1 = 1$.

We can easily calculate the values of f at all of the important points marked in the graphs above. Let us plot the points of f corresponding to them.



Those extra points plotted between $x = 0$ and $x = \pi$ show us the behavior of this function f. Notice that between $x = 0$ and $x = \frac{\pi}{2}$, the graph increases to a peak, then decreases in a very sinusoidal manner. If we continue this with standard values and "connect the dots", we end up with the following graph.



We've ended up with another periodic function that looks like a stretched and shifted version of sine or cosine.

Let's try again using division instead of addition.

Creating a New Function: Tangent

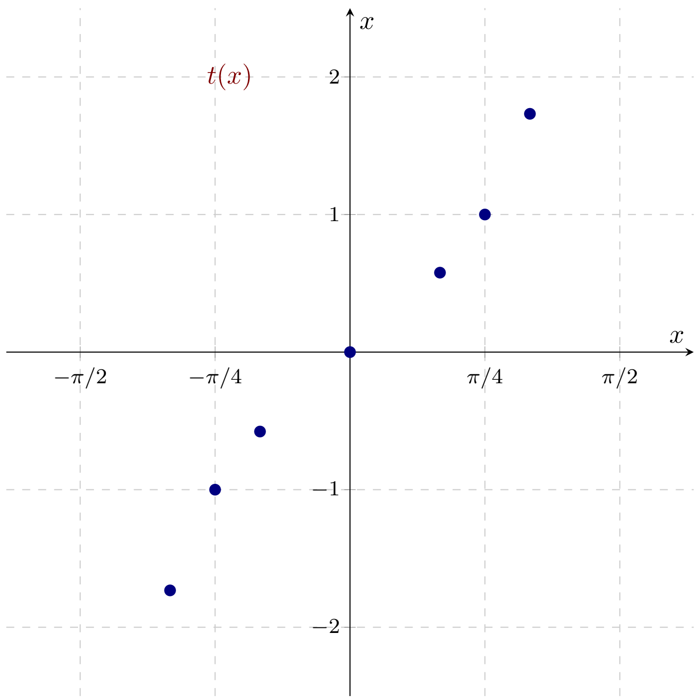
Let's set $t(x) = \dfrac{\sin(x)}{\cos(x)}$.

Notice that this function is undefined at all x-values with $\cos(x) = 0$. That means the function t is not defined for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

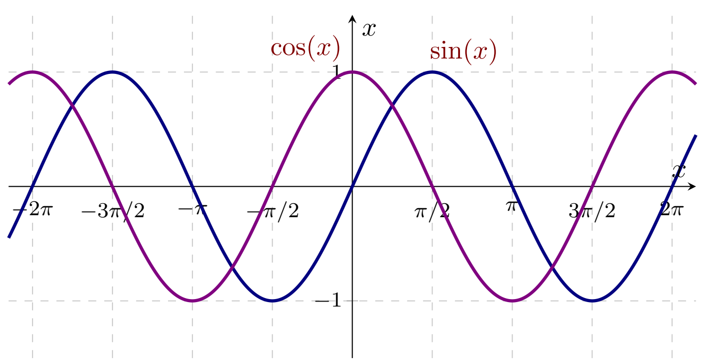
We can calculate values of t for other inputs. $t(0) = \dfrac{\sin(0)}{\cos(0)} = \dfrac{0}{1} = 0$. The following table lists some of the other values arising from this division. Notice that in this table we have chosen to rationalize the denominators of the fractions that have appeared. That is, we have written $\frac{1}{\sqrt{2}}$ as $\frac{\sqrt{2}}{2}$, by multiplying the fraction by 1 written in the form $\frac{\sqrt{2}}{\sqrt{2}}$. Similarly $\frac{1}{\sqrt{3}}$ is written as $\frac{\sqrt{3}}{3}$. If you do not recognize the values in this table, that is ok. We will be studying these functions in more detail later.

x	$\sin(x)$	$\cos(x)$	$t(x) = \frac{\sin(x)}{\cos(x)}$
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$-\frac{\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

If we plot these points, we find the following graph.



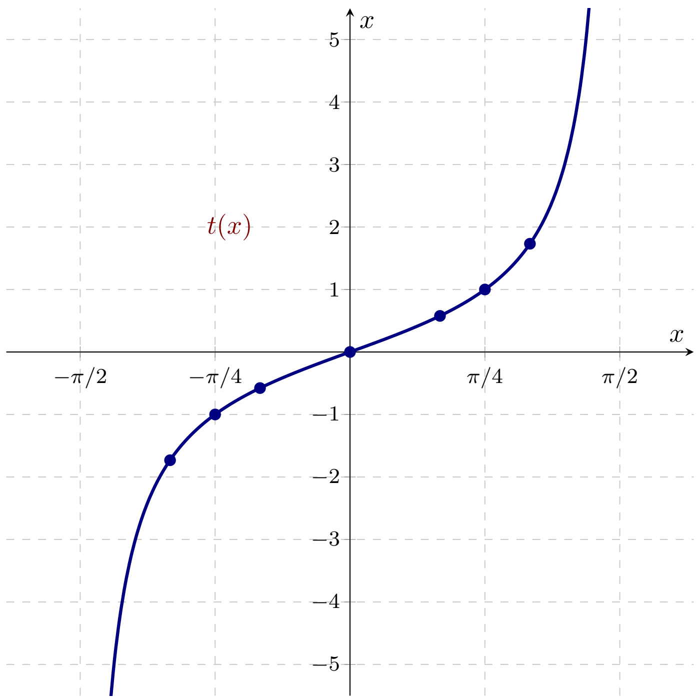
Let's think about what happens if x is a number really close to $\frac{\pi}{2}$, but just a little bit smaller than $\frac{\pi}{2}$. Notice from the graphs that the value of $\sin(x)$ will be a positive number that is really close to 1 and the value of $\cos(x)$ will be really close to 0 but still positive.



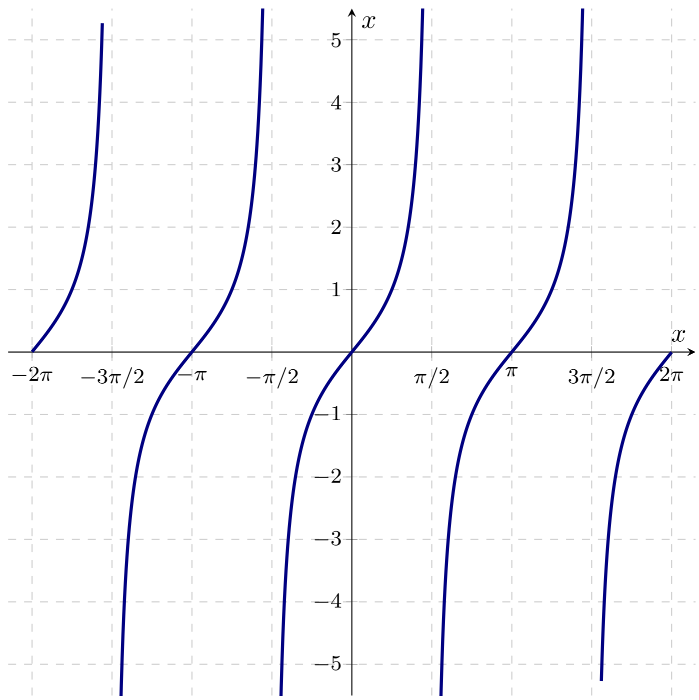
What happens if we take 1 and divide it by a small positive number? Let's look at a table of values to see.

z	$\frac{1}{z}$
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{10}$	10
$\frac{1}{100}$	100
$\frac{1}{1000}$	1000

Notice that as the numbers $z = 1,\ 1/2,\ 1/3,\ \ldots$ got smaller and smaller, the values of $\frac{1}{z} = 1,\ 2,\ 3,\ \ldots$ got larger and larger? That is the same thing we are noticing in the graph of the function t we are building above. For values of x really close to $\pi/2$, but still less than $\pi/2$, the value of $t(x)$ is basically 1 divided by a very small positive number. This table of values tells us that the smaller that denominator gets, the larger the fraction becomes. Adding this behavior to the graph of t gives the following.



By repeating similar calculations for other standard inputs, we arrive at the following graph.



You may recognize this, it will be one of our list of famous functions. This is tangent:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

As you can see from the graph, $\tan(x)$ is an odd, periodic function with period π (not 2π like sine and cosine).

4.2 Logarithms

Learning Objectives

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4.2.1 Definition of Logarithms

In Section 3-2-2, we briefly introduced the concept of *inverse functions*. Recall that for a one-to-one function f, we can define the inverse function f^{-1}. If we think of f as a process that takes some input x and produces some output $f(x)$, then providing $f(x)$ as an input to f^{-1} produces the original input x, and vice versa. Symbolically, we wrote that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

For a more down-to-earth example, consider the function b, which has as its domain the set of all students in the class. The rule for b is that if x is a student, then $b(x)$ is the student's birthdate. The inverse function b^{-1}, then, takes a birthdate and outputs the student who was born on that date. Notice that the domain of b^{-1} is not the set of all days of the year! It is only able to be defined for dates that are someone's birthday, that is, dates that are in the range of b. This highlights an important property of functions and their inverses: the range of f is the domain of f^{-1}!

Summary

- a

4.2.2 Properties of Logarithms

The key to understanding logarithms is through their relationship with exponential functions. Since $f(x) = \log_b(x)$ is the inverse function to $g(x) = b^x$, many of the properties of exponential functions can be translated into properties of logarithms. In this section, we'll try to discover these and find several other interesting properties of logarithms along with way.

We highlight several important principles from our previous discussion of inverse functions:

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

Inverse Property of Logarithms

An important fact to recall is that the range of the function $g(x) = b^x$ is $(0, \infty)$, the set of all positive real numbers. This means that any positive real number can be written as the output of the exponential function with base b. Let's fix $b = 10$ and try to write the number 17 as an output of the function $g(x) = 10^x$. If 17 is an output of g, then $17 = 10^x$ for some real number x. Taking log of both sides of this equation, we find that $\log(17) = \log(10^x)$.

Now we use the most important property of logarithms: the logarithms and exponential of the same base are inverses. With our base being set to 10, this tells us that $\log(10^x) = x$. It is important to remember that even though our notation for the exponential function writes its input as an exponent, and not by wrapping it in parenthesis, x is the input to the exponential function in 10^x.

Returning to our original quest to write 17 as an output of the exponential with base 10, we use the inverse property of logarithms to say that $\log(17) = x$, and therefore,

$$17 = 10^{\log(17)}.$$

Another way to see this is by using the fact that the function $g(x) = 10^x$ is the inverse of $f(x) = \log(x)$.

There was nothing special about 10 and 17 in what we just showed, so this allows us to arrive at a very general way to write positive real numbers as exponentials.

> If x and b are positive real numbers, we can write $x = b^{\log_b(x)}$.

Another way to understand this is to remember the definition of the logarithm. $\log_b(x)$ is precisely the power to which you have to raise b in order to obtain x.

Finally, this can also be viewed as a statement about inverse functions. If $f(x) = \log_b(x)$, then $f^{-1}(x) = b^x$. In this setup, the statement $f^{-1}(f(x)) = x$ becomes $b^{\log_b(x)} = x$.

Product Property of Logarithms

You might think that the method in the previous section of writing positive real numbers as exponentials unnecessarily complicates things, but we can use it to adapt properties of exponents into properties of logarithms.

Recall that multiplying exponential expressions of the same base results in another exponential expression: in symbols,

$$b^u \cdot b^v = b^{u+v}$$

for any real numbers u and v.

Let's see if we can use this fact, again restricting our attention to $b = 10$. Since 2 and 3 are positive real numbers, we can write $2 = 10^{\log(2)}$ and $3 = 10^{\log(3)}$. Then,

$$\log(6) = \log(2\cdot 3) = \log(10^{\log(2)} \cdot 10^{\log(3)}) = \log(10^{\log(2)+\log(3)}) = \log(2)+\log(3).$$

Notice again how we used the fact that the logarithm and exponential with base 10 are inverses! There's nothing special about 2 and 3, so for any positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$. Even more, there's nothing special about base 10, allowing us to come up with a general rule.

> If x, y, and b are positive real numbers, then $\log_b(xy) = \log_b(x)+\log_b(y)$.

Quotient Property of Logarithms

Now that we've dealt with multiplication, it makes sense to deal with division. If x and y are positive real numbers, we can think about the quotient x/y as a product: $x \cdot (1/y)$. What's more, we can write $1/y$ as a power of y: $1/y = y^{-1}$. Using the product property of logarithms from the previous section, we can conclude that $\log_b(x/y) = \log_b(x) + \log_b(y^{-1})$.

It would be really nice if there was a nice relationship between $\log_b(y^{-1})$ and $\log_b(y)$. Indeed, there is! Using the definition of the logarithm, $\log_b(y)$ is the power to which you have to raise b to obtain y, but to obtain y^{-1}, we can use the negative power. As an example, note that $\log(1000) = \log(10^3) = 3$, but

$\log\left(\frac{1}{1000}\right) = \log(10^{-3}) = -3$. In general,

$$\log_b(y^{-1}) = -\log_b(y).$$

Combining this with our previous work, we obtain the following quotient property of logarithms.

> If x, y, and b are positive real numbers, then $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Power Property of Logarithms

Something else you might remember about exponents is that repeated exponentiation is the same thing as multiplying exponents. For example, $(7^3)^2 = 7^{(3\cdot 2)} = 7^6$ (check this yourself!). In words, this says that raising 7 to the 3rd power, then raising that result to the 2nd power is the same as raising 7 to the $3 \cdot 2 = 6$th power. Since $7^3 = 343$, $\log_7(343) = 3$. So in the language of logarithms, the above says that $\log_7(343^2) = 2 \cdot \log_7(343)$.

In general,

$$(b^u)^v = b^{u\cdot v}$$

for all real numbers b, u, and v.

Let's see if this fact has any consequences for logarithms! Recall that for positive b and x, $\log_b(x^u)$ is the power to which we need to raise b in order to obtain x^u. However, another way to obtain x^u is to raise b to the power $\log_b(x)$ (yielding x) and then raise that result to the power u. Since repeated exponentiation is the same thing as multiplying exponents, this amounts to raising b to the power $u\log_b(x)$. In symbols, we've shown that

> If x and b are positive real numbers, and u is a real number, then $\log_b(x^u) = u\log_b(x)$.

In essence, taking the logarithm of a power of x is the same thing as multiplying the logarithm of x by the power. An intuitive way to think about this property is in the context of the product property from above. Since logarithms "turn multiplication into addition" and exponentiation is repeated multiplication, logarithms should "turn exponentiation into repeated addition", that is, multiplication. As an example, notice that

$$\begin{aligned}
\log_2(3^4) &= \log_2(3^2 \cdot 3^2) \\
&= \log_2(3^2) + \log_2(3^2) \\
&= \log_2(3 \cdot 3) + \log_2(3 \cdot 3) \\
&= \log_2(3) + \log_2(3) + \log_2(3) + \log_2(3) \\
&= 4 \log_2(3).
\end{aligned}$$

The above calculation uses the product property to arrive at the same conclusion as the power property.

Change-of-Base Formula

One important thing to recognize is that logarithms can have any positive number as their base. Sometimes, when doing calculations, it may be preferable to use one base over another. The good news is that any logarithm can be computed using this preferred base.

As an example, consider the quantity $\log_3(7)$. Many calculators are unable to directly calculate logarithms with a base other than e or 10, so let's convert this into a natural logarithm (logarithm with base e). Rewriting 7 as $3^{\log_3(7)}$ using the inverse property of logarithms, we see that $\ln(7) = \ln(3^{\log_3(7)})$. Now, using the power property of logarithms, we see that $\ln(3^{\log_3(7)}) = \log_3(7) \cdot \ln(3)$. This gives us the equality $\ln(7) = \log_3(7) \cdot \ln(3)$, so dividing both sides by $\ln(3)$, $\log_3(7) = \dfrac{\ln(7)}{\ln(3)}$. If you have an aversion to $\log_3$ and a fondness for ln, then this allows you to calculate $\ln(7)/\ln(3)$ instead of $\log_3(7)$.

Of course, there's nothing special about 3, 7, and the natural logarithm. In general, we have the following formula.

> If a, b, and x are positive real numbers, then $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

Logarithm Properties in Action

Example 36. *Say* $\log_b(3)$ *is approximately* 0.388 *and* $\log_b(2)$ *is approximately* 0.245*. Using the properties of logarithms, approximate* $\log_b(108)$.

Explanation To use the properties of logarithms, we can make use of the factorization of 108: $108 = 4 \cdot 27 = 2^2 \cdot 3^3$. Using the product property of logarithms, $\log_b(108) = \log_b(2^2 \cdot 3^3) = \log_b(2^2) + \log_b(3^3)$. Now we can apply the product property of logarithms to simplify each term. We conclude that $\log_b(2^2) + \log_b(3^3) = 2\log_b(2) + 3\log_b(3) = 2(0.388) + 3(0.245) = 1.511$.

Therefore, $\log_b(108)$ is approximately 1.511.

Example 37. *Use the properties of logarithms to write* $5\log_5(u) - \dfrac{1}{3}\log_5(v) + \log_5(v)$ *as a single logarithm with coefficient 1. Simplify as much as possible.*

Explanation We can first use the power property to rewrite $5\log_5(u) = \log_5(u^5)$ and $\dfrac{1}{3}\log_5(v) = \log_5(v^{1/3})$. Then we can use the product and quotient properties to combine the terms of the expression.

$$\begin{aligned}
\log_5(u^5) - \log_5(v^{1/3}) + \log_5(v) &= \log_5\left(\frac{u^5}{v^{1/3}}\right) + \log_5(t) \\
&= \log_5\left(\frac{u^5 v}{v^{1/3}}\right) \\
&= \log_5(u^5 v^{2/3})
\end{aligned}$$

There are other ways to approach this problem as well. See if you can find another way to do this problem!

Summary

If x, y, and b are positive real numbers,

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^u) = u\log_b(x)$ for all real numbers u.
- $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ for all positive real numbers a.

4.2.3 Solving Logarithmic Equations

Using Inverses to Solve Equations

Now that we have an understanding of the properties of logarithms, we're prepared to solve equations involving logarithms and exponential functions. Before we do that, however, let's discuss a method of solving equations that you're already familiar with.

Consider the equation
$$x + 2 = 7.$$

You may have already found that the solution is $x = 5$, but let's think about the process of finding the solution.

Our general plan when solving equations is to isolate the variable we're solving for. In this case, we'd like to isolate x by itself on one side of the equation. However, x is not by itself: it's contained in a sum! Naturally, to undo the addition of 2, we subtract 2 from both sides and obtain $x = 5$. The key here is that it was stuck in some operation, and in order to "access" the x, we had to undo that operation.

We can also view this process in the context of functions. Let f be a function defined by $f(x) = x+2$. Then, our equation becomes $f(x) = 7$. In the language of functions, "undoing" f corresponds to applying the inverse function f^{-1}. In this case, $f^{-1}(x) = x - 2$. By applying f^{-1} to both sides of our original equation, we find that

$$\begin{aligned} f(x) &= 7 \\ f^{-1}(f(x)) &= f^{-1}(7) \\ x &= 7 - 2 \\ x &= 5. \end{aligned}$$

This may seem like an awfully strange way to subtract 2, but it has the benefit of being usable for any invertible function.

For example, say we want to solve the equation $\dfrac{x+1}{x} = 4$. If we define a function g by $g(x) = \dfrac{x+1}{x}$, our equation becomes $g(x) = 4$. We can find that the inverse is defined by $g^{-1}(x) = \dfrac{1}{x-1}$. Therefore,

$$\begin{aligned} g(x) &= 4 \\ g^{-1}(g(x)) &= g^{-1}(4) \\ x &= \frac{1}{4-1} \\ x &= \frac{1}{3} \end{aligned}$$

yields the solution to the equation.

Since we had to do quite a bit of work to find the equation for g^{-1} in the above scenario, this method may not be useful in that context. However, there are many functions for which we already know the inverse! For example, the inverse function of $h(x) = x^3$ is $h^{-1}(x) = \sqrt[3]{x}$. Therefore, if we want to solve $h(x) = 343$, we can apply h^{-1} on both sides to find that

$$\begin{aligned} h^{-1}(h(x)) &= h^{-1}(343) \\ x &= 7. \end{aligned}$$

Another important example of inverse functions that we know instantly comes from logarithms! If $f(x) = b^x$, then we know from our previous discussion that $f^{-1}(x) = \log_b(x)$. This is the definition of the logarithm, and looking at solving equations from the point of view of applying inverses is key to solving logarithmic and exponential equations.

For example, if we want to solve the equation $\log(2t-5) = 7$, we can define $f(x) = \log(x)$, so $f^{-1}(x) = 10^x$. This means our equation is $f(2t-5) = 7$. Therefore,

$$\begin{aligned} f(2t-5) &= 7 \\ f^{-1}(f(2t-5)) &= f^{-1}(7) \\ 2t-5 &= 10^7 \\ 2t &= 1000000 + 5 \\ t &= \frac{1000005}{2} \end{aligned}$$

yields the solution to the equation.

Exponential Equations

Example 38. *Solve the equation* $-4^{x-1} + 6 = 3$.

Explanation Notice that the variable we're solving for in this equation is located in the exponent of the exponential expression 4^{x-1}. Whenever this occurs, we call the equation an *exponential equation*.

If we define a function by $f(x) = 4^x$, then our equation becomes $-f(x-1)+6 = 2$. In order to solve this equation, we must use the inverse function: $f^{-1}(x) = \log_4(x)$. However, before we can apply this to both sides of the equation, we need to isolate $f(x-1)$ like so:

$$\begin{aligned} -f(x-1)+6 &= 3 \\ -f(x-1) &= -3 \\ f(x-1) &= 3. \end{aligned}$$

Now we can take f^{-1} of both sides of the equation and obtain

$$\begin{aligned} f^{-1}(f(x-1)) &= f^{-1}(3) \\ x-1 &= \log_4(3) \\ x &= \log_4(3)+1. \end{aligned}$$

Therefore, the solution to the equation $-4^x + 6 = 3$ is $\log_4(3)+1$. Your first instinct might be that this doesn't seem like a solution, since there's still a logarithm in our expression! However, there is no nicer way to write the number $\log_4(3)$. If you were to plug this into a calculator, you would get a decimal approximation to the value of $\log_4(3)$, but any decimal approximation loses some information, so the exact value of the solution is $\log_4(3)+1$.

The process of writing out a function $f(x) = 4^x$ and then taking inverses may seem unnecessary, and indeed, there's no need to actually be so explicit when doing your own calculations. For example, the work

$$\begin{aligned} -4^{x-1}+6 &= 3 \\ -4^{x-1} &= -3 \\ 4^{x-1} &= 3 \\ \log_4(4^{x-1}) &= \log_4(3) \\ x-1 &= \log_4(3) \\ x &= \log_4(3)+1 \end{aligned}$$

would be perfectly sufficient, and is usually how work for this kind of problem would be written. However, it must be emphasized that solving exponential equations involves more than just the basic operations of addition, subtraction, multiplication, and division. We now need to involve the process of taking logarithms of both sides of the equation.

