Topics in Algorithms - Assignment 2

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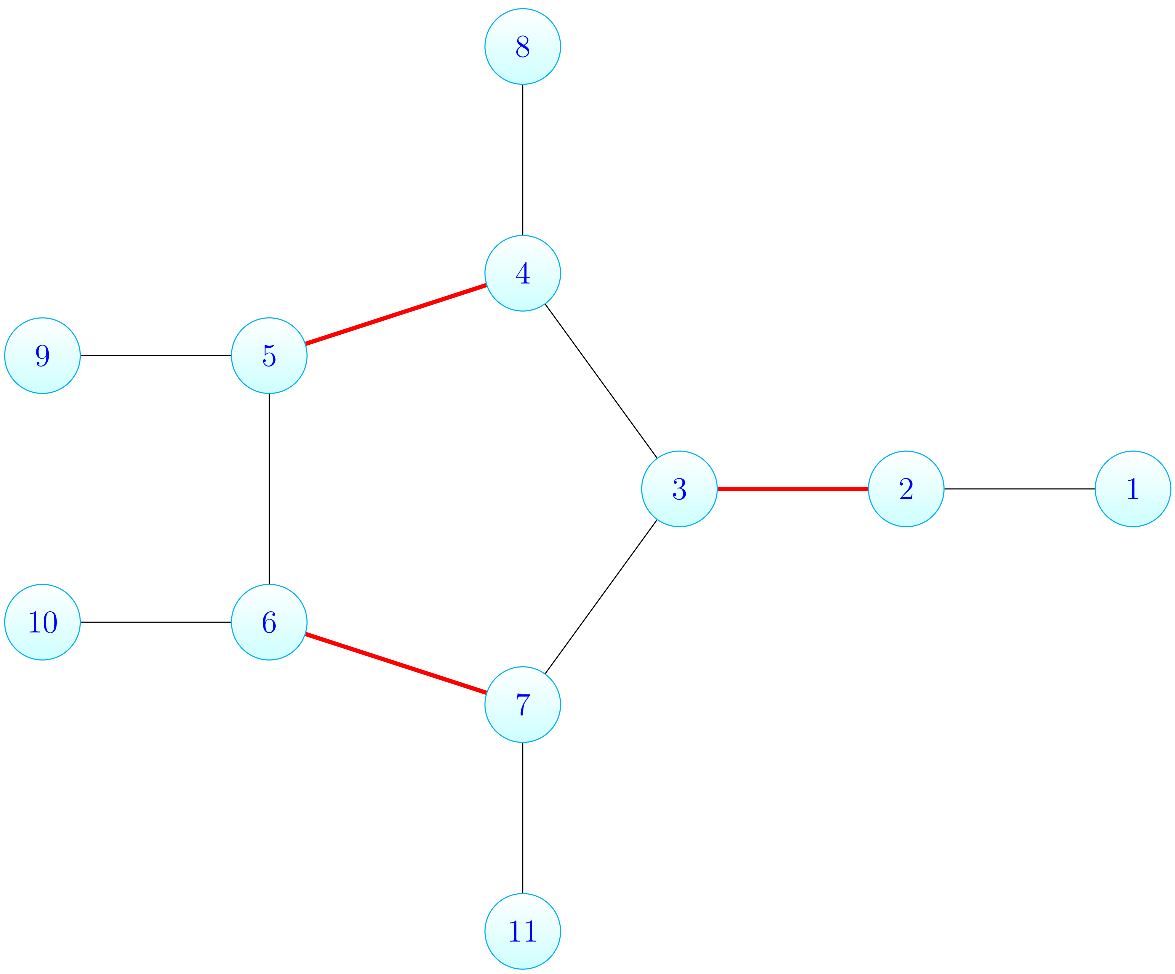
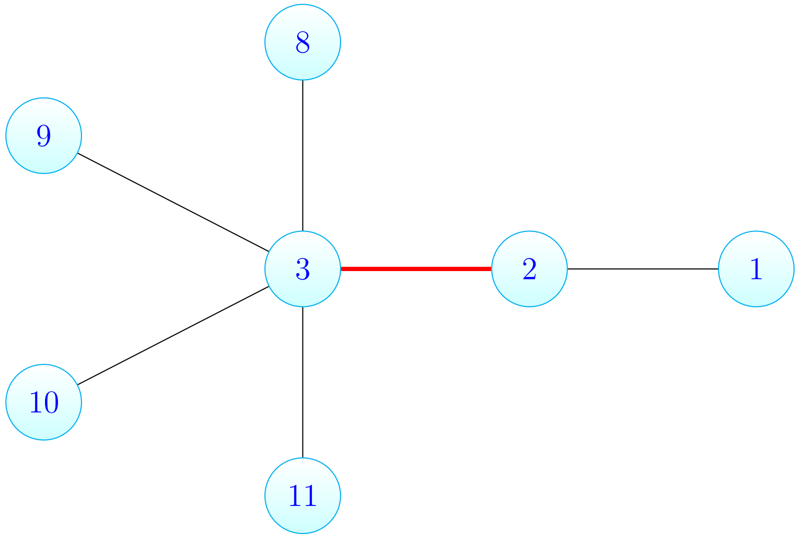
Following are the exercises from http://math.mit.edu/ goemans/18453S17/matching-nonbip-notes.pdf

