Proof And Types: Assignment 1

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1.8

Prove the Weak Church Rosser Theorem. For all $M_1, M_2, M_3 \in \Lambda$, if $M_1 \to_\beta M_2$ and $M_1 \to_\beta M_3$ then there is a term $M_4 \in \Lambda$ such that $M_2 \twoheadrightarrow_\beta M_4$ and $M_3 \twoheadrightarrow_\beta M_4$.

Proof. We first prove a sequence a lemmas.
Lemma 1: If $x \neq y$ and $x \notin FV(L)$ then

$$M[x := N][y := L] = M[y := L][x := N[y := L]]$$

Proof. By induction on M.

- *Case*: $M = x$, since $x \neq y$

$$\begin{aligned} M[x := N][y := L] &= x[x := N][y := L] \\ &= N[y := L] \\ &= x[x := N[y := L]] \\ &= x[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- *Case*: $M = y$

$$\begin{aligned} M[x := N][y := L] &= L \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- *Case*: $M = x$ where $z \neq x$ and $z \neq y$

$$\begin{aligned} M[x := N][y := L] &= z \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- *Case*: $M = \lambda z.P$ where $z \neq x$ and $z \neq y$

$$\begin{aligned} M[x := N][y := L] &= \lambda z.P[x := N][y := L] \\ &= \lambda z.P[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

- *Case*: $M = PQ$

$$\begin{aligned} M[x := N][y := L] &= (PQ)[x := N][y := L] \\ &= (P[x := N][y := L])(Q[x := N][y := L]) \\ &= (P[y := L][x := N[y := L]])(Q[y := L][x := N[y := L]]) \\ &= (PQ)[y := L][x := N[y := L]] \\ &= M[y := L][x := N[y := L]] \end{aligned}$$

□

Lemma 2: Assume that $P, P' \in \Lambda$ are such that $P \twoheadrightarrow_\beta P'$ then for all $x \in V$ and $Q \in \Lambda$:
(i) $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$
(ii) $PQ \twoheadrightarrow_\beta P'Q$
(iii) $QP \twoheadrightarrow_\beta QP'$

Proof. By induction on the derivation of $P \twoheadrightarrow_\beta P'$.

- *Case*: $P \rightarrow_\beta P'$ then clearly, $\lambda x.P \rightarrow_\beta \lambda x.P'$ and $PQ \rightarrow_\beta P'Q$ and $QP \rightarrow_\beta QP'$

- *Case*: there exists P'' such that $P \twoheadrightarrow_\beta P''$ and $P'' \rightarrow_\beta P'$ then by inductive hypothesis, $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$ and $\lambda x.P'' \rightarrow_\beta \lambda x.P'$ and so $\lambda x.P \twoheadrightarrow_\beta \lambda x.P'$.

  Again, by inductive hypothesis, $PQ \twoheadrightarrow_\beta P''Q$ and $P''Q \rightarrow_\beta P'Q$ and so $PQ \twoheadrightarrow_\beta P'Q$.

  Similarly, by inductive hypothesis, $QP \twoheadrightarrow_\beta QP''$ and $QP'' \rightarrow_\beta QP'$ and so $QP \twoheadrightarrow_\beta QP'$.

- *Case*: $P = P'$ then clearly, $\lambda x.P = \lambda x.P'$ and $PQ = P'Q$ and $QP = QP'$

$\square$

Lemma 3: For all $P, P', Q \in \Lambda$ if $P \to_\beta P'$ then also $P[x := Q] \to_\beta P'[x := Q]$

Proof. By induction on the derivation of $P \to_\beta P'$.

- *Case*: $P = (\lambda y.R)S$ and $P' = R[y := S]$. Then we get

$$\begin{array}{rcrl} P[x := Q] & = & ((\lambda y.R)S)[x := Q] & \\ & = & (\lambda y.R[x := Q])(S[x := Q]) & \\ & \to_\beta & R[x := Q][y := S[x := Q]] & \\ & = & R[y := S][x := Q]] & \text{(by lemma 1)} \\ & = & P'[x := Q] & \end{array}$$

- *Case*: $P = \lambda y.R$ and $P' = \lambda y.R'$ such that $R \to_\beta R'$. Then we get

$$\begin{array}{rcr} P[x := Q] & = & (\lambda y.R)[x := Q] \\ & = & \lambda y.R[x := Q] \\ & \to_\beta & \lambda y.R'[x := Q] \\ & = & (\lambda y.R')[x := Q] \\ & = & P'[x := Q] \end{array}$$

- *Case*: $P = RS$ and $P' = R'S$ such that $R \to_\beta R'$. Then we get

$$\begin{array}{rcr} P[x := Q] & = & (RS)[x := Q] \\ & = & R[x := Q]S[x := Q] \\ & \to_\beta & R'[x := Q]S[x := Q] \\ & = & (R'S)[x := Q] \\ & = & P'[x := Q] \end{array}$$

- *Case*: $P = RS$ and $P' = RS'$ such that $S \to_\beta S'$. Then we get

$$\begin{array}{rcr} P[x := Q] & = & (RS)[x := Q] \\ & = & R[x := Q]S[x := Q] \\ & \to_\beta & R[x := Q]S'[x := Q] \\ & = & (RS')[x := Q] \\ & = & P'[x := Q] \end{array}$$

$\square$

Lemma 4: For all $P, Q, Q' \in \Lambda$ if $Q \to_\beta Q'$ then $P[x := Q] \twoheadrightarrow_\beta P[x := Q']$

Proof. By induction on structure of P.

- *Case*: $P = x$ then $P[x := Q] = Q \rightarrow_\beta Q' = P[x := Q']$

- *Case*: $P = y$ then $P[x := Q] = y \rightarrow_\beta y = P[x := Q']$

- *Case*: $P = \lambda y.P'$ then

$$\begin{array}{rcr} P[x := Q] & = & (\lambda y.P')[x := Q] \\ & = & \lambda y.P'[x := Q] \\ & \rightarrow_\beta & \lambda y.P'[x := Q'] \\ & = & (\lambda y.P')[x := Q'] \\ & = & P[x := Q'] \end{array}$$

- *Case*: $P = P_1 P_2$ then

$$\begin{array}{rcr} P[x := Q] & = & (P_1 P_2)[x := Q] \\ & = & (P_1[x := Q])(P_2[x := Q]) \\ & \rightarrow_\beta & (P_1[x := Q'])(P_2[x := Q]) \\ & \rightarrow_\beta & (P_1[x := Q'])(P_2[x := Q']) \\ & = & (P_1[x := Q'])(P_2[x := Q']) \\ & = & (P_1 P_2)[x := Q'] \\ & = & P[x := Q'] \end{array}$$

$\square$

Now, we can finally present the proof of Weak Church Rosser property.
We proceed by induction on derivation of $M_1 \rightarrow_\beta M_2$.

- *Case*: $M_1 = (\lambda x.P)Q$ and $M_2 = P[x := Q]$. Then

  Either $M_3 = (\lambda x.P')Q$ such that $P \rightarrow_\beta P'$. Then set $M_4 = P'[x := Q]$ and note that both $M_2 \twoheadrightarrow_\beta M_4$ (by Lemma 3) and $M_3 \twoheadrightarrow_\beta M_4$ (by Lemma 2).

  Or either $M_3 = (\lambda x.P)Q'$ such that $Q \rightarrow_\beta Q'$. Then set $M_4 = P[x := Q']$ and note that both $M_2 \twoheadrightarrow_\beta M_4$ (by Lemma 4) and $M_3 \twoheadrightarrow_\beta M_4$ (by substitution).

- *Case*: $M_1 = \lambda x.P$ and $M_2 = \lambda x.P'$ because $P \rightarrow_\beta P'$. Then M_3 must be of the form $\lambda x.P''$ such that $P \rightarrow_\beta P''$. Then by inductive hypothesis, there exists a term Q such that $P' \rightarrow_\beta Q$ and $P'' \rightarrow_\beta Q$. Hence setting $M_4 = \lambda x.Q$ we get the sought term.

- *Case*: $M_1 = PQ$ and $M_2 = P'Q$ such that $P \rightarrow_\beta P'$ then

  Either $M_3 = P''Q$ such that $P \rightarrow_\beta P''$. By inductive hypothesis, there exists a term R such that $P' \rightarrow_\beta R$ and $P'' \rightarrow_\beta R$. Hence $M_4 = RQ$ is the sought term.

  Or $M_3 = PQ'$ such that $Q \rightarrow_\beta Q'$. Then note that both $M_2 \rightarrow_\beta P'Q'$ and $M_3 \rightarrow_\beta P'Q'$. Hence $M_4 = P'Q'$ as desired.

$\square$

1.14

Consider the fixed point combinator

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

Show that $F(YF) =_\beta YF$ holds for all F.

Proof.

$$\begin{array}{ccr} YF & = & (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))F \\ & \to_\beta & ((\lambda x.f(xx))(\lambda x.f(xx)))[f := F] \\ & = & (\lambda x.F(xx))(\lambda x.F(xx)) \\ & \to_\beta & (F(xx))[x := (\lambda x.F(xx))] \\ & = & F((\lambda x.F(xx))(\lambda x.F(xx))) \end{array}$$

And also,

$$\begin{array}{ccr} F(YF) & = & F\,((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))F) \\ & \to_\beta & F\,(((\lambda x.f(xx))(\lambda x.f(xx)))[f := F]) \\ & = & F\,((\lambda x.F(xx))(\lambda x.F(xx))) \end{array}$$

Hence, $YF =_\beta F(YF)$ □

2.5

Show that intuitionistic propositional logic is consistent, that is $\nvdash \bot$.

Proof. Suppose not, that is $\vdash \bot$.
Now, using soundness theorem for Natural Deduction logic system, we know that for every Heyting Algebra $\mathcal{H}$ and every valuation v on $\mathcal{H}$, $v(\bot) = 1$.
But consider, the usual boolean algebra on $(0, 1)$ (which is a Heyting algebra) and the valuation v: PV $\to \mathcal{H}$ where $v(x) = 0\ \forall x \in$ PV.
According to the definition, $\widetilde{v}(\bot) = 0$, where $\widetilde{v}$ is the extension of v to all the well-formed formulas. But that leads to a contradiction. □

2.6

Let $(A, \vee, \wedge)$ be an algebra with two binary operations (written in infix notation). Assume that the equations (i)-(iv) of Lemma 2.3.4 hold for all elements of A. Define $a \leq b$ by $a \vee b = b$. Prove that $(A, \leq)$ is a lattice, where suprema and infima are given by $\vee$ and $\wedge$.

