Random Trees and Effective Resistance

Kishlaya Jaiswal

August 14, 2020

Abstract

We look at a graph as an electric network to present a simple proof for the well known relation between effective resistance of an edge and the chances of that edge being in a uniformly random spanning tree.
Advisor: *Samir Datta*

1 Introduction

Let $G = (V, E)$ be a given undirected graph. Sampling a uniformly random spanning tree T is a well-studied problem and the question we ask is: given an edge $e \in E$, what are the chances that a random spanning tree has this edge e.

We consider an electric circuit where each edge in G is replaced with a resistor of 1Ω. To setup a potential difference, we supply external current i_{ext} at nodes; then potential p at each node is given by: $i_{ext} = Lp$ where L denotes the Laplacian of the graph.

Effective resistance of an edge $e = (a, b)$, denoted by $R_{\text{eff}}(e)$, is defined as potential difference across e when a unit current is inducted at a and taken out at b.

So we consider the particular vector $i_{ext} = x_e$ where $x_e(a) = 1$, $x_e(b) = -1$ and $x_e(c) = 0$ for all $c \neq a, b$. Since $x_e \perp \mathbf{1}$, we know that a solution for p exists and can be given by $p = L^\dagger x_e$ where $L^\dagger$ denotes the psuedo-inverse of L. Therefore, potential difference between a and b is $p(a) - p(b) = x_e^T p = x_e^T L^\dagger x_e$ and so $R_{\text{eff}}(e) = p(a) - p(b) = x_e^T L^\dagger x_e$

It turns out that:

$$\mathbb{P}[e \in T] = R_{\text{eff}}(e)$$

Given these values $\mathbb{P}(e \in T)$, we can use them to sample a random spanning tree as follows [Gué83]:

Algorithm 1: Sampling a uniformly random spanning tree

$T = \{\}$
for $e \in E$ **do**
  Let p be the probability that e is in a random spanning tree of G
  Flip a biased coin whose chance of heads is p
  if *heads* **then**
    $T = T \cup \{e\}$
    $G = G \big/ \{e\}$ (contract e)
  else
    $G = G \setminus \{e\}$ (delete e)
  end
end
Return T

But directly calculating these probabilities require enumerating all spanning trees (hard), whereas computing $R_{\text{eff}}(e)$ only involves multiplying vectors (easy).

The proof we discuss here is from [Gha15]. In what follows, we shall assume that the graph is connected (if the graph is not connected then no spanning tree exists).

2 Matrix Tree Theorem

Matrix Tree Theorem [GR01] counts the number of spanning trees of G in terms of the Laplacian of the graph. That is, let $0 < \lambda_1 \leq \cdots \lambda_n$ be the eigen values of L, then

$$\#\text{spanning trees of } G = \frac{1}{n}\lambda_1 \ldots \lambda_n$$

We define

$$\bar{L} = L + \frac{1}{n}J$$

$\bar{L}\mathbf{1} = \mathbf{1}$ and for any other eigenvector v (of L) of non-zero eigenvalue, since $v \perp \mathbf{1}$, v is also an eigenvector of $\bar{L}$ with same eigenvalue. Hence $\{1, \lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $\bar{L}$ and so we can re-state the Matrix Tree Theorem as:

$$\#\text{spanning trees of } G = \frac{1}{n}\det(\bar{L})$$

3 Main Result

Lemma 3.1. *Given a positive-definite symmetric matrix $M \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$*

$$\det(M + xx^T) = \det(M)(1 + x^T M^{-1}x)$$

Proof. Since M is positive-definite symmetric, there exists a unique positive-definite symmetric matrix $M^{1/2}$ such that $(M^{1/2})^2 = M$

$$\begin{aligned}\det(M + xx^T) &= \det\left(M^{1/2}(I + M^{-1/2}xx^T M^{-1/2})M^{1/2}\right)\\ &= \det(M)\det(I + M^{-1/2}xx^T M^{-1/2})\\ &= \det(M)\det(I + yy^T)\end{aligned}$$

where $y = M^{-1/2}x$. Notice that solutions of $y^T v = 0$ gives $n-1$ eigenvectors of $I + yy^T$ with eigenvalue 1 and from trace computation we get the last eigenvalue is $1+y^Ty$. Hence $\det(I+yy^T) = 1 + y^Ty = 1 + x^T M^{-1}x$ and we are done. □

Since G is connected, $\bar{L}$ is a positive definite symmetric matrix and $\det(\bar{L}) > 0$. We also note that Laplacian of $G - \{e\}$ is simply $L - x_e x_e^T$. Therefore, #spanning trees of G not containing e = #spanning trees of $G \setminus \{e\} = \frac{1}{n}\det(\bar{L} - x_e x_e^T)$. Finally we have,

$$\begin{aligned}\mathbb{P}[e \in T] &= 1 - \mathbb{P}[e \notin T]\\ &= 1 - \frac{\det(\bar{L} - x_e x_e^T)}{\det(\bar{L})}\\ &= 1 - \frac{\det(\bar{L})(1 - x_e^T \bar{L}^{-1}x_e)}{\det(\bar{L})}\\ &= x_e^T \bar{L}^{-1}x_e\\ &= x_e^T L^{\dagger}x_e\end{aligned}$$

as $L = \bar{L}$ when restricted to the subspace perpendicular to $\mathbf{1}$. Thus, $\mathbb{P}[e \in T] = x_e^T L^\dagger x_e = R_{\text{eff}}(e)$ $\square$

4 Further Extensions

We can further extend this result by asking the probability of $F \subseteq T$ where F is any subset of edges. In this case [BP93] showed that $\mathbb{P}[F \subseteq T] = \det(Y_F)$ where Y is a $E \times E$ matrix such that $Y(e, f) = x_e^T L^\dagger x_f$. This can be easily proved by inducting on size of F and using Cauchy-Binet formula (note that the base case $|F| = 1$ is what we have proved above).

When G is weighted undirected graph, in which case resistance of each edge is inverse of it's weight then we have $\mathbb{P}[e \in T] = w(e)R_{\text{eff}}(e)$ where T is sampled with probability proportional to $\prod_{e \in T} w(e)$. The proof is similar with the modification that we work with the weighted Laplacian and use Matrix Tree theorem for weighted graphs.

References

[BP93] Robert Burton and Robin Pemantle. "Local Characteristics, Entropy and Limit Theorems for Spanning Trees and Domino Tilings Via Transfer-Impedances." In: *Ann. Probab.* 21.3 (July 1993), pp. 1329–1371. DOI: 10.1214/aop/1176989121. URL: https://doi.org/10.1214/aop/1176989121.

[Gha15] Shayan Oveis Gharan. *Recent Developments in Approximation Algorithms*. University Lecture. 2015. URL: https://homes.cs.washington.edu/~shayan/courses/approx/.

[GR01] C. Godsil and G.F. Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics. Springer New York, 2001. ISBN: 9780387952208. URL: https://books.google.co.in/books?id=gk60QgAACAAJ.

[Gué83] A Guénoche. "Random spanning tree." In: *Journal of Algorithms* 4.3 (1983), pp. 214–220. ISSN: 0196-6774. DOI: https://doi.org/10.1016/0196-6774(83)90022-6. URL: http://www.sciencedirect.com/science/article/pii/0196677483900226.