

# Proof And Types: Assignment 1

Kishlaya Jaiswal

October 13, 2019

## 1.8

Prove the Weak Church Rosser Theorem. For all  $M_1, M_2, M_3 \in \Lambda$ , if  $M_1 \rightarrow_\beta M_2$  and  $M_1 \rightarrow_\beta M_3$  then there is a term  $M_4 \in \Lambda$  such that  $M_2 \rightarrow_\beta M_4$  and  $M_3 \rightarrow_\beta M_4$ .

*Proof.* We first prove a sequence of lemmas.

**Lemma 1:** If  $x \neq y$  and  $x \notin FV(L)$  then

$$M[x := N][y := L] = M[y := L][x := N[y := L]]$$

*Proof.* By induction on  $M$ .

- Case:  $M = x$ , since  $x \neq y$

$$\begin{aligned} M[x := N][y := L] &= x[x := N][y := L] \\ &= N[y := L] \\ &= x[x := N[y := L]] \\ &= x[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- Case:  $M = y$

$$\begin{aligned} M[x := N][y := L] &= y \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- Case:  $M = x$  where  $z \neq x$  and  $z \neq y$

$$\begin{aligned} M[x := N][y := L] &= & z \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- Case:  $M = \lambda z.P$  where  $z \neq x$  and  $z \neq y$

$$\begin{aligned} M[x := N][y := L] &= \lambda z.P[x := N][y := L] \\ &= \lambda z.P[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- Case:  $M = PQ$

$$\begin{aligned} M[x := N][y := L] &= (PQ)[x := N][y := L] \\ &= (P[x := N][y := L])(Q[x := N][y := L]) \\ &= (P[y := L][x := N[y := L]])(Q[y := L][x := N[y := L]]) \\ &= (PQ)[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

□

**Lemma 2:** Assume that  $P, P' \in \Lambda$  are such that  $P \twoheadrightarrow_\beta P'$  then for all  $x \in V$  and  $Q \in \Lambda$ :

- (i)  $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$
- (ii)  $PQ \twoheadrightarrow_\beta P'Q$
- (iii)  $QP \twoheadrightarrow_\beta QP'$

*Proof.* By induction on the derivation of  $P \twoheadrightarrow_\beta P'$ .

- Case:  $P \rightarrow_\beta P'$  then clearly,  $\lambda x.P \rightarrow_\beta \lambda x.P'$  and  $PQ \rightarrow_\beta P'Q$  and  $QP \rightarrow_\beta QP'$
- Case: there exists  $P''$  such that  $P \twoheadrightarrow_\beta P''$  and  $P'' \rightarrow_\beta P'$  then by inductive hypothesis,  $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$  and  $\lambda x.P'' \rightarrow_\beta \lambda x.P'$  and so  $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$ .  
Again, by inductive hypothesis,  $PQ \twoheadrightarrow_\beta P''Q$  and  $P''Q \rightarrow_\beta P'Q$  and so  $PQ \twoheadrightarrow_\beta P'Q$ .  
Similarly, by inductive hypothesis,  $QP \twoheadrightarrow_\beta QP''$  and  $QP'' \rightarrow_\beta QP'$  and so  $QP \twoheadrightarrow_\beta QP'$ .
- Case:  $P = P'$  then clearly,  $\lambda x.P = \lambda x.P'$  and  $PQ = P'Q$  and  $QP = QP'$

□

**Lemma 3:** For all  $P, P', Q \in \Lambda$  if  $P \rightarrow_\beta P'$  then also  $P[x := Q] \rightarrow_\beta P'[x := Q]$

*Proof.* By induction on the derivation of  $P \rightarrow_\beta P'$ .

- Case:  $P = (\lambda y.R)S$  and  $P' = R[y := S]$ . Then we get

$$\begin{aligned}
 P[x := Q] &= ((\lambda y.R)S)[x := Q] \\
 &= (\lambda y.R[x := Q])(S[x := Q]) \\
 &\rightarrow_\beta R[x := Q][y := S[x := Q]] \\
 &= R[y := S][x := Q] && \text{(by lemma 1)} \\
 &= P'[x := Q]
 \end{aligned}$$

- Case:  $P = \lambda y.R$  and  $P' = \lambda y.R'$  such that  $R \rightarrow_\beta R'$ . Then we get

$$\begin{aligned}
 P[x := Q] &= (\lambda y.R)[x := Q] \\
 &= \lambda y.R[x := Q] \\
 &\rightarrow_\beta \lambda y.R'[x := Q] \\
 &= (\lambda y.R')[x := Q] \\
 &= P'[x := Q]
 \end{aligned}$$

- Case:  $P = RS$  and  $P' = R'S$  such that  $R \rightarrow_\beta R'$ . Then we get

$$\begin{aligned}
 P[x := Q] &= (RS)[x := Q] \\
 &= R[x := Q]S[x := Q] \\
 &\rightarrow_\beta R'[x := Q]S[x := Q] \\
 &= (R'S)[x := Q] \\
 &= P'[x := Q]
 \end{aligned}$$

- Case:  $P = RS$  and  $P' = RS'$  such that  $S \rightarrow_\beta S'$ . Then we get

$$\begin{aligned}
 P[x := Q] &= (RS)[x := Q] \\
 &= R[x := Q]S[x := Q] \\
 &\rightarrow_\beta R[x := Q]S'[x := Q] \\
 &= (RS')[x := Q] \\
 &= P'[x := Q]
 \end{aligned}$$

□

**Lemma 4:** For all  $P, Q, Q' \in \Lambda$  if  $Q \rightarrow_\beta Q'$  then also  $P[x := Q] \rightarrow_\beta P[x := Q']$

*Proof.* By induction on structure of  $P$ .