Proof. Lemma 2.3.4 says that the following are valid:

- $a \vee a = a$ and $a \wedge a = a$
- $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
- $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$

Define $\leq$ relation on A as: $a \leq b$ if $a \vee b = b$.
Using these, we shall first show that our $(A, \leq)$ is a poset:
Reflexive: since we have $a \vee a = a$ (from the lemma) $\implies a \leq a$
Anti-symmetric: Suppose $a \leq b$ and $b \leq a$ then we have $a \vee b = a$ and $b \vee a = a$. Using the lemma, $a = b \vee a = a \vee b = b$
Transitive: Suppose $a \leq b$ and $b \leq c$ then we have $a \vee b = b$ and $b \vee c = c$. Hence, $b \vee c = (a \vee b) \vee (b \vee c) \implies c = a \vee (b \vee b) \vee c = a \vee (b \vee c) = a \vee c \implies a \leq c$
Now, we show that the operation
$\vee$ **is suprema:** Suppose c is an upper bound on a and b then $a \leq c$ and $b \leq c \implies a \vee c = c$ and $b \vee c = c$ then we get $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$. Hence $a \vee b \leq c$
Before proving infima, we show that $a \vee b = b$ and $a \wedge b = a$ are equivalent.
$b = a \vee b \implies a \wedge b = a \wedge (a \vee b) = (a \vee b) \wedge a = a$
$a = a \wedge b \implies a \vee b = (a \wedge b) \vee b = (b \wedge a) \vee b = b$
$\wedge$ **is infima:** Suppose c is a lower bound on a and b then $c \leq a$ and $c \leq b \implies c \wedge a = c$ and $c \wedge b = c$ then we get $c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c$. Hence $c \leq a \wedge b$ □

2.8

Let A be a lattice satisfying $(a \vee b) \wedge c \leq (a \wedge c) \vee (b \wedge c)$, for all a, b, c. Show that A is distributive.

Proof. We shall first show that:

$$(a \wedge c) \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Then from anti-symmetric property of $\leq$, the result follows.
For that, we recall the following two properties:
Property 1: $a \leq c$ and $b \leq c \implies a \vee b \leq c$
Property 2: $c \leq a$ and $c \leq b \implies c \leq a \wedge b$
Observe

$$a \wedge c \leq a \leq a \vee b \;\&\; a \wedge c \leq c \implies a \wedge c \leq (a \vee b) \wedge c \text{ (using property 2)}$$

Similarly,

$$b \wedge c \leq b \leq a \vee b \;\&\; b \wedge c \leq c \implies b \wedge c \leq (a \vee b) \wedge c \text{ (using property 2)}$$

Now using, property 1 on the above two inequalities, we get our desired result.

Now we show the other distributive lattice property holds, namely:

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

First note that, $a \wedge b \leq a \leq a \vee c$ and $a \wedge b \leq b \leq b \vee c$. Hence $a \wedge b \leq (a \vee c) \wedge (b \vee c)$. Next, $c \leq a \vee c$ and $c \leq b \vee c$ and so $c \leq (a \vee c) \wedge (b \vee c)$. Therefore, we have established. $(a \wedge b) \vee c \leq (a \vee c) \wedge (b \vee c)$.
For the other inequality, note that, $(a \wedge b) \vee c = (a \wedge b) \vee (a \wedge c) \vee c \geq (a \wedge (b \vee c)) \vee c = (a \wedge (b \vee c)) \vee (c \wedge (b \vee c)) \geq (a \vee c) \wedge (b \vee c)$, where both the inequalities follow from the assumption given in the problem.

□

2.13

Assume that $\mathcal{B}, v \not\models \varphi$, for some Boolean algebra $\mathcal{B}$ and some φ and v. Show that there exists a prime filter F in $\mathcal{B}$, with $v(\neg\varphi) \in F$. Then define a binary valuation by $w(p) = 1$ iff $v(p) \in F$ and show that $w(\varphi) = 0$.

Proof. Recall the following result:
Theorem: Let F be a proper filter in $\mathcal{H}$ and $a \notin F$. There exists a prime filter G such that $F \subseteq G$ and $a \notin G$
Now, we state and prove another result regarding prime filters.
Proposition: Let G be a prime filter of a boolean algebra. Then $\alpha \in G$ iff $-\alpha \notin G$

Proof. : Let $\alpha \in G$. Then since $\alpha \wedge -\alpha = 0$ and $0 \notin G$ (as G is proper), we get that $-\alpha \notin G$. Now suppose, $-\alpha \notin G$. Since $-\alpha \vee \alpha = 1$ and $1 \in G$, by the property of prime filters we get that $\alpha \in G$. □

Now we note that, since $v \not\models \varphi \implies v(\varphi) \neq 1 \implies -v(\varphi) \neq 0$ as $\mathcal{B}$ is a Boolean algebra so $v(\varphi) \vee -v(\varphi) = 1$
Let $F = \{a : -v(\varphi) \leq a\}$. Then F is a proper filter because:

- if $a, b \in F$ then $-v(\varphi) \leq a$ and $-v(\varphi) \leq b \implies -v(\varphi) \leq a \wedge b \implies a \wedge b \in F$
- if $a \in F$ then $-v(\varphi) \leq a$ and so $-v(\varphi) \leq b$, $\forall a \leq b$
- Since $-v(\varphi) \neq 0 \implies 0 \notin F$

