Graph Theory - Homework 1

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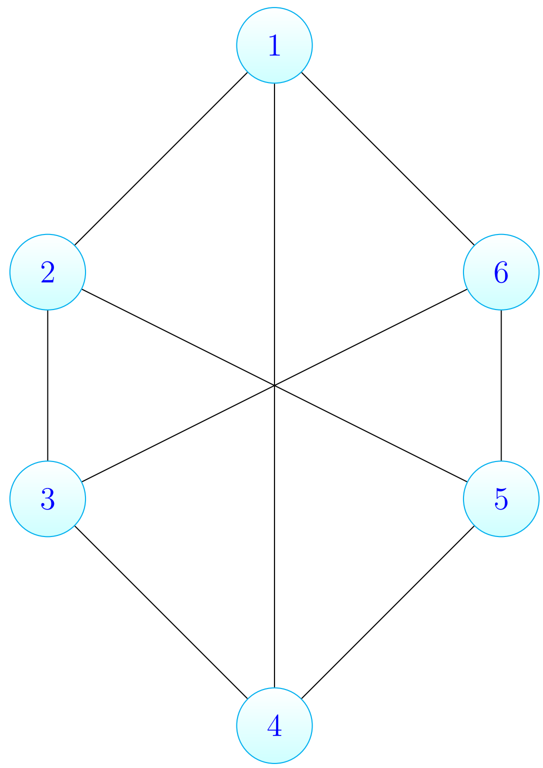
Exercise 1

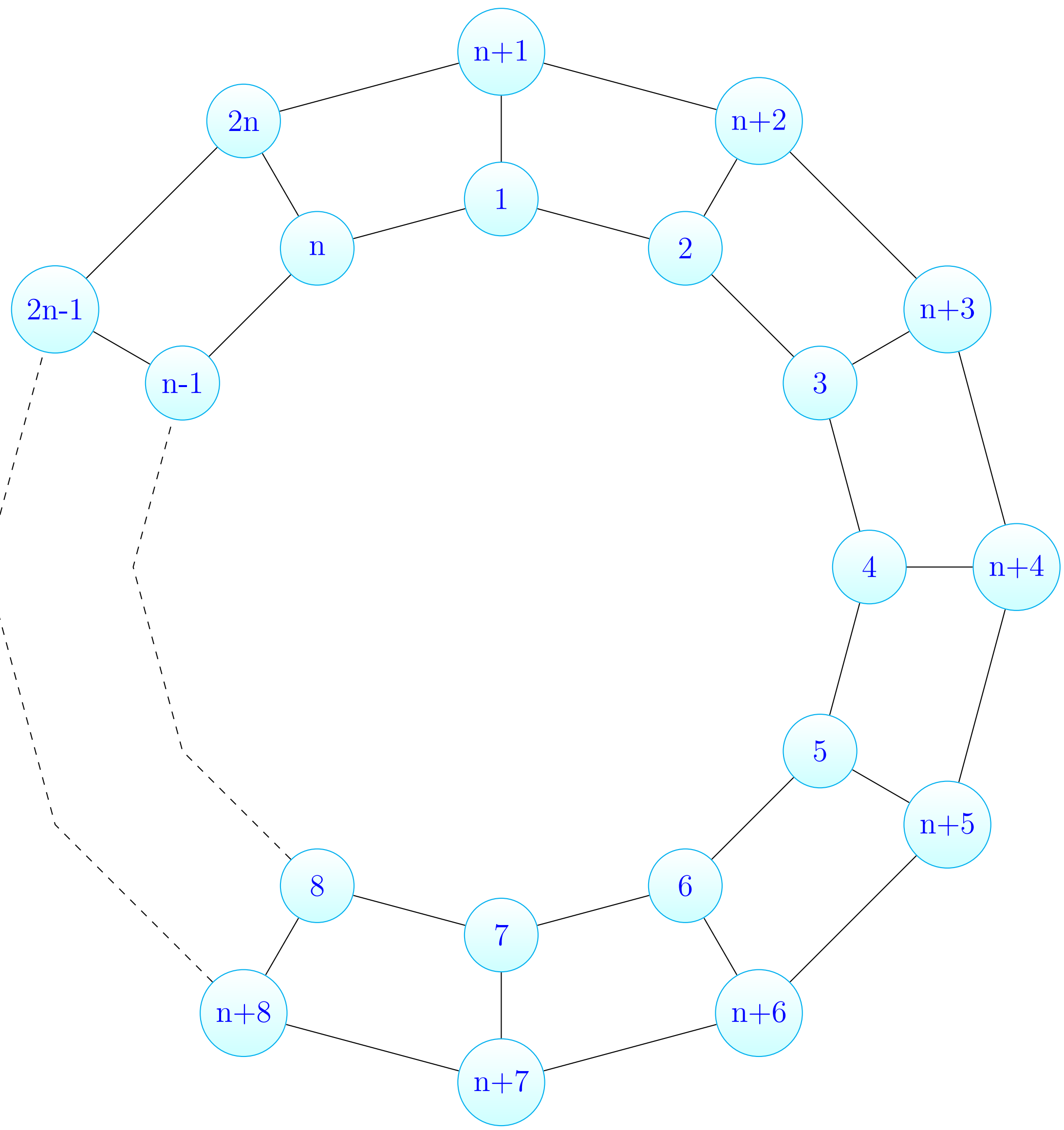
Proof. Since G is not connected, there are atleast two non-empty disconnected components. Let u and v be two vertices in $\bar{G}$.
Case 1: If u and v are in different connected components in G, then there is an edge $(u, v) \in \bar{G}$ and so there is a path between u and v in $\bar{G}$
Case 2: If u and v are in same connected component in G. Consider w in a separate connected component of G. Hence there are edges (u, w) and (v, w) in $\bar{G}$. