Quantum Computing - Assignment 3

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Exercise 1

Let $|\psi\rangle_{AB} \in H_A \otimes H_B$ be a pure state. Assume wlog $dim(H_A) \geq dim(H_B) = m$. By Schmidt decomposition there exist orthonormal bases $\{u_i \mid i \leq m\} \subset H_A$ and $\{v_i \mid i \leq m\} \subset H_B$ such that

$$|\psi\rangle_{AB} = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$$

Claim $|\psi\rangle_{AB}$ is entangled iff more than one Schmidt coefficients $\{\alpha_i \mid i \leq m\}$ are non-zero

Proof. Firstly, it is clear that atleast one Schmidt coefficient is non-zero because otherwise $|\psi\rangle_{AB} = 0$ is not a valid pure state.

So suppose exactly one Schmidt coefficient α_i is non-zero, then

$$|\psi\rangle_{AB} = \alpha_i |u_i\rangle_A \otimes |v_i\rangle_B$$

is a product state.
Conversely, suppose more than one Schmidt coefficient is non-zero, say α_i and α_j where $i \neq j$. We need to show that $|\psi\rangle_{AB}$ is entangled. For the sake of contradiction assume that $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\varphi\rangle_B$ is a product state. Then we have

$$\begin{aligned} |\phi\rangle_A \otimes |\varphi\rangle_B &= \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle \\ \implies \big(\langle u_i| \otimes \langle v_i|\big)\big(|\phi\rangle_A \otimes |\varphi\rangle_B\big) &= \big(\langle u_i| \otimes \langle v_i|\big)\left(\sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle\right) \\ \implies \langle u_i|\phi\rangle \langle v_i|\varphi\rangle &= \alpha_i \end{aligned}$$

Similarly, $\langle u_j|\phi\rangle \langle v_j|\varphi\rangle = \alpha_j$. Furthermore,

$$\begin{aligned} |\phi\rangle_A \otimes |\varphi\rangle_B &= \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle \\ \implies \big(\langle u_i| \otimes \langle v_j|\big)\big(|\phi\rangle_A \otimes |\varphi\rangle_B\big) &= \big(\langle u_i| \otimes \langle v_j|\big)\left(\sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle\right) \\ \implies \langle u_i|\phi\rangle \langle v_j|\varphi\rangle &= 0 \end{aligned}$$

Thus, we get $\alpha_i\alpha_j = \langle u_i|\phi\rangle \langle v_i|\varphi\rangle \langle u_j|\phi\rangle \langle v_j|\varphi\rangle = 0$. That is both α_i and α_j can't be non-zero, leading us to a contradiction.

$\square$

Exercise 2

Proof. We will prove that Trace is a commutative operator, that is $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_i \langle i| AB |i\rangle \\
&= \sum_i \langle i| A \left(\sum_j |j\rangle \langle j| \right) B |i\rangle \\
&= \sum_i \sum_j \langle i| A |j\rangle \langle j| B |i\rangle \\
&= \sum_j \sum_i \langle j| B |i\rangle \langle i| A |j\rangle \\
&= \sum_j \langle j| B \left(\sum_i |i\rangle \langle i| \right) A |j\rangle \\
&= \sum_j \langle j| BA |j\rangle \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

Now $\mathrm{Tr}(A(BC)) = \mathrm{Tr}((BC)A)$ and $\mathrm{Tr}(B(CA)) = \mathrm{Tr}((CA)B)$. Since matrix multiplication is associative we get $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$

$$\mathrm{Tr}_B(|x_1\rangle \langle x_2|_A \otimes |y_1\rangle \langle y_2|_B) = \sum_i I_A \otimes \langle i|_B \left(|x_1\rangle \langle x_2|_A \otimes |y_1\rangle \langle y_2|_B \right) I_A \otimes |i\rangle_B$$

Using the rule $(A \otimes B).(C \otimes D) = AB \otimes CD$, we get

$$\begin{aligned}
\mathrm{Tr}_B(|x_1\rangle \langle x_2|_A \otimes |y_1\rangle \langle y_2|_B) &= \sum_i |x_1\rangle \langle x_2|_A \otimes \langle i|_B |y_1\rangle \langle y_2|_B |i\rangle_B \\
&= |x_1\rangle \langle x_2|_A \otimes \sum_i \langle i|_B |y_1\rangle \langle y_2|_B |i\rangle_B \\
&= |x_1\rangle \langle x_2|_A \otimes \mathrm{Tr}(|y_1\rangle \langle y_2|_B) \\
&= |x_1\rangle \langle x_2|_A \otimes \mathrm{Tr}(\langle y_2|y_1\rangle) \text{ (Trace is commutative)} \\
&= |x_1\rangle \langle x_2|_A \langle y_2|y_1\rangle
\end{aligned}$$

$\square$

Exercise 3

Let $N : \mathscr{L}(H) \to \mathscr{L}(H)$, $N(\rho) = (1-p)\rho + pZ\rho Z$. We will show that N is a quantum channel, that is N is a linear, trace-preserving and completely positive operator.

Claim N is linear

Proof.

$$\begin{aligned}
N(\rho_1 + \lambda\rho_2) &= (1-p)(\rho_1 + \lambda\rho_2) + pZ(\rho_1 + \lambda\rho_2)Z \\
&= (1-p)(\rho_1) + (1-p)(\lambda\rho_2) + pZ(\rho_1)Z + pZ(\lambda\rho_2)Z \\
&= ((1-p)\rho_1 + pZ\rho_1 Z) + \lambda((1-p)\rho_2 + pZ\rho_2 Z) \\
&= N(\rho_1) + \lambda N(\rho_2)
\end{aligned}$$

$\square$

Claim N is trace preserving

Proof.

$$\begin{aligned}\operatorname{Tr}(N(\rho)) &= \operatorname{Tr}((1-p)\rho + pZ\rho Z) \\ &= (1-p)\operatorname{Tr}(\rho) + p\operatorname{Tr}(Z\rho Z) \\ &= (1-p)\operatorname{Tr}(\rho) + p\operatorname{Tr}(Z^2\rho) \\ &= (1-p)\operatorname{Tr}(\rho) + p\operatorname{Tr}(\rho) \\ &= \operatorname{Tr}(\rho)\end{aligned}$$

$\square$

Claim $\forall E$, $\mathrm{Id}_E \otimes N : \mathscr{L}(H_E \otimes H) \to \mathscr{L}(H_E \otimes H)$ is a positive operator

Proof. Let $\theta \in \mathscr{L}(H_E \otimes H)$ be any PSD matrix. Want to show that $(\mathrm{Id}_E \otimes N)(\theta)$ is also PSD matrix. Since θ is a PSD matrix, we have $\forall x\ \langle x|\theta|x\rangle \geq 0$
Identify $\mathscr{L}(H_E \otimes H) \cong \mathscr{L}(H_E) \otimes \mathscr{L}(H)$ and fix any basis $\{\sigma_i \mid i \leq m\}$ for $\mathscr{L}(H_E)$ and $\{\rho_j \mid j \leq n\}$ for $\mathscr{L}(H)$, then $\theta = \sum_{i,j} c_{ij}(\sigma_i \otimes \rho_j)$ for some $c_{ij} \in \mathbb{C}$.

$$(\mathrm{Id}_E \otimes N)(\theta) = \sum_{i,j} c_{ij}(\mathrm{Id}_E \otimes N)(\sigma_i \otimes \rho_j) = \sum_{i,j} c_{ij}\sigma_i \otimes N(\rho_j)$$

$$\begin{aligned}\langle x|\,\mathrm{Id}_E \otimes N(\theta)|x\rangle &= \sum_{i,j} c_{ij}\,\langle x|\sigma_i \otimes N(\rho_j)|x\rangle \\ &= \sum_{i,j} c_{ij}(1-p)\,\langle x|\sigma_i \otimes \rho_j|x\rangle + \sum_{i,j} c_{ij}p\,\langle x|\sigma_i \otimes Z\rho_j Z|x\rangle\end{aligned}$$