We can apply this theorem to F and $a = 0$ to get a prime filter G such that $-v(\varphi) \in F \subseteq G$. Now,

$$v(\neg\varphi) = v(\varphi \to \bot) = v(\varphi) \implies v(\bot) = v(\varphi) \implies 0 = -v(\varphi) \in G$$

Claim: $v(\varphi) \notin G$

Proof. : Suppose $v(\varphi) \in G \implies v(\varphi) \wedge -v(\varphi) = 0 \in G$ But G being a filter, $0 \leq a \in G$ $\forall a \in G$. Thus $G = \mathcal{H}$ and so G is not a proper filter hence is not a prime filter, which is a contradiction. $\square$

Now, to show that $w(\varphi) = 0$, we induct on the size of the formula φ:

- For $\varphi = p$, we have $w(p) = 0$ by definition of w as $v(p) \notin G$ (by the above claim).
- For $\varphi = \alpha \vee \beta$, we note that $\alpha \leq \alpha \vee \beta$ and so if $\alpha \in G \implies \alpha \vee \beta = \varphi \in G$. Thus, $\alpha \notin G$. By inductive hypothesis, $w(\alpha) = 0$. Similarly, $w(\beta) = 0$. Hence, $w(\varphi) = w(\alpha \vee \beta) = w(\alpha) \vee w(\beta) = 0 \vee 0 = 0$
- For $\varphi = \alpha \wedge \beta$. If $v(\alpha) \in G$ and $v(\beta) \in G$ then $v(\alpha) \wedge v(\beta) \in G \implies v(\varphi) \in G$. Hence we can assume WLOG $v(\alpha) \notin G$. So, by inductive hypothesis $w(\alpha) = 0$. Thus, $w(\varphi) = w(\alpha) \wedge w(\beta) = 0 \wedge w(\beta) = 0$
- For $\varphi = \alpha \to \beta$. So, $w(\varphi) = w(\alpha) \Rightarrow w(\beta)$. But for boolean algebras, we know that $c \Rightarrow d = -c \vee d$. So $w(\varphi) = -w(\alpha) \vee w(\beta)$. Similarly, $v(\varphi) = -v(\alpha) \vee v(\beta) \implies -v(\alpha) \notin G$ and $v(\beta) \notin G$ as $v(\varphi) \notin G$. Hence $v(\alpha) \in G$ (by above proposition) $\implies w(\alpha) = 1 \implies -w(\alpha) = 0$ and $w(\beta) = 0$ (by inductive hypothesis).

Hence, $w(\varphi) = 0$ $\square$

2.14

Let $\mathcal{B}_0$ be a Boolean algebra with $0 \neq 1$, and let $\mathbb{B}$ be the two-element Boolean algebra of truth values. Show that the following three conditions are equivalent:
(i) $\mathbb{B} \vDash \varphi$
(ii) $\mathcal{B}_0 \vDash \varphi$
(iii) $\mathcal{B} \vDash \varphi$, for all Boolean algebras $\mathcal{B}$

Proof. We show the following sequence:
(i) $\implies$ (iii):
We prove the contrapositive. Suppose there exists some Boolean algebra $\mathcal{B}$ and a valuation v in $\mathcal{B}$ such that $\mathcal{B}, v \not\models \varphi$, then using the previous exercise, there exists a $0-1$ valuation w such that $w \not\models \varphi$. Hence $\mathbb{B} \not\models \varphi$.
(iii) $\implies$ (ii):
Since $\mathcal{B} \models \varphi$ $\forall \mathcal{B}$, setting $\mathcal{B} = \mathcal{B}_0$, gives us the desired result.
(ii) $\implies$ (i):
Consider any valuation v in $\mathbb{B}$. We want to show that $v \models \varphi$. But since $\mathbb{B} \subseteq \mathcal{B}_0$, we can regard v as $\mathcal{B}_0$ valuation. But since $\mathcal{B}_0, w \models \varphi$ $forall$ valuations w, in particular, setting $w = v$, we get the desired result.

$\square$

2.21 (i)

$((p \to q) \to p) \to \neg\neg p$ is VALID

Proof.

$$\begin{array}{rcrr}
(p \to q) \to p, \neg p, p & \vdash & \bot & (\to E) \\
(p \to q) \to p, \neg p, p & \vdash & q & (\bot E) \\
(p \to q) \to p, \neg p & \vdash & p \to q & (\to I) \\
(p \to q) \to p, \neg p & \vdash & p & (\to E) \\
(p \to q) \to p, \neg p & \vdash & \bot & (\to E) \\
(p \to q) \to p & \vdash & \neg\neg p & (\to I) \\
 & \vdash & ((p \to q) \to p) \to \neg\neg p & (\to I)
\end{array}$$

□

2.21 (ii)

$((((p \to q) \to p) \to p) \to q) \to q$ is VALID

Proof. Let $\Gamma = \{(((p \to q) \to p) \to p) \to q, (p \to q) \to p\}$. Then

$$\begin{array}{rcrr}
\Gamma, p & \vdash & p & (Ax) \\
\Gamma, p & \vdash & ((p \to q) \to p) \to p & (\to I) \\
\Gamma, p & \vdash & (((p \to q) \to p) \to p) \to q) & (Ax) \\
\Gamma, p & \vdash & q & (\to E) \\
\Gamma & \vdash & p \to q & (\to I) \\
\Gamma & \vdash & p & (\to E) \\
(((p \to q) \to p) \to p) \to q & \vdash & ((p \to q) \to p) \to p & (\to I) \\
(((p \to q) \to p) \to p) \to q & \vdash & q & (\to E) \\
 & \vdash & ((((p \to q) \to p) \to p) \to q) \to q & (\to I)
\end{array}$$