Example 39. *Solve the equation* $3^x = 5^{2-x}$.

Explanation At first glance, this problem seems fundamentally different from the previous example. Instead of dealing with an exponential function with one base, we're dealing with two different bases: 3 and 5.

However, recall from the previous section that any positive real number can be written as a power of any number we want. In this case, 3 can be written as $3 = 5^{\log_5(3)}$. Therefore, our equation becomes

$$5^{x\log_5(3)} = 5^{2-x}.$$

Dividing both sides by 5^{2-x} yields

$$\begin{aligned}\frac{5^{x\log_5(3)}}{5^{2-x}} &= 1\\ 5^{x\log_5(3)-(2-x)} &= 1.\end{aligned}$$

Now, since our variable is trapped in the exponent, we're in the situation from before! If $f(x) = 5^x$, then our equation has become $f(x\log_5(3) - (2-x)) = 1$. To solve this, we can take $f^{-1} = \log_5$ of both sides of the equation and do some more algebra to isolate x.

$$\begin{aligned}\log_5(5^{x\log_5(3)-(2-x)}) &= \log_5(1)\\ x\log_5(3) - (2-x) &= 0\\ x\log_5(3) + x - 2 &= 0\\ x\log_5(3) + x &= 2\\ x(\log_5(3) + 1) &= 2\\ x &= \frac{2}{\log_5(3)+1}.\end{aligned}$$

Logarithmic Equations

Example 40. *Solve the equation* $5\log_2(x+3) = -2$.

Explanation To start off, divide both sides by 5 to isolate the $\log_2(x+3)$ on the left-hand side.

$$\log_2(x+3) = -\frac{2}{5}$$

Next, we apply the inverse of $\log_2$ to both sides of the equation, obtaining

$$\begin{aligned}2^{\log_2(x+3)} &= 2^{-2/5}\\ x+3 &= \frac{1}{\sqrt[5]{4}}\\ x &= \frac{1}{\sqrt[5]{4}} - 3.\end{aligned}$$

Since logarithms are not always defined (their domain is only positive real numbers), we should check that plugging in our solution for x does not result in any part of our original equation being undefined. In our case, this amounts to checking that $\log_2(x+3)$ is defined, that is, that $x+3$ is positive. Since $x+3 = \frac{1}{\sqrt[5]{4}} > 0$, our solution is $\frac{1}{\sqrt[5]{4}} - 3$.

Example 41. *Solve the equation* $\log_6(x) = 1 - \log_6(x-1)$.

Explanation As in Example 2, there appear to be too many functions going on here. However, if we add $\log_6(x-1)$ to both sides of the equation and use the product property of logarithms, we obtain:

$$\begin{aligned} \log_6(x) + \log_6(x-1) &= 1 \\ \log_6(x(x-1)) &= 1. \end{aligned}$$

Next, we can apply the inverse function of $\log_6$, which is given by $f(x) = 6^x$. Doing so, we see that

$$\begin{aligned} 6^{\log_6(x(x-1))} &= 6^1 \\ x(x-1) &= 6 \\ x^2 - x - 6 &= 0. \end{aligned}$$

This results in a quadratic equation! This is something we know how to solve. By using our preferred method, we find that $x = 3$ or $x = -2$.

We're not done yet, however! We need to check that these x-values don't cause any logarithms in our original equation to be undefined. Note that $\log_6(-2)$ is undefined, since -2 is negative, so $x = -2$ is not a solution to our equation. Since $\log_6(3)$ and $\log_6(3-1)$ are both defined, $x = 3$ is our only solution.

Summary

- When solving exponential equations, our strategy is to isolate a single exponential on one side of the equation, then apply a logarithm to both sides to undo the exponential.
- When solving logarithmic equations, our strategy is to isolate a single logarithm on one side of the equation, then apply an exponential function to both sides to undo the logarithm.
- Since the domain of logarithms is only $(0, \infty)$, we need to check that our solutions do not make our original logarithmic equations undefined.

4.3 Inverse Trigonometric Functions

Learning Objectives

- –
  –
  –
- –

4.3.1 Inverse Cosine

Motivating Questions

- Is it possible for a periodic function that fails the Horizontal Line Test to have an inverse?
- For the restricted cosine function, how do we define the corresponding arccosine function?
- What are the key properties of arccosine?

Introduction

In our prior work with inverse functions, we learned several important principles, including

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f. We call g the inverse of f, and write $g = f^{-1}$.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that writing "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

The trigonometric function $g(t) = \cos(t)$ is periodic, so it fails the horizontal line test. Hence, considering this function on its full domain, it does not have an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings. For instance, we may want to say both

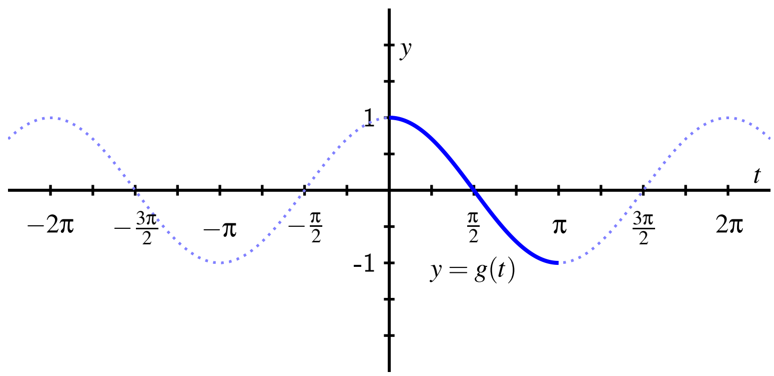
$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) \quad \text{and} \quad \frac{\pi}{6} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

depending on the context in which we are considering the relationship between the angle and side length.

It is also helpful to contextualize the importance of finding an angle in terms of a known value of a trigonometric function. Suppose we know the following information about a right triangle: one leg has length 2.5, and the hypotenuse has length 4. If we let θ be the angle adjacent to the side of length 2.5, it follows that $\cos(\theta) = \frac{2.5}{4}$. We naturally want to use the inverse of the cosine function to solve the most recent equation for θ. But the cosine function does not have an inverse function, so how can we address this situation?

While the original trigonometric function $g(t) = \cos(t)$ does not have an inverse function, we can instead consider a restricted version of the function that does. We thus investigate how we can think differently about the trigonometric functions so that we can discuss inverses in a meaningful way.

Consider the plot of the standard cosine function on $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ with the portion on $[0, \pi]$ emphasized below.



Exploration Let g be the function whose domain is $0 \le t \le \pi$ and whose outputs are determined by the rule $g(t) = \cos(t)$.
The key observation here is that g is defined in terms of the cosine function, but because it has a different domain, it is *not* the cosine function.

a. What is the domain of g?

b. What is the range of g?

c. Does g pass the horizontal line test? Why or why not?

d. Explain why g has an inverse function, g^{-1}, and state the domain and range of g^{-1}.

e. We know that $g\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. What is the exact value of $g^{-1}\left(\frac{\sqrt{2}}{2}\right)$? How about the exact value of $g^{-1}\left(-\frac{\sqrt{2}}{2}\right)$?

f. Determine the exact values of $g^{-1}\left(-\frac{1}{2}\right)$, $g^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $g^{-1}(0)$, and $g^{-1}(-1)$. Use proper notation to label your results.

The Arccosine Function

For the cosine function restricted to the domain $[0, \pi]$ that we considered above, the function is strictly decreasing on its domain and thus passes the Horizontal Line Test. Therefore, this restricted version of the cosine function has an inverse function; we will call this inverse function the *arccosine* function.

Definition Let $y = g(t) = \cos(t)$ be defined on the domain $[0, \pi]$, and observe $g : [0, \pi] \to [-1, 1]$. For any real number y that satisfies $-1 \le y \le 1$, the **arccosine of y**, denoted

$$\arccos(y)$$

is the angle t satisfying $0 \le t \le \pi$ such that $\cos(t) = y$. Note that we use $t = \cos^{-1}(y)$ interchangeably with with $t = \arccos(y)$.

In particular, we note that the output of the arccosine function is an angle. Recall that in the context of the unit circle, an angle measured in radians and the corresponding arc length along the unit circle are numerically equal. This is the origin of the "arc" in "arccosine": given a value $-1 \le y \le 1$, the arccosine function produces the corresponding *arc* (measured counterclockwise from $(1, 0)$) such that the cosine of that arc is y.

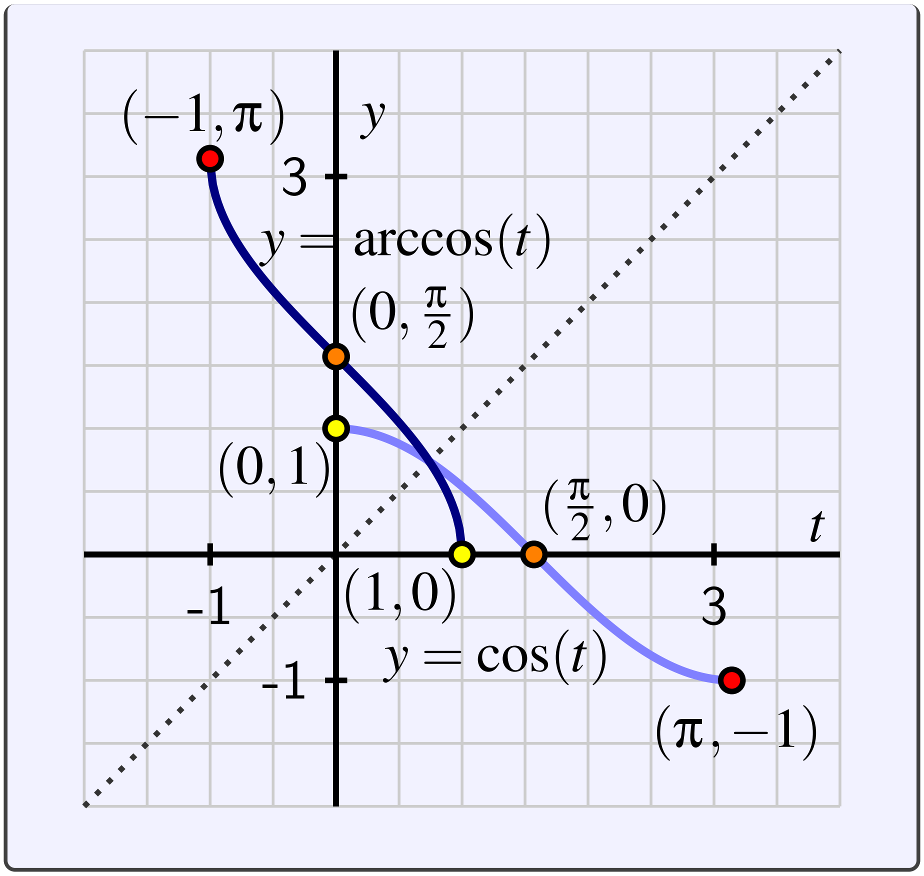
For any function with an inverse function, the inverse function reverses the process of the original function. Thus, given $y = \cos(t)$, we can read this statement as saying "y is the cosine of the angle t". Changing perspective and writing the equivalent statement, $t = \arccos(y)$, we read this statement as "t is the angle whose cosine is y". Just as $y = f(t)$ and $t = f^{-1}(y)$ mean the same thing for a function and its inverse in general. To summarize, both expressions

$$y = \cos(t) \text{ and } t = \arccos(y)$$

mean the same thing for any angle t that satisfies $0 \le t \le \pi$. We read $t = \cos^{-1}(y)$ as "t is the angle whose cosine is y" or "t is the inverse cosine of y". Key properties of the arccosine function can be summarized as follows.

Properties of the arccosine function.

- The restricted cosine function, $y = g(t) = \cos(t)$, is defined on the domain $[0, \pi]$ with range $[-1, 1]$. This function has an inverse function that we call the arccosine function, denoted $t = g^{-1}(y) = \arccos(y)$.
- The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with range $[0, \pi]$.
- The arccosine function is always decreasing on its domain.
- Below we have a plot of the restricted cosine function (in light blue) and its corresponding inverse, the arccosine function (in dark blue).



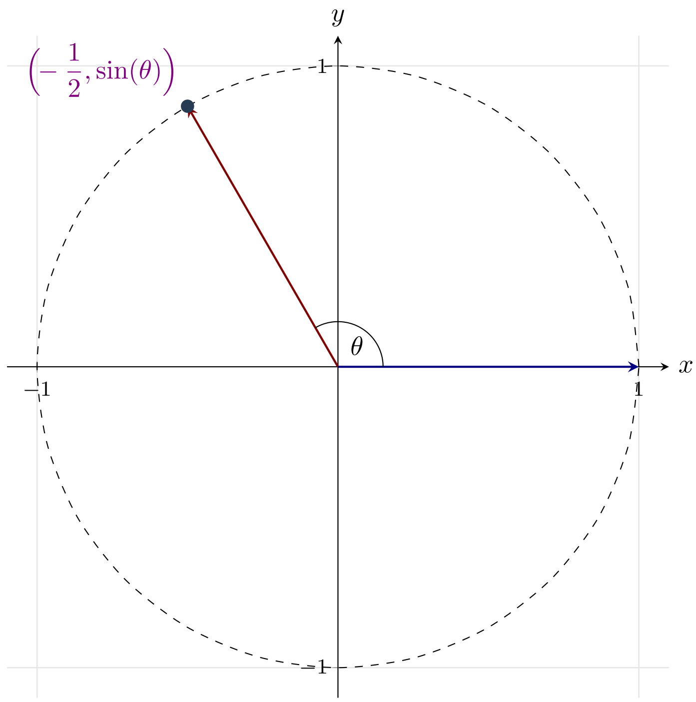
Just as the natural logarithm function allowed us to rewrite exponential equations in an equivalent way (for instance, $y = e^t$ and $t = \ln(y)$ give the same information), the arccosine function allows us to do likewise for certain angles and cosine outputs. For instance, saying $\cos\left(\frac{\pi}{2}\right) = 0$ is the same as writing $\frac{\pi}{2} = \arccos(0)$, which reads "$\frac{\pi}{2}$ is the angle whose cosine is 0". Indeed, these relationships are reflected in the plot above, where we see that any point (a, b) that lies on the graph of $y = \cos(t)$ corresponds to the point (b, a) that lies on the graph of $y = \arccos(t)$.

Exploring Arccosine

Example 42. *Use the special points on the unit circle to determine the exact values of each of the following numerical expressions. Do so without using a computational device.*

(a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

Explanation We start by finding $\arccos\left(-\frac{1}{2}\right)$. Remember that for x in $[-1,1]$, $\arccos(x)$ is the angle θ in $[0,\pi]$ such that $\cos(\theta)=x$. Hence we are looking for the value of θ corresponding to the point on the upper hemisphere of the unit circle with x-value $-\frac{1}{2}$.



Hence, θ is $\frac{2\pi}{3}$, and we now see that

$$\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arccosine. Since cosine is a periodic function, it fails the horizontal line test. However, if we *restrict* cosine to a portion of its domain on which it is only decreasing, $[0,\pi]$, then we may define a function g on this domain such that $g(x)=\cos(x)$ for x in $[0,\pi]$. Arccosine is defined as the inverse of this function g. Therefore, g is the inverse of arccosine. Thus, in practice, cosine is the inverse of arccosine.

A word of caution: arccosine is only the inverse of restricted cosine, as we will demonstrate with the next example.

(b) $\arccos\left(\cos\left(\frac{7\pi}{6}\right)\right)$

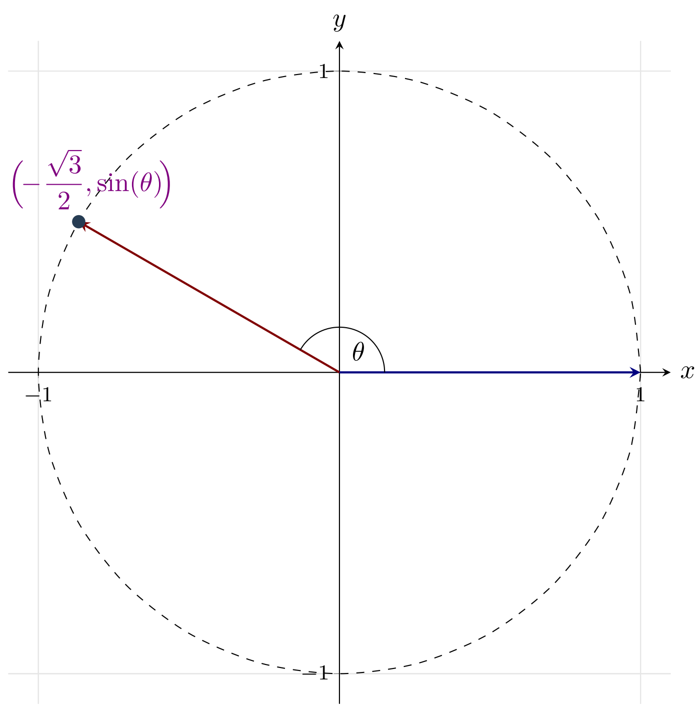
Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{7\pi}{6}$ since arccosine is the inverse of cosine.

But, wait!

Remember, we had to restrict the domain of cosine in order to define an inverse function, which we called arccosine. Arccosine is the inverse of the *restricted* cosine function, whose domain is $[0, \pi]$. $\frac{7\pi}{6}$ is larger than π, so it is not within the domain of this restricted cosine.

Thus, we begin by simplifying $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

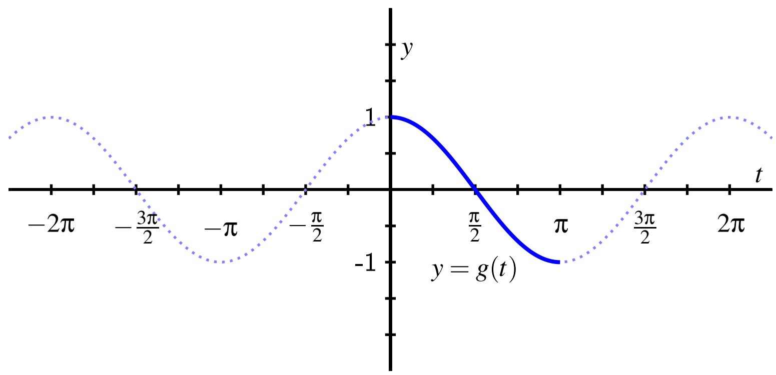
Now, when we consider $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, we will once again recall the unit circle. We are looking at the upper hemisphere, but this time we want to find the angle θ in $[0, \pi]$ that corresponds to the point with x-value $-\frac{\sqrt{3}}{2}$.



Hence, θ is $\frac{5\pi}{6}$, and we now see that

$$\arccos\left(\cos\left(\frac{7\pi}{6}\right)\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Now, let's look again at the graph of cosine. Here we highlight $g : [0, \pi] \to [0, \pi]$ defined by $y = g(x) = \cos(x)$, the restricted cosine function. We may use the symmetry of the graph of cosine to help find the appropriate values for arccosine.



(c) *We can also solve trig equations as in Section 10-3 Some Applications of Trig Functions:* $4\arccos(x) - 3\pi = 0$.

Explanation We start by isolating the arccosine term so that our equation is now

$$\arccos(x) = \frac{3\pi}{4}.$$

We observe that $\frac{3\pi}{4}$ is in the range of arccosine, so we may use the fact that cosine is the inverse of arccosine. Thus, $\arccos(x) = \frac{3\pi}{4}$ is equivalent to

$$\cos(\arccos(x)) = \cos\left(\frac{3\pi}{4}\right).$$

This is further equivalent to $x = -\frac{2}{2}$.

Summary

- Any function that fails the Horizontal Line Test cannot have an inverse function. However, for a periodic function that fails the horizontal line test, if we restrict the domain of the function to an interval that is the length of a single period of the function, we then determine a related function that does, in fact, have an inverse function. This makes it possible for us to develop the inverse function of the restricted cosine function.

- We choose to define the restricted cosine function on the domain

$[0, \pi]$. On this interval, the restricted function is strictly decreasing, and thus has an inverse function. The restricted cosine function has range $[-1, 1]$.

4.3.2 Other Inverse Trig Functions

Motivating Questions

- For the restricted sine and tangent functions, how do we define the corresponding arcsine and arctangent functions?
- What are the key properties of arcsine and arctangent?

Introduction

In the last section we defined *arccosine*, the inverse for cosine restricted to a single period. In this section we will explore the definition of similar inverse functions on restricted domains of sine and tangent.

As we recalled last time,

- A function f has an inverse function if and only if there exists a function g that undoes the work of f: that is, there is some function g, the inverse of f, for which $g(f(x)) = x$ for each x in the domain of f, and $f(g(y)) = y$ for each y in the range of f.
- A function f has an inverse function if and only if the graph of f passes the *Horizontal Line Test*.
- When f has an inverse, we know that "$y = f(t)$" and "$t = f^{-1}(y)$" are two different perspectives on the same statement.

As with the cosine function, the trigonometric functions $f(t) = \sin(t)$ and $h(t) = \tan(t)$ are periodic, so they fail the horizontal line test. Hence, considering these functions on their full domains, neither has an inverse function. At the same time, it is reasonable to think about changing perspective and viewing angles as outputs in certain restricted settings, as we did with cosine.

The Arcsine Function

We can develop an inverse function for a restricted version of the sine function in a similar way. As with the cosine function, we need to choose an interval on which the sine function is always increasing or always decreasing in order to have the function pass the horizontal line test. The standard choice is the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on which $f(t) = \sin(t)$ is increasing and attains all of the values in the range of the sine function. Thus, we consider $f(t) = \sin(t)$ so that $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \left[-1, 1\right]$ and use this restricted function to define the corresponding arcsine function.

Definition Let $y = f(t) = \sin(t)$ be defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and observe $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \left[-1, 1\right]$. For any real number y that satisfies $-1 \leq y \leq 1$, the **arcsine of y**, denoted

$$\arcsin(y)$$

is the angle t satisfying $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ such that $\sin(t) = y$. Note that we use $t = \sin^{-1}(y)$ interchangeably with with $t = \arcsin(y)$.

