

Linear Equations: Slope

We explore the slope of lines.

We observed that a constant rate of change between points produces a linear relationship, whose graph is a straight line. Such a constant rate of change has a special name, **slope**, and we'll explore slope in more depth here.

Definition 1. *When x and y are two variables where the rate of change between any two points is always the same, we call this common rate of change the **slope**. Since having a constant rate of change means the graph will be a straight line, its also called the **slope of the line**.*

Considering the definition for **rate of change**, this means that when x and y are two variables where the rate of change between any two points is always the same, then you can calculate slope, m , by finding two distinct data points (x_1, y_1) and (x_2, y_2) , and calculating

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

A slope is a rate of change. So if there are units for the horizontal and vertical variables, then there will be units for the slope. The slope will be measured in

$$\frac{\text{vertical units}}{\text{horizontal units}}$$

Definition 2. *If the slope is constant and nonzero, we say that there is a **linear relationship** between x and y . When the slope is 0, we say that y is **constant** with respect to x .*

Here are some linear scenarios with different slopes. As you read each scenario, note how a slope is more meaningful with units.

- If a tree grows 2.5 feet every year, its rate of change in height is the same from year to year. So the height and time have a linear relationship where the slope is 2.5 ft/yr.
- If a company loses 2 million dollars every year, its rate of change in reserve funds is the same from year to year. So the company's reserve funds and time have a linear relationship where the slope is -2 million dollars per year.

Learning outcomes:
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- If Sakura is an adult who has stopped growing, her rate of change in height is the same from year to year—it's zero. So the slope is 0 in/yr. Sakura's height is constant with respect to time.

Remark 1. A useful phrase for remembering the definition of slope is “rise over run.” Here, “rise” refers to “change in y ”, Δy , and “run” refers to “change in x ”, Δx . Be careful though. As we have learned, the horizontal direction comes first in mathematics, followed by the vertical direction. The phrase “rise over run” reverses this. (It's a bit awkward to say, but the phrase “run under rise” puts the horizontal change first.)

Example 1. On Dec. 31, Yara had only \$50 in her savings account. For the new year, she resolved to deposit \$20 into her savings account each week, without withdrawing any money from the account.

Yara keeps her resolution, and her account balance increases steadily by \$20 each week. That's a constant rate of change, so her account balance and time have a linear relationship with slope of $20 \frac{\text{dollars}}{\text{week}}$.

Explanation. We can model the balance, y , in dollars, in Yara's savings account x weeks after she started making deposits with an equation. Since Yara started with \$50 and adds \$20 each week, then x weeks after she started making deposits,

$$y = 50 + 20x$$

where y is a dollar amount. Notice that the slope,

$$20 \frac{\text{dollars}}{\text{week}}$$

, serves as the multiplier for x weeks.

We can also consider Yara's savings using a table

	x (weeks since Dec 31)	y (savings account balance in dollars)	
	0	50	
+1 →	1	70	← +20
+1 →	2	90	← +20
+2 →	4	130	← +40
+3 →	7	190	← +60
+5 →	12	290	← +100

In first few rows of the table, we see that when the number of weeks x increases by 1, the balance y increases by 20. The row-to-row rate of change is

$$\frac{20 \text{ dollars}}{1 \text{ week}} = 20 \frac{\text{dollars}}{\text{week}},$$

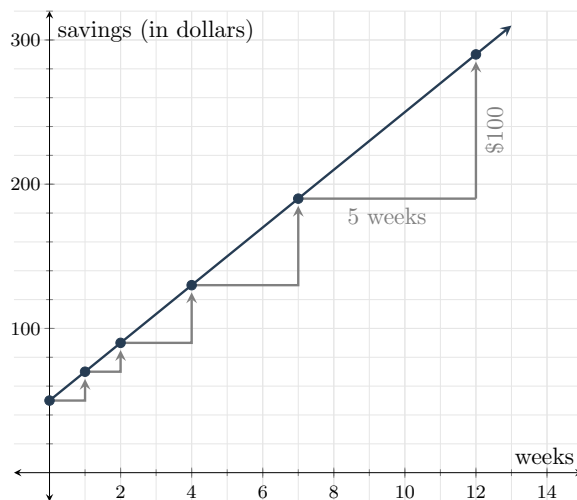
the slope. In any table for a linear relationship, whenever x increases by 1 unit, y will increase by the slope.

In further rows, notice that as row-to-row change in x increases, row-to-row change in y increases proportionally to preserve the constant rate of change. Looking at the change in the last two rows of the table, we see x increases by 5 and y increases by 100, which gives a rate of change of

$$\frac{100 \text{ dollars}}{5 \text{ week}} = 20 \frac{\text{dollars}}{\text{week}},$$

the value of the slope again.