Exercise 2.1

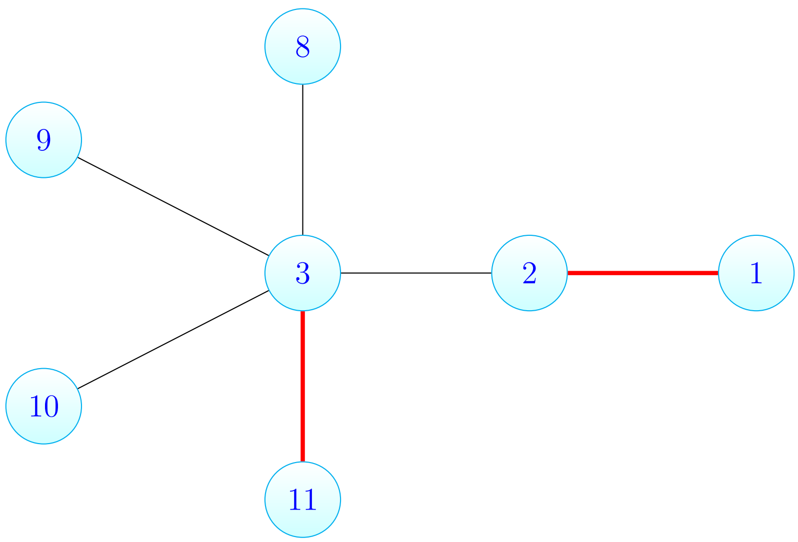
Consider the following graph:



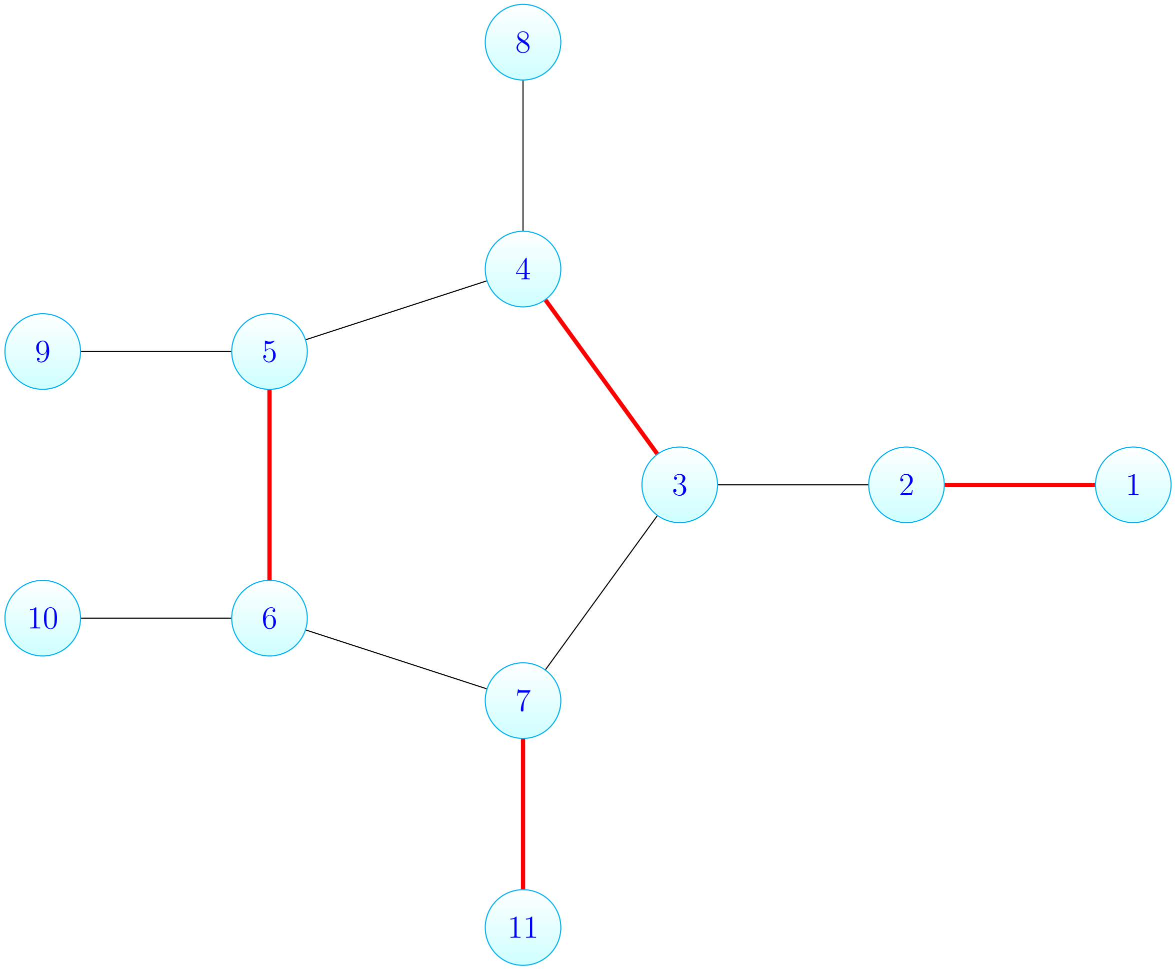
When we shrink the blossom, we would get



An augmenting path in the graph G/B is: $1 \to 2 \to 3 \to 11$, which gives the following maximum matching M^*/B in G/B



But this leads to the augmenting path $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 11$ in G and gives the following matching M^* (which is not a maximum matching):



This does not contradict theorem 2.2 because if M^* were to be a maximum matching in G, then the theorem says: *Let B be a blossom with respect to M^* . Then M^* is a maximum size matching in G if and only if M^*/B is a maximum size matching in G/B.* Notice that B was a blossom with respect to initial matching M and not M^* and hence the hypothesis of the theorem is not satisfied.

Exercise 2.2

Recall **Berge's lemma**: a matching M in a graph G is maximum if and only if there is no augmenting path wrt M in G.
Further note that if a matching matches the vertices in S, then augmenting this matching along any augmenting path will still keep the vertices in S matched.
Therefore, suppose a matching M covers S; if M is already maximum then we are done. Otherwise, by Berge's lemma there is an augmenting path wrt to M. Augment along this path to get a new matching M'. Then $|M'| > |M|$ and M' also covers S. Thus, repeating this $O(|V|)$ times, we get a maximum matching M^* which covers S.