Problem 1 *The goal of this activity is to understand key properties of the arcsine function in a way similar to our discussion of the arccosine function in the previous section. We will use our deductive reasoning skills a la Sherlock Holmes to build off our discussion from the last section.*

(a) *Using the definition of arcsine given above, what are the domain and range of the arcsine function?*

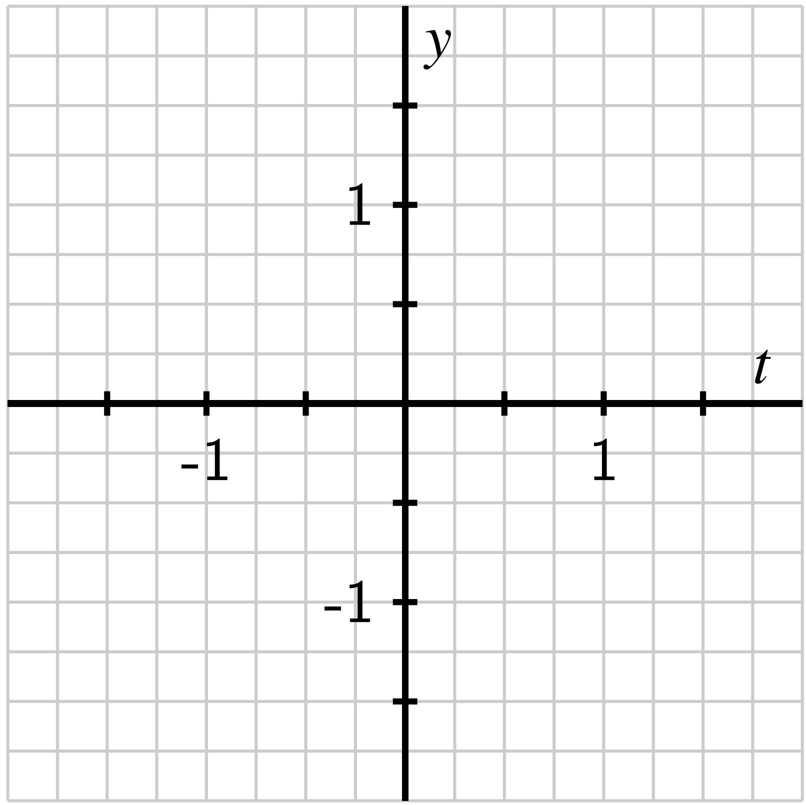
- *The domain of arcsine is* $\boxed{?}$.
- *The range of arcsine is* $\boxed{?}$.

(b) *Determine the following values exactly:*

- $\arcsin(-1) = \boxed{?}$
- $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = \boxed{?}$
- $\arcsin(0) = \boxed{?}$
- $\arcsin\left(\frac{1}{2}\right) = \boxed{?}$
- $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \boxed{?}$

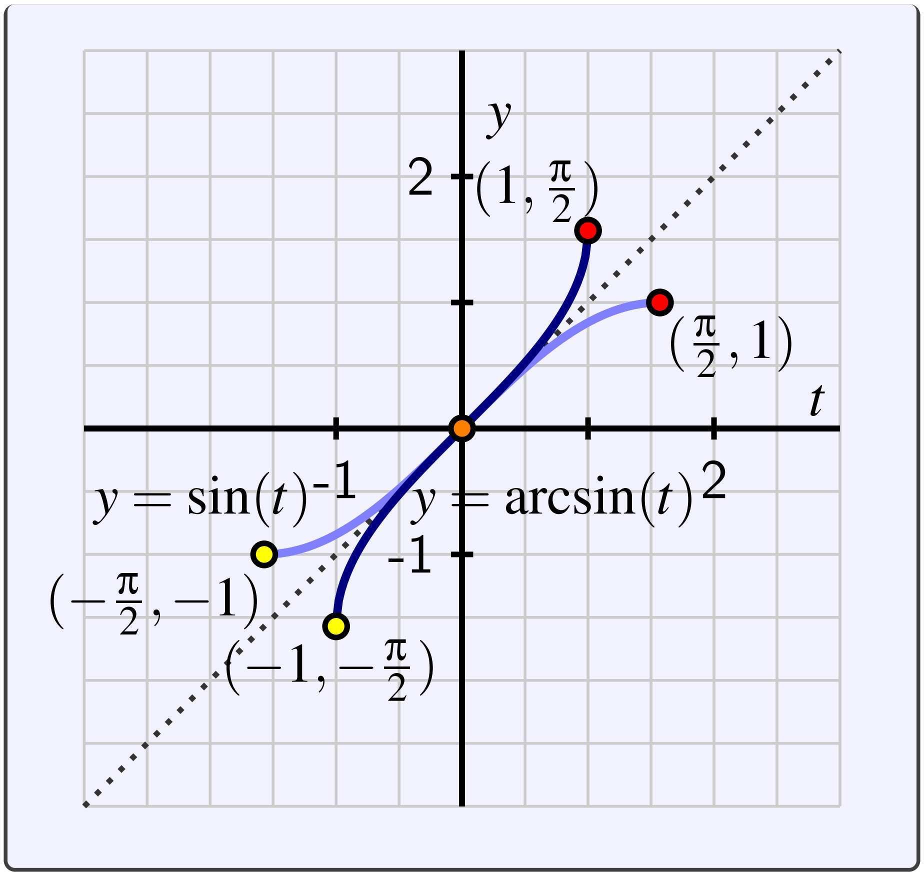
(c) *On the axes provided below, sketch a careful plot of the restricted sine function on the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *along with its corresponding inverse, the arcsine function. Label at least three points on each curve so that each point on the sine graph corresponds to a point on the arcsine graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

(d) *True or false:* $\arcsin(\sin(5\pi)) = 5\pi$*? (true/false)*
Write a complete sentence to explain your reasoning.



Properties of the arcsine function.

- The restricted sine function, $y = f(t) = \sin(t)$, is defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with range $[-1, 1]$. This function has an inverse function that we call the arcsine function, denoted $t = f^{-1}(y) = \arcsin(y)$.
- The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The arcsine function is always increasing on its domain.
- Below we have a plot of the restricted sine function (in light blue) and its corresponding inverse, the arcsine function (in dark blue).



Exploring Arcsine

Example 43. *Let's solve the following equations analytically, then we can consider the graph of arcsine.*

(a) $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

Explanation We start by finding $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$. Remember that for x in $[-1, 1]$, $\arcsin(x)$ is the value y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = x$. Hence, y is $-\frac{\pi}{4}$, and we now see that

$$\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Now, if you're thinking, "Hey, we didn't need that extra step!" Then you would be correct. But ***why*** didn't we need that final step?

Let's recall how we defined arcsine. Since sine is a periodic function, it fails the horizontal line test. However, if we *restrict* sine to a portion of its domain on which it is only increasing, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then we may define a function f on this domain such that $f(x) = \sin(x)$ for x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Arcsine then is defined as the inverse of this function f. Therefore, f is the inverse of arcsine. Thus, in practice, sine is the inverse of arcsine.

A word of caution: As was the case with arccosine and cosine, arcsine is only the inverse of restricted sine. We will illustrate this with the next example.

(b) $\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{4}$ since arcsine is the inverse of sine.

Hold those horses!

Remember, we had to restrict the domain of sine in order to define an inverse function, which we called arcsine. Arcsine is the inverse of the *restricted* sine function, whose domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\frac{5\pi}{4}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted sine function.

Thus, we begin by simplifying $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Now, let's consider $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$, recalling again the *range* of arcsine. We are looking for the value of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = -\frac{\sqrt{2}}{2}$.

Hence, y is $-\frac{\pi}{4}$, and we now see that

$$\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

[graph for arcsine?]

(c) $\arcsin(2x) = \frac{\pi}{3}$

Explanation First, we observe that $\frac{\pi}{3}$ is in the range of arcsine, so there should be a solution. We will now use the fact that sine is the inverse of arcsine to reduce this to a linear equation.

$$\begin{aligned} \arcsin(2x) &= \frac{\pi}{3} \\ \sin(\arcsin(2x)) &= \sin(\frac{\pi}{3}) \end{aligned}$$

Thus, we have

$$2x = \frac{\sqrt{3}}{2},$$

which is equivalent to $x = \dfrac{\sqrt{3}}{4}$.

[Insert graph here?]

The Arctangent Function

Finally, we develop an inverse function for a restricted version of the tangent function. We choose the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on which $h(t) = \tan(t)$ is increasing and attains all of the values in the range of the tangent function.

Definition Let $y = h(t) = \tan(t)$ be defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and observe $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \left(-\infty, \infty\right)$. For any real number y, the **arctangent of y**, denoted

$$\arctan(y)$$

is the angle t satisfying $-\frac{\pi}{2} < t < \frac{\pi}{2}$ such that $\tan(t) = y$. Note that we use $t = \tan^{-1}(y)$ interchangeably with with $t = \arctan(y)$.

Problem 2 *Let us once again channel our inner Sherlock Holmes to understand key properties of the arctangent function.*

(a) *Using the definition given above, what are the domain and range of the arctangent function?*

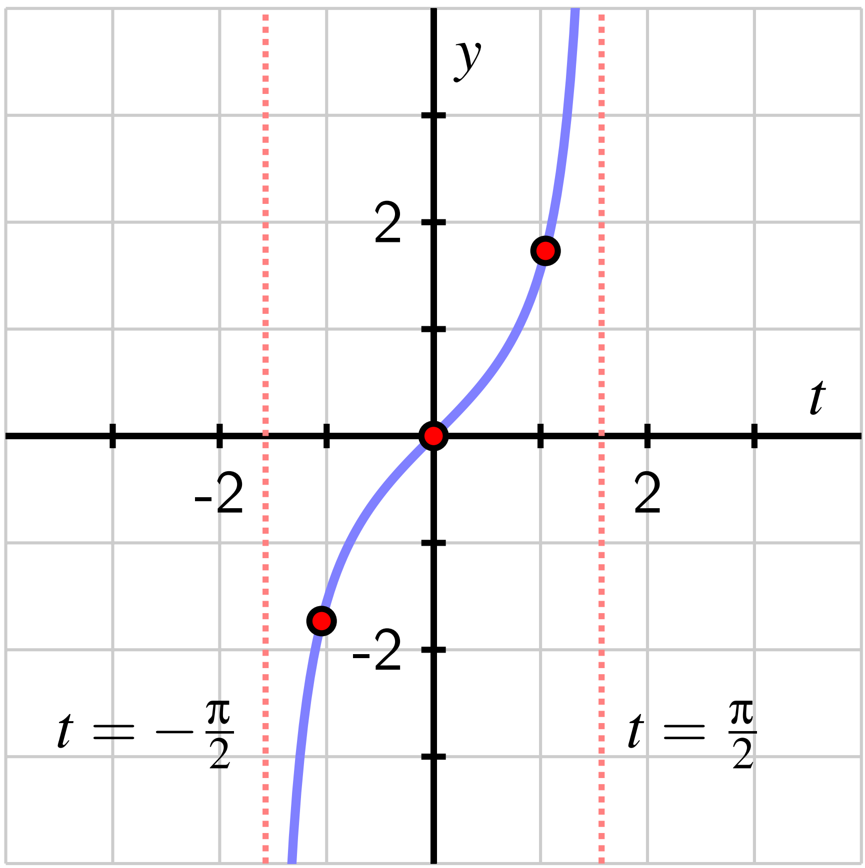
- *The domain of arctangent is* $\boxed{?}$.
- *The range of arctangent is* $\boxed{?}$.

(b) *Determine the following values exactly:*

- $\arctan(-\sqrt{3}) = \boxed{?}$
- $\arctan(-1) = \boxed{?}$
- $\arctan(0) = \boxed{?}$
- $\arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{?}$.

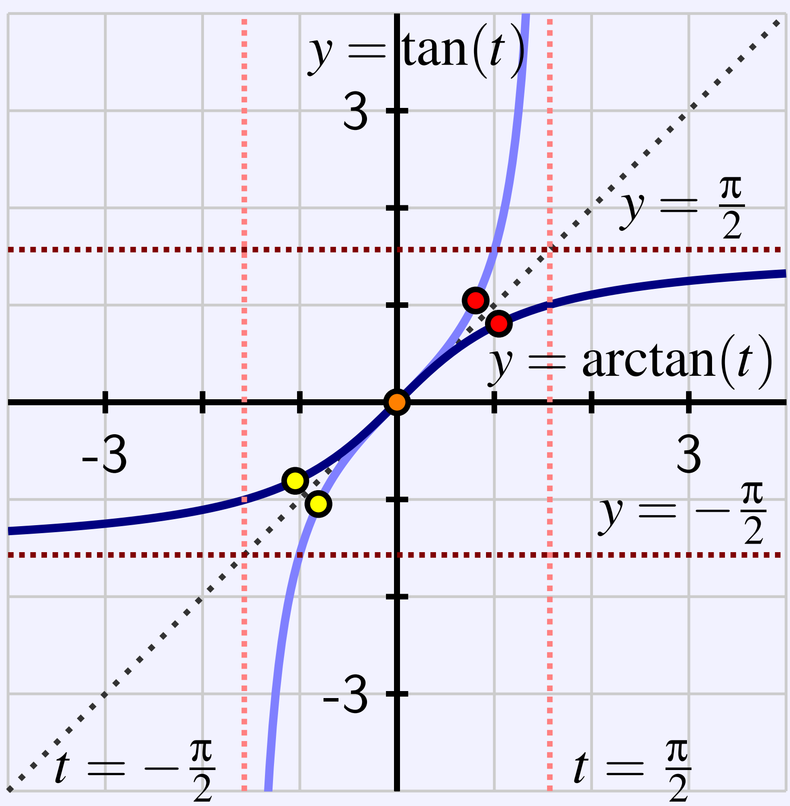
(c) *The restricted tangent function on the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is plotted below. On the same axes, sketch its corresponding inverse function (arctangent). Label at least three points on each curve so that each point on the tangent graph corresponds to a point on the arctangent graph. In addition, sketch the line* $y = t$ *to demonstrate how the graphs are reflections of one another across this line.*

(d) *Complete the following sentence: "as* t *increases without bound,* $\arctan(t)$ *. . ." (increases without bound/decreases without bound/increases toward* $\frac{\pi}{2}$*/decreases toward* $-\frac{\pi}{2}$*)*



Properties of the arctangent function.

- The restricted tangent function, $y = h(t) = \tan(t)$, is defined on the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with range $(-\infty, \infty)$. This function has an inverse function that we call the arctangent function, denoted $t = h^{-1}(y) = \arctan(y)$.
- The domain of $y = h^{-1}(t) = \arctan(t)$ is $(-\infty, \infty)$ with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The arctangent function is always increasing on its domain.
- Below we have a plot of the restricted tangent function (in light blue) and its corresponding inverse, the arctangent function (in dark blue).



Exploring Arctangent

Example 44. *Let's solve the following equations analytically, then we can consider the graph of arctangent.*

(a)

(b) $\tan(\arctan(-\sqrt{3}))$

Explanation We start by finding $\arctan(-\sqrt{3})$. Remember that for x in $(-\infty, \infty)$, $\arctan(x)$ is the value y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = x$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\tan(\arctan(-\sqrt{3})) = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

Now, I know you're thinking, "Hey, why do you keep making us do an extra step?" It's because it is imperative that you ***consider the range*** of the arc trig functions. These are considerations you will also need to make when we start combining different trig functions with the inverses of others (say sine of arctangent of a value).

Let's recall how we defined arctangent. Since tangent is a periodic function, it fails the horizontal line test. However, if we *restrict* tangent to a single period (note tangent only increasing), $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then we may define a function h on this domain such that $h(x) = \tan(x)$ for x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Arctangent then is defined as the inverse of this function h. Therefore, h is the inverse of arctangent. Thus, in practice, tangent is the inverse of arctangent.

A word of caution: As was the case with the previous two trig functions and their respective inverses, arctangent is only the inverse of restricted tangent. We will illustrate this with the next example.

(c) $\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right)$

Explanation It may be tempting to take a look at this expression and conclude that the solution is $\frac{5\pi}{3}$ since arctangent is the inverse of tangent.

But, wait!

Remember, we had to restrict the domain of tangent in order to define an inverse function, which we called arctangent. Arctangent is the inverse of the *restricted* tangent function, whose domain is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\frac{5\pi}{3}$ is larger than $\frac{\pi}{2}$, so it is not within the domain of this restricted tangent function.

Thus, we begin by simplifying $\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$.

Now, let's consider $\arctan(-\sqrt{3})$, recalling again the *range* of arctangent. We are looking for the value of y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = -\sqrt{3}$.

Hence, y is $-\frac{\pi}{3}$, and we now see that

$$\arctan\left(\tan\left(\frac{5\pi}{3}\right)\right) = \arctan(-\sqrt{3}) = -\frac{\pi}{3}.$$

[graph for arctangent?]

(d) $4\arctan^2(x) - 3\pi\arctan(x) - \pi^2 = 0$
Explanation We will treat this like a quadratic equation to begin, as we did in Section 10-3 Some Applications of Trig Functions.

Let $y = \arctan(x)$, then we have a standard quadratic equation: $4y^2 - 3\pi y - \pi^2 = 0$. Factoring, we see that this is equivalent to

$$(4y + \pi)(y - \pi) = 0.$$

This has two solutions: $y = -\frac{\pi}{4}$ and $y = \pi$. In other words, we now simply solve (a) $\arctan(x) = -\frac{\pi}{4}$ and (b) $\arctan(x) = \pi$. π is not in the range of arctangent, so (b) does not have a solution. Hence, this cannot be a solution to our equation, and we must look at (a). $-\frac{\pi}{4}$ is in the range of arctangent, so the solution to (a) will be a solution to our original equation.

Since tangent is the inverse to arctangent, the equation (a) is equivalent to

$$\tan(\arctan(x)) = \tan\left(-\frac{\pi}{4}\right),$$

which is further equivalent to $x = -1$

Summary

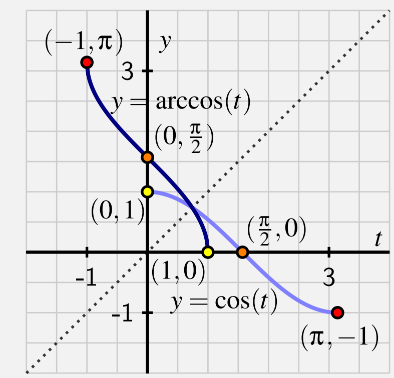
- We choose to define the restricted cosine, sine, and tangent functions on the respective domains $[0, \pi]$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $(-\frac{\pi}{2}, \frac{\pi}{2})$. On each such interval, the restricted function is strictly decreasing (cosine) or strictly increasing (sine and tangent), and thus has an inverse function. The restricted sine and cosine functions each have range $[-1, 1]$, while the restricted tangent's range is the set of all real numbers. We thus define the inverse function of each as follows:

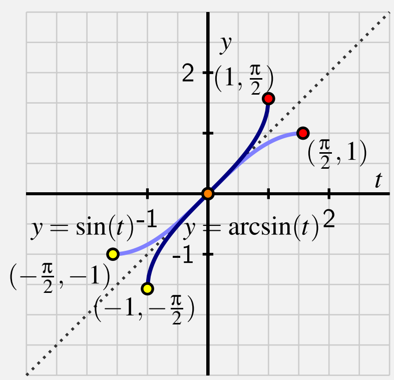
  i. For any y such that $-1 \leq y \leq 1$, the arccosine of y (denoted $\arccos(y)$) is the angle t in the interval $[0, \pi]$ such that $\cos(t) = y$. That is, t is the angle whose cosine is y.

ii. For any y such that $-1 \le y \le 1$, the arcsine of y (denoted $\arcsin(y)$) is the angle t in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(t) = y$. That is, t is the angle whose sine is y.

iii. For any real number y, the arctangent of y (denoted $\arctan(y)$) is the angle t in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan(t) = y$. That is, t is the angle whose tangent is y.

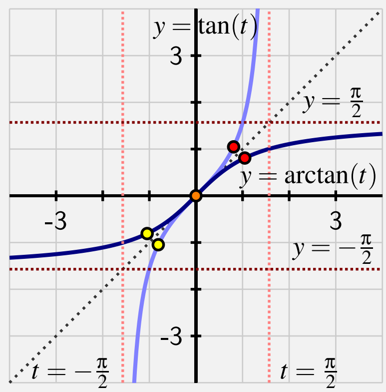
- The domain of $y = g^{-1}(t) = \arccos(t)$ is $[-1, 1]$ with corresponding range $[0, \pi]$, and the arccosine function is always decreasing. These facts correspond to the domain and range of the restricted cosine function and the fact that the restricted cosine function is decreasing on $[0, \pi]$.





The domain of $y = f^{-1}(t) = \arcsin(t)$ is $[-1, 1]$ with corresponding range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and the arcsine function is always increasing. These facts correspond to the domain and range of the restricted sine function and the fact that the restricted sine function is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The domain of $y = f^{-1}(t) = \arctan(t)$ is the set of all real numbers with corresponding range $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the arctangent function is always increasing. These facts correspond to the domain and range of the restricted tangent function and the fact that the restricted tangent function is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.



4.3.3 Applications of Inverse Trigonometry

Motivating Questions

- How can we use inverse trigonometric functions to determine missing angles in right triangles?
- What other situations may require us to use inverse trignometric functions?

Introduction

When we learned about trig functions in Section 10, we observed that in any right triangle, if we know the measure of one additional angle and the length of one additional side, we can determine all of the other parts of the triangle. With the inverse trigonometric functions that we developed in the last two sections, we are now also able to determine the missing angles in any right triangle where we know the lengths of two sides.

While the original trigonometric functions take a particular angle as input and provide an output that can be viewed as the ratio of two sides of a right triangle, the inverse trigonometric functions take an input that can be viewed as a ratio of two sides of a right triangle and produce the corresponding angle as output. Indeed, it's imperative to remember that statements such as

$$\arccos(x) = \theta \text{ and } \cos(\theta) = x$$

say the exact same thing from two different perspectives, and that we read "$\arccos(x)$" as "the angle whose cosine is x".

Exploration Consider a right triangle that has one leg of length 3 and another leg of length $\sqrt{3}$. Let θ be the angle that lies opposite the shorter leg. Sketch a labeled picture of the triangle.

(a) What is the exact length of the triangle's hypotenuse?

(b) What is the exact value of $\sin(\theta)$?

(c) Rewrite your equation from (b) using the arcsine function in the form $\arcsin(\square) = \Delta$, where $\square$ and Δ are numerical values.

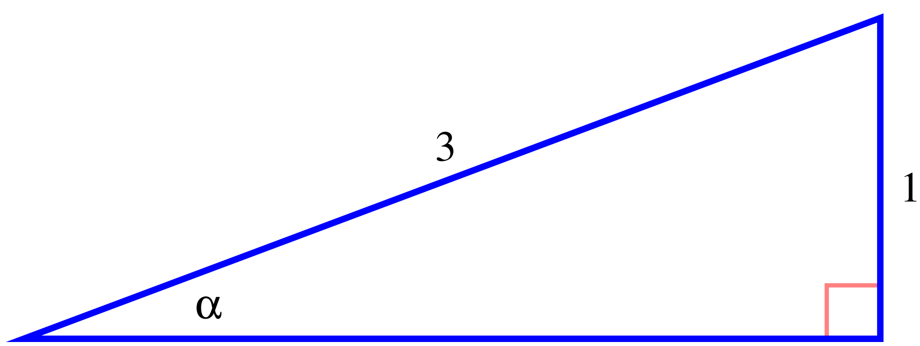
(d) What special angle from the unit circle is θ?