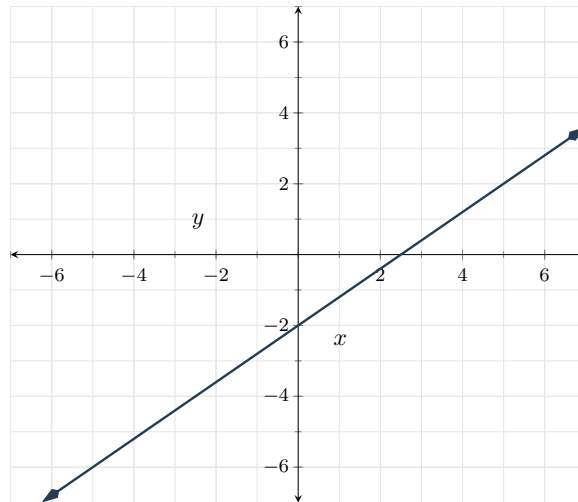
We can see this constant rate of change on the graph by drawing in **slope triangles** between points on the graph, showing the change in x as a horizontal distance and the change in y as a vertical distance.



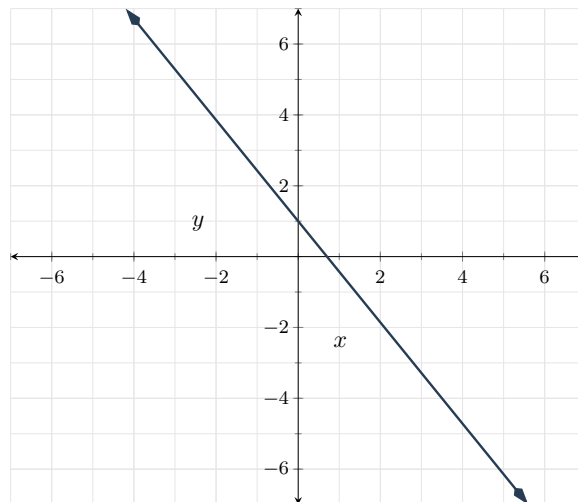
The Relationship Between Slope and Increase/Decrease

In a linear relationship, as the x -value increases (in other words as you read its graph from left to right):

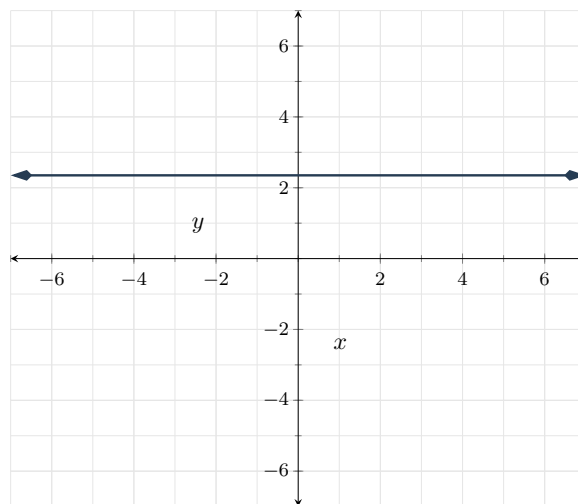
- if the y -values increase (in other words, the line goes upward), its slope is positive.



- if the y -values decrease (in other words, the line goes downward), its slope is negative.



- if the y -values don't change (in other words, the line is flat, or horizontal), its slope is 0.

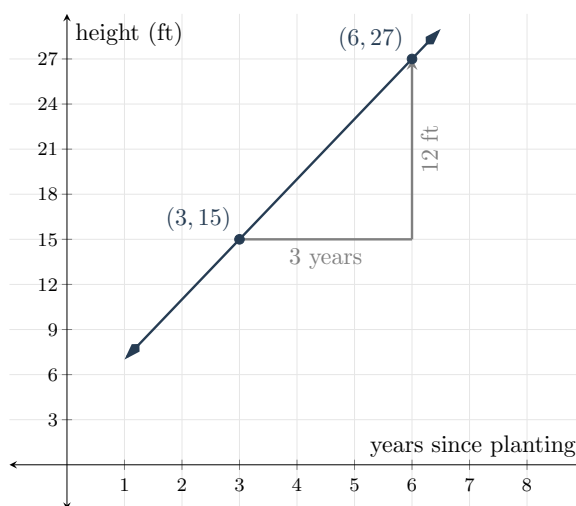


Finding the Slope by Two Given Points

Whenever you know two points on a line, you can find the slope of the line directly from the definition of slope.

Example 2. *Your neighbor planted a sapling from Portland Nursery in his front yard. Ever since, for several years now, it has been growing at a constant rate. By the end of the third year, the tree was 15 ft tall; by the end of the sixth year, the tree was 27 ft tall. What's the tree's rate of growth (i.e. the slope)?*

Explanation. *We could sketch a graph for this scenario, and include a slope triangle. If we did that, it would look like:*



We don't actually need the picture, though, to find the slope. From the definition of slope, we have that

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

We know that after 3 yr, the height is 15 ft. As an ordered pair, that information gives us the point (3,15) which we can label as (x_1, y_1) . Similarly, the background information tells us to consider (6,27), which we label as (x_2, y_2) . Here, x_1 and y_1 represent the first point's x -value and y -value, and x_2 and y_2 represent the second point's x -value and y -value.

Substituting in our values for $x_1 = 3$, $y_1 = 15$, $x_2 = 6$, and $y_2 = 27$ into our definition of slope, we have

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{27 - 15}{6 - 3} = \frac{12\text{ft}}{3\text{yr}} = \boxed{4} \frac{\text{ft}}{\text{given yr}}$$