Furthermore, here is a lambda term of the given type:

$$\lambda x^{(((p\to q)\to p)\to p)\to q}.x(\lambda y^{(p\to q)\to p}.y(\lambda z^{p}.x(\lambda u^{(p\to q)\to p}.z)))$$

□

2.21 (iii)

$\neg p \vee \neg\neg p$ is NOT VALID

Proof. In the subsequent sections, we shall consider the Heyting algebra $\mathcal{H} = \mathbb{R}$ with all open subsets.
Consider the valuation v on $\mathcal{H}$ defined as follows: $v(p) = (0, , \infty)$. Then we get that

$$\begin{array}{rcr} v(\neg p) & = & (-\infty, 0) \\ v(\neg\neg p) & = & (0, \infty) \\ v(\neg p \vee \neg\neg p) & = & \mathbb{R} \setminus \{0\} \end{array}$$

□

2.21 (iv)

$\neg p \vee \neg q \to \neg(p \wedge q)$ is VALID

Proof. Consider any valuation v on $\mathcal{H}$. Let $v(p) = P \subseteq \mathbb{R}$ and $v(q) = Q \subseteq \mathbb{R}$. Then we get that

$$\begin{array}{rcr} v(\neg p) & = & \overline{P} \\ v(\neg q) & = & \overline{Q} \\ v(\neg p \vee \neg q) & = & \overline{P} \cup \overline{Q} \\ v(p \wedge q) & = & P \cap Q \\ v(\neg(p \wedge q)) & = & \overline{P \cap Q} \\ v(\neg p \vee \neg q \to \neg(p \wedge q)) = Int[\overline{\overline{P} \cup \overline{Q}} \cup \overline{P \cap Q}] & = & Int[(P \cap Q) \cup \overline{(P \cap Q)}] = \mathbb{R} \end{array}$$

□

2.21 (v)

$(p \to p \wedge q) \vee (q \to p \wedge q)$ is NOT VALID

Proof. Consider the valuation v on $\mathcal{H}$ defined as follows: $v(p) = (0, \infty)$ and $v(q) = (-\infty, 0)$. Then we get that

$$\begin{array}{rcr} v(p \wedge q) & = & \phi \\ v(p \to p \wedge q) & = & (-\infty, 0) \\ v(q \to p \wedge q) & = & (0, \infty) \\ v((p \to p \wedge q) \vee (q \to p \wedge q)) & = & \mathbb{R} \setminus \{0\} \end{array}$$

□

2.21 (vi)

$(p \to q \vee r) \to (p \to q) \vee r$ is NOT VALID

Proof. Consider the valuation v on $\mathcal{H}$ defined as follows: $v(p) = (0, \infty)$ and $v(q) = \phi$ and $v(r) = (0, 1)$. Then we get that

$$\begin{array}{rcr} v(q \vee r) & = & (0,1) \\ v(p \to q \vee r) & = & Int[(-\infty, 0] \cup (0,1)] = (-\infty, 1) \\ v(p \to q) & = & Int[(-\infty, 0]] = (-\infty, 0) \\ v((p \to q) \vee r) & = & (-\infty, 0) \cup (0, 1) \\ v((p \to q \vee r) \to (p \to q) \vee r) & = & Int[[1, \infty) \cup (-\infty, 0) \cup (0, 1)] = \mathbb{R} \setminus \{0\} \end{array}$$

□

2.21 (vii)

$(p \to q \vee r) \to (p \to q) \vee (p \to r)$ is NOT VALID

Proof. Consider the valuation v on $\mathcal{H}$ defined as follows: $v(p) = \mathbb{R} \setminus \{0\}$ and $v(q) = (-\infty, 0)$ and $v(r) = (0, \infty)$. Then we get that

$$\begin{array}{rcrcr} v(p \to q \vee r) & = & Int[\{0\} \cup \mathbb{R} \setminus \{0\}] & = & \mathbb{R} \\ v(p \to q) & = & Int[\{0\} \cup (-\infty, 0)] & = & (-\infty, 0) \\ v(p \to r) & = & Int[\{0\} \cup (0, \infty)] & = & (0, \infty) \\ v((p \to q) \vee (p \to r)) & & & = & \mathbb{R} \setminus \{0\} \\ v((p \to q \vee r) \to (p \to q) \vee (p \to r)) & = & Int[\phi \cup \mathbb{R} \setminus \{0\}] & = & \mathbb{R} \setminus \{0\} \end{array}$$

□

2.21 (viii)

$((p \vee \neg p) \to \neg q) \to \neg q$ is VALID

Proof.

$$
\begin{array}{rcrr}
(p \vee \neg p) \rightarrow \neg q, q, p & \vdash & p \vee \neg p & (\vee I) \\
(p \vee \neg p) \rightarrow \neg q, q, p & \vdash & \neg q & (\rightarrow E) \\
(p \vee \neg p) \rightarrow \neg q, q, p & \vdash & \perp & (\rightarrow E) \\
(p \vee \neg p) \rightarrow \neg q, q & \vdash & \neg p & (\rightarrow I) \\
(p \vee \neg p) \rightarrow \neg q, q & \vdash & p \vee \neg p & (\vee I) \\
(p \vee \neg p) \rightarrow \neg q, q & \vdash & \neg q & (\rightarrow E) \\
(p \vee \neg p) \rightarrow \neg q, q & \vdash & \perp & (\rightarrow E) \\
(p \vee \neg p) \rightarrow \neg q & \vdash & \neq q & (\rightarrow I) \\
 & \vdash & ((p \vee \neg p) \rightarrow \neg q) \rightarrow \neg q & (\rightarrow I)
\end{array}
$$