Evaluating Inverse Trigonometric Functions

Like the trigonometric functions themselves, there are a handful of important values of the inverse trigonometric functions that we can determine exactly without the aid of a computer. For instance, we know from the unit circle that $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$, $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$, and $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$. In these evaluations, we have to be careful to remember that the range of the arccosine function is $[0, \pi]$, while the range of the arcsine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and the range of the arctangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, in order to ensure that we choose the appropriate angle that results from the inverse trigonometric function. This is why our emphasis is now turning to the *graphs* of these functions.

In addition, there are many other values at which we may wish to know the angle that results from an inverse trigonometric function. To determine such values, one can use a computational device (such as *Desmos*) in order to evaluate the function; however, in this class we leave it in the form $\arccos(a)$, as this is the exact value.

Example 45. *Consider the right triangle pictured below and assume we know that the vertical leg has length* 1 *and the hypotenuse has length* 3*. Let* α *be the angle opposite the known leg. Determine exact values for all of the remaining parts of the triangle.*



Explanation Because we know the hypotenuse and the side opposite α, we observe that $\sin(\alpha) = \frac{1}{3}$. Rewriting this statement using inverse function notation, we have equivalently that $\alpha = \arcsin\left(\frac{1}{3}\right)$, which is the exact value of α. Since this is not one of the known special angles on the unit circle, we leave it in this form.

We can now find the remaining leg's length and the remaining angle's measure. If we let x represent the length of the horizontal leg, by the Pythagorean Theorem we know that

$$x^2 + 1^2 = 3^2,$$

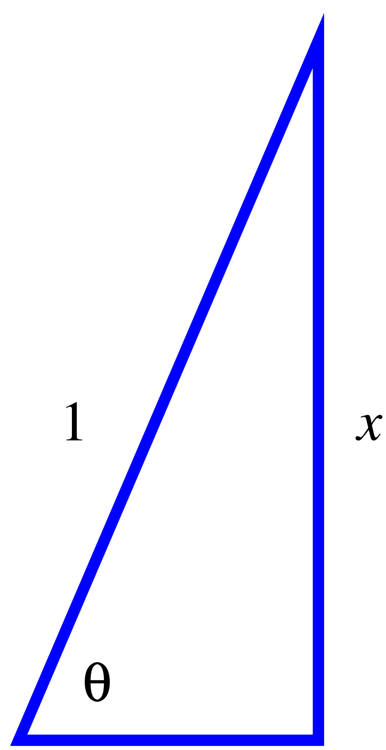
and thus $x^2 = 8$ so $x = \sqrt{8}$. Calling the remaining angle β, since $\alpha + \beta = \dfrac{\pi}{2}$, it follows that

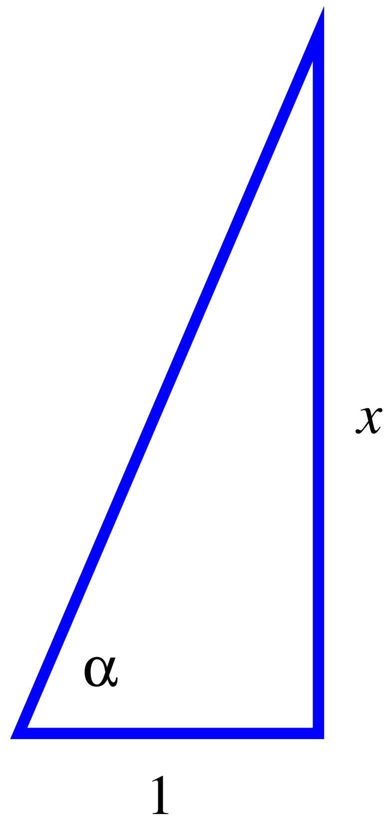
$$\beta = \frac{\pi}{2} - \arcsin\left(\frac{1}{3}\right).$$

Example 46. *Let's consider the composite function $h(x) = \cos(\arcsin(x))$.*
Does it makes sense to consider this function? Let's think ...
This function makes sense to consider since the arcsine function has range $[-1, 1]$, on which we may evaluate the cosine function. In the questions that follow, we investigate how to express h without using trigonometric functions at all.

(a) *What is the domain of h? The range of h?*

(b) *Since the arcsine function produces an angle, let's say that $\theta = \arcsin(x)$, so that θ is the angle whose sine is x. By definition, we can picture θ as an angle in a right triangle with hypotenuse 1 and a vertical leg of length x, as shown in the image on the left below. Use the Pythagorean Theorem to determine the length of the horizontal leg as a function of x.*





The right triangle on the left corresponds to the angle $\theta = \arcsin(x)$. The right triangle on the right corresponds to the angle $\alpha = \arctan(x)$.

(c) *What is the value of* $\cos(\theta)$ *as a function of* x*? What have we shown about* $h(x) = \cos(\arcsin(x))$*?*

(d) *How about the function* $p(x) = \cos(\arctan(x))$*? How can you reason similarly to write* p *in a way that doesn't involve any trigonometric functions at all? (Hint: let* $\alpha = \arctan(x)$ *and consider the right triangle on the right above.)*

Using Inverse Trig in Applied Contexts

Now that we have developed the (restricted) sine, cosine, and tangent functions and their respective inverses, in any setting in which we have a right triangle together with one side length and any one additional piece of information (another side length or a non-right angle measurement), we can determine all of the remaining pieces of the triangle. In the example that follows and the homework, we explore these possibilities in a variety of different applied contexts.

Example 47. *A roof is being built with a "7-12 pitch." This means that the roof rises* 7 *inches vertically for every* 12 *inches of horizontal span; in other words, the slope of the roof is* $\frac{7}{12}$.

(a) *What is the exact measure of the angle the roof makes with the horizontal?*
Explanation

(b) *What is the exact measure of the angle at the peak of the roof (made by the front and back portions of the roof that meet to form the ridge)?*
Explanation

Further Exploration

We can now use trigonometry to find angles of right triangles if we know the side lengths and side lengths of right triangles if we know the angles. You might be wondering, "What about triangles that are not right triangles? Can we use trig to learn anything about those?" It turns out that the Law of Sines and the Law of Cosines gives use a way to analyze other triangles beyond just right triangles using trig functions. For more information about this topic, see Laws of Sines and Cosines by Katherine Yoshiwara[7] .

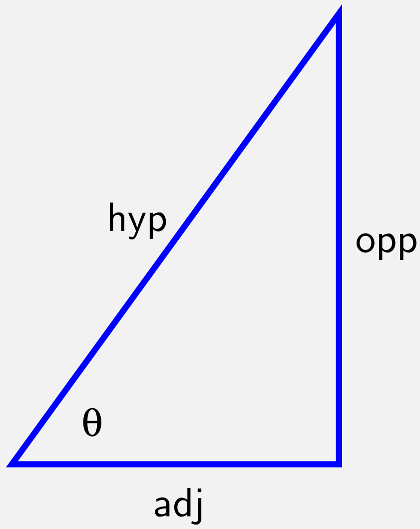
Summary

Anytime we know two side lengths in a right triangle, we can use one of the inverse trigonometric functions to determine the measure of one of the non-right angles. For instance, if we know the values of opp and adj in the triangle pictured below, then since

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}},$$

it follows that $\theta = \arctan\left(\frac{\text{opp}}{\text{adj}}\right)$.

[7]See Laws of Sines and Cosines by Katherine Yoshiwara at https://yoshiwarabooks.org/trig/chap3.html



If we instead know the hypotenuse and one of the two legs, we can use either the arcsine or arccosine function accordingly.

Part 5

Working with More Variables

5.1 Linear Systems of Equations

Learning Objectives

- From Systems to Solutions
  - What is a system of equations?
  - What is a solution to a system?
  - Solving systems via graphs
- Solving Systems Algebraically
  - Eliminating Variables
  - Substitution Method
- Applications of Systems of Equations
  - Word problems
  - Mixture Problems

5.1.1 From Systems to Solutions

Motivating Questions

- What is a system of equations?
- What is a solution to a system?
- How can we solve systems of equations using graphs?

Introduction

We have already seen many techniques for solving equations. Until now, however, we have only solved equations of the form $f(x) = 0$ for the variable x. In this section, we will consider equations with more than one variable and discuss how to solve them.

Consider a peculiar grocery store where the prices of all the items for sale are not listed, and you only find out the total cost of your purchase. Say you buy 6 mangos and 3 bananas, and your total cost is 9 dollars. Assume all the mangos cost the same amount and all the bananas cost the same amount. Without making any more purchases, is it possible find out how much a mango and a banana cost on their own?

Let's create an equation to describe this situation. Let x be a variable representing the cost of a mango, and let y be a variable representing the cost of a banana. Then, the equation

$$6x + 3y = 9$$

represents that buying 6 mangos at a cost of x dollars and 3 bananas at a cost of y dollars yields a total cost of 9 dollars.

You might have noticed that plugging $x = 1$ and $y = 1$ into the equation gives us a true statement, so you might conclude that mangos and bananas both cost 1 dollar. However, notice that plugging $x = 1.20$ and $y = 0.60$ into the equation also gives us a true statement, so it's also possible that mangos cost \$1.20 and bananas cost \$0.60. Even more worrying is that $x = 0$ and $y = 3$ also gives us a solution to the equation: is this store peculiar enough to be giving away mangos for free and charging \$3 per banana?

Examining the equation we set up can give us more insight. Let's rearrange the equation to solve for y in terms of x:

$$\begin{aligned} 6x + 3y &= 9 \\ 3y &= 9 - 6x \\ y &= 3 - 2x. \end{aligned}$$

Now it becomes clearer what's going on. Whatever x is, we can find a value of y that satisfies our original equation. No matter the cost of a single mango, there's a way to price the bananas so that our equation is true! This means it's impossible to find the price of a single mango or a single banana with the information you've been given. We need more data!

Systems of linear equations

In order to collect more information, you go back to the store and buy 2 mangos and 2 banana for a total cost of \$3.20. This can be modeled by the equation

$$2x + 2y = 3.2.$$

Keep in mind that this x and y are the same x and y from before, so in order to find the cost of a mango and a banana, we must find x and y that satisfy both equations

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}$$

at the same time. This coupling of two (or more) linear equations is called a *system of linear equations*.

Definition A **linear equation of two variables** is an equation of the form

$$a_1 x + a_2 y = c,$$

where a_1, a_2, and c are real numbers and at least one of a_1 and a_2 is nonzero.
A **system of linear equations of two variables** is a collection of two or more linear equations of two variables.
We say a **solution** to a system of linear equations of two variables is a point (x, y) satisfying all equations in the system.

It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called *consistent*, those with no solution are called *inconsistent*.

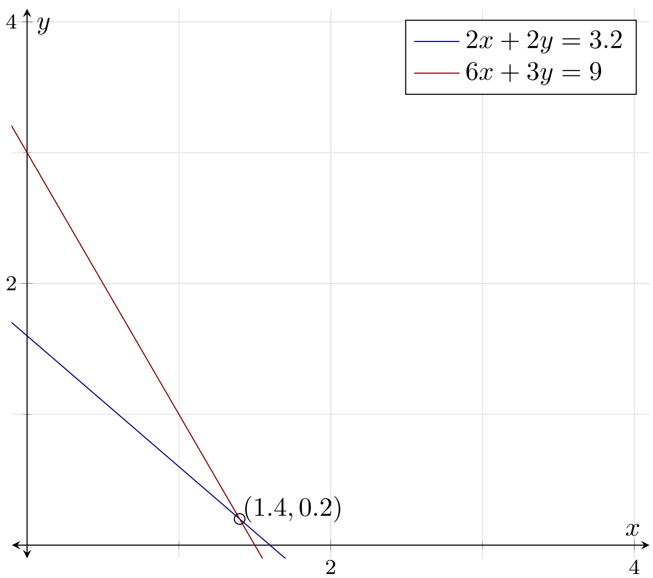
The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. Note that we can still have systems of non-linear equations, but they can be much more difficult to solve.

Finding solutions graphically

Let's return to our example from earlier and try to find a solution to the system of linear equation:

$$\begin{cases} 6x + 3y = 9 \\ 2x + 2y = 3.2 \end{cases}.$$

We want to find x and y satisfying both equations in the system. If x and y satisfy $6x + 3y = 9$, then the point (x, y) lies on the graph of $6x + 3y = 9$. Similarly, if x and y satisfy $2x+2y = 3.2$, then the point (x, y) lies on the graph of $2x + 2y = 3.2$. Therefore, to find any solutions, we can look at the graphs of $6x + 3y = 9$ and $2x + 2y = 3.2$, and see if there are any points that lie at the intersection of the two graphs:



By inspecting the graph, we see that these two lines intersect only at $(1.4, 0.2)$, so the only solution to the system is $x = 1.4$ and $y = 0.2$.

In context, this means that mangos cost \$1.40 each and bananas cost \$0.20 each. Note that in order to have exactly one solution to our system of linear equations in two variables, we needed the system to have two equations.

Note that not every system of linear equations will have one solution. If the graphs of the two equations are parallel, they will never intersect, so there won't be any solutions. Additionally, if the two equations are represented by the same graph, there will be infinitely many intersection points, and therefore, infinitely many solutions.

Next, we will see some methods for solving systems of equations algebraically.

5.1.2 Solving Systems of Equations Algebraically

Substitution

In the previous section, we focused on solving systems of equations by graphing. In addition to being time consuming, graphing can be an awkward method to determine the exact solution when the solution has large numbers, fractions, or decimals. There are two symbolic methods for solving systems of linear equations, and in this section we will use one of them: substitution.

Example 48. *In 2014, the New York Times posted the following about the movie, "The Interview":*

> *"The Interview" generated roughly* \$15 *million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday. Sony did not say how much of that total represented* \$6 *digital rentals versus* \$15 *sales. The studio said there were about two million transactions overall.*

A few days later, Joey Devilla cleverly pointed out in his blog, that there is enough information given to find the amount of sales versus rentals.

Explanation Using algebra, we can write a system of equations and solve it to find the two quantities. Although since the given information uses approximate values, the solutions we will find will only be approximations too.

First, we will define variables. We need two variables, because there are two unknown quantities: how many sales there were and how many rentals there were. Let r be the number of rental transactions and let s be the number of sales transactions.

If you are unsure how to write an equation from the background information, use the units to help you. The units of each term in an equation must match because we can only add like quantities. Both r and s are in transactions. The article says that the total number of transactions is 2 million. So our first equation will add the total number of rental and sales transactions and set that equal to 2 million. Our equation is:

$$(r\,\text{transactions}) + (s\,\text{transactions}) = 2{,}000{,}000\,\text{transactions}$$

Without the units:

$$r + s = 2{,}000{,}000$$

The price of each rental was \$6. That means the problem has given us a *rate* of $6\,\dfrac{\text{dollars}}{\text{transaction}}$ to work with. The rate unit suggests this should be multiplied by something measured in transactions. It makes sense to multiply by r, and then the number of dollars generated from rentals was $6r$. Similarly, the price of each sale was \$15, so the revenue from sales was $15s$. The total revenue was \$15 million, which we can represent with this equation:

$$\left(6\,\frac{\text{dollars}}{\text{transaction}}\right)(r\text{ transactions}) + \left(15\,\frac{\text{dollars}}{\text{transaction}}\right)(s\text{ transactions}) = \$15{,}000{,}000$$

Without the units:

$$6r + 15s = 15{,}000{,}000$$

Here is our system of equations:

$$\begin{array}{rcrcr} r & + & s & = & 2{,}000{,}000 \\ 6r & + & 15s & = & 15{,}000{,}000 \end{array}$$

To solve the system, we will use the **substitution** method. The idea is to use *one* equation to find an expression that is equal to r but, cleverly, does not use the variable "r." Then, substitute this for r into the *other* equation. This leaves you with *one* equation that only has *one* variable.

The first equation from the system is an easy one to solve for r:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ r &= 2{,}000{,}000 - s \end{aligned}$$

This tells us that the expression $2{,}000{,}000 - s$ is equal to r, so we can *substitute* it for r in the second equation:

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6(\color{blue}{2{,}000{,}000 - s}) + 15s &= 15{,}000{,}000 \end{aligned}$$

Now we have an equation with only one variable, s, which we will solve for:

$$\begin{aligned} 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 - 6s + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 + 9s &= 15{,}000{,}000 \\ 9s &= 3{,}000{,}000 \\ \frac{9s}{\color{blue}{9}} &= \frac{3{,}000{,}000}{\color{blue}{9}} \\ s &= 333{,}333.\overline{3} \end{aligned}$$

At this point, we know that $s = 333{,}333.\overline{3}$. This tells us that out of the 2 million transactions, roughly 333,333 were from online sales. Recall that we solved the first equation for r, and found $r = 2{,}000{,}000 - s$.

$$\begin{aligned} r &= 2{,}000{,}000 - s \\ r &= 2{,}000{,}000 - \color{blue}{333{,}333.\overline{3}} \\ r &= 1{,}666{,}666.\overline{6} \end{aligned}$$

To check our answer, we will see if $s = 333{,}333.\overline{3}$ and $r = 1{,}666{,}666.\overline{6}$ make the original equations true:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ 1{,}666{,}666.\overline{6} + 333{,}333.\overline{3} &= 2{,}000{,}000 \\ 2{,}000{,}000 &= 2{,}000{,}000 \\ 6r + 15s &= 15{,}000{,}000 \\ 6\left(1{,}666{,}666.\overline{6}\right) + 15\left(333{,}333.\overline{3}\right) &= 15{,}000{,}000 \\ 10{,}000{,}000 + 5{,}000{,}000 &= 15{,}000{,}000 \end{aligned}$$

In summary, there were roughly 333,333 copies sold and roughly 1,666,667 copies rented.

Elimination

We just learned how to solve a system of linear equations using substitution above. Now, we will learn a second symbolic method for solving systems of linear equations.

Example 49. *Alicia has* \$1000 *to give to her two grandchildren for New Year's. She would like to give the older grandchild* \$120 *more than the younger grandchild, because that is the cost of the older grandchild's college textbooks this term. How much money should she give to each grandchild?*

Explanation To answer this question, we will demonstrate a new technique. You may have a very good way for finding how much money Alicia should give to each grandchild, but right now we will try to see this new method.

Let A be the dollar amount she gives to her older grandchild, and B be the dollar amount she gives to her younger grandchild. (As always, we start solving a word problem like this by defining the variables, including their units.) Since the total she has to give is \$1000, we can say that $A + B = 1000$. And since she wants to give \$120 more to the older grandchild, we can say that $A - B = 120$. So we have the system of equations:

$$\begin{aligned} A + B &= 1000 \\ A - B &= 120 \end{aligned}$$

We could solve this system by substitution as we learned previously but there is an easier method. If we add together the *left* sides from the two equations, it should equal the sum of the *right* sides:

$$\begin{array}{rcl} A + B & = & 1000 \\ +A - B & & +120 \end{array}$$

So we have:

$$2A = 1120$$

Note that the variable B is eliminated. This happened because the $+B$ and the $-B$ perfectly cancel each other out when they are added. With only one variable left, it doesn't take much to finish:

$$\begin{aligned} 2A &= 1120 \\ A &= 560 \end{aligned}$$

To finish solving this system of equations, we need the value of B. For now, an easy way to find B is to substitute in our value of A into one of the original equations:

$$\begin{aligned} A + B &= 1000 \\ {\color{blue}560} + B &= 1000 \\ B &= 440 \end{aligned}$$

To check our work, substitute $A = 560$ and $B = 440$ into the original equations:

$$\begin{aligned} A + B &= 1000 \\ {\color{blue}560} + {\color{blue}440} &= 1000 \\ 1000 &= 1000 \\ A - B &= 120 \\ {\color{blue}560} - {\color{blue}440} &= 120 \\ 120 &= 120 \end{aligned}$$

This confirms that our solution is correct. In summary, Alicia should give \$560 to her older grandchild, and \$440 to her younger grandchild.

This method for solving the system of equations in the example above worked because B and $-B$ add to zero. Once the B-terms were eliminated we were able to solve for A. This method is called the **elimination method**. Some people call it the **addition method**, because we added the corresponding sides from the two equations to eliminate a variable.

If neither variable can be immediately eliminated, we can still use this method but it will require that we first adjust one or both of the equations. Let's look at an example where we need to adjust one of the equations.

Example 50. *Solve the system of equations using the elimination method.*

$$\begin{array}{rcrcr} 3x & - & 4y & = & 2 \\ 5x & + & 8y & = & 18 \end{array}$$

Explanation To start, we want to see whether it will be easier to eliminate x or y. We see that the coefficients of x in each equation are 3 and 5, and the coefficients of y are -4 and 8. Because 8 is a multiple of 4 and the coefficients already have opposite signs, the y variable will be easier to eliminate.

To eliminate the y terms, we will multiply each side of the first equation by 2 so that we will have $-8y$. We can call this process scaling the first equation by 2.

$$\begin{array}{rcrcr} 2 \cdot (3x & - & 4y) & = & 2 \cdot (2) \\ 5x & + & 8y & = & 18 \end{array}$$

$$\begin{array}{rcrcr} 6x & - & 8y & = & 4 \\ 5x & + & 8y & = & 18 \end{array}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{rcr} 6x - 8y & = & 4 \\ +5x + 8y & & +18 \end{array}$$

So we have:

$$\begin{aligned} 11x &= 22 \\ x &= 2 \end{aligned}$$

To solve for y, we can substitute 2 for x into either of the original equations or the new one. We use the first original equation, $3x - 4y = 2$:

$$\begin{aligned} 3x - 4y &= 2 \\ 3(2) - 4y &= 2 \\ 6 - 4y &= 2 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Our solution is $x = 2$ and $y = 1$. We will check this in both of the original equations:

$$\begin{aligned} 5x + 8y &= 18 \\ 5(2) + 8(1) &= 18 \\ 10 + 8 &= 18 \\ 3x - 4y &= 2 \\ 3(2) - 4(1) &= 2 \\ 6 - 4 &= 2 \end{aligned}$$

The solution to this system is $(2, 1)$ and the solution set is $\{(2, 1)\}$.

5.1.3 Applications of Systems of Equations

Example 51. *Two Different Interest Rates Notah made some large purchases with his two credit cards one month and took on a total of* \$8,400 *in debt from the two cards. He didn't make any payments the first month, so the two credit card debts each started to accrue interest. That month, his Visa card charged* 2% *interest and his Mastercard charged* 2.5% *interest. Because of this, Notah's total debt grew by* \$178. *How much money did Notah charge to each card?*
Explanation To start, we will define two variables based on our two unknowns. Let v be the amount charged to the Visa card (in dollars) and let m be the amount charged to the Mastercard (in dollars).

