Graph Theory; Second Set of assignment problems

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Assume that we are dealing with simple graphs.

Let G be a triangle-free simple graph. Let $f : V(G) \to [0,1]$ be defined such that $\sum_{v \in V} f(v) = 1$. Over the set of all such functions, we wish to maximize the sum $S = \sum_{(xy) \in E(G)} f(x)f(y)$. Prove the following:

(a) If x and y are not adjacent and if $\sum_{z \sim x} f(z) \geq \sum_{z \sim y} f(z)$, then any (small) change in $f(x)$ (from $f(x)$ to $f(x) + \epsilon$) and in $f(y)$ (from $f(y)$ to $f(y) - \epsilon$) changes S to S' where $S' \geq S$.

(b) Show that in the above argument we can actually take ϵ to be as large as $f(y)$.

(c) Show that through a sequence of steps described as above, we obtain largest value of S when, for only two adjacent vertices v and w, we have $f(v)$ and $f(w)$ both non-zero.

(d) Show that by taking $f(v) = f(w) = \frac{1}{2}$, S is maximized with the value $\frac{1}{4}$.

(e) Show that by letting $f(x) = \frac{1}{n}$, for all x, we get one value of S to be $\frac{e(G)}{n^2}$.

(f) Use the last two parts to prove Turan's theorem: $e(G) \leq \frac{n^2}{4}$.