Exercise 2.3

(a)

By Tutte's theorem, given any maximum matching M, we have

$$|M| = |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \frac{1}{2}\left(|V| + |U| - o(G \setminus U)\right) (\spadesuit)$$

And after removing U, any maximum matching can M can use atmost $k_i = \left\lfloor \frac{|K_i|}{2} \right\rfloor$ many edges from K_i, therefore it is certain that M uses atmost k_i edges from $G[K_i]$. If it uses strictly less than k_i edges from $G[K_i]$ then note that size of M is also strictly less than $|U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor$ which is a contradiction to $(\spadesuit)$.
Hence the vertices in $G[K_i]$ with even $|K_i|$ are perfectly matched whereas with odd $|K_i|$ are perfectly matched except for one vertex.

(b)

First, suppose some vertex in $u \in U$ remains unmatched in M. Then even if we remove this vertex u from the graph, the number of odd components after removing $U \setminus \{u\}$ remains unchanged, so for the new graph thus obtained, we have the following inequality:

$$|M| \leq \frac{1}{2}((|V| - 1) + (|U| - 1) - o(G \setminus U)) = \frac{1}{2}(|V| + |U| - o(G \setminus U)) - 1$$

which is again a contradiction to $(\spadesuit)$.
Now suppose that u is matched to a vertex in $v \in U$. Then if we remove the vertices $\{u, v\}$ from the graph, then the number of odd components after removing $U \setminus \{u, v\}$ remains unchanged and also the size of M decreases by 1, so for the new graph thus obtained, we have the following inequality:

$$|M| - 1 \leq \frac{1}{2}((|V| - 2) + (|U| - 2) - o(G \setminus U)) = \frac{1}{2}(|V| + |U| - o(G \setminus U)) - 2$$

which is again a contradiction to $(\spadesuit)$.
Thus, each vertex $u \in U$ is matched to a vertex not in U but since all the vertices in even components are matched among themselves, the only possibility is that u is matched to a vertex in some odd component K_i.

(c)

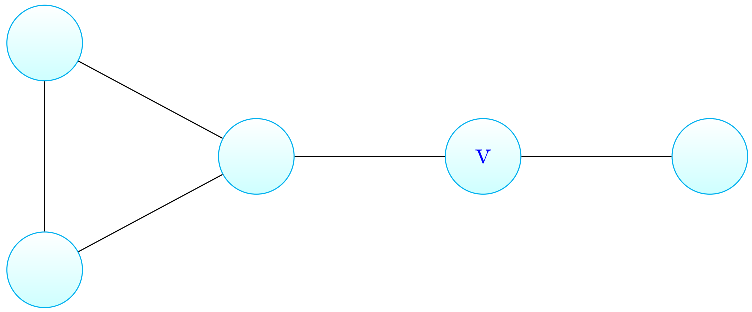
From the above two results, we get that

- Every vertex in U is matched
- Every vertex in an even component K_i is matched

Therefore, if some $v \in V$ is unmatched, then it has to belong to some odd component K_i.

Exercise 2.4

Yes there could be several minimizers in the Tutte Berge formula. A simple example is that of two vertices joined by a single edge, in which case maximum matching is of size 1 and we could choose U to be either of the vertices to get $1 = \frac{1}{2}(2 + 1 - 1)$.
Another example with different sizes of U: Both $U = \phi$ and $U = \{v\}$ give $2 = \frac{1}{2}(5 + 0 - 1) = \frac{1}{2}(5 + 1 - 2)$ in the given graph below:



Exercise 2.5

For a graph G and $S \subseteq V(G)$, define *deficiency of* S $def(S) = o(G \setminus S) - |S|$. And $def(G) = \max_{S \subseteq V} def(S)$.
Therefore, Tutte-Berge formula now translates to: for any maximum matching M, $|M| = \frac{1}{2}(|V| - def(G))$

Parity Lemma Let $|V| = n$ and $S \subseteq V$, then $def(S) \equiv n(\mod 2)$

Proof. Counting vertices in S and the components in $G \setminus S$, we have $|S| + o(G \setminus S) \equiv n(\mod 2)$. □

Lemma 2 Let T be a maximal set of maximum deficiency in a graph G. Then every component of $G \setminus T$ is odd and factor-critical.

Proof. Let C be any component of $G \setminus T$ and $u \in C$, then $\forall S \subseteq C - u$, we have

$$\begin{aligned} def_G(T \cup u \cup S) &= o(G - T - u - S) - (|T| + 1 + |S|) \\ &= (o(G \setminus T) - 1 + o(C - u - S)) - (|T| + 1 + |S|) \\ &= o(G \setminus T) - |T| + o(C - u - S) - |S| - 2 \\ &= def_G(T) + def_{C-u}(S) - 2 \\ \implies def_{C-u}(S) &= def_G(T \cup u \cup S) - def_G(T) + 2 \end{aligned}$$

By our choice of T, $def_G(T \cup u \cup S) < def_G(T)$ and by Parity lemma, $def_G(T \cup u \cup S)$ and $def_G(T)$ have same parity. Thus $def_{C-u}(S) \leq 0$, $\forall S \subseteq C - u$ and hence $C - u$ has a perfect matching (Tutte's theorem) and so C is critical. It follows that C is of odd-size.
(Alternatively, if C is a component of even size, then adding to T any leaf of a spanning tree of C creates a larger set with the same deficiency as T which contradicts the choice of T. And hence all components are odd). □

Next for any $T \subseteq V$, we define auxillary bipartite graph $H(T)$ by contracting each component of $G \setminus T$ to a single vertex and deleting edges within T. Let U denote the set of contracted vertices of components then $H(T)$ is a (T, U) bipartite graph with an edge $t - u$ for $t \in T, u \in U$ iff t has a neighbour in G in the component of $G \setminus T$ corresponding to u.

Lemma 3 If T is a maximal set of maximum deficiency in a graph G. then $H(T)$ contains a matching that covers T.