But we can re-write

$$\begin{aligned}\sigma \otimes Z\rho Z &= (I_E \otimes Z)(\sigma \otimes \rho)(I_E \otimes Z) \\ \implies \langle x|\sigma \otimes Z\rho Z|x\rangle &= \langle y|\sigma \otimes \rho|y\rangle\end{aligned}$$

where $|y\rangle = (I_E \otimes Z)\,|x\rangle$ (because $I_E \otimes Z$ is Hermitian). Hence we get

$$\langle x|\,\mathrm{Id}_E \otimes N(\theta)|x\rangle = (1-p)\sum_{i,j} c_{ij}\,\langle x|\sigma_i \otimes \rho_j|x\rangle + p\sum_{i,j} c_{ij}\,\langle y|\sigma_i \otimes \rho_j|y\rangle = (1-p)\,\langle x|\theta|x\rangle + p\,\langle y|\theta|y\rangle \geq 0$$

that is N is a completely positive operator. $\square$

We begin by noting the following relations:

$$ZXZ = -X, ZYZ = -Y$$

because

- $ZXZ\,|0\rangle = ZX\,|0\rangle = Z\,|1\rangle = -\,|1\rangle = -X\,|0\rangle$ and $ZXZ\,|1\rangle = -ZX\,|1\rangle = -Z\,|0\rangle = -\,|0\rangle = -X\,|1\rangle$
- $ZYZ\,|0\rangle = ZY\,|0\rangle = iZ\,|1\rangle = -i\,|1\rangle = -Y\,|0\rangle$ and $ZYZ\,|1\rangle = -ZY\,|1\rangle = iZ\,|0\rangle = i\,|0\rangle = -Y\,|1\rangle$

Thus, we get $N(X) = (1-p)X + pZXZ = (1-2p)X$ and $N(Y) = (1-p)Y + pZYZ = (1-2p)Y$ and $N(Z) = (1-p)Z + pZZZ = Z$. Hence

$$\begin{aligned}N\left(\frac{1}{2}(I + r_xX + r_yY + r_zZ)\right) &= \frac{1}{2}(N(I) + r_xN(X) + r_yN(Y) + r_zN(Z)) \\ &= \frac{1}{2}(I + r_x(1-2p)X + r_y(1-2p)Y + r_zZ)\end{aligned}$$

Exercise 4

Claim Let A be any linear operator over a $\mathbb{C}-$vector space, then we can write $A = B + iC$ for some B, C Hermitian operators

Proof. Consider $B = (A + A^\dagger)/2$ and $C = (A - A^\dagger)/2i$ then $A = B + iC$ is clear.
Moreover, $B^\dagger = (A^\dagger + A)/2 = B$ and $C^\dagger = -(A^\dagger - A)/2i = C$, as desired. $\square$

Claim Let A be a Hermitian operator then $\forall v$, $\langle v|A|v\rangle \in \mathbb{R}$

Proof. Using spectral theorem, we know that there exists an orthonormal eigenbasis $\{v_i \mid i \leq n\}$ such that $Av_i = \lambda_i v_i$ where $\lambda_i \in \mathbb{R}$, $\forall i \leq n$
Now for any v, write $v = \sum_i c_i v_i$ and then

$$\langle v|A|v\rangle = \sum_i |c_i|^2 \lambda_i \in \mathbb{R}$$

$\square$

Claim Let A be any linear operator over a $\mathbb{C}-$vector space such that $\forall v$, $\langle v|A|v\rangle \in \mathbb{R}$. Then A is a Hermitian operator.

Proof. We write $A = B + iC$ with B, C Hermitian.
Since B, C are Hermitian, $\langle v|B|v\rangle \in \mathbb{R}$ and $\langle v|C|v\rangle \in \mathbb{R}$. Therefore $\forall v$

$$\langle v|A|v\rangle = \langle v|B|v\rangle + i\,\langle v|C|v\rangle \in \mathbb{R} \implies \langle v|C|v\rangle = 0$$

Hence $C = 0 \implies A = B$ which is Hermitian. $\square$

Finally, let A be a positive operator, which means $\forall v$, $\langle v|A|v\rangle \geq 0$. That is $\forall v$, $\langle v|A|v\rangle \in \mathbb{R}$. Using above claim, we get that A is Hermitian.