Paths, Cycles and Permanent

A bridge between Combinatorics and Algebra

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Disjoint Paths

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- A randomized poly-time algorithm for shortest $DP(2)$ was presented by [Björklund, Husfeldt]

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- A parallel algorithm to compute permanent $\pmod{2^k}$ was discovered by [Braverman, Kulkarni & Roy]

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- Given a weighted directed graph (G, w) with vertices labelled $\{v_1, \ldots, v_n\}$, adjacency matrix A_G of G is defined as
$$(A_G)_{ij} = \begin{cases} w(e) & \text{if } e = (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

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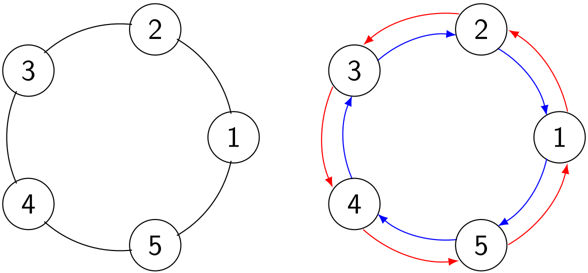
- Denote by $w(C) = \prod_{e \in C} w(e)$ the weight of cycle cover then

$$\mathrm{perm}(A_G) = \sum_{\text{cycle covers } C} w(C)$$

The Bridge

Undirected Graphs



- We view undirected graph as a directed graph with each edge $\{u, v\}$ replaced with two directed edges (u, v) and (v, u) with $w((u, v)) = w((v, u)) = w(\{u, v\})$ (*symmetrical* weights)

Applications

Shortest 2-disjoint paths [BH]

- SDP(2): Given a undirected graph G and 2 pairs of vertices $\{s_i, t_i \mid i \in \{1, 2\}\}$, find disjoint paths between each pair such that the sum of the lengths of these paths is minimum.

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- To force cycles (in a cycle cover) to pass through the given terminals, we introduce *pattern graphs*.
- A *pattern* P is an ordered pairing of the given terminals.
- A pattern graph G_P is same as G but such that if $(u, v) \in P$ then all outgoing edges from u, except edge (u, v), are deleted.

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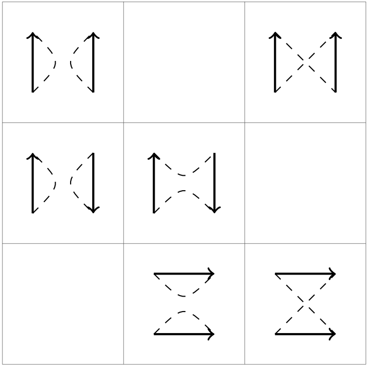
$$P_0 \quad \begin{array}{cc} t_1 & t_2 \\ \uparrow & \uparrow \\ s_1 & s_2 \end{array}$$

$$P_1 \quad \begin{array}{cc} t_1 & t_2 \\ \uparrow & \downarrow \\ s_1 & s_2 \end{array}$$

$$P_2 \quad \begin{array}{ccc} t_1 & \rightarrow & t_2 \\ & & \\ s_1 & \rightarrow & s_2 \end{array}$$



Smallest exponent with non-zero coefficient in $\mathrm{perm}(A_{P_0}) + \mathrm{perm}(A_{P_1}) - \mathrm{perm}(A_{P_2})$ gives length of shortest 2-disjoint paths

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- We were able to find algorithms for shortest disjont cycle $SDC(1, k)$ shortest disjoint cycle and shortest 2-disjoint cycles $SDC(2, k)$, for all $k \geq 1$
- Although our algorithm for $SDC(1, k)$ is already reminiscient in the work of [Wahlström] on finding a cycle through k-vertices

Applications

Shortest cycle [Wahlström]

- For each binary sequence $b = (b_1 b_2 \dots b_{k-1})$ consider the pattern P_b defined as: $(s_1, t_1) \in P_b$ and $b_{i-1} = 0 \implies (s_i, t_i) \in P_b$ else $(t_i, s_i) \in P_b$

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- Exponent of smallest term in $\sum_b \mathrm{perm}(A_{P_b}) - \sum_c \mathrm{perm}(A_{Q_c}) \pmod 4$ gives the weight of shortest 2-disjoint cycles separating (s_1, t_1) and (s_2, t_2)

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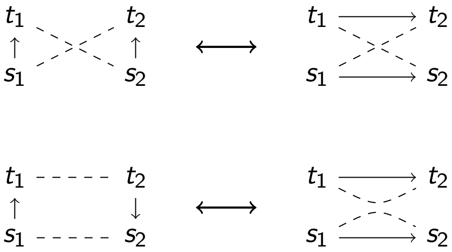

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- NC^i is the class of decision problems solvable in time $O(\log^i n)$ on a parallel computer with a polynomial number of processors, or the class of decision problems decidable by uniform boolean circuits with a polynomial number of gates of fan-in 2 and depth $O(\log^i n)$

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- $\oplus\mathrm{L} \subseteq \mathrm{NC}^2$ is the class of decision problems solvable by an NL machine such that:
  - If the answer is 'yes', then the number of accepting paths is odd.
  - If the answer is 'no', then the number of accepting paths is even.