□

2.21 (ix)

$(p \rightarrow \neg p) \rightarrow \neg(\neg p \rightarrow p)$ is VALID

Proof.

$$
\begin{array}{rcrr}
p, p \rightarrow \neg p, \neg p \rightarrow p & \vdash & \neg p & (\rightarrow E) \\
p, p \rightarrow \neg p, \neg p \rightarrow p & \vdash & \perp & (\rightarrow E) \\
p \rightarrow \neg p, \neg p \rightarrow p & \vdash & \neg p & (\rightarrow I) \\
p \rightarrow \neg p, \neg p \rightarrow p & \vdash & p & (\rightarrow E) \\
p \rightarrow \neg p, \neg p \rightarrow p & \vdash & \perp & \text{(from previous two steps)} \\
p \rightarrow \neg p & \vdash & (\neg p \rightarrow p) \rightarrow \perp & (\rightarrow I) \\
 & \vdash & (p \rightarrow \neg p) \rightarrow ((\neg p \rightarrow p) \rightarrow \perp) & (\rightarrow I) \\
 & \vdash & (p \rightarrow \neg p) \rightarrow \neg(\neg p \rightarrow p) &
\end{array}
$$

□

2.23 (i)

$\neg\neg(\varphi \rightarrow \psi) \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$ is VALID

Proof. Let $\Gamma = \{\neg\neg(\varphi \rightarrow \psi), \neg\neg\varphi, \neg\psi\}$. Suffices to show $\Gamma \vdash \perp$.

$$
\begin{array}{rcrr}
\Gamma, \varphi \to \psi, \varphi & \vdash & \psi & (\to E) \\
\Gamma, \varphi \to \psi, \varphi & \vdash & \bot & (\to E) \\
\Gamma, \varphi \to \psi & \vdash & \neg\varphi & (\to I) \\
\Gamma, \varphi \to \psi & \vdash & \bot & (\to E) \\
\Gamma & \vdash & \neg(\varphi \to \psi) & (\to I) \\
\Gamma & \vdash & \bot & (\to E)
\end{array}
$$

□

2.23 (ii)

$\neg\neg(\varphi \wedge \psi) \to (\neg\neg\varphi \wedge \neg\neg\psi)$ is VALID

Proof.

$$
\begin{array}{rcrr}
\neg\neg(\varphi \wedge \psi), \neg\varphi, \varphi \wedge \psi & \vdash & \varphi & (\wedge E) \\
\neg\neg(\varphi \wedge \psi), \neg\varphi, \varphi \wedge \psi & \vdash & \bot & (\to E) \\
\neg\neg(\varphi \wedge \psi), \neg\varphi & \vdash & \neg(\varphi \wedge \psi) & (\to I) \\
\neg\neg(\varphi \wedge \psi), \neg\varphi & \vdash & \bot & (\to E) \\
\neg\neg(\varphi \wedge \psi) & \vdash & \neg\neg\varphi & (\to I)
\end{array}
$$

Similarly,

$$
\begin{array}{rcrr}
\neg\neg(\varphi \wedge \psi), \neg\psi, \varphi \wedge \psi & \vdash & \psi & (\wedge E) \\
\neg\neg(\varphi \wedge \psi), \neg\psi, \varphi \wedge \psi & \vdash & \bot & (\to E) \\
\neg\neg(\varphi \wedge \psi), \neg\psi & \vdash & \neg(\varphi \wedge \psi) & (\to I) \\
\neg\neg(\varphi \wedge \psi), \neg\psi & \vdash & \bot & (\to E) \\
\neg\neg(\varphi \wedge \psi) & \vdash & \neg\neg\psi & (\to I)
\end{array}
$$

Hence, $\neg\neg(\varphi \wedge \psi) \vdash \neg\neg\varphi \wedge \neg\neg\psi \implies \neg\neg(\varphi \wedge \psi) \to (\neg\neg\varphi \wedge \neg\neg\psi)$ □

2.23 (iii)

$\neg\neg(\varphi \vee \psi) \to (\neg\neg\varphi \vee \neg\neg\psi)$ is NOT VALID

Proof. Take $\varphi = (-\infty, 0)$ and $\psi = (0, \infty)$. Then we get:

$$\begin{array}{rcr}
v(\varphi \vee \psi) & = & \mathbb{R} \setminus \{0\} \\
v(\neg(\varphi \vee \psi)) & = & \phi \\
v(\neg\neg(\varphi \vee \psi)) & = & \mathbb{R} \\
v(\neg\neg\varphi) & = & (-\infty, 0) \\
v(\neg\neg\psi) & = & (0, \infty) \\
v(\neg\neg\varphi \vee \neg\neg\psi) & = & \mathbb{R} \setminus \{0\} \\
v(\neg\neg(\varphi \vee \psi) \to (\neg\neg\varphi \vee \neg\neg\psi)) & = & \mathbb{R} \setminus \{0\}
\end{array}$$

□

2.23 (iv)