- Case:  $P = x$  then  $P[x := Q] = Q \rightarrow_\beta Q' = P[x := Q']$
- Case:  $P = y$  then  $P[x := Q] = y \rightarrow_\beta y = P[x := Q']$
- Case:  $P = \lambda y.P'$  then

$$\begin{aligned}
P[x := Q] &= (\lambda y.P')[x := Q] \\
&= \lambda y.P'[x := Q] \\
&\rightarrow_\beta \lambda y.P'[x := Q'] \\
&= (\lambda y.P')[x := Q'] \\
&= P[x := Q']
\end{aligned}$$

- Case:  $P = P_1P_2$  then

$$\begin{aligned}
P[x := Q] &= (P_1P_2)[x := Q] \\
&= (P_1[x := Q])(P_2[x := Q]) \\
&\rightarrow_\beta (P_1[x := Q'])(P_2[x := Q]) \\
&\rightarrow_\beta (P_1[x := Q'])(P_2[x := Q']) \\
&= (P_1[x := Q'])(P_2[x := Q']) \\
&= (P_1P_2)[x := Q'] \\
&= P[x := Q']
\end{aligned}$$

□

Now, we can finally present the proof of Weak Church Rosser property.

We proceed by induction on derivation of  $M_1 \rightarrow_\beta M_2$ .

- Case:  $M_1 = (\lambda x.P)Q$  and  $M_2 = P[x := Q]$ . Then  
Either  $M_3 = (\lambda x.P')Q$  such that  $P \rightarrow_\beta P'$ . Then set  $M_4 = P'[x := Q]$  and note that both  $M_2 \rightarrow_\beta M_4$  (by Lemma 3) and  $M_3 \rightarrow_\beta M_4$  (by Lemma 2).  
Or either  $M_3 = (\lambda x.P)Q'$  such that  $Q \rightarrow_\beta Q'$ . Then set  $M_4 = P[x := Q']$  and note that both  $M_2 \rightarrow_\beta M_4$  (by Lemma 4) and  $M_3 \rightarrow_\beta M_4$  (by substitution).
- Case:  $M_1 = \lambda x.P$  and  $M_2 = \lambda x.P'$  because  $P \rightarrow_\beta P'$ . Then  $M_3$  must be of the form  $\lambda x.P''$  such that  $P \rightarrow_\beta P''$ . Then by inductive hypothesis, there exists a term  $Q$  such that  $P' \rightarrow_\beta Q$  and  $P'' \rightarrow_\beta Q$ . Hence setting  $M_4 = \lambda x.Q$  we get the sought term.
- Case:  $M_1 = PQ$  and  $M_2 = P'Q$  such that  $P \rightarrow_\beta P'$  then  
Either  $M_3 = P''Q$  such that  $P \rightarrow_\beta P''$ . By inductive hypothesis, there exists a term  $R$  such that  $P' \rightarrow_\beta R$  and  $P'' \rightarrow_\beta R$ . Hence  $M_4 = RQ$  is the sought term.  
Or  $M_3 = PQ'$  such that  $Q \rightarrow_\beta Q'$ . Then note that both  $M_2 \rightarrow_\beta P'Q'$  and  $M_3 \rightarrow_\beta P'Q'$ . Hence  $M_4 = P'Q'$  as desired.

□

## 1.14

Consider the fixed point combinator

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

Show that  $F(YF) =_\beta YF$  holds for all  $F$ .

*Proof.*

$$\begin{aligned} YF &= (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))F \\ &\rightarrow_\beta ((\lambda x.f(xx))(\lambda x.f(xx)))[f := F] \\ &= (\lambda x.F(xx))(\lambda x.F(xx)) \\ &\rightarrow_\beta (F(xx))[x := (\lambda x.F(xx))] \\ &= F((\lambda x.F(xx))(\lambda x.F(xx))) \end{aligned}$$

And also,

$$\begin{aligned} F(YF) &= F((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))F) \\ &\rightarrow_\beta F(((\lambda x.f(xx))(\lambda x.f(xx)))[f := F]) \\ &= F((\lambda x.F(xx))(\lambda x.F(xx))) \end{aligned}$$

Hence,  $YF =_\beta F(YF)$  □

## 2.5

Show that intuitionistic propositional logic is consistent, that is  $\not\vdash \perp$ .

*Proof.* Suppose not, that is  $\vdash \perp$ .

Now, using soundness theorem for Natural Deduction logic system, we know that for every Heyting Algebra  $\mathcal{H}$  and every valuation  $v$  on  $\mathcal{H}$ ,  $v(\perp) = 1$ .

But consider, the usual boolean algebra on  $(0, 1)$  (which is a Heyting algebra) and the valuation  $v: PV \rightarrow \mathcal{H}$  where  $v(x) = 0 \ \forall x \in PV$ .

According to the definition,  $\tilde{v}(\perp) = 0$ , where  $\tilde{v}$  is the extension of  $v$  to all the well-formed formulas. But that leads to a contradiction. □

## 2.6

Let  $(A, \vee, \wedge)$  be an algebra with two binary operations (written in infix notation). Assume that the equations (i)-(iv) of Lemma 2.3.4 hold for all elements of  $A$ . Define  $a < b$  by  $a \vee b = b$ . Prove that  $(A, \leq)$  is a lattice, where suprema and infima are given by  $\vee$  and  $\wedge$ .

*Proof.* Lemma 2.3.4 says that the following are valid:

- $a \vee a = a$  and  $a \wedge a = a$
- $a \vee b = b \vee a$  and  $a \wedge b = b \wedge a$
- $(a \vee b) \vee c = a \vee (b \vee c)$  and  $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $(a \vee b) \wedge a = a$  and  $(a \wedge b) \vee a = a$

Define  $\leq$  relation on  $A$  as:  $a \leq b$  if  $a \vee b = b$ .