Proof. For $S \subseteq T$, all vertices of $U - N_{H(T)}(S)$ are odd components of $G \setminus (T \setminus S)$. By the choice of T , we have $(|U| - |N_H(S)|) - |T \setminus S| \leq def(T \setminus S) \leq def(T)$. Since $def(T) = |U| - |T|$, the inequality simplifies to $|S| \leq |N_H(S)|$. Thus Hall's Condition holds, and $H(T)$ has a matching that covers T. □

Now let T be a maximal set of deficiency $def(G)$. If we show that $def(C) = def(T)$, then for any maximum matching M:

$$|M| = \frac{1}{2}(n - def(G)) = \frac{1}{2}(n - def(C)) = \frac{1}{2}(|V| + |C| - o(G \setminus C))$$

and hence C is a minimizer in the Tutte-Berge formula.
We begin by noting that T is a minimizer of the Tutte-Berge formula and so $2|M| = n - def(G)$ implies that exactly $def(G)$ vertices are not covered by M. Furthermore. we have proved in exercise 2.3 that M matches each vertex of T with a distinct component of $G \setminus T$ (which are all odd - lemma 2) and also since M is maximum and each of these components is factor-critical (lemma 2), M matches each of these components near-perfectly. Now consider the bipartite-graph $H(T)$. By lemma 3, there is a T-saturating matching and therefore Hall's theorem says $|N_{H(T)}(S)| \geq |S|$, $\forall S \subseteq T$. Because $|N_{H(T)}(\phi)| = |\phi|$, we consider a maximal subset R for which the equality is achieved in Hall's condition, so $|N_{H(T)}(R)| = |R|$. Let R' be the union of all vertices of the components of $N_{H(T)}(R)$.

Claim $D = R \cup R'$, $C = T \setminus R$ and $B = V \setminus (T \cup R')$.

Proof. We observe that D is defined to be the set of *essential* vertices which doesn't have any neighbours in *inessential* vertices. We show that the same property holds for $R \cup R'$ as well. Since every maximum matching M matches T, and hence R, we conclude that R is a subset of essential vertices and so is R' as any maximum matching matches R to distinct components of $G[R']$. Furthermore, both R and R' don't have neighbour in other components of $G \setminus T$, that is $R \cup R'$ doesn't have a neighbours in *inessential* vertices (as T only consists of essential vertices). Therefore, $D = R \cup R'$.
Let $H' = H(T) - (R \cup N_{H(T)}(R))$. For any $S \subset T \setminus R$, we have $|N_{H(T)}(S)| > |S|$ (because if there was equality for some set, then we could have added it to R thereby contradicting its maximality). This means even if we delete a vertex v from $N_{H'}(T \setminus R)$, Hall's condition still holds and so it has a $T \setminus R$ saturating matching, which omits v. This tells us all vertices of $V \setminus (T \cup R')$ are inessential. This can be seen as follows: fix a vertex $x \in V \setminus (T \cup R')$. Let C be the component to which it belongs. By lemma 2, C is factor-critical. So if we delete x,

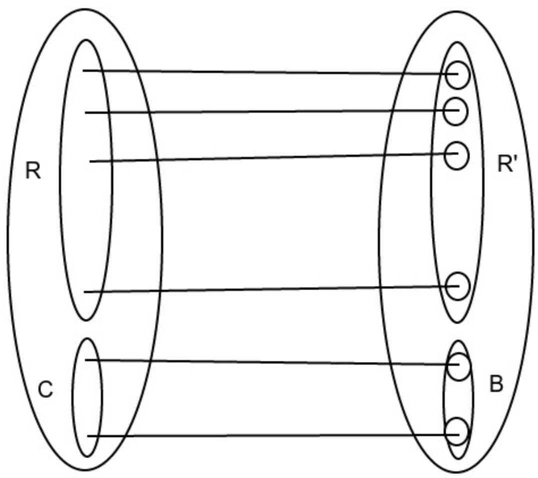


Figure 1: Gallai Decomposition

then we can find a perfect matching in $C - x$. So we delete this component C and note that it corresponds to deleting a vertex $v \in N_{H'}(T \setminus R)$ which has a maximum matching already. Therefore, we get that $V \setminus (T \cup R') \subseteq B$ but note that T only consists of essential vertices. And so $B = V \setminus (T \cup R')$ which implies $C = T \setminus R$. $\square$

Finally we have $def(T) = o(G \setminus T) - |T| = (o(G[R']) + o(G[B])) - (|R| + |C|) = (|R| + o(G[B])) - (|R| + |C|) = o(G[B]) - |C|$. Since $G[D]$ has a perfect matching, all it's components are even in size. Therefore the only odd components of $o(G \setminus C) = o(G[B])$. Thus, $def(T) = o(G \setminus C) - |C| = def(C)$.

Exercise 2.7

(1)

Fix any minimizer U. Let $u \in U$ if possible. Since G is factor critical, there is maximum (perfect) matching which leaves u unmatched, which contradicts $2,3(b)-$ any maximum matching M covers U.
Therefore, $U = \phi$

(3a)

We prove by induction on number of ears that if G has an odd ear decomposition then G is factor-critical.

Inductive hypothesis: Given G with $n-$odd ear decomposition ($n \geq 0$) then G is factor critical

Base case: $n = 0$. This means that G is just an odd cycle. Then it is clear if we remove any vertex v, then we are left with an odd-length path, which has a perfect matching.
Suppose it is true for some $n \geq 0$. Consider G with $(n+1)$ odd ears. Now we remove a vertex v from G. Let the last year be $v_0 - v_1 - \cdots - v_{2k+1}$, then we have two cases:

- *v belongs internally to the last year.* that is $v \in \{v_1, \ldots, v_{2k}\}$, then note that after removing this vertex, we will left with an odd path and an even path in the last year. WLOG say that $v_0 - \cdots - v_{i-1}$ is an even-length path and $v_{i+1} - \cdots - v_{2k}$ is an odd-length path. Then we can add edges $(v_1, v_2), (v_3, v_4), \ldots$ to our matching. Furthermore, we can also add $(v_{2k}, v_{2k-1}), (v_{2k-2}, v_{2k-3}), \ldots$ to our matching but this will remove the vertex v_{2k} from the second-last ear. Now consider the graph with only the first n ears, with the vertex v_{2k} removed. By inductive hypothesis this has a perfect matching. Hence combing with the above mentioned edges, we get a perfect matching for G with v removed.