To determine our equations, notice that we are given two different totals. We will use these to form our two equations. The total amount charged is \$8,400 so we have:

$$(v\,\text{dollars}) + (m\,\text{dollars}) = \$8400$$

Or without units:

$$v + m = 8400$$

The other total we were given is the total amount of interest, \$178, which is also in dollars. The Visa had v dollars charged to it and accrues 2% interest. So $0.02v$ is the dollar amount of interest that comes from using this card. Similarly, $0.025m$ is the dollar amount of interest from using the Mastercard. Together:

$$0.02(v\,\text{dollars}) + 0.025(m\,\text{dollars}) = \$178$$

Or without units:

$$0.02v + 0.025m = 178$$

As a system, we write:

$$\begin{array}{ccccc} v & + & m & = 8400 \\ 0.02v & + & 0.025m & = 178 \end{array}$$

To solve this system by substitution, notice that it will be easier to solve for one of the variables in the first equation. We'll solve that equation for v:

$$\begin{aligned} v + m &= 8400 \\ v &= 8400 - m \end{aligned}$$

Now we will substitute $8400 - m$ for v in the second equation:

$$\begin{aligned}
0.02v + 0.025m &= 178 \\
0.02(8400 - m) + 0.025m &= 178 \\
168 - 0.02m + 0.025m &= 178 \\
168 + 0.005m &= 178 \\
0.005m &= 10 \\
\frac{0.005m}{0.005} &= \frac{10}{0.005} \\
m &= 2000
\end{aligned}$$

Lastly, we can determine the value of v by using the earlier equation where we isolated v:

$$\begin{aligned}
v &= 8400 - m \\
v &= 8400 - 2000 \\
v &= 6400
\end{aligned}$$

In summary, Notah charged \$6400 to the Visa and \$2000 to the Mastercard. We should check that these numbers work as solutions to our original system ¡em¿and¡/em¿ that they make sense in context. (For instance, if one of these numbers were negative, or was something small like \$0.50, they wouldn't make sense as credit card debt.)

Mixture Problems

The next two examples are called **mixture problems**, because they involve mixing two quantities together to form a combination and we want to find out how much of each quantity to mix.

Example 52. *Mixing Solutions with Two Different Concentrations LaVonda is a meticulous bartender and she needs to serve* 600 *milliliters of Rob Roy, an alcoholic cocktail that is* 34% *alcohol by volume. The main ingredients are scotch that is* 42% *alcohol and vermouth that is* 18% *alcohol. How many milliliters of each ingredient should she mix together to make the concentration she needs?*
Explanation The two unknowns are the quantities of each ingredient. Let s be the amount of scotch (in mL) and let v be the amount of vermouth (in mL).

One quantity given to us in the problem is 600 mL. Since this is the total volume of the mixed drink, we must have:

$$(s\,\mathrm{mL}) + (v\,\mathrm{mL}) = 600\,\mathrm{mL}$$

Or without units:

$$s + v = 600$$

To build the second equation, we have to think about the alcohol concentrations for the scotch, vermouth, and Rob Roy. It can be tricky to think about percentages like these correctly. One strategy is to focus on the *amount* (in mL) of *alcohol* being mixed. If we have s milliliters of scotch that is 42% alcohol, then $0.42s$ is the actual *amount* (in mL) of alcohol in that scotch. Similarly, $0.18v$ is the amount of alcohol in the vermouth. And the final cocktail is 600 mL of liquid that is 34% alcohol, so it has $0.34(600) = 204$ milliliters of alcohol. All this means:

$$0.42(s\,\mathrm{mL}) + 0.18(v\,\mathrm{mL}) = 204\,\mathrm{mL}$$

Or without units:

$$0.42s + 0.18v = 204$$

So our system is:

$$\begin{aligned} s \quad &+ \quad v &&= 600 \\ 0.42s \quad &+ \quad 0.18v &&= 204 \end{aligned}$$

To solve this system, we'll solve for s in the first equation: $\begin{aligned} s+v &= 600 \\ s &= 600 - v \end{aligned}$

And then substitute s in the second equation with $600 - v$:

$$\begin{aligned} 0.42s + 0.18v &= 204 \\ 0.42(600 - v) + 0.18v &= 204 \\ 252 - 0.42v + 0.18v &= 204 \\ 252 - 0.24v &= 204 \\ -0.24v &= -48 \\ \frac{-0.24v}{-0.24} &= \frac{-48}{-0.24} \\ v &= 200 \end{aligned}$$

As a last step, we will determine s using the equation where we had isolated s:

$$\begin{aligned} s &= 600 - v \\ s &= 600 - 200 \\ s &= 400 \end{aligned}$$

In summary, LaVonda needs to combine 400 mL of scotch with 200 mL of vermouth to create 600 mL of Rob Roy that is 34% alcohol by volume.

As a check for the previous example, we can use estimation to see that our solution is reasonable. Since LaVonda is making a 34% solution, she would need to use more of the 42% concentration than the 18% concentration, because 34% is closer to 42% than to 18%. This agrees with our answer because we found that she needed 400 mL of the 42% solution and

200 mL of the 18% solution. This is an added check that we have found reasonable answers.

Example 53. *Mixing a Coffee Blend Desi owns a coffee shop and they want to mix two different types of coffee beans to make a blend that sells for* \$12.50 *per pound. They have some coffee beans from Columbia that sell for* \$9.00 *per pound and some coffee beans from Honduras that sell for* \$14.00 *per pound. How many pounds of each should they mix to make* 30 *pounds of the blend?*
Explanation Before we begin, it may be helpful to try to estimate the solution. Let's compare the three prices. Since \$12.50 is between the prices of \$9.00 and \$14.00, this mixture is possible. Now we need to estimate the amount of each type needed. The price of the blend (\$12.50 per pound) is closer to the higher priced beans (\$14.00 per pound) than the lower priced beans (\$9.00 per pound). So we will need to use more of that type. Keeping in mind that we need a total of 30 pounds, we roughly estimate 20 pounds of the \$14.00 Honduran beans and 10 pounds of the \$9.00 Columbian beans. How good is our estimate? Next we will solve this exercise exactly.

To set up our system of equations we define variables, letting C be the amount of Columbian coffee beans (in pounds) and H be the amount of Honduran coffee beans (in pounds).

The equations in our system will come from the total amount of beans and the total cost. The equation for the total amount of beans can be written as:

$$(C\,\mathrm{lb}) + (H\,\mathrm{lb}) = 30\,\mathrm{lb}$$

Or without units:

$$C + H = 30$$

To build the second equation, we have to think about the cost of all these beans. If we have C pounds of Columbian beans that cost \$9.00 per pound, then $9C$ is the cost of those beans in dollars. Similarly, $14H$ is the cost of the Honduran beans. And the total cost is for 30 pounds of beans priced at \$12.50 per pound, totaling $12.5(30) = 37.5$ dollars. All this means:

$$\left(9\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(C\,\mathrm{lb}) + \left(14\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(H\,\mathrm{lb}) = \left(12.50\,\tfrac{\mathrm{dollars}}{\mathrm{lb}}\right)(30\,\mathrm{lb})$$

Or without units and carrying out the multiplication on the right:

$$9C + 14H = 37.5$$

Now our system is:

$$\begin{array}{rcrl} C & + & H & = 30 \\ 9C & + & 14H & = 37.50 \end{array}$$

To solve the system, we'll solve the first equation for C:

$$\begin{aligned} C + H &= 30 \\ C &= 30 - H \end{aligned}$$

Next, we'll substitute C in the second equation with $30 - H$:

$$\begin{aligned}
9C + 14H &= 375 \\
9(30 - H) + 14H &= 375 \\
270 - 9H + 14H &= 375 \\
270 + 5H &= 375 \\
5H &= 105 \\
H &= 21
\end{aligned}$$

Since $H = 21$, we can conclude that $C = 9$.

In summary, Desi needs to mix 21 pounds of the Honduran coffee beans with 9 pounds of the Columbian coffee beans to create this blend. Our estimate at the beginning was pretty close, so we feel this answer is reasonable.

5.2 Non-linear Systems

Learning Objectives

- Famous Formulas
  - Reviewing famous functions
  - Introducing conic sections and their formulas
- Solving Non-linear Systems Graphically
  - What is a non-linear system?
  - Finding solutions graphically
  - What can we say about when solutions exist?
- Eliminating Variables
  - Algebra of reducing multivariable systems to a single equation
  - Systems created from functions
  - Reviewing some algebra and misconceptions

5.2.1 Famous Formulas

Introduction

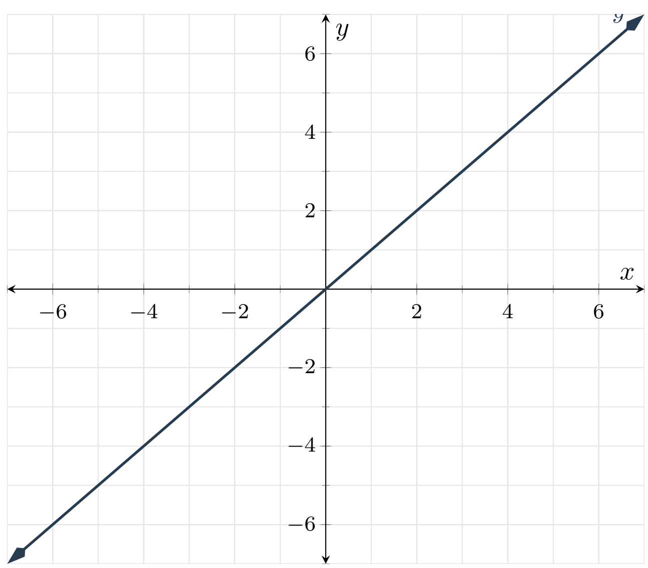
As a review, we go over the list of famous functions from earlier. Then, we move to a discussion of conic sections.

Linear Functions

Recall that the graph of a linear function is a line.

Example 54. *A prototypical example of a linear function is*

$$y = x.$$



Important Values of $y = x$	
x	y
-2	-2
-1	-1
0	0
1	1
2	2

In general, linear functions can be written as $y = mx + b$ where m and b can be any numbers. We learned that m represents the slope, and b is the y-coordinate of the y-intercept. You can play with changing the values of m and b on the graph using Desmos and see how that changes the line.

Desmos link: `https://www.desmos.com/calculator/japnhapzvn`

Table 1: Properties of Linear Functions $y = mx + b$

Periodic?	If $m = 0$
Odd?	If $b = 0$
Even?	If $m = 0$
One-to-one/invertible	If $b = 0$

Note that any real number can be plugged into $f(x) = mx + b$, so the domain of linear functions is $(-\infty, \infty)$. Unless $m = 0$, we can find a y such that $y = mx + b$, so the range of linear functions with $m \neq 0$ is $(-\infty, \infty)$. If $m = 0$, then the only output of the linear function is b, so its range is $\{b\}$.

Table 2: Domain and Range of Linear Functions $y = mx + b$

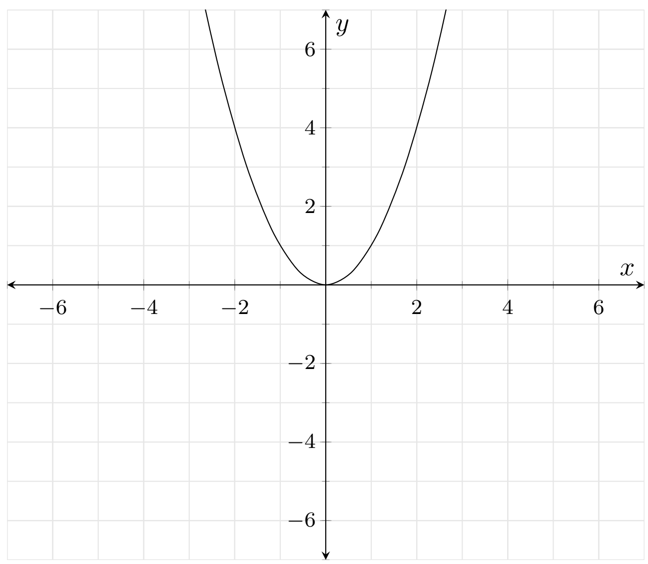
Domain	$(-\infty, \infty)$
Range	If $m \neq 0$, $(-\infty, \infty)$; if $m = 0$, $\{b\}$

Quadratic Functions

Recall that the graph of a quadratic function is a parabola.

Example 55. *A prototypical example of a quadratic function is*

$$y = x^2.$$



Important Values of $y = x^2$	
x	y
-2	4
-1	1
0	0
1	1
2	4

In general, quadratic functions can be written as $y = ax^2 + bx + c$ where a, b, and c can be any numbers. You can play with changing the values of a, b, and c on the graph using Desmos and see how that changes the parabola.

Desmos link: `https://www.desmos.com/calculator/nmlghfrws9`

Note that any real number can be plugged into $f(x) = ax^2+bx+c$, so the domain of quadratic functions is $(-\infty, \infty)$. In Chapter 4, we saw that all quadratic

Table 3: Properties of Quadratic Functions $y = ax^2 + bx + c$

Periodic?	If $a = 0$ and $b = 0$
Odd?	If $a = 0$, $b = 0$, and $c = 0$
Even?	If $b = 0$
One-to-one/invertible	If $a = 0$ and $c = 0$

functions have a vertex form $f(x) = d(x-h)^2 + k$, where the vertex is at (h, k). If $d > 0$, all points above the vertex, that is $[k, \infty)$ are in the range of the quadratic, and if $d < 0$, all points below the vertex, that is $(\infty, k]$ are in the range of the quadratic.

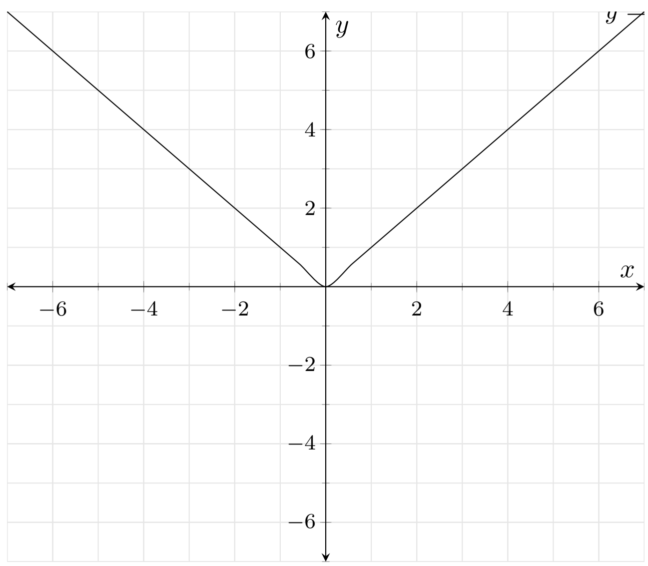
Table 4: Domain and Range of Quadratic Functions $y = d(x-h)^2 + k$

Domain	$(-\infty, \infty)$
Range	If $d > 0$, $[k, \infty)$; if $d < 0$, $(\infty, k]$

Absolute Value

Another important type of function is the absolute value function. This is the function that takes all y-values and makes them positive. The absolute value function is written as

$$y = |x|.$$



Important Values of $y = \|x\|$	
x	y
-2	2
-1	1
0	0
1	1
2	2

Table 5: Properties of The Absolute Value Function $y = |x|$

Periodic?	No
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that any real number has an absolute value, so the domain of the absolute value function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the absolute value function.

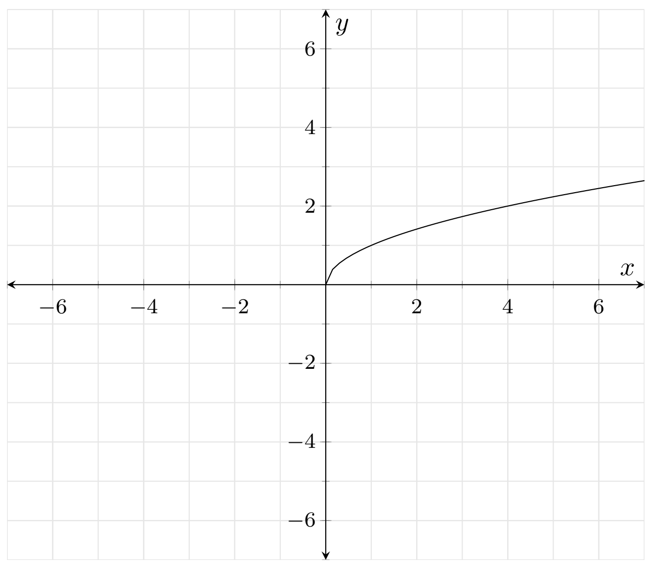
Table 6: Domain and Range of The Absolute Value Function $y = |x|$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

Square Root

Another famous function is the square root function,

$$y = \sqrt{x}.$$



Important Values of $y = \sqrt{x}$	
x	y
0	0
1	1
4	2
9	3
25	5

Table 7: Properties of The Square Root Function $y = \sqrt{x}$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that only non-negative numbers have square roots, so the domain of the square root function is $[0, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the square root function. Algebraically, we can say that for any non-negative y, $\sqrt{(y^2)} = y$, so y is in the range of the square root function.

Table 8: Domain and Range of The Square Root Function $y = \sqrt{x}$

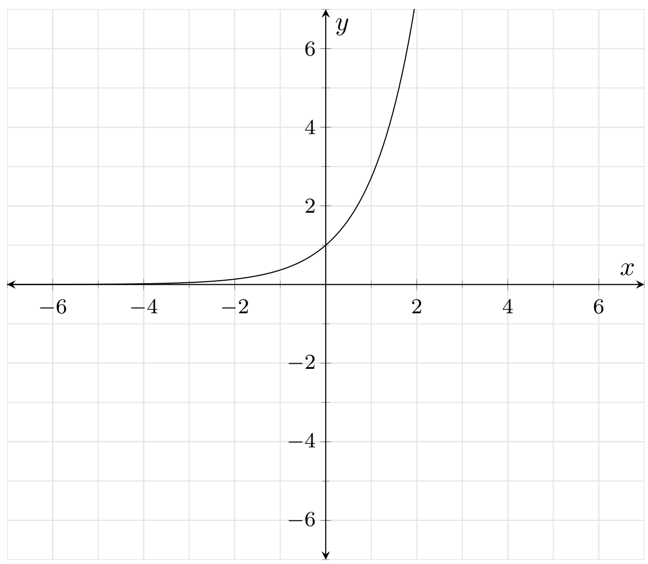
Domain	$[0, \infty)$
Range	$[0, \infty)$

Exponential

Another famous function is the exponential growth function,

$$y = e^x.$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828..$



Important Values of $y = e^x$	
x	y
0	1
1	e
−1	$\frac{1}{e}$

In general, we can talk about exponential functions of the form $y = b^x$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/qsmvb7tiex`

Note that the domain of the exponential functions is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that all non-negative numbers are in the range of the exponential functions.

Table 9: Properties of The Exponential Functions $y = b^x$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Table 10: Domain and Range of The Exponential Functions $y = b^x$

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$

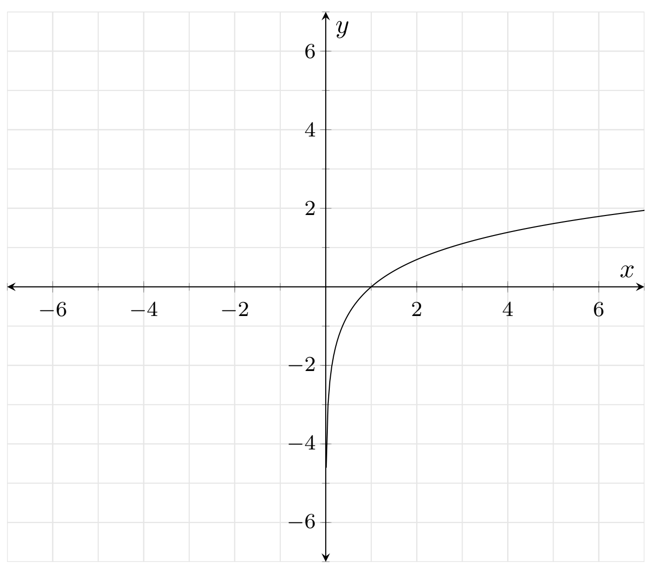
Logarithm

Another group of famous functions are logarithms.

Example 56. *The most famous logarithm function is*

$$y = \ln(x) = \log_e(x).$$

Here e is the mathematical constant known as Euler's number. $e \approx 2.71828$.



Important Values of $y = ln(x)$	
x	y
0	undefined
$\frac{1}{e}$	-1
1	0
e	1

You may notice that the table of values for $y = \ln(x)$ and $y = e^x$ are similiar. This is becase these two functions are interconnected. We will explore this more later in the course.

In general, we can talk about logarithmic functions of the form $y = \log_b(x)$ where b is a positive number not equal to 1. You can play with changing the values of b on the graph using Desmos and see how that changes the graph. Pay particular attention to the difference between $b > 1$ and $0 < b < 1$.

Desmos link: `https://www.desmos.com/calculator/lxllnpdi6w`

Table 11: Properties of The Logarithm Functions $y = \log_b(x)$

Periodic?	No
Odd?	No
Even?	No
One-to-one/invertible	Yes

Note that since the logarithm is the inverse of the exponential, the domain of the logarithms is the range of the exponentials: $[0, \infty)$. Furthermore, the range of the logarithms is the range of the exponentials: $(-\infty, \infty)$.

Table 12: Domain and Range of The Logarithms $y = \log_b(x)$

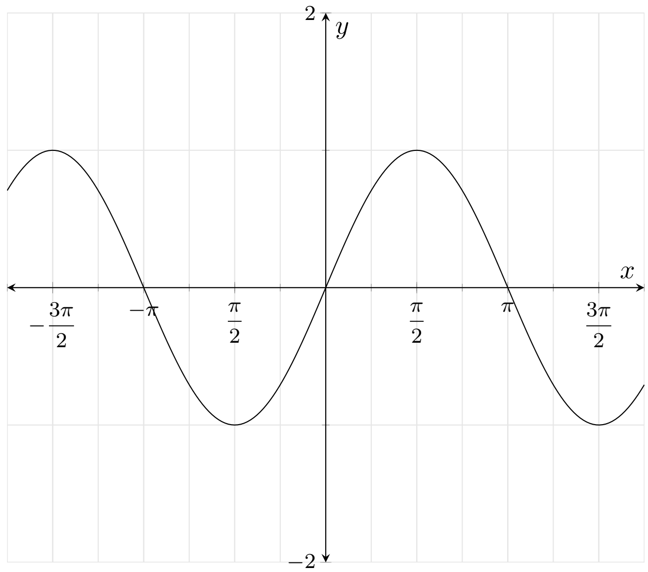
Domain	$[0, \infty)$
Range	$(-\infty, \infty)$

Sine

Another important function is the sine function,

$$y = \sin(x).$$

This function comes from trigonometry. In the table below we will use another mathematical constant, π ("pi" pronounced pie). $\pi \approx 3.14159$.