Using these, we shall first show that our  $(A, \leq)$  is a poset:

**Reflexive:** since we have  $a \vee a = a$  (from the lemma)  $\implies a \leq a$

**Anti-symmetric:** Suppose  $a \leq b$  and  $b \leq a$  then we have  $a \vee b = a$  and  $b \vee a = a$ . Using the lemma,  $a = b \vee a = a \vee b = b$

**Transitive:** Suppose  $a \leq b$  and  $b \leq c$  then we have  $a \vee b = b$  and  $b \vee c = c$ . Hence,  $b \vee c = (a \vee b) \vee (b \vee c) \implies c = a \vee (b \vee b) \vee c = a \vee (b \vee c) = a \vee c \implies a \leq c$

Now, we show that the operation

$\vee$  **is suprema:** Suppose  $c$  is an upper bound on  $a$  and  $b$  then  $a \leq c$  and  $b \leq c \implies a \vee c = c$  and  $b \vee c = c$  then we get  $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$ . Hence  $a \vee b \leq c$

Before proving infima, we show that  $a \vee b = b$  and  $a \wedge b = a$  are equivalent.

$$b = a \vee b \implies a \wedge b = a \wedge (a \vee b) = (a \vee b) \wedge a = a$$

$$a = a \wedge b \implies a \vee b = (a \wedge b) \vee b = (b \wedge a) \vee b = b$$

$\wedge$  **is infima:** Suppose  $c$  is a lower bound on  $a$  and  $b$  then  $c \leq a$  and  $c \leq b \implies c \wedge a = c$  and  $c \wedge b = c$  then we get  $c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c$ . Hence  $c \leq a \wedge b$   $\square$

## 2.8

Let  $A$  be a lattice satisfying  $(a \vee b) \wedge c \leq (a \wedge c) \vee (b \wedge c)$ , for all  $a, b, c$ . Show that  $A$  is distributive.

*Proof.* We shall first show that:

$$(a \wedge c) \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Then from anti-symmetric property of  $\leq$ , the result follows.

For that, we recall the following two properties:

**Property 1:**  $a \leq c$  and  $b \leq c \implies a \vee b \leq c$

**Property 2:**  $c \leq a$  and  $c \leq b \implies c \leq a \wedge b$

Observe

$$a \wedge c \leq a \leq a \vee b \text{ \& } a \wedge c \leq c \implies a \wedge c \leq (a \vee b) \wedge c \text{ (using property 2)}$$

Similarly,

$$b \wedge c \leq b \leq a \vee b \text{ \& } b \wedge c \leq c \implies b \wedge c \leq (a \vee b) \wedge c \text{ (using property 2)}$$

Now using, property 1 on the above two inequalities, we get our desired result.

Now we show the other distributive lattice property holds, namely:

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

First note that,  $a \wedge b \leq a \leq a \vee c$  and  $a \wedge b \leq b \leq b \vee c$ . Hence  $a \wedge b \leq (a \vee c) \wedge (b \vee c)$ . Next,  $c \leq a \vee c$  and  $c \leq b \vee c$  and so  $c \leq (a \vee c) \wedge (b \vee c)$ . Therefore, we have established.  $(a \wedge b) \vee c \leq (a \vee c) \wedge (b \vee c)$ .

For the other inequality, note that,  $(a \wedge b) \vee c = (a \wedge b) \vee (a \wedge c) \vee c \geq (a \wedge (b \vee c)) \vee c = (a \wedge (b \vee c)) \vee (c \wedge (b \vee c)) \geq (a \vee c) \wedge (b \vee c)$ , where both the inequalities follow from the assumption given in the problem. □

## 2.13

Assume that  $\mathcal{B}, v \not\models \varphi$ , for some Boolean algebra  $\mathcal{B}$  and some  $\varphi$  and  $v$ . Show that there exists a prime filter  $F$  in  $\mathcal{B}$ , with  $v(\neg\varphi) \in F$ . Then define a binary valuation by  $w(p) = 1$  iff  $v(p) \in F$  and show that  $w(\varphi) = 0$ .

*Proof.* Recall the following result:

**Theorem:** Let  $F$  be a proper filter in  $\mathcal{H}$  and  $a \notin F$ . There exists a prime filter  $G$  such that  $F \subseteq G$  and  $a \notin G$

Now, we state and prove another result regarding prime filters.

**Proposition:** Let  $G$  be a prime filter of a boolean algebra. Then  $\alpha \in G$  iff  $-\alpha \notin G$

*Proof.* : Let  $\alpha \in G$ . Then since  $\alpha \cap -\alpha = 0$  and  $0 \notin G$  (as  $G$  is proper), we get that  $-\alpha \notin G$ . Now suppose,  $-\alpha \notin G$ . Since  $-\alpha \cup \alpha = 1$  and  $1 \in G$ , by the property of prime filters we get that  $\alpha \in G$ . □

Now we note that, since  $v \not\models \varphi \implies v(\varphi) \neq 1 \implies -v(\varphi) \neq 0$  as  $\mathcal{B}$  is a Boolean algebra so  $v(\varphi) \vee -v(\varphi) = 1$

Let  $F = \{a : -v(\varphi) \leq a\}$ . Then  $F$  is a proper filter because:

- if  $a, b \in F$  then  $-v(\varphi) \leq a$  and  $-v(\varphi) \leq b \implies -v(\varphi) \leq a \wedge b \implies a \wedge b \in F$
- if  $a \in F$  then  $-v(\varphi) \leq a$  and so  $-v(\varphi) \leq b, \forall a \leq b$
- Since  $-v(\varphi) \neq 0 \implies 0 \notin F$

We can apply this theorem to  $F$  and  $a = 0$  to get a prime filter  $G$  such that  $-v(\varphi) \in F \subseteq G$ . Now,

$$v(\neg\varphi) = v(\varphi \rightarrow \perp) = v(\varphi) \implies v(\perp) = v(\varphi) \implies 0 = -v(\varphi) \in G$$

