Quantum Computing - Assignment 1

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Exercise 1

Proof. Observe that

$$\| |\psi\rangle \|^2 = \langle\psi|\psi\rangle = \big(|\psi\rangle \big)^\dagger \big(|\psi\rangle \big)$$

U is unitary that is $U^\dagger U = I$, and hence

$$\|U |\psi\rangle\|^2 = \big(U |\psi\rangle \big)^\dagger \big(U |\psi\rangle \big) = \langle\psi| U^\dagger U |\psi\rangle = \langle\psi| I |\psi\rangle = \langle\psi|\psi\rangle = \| |\psi\rangle \|^2$$

Since $\|.\| \geq 0 \implies \|U |\psi\rangle\| = \| |\psi\rangle \|$ $\square$

Exercise 2

Proof.

$$\begin{aligned} [X, Z] |0\rangle &= (XZ - ZX) |0\rangle = X |0\rangle - Z |1\rangle = |1\rangle + |1\rangle = 2 |1\rangle \\ [X, Z] |1\rangle &= (XZ - ZX) |1\rangle = -X |1\rangle - Z |0\rangle = -|0\rangle - |0\rangle = -2 |0\rangle \end{aligned}$$

Hence $[X, Z] = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ $\square$

Exercise 3

Proof.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \text{ and } X^\dagger X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \implies Y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y \text{ and } Y^\dagger Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z \text{ and } Z^\dagger Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus Pauli matrices are Hermitian and Unitary. And,

$$X \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, X \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$Y \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, Y \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = -\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

$$Z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, Z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\square$

Thus the eigenvalues are of Pauli matrices are ± 1.

Exercise 4

Proof.

$$HXH|0\rangle = HX|+\rangle = H|+\rangle = |0\rangle$$
$$HXH|1\rangle = HX|-\rangle = H(-|-\rangle) = -|1\rangle$$

And hence $HXH = Z$

$$HZH|0\rangle = HZ|+\rangle = H|-\rangle = |1\rangle$$
$$HZH|1\rangle = HZ|-\rangle = H|+\rangle = |0\rangle$$

And hence $HZH = X$ $\square$

Exercise 5

Proof. From the above exercise 3, we know that $|+\rangle$ is an eigenvector of X with eigenvalue 1 and $|-\rangle$ is an eigenvector of X with eigenvalue -1, that is $X = |+\rangle\langle+| - |-\rangle\langle-|$
Hence $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ is an eigenbasis for X and so $|+\rangle\langle+|$ and $|-\rangle\langle-|$ are the measurement operators corresponding to a measurement of X observable. $\square$

Exercise 6

Proof. First we check that $\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|{+}{+}\rangle + |{-}{-}\rangle\big)$ indeed.

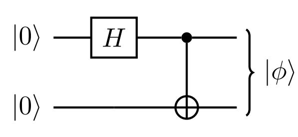
$$\begin{aligned}|{+}{+}\rangle + |{-}{-}\rangle &= \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle) = |00\rangle + |11\rangle\end{aligned}$$

So it suffices to show that $\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$ is an entangled state.

Suppose not and so it can be written as $|\psi\rangle \otimes |\phi\rangle$ where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$.
Then $|\psi\rangle \otimes |\phi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$

Since $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is a linearly independent set in $\mathbb{C}^4$, we get $\alpha\delta = \beta\gamma = 0$ and $\alpha\gamma = \beta\delta \neq 0$ whose solution doesn't exist. Hence $\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$ is an entangled state. $\square$

Exercise 7



Proof. We start with $|\psi\rangle = |00\rangle$ state.

Applying $H \otimes I$ to $|\psi\rangle$, we get $\frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)|0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |10\rangle\big)$

Applying controlled-NOT gate (where first qubit is the control and second qubit is target) to this, we finally get $|\phi\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$ as required. $\square$