- *v doesn't belong internally to the last year.* In this case, we consider the graph with the last year $(v_1, \ldots, v_{2k})$ removed. Now when we remove the vertex v then this graph has n ears with one vertex removed, which has a perfect matching by inductive hypothesis. And since $v_1 - \cdots - v_{2k}$ is an odd-length path, it admits a perfect matching. Hence combining these two matchings we get a perfect matching for the entire graph with v removed.

(3b)

Suppose G is factor critical. We want to show that G has an odd-ear decomposition.
We claim that G must be connected. Suppose not, then let K be any connected-component such that $G \setminus K \neq \phi$. We first note K cannot be even-sized connected-component, because $K \setminus \{v\}$ for any $v \in K$ doesn't admit a perfect matching. So K is an odd-sized component but then choose $v \in G \setminus K$, and so $G \setminus \{v\}$ has a perfect matching which leads to a contradiction as K (being odd in size) cannot admit a perfect matching.
To proceed with our main proof, we shall use structural induction on G.
Denote by M_v the perfect matching obtained after removing v from G. Consider any edge (u, v) in G and consider $M_u \oplus M_v$. It contains an alternating even-length path between u and v. Thus alongwith the (u, v) edge, we get an odd cycle in G. If G is exactly equal to this odd cycle obtained then we are done (odd ear decomposition with 0 ears).
Otherwise, we fix a vertex v of this odd cycle. Suppose we have already found k ears so far. Let H be the subgraph induced by this odd cycle and k ears. We shall also assume that no edge in M_v crosses H (and we will maintain this invariant in our inductive step).
Since G was connected, there exists an edge (a, b) crossing H such that $a \in H$ and $b \notin H$. By our above assumption $(a, b) \notin M_v$. Now $M_b \oplus M_v$ contains an even-length alternating path from b to v. This path crosses H because $v \in H$ and $b \notin H$, so let (x, y) be the first edge on this path which crosses H (where $x \notin H, y \in H$). Again by our assumption $(x, y) \notin M_v$. Now we note this $b \longrightarrow x$ subpath will be of odd-length. Hence the path $a \to b \longrightarrow x \to y$ is

an odd length path whose internal vertices do not lie in H and hence we have found a new odd-ear to be added to H and we are done.
(It should be noted that there the invariant "no edge in M_v crosses H" is still maintained because any other edge incident on this new ear cannot be from M_v as this ear itself alternates between edges from M_v and M_b. And this same reasoning applies in the base case as well).

(2)

We shall use the notation and algorithm introduced in the given paper.
Note that we already showed that any factor-critical graph G must be connected (in part $(3b)$). And next note that if G is connected then after shrinking a blossom also, G' remains connected (♦).
Another observation that follows from the previous part (3) is that if G is factor-critical then after shrinking any odd-cycle, G' remains factor-critical. This is because if G is factor-critical then starting with the given odd-cycle we can find an odd-ear decomposition (as illustrated in the previous solution). Hence after shrinking this odd-cycle, the first ear becomes an odd-cycle and the subsequent ears remain as it is. And so G' has an odd-ear decomposition implying G' is factor-critical.
Therefore, consider the last step of the given Edmonds algorithm. Using previous part (1), we know that $U = \phi$ is a minimizer in the Tutte-Burge formula and according to the algorithm: $ODD = U = \phi$. Now after removing $U = \phi$ from G', we are left with a single connected component (because G' is connected (♦)). But after removing $U = ODD$ vertices, all the remaining vertices are $EVEN$ vertices with no edges between them. Thus, there can only be a single $EVEN$ vertex remaning (otherwise G' would be disconnected).
Hence the Edmonds algorithm terminates with a single vertex.

Following are the exercises from http://math.mit.edu/g̃oemans/18453S17/flowscuts.pdf

Exercise 4.1

We first do some basic reductions.
1. We can assume that all entries are fractional (non-integer). This is because, we can choose a small enough $\epsilon > 0$ and let $A = (a_{ij})$ then we can replace it with:

$$A = \begin{pmatrix} a_{1,1} + \epsilon & a_{1,2} + \epsilon & \cdots & a_{1,n-1} + \epsilon & a_{1,n} - (n-1)\epsilon \\ a_{2,1} + \epsilon & a_{2,2} + \epsilon & \cdots & a_{2,n-1} + \epsilon & a_{2,n} - (n-1)\epsilon \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m-1,1} + \epsilon & a_{m-1,2} + \epsilon & \cdots & a_{m-1,n-1} + \epsilon & a_{m-1,n} - (n-1)\epsilon \\ a_{m,1} - (m-1)\epsilon & a_{m,2} - (m-1)\epsilon & \cdots & a_{m,n-1} - (m-1)\epsilon & a_{m,n} + (m-1)(n-1)\epsilon \end{pmatrix}$$

Note that the row sums and column sums are unaltered.
2. We can assume all the entries $0 < a_{ij} < 1$. This is because we can write $A = I + F$ where I is the integral part of each entry and F is fractional part of each entry. Then since the row sum was integral, means row sum of both I and F are integral, means row sum of F is

integral. Similarly, for column sums as well.

Hence we have a matrix A whose all entries are between 0 and 1 such that all row and column sums are integer. We construct a graph $G = (V, E)$ as follows: denote i^{th} row by a vertex $v_i \in A$ and j^{th} column by another vertex $u_j \in B$. Then $V = \{s\} \cup A \cup B \cup \{t\}$ and $E = (\{s\} \times A) \cup (A \times B) \cup (B \times \{t\})$. Edge capacities are as follows:

$$c(u,v) = \begin{cases} r_i, & \text{if } u = s, v = v_i \\ c_j, & \text{if } v = t, u = u_j \\ 1, & \text{otherwise} \end{cases}$$

where r_i denote the i^{th} row sum and c_j denote the j^{th} column sum.
We run Ford-Fulkerson algorithm on this graph G to obtain a max-flow. *Observations*:

1. Max flow value $\leq \sum_i r_i$ (sum of all edge capacities going out of s)
2. Set $g(s, v_i) = r_i$, $g(v_i, u_j) = a_{ij}$, and $g(u_j, t) = c_j$, then the value of this flow is $\sum_i r_i$. (g is a possible flow because of the given condition on A about row sums and columns sums). And hence the max-flow value is $\sum_i r_i$.
3. If the capacities are integral, then note that in the Ford-Fulkerson algorithm, we start with flow value 0 and in each iteration we increment the flow value by the minimum capacity edge on the path, which is an integer. Thus, there exists an integral max-flow f.