**Claim:**  $v(\varphi) \notin G$

*Proof.* : Suppose  $v(\varphi) \in G \implies v(\varphi) \wedge -v(\varphi) = 0 \in G$ . But  $G$  being a filter,  $0 \leq a \in G \implies \forall a \in G$ . Thus  $G = \mathcal{H}$  and so  $G$  is not a proper filter hence is not a prime filter, which is a contradiction.  $\square$

Now, to show that  $w(\varphi) = 0$ , we induct on the size of the formula  $\varphi$ :

- For  $\varphi = p$ , we have  $w(p) = 0$  by definition of  $w$  as  $v(p) \notin G$  (by the above claim).
- For  $\varphi = \alpha \vee \beta$ , we note that  $\alpha \leq \alpha \vee \beta$  and so if  $\alpha \in G \implies \alpha \vee \beta = \varphi \in G$ . Thus,  $\alpha \notin G$ . By inductive hypothesis,  $w(\alpha) = 0$ . Similarly,  $w(\beta) = 0$ . Hence,  $w(\varphi) = w(\alpha \vee \beta) = w(\alpha) \vee w(\beta) = 0 \vee 0 = 0$ .
- For  $\varphi = \alpha \wedge \beta$ . If  $v(\alpha) \in G$  and  $v(\beta) \in G$  then  $v(\alpha) \wedge v(\beta) \in G \implies v(\varphi) \in G$ . Hence we can assume WLOG  $v(\alpha) \notin G$ . So, by inductive hypothesis  $w(\alpha) = 0$ . Thus,  $w(\varphi) = w(\alpha \wedge \beta) = 0 \wedge w(\beta) = 0$ .
- For  $\varphi = \alpha \rightarrow \beta$ . So,  $w(\varphi) = w(\alpha) \Rightarrow w(\beta)$ . But for boolean algebras, we know that  $c \Rightarrow d = -c \vee d$ . So  $w(\varphi) = -w(\alpha) \vee w(\beta)$ . Similarly,  $v(\varphi) = -v(\alpha) \vee v(\beta) \implies -v(\alpha) \notin G$  and  $v(\beta) \notin G$  as  $v(\varphi) \notin G$ . Hence  $v(\alpha) \in G$  (by above proposition)  $\implies w(\alpha) = 1 \implies -w(\alpha) = 0$  and  $w(\beta) = 0$  (by inductive hypothesis).

Hence,  $w(\varphi) = 0$   $\square$

## 2.14

Let  $\mathcal{B}_0$  be a Boolean algebra with  $0 \neq 1$ , and let  $\mathbb{B}$  be the two-element Boolean algebra of truth values. Show that the following three conditions are equivalent:

- (i)  $\mathbb{B} \models \varphi$
- (ii)  $\mathcal{B}_0 \models \varphi$
- (iii)  $\mathcal{B} \models \varphi$ , for all Boolean algebras  $\mathcal{B}$

*Proof.* We show the following sequence:

(i)  $\implies$  (iii):

We prove the contrapositive. Suppose there exists some Boolean algebra  $\mathcal{B}$  and a valuation  $v$  in  $\mathcal{B}$  such that  $\mathcal{B}, v \not\models \varphi$ , then using the previous exercise, there exists a 0 – 1 valuation  $w$  such that  $w \not\models \varphi$ . Hence  $\mathbb{B} \not\models \varphi$ .

(iii)  $\implies$  (ii):

Since  $\mathcal{B} \models \varphi \forall \mathcal{B}$ , setting  $\mathcal{B} = \mathcal{B}_0$ , gives us the desired result.

(ii)  $\implies$  (i):

Consider any valuation  $v$  in  $\mathbb{B}$ . We want to show that  $v \models \varphi$ . But since  $\mathbb{B} \subseteq \mathcal{B}_0$ , we can regard  $v$  as  $\mathcal{B}_0$  valuation. But since  $\mathcal{B}_0, w \models \varphi$  for all valuations  $w$ , in particular, setting  $w = v$ , we get the desired result.  $\square$



### 2.21 (i)

$((p \rightarrow q) \rightarrow p) \rightarrow \neg\neg p$  is VALID

*Proof.*

$(p \rightarrow q) \rightarrow p, \neg p, p$	$\vdash$	$\perp$	$(\rightarrow E)$
$(p \rightarrow q) \rightarrow p, \neg p, p$	$\vdash$	$q$	$(\perp E)$
$(p \rightarrow q) \rightarrow p, \neg p$	$\vdash$	$p \rightarrow q$	$(\rightarrow I)$
$(p \rightarrow q) \rightarrow p, \neg p$	$\vdash$	$p$	$(\rightarrow E)$
$(p \rightarrow q) \rightarrow p, \neg p$	$\vdash$	$\perp$	$(\rightarrow E)$
$(p \rightarrow q) \rightarrow p$	$\vdash$	$\neg\neg p$	$(\rightarrow I)$
	$\vdash$	$((p \rightarrow q) \rightarrow p) \rightarrow \neg\neg p$	$(\rightarrow I)$

□

### 2.21 (ii)

$((((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow q$  is VALID

*Proof.* Let  $\Gamma = \{((p \rightarrow q) \rightarrow p) \rightarrow p \rightarrow q, (p \rightarrow q) \rightarrow p\}$ . Then

$\Gamma, p$	$\vdash$	$p$	$(Ax)$
$\Gamma, p$	$\vdash$	$((p \rightarrow q) \rightarrow p) \rightarrow p$	$(\rightarrow I)$
$\Gamma, p$	$\vdash$	$((p \rightarrow q) \rightarrow p) \rightarrow p$	$(Ax)$
$\Gamma p$	$\vdash$	$q$	$(\rightarrow E)$
$\Gamma$	$\vdash$	$p \rightarrow q$	$(\rightarrow I)$
$\Gamma$	$\vdash$	$p$	$(\rightarrow E)$
$((p \rightarrow q) \rightarrow p) \rightarrow p \rightarrow q$	$\vdash$	$((p \rightarrow q) \rightarrow p) \rightarrow p$	$(\rightarrow I)$
$((p \rightarrow q) \rightarrow p) \rightarrow p \rightarrow q$	$\vdash$	$q$	$(\rightarrow E)$
	$\vdash$	$((p \rightarrow q) \rightarrow p) \rightarrow p \rightarrow q$	$(\rightarrow I)$