Important Values of $y = \sin(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Note that the domain of the sine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 13: Properties of The Sine Function $y = \sin(x)$

Periodic?	Yes, with period 2π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Table 14: Domain and Range of The Sine Function $y = \sin(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

In general, we can consider $y = a\sin(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

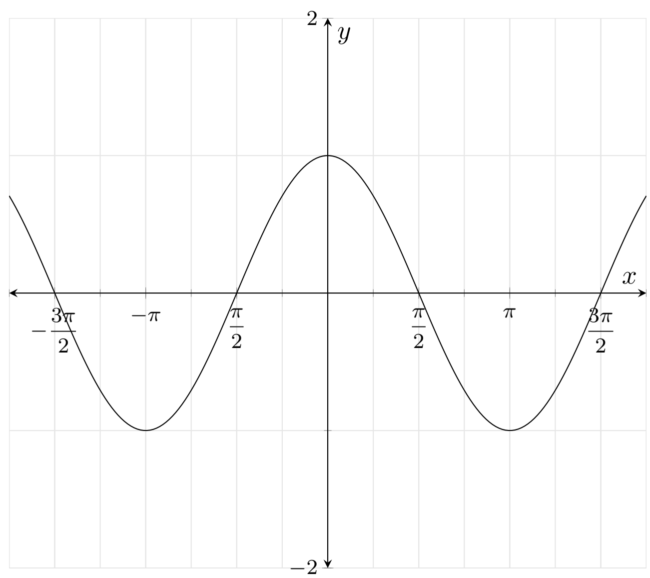
Desmos link: https://www.desmos.com/calculator/vkxzcfv2aq

Cosine

A function introduced in Section 3-2 is the cosine function,

$$y = \cos(x).$$

As with sine, the cosine function comes from trigonometry. In the table below we will again use π.



Important Values of $y = \cos(x)$	
x	y
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

As mentioned earlier, the cosine function is even and periodic with period 2π. Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 15.

Table 15: Properties of The Cosine Function $y = \cos(x)$

Periodic?	Yes, with period 2π
Odd?	No
Even?	Yes
One-to-one/invertible	No

Note that the domain of the cosine function is $(-\infty, \infty)$. Furthermore, by looking at the graph, we can see that its range is $[-1, 1]$

Table 16: Domain and Range of The Cosine Function $y = \cos(x)$

Domain	$(-\infty, \infty)$
Range	$[-1, 1]$

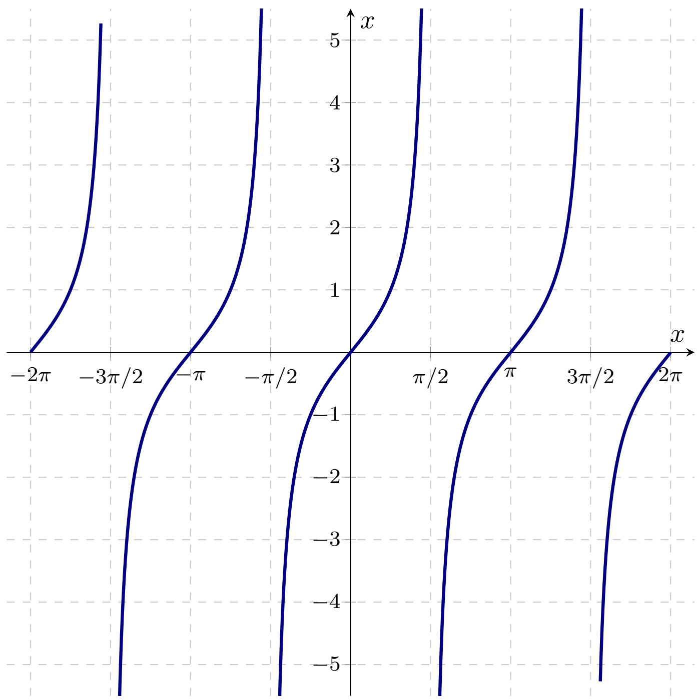
In general, we can consider $y = a\cos(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

Desmos link: `https://www.desmos.com/calculator/kvmz1kt19n`

Tangent

A function introduced in Section 4-1 is the tangent function,

$$y = \tan(x).$$



Important Values of $y = \tan(x)$	
x	y
$-\pi$	0
$-\frac{\pi}{2}$	undefined
0	0
$\frac{\pi}{2}$	undefined
π	0
$\frac{3\pi}{2}$	undefined
2π	0

As mentioned earlier, the tangent function is odd and periodic with period π.

Since it is periodic, however, it cannot be one-to-one, since its values repeat. We summarize some information in Table 17.

Table 17: Properties of The Tangent Function $y = \tan(x)$

Periodic?	Yes, with period π
Odd?	Yes
Even?	No
One-to-one/invertible	No

Note that the domain of the tangent function is all real numbers except for odd multiples of $\frac{\pi}{2}$, since tangent is undefined at those places. Furthermore, by looking at the graph, we can see that its range is $(-\infty, \infty)$.

Table 18: Domain and Range of The Tangent Function $y = \tan(x)$

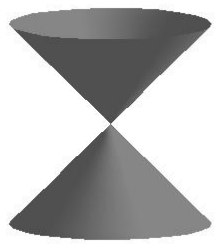
Domain	$\cdots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots$
Range	$(-\infty, \infty)$

In general, we can consider $y = a\tan(bx)$. You can play with changing the values of a and b on the graph using Desmos and see how that changes the graph.

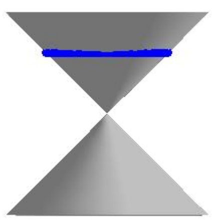
Desmos link: `https://www.desmos.com/calculator/1je3xt6hag`

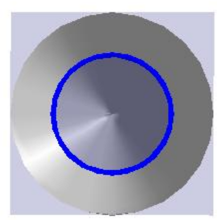
Conic Sections

In this section, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

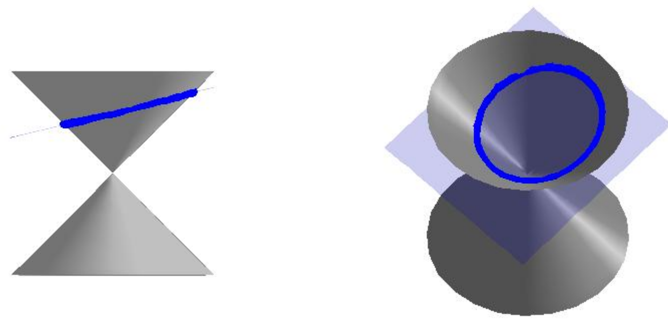


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

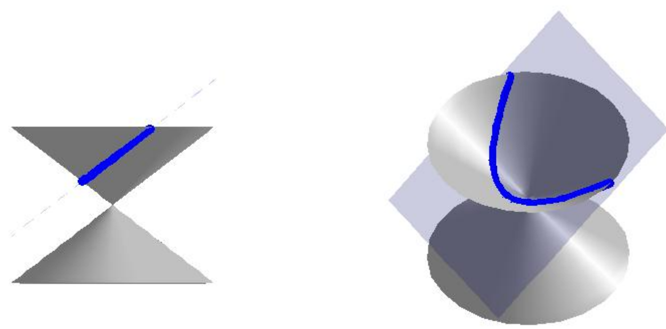




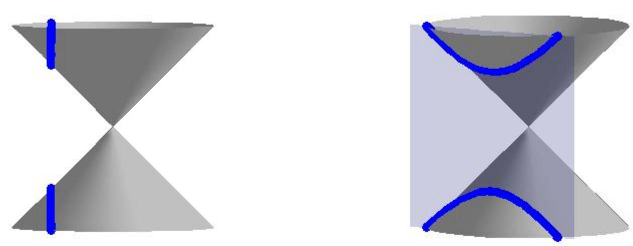
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

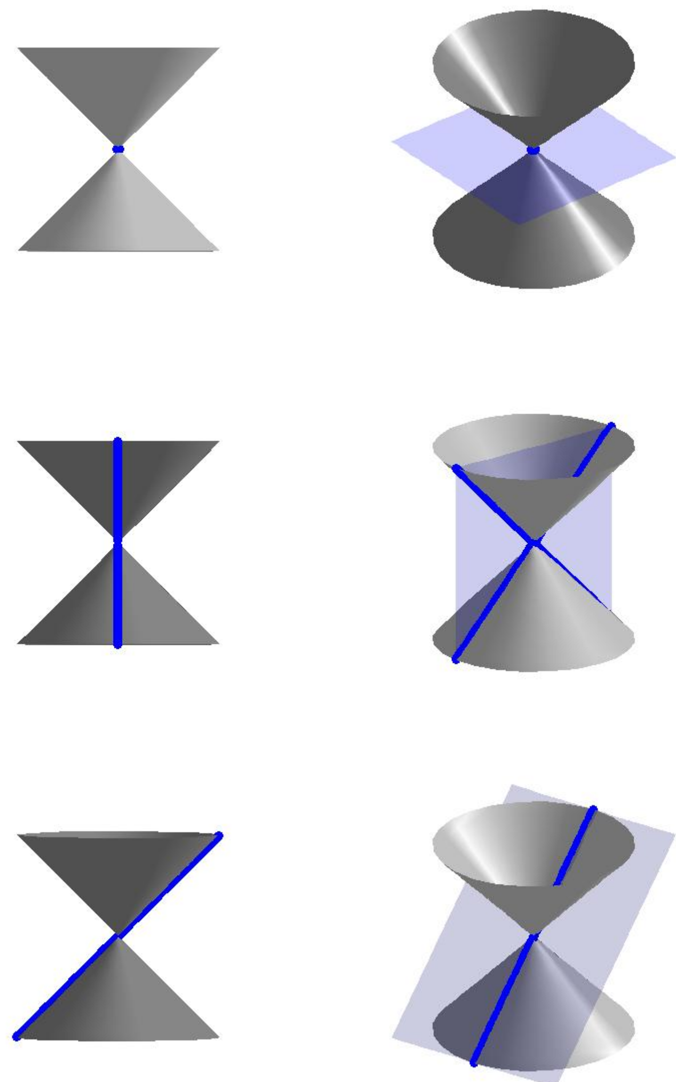


If we slice the cone with a vertical plane, we get a **hyperbola**.



For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

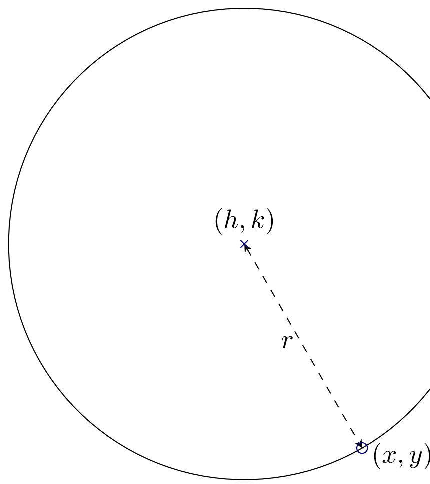


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. It's not necessary to memorize the description of each conic section. We'd just like you to understand that each conic section is the graph of an equation which can be rearranged into a certain *standard equation*. This standard equation is useful, because it allows us to say something about various geometric properties of the graph. In addition, we will only discuss conic sections centered at the origin.

Circles

Recall from Geometry that a circle can be determined by fixing a point (called the *center*) and a positive number (called the *radius*) as follows.

Definition A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



We express this relationship algebraically using the Distance Formula as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Definition **The standard equation of a circle** with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

This is the first example of a standard equation. If we are given a standard equation of a circle, we can easily find its center and its radius, which is all that we need to be able to draw the circle in the xy-plane. In other courses, we would spend a lot of time taking an equation, converting it into a standard equation, recognizing it as the standard equation of a conic section, and then using the information provided by the standard equation to graph the relation. For our purposes, we only need to know that this process can be done.

We close this section with the most important circle in all of mathematics: the *unit circle*.

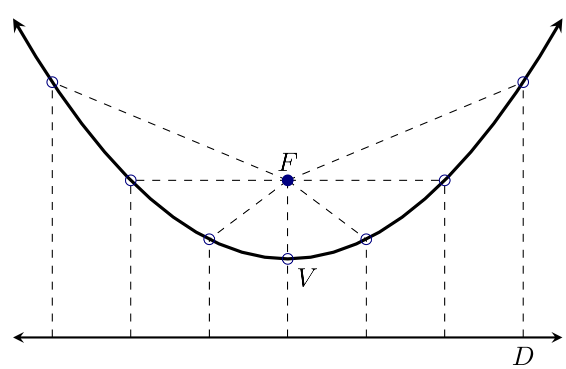
> **Definition** The **unit circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the unit circle is $x^2 + y^2 = 1$.

As you will soon see, the unit circle is central to the study of trigonometry.

Parabolas

We know that parabolas are the graphs of quadratic functions. To our surprise and delight, we may also define parabolas in terms of distance.

> **Definition** Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola. The **vertex** is the point on the parabola closest to the focus.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the vertex. Notice that the focus F is not actually a point on the parabola, but only serves to help in its construction.

As with circles, there is a standard equation for parabolas.

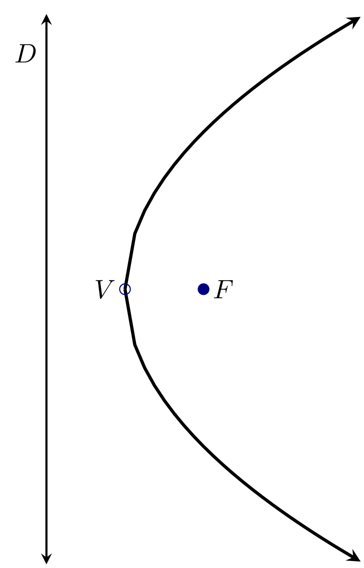
Definition The **standard equation of a parabola** which opens up or down with vertex (h, k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards. The **focal length** of the parabola is the distance from the focus to the vertex.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

Recall from our earlier discussion of inverse functions that interchanging the roles of x and y results in reflecting the graph across the line $y = x$. Therefore, if we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices (plural of 'directrix') of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Definition **The standard equation of a parabola** that opens to the left or right with vertex (h, k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

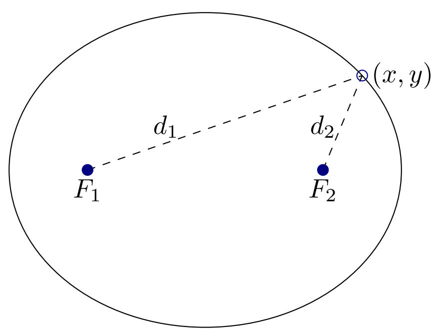
If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Ellipses

In the definition of a circle, we fixed a point called the *center* and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

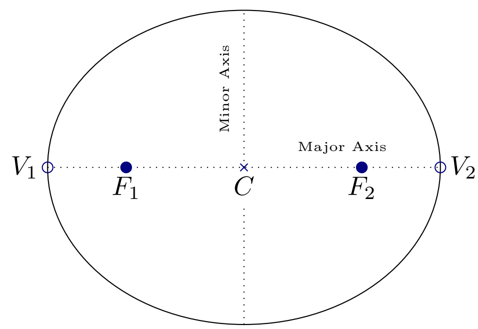
> **Definition** Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** (the plural of 'focus') of the ellipse.

In the figure below, d_1 is the distance from (x, y) to F_1, and d_2 is the distance from (x, y) to F_2. Since (x, y) is on the ellipse, $d_1 + d_2 = d$ for some fixed d.



We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse. Notice again that the foci are not actually points on the ellipse, but only serve to help in its construction.

The *center* of the ellipse is the midpoint of the line segment connecting the two foci. The *major axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The *minor axis* of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The major axis is always the longer of the two segments. The *vertices* of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



There is also a standard equation for ellipses.

Definition For positive unequal numbers a and b, **the standard equation of an ellipse** with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle and divide both sides by r^2, we get

Definition **The alternate standard equation of a circle** with center (h, k) and radius $r > 0$ is

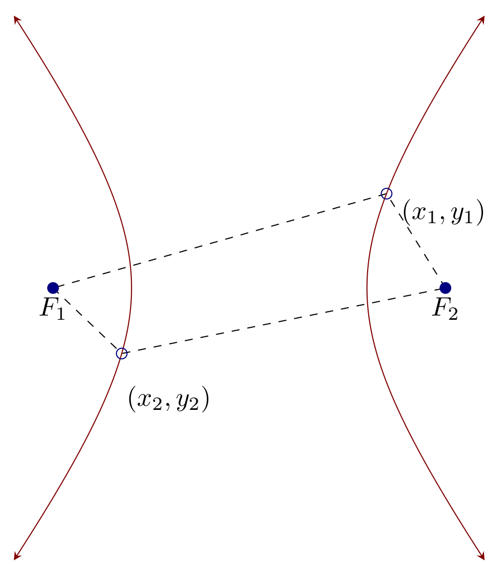
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Notice the similarity between the two equations. Both involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both equations as shifts and stretches of the unit circle $x^2+y^2=1$. Replacing x with $(x-h)$ and y with $(y-k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.

Hyperbolas

In the definition of an ellipse, we fixed two points called foci and looked at points whose distances to the foci always *added* to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced *added* with *subtracted*. The answer is a hyperbola.

Definition Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



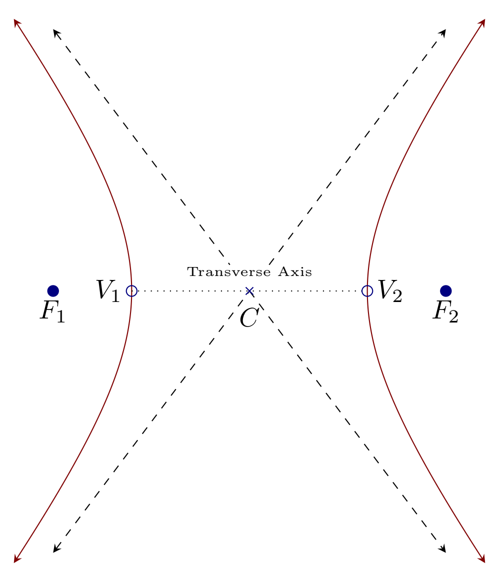
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) = d$$

and

$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) = d$$

Note that the hyperbola has two parts, called *branches*. The *center* of the hyperbola is the midpoint of the line segment connecting the two foci. The *transverse axis* of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The *vertices* of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, there are lines called *asymptotes* which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



The above hyperbola has center C, foci F_1 and F_2, and vertices V_1 and V_2. The asymptotes are represented by dashed lines.

The *conjugate axis* of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes.

As with all the other conic sections, we have a standard equation for hyperbolas.

Definition For positive numbers a and b, **the equation of a hyperbola** opening left and right with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

Definition For positive numbers a and b, **the equation of a hyperbola** opening upwards and downwards with center (h, k) is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a *difference* of squares where the circle and ellipse formulas both involve the *sum* of squares.

5.2.2 Solving Non-linear Systems Graphically

Motivating Questions

- What is a non-linear system of equations?
- How can we find solutions graphically?
- What can we say about when solutions exist?

Introduction

In this section, we study systems of non-linear equations. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far.

What are non-linear systems of equations?

The key to identifying non-linear equations is to note that the variables involved are not necessarily to the first power, and the coefficients of the variables may not just be real numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. An example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 - 9y^2 & = 36 \end{cases}.$$

Note that this system is non-linear because the variables x and y are raised to the second power.

Another example of a non-linear system of equations is given by

$$\begin{cases} x^2 + y^2 & = 4 \\ y - 2x & = 0 \end{cases}.$$

Even though y and x are both raised to the first power in the second equation above, the first equation still contains second powers of variables, so this is a non-linear system.

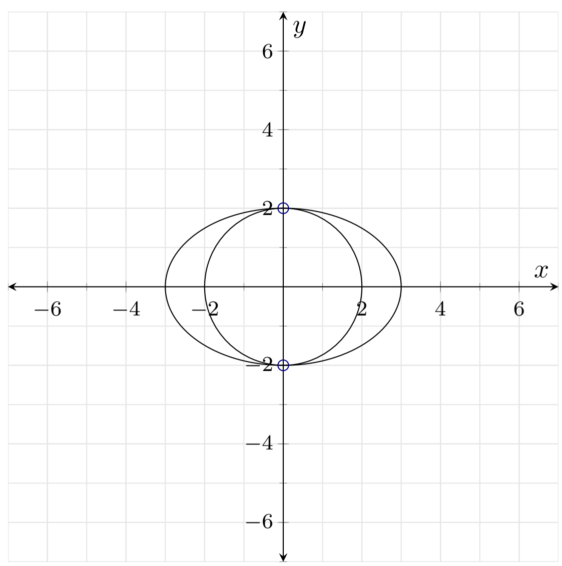
Solving systems graphically

Finding solutions to non-linear systems is the same concept as finding solutions to linear systems. This means that we can also think about finding solutions as finding intersections points of the graphs of the equations in our system.

Example 57. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 + 9y^2 &= 36 \end{cases}.$$

Explanation We sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ can be recognized as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4. This is illustrated in the figure below.



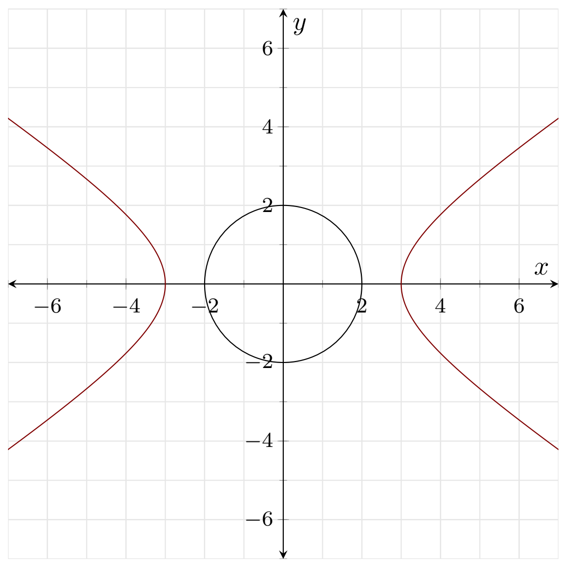
We see from the figure that the two graphs intersect at their y-intercepts only, $(0, 2)$ and $(0, -2)$. Recalling that points of intersection correspond to solutions to the system of equations, $(0, 2)$ and $(0, -2)$ are the only solutions to the system.

Example 58. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 - 9y^2 &= 36 \end{cases}.$$

Explanation First, notice that this system only differs from the previous example in that it has a minus sign in front of the $9y^2$ in the bottom equation.

We again sketch the graphs of both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, as in the previous example. However, the graph of $4x^2 - 9y^2 = 36$, when written in the standard form $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ can be recognized as a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. This is illustrated in the figure below.



We see that the circle and the hyperbola have no points in common. Recalling that points of intersection correspond to solutions to the system of equations, we say that the system has no solutions.

Note that we can characterize systems of nonlinear equations as being consistent or inconsistent, just like their linear counterparts. Unlike systems of linear equations, however, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. Secondly, as we have seen above, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the graphs of equations in a non-linear system may not be easily visualized, it sometimes pays to take advantage of visualization when you are able.

5.2.3 Eliminating Variables

Solving Non-linear Systems Algebraically

Algebraically, we can use the methods of substitution and elimination outlined in Section 8.1 to solve non-linear systems of equations. However, we need to exercise care when solving non-linear systems, especially since the operations involved may not always result in valid solutions!