Finally, consider the matrix A' such that $A'_{ij} = f(v_i, u_j)$, flow value along edge (v_i, u_j), then

- Row sums and column sums of A and A' are identical (because of flow conservation for each internal vertex)
- $A'_{ij} = 0$ (or 1), and so $A'_{ij} = \lfloor A_{ij} \rfloor$ (or $\lceil A_{ij} \rceil$) as $0 < A_{ij} < 1$ (from our initial assumption).

Alternate algorithm: Here's a more direct approach to find the values of A' (assuming $0 < A_{ij} < 1$):

Algorithm 1: Rounding each entry of A

```
Result: 0-1 matrix
for i = 0; i < n; i++ do
    for j = 0; j < r_i do
        if c_j == 0 then
            continue;
        end
        a_ij ← 1;
        c_j ← c_j − 1;
        j ← j + 1;
    end
end
```

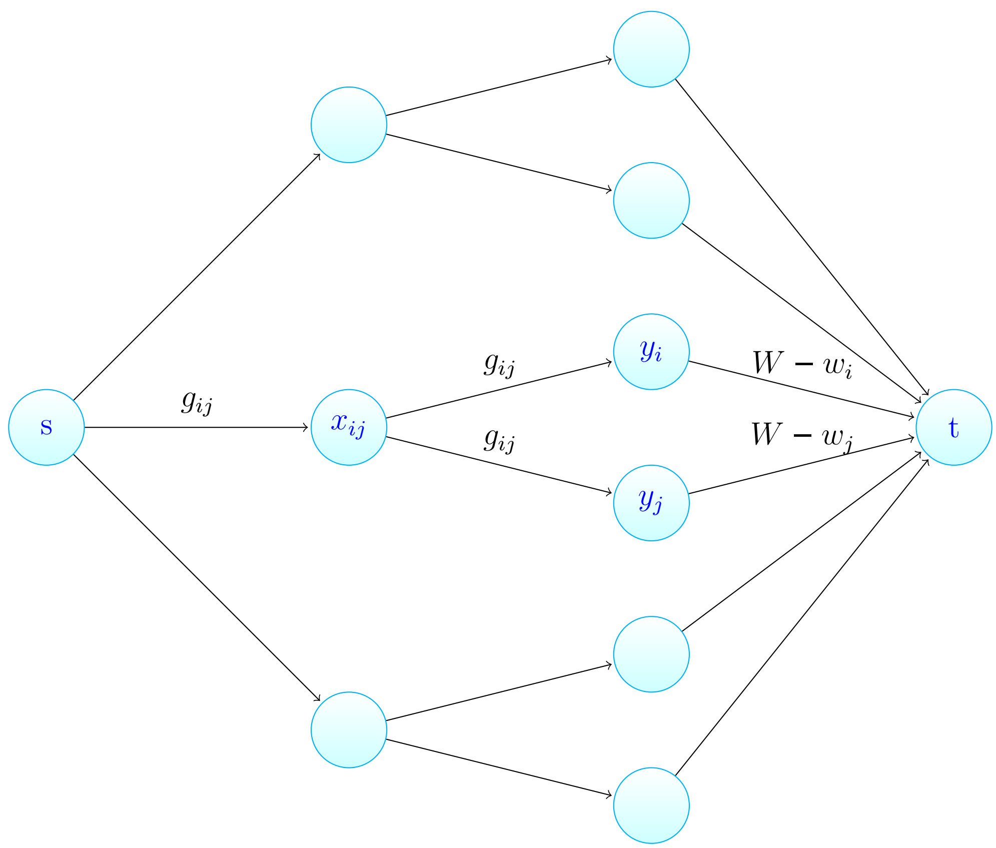
(This algorithm demonstrates that the values a_{ij} don't matter. We are just finding some $0/1$ matrix such each row and column has a specified number of $1's$ only - assuming existence of a solution).

4.2

We construct a graph $G = (V, E)$ as follows: $V = \{s\} \cup \{x_{ij} \mid 1 \leq i < j \leq n-1\} \cup \{y_i \mid 1 \leq i \leq n-1\} \cup \{t\}$ and add edges:

- (s, x_{ij}) of capacity g_{ij}, $\forall 1 \leq i < j \leq n-1$
- (x_{ij}, y_i), (x_{ij}, y_j), both of capacity g_{ij}, $\forall 1 \leq i < j \leq n-1$
- (y_i, t) of capacity $W - w_i$, $\forall 1 \leq i < j \leq n-1$

where $W = w_n + \sum_{i=1}^{n-1} g_{i,n}$, denotes the total number of games team n can win. (We remark here that if there is a possible outcome of games such that team n has at least as many victories as all the other teams, then there is also a possible outcome in which team n has as many victories as all other teams but team n wins all the games to be played by them).



Here x_{ij} corresponds to the game between teams i and j and y_i corresponds to team i. If there are x games played between team i and j, then x units of flow in x_{ij}, is divided among y_i and y_j depending upon which team won how many games. Finally, the flow from y_i to t

corresponds to the number of games won by team i, which can be no more than $W - w_i$, for team n to have at least as many victories as all the other team.
Thus, team n has at least as many victories as all the other team iff all games have an outcome, that is all the edges out of s are saturated, implying the maximum flow is of size $\sum_{1 \le i < j \le n-1} g_{ij}$.