Furthermore, here is a lambda term of the given type:

$$\lambda x^{(((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow q} . x(\lambda y^{(p \rightarrow q) \rightarrow p} . y(\lambda z^p . x(\lambda u^{(p \rightarrow q) \rightarrow p} . z)))$$

□

### 2.21 (iii)

$\neg p \vee \neg \neg p$  is NOT VALID

*Proof.* In the subsequent sections, we shall consider the Heyting algebra  $\mathcal{H} = \mathbb{R}$  with all open subsets.

Consider the valuation  $v$  on  $\mathcal{H}$  defined as follows:  $v(p) = (0, \infty)$ . Then we get that

$$\begin{aligned} v(\neg p) &= (-\infty, 0) \\ v(\neg \neg p) &= (0, \infty) \\ v(\neg p \vee \neg \neg p) &= \mathbb{R} \setminus \{0\} \end{aligned}$$

□

### 2.21 (iv)

$\neg p \vee \neg q \rightarrow \neg(p \wedge q)$  is VALID

*Proof.* Consider any valuation  $v$  on  $\mathcal{H}$ . Let  $v(p) = P \subseteq \mathbb{R}$  and  $v(q) = Q \subseteq \mathbb{R}$ . Then we get that

$$\begin{aligned} v(\neg p) &= \overline{P} \\ v(\neg q) &= \overline{Q} \\ v(\neg p \vee \neg q) &= \overline{P} \cup \overline{Q} \\ v(p \wedge q) &= P \cap Q \\ v(\neg(p \wedge q)) &= \overline{P \cap Q} \\ v(\neg p \vee \neg q \rightarrow \neg(p \wedge q)) &= \text{Int}[\overline{\overline{P} \cup \overline{Q} \cup P \cap Q}] = \text{Int}[(P \cap Q) \cup \overline{(P \cap Q)}] = \mathbb{R} \end{aligned}$$

□

### 2.21 (v)

$(p \rightarrow p \wedge q) \vee (q \rightarrow p \wedge q)$  is NOT VALID

*Proof.* Consider the valuation  $v$  on  $\mathcal{H}$  defined as follows:  $v(p) = (0, \infty)$  and  $v(q) = (-\infty, 0)$ . Then we get that

$$\begin{aligned} v(p \wedge q) &= \emptyset \\ v(p \rightarrow p \wedge q) &= (-\infty, 0) \\ v(q \rightarrow p \wedge q) &= (0, \infty) \\ v((p \rightarrow p \wedge q) \vee (q \rightarrow p \wedge q)) &= \mathbb{R} \setminus \{0\} \end{aligned}$$

□

### 2.21 (vi)

$(p \rightarrow q \vee r) \rightarrow (p \rightarrow q) \vee r$  is NOT VALID

*Proof.* Consider the valuation  $v$  on  $\mathcal{H}$  defined as follows:  $v(p) = (0, \infty)$  and  $v(q) = \phi$  and  $v(r) = (0, 1)$ . Then we get that

$$\begin{aligned}
 v(q \vee r) &= (0, 1) \\
 v(p \rightarrow q \vee r) &= \text{Int}[(-\infty, 0] \cup (0, 1)] = (-\infty, 1) \\
 v(p \rightarrow q) &= \text{Int}[(-\infty, 0]] = (-\infty, 0) \\
 v((p \rightarrow q) \vee r) &= (-\infty, 0) \cup (0, 1) \\
 v((p \rightarrow q \vee r) \rightarrow (p \rightarrow q) \vee r) &= \text{Int}[[1, \infty) \cup (-\infty, 0) \cup (0, 1)] = \mathbb{R} \setminus \{0\}
 \end{aligned}$$

□

### 2.21 (vii)

$(p \rightarrow q \vee r) \rightarrow (p \rightarrow q) \vee (p \rightarrow r)$  is NOT VALID

*Proof.* Consider the valuation  $v$  on  $\mathcal{H}$  defined as follows:  $v(p) = \mathbb{R} \setminus \{0\}$  and  $v(q) = (-\infty, 0)$  and  $v(r) = (0, \infty)$ . Then we get that

$$\begin{aligned}
 v(p \rightarrow q \vee r) &= \text{Int}[\{0\} \cup \mathbb{R} \setminus \{0\}] = \mathbb{R} \\
 v(p \rightarrow q) &= \text{Int}[\{0\} \cup (-\infty, 0)] = (-\infty, 0) \\
 v(p \rightarrow r) &= \text{Int}[\{0\} \cup (0, \infty)] = (0, \infty) \\
 v((p \rightarrow q) \vee (p \rightarrow r)) &= \mathbb{R} \setminus \{0\} \\
 v((p \rightarrow q \vee r) \rightarrow (p \rightarrow q) \vee (p \rightarrow r)) &= \text{Int}[\phi \cup \mathbb{R} \setminus \{0\}] = \mathbb{R} \setminus \{0\}
 \end{aligned}$$