$(\neg\neg\varphi \to \neg\neg\psi) \to \neg\neg(\varphi \to \psi)$ is VALID

Proof. Let $\Gamma = \{\neg\neg\varphi \to \neg\neg\psi, \neg(\varphi \to \psi)\}$. Suffices to show $\Gamma \vdash \bot$.

$$\begin{array}{rcrr}
\Gamma, \varphi, \neg\varphi & \vdash & \bot & (\to E) \\
\Gamma, \varphi, \neg\varphi & \vdash & \psi & (\bot E) \\
\Gamma, \neg\varphi & \vdash & \varphi \to \psi & (\to I) \\
\Gamma, \neg\varphi & \vdash & \bot & (\to E) \\
\Gamma & \vdash & \neg\neg\varphi & (\to E) \\
\Gamma & \vdash & \neg\neg\psi & (\to E)
\end{array}$$

Furthermore,

$$\begin{array}{rcrr}
\Gamma, \varphi, \psi & \vdash & \psi & (Ax) \\
\Gamma, \psi & \vdash & \varphi \to \psi & (\to I) \\
\Gamma, \psi & \vdash & \bot & (\to E) \\
\Gamma & \vdash & \neg\psi & (\to E)
\end{array}$$

Hence, $\Gamma \vdash \bot$. □

2.23 (v)

$(\neg\neg\varphi \wedge \neg\neg\psi) \to \neg\neg(\varphi \wedge \psi)$ is VALID

Proof. Let $\Gamma = \{\neg\neg\varphi \wedge \neg\neg\psi, \neg(\varphi \to \psi)\}$. Suffices to show $\Gamma \vdash \bot$.

Γ, φ	$\vdash$	$\varphi \wedge \psi$	$(\wedge I)$
Γ, φ	$\vdash$	$\bot$	$(\to E)$
Γ	$\vdash$	$\neg\varphi$	$(\to I)$
Γ	$\vdash$	$\neg\neg\varphi$	(Ax)
Γ	$\vdash$	$\bot$	$(\to E)$

□

2.23 (vi)

$(\neg\neg\varphi \vee \neg\neg\psi) \to \neg\neg(\varphi \vee \psi)$ is VALID

Proof. Let $\Gamma = \{\neg\neg\varphi \vee \neg\neg\psi, \neg(\varphi \to \psi)\}$. Suffices to show $\Gamma \vdash \bot$.
For that, we show the following two derivations:

$$\Gamma, \neg\neg\varphi \vdash \bot$$

$$\Gamma, \neg\neg\psi \vdash \bot$$

Then by $(\vee E)$, it follows that, $\Gamma \vdash \bot$.

$\Gamma, \neg\neg\varphi, \varphi$	$\vdash$	$\varphi \vee \psi$	$(\vee I)$
$\Gamma, \neg\neg\varphi, \varphi$	$\vdash$	$\bot$	$(\to E)$
$\Gamma, \neg\neg\varphi$	$\vdash$	$\neg\varphi$	$(\to I)$
$\Gamma, \neg\neg\varphi$	$\vdash$	$\neg\neg\varphi$	(Ax)
$\Gamma, \neg\neg\varphi$	$\vdash$	$\bot$	$(\to E)$

$\Gamma, \neg\neg\psi, \psi$	$\vdash$	$\varphi \vee \psi$	$(\vee I)$
$\Gamma, \neg\neg\psi, \psi$	$\vdash$	$\bot$	$(\to E)$
$\Gamma, \neg\neg\psi$	$\vdash$	$\neg\psi$	$(\to I)$
$\Gamma, \neg\neg\psi$	$\vdash$	$\neg\neg\psi$	(Ax)
$\Gamma, \neg\neg\psi$	$\vdash$	$\bot$	$(\to E)$

□

2.32

A state c in a Kripke model C determines p iff either $c \Vdash p$ or $c \Vdash \neg p$. Define a binary valuation v_c by $v_c(p) = 1$ iff $c \Vdash p$. Show that if c determines all propositional variables in φ then $v_c(\varphi) = 1$ implies $c \Vdash \varphi$. Conclude that a formula is a classical tautology if and only if it is forced in all one-element models.

Proof. We proceed by induction on structure of φ:

- $\varphi = p$. Then since c determines all the propositional variables in φ, we get: $v_c(\varphi) = 1 \implies v_c(p) = 1 \implies c \Vdash p \implies c \Vdash \varphi$
- $\varphi = \alpha \vee \beta$. Then $v_c(\varphi) = v_c(\alpha \vee \beta) = v_c(\alpha) \vee v_c(\beta) = 1 \implies v_c(\alpha) = 1$ or $v_c(\beta) = 1$, so by inductive hypothesis, $c \Vdash \alpha$ or $c \Vdash \beta$ respectively. And hence, $c \Vdash \varphi$
- $\varphi = \alpha \wedge \beta$. Then $v_c(\varphi) = v_c(\alpha \wedge \beta) = v_c(\alpha) \wedge v_c(\beta) = 1 \implies v_c(\alpha) = 1$ and $v_c(\beta) = 1$, so by inductive hypothesis, $c \Vdash \alpha$ and $c \Vdash \beta$. And hence, $c \Vdash \varphi$
- $\varphi = \alpha \to \beta$. So, $v_c(\varphi) = v_c(\alpha \to \beta) = v_c(\alpha) \Rightarrow v_c(\beta)$. But since, v_c is a valuation in a boolean algebra, $v_c(\alpha) \Rightarrow v_c(\beta) = -v_c(\alpha) \vee v_c(\beta) = v_c(\neg\alpha) \vee v_c(\beta)$. Thus, $v_c(\varphi) = v_c(\neg\alpha) \vee v_c(\beta) = 1$. Now suppose, $v_c(\beta) = 1 \implies c \Vdash \beta$ and so by monotonicity property, $\forall\ c' \geq c$, $c' \Vdash \beta$, so for any $c' \geq c$ with $c' \Vdash \alpha$, we have that $c' \Vdash \beta$. Hence $c \Vdash \alpha \to \beta \implies c \Vdash \varphi$. Now suppose $v_c(\beta) \neq 1$, then $v_c(\neg\alpha) = 1 \implies c \Vdash \neg\alpha \implies \forall\ c' \geq c$, we have $c' \nVdash \alpha$. Hence, the statement $\forall\ c' \geq c$ with $c' \Vdash \alpha \implies c' \Vdash \beta$ holds vacuously true. Therefore, we get $c \Vdash \varphi$.