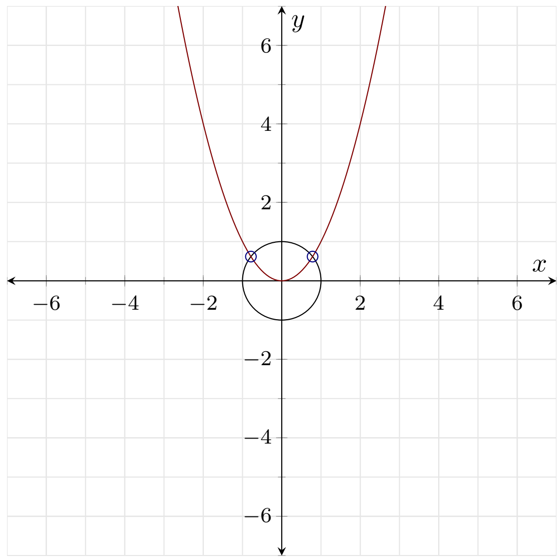
For example, consider the system given by

$$\begin{cases} y - x^2 &= 0 \\ x^2 + y^2 &= 1 \end{cases}.$$

Let's try to use substitution here. From the top equation, we can see that $y = x^2$. Substituting this into the bottom equation results in $x^2 + (x^2)^2 = 1$, or $x^4 + x^2 = 1$, which we can rewrite as $x^4 + x^2 - 1 = 0$. We can now use the quadratic formula on $x^4 + x^2 - 1$ to find that $x^2 = \frac{-1 \pm \sqrt{5}}{2}$. Taking a square root, we find that $x = \pm\sqrt{\frac{-1 \pm \sqrt{5}}{2}}$ are possible values of x. Note that there are actually *four* separate possible values of x, one for each choice of plus or minus in the expression above: $\sqrt{\frac{-1+\sqrt{5}}{2}}$, $\sqrt{\frac{-1-\sqrt{5}}{2}}$, $-\sqrt{\frac{-1+\sqrt{5}}{2}}$, and $-\sqrt{\frac{-1-\sqrt{5}}{2}}$.

However, $\frac{-1-\sqrt{5}}{2}$ is actually negative! Since the square root of a negative number is not a real number, $\sqrt{\frac{-1-\sqrt{5}}{2}}$ and $-\sqrt{\frac{-1-\sqrt{5}}{2}}$ are not valid x-values of a solution to this system. The other two solutions are fine. Therefore, keeping in mind that $y = x^2$, the solutions to our system are given by $\left(\sqrt{\frac{-1+\sqrt{5}}{2}}, \frac{-1+\sqrt{5}}{2}\right)$ and $\left(-\sqrt{\frac{-1+\sqrt{5}}{2}}, \frac{-1+\sqrt{5}}{2}\right)$. We can plug these back into the original equations to make sure that they satisfy both.

Taking a look at the graphs of these equations should shed some light on what's happening here.



The only intersection points of the two graphs have a positive y-coordinate. This could have tipped us off earlier that some of the x-values we got wouldn't be valid. Indeed, from the first equation, we have that $x = \sqrt{y}$, and this ensures that y must be positive.

The above example illustrates the importance of always checking that the solutions you find are real numbers, and also checking that the solutions you find are actually solutions to the system.

Eliminating Variables

Now we illustrate the method of elimination, which can be used when you notice that the equations in the system have like terms. The difference from before is that we now may have non-linear terms that we can eliminate.

Let's apply this technique to a system we saw previously.

Example 59. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 & = 4 \\ 4x^2 + 9y^2 & = 36 \end{cases}.$$

Explanation We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 & = -16 \\ 4x^2 + 9y^2 & = 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $5y^2 = 20$. From here, we see that $y^2 = 4$, so $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution set is thus $\{(0, 2), (0, -2)\}$.

Example 60. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y^2 &= 4 \\ 4x^2 - 9y^2 &= 36 \end{cases}.$$

Explanation We proceed as before to eliminate one of the variables. We can multiply the top equation by -4, so we get the equivalent system of equations

$$\begin{cases} -4x^2 - 4y^2 &= -16 \\ 4x^2 - 9y^2 &= 36 \end{cases}.$$

Now we can eliminate the x^2 terms to obtain $-13y^2 = 20$. From here, we see that $y^2 = -\dfrac{20}{13}$. Since the square root of a negative number is not a real number, we see that there are no real values of y that solve this equation. Therefore, we conclude that this system has no solution. Recall that a system that has no solution is called inconsistent.

Example 61. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + 2xy - 16 &= 0 \\ y^2 + 2xy - 16 &= 0 \end{cases}.$$

Explanation At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome xy term, and the constant term as well, by elimination and doing so we get a more tractable relationship between x and y. We can multiply the top equation by -1, so we get the equivalent system of equations

$$\begin{cases} -x^2 - 2xy + 16 &= 0 \\ y^2 + 2xy - 16 &= 0 \end{cases}.$$

Eliminating, we find that $y^2 - x^2 = 0$, so $y^2 = x^2$, and $y = \pm x$. Substituting $y = x$ into the top equation, we get $x^2 + 2x^2 - 16 = 0$, so that $x^2 = \dfrac{16}{3}$ or $x = \pm\dfrac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into the top equation, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$, which gives no real solutions. Substituting each of $x = \pm\dfrac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution set $\left\{\left(\dfrac{4\sqrt{3}}{3}, \dfrac{4\sqrt{3}}{3}\right), \left(-\dfrac{4\sqrt{3}}{3}, -\dfrac{4\sqrt{3}}{3}\right)\right\}$. Try plugging these into the original system to see that they are actually solutions. Verifying this graphically would be a fun exercise, but we leave that up to you.

Some Common Issues and Techniques

Example 62. *Find all solutions to the following system of equations:*

$$\begin{cases} x^2 + y & = 12 \\ 3xy & = 0 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since the second equation contains an xy term. Since we can't see any like terms in the two equations, it makes sense to try to use substitution. We might be tempted to divide both sides of the bottom equation by $3x$, so as to isolate y, but as always with division, we need to be careful! Indeed, $x = 0$ is still a possibility, so we cannot divide through by $3x$, since we'd then be dividing by 0.

Instead, it helps to think about what it means for the product of two numbers to equal 0. In fact, the product of two nonzero numbers can never be 0. In our situation, we know that $3xy = 0$, so either $3x = 0$ or $y = 0$.

If $3x = 0$, then dividing by 3 (since $3 \neq 0$) gives us $x = 0$. We can plug that into the top equation and find that $0^2 + y = 12$, so $y = 12$. We can then check that $(0, 12)$ is a solution to our original system.

If $y = 0$, we can plug that into the top equation to find that $x^2 + 0 = 12$. Solving for x yields $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. We can then check that $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$ are solutions to the system.

Our final solution set is $\{(2\sqrt{3}, 0), (-2\sqrt{3}, 0), (0, 12)\}$.

Example 63. *Find all solutions to the following system of equations:*

$$\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} & = 1 \\ \\ \dfrac{3}{x} + \dfrac{2}{y} & = -1 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$ then the system becomes

$$\begin{cases} 4u + 3v & = 1 \\ 3u + 2v & = -1 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = -5$ and $v = 7$. Therefore, $x = \dfrac{1}{u} = -\dfrac{1}{5}$ and $y = \dfrac{1}{v} = \dfrac{1}{7}$, and our solution set is $\left\{\left(-\dfrac{1}{5}, \dfrac{1}{7}\right)\right\}$.

We say that the original system is linear in form because its equations are not linear, but a few substitutions reveal a structure that we can treat like a system of linear equations. However, the substitutions may introduce some complexity, as seen in the following example.

Example 64. *Find all solutions to the following system of equations:*

$$\begin{cases} 5e^x + 3e^{2y} & = 1 \\ \\ 3e^x + e^{2y} & = 2 \end{cases}.$$

Explanation Notice that this is, in fact, a non-linear system, since both equations divide by the variables we're using.

If we define new (but related) variables by letting $u = e^x$ and $v = e^{2y}$ then the system becomes

$$\begin{cases} 5u + 3v & = 1 \\ 3u + v & = 2 \end{cases}.$$

This associated system of linear equations can then be solved using any of the techniques you've learned earlier to find that $u = \dfrac{3}{4}$ and $v = -\dfrac{1}{4}$. Therefore, $x = \ln(u) = \ln\left(\dfrac{3}{4}\right)$ and $y = \dfrac{\ln(v)}{2} = \dfrac{\ln\left(-\frac{1}{4}\right)}{7}$. However, the nonlinearity of the system throws us a wrench! Logarithms are not defined on negative numbers, so $\dfrac{\ln\left(-\frac{1}{4}\right)}{7}$ does not exist, and there is actually no value of y that satisfies both equations. Therefore, this system does not have a solution.

Exploration Consider the following system.

$$\begin{cases} 4\ln(x) + 3y^2 & = 1 \\ 3\ln(x) + 2y^2 & = -1 \end{cases}.$$

(a) Is the system linear in form?

(b) If so, make substitutions by defining variables u and v so that the system in terms of u and v is linear. What is u? What is v? What is our new associated linear system?

(c) What is the solution set to our associated linear system?

(d) What is the solution set to our original system?

5.3 Applications of Systems

Learning Objectives

- Applications of Systems
  - Word problems with extraneous variables
  - Word problems similar to related rates or optimization

5.3.1 Applications of Systems

Introduction

Suppose a rectangle has width w and length l, with area 24 and perimeter 20. The area of the rectangle is wl and the perimeter is given by $2w+2l$, giving the following system of equations

$$\begin{cases} wl &= 24 \\ 2w+2l &= 20. \end{cases}$$

Since the first equation here is not a linear equation, this is a nonlinear system of equations. If we want to find the dimensions of the corresponding rectangle, we must solve this system. Since calculations of areas and volumes are nonlinear in general, situations involving geometric shapes often result in nonlinear systems of equations.

To solve this system, we will start by dividing both sides of the second equation by 2, to obtain the following equivalent system.

$$\begin{cases} wl &= 24 \\ w+l &= 10. \end{cases}$$

If this second equation is satisfied, that means $l = 10 - w$, which can be substituted into the top equation to eliminate the variable l.

$$\begin{aligned} wl &= 24 \\ w(10-w) &= 24 \\ 10w - w^2 &= 24 \\ w^2 - 10w + 24 &= 0 \\ (w-6)(w-4) &= 0. \end{aligned}$$

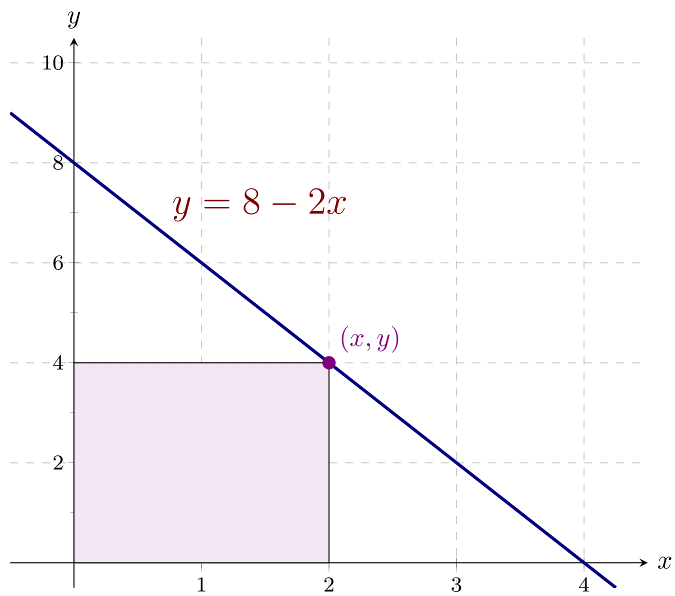
The $w-6$ factor gives a solution of $w = 6$, and the $w-4$ factor gives a solution of $w = 4$.

Looking back at $l = 10 - w$ we see that if $w = 6$, then $l = 10 - 6 = 4$ and if $w = 4$ then $l = 10 - 4 = 6$.

There are two possible rectangles: One with width 6 and length 4, and the other with width 4 and length 6.

Applications of Systems

Exercise 3 *A rectangle is drawn in the first quadrant with one side along the x-axis, one side along the y-axis, the lower left corner at the origin, and* *upper right corner on the graph of the equation $y = 8 - 2x$. Denote this* upper right *vertex as (x, y). Find the coordinates of the point (x, y) if the area of the rectangle is $\frac{15}{2}$.*



Explanation Since (x, y) are the coordinates of the upper right vertex, this tells us that x and y are both positive. It also tells us that the distance from the x-axis is y, and the distance from the y-axis is x. In other words, the height of the rectangle is just y, and the width of the rectangle is x. In terms of x and y, the area is given by xy, giving us one equation $xy = \frac{15}{2}$.

Since the upper right corner (x, y) is on the graph of the line, we also know that $y = 8 - 2x$. This leaves us with the following system: This gives a system of nonlinear equations

$$\begin{cases} xy &= \frac{15}{2} \\ y &= 8 - 2x. \end{cases}$$

This bottom equation is already solved for y, so the easiest way to eliminate a variable would be to substitute it into the y in the top equation.

$$\begin{aligned} xy &= \frac{15}{2} \\ x(8-2x) &= \frac{15}{2} \\ 8x - 2x^2 &= \frac{15}{2} \\ 2x^2 - 8x &= -\frac{15}{2} \\ x^2 - 4x &= -\frac{15}{4}. \end{aligned}$$

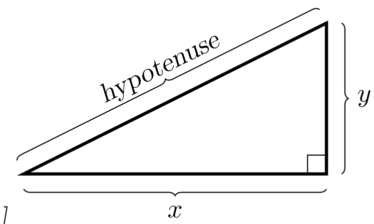
We'll solve this equation by completing the square.

$$\begin{aligned} x^2 - 4x &= -\frac{15}{4} \\ x^2 - 4x + 4 &= -\frac{15}{4} + 4 \\ x^2 - 4x + 4 &= \frac{1}{4} \\ (x-2)^2 &= \frac{1}{4} \\ x - 2 &= \pm\sqrt{\frac{1}{4}} \\ x - 2 &= \pm\frac{1}{2} \\ x &= 2 \pm \frac{1}{2} \\ x &= \frac{3}{2}, \frac{5}{2} \end{aligned}$$

If $x = \frac{3}{2}$ then $y = 8 - 2\left(\frac{3}{2}\right) = 5$, and if $x = \frac{5}{2}$ then $y = 8 - 2\left(\frac{5}{2}\right) = 3$.

There are two possibilities. One has coordinates $\left(\frac{3}{2}, 5\right)$ and the other has coordinates $\left(\frac{5}{2}, 3\right)$.

Exercise 4 *A right triangle has hypotenuse of length $13m$ and area $30m^2$. Find the lengths of the two legs of the triangle.*



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Explanation

Since this is a right triangle, the Pythagorean Theorem tells us that $x^2 + y^2 = 13^2 = 169$. The area of a triangle is given by $\frac{1}{2} \times \text{base} \times \text{height}$ which means $\frac{1}{2}xy = 30$, or equivalently $xy = 60$.

This gives a system of nonlinear equations

$$\begin{cases} x^2 + y^2 & = 169 \\ xy & = 60. \end{cases}$$

A direct way to solve this system of equations would be to solve the bottom equation for y, giving $y = \frac{60}{x}$, and substitute this into the top equation eliminating the y variable. After simplification that will yield a degree 4 polynomial equation to solve for x. Instead of following that method, we will make use of a different algebraic trick.

We know that there is a difference between $x^2 + y^2$ and $(x + y)^2$. If we multiply out $(x + y)^2$ we get $x^2 + 2xy + y^2$. That means if we add $2xy$ to $x^2 + y^2$, it becomes $(x + y)^2$. To make use of that, we need to know the value of $2xy$ so we can add it to the other side of our top equation. Notice that the bottom equation of our system $xy = 60$ means that $2xy = 2(60) = 120$. If the bottom equation is satisfied, the top equation can be rewritten as:

$$\begin{aligned} x^2 + y^2 &= 169 \\ x^2 + y^2 + 2xy &= 169 + 2xy \\ x^2 + 2xy + y^2 &= 169 + 120 \\ (x + y)^2 &= 289. \end{aligned}$$

Taking square roots of both sides gives $|x + y| = \sqrt{289} = 17$. That is, $x + y = \pm 17$. Neither x nor y can be negative (since they denote lengths of the sides of this triangle), this results in $x + y = 17$. Our system of equations is equivalent to:

$$\begin{cases} x + y & = 17 \\ xy & = 60. \end{cases}$$

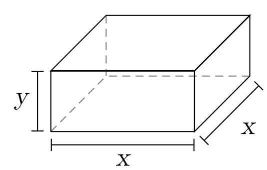
Now that we've been able to simplify the first equation, we will proceed with substitution as mentioned above. If $y = \dfrac{60}{x}$ is satisfied, then the top equation gives:

$$\begin{aligned} x + y &= 17 \\ x + \frac{60}{x} &= 17 \\ x\left(x + \frac{60}{x}\right) &= x(17) \\ x^2 + 60 &= 17x \\ x^2 - 17x + 60 &= 0 \\ (x-12)(x-5) &= 0. \end{aligned}$$

The $x-12$ factor yields a solution of $x = 12$, and the $x-5$ factor gives a solution of $x = 5$. Using $y = \dfrac{60}{x}$ we find that if $x = 12$ then $y = \dfrac{60}{12} = 5$, and if $x = 5$ then $y = \dfrac{60}{5} = 12$.

The two legs of the triangle have lengths $5m$ and $12m$.

Exercise 5 *Suppose we have a box with square base, as illustrated below, constructed to have volume $8cm^3$ and surface area $24cm^2$. Call the side-lengths of the base as x, and the height of the box as y.*



Find the dimensions of the box.

Explanation We know that the volume of the box is given by x^2y (length $\times$ width $\times$ height), giving the nonlinear equation $x^2y = 8$. The surface of the box consists of six rectangles. The top and bottom each have area x^2, and the four sides each have area xy. The full surface area of the box is given by $2x^2 + 4xy$. This setup gives us a system of two nonlinear equations with two unknowns:

$$\begin{cases} x^2y &= 8 \\ 2x^2 + 4xy &= 24. \end{cases}$$

If we want to find the dimensions of the box, we will have to solve this system of equations.

As you have seen in the previous section, solving systems of nonlinear equations involves finding a way to eliminate one of the variables by performing operations on the two equations and/or using substutition. In the case of these equations, notice that the variable y only occurs in a single term in each equation. In the top equation there is an x^2y term, while in the bottom equation there is an xy term. These are not like terms, so we will need to deal with that. Let us begin by multiplying both sides of the second equation by x. This gives the system

$$\begin{cases} x^2y & = 8 \\ 2x^3 + 4x^2y & = 24x. \end{cases}$$

Since no solution has x-coordinate equal to 0, this system is equivalent to the original one. This modification has given that the y variable appears in like terms in boh equations. Substituting $x^2y = 8$ into the new bottom equation gives:

$$\begin{aligned} 2x^3 + 4x^2y &= 24x \\ 2x^3 + 4\left(x^2y\right) &= 24x \\ 2x^3 + 4\,(8) &= 24x \\ 2x^3 + 32 &= 24x \\ 2x^3 - 24x + 32 &= 0 \\ x^3 - 12x + 16 &= 0. \end{aligned}$$

That is, if the $x^2y = 8$ equation is satisfied, then the bottom equation of the system is equivalent to $x^3 - 12x + 16 = 0$. This is a polynomial equation in the single variable, x. (Notice that if we had taken the original top equation $x^2y = 8$, solved it for y to obtain $y = \dfrac{8}{x^2}$, and substituted that into the original bottom equation, we would have arrived at this exact same result.)

Notice that $2^3 - 12(2) + 16 = 8 - 24 + 16 = 0$. That means $x = 2$ is a solution to this cubic equation, and that $x - 2$ is a factor of the polynomial $x^3 - 12x + 16$. By long-division we can find that $x^3 - 12x + 16 = (x - 2)(x^2 + 2x - 8)$. Since $x^2 + 2x - 8 = (x + 4)(x - 2)$ we see that $x^3 - 12x + 16 = (x - 2)^2(x + 4)$. The zeroes of this polymial are $x = 2$ and $x = -4$. Since x represents a length of the side of the box, the $x = -4$ solution is extraneous and should be dropped.

The only solution to the system has $x = 2$. Looking back at the first equation of the system:

$$\begin{aligned} x^2y &= 8 \\ (2)^2y &= 8 \\ 4y &= 8 \\ y &= 2. \end{aligned}$$

The solution is for $(x, y) = (2, 2)$. Since the question asks us to find the dimensions of the box, we say that the box is $2cm \times 2cm \times 2cm$. That is, it's a cube with side length $2cm$.

Part 6

Preparing for Calculus

6.1 Average Rate of Change: Difference Quotients

Learning Objectives

- Secant Lines
  - Definition
  - Finding secant lines
  - Applications
- Difference Quotients
  - Average rate of change when one or both points are given as letters
  - Simplify with algebra (early examples)
  - Finding slopes of secant lines

6.1.1 Average Rate of Change and Secant Lines

Motivating Questions

- What does a line passing through two points of a function represent?
- How does this inform our understanding of the function?

Introduction

We begin by recalling the definitions of *average rate of change* of a function and *secant line* to the graph of a function.

Definition For a function f defined on an interval $[a, b]$,

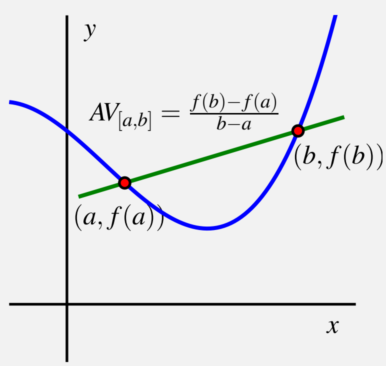
- the **average rate of change of f on** $[a, b]$ is the quantity

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

- a **secant line** to the graph of f is a line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$.

  Recall: The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

This is illustrated in the figure below, where the green line (between the red points on the graph) is the secant line of f from $(a, f(a))$ to $(b, f(b))$.