□

### 2.21 (viii)

$((p \vee \neg p) \rightarrow \neg q) \rightarrow \neg q$  is VALID

*Proof.*

$(p \vee \neg p) \rightarrow \neg q, q, p$	$\vdash$	$p \vee \neg p$	$(\vee I)$
$(p \vee \neg p) \rightarrow \neg q, q, p$	$\vdash$	$\neg q$	$(\rightarrow E)$
$(p \vee \neg p) \rightarrow \neg q, q, p$	$\vdash$	$\perp$	$(\rightarrow E)$
$(p \vee \neg p) \rightarrow \neg q, q$	$\vdash$	$\neg p$	$(\rightarrow I)$
$(p \vee \neg p) \rightarrow \neg q, q$	$\vdash$	$p \vee \neg p$	$(\vee I)$
$(p \vee \neg p) \rightarrow \neg q, q$	$\vdash$	$\neg q$	$(\rightarrow E)$
$(p \vee \neg p) \rightarrow \neg q, q$	$\vdash$	$\perp$	$(\rightarrow E)$
$(p \vee \neg p) \rightarrow \neg q$	$\vdash$	$\neq q$	$(\rightarrow I)$
	$\vdash$	$((p \vee \neg p) \rightarrow \neg q) \rightarrow \neg q$	$(\rightarrow I)$

□

## 2.21 (ix)

$(p \rightarrow \neg p) \rightarrow \neg(\neg p \rightarrow p)$  is VALID

*Proof.*

$p, p \rightarrow \neg p, \neg p \rightarrow p$	$\vdash$	$\neg p$	$(\rightarrow E)$
$p, p \rightarrow \neg p, \neg p \rightarrow p$	$\vdash$	$\perp$	$(\rightarrow E)$
$p \rightarrow \neg p, \neg p \rightarrow p$	$\vdash$	$\neg p$	$(\rightarrow I)$
$p \rightarrow \neg p, \neg p \rightarrow p$	$\vdash$	$p$	$(\rightarrow E)$
$p \rightarrow \neg p, \neg p \rightarrow p$	$\vdash$	$\perp$	(from previous two steps)
$p \rightarrow \neg p$	$\vdash$	$(\neg p \rightarrow p) \rightarrow \perp$	$(\rightarrow I)$
	$\vdash$	$(p \rightarrow \neg p) \rightarrow ((\neg p \rightarrow p) \rightarrow \perp)$	$(\rightarrow I)$
	$\vdash$	$(p \rightarrow \neg p) \rightarrow \neg(\neg p \rightarrow p)$	

□

## 2.23 (i)

$\neg\neg(\varphi \rightarrow \psi) \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$  is VALID

*Proof.* Let  $\Gamma = \{\neg\neg(\varphi \rightarrow \psi), \neg\neg\varphi, \neg\psi\}$ . Suffices to show  $\Gamma \vdash \perp$ .

$\Gamma, \varphi \rightarrow \psi, \varphi$	$\vdash$	$\psi$	$(\rightarrow E)$
$\Gamma, \varphi \rightarrow \psi, \varphi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\Gamma, \varphi \rightarrow \psi$	$\vdash$	$\neg\varphi$	$(\rightarrow I)$
$\Gamma, \varphi \rightarrow \psi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\Gamma$	$\vdash$	$\neg(\varphi \rightarrow \psi)$	$(\rightarrow I)$
$\Gamma$	$\vdash$	$\perp$	$(\rightarrow E)$

□

### 2.23 (ii)

$\neg\neg(\varphi \wedge \psi) \rightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$  is VALID

*Proof.*

$\neg\neg(\varphi \wedge \psi), \neg\varphi, \varphi \wedge \psi$	$\vdash$	$\varphi$	$(\wedge E)$
$\neg\neg(\varphi \wedge \psi), \neg\varphi, \varphi \wedge \psi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\neg\neg(\varphi \wedge \psi), \neg\varphi$	$\vdash$	$\neg(\varphi \wedge \psi)$	$(\rightarrow I)$
$\neg\neg(\varphi \wedge \psi), \neg\varphi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\neg\neg(\varphi \wedge \psi)$	$\vdash$	$\neg\neg\varphi$	$(\rightarrow I)$

Similarly,

$\neg\neg(\varphi \wedge \psi), \neg\psi, \varphi \wedge \psi$	$\vdash$	$\psi$	$(\wedge E)$
$\neg\neg(\varphi \wedge \psi), \neg\psi, \varphi \wedge \psi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\neg\neg(\varphi \wedge \psi), \neg\psi$	$\vdash$	$\neg(\varphi \wedge \psi)$	$(\rightarrow I)$
$\neg\neg(\varphi \wedge \psi), \neg\psi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\neg\neg(\varphi \wedge \psi)$	$\vdash$	$\neg\neg\psi$	$(\rightarrow I)$

Hence,  $\neg\neg(\varphi \wedge \psi) \vdash \neg\neg\varphi \wedge \neg\neg\psi \implies \neg\neg(\varphi \wedge \psi) \rightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$

□

### 2.23 (iii)

$\neg\neg(\varphi \vee \psi) \rightarrow (\neg\neg\varphi \vee \neg\neg\psi)$  is NOT VALID

*Proof.* Take  $\varphi = (-\infty, 0)$  and  $\psi = (0, \infty)$ . Then we get:

$$\begin{array}{lll}
v(\varphi \vee \psi) & = & \mathbb{R} \setminus \{0\} \\
v(\neg(\varphi \vee \psi)) & = & \phi \\
v(\neg\neg(\varphi \vee \psi)) & = & \mathbb{R} \\
v(\neg\neg\varphi) & = & (-\infty, 0) \\
v(\neg\neg\psi) & = & (0, \infty) \\
v(\neg\neg\varphi \vee \neg\neg\psi) & = & \mathbb{R} \setminus \{0\} \\
v(\neg\neg(\varphi \vee \psi) \rightarrow (\neg\neg\varphi \vee \neg\neg\psi)) & = & \mathbb{R} \setminus \{0\}
\end{array}$$

□

### 2.23 (iv)

$(\neg\neg\varphi \rightarrow \neg\neg\psi) \rightarrow \neg\neg(\varphi \rightarrow \psi)$  is VALID