It suffices to give a necessary and sufficient condition for team n to lose the game, which is as follows: there exists a subset $R \subseteq \{1, \ldots, n-1\}$ such that $W < \frac{w(R)+g(R)}{|R|}$ where $w(R) = \sum_{i \in R} w_i$ and $g(R) = \sum_{i.j \in R, i<j} g_{ij}$.
To prove this, suppose that team n loses, which means that the flow is of strictly lesser value and so all the edges out of s are not saturated. So we find the min cut (S, T) (S is nodes reachable from s in the residual graph) in this graph and let R be the team nodes y_j which are reachable from s in the residual graph. We claim that:

1. R is non-empty. Because all edges out of s are not-saturated implies there is a x_{ij} such that $s \to x_{ij}$ is an edge in the residual graph. And since total incoming flow in x_{ij} is strictly less than g_{ij}, it is clear that $x_{ij} \to y_i$ and $x_{ij} \to y_j$ are both edges in the residual graph, as both have capacity g_{ij} and cannot get saturated.

2. $x_{ij} \in S$ iff $y_i, y_j \in R$. If $x_{ij} \in S$ then from the above argument it follows that both $y_i, y_j \in R$. Conversely if $y_i, y_j \in R$ and $x_{ij} \notin S$ then adding this vertex to S decreases the capacity of the cut.

Finally, we compare the cuts (S, T) and $(\{s\}, G \setminus \{s\})$.

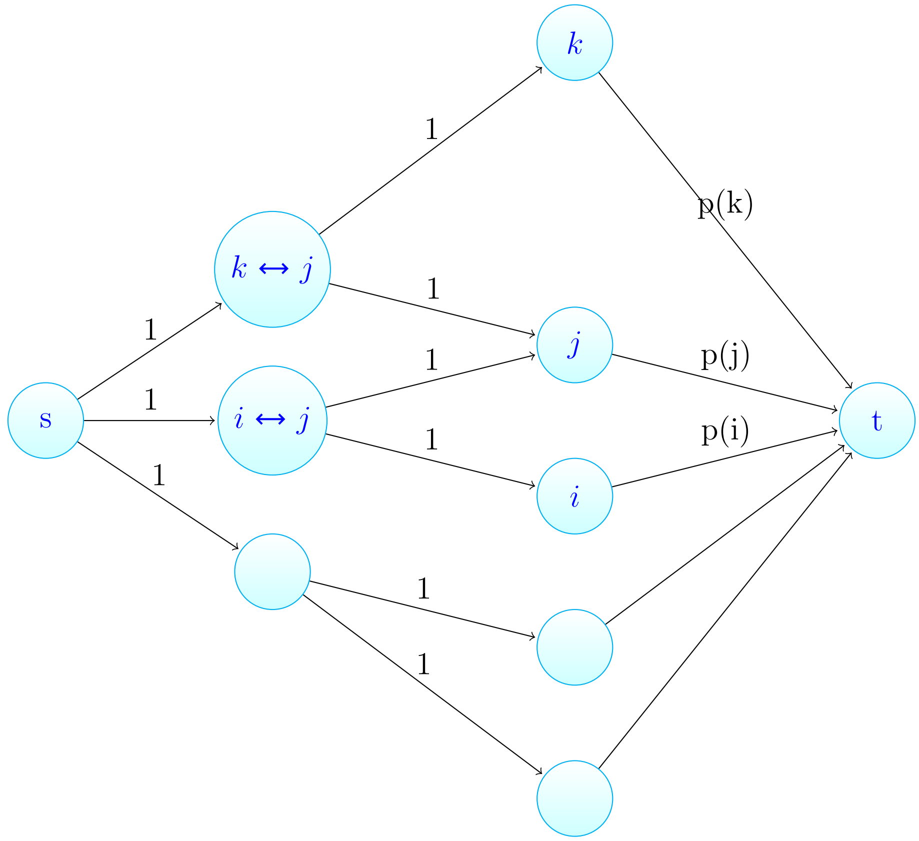
$$\begin{aligned}
& c(\{s\}, G \setminus \{s\}) && > && c(S,T) \\
\iff & \sum_{1 \le i < j \le n-1} g_{ij} && > && c(S,T) \\
\iff & \sum_{1 \le i < j \le n-1} g_{ij} && > && \sum_{i,j \notin R} g_{ij} + \sum_{i \in R} (W - w_i) \\
\iff & \sum_{i,j \in R} g_{ij} && > && |R|W - \sum_{i \in R} w_i \\
\iff & w(R) + r(R) && > && |R|W \\
\iff & W && < && \frac{w(R) + r(R)}{|R|}
\end{aligned}$$

4.3

Let $G = (V, E)$ be the given graph. We construct a graph $G' = (V', E')$ as follows: $V' = \{s\} \cup \{i \leftrightarrow j \mid \{i, j\} \in E\} \cup \{i \mid i \in V\} \cup \{t\}$ and add edges:

- $(s, i \leftrightarrow j)$ of capacity 1, $\forall \{i, j\} \in E$
- $(i \leftrightarrow j, i)$, $(i \leftrightarrow j, j)$, both of capacity 1, $\forall \{i, j\} \in E$

- (i, t) of capacity $p(i)$, $\forall i \in V$



(a)

Here if the node $i \leftrightarrow j$ has an incoming unit flow then either it will be directed towards i or j, which will determine the orientation of this edge as: if it's directed towards i then orient $j \rightarrow i$ and if it's directed towards j then orient $i \rightarrow j$. And if $i \leftrightarrow j$ has an incoming zero flow then that means whatever the orientation of this edge be, it will violate some constraint. Next, the incoming unit flow in each $i \in V$ contributes to the in-degree of i and since the capacity of (i, t) edge is $p(i)$, we can't have more than $p(i)$ units of flow incoming into i. Thus, the max flow in this graph is of size $|E|$ (that is all the outgoing edges from s are saturated) iff there exists a possible orientation of the edges such that all in-degree constraints are satisfied.