Recall that given two points (x_0, y_0) and (x_1, y_1) in the plane, with $x_0 \neq x_1$, we can find the equation of the line passing through them by using the slope ("rise-over-run"):

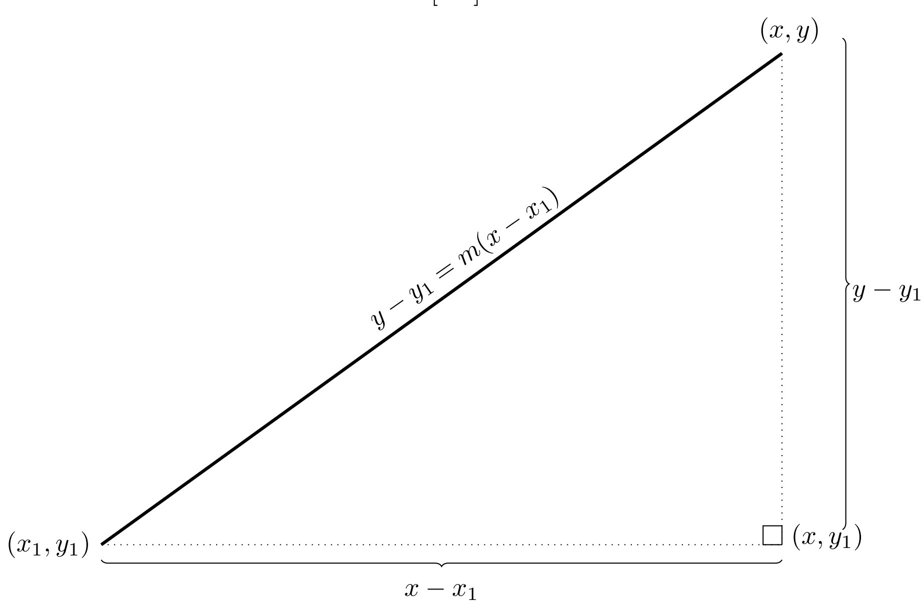
$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Then the line equation is given by $y - y_0 = m(x - x_0)$, simply because any given point (x, y) in such line must realize the *same* slope:

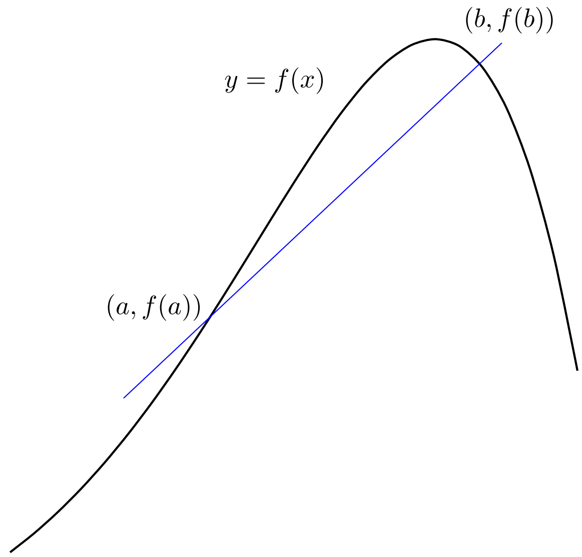
$$m = \frac{y - y_0}{x - x_0}.$$

Of course, one may also use the point (x_1, y_1) instead of (x_0, y_0) and consider the equation $y - y_1 = m(x - x_1)$, as it describes the same line.

[3in]



With this in place, we'll focus on the situation where two such points lie in the graph of some function $y = f(x)$.



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Definitions and examples

> **Definition:** Consider a function $y = f(x)$. A line passing through two points $(a, f(a))$ and $(b, f(b))$, with $a \neq b$, in the graph of $y = f(x)$, is called a **secant line** to the graph.
> **Recall:** The slope of a secant line is the average rate of change of the function on the interval $[a, b]$.

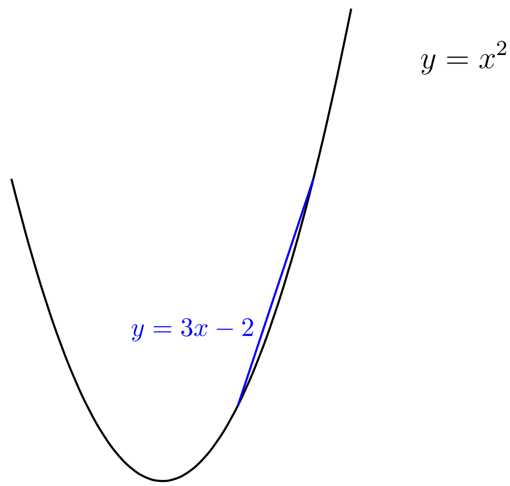
Example 65. *On the following situations, given a function $y = f(x)$ and two points in the graph, find the equation of the secant line they determine.*

a. $f(x) = x^2$, *points* $(1, f(1))$ *and* $(2, f(2))$.

   Explanation First, we have that $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$, so the points given are actually $(1, 1)$ and $(2, 4)$. So

   $$m = \frac{4-1}{2-1} = 3$$

   means that the line equation we're looking for is $y - 1 = 3(x - 1)$, which may be rewritten simply as $y = 3x - 2$.



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b. $f(x) = \sin x$, *points* $(\pi/4, f(\pi/4))$ *and* $(3\pi/4, f(3\pi/4))$.

Explanation This time, we have that $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ and also that $f(3\pi/4) = \sin(3\pi/4) = \sqrt{2}/2$, so the points given were in fact $(\pi/4, \sqrt{2}/2)$ and $(3\pi/4, \sqrt{2}/2)$. Hence the slope of the secant line is

$$m = \frac{\sqrt{2}/2 - \sqrt{2}/2}{3\pi/4 - \pi/4} = \frac{0}{\pi/2} = 0,$$

so the line equation $y - \sqrt{2}/2 = 0(x - \pi/4)$ boils down to $y = \sqrt{2}/2$. Again, you should see $y = \sqrt{2}/2$ not as a single value of y, but as a line equation for which it just happens that x does not appear — thus describing a horizontal line.

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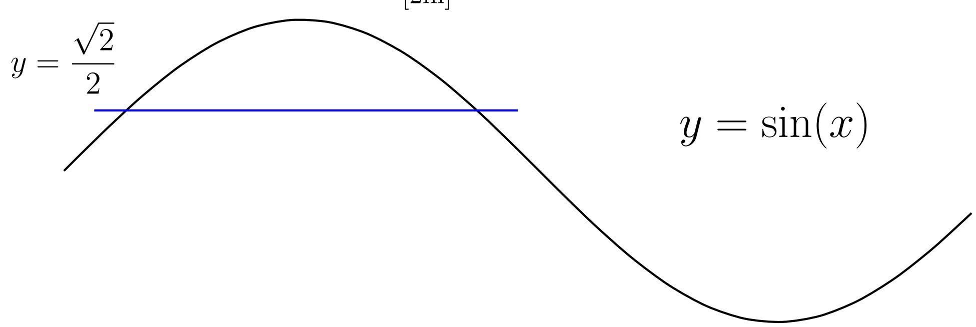


c. $f(x) = 2x + 3$, *points* $(-1, f(-1))$ *and* $(3, f(3))$.

Explanation Now, we have $f(-1) = 2(-1) + 3 = 1$ and $f(3) = 2 \cdot 3 + 3 = 9$, so the points given were $(-1, 1)$ and $(3, 9)$. The slope these points determine is

$$m = \frac{9-1}{3-(-1)} = \frac{8}{4} = 2,$$

so we obtain $y - 1 = 2(x - (-1))$, which can be rewritten as $y = 2x + 3$. This is not a coincidence! The secant line to the graph of a line must be the line itself. This is because a line is determined by two points, and since both the original line and the secant line must share the two given points, they must be, in fact, equal.

6.1.2 Slopes of Secant Lines as a Function of h

Motivating Questions

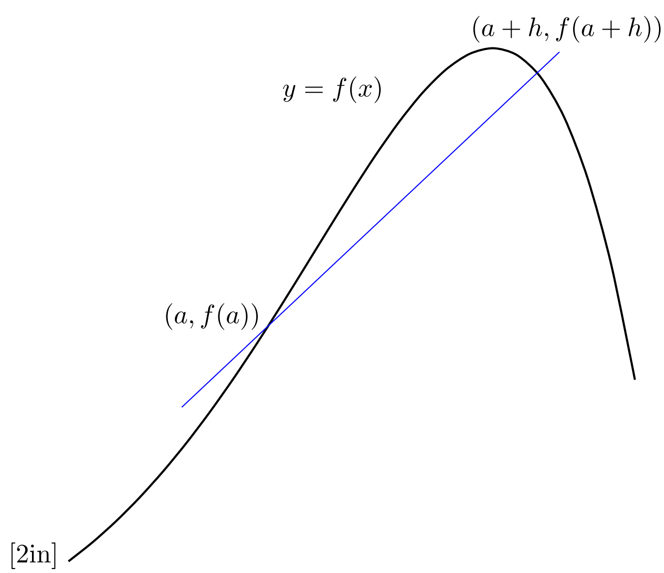
- How can we rewrite the average rate of change of a function in terms of the horizontal distance, h, between the points? Why would we want to do that?
- What are some algebra techniques that allow us to simplify the average rate of change for an arbitrary h?
- What does it tell us when we put in small values for h?

Introduction

We have discussed a secant line to the graph of a function $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$, and the fact that the slope, m, of this line is the average rate of change of the function f on the interval $[a, b]$, $AV_{[a,b]}$. Furthermore, recall that we can let $h = b - a$, so that the slope expression becomes

$$m = \frac{f(a+h) - f(a)}{h},$$

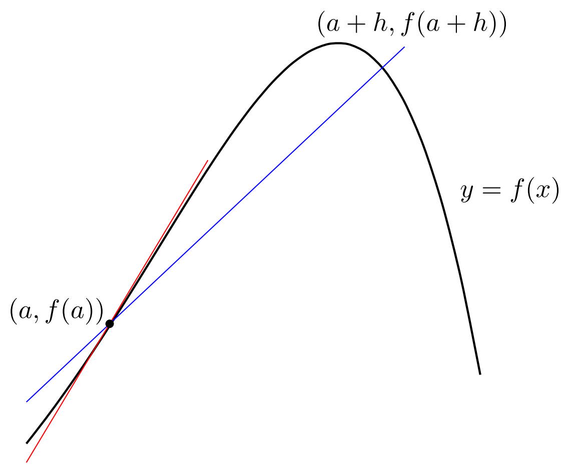
where h is the horizontal distance between the points.



One of the main objectives of Calculus is to understand instantaneous rates of change, as opposed to average rates of change. Namely, what is the behavior of the expression

$$\frac{f(a+h)-f(a)}{h}$$

when h gets very small? Geometrically, making h become very small is making the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ approach a certain line — the tangent line to the graph of $y = f(x)$ at the point a. This is demonstrated in the figure below, where the secant line is blue, and the line tangent to the graph at a is in red.



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Follow the link below to an example Desmos graph where you can see the effect of changing the value of h on the secant line in real-time. Desmos link: `https://www.desmos.com/calculator/f6fh2wkrrn`

The slope of such a tangent line, when it exists, is called the derivative of f at a. Here, we'll discuss difference quotients and several examples, to prepare you to learn those things in more detail in a future Calculus class.

Definitions and examples

Definition: The difference quotient of a function $y = f(x)$ at a point a of its domain is the quantity

$$\frac{f(a+h)-f(a)}{h},$$

i.e., the average rate of change of f on the interval $[a, a+h]$.

Example 66. *Find the difference quotients of the following functions, at the given point.*

a. $f(x) = x^2$, $a = 2$.

Explanation Let's evaluate it directly:

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{(2+h)^2-2^2}{h} = \frac{2^2+4h+h^2-2^2}{h}\\ &= \frac{4h+h^2}{h} = h+4.\end{aligned}$$

We thus have an equation for the slope of the secant line from $(2, f(2))$ to $(2+h, f(2+h))$:

$$AV_{[2,2+h]} = m = h + 4.$$

Recall from Example 1(a) of Section 12-3-1, that we calculated the slope of the secant line from $(1, f(1))$ to $(2, f(2))$. Letting $h = -1$, we see that this gives the same answer for the slope of that secant line.

Furthermore, if we now let $h \to 0$, this expression, $\dfrac{f(2+h)-f(2)}{h} \to 4$.

b. $f(x) = \sin(x)$, $a = \dfrac{\pi}{3}$.

Explanation Again, we evaluate directly:

$$\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} = \frac{\sin(\frac{\pi}{3}+h)-\sin(\frac{\pi}{3})}{h}$$

Recognizing that we cannot further simplify this expression in its current form, we replace $\sin((\pi/3)+h)$ using the sine sum expression:

$$\begin{aligned}\frac{\sin(\frac{\pi}{3}+h)-\sin(\frac{\pi}{3})}{h} &= \frac{\sin(\frac{\pi}{3})\cos(h)+\cos(\frac{\pi}{3})\sin(h)-\frac{\sqrt{3}}{2}}{h}\\ &= \frac{1}{2h}\big(\sqrt{3}(\cos(h)-1)+\sin(h)\big)\end{aligned}$$

This gives a less easy to visualize equation for the slope of the secant line from $(a, f(a))$ to $(a+h, f(a+h))$, for $a = \dfrac{\pi}{3}$:

$$AV_{[\frac{\pi}{3},\frac{\pi}{3}+h]} = m = \frac{1}{2h}\big(\sqrt{3}(\cos(h)-1)+\sin(h)\big).$$

However, consider $h = \dfrac{\pi}{2}$, so that we are looking at the secant line from $(\pi/3, f(\pi/3))$ to $(5\pi/6, f(5\pi/6))$. We then see that

$$\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} = \frac{1}{2}\cdot\frac{\pi}{2}\big(\sqrt{3}(\cos\left(\frac{\pi}{2}\right)-1)+\sin\left(\frac{\pi}{2}\right)\big) = 0,$$

as before.

Furthermore, what happens as we let $h \to 0$. Consider $h = \dfrac{\pi}{6}$, then we have

$$\begin{aligned}\frac{f(\frac{\pi}{3}+h)-f(\frac{\pi}{3})}{h} &= \frac{1}{2}\cdot\frac{6}{\pi}(\sqrt{3}(\cos\left(\frac{\pi}{6}\right)-1)+\sin\left(\frac{\pi}{6}\right)) \\ &= \frac{3}{\pi}\Big(\sqrt{3}\Big(\frac{\sqrt{3}}{2}-1\Big)+\frac{1}{2}\Big) \\ &= \frac{6-3\sqrt{3}}{\pi},\end{aligned}$$

Note that this is greater than 0. Think about the graph of $y = sin(x)$. It is increasing on the interval $[\dfrac{\pi}{3}, \dfrac{\pi}{2}]$.

Follow the Desmos link to explore more initial values of x and see what happens as you adjust h smaller and smaller to zero. Desmos link: `https://www.desmos.com/calculator/xbi081bx3w`

c. $f(x) = 2x + 3,\ a = -1.$

Explanation Evaluate directly:

$$\frac{f(-1+h)-f(-1)}{h} = \frac{2(h-1)+3-(2(-1)+3)}{h} = \frac{2h-2+2}{h} = 2$$

Observe that the equation for the slope of the secant line from $(-1, f(-1))$ to $(-1+h, f(-1+h))$ is simply

$$AV_{[-1,-1+h]} = m = 2.$$

Recall from Example 1(c) of Section 12-3-1, that the equation of the secant line from $(-1, f(-1))$ to $(3, f(3))$ was simply $f(x)$. Why was this?

Now, this tells us that regardless of the points we choose, the secant line between them will have the same slope and equation as the line itself.

6.1.3 Algebra of Secant Lines

Motivating Questions

- What are some algebra techniques that allow us to simplify the equation of a secant line?
- Why is this important?

Introduction

Given the graph of a function $y = f(x)$, we have discussed methods to determine the slope of the secant line between two points, $(a, f(a))$ and $(b, f(b))$, on the graph. We know that this slope represents the average rate of change of the function f on the interval $[a, b]$, denoted by $AV_{[a,b]}$. Both of these can be rewritten by letting $b = h - a$, so that we have the value h representing the horizontal distance between the points. This means that as $h \to 0$, the secant line, or the average rate of change of the function, approaches a value known as the slope of the tangent line of f at a. This will be discussed extensively in future calculus courses, but in this section we will focus on tools to simplify the expression $AV_{[a,a+h]}$, as they are essential to calculating this limit.

Definitions and examples

Recall the special formula for difference of squares, $a^2 - b^2 = (a-b)(a+b)$. For non-square values of a and b we can use the same idea to rationalize differences (or sums) of square roots through multiplication by the corresponding sum (or difference), which we call the *conjugate*. Given any expression $\sqrt{a} \pm \sqrt{b}$, a, b real numbers, the conjugate of this expression is $\sqrt{a} \mp \sqrt{b}$. Multiplying such an expression by its conjugate rationalizes it through the distributive property: $(\sqrt{a} + \sqrt{b}) \cdot (\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b} - (\sqrt{b})^2 = a - b$

Definition: Given any difference of positive values $a - b$, we know from the difference of squares, that $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$.
The sum $\sqrt{a} + \sqrt{b}$ is the *conjugate* of the difference $\sqrt{a} - \sqrt{b}$. Likewise, the difference $\sqrt{a} - \sqrt{b}$ is the *conjugate* of the sum $\sqrt{a} + \sqrt{b}$.
Multiplying such an expression by its conjugate will rationalize the expression.

Note that this is one of the most important tools in your simplification toolbox. Other tools include simplifying polynomials and fractions (finding the common denominator), moving coefficients inside or outside the square root, and the trigonometric identities introduced in Section 10-2.

Example 67. *For the following, find the difference quotient. Simplify as much as possible*

(a) $f(x) = \sqrt{x}$, $x \geq 0$

Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function f since its domain is $[0, \infty)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{1}$$

Observe that we cannot combine any terms in (1), but the numerator is of the form $\sqrt{a} - \sqrt{b}$. Hence, we will multiply by the conjugate to rationalize the numerator:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \tag{2}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \tag{3}$$

Remember, in (2), that in order to avoid changing the value of the expression, we must multiply by the conjugate over itself, i.e., multiply by 1. Then (3) has a difference of squares in the numerator and is equal to

$$\frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}},$$

by cancelling out the h in the numerator and the denominator.

This expression that we have found is now in a form that allows us to consider what happens when $h \to 0$ by removing the h from the denominator.

(b) $g(x) = \sqrt{8 - 2x}$, $x \geq 4$.

Explanation We consider $h > 0$ to avoid any potential undefined values plugged into our function g since its domain is $[4, \infty)$.

$$\frac{g(x+h) - g(x)}{h} = \frac{\sqrt{8 - 4(x+h)} - \sqrt{8 - 4x}}{h} \tag{4}$$

Once again, we cannot combine any terms in the numerator of (4), so we will multiply by the conjugate to rationalize the numerator, hoping we will be able to simplify the equation. (4) is equal to

$$\begin{aligned}
\frac{(\sqrt{8-4(x+h)}-\sqrt{8-4x})}{h}\cdot\frac{(\sqrt{8-4(x+h)}+\sqrt{8-4x})}{(\sqrt{8-4(x+h)}+\sqrt{8-4x})} \\
&= \frac{(\sqrt{8-4(x+h)})^2-(\sqrt{8-4x})^2}{h(\sqrt{8-4(x+h)}+\sqrt{8-4x})} \\
&= \frac{(8-4x-4h)-(8-4x)}{h(\sqrt{4(2-(x+h))}+\sqrt{4(2-x)})} \\
&= \frac{-4h}{2h(\sqrt{2-(x+h)}+\sqrt{2-x})}
\end{aligned}$$

Now, we simply cancel the $2h$ in the numerator and the denominator, giving

$$\frac{g(x+h)-g(x)}{h} = \frac{-2}{\sqrt{2-(x+h)}+\sqrt{2-x}}.$$

(c) $f(x) = \cos(2x)$
Explanation Note that $\cos(z)$ is defined for all real numbers z, so we need not worry about the values of x and h plugged into the difference quotient formula.

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(2(x+h))-\cos(2x)}{h}$$

Now, we can expand out using the summation formula for cosine:

$$\begin{aligned}
\frac{\cos(2x)\cos(2h)-\sin(2x)\sin(2h)-\cos(2x)}{h} &= \frac{\cos(2x)(\cos(2h)-1)-\sin(2x)\sin(2h)}{h} \\
&= \cos(2x)\frac{\cos(2h)-1}{h} - \sin(2x)\frac{\sin(2h)}{h}
\end{aligned}$$

Here we can plug in decreasing values of h for cosine and sine and start to notice a pattern..
Explore this further by changing the x and h values in the following Desmos graph: Desmos link: https://www.desmos.com/calculator/lf9wkgurwz

(d) $f(x) = |x-1|$
Explanation We will consider two regions and ranges of h: (1) $x \in (-\infty, 1)$ with $h < 0$ and (2) $x \in (1, \infty)$ with $h > 0$.

Let's start with region (1), where $x < 1$ and $h < 0$. From this, we know that $x + h - 1 < x - 1 < 0$, so $|x-1| = -(x-1)$. Hence we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{|x+h-1|-|x-1|}{h} \\
&= \frac{-x-h+1+(x-1)}{h} \\
&= \frac{-h}{h} = -1
\end{aligned}$$

Alternatively, if we consider region (2), where $x > 1$ and $h > 0$, then we have $x + h - 1 > x - 1 > 0$, so that

$$\begin{aligned}\frac{|x+h+1| - |x+1|}{h} &= \frac{x+h-1-(x-1)}{h} \\ &= \frac{h}{h} = 1\end{aligned}$$

Notice that this is not as clear-cut if we consider say $x < 1$ and $h > 0$. Then we would need to consider if h is large enough that $x + h > 1$. Let's explore this some more.

Let $x < 1$ and $h > 0$. Further, assume $h > 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h} \\ &= \frac{x+h+1+(x-1)}{h} \\ &= \frac{2x+h}{h}\end{aligned}$$

Alternatively, if $h < 1 - x$, then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{|x+h-1| - |x-1|}{h} \\ &= \frac{-x-h+1+(x-1)}{h} \\ &= \frac{-h}{h} = -1\end{aligned}$$

As when $x < 1$ and $h < 0$.

(e) $g(x) = \dfrac{2x}{x^2+3}$

Explanation First note that the denominator of our function g is greater than zero for all real values of x, so the function is defined for all real numbers. Thus, we may calculate the difference quotient without concern for input values of x and h.

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{2(x+h)}{(x+h)^2+3} - \frac{2x}{x^2+3}}{h}$$

This expression for the difference quotient looks rather messy, so let's find the common denominator and see if we can cancel out some terms in the numerator by combining the fractions. We will leave the terms in the denominator in their current format, but multiply out the $(x+h)^2$ in the numerator for ease of simplification.

Note that the common denominator is $((x+h)^2+3)(x^2+3)$. Then we have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{(2x+2h)(x^2+3)-2x(x^2+2xh+h^2+3)}{((x+h)^2+3)(x^2+3)}}{h}\\ &= \frac{2x^3+2x^2h+6x+6h-(2x^3+4x^2h+2xh^2+6x)}{h\big((x+h)^2+3\big)(x^2+3)}\end{aligned}$$

Observe that we have combined the denominators now and have many common terms in the numerator that can be subtracted from each other, so that

$$\frac{g(x+h)-g(x)}{h} = \frac{-2x^2h+6h-2xh^2}{h\big((x+h)^2+3\big)(x^2+3)}.$$

Now, all the terms in the numerator have a factor of h, so we can cancel the h in the numerator and denominator for a final, simplified difference quotient of

$$\frac{-2x^2-2xh+6}{\big((x+h)^2+3\big)(x^2+3)}.$$

Summary Useful tools for simplification:

- Simplifying polynomials.
- Simplifying fractions by finding common denominators.
- Multiplying by the conjugate to rationalize the numerator.
- Considering regions for absolute value functions.

Part 7

Back Matter

Index