*Proof.* Let  $\Gamma = \{\neg\neg\varphi \rightarrow \neg\neg\psi, \neg(\varphi \rightarrow \psi)\}$ . Suffices to show  $\Gamma \vdash \perp$ .

$$\begin{array}{lll}
\Gamma, \varphi, \neg\varphi & \vdash & \perp & (\rightarrow E) \\
\Gamma, \varphi, \neg\varphi & \vdash & \psi & (\perp E) \\
\Gamma, \neg\varphi & \vdash & \varphi \rightarrow \psi & (\rightarrow I) \\
\Gamma, \neg\varphi & \vdash & \perp & (\rightarrow E) \\
\Gamma & \vdash & \neg\neg\varphi & (\rightarrow E) \\
\Gamma & \vdash & \neg\neg\psi & (\rightarrow E)
\end{array}$$

Furthermore,

$$\begin{array}{lll}
\Gamma, \varphi, \psi & \vdash & \psi & (Ax) \\
\Gamma, \psi & \vdash & \varphi \rightarrow \psi & (\rightarrow I) \\
\Gamma, \psi & \vdash & \perp & (\rightarrow E) \\
\Gamma & \vdash & \neg\psi & (\rightarrow E)
\end{array}$$

Hence,  $\Gamma \vdash \perp$ .

□

### 2.23 (v)

$(\neg\neg\varphi \wedge \neg\neg\psi) \rightarrow \neg\neg(\varphi \wedge \psi)$  is VALID

*Proof.* Let  $\Gamma = \{\neg\neg\varphi \wedge \neg\neg\psi, \neg(\varphi \rightarrow \psi)\}$ . Suffices to show  $\Gamma \vdash \perp$ .

$\Gamma, \varphi$	$\vdash$	$\varphi \wedge \psi$	$(\wedge I)$
$\Gamma, \varphi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\Gamma$	$\vdash$	$\neg\varphi$	$(\rightarrow I)$
$\Gamma$	$\vdash$	$\neg\neg\varphi$	$(Ax)$
$\Gamma$	$\vdash$	$\perp$	$(\rightarrow E)$

□

## 2.23 (vi)

$(\neg\neg\varphi \vee \neg\neg\psi) \rightarrow \neg\neg(\varphi \vee \psi)$  is VALID

*Proof.* Let  $\Gamma = \{\neg\neg\varphi \vee \neg\neg\psi, \neg(\varphi \rightarrow \psi)\}$ . Suffices to show  $\Gamma \vdash \perp$ .

For that, we show the following two derivations:

$$\Gamma, \neg\neg\varphi \vdash \perp$$

$$\Gamma, \neg\neg\psi \vdash \perp$$

Then by  $(\vee E)$ , it follows that,  $\Gamma \vdash \perp$ .

$\Gamma, \neg\neg\varphi, \varphi$	$\vdash$	$\varphi \vee \psi$	$(\vee I)$
$\Gamma, \neg\neg\varphi, \varphi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\Gamma, \neg\neg\varphi$	$\vdash$	$\neg\varphi$	$(\rightarrow I)$
$\Gamma, \neg\neg\varphi$	$\vdash$	$\neg\neg\varphi$	$(Ax)$
$\Gamma, \neg\neg\varphi$	$\vdash$	$\perp$	$(\rightarrow E)$

$\Gamma, \neg\neg\psi, \psi$	$\vdash$	$\varphi \vee \psi$	$(\vee I)$
$\Gamma, \neg\neg\psi, \psi$	$\vdash$	$\perp$	$(\rightarrow E)$
$\Gamma, \neg\neg\psi$	$\vdash$	$\neg\psi$	$(\rightarrow I)$
$\Gamma, \neg\neg\psi$	$\vdash$	$\neg\neg\psi$	$(Ax)$
$\Gamma, \neg\neg\psi$	$\vdash$	$\perp$	$(\rightarrow E)$

□

## 2.32

A state  $c$  in a Kripke model  $C$  determines  $p$  iff either  $c \Vdash p$  or  $c \Vdash \neg p$ . Define a binary valuation  $v_c$  by  $v_c(p) = 1$  iff  $c \Vdash p$ . Show that if  $c$  determines all propositional variables in  $\varphi$  then  $v_c(\varphi) = 1$  implies  $c \Vdash \varphi$ . Conclude that a formula is a classical tautology if and only if it is forced in all one-element models.

*Proof.* We proceed by induction on structure of  $\varphi$ :

- $\varphi = p$ . Then since  $c$  determines all the propositional variables in  $\varphi$ , we get:  $v_c(\varphi) = 1 \implies v_c(p) = 1 \implies c \Vdash p \implies c \Vdash \varphi$
- $\varphi = \alpha \vee \beta$ . Then  $v_c(\varphi) = v_c(\alpha \vee \beta) = v_c(\alpha) \vee v_c(\beta) = 1 \implies v_c(\alpha) = 1$  or  $v_c(\beta) = 1$ , so by inductive hypothesis,  $c \Vdash \alpha$  or  $c \Vdash \beta$  respectively. And hence,  $c \Vdash \varphi$
- $\varphi = \alpha \wedge \beta$ . Then  $v_c(\varphi) = v_c(\alpha \wedge \beta) = v_c(\alpha) \wedge v_c(\beta) = 1 \implies v_c(\alpha) = 1$  and  $v_c(\beta) = 1$ , so by inductive hypothesis,  $c \Vdash \alpha$  and  $c \Vdash \beta$ . And hence,  $c \Vdash \varphi$
- $\varphi = \alpha \rightarrow \beta$ . So,  $v_c(\varphi) = v_c(\alpha \rightarrow \beta) = v_c(\alpha) \Rightarrow v_c(\beta)$ . But since,  $v_c$  is a valuation in a boolean algebra,  $v_c(\alpha) \Rightarrow v_c(\beta) = \neg v_c(\alpha) \vee v_c(\beta) = v_c(\neg \alpha) \vee v_c(\beta)$ . Thus,  $v_c(\varphi) = v_c(\neg \alpha) \vee v_c(\beta) = 1$ . Now suppose,  $v_c(\beta) = 1 \implies c \Vdash \beta$  and so by monotonicity property,  $\forall c' \geq c, c' \Vdash \beta$ , so for any  $c' \geq c$  with  $c' \Vdash \alpha$ , we have that  $c' \Vdash \beta$ . Hence  $c \Vdash \alpha \rightarrow \beta \implies c \Vdash \varphi$ . Now suppose  $v_c(\beta) \neq 1$ , then  $v_c(\neg \alpha) = 1 \implies c \Vdash \neg \alpha \implies \forall c' \geq c$ , we have  $c' \nVdash \alpha$ . Hence, the statement  $\forall c' \geq c$  with  $c' \Vdash \alpha \implies c' \Vdash \beta$  holds vacuously true. Therefore, we get  $c \Vdash \varphi$ .