(b)

Suppose that graph cannot be oriented, then the max flow is of size strictly less than $|E|$. We consider the min-cut (S, T) (S is nodes reachable from s in the residual graph). Since some outgoing edges from s are not saturated so $S \setminus \{s\}$ is non-empty.

We note that $i, j \in S$ iff $i \leftrightarrow j \in S$, because if $i \leftrightarrow j \in S$ then $(s) \longrightarrow (i \leftrightarrow j) \longrightarrow (j)$ and $(s) \longrightarrow (i \leftrightarrow j) \longrightarrow (i)$ are paths in the residual graph. Conversely, if $i, j \in S$ and $(i \leftrightarrow j) \notin S$ then it can be added to S and reduce the cut size.
Let R be the set of vertex nodes i reachable from s. Since $S \setminus \{s\}$ is non-empty, R is also non-empty because of the above-mentioned fact.
Finally, we compare the cuts (S, T) and $(\{s\}, V' \setminus \{s\})$.

$$c(\{s\}, V' \setminus \{s\}) > c(S, T)$$

But $c(\{s\}, V' \setminus \{s\}) = |E|$ and $c(S, T) = |\#\{i \leftrightarrow j \mid i \notin R \wedge j \notin R\}| + \sum_{v \in R} p(v)$ (edges going from s to $i \leftrightarrow j$ and edges going from i to t).
But $|E| - |\#\{i \leftrightarrow j \mid i \notin R, j \notin R\}| = |\#\{i \leftrightarrow j \mid i \in R \vee j \in R\}| = |E(R)|$. Hence, $|E(R)| > \sum_{v \in R} p(v)$

4.4

Theorem Given a digraph G and $s, t \in G$ with all edge capacities 1. There is a flow of value k from s to t iff there are k-disjoint paths between s and t.

Proof. Suppose there are k-disjoint paths between s and t, then set $f(e) = 1$ for all the edges on these paths and $f(e) = 0$ otherwise. Since paths were disjoint, this gives us a flow of value k. □

Conversely, suppose there is a flow of value k, Choose a vertex v such that $f(s, v) = 1$. By conservation there must be a vertex w such that $f(v, w) = 1$. Extending this way until we reach t, everytime choosing a new edge, we get a path from s to t. But now note that there are k vertices such that $f(s, v) = 1$. So for each such vertex we can perform the above process and obtain k edge-disjoint paths from s to t.

Now, we are given an undirected graph G. We find it's *maximum adjacency ordering* $v_1, v_2, \ldots, v_n$. Using claim 4.6 (from the paper), we conclude that $(\{v_1, \ldots, v_{n-1}\}, \{v_n\})$ is a (v_{n-1}, v_n) cut. But since the degree of each vertex is atleast k, this cut size is also atleast k.
Finally, we convert graph G into a digraph by replacing each edge $\{u, v\}$ with two edges (u, v) and (v, u) all of capacity 1. Now by our above argument, the (v_{n-1}, v_n) min cut size is $\geq k$ and so the max-flow is $\geq k$. Using above-mentioned theorem, we get that there are atleast k edge-disjoint paths between v_n and v_{n-1}.

4.5

Want to show that

$$u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cup B)) + u(\delta(A \cap B))$$

where $u(\delta(S)) = sum_{e \in \delta(S)} u(e)$

Since the capacities are non-negative, it suffices to show that each term on the right also appears on the left (same number of times).
Therefore, $u(\delta(A \cup B)) + u(\delta(A \cap B)) = \sum_{e \in \delta(A \cup B)} u(e) + \sum_{f \in \delta(A \cap B)} u(f) = \sum_{e \in \delta(A \Delta B)} u(e) + 2 \sum_{f \in \delta(A \cap B)} u(f)$. Now each term $u(e)$ where $e \in \delta(A \Delta B)$ is also counted once either for $e \in A$ or $e \in B$ on the left. Similarly, each term $u(f)$ where $f \in A \cap B$ is counted twice for both $f \in A$ and $f \in B$. So $u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cup B)) + u(\delta(A \cap B))$.

4.6

Suppose f is submodular then consider $A = S \cup \{e\}, B = T$:

$$f(S \cup \{e\}) + f(T) \geq f(T \cup \{e\}) + f(S)$$

because $(S \cup \{e\}) \cup T = T \cup \{e\}$ and $(S \cup \{e\}) \cap T = (S \cap T) \cup (T \cap \{e\}) = S$ as $S \subseteq T$ and $e \notin T$. After re-arranging, we get that submodularity implies diminishing returns.

Conversely, let $A \setminus B = \{a_1, a_2, \ldots, a_n\}$ then define $A_0 = A \cap B$, $A_i = A_{i-1} \cup \{a_i\}$, $\forall i \geq 1$ and $B_0 = B$, $B_i = B_{i-1} \cup \{a_i\}$, $\forall i \geq 1$.
Now we note that, $A_i \subseteq B_i$. Proof by induction: $A \cap B = A_0 \subseteq B_0 = B$. Assuming $A_{i-1} \subseteq B_{i-1}$, $A_{i-1} \cup \{a_i\} = A_i \subseteq B_i = B_{i-1} \cup \{a_i\}$.
Furthermore, $a_{i+1} \notin B_i$ by construction. Hence we can apply property of diminishing returns on these sets as:

$$\begin{aligned}
f(A_1) - f(A_0) &\geq f(B_1) - f(B_0) \\
f(A_2) - f(A_1) &\geq f(B_2) - f(B_1) \\
&\vdots \\
f(A_n) - f(A_{n-1}) &\geq f(B_n) - f(B_{n-1})
\end{aligned}$$

Adding them up, we get $f(A_n) - f(A_0) \geq f(B_n) - f(B_0)$. But $f(A_n) = A$ and $f(B_n) = f(A \cup B)$. Hence

$$f(A) - f(A \cap B) \geq f(A \cup B) - f(B)$$

After re-arranging, we get that diminishing returns implies submodularity.

[References: Douglas B. West notes on Gallai Edmonds Structure Theorem for problem 2.5 and discussion with Sricharan AR and Satya P. Nayak for problem 4.4]