Now, we show that a formula is a classical tautology iff it is forced in every one-element model.
For that, we first show that every one-element model uniquely determines a valuation.
Suppose $(\mathcal{C} = \{c\}, \leq, \Vdash)$ is a one-element model. Let P be the set of all those propositional variables which are forced in c. Consider $q \notin P$. Then note that the statement $\forall\ c' \geq c$ describes only c and $c \nVdash q$. Hence $c \Vdash \neg q$. Thus, $v_c(p) = 1$ iff $p \in P$, defines a unique valuation.
Now, let φ be a classical tautology. Let $(\mathcal{C} = \{c\}, \leq, \Vdash)$ be a one-element model. Then it defines a unique valuation v_c. Hence, $v_c \vDash \varphi \implies v_c(\varphi) = 1 \implies c \Vdash \varphi$
Conversely, suppose φ is a formula which is forced in all one-element models. Then consider any valuation v. Define $(\mathcal{C} = \{c\}, \leq, \Vdash)$ such that $c \Vdash p$ iff $v(p) = 1$ for all propositional variables p. This is a clearly valid Kripke model. Hence, $c \Vdash \varphi$. Now, we observe that $v_c = v$ as they agree on all propositional variables. Suppose that $v_c(\varphi) = 0 \implies v_c(\neg\varphi) = 1 \implies c \Vdash \neg\varphi$ which is a contradiction. Hence $v(\varphi) = 1$. And thus $v \vDash \varphi$. □

2.34

Prove Glivenko's Theorem: A double negation of a classical tautology is intuitionistically valid.

Proof. Let φ be a classical tautology. We want to show that $\neg\neg\varphi$ is an intuitionistic tautology. For that, let $(\mathcal{C}, \leq, \Vdash)$ be any Kripke model. Let P be the set of all propositional variables appearing in φ. Consider $c \in \mathcal{C}$. Suppose c doesn't determine some propositional variable $p \in P$ so in particular, since $c \nVdash \neg p$ implies that there exists a state $c' \geq c$ such that $c' \Vdash p$. Since P is finite, we can find a state $c' \geq c$ which determines p $\forall p \in P$. Now consider the valuation $v_{c'}$, defined by c'. Since φ is a tautology, $v_{c'}(\varphi) = 1$ and hence $c' \Vdash \varphi$ by the previous result. Thus $c \nVdash \neg\varphi$.
Therefore, given $c \in \mathcal{C}$, $\forall c' \geq c$, since $c' \nVdash \neg\varphi$, we conclude that $c \Vdash \neg\neg\varphi$. $\square$

3.12

What is wrong with the following reduction of problem (vi) to problem (i):
To answer $? \vdash M : \tau$ *ask the question* $? \vdash \lambda yz.y(zM)(zt_\tau)$

Proof. This only guarantees that the type of M is an instance of τ. $\square$

3.15

Show that strong normalization for $(\lambda_\rightarrow)$ a la Curry implies strong normalization for $(\lambda_\rightarrow)$ a la Church, and conversely.

Proof. Suppose strong normalization holds for $(\lambda_\rightarrow)$ a la Curry. For the sake of contradiction assume it doesn't hold for $(\lambda_\rightarrow)$ a la Church. That implies there is a term M such that it admits an infinite reduction sequence:

$$M \rightarrow_\beta M_1 \rightarrow_\beta M_2 \rightarrow_\beta \cdots \rightarrow_\beta M_n \rightarrow_\beta \cdots$$

But now, we can replace, each variable x_σ by a fresh variable x and consider the above sequence in Curry-typed lambda calculus, which gives us a contradiction.

Conversely, suppose strong normalization holds for $(\lambda_\rightarrow)$ a la Church. For the sake of contradiction assume it doesn't hold for $(\lambda_\rightarrow)$ a la Curry. That implies there is a term M such that it admits an infinite reduction sequence:

$$M \rightarrow_\beta M_1 \rightarrow_\beta M_2 \rightarrow_\beta \cdots \rightarrow_\beta M_n \rightarrow_\beta \cdots$$

Fix a type for the term M. Then each M_n inhabits the same type. But now, we can replace, each variable x by a type-annotated variable x_σ where σ is the type of the variable x. and we get an infinite sequence in Church-typed lambda calculus, which gives us a contradiction. $\square$