Now, we show that a formula is a classical tautology iff it is forced in every one-element model.

For that, we first show that every one-element model uniquely determines a valuation.

Suppose  $(\mathcal{C} = \{c\}, \leq, \Vdash)$  is a one-element model. Let  $P$  be the set of all those propositional variables which are forced in  $c$ . Consider  $q \notin P$ . Then note that the statement  $\forall c' \geq c$  describes only  $c$  and  $c \nVdash q$ . Hence  $c \Vdash \neg q$ . Thus,  $v_c(p) = 1$  iff  $p \in P$ , defines a unique valuation.

Now, let  $\varphi$  be a classical tautology. Let  $(\mathcal{C} = \{c\}, \leq, \Vdash)$  be a one-element model. Then it defines a unique valuation  $v_c$ . Hence,  $v_c \models \varphi \implies v_c(\varphi) = 1 \implies c \Vdash \varphi$

Conversely, suppose  $\varphi$  is a formula which is forced in all one-element models. Then consider any valuation  $v$ . Define  $(\mathcal{C} = \{c\}, \leq, \Vdash)$  such that  $c \Vdash p$  iff  $v(p) = 1$  for all propositional variables  $p$ . This is a clearly valid Kripke model. Hence,  $c \Vdash \varphi$ . Now, we observe that  $v_c = v$  as they agree on all propositional variables. Suppose that  $v_c(\varphi) = 0 \implies v_c(\neg \varphi) = 1 \implies c \Vdash \neg \varphi$  which is a contradiction. Hence  $v(\varphi) = 1$ . And thus  $v \models \varphi$ .  $\square$

## 2.34

Prove Glivenko's Theorem: A double negation of a classical tautology is intuitionistically valid.



*Proof.* Let  $\varphi$  be a classical tautology. We want to show that  $\neg\neg\varphi$  is an intuitionistic tautology. For that, let  $(\mathcal{C}, \leq, \Vdash)$  be any Kripke model. Let  $P$  be the set of all propositional variables appearing in  $\varphi$ . Consider  $c \in \mathcal{C}$ . Suppose  $c$  doesn't determine some propositional variable  $p \in P$  so in particular, since  $c \not\Vdash \neg p$  implies that there exists a state  $c' \geq c$  such that  $c' \Vdash p$ . Since  $P$  is finite, we can find a state  $c' \geq c$  which determines  $p \forall p \in P$ . Now consider the valuation  $v_{c'}$ , defined by  $c'$ . Since  $\varphi$  is a tautology,  $v_{c'}(\varphi) = 1$  and hence  $c' \Vdash \varphi$  by the previous result. Thus  $c \not\Vdash \neg\varphi$ .

Therefore, given  $c \in \mathcal{C}$ ,  $\forall c' \geq c$ , since  $c' \not\Vdash \neg\varphi$ , we conclude that  $c \Vdash \neg\neg\varphi$ .  $\square$

### 3.12

What is wrong with the following reduction of problem (vi) to problem (i):

To answer  $? \vdash M : \tau$  ask the question  $? \vdash \lambda yz.y(zM)(zt_\tau)$

*Proof.* This only guarantees that the type of  $M$  is an instance of  $\tau$ .  $\square$

### 3.15

Show that strong normalization for  $(\lambda_{\rightarrow})$  a la Curry implies strong normalization for  $(\lambda_{\rightarrow})$  a la Church, and conversely.

*Proof.* Suppose strong normalization holds for  $(\lambda_{\rightarrow})$  a la Curry. For the sake of contradiction assume it doesn't hold for  $(\lambda_{\rightarrow})$  a la Church. That implies there is a term  $M$  such that it admits an infinite reduction sequence:

$$M \rightarrow_{\beta} M_1 \rightarrow_{\beta} M_2 \rightarrow_{\beta} \cdots \rightarrow_{\beta} M_n \rightarrow_{\beta} \cdots$$

But now, we can replace, each variable  $x_{\sigma}$  by a fresh variable  $x$  and consider the above sequence in Curry-typed lambda calculus, which gives us a contradiction.

Conversely, suppose strong normalization holds for  $(\lambda_{\rightarrow})$  a la Church. For the sake of contradiction assume it doesn't hold for  $(\lambda_{\rightarrow})$  a la Curry. That implies there is a term  $M$  such that it admits an infinite reduction sequence:

$$M \rightarrow_{\beta} M_1 \rightarrow_{\beta} M_2 \rightarrow_{\beta} \cdots \rightarrow_{\beta} M_n \rightarrow_{\beta} \cdots$$

But now, we can replace, each variable  $x$  by a type-annotated variable  $x_{\sigma}$  where  $\sigma$  is the type of the variable  $x$  for each term. and consider the above sequence in Church-typed lambda calculus, which gives us a contradiction.  $\square$