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- Shall reduce odd case to even case by matrix perturbation

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- Then

$$\mathrm{perm}(A') = \sum_j \left(\sum_i v_i a_{ij} \right) \mathrm{perm}(A[\hat{1}, \hat{j}])$$

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Even case

- Ha! We have a double summation so we can evaluate this sum in two ways

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- Let A_i denote the matrix A but with the first row replaced with i^{th} row then

$$\mathrm{perm}(A') = \sum_i v_i \mathrm{perm}(A_i) = \mathrm{perm}(A) + \sum_{i>1} v_i \mathrm{perm}(A_i)$$

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- Since $A^T v = 0 \pmod 2 \implies \sum_i v_i a_{ij} = 2b_j$ and so $\mathrm{perm}(A') \pmod 4 = 2\left(\sum_j b_j \mathrm{perm}(A[\hat{1},\hat{j}]) \pmod 2\right)$
- But A_i has two equal rows and we exploit this as follows:

$$\mathrm{perm}(A_i) = \sum_{j,k} a_{ij} a_{ik} \mathrm{perm}(A[\widehat{\{1,i\}},\widehat{\{j,k\}}])$$

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- Re-write this as $\mathrm{perm}(A_i) = 2\left(\sum_{j<k} a_{ij} a_{ik} \mathrm{perm}(A[\widehat{\{1,i\}}, \widehat{\{j,k\}}])\right)$

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$$
\begin{aligned}
&\operatorname{perm}(A) \pmod 4 \\
&\quad = 2\left(\sum_{j=1}^{n} b_j \operatorname{perm}(A[\widehat{\{1\}}, \widehat{\{j\}}]) \pmod 2\right) \\
&\quad - 2\sum_{i=2}^{n} v_i \left(\sum_{\substack{j,k=1 \\ j<k}}^{n} a_{ij}a_{ik} \operatorname{perm}(A[\widehat{\{1,i\}}, \widehat{\{j,k\}}]) \pmod 2\right)
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- $\exists j$ such that $\mathrm{perm}(A[\hat{i},\hat{j}]) \not\equiv 0 \pmod 2$
- Increment a_{ij} by 1 and call the resulting matrix C. So we get

$$\mathrm{perm}(C) = \mathrm{perm}(A) + \mathrm{perm}(A[\hat{i},\hat{j}]) \equiv 0 \pmod 2$$

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- This gives us a sequential algorithm for computing $\mathrm{perm}(A) \pmod 4$

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- Thus $QA = LU$ ($Q = P^{-1}$)
- Q is also a permutation matrix and so QA is just the matrix A with it's rows permuted. Hence $\mathrm{perm}(A) = \mathrm{perm}(QA)$

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- Required permutation is $Q = (n, s_n) \dots (2, s_2)(1, s_1)$ such that all leading principal minors of QA are non-zero

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- Need a notion of modulo 4 arithmetic extending this field structure

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- Can we do better?

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- Let $f(x) = \sum_{i=0}^{d} c_i x^i$ be an integer polynomial such that $q > d + 2$ then $\sum_{a \in \mathbb{F}^*} a^{q-1-t} f(a) = c_t \pmod 2$

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- We observe that if we instead do the computations over $\mathfrak{R}$ then $\sum_{a\in\mathbb{F}^*} a^m = \begin{cases} 2\alpha_m \text{ if } q-1 \nmid m \\ 2\beta_m + 1 \text{ otherwise} \end{cases}$ where $\alpha_m, \beta_m \in \mathfrak{R}$

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$\begin{cases} 4\alpha_m^2 \text{ if } q-1 \nmid m \\ 4\beta_m^2 + 4\beta_m + 1 \text{ otherwise} \end{cases} = \begin{cases} 0 \pmod 4 \text{ if } q-1 \nmid m \\ 1 \pmod 4 \text{ otherwise} \end{cases}$

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Theorem

$$\sum_{a_1,\ldots,a_{2^{k-1}}\in\mathbb{F}^*}(a_1\cdots a_{2^{k-1}})^m = \begin{cases} 0 \pmod{2^k} \textit{ if } q-1 \nmid m \\ 1 \pmod{2^k} \textit{ otherwise} \end{cases}$$

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- Compute over $\mathfrak{R} \pmod{2^k}$:

$$\sum_{a_1, a_2, \ldots \in \mathbb{F}^*} (a_1 a_2 \ldots)^{q-1-t} \mathrm{perm}(A(a_1 a_2 \ldots)) = c_t \pmod{2^k}$$

