Graph Theory - Homework 3

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February 15, 2020

Exercise 1

Proof. .
(a) Since, $|T - e + e'| = |T| = |V| - 1$, it suffices to show that the induced subgraph on $T - e + e'$ is connected for some $e' \in T' - T$. If $e = (x, y)$, then it suffices to show that there is a path between $x \rightsquigarrow y$ in $T - e + e'$.
Consider $T \setminus \{e\}$. It has two connected components X, Y such that $X \sqcup Y = V(T)$ and $x \in X, y \in Y$. Now since $X \sqcup Y = V(T')$, there are vertices $x' \in X, y' \in Y$ such that $e' = (x', y') \in T' - T$. Thus, $x \rightsquigarrow x' \xrightarrow{e'} y' \rightsquigarrow y$ is a path between x, y in $T - e + e'$.

(b) Since, $|T' + e - e'| = |T'| = |V| - 1$, it suffices to show that the induced subgraph on $T' + e - e'$ has no cycles for some $e' \in T' - T$.
Consider $T' + e$. It contains a cycle C passing through the edge $e = (x, y)$. Let $e' \in T' - T$ be any other edge on this cycle C. We claim that $T' + e - e'$ has no cycles. For the sake of contradiction, suppose it has a cycle C'. If $e \notin C'$ then $C' \subseteq T'$ which is a contradiction to T' being a tree. Otherwise if $e \in C'$, then consider the path $C' \setminus \{e\}$ between x, y in T'. But then $C \setminus \{e\}$ is also a path between x, y in T', which is clearly different from $C' \setminus \{e\}$ as $e' \in C$ but $e' \notin C'$. But a tree has a unique path between any two vertices. □

Exercise 2

Proof. We use Kirchoff's theorem which says that number of spanning trees of a connected graph G is given by:

$$t(G) = \frac{1}{n}\lambda_1\lambda_2 \dots \lambda_{n-1}$$

where λ_i are the non-zero eigenvalues of laplacian $L(G)$.
We have $G = K_{m,n}$, $L(G) = D(G) - A(G)$

$$L(G) = \begin{pmatrix} nI & -J \\ -J & mI \end{pmatrix}$$

Let $v = (v_1 \dots v_m, u_1 \dots u_n)^T$, then $L(G)v = \lambda v$ gives us following two equations:

$$(n - \lambda)v_i = \sum u_j$$

$$(m - \lambda)u_j = \sum v_i$$

.

Case 1: $\lambda = n$. $u_1 = u_2 = \dots u_n = 0$ and $\sum v_i = 0$. Hence $v = (v_1 \dots v_m, 0 \dots 0)^T$ is an eigenvector with eigenvalue n. This gives us that $\lambda = n$ has an eigenspace of dimension $m - 1$.

Case 2: $\lambda = m$. $v_1 = v_2 = \dots v_m = 0$ and $\sum v_j = 0$. Hence $v = (0 \dots 0, u_1 \dots u_n)^T$ is an eigenvector with eigenvalue m. This gives us that $\lambda = m$ has an eigenspace of dimension $n - 1$.

Case 3: $\lambda \neq m, n$ Then we have $v_1 = v_2 = \dots = v_m$ and $u_1 = u_2 = \dots = u_n$. Thus

$$u_1 = \frac{mv_1}{m - \lambda}; v_1 = \frac{nu_1}{n - \lambda} = \left(\frac{n}{n - \lambda}\right)\left(\frac{mv_1}{m - \lambda}\right)$$

If $v_1 = 0 \implies v_i = u_j = 0$. Hence

$$\left(\frac{n}{n - \lambda}\right)\left(\frac{m}{m - \lambda}\right) = 1 \implies \lambda = 0, m + n$$

(Observation: Case 3 was unnecessary as we know $\sum \lambda = 2mn$ and from the first two cases we already have that $n(m - 1) + m(n - 1) = 2mn - (m + n)$, so the last eigenvalue has to be $m + n$)

Finally,

$$t(G) = \frac{1}{m + n} n^{m-1} m^{n-1} (m + n) = n^{m-1} m^{n-1}$$

□