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- $x^{2.3^l} + x^{3^l} + 1$ is irreducible over $\mathbb{Z}_2$ for all $l \geq 0$ [JH van Lint] [HV]
- Choose $l = \lceil \frac{\log \log N}{\log 3} \rceil$ such that $2^{2.3^l} > N + 2$ where $N = poly(n)$ (and $n =$size of matrix)

Example

- Let $A = \begin{pmatrix} 1 & x+1 & x+2 \\ x & x^2 & x^2+x \\ x^2 & 3 & x^2+3 \end{pmatrix}$, $p(x) = x^6 + x^3 + 1$ be the irreducible polynomial and we want to evaluate $\mathrm{perm}(A) \pmod 4$ over the ring $\mathfrak{R} = \mathbb{Z}[x]/(p(x))$.

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- Direct computation gives us $\mathrm{perm}(A) = 2x^5 + 6x^4 + 2x^3 + 12x^2 + 12x$. Now we demonstrate the steps taken by our algorithm.

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- **Step 1**: We start by evaluating $\mathrm{perm}(A) \pmod 2$. We directly notice here that $\det(A) = 0 \implies \mathrm{perm}(A) \equiv 0 \pmod 2$

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- $\begin{pmatrix} x^3+1 \\ x^5+x \\ 1 \end{pmatrix}$ is a solution to the above equations

Example

- **Step 3**: For each $j = 1, 2, 3$, we find b_j such that $\sum_i v_i a_{ij} = 2b_j \pmod 4$

$$
\begin{aligned}
j = 1: &\quad (x^3+1) + x(x^5+x) + x^2 = 2x^2 \\
j = 2: &\quad (x+1)(x^3+1) + x^2(x^5+x) + 3 = 2x^3 \\
j = 3: &\quad (x+2)(x^3+1) + (x^2+x)(x^5+x) + x^2 + 3 = 2x^3 + 2x^2
\end{aligned}
$$

Example

- **Step 4**: We have the formula

$$\mathrm{perm}(A) \pmod 4 = 2\left(\sum_{j=1}^{3} b_j \mathrm{perm}(A[\widehat{\{3\}}, \widehat{\{j\}}]) \pmod 2\right) - 2\sum_{i=1}^{2} v_i \left(\sum_{\substack{j,k=1 \\ j<k}}^{3} t_{jk} \pmod 2\right)$$

where

$$t_{jk} = a_{ij} a_{ik} \mathrm{perm}(A[\widehat{\{3, i\}}, \widehat{\{j, k\}}])$$

Example

- **Step 4.1**:

$$\mathrm{perm}(A[\widehat{\{3\}}, \widehat{\{1\}}]) = \mathrm{perm}\begin{pmatrix} x+1 & x+2 \\ x^2 & x^2+x \end{pmatrix} = x \pmod 2$$

$$\mathrm{perm}(A[\widehat{\{3\}}, \widehat{\{2\}}]) = \mathrm{perm}\begin{pmatrix} 1 & x+2 \\ x & x^2+x \end{pmatrix} = x \pmod 2$$

$$\mathrm{perm}(A[\widehat{\{3\}}, \widehat{\{3\}}]) = \mathrm{perm}\begin{pmatrix} 1 & x+1 \\ x & x^2 \end{pmatrix} = x \pmod 2$$

$$\implies \sum_{j=1}^{3} b_j \mathrm{perm}(A[\widehat{\{3\}}, \widehat{\{j\}}]) = 0 \pmod 2$$

Example

- **Step 4.2**:

$$\sum_{\substack{j,k=1\\ j<k}}^{3} a_{1j}a_{1k}\mathrm{perm}(A[\widehat{\{1,3\}},\widehat{\{j,k\}}]) = x^3+x^2+x \pmod 2$$

$$\sum_{\substack{j,k=1\\ j<k}}^{3} a_{2j}a_{2k}\mathrm{perm}(A[\widehat{\{2,3\}},\widehat{\{j,k\}}]) = x^4+x^3+x^2 \pmod 2$$

$$\sum_{i=1}^{2} v_i \left(\sum_{\substack{j,k=1\\ j<k}}^{3} a_{ij}a_{ik}\mathrm{perm}(A[\widehat{\{3,i\}},\widehat{\{j,k\}}]) \pmod 2 \right) = x^5+x^4+x^3 \pmod 4$$

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- Therefore, $\mathrm{perm}(A) \pmod 4 = 2x^5 + 2x^4 + 2x^3$ which matches with our direct computation.

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- $SDC(l, k)$ for other values of $l \geq 3$?
- Permanent over rings with characteristic 2^k, $k \geq 2$?

Thank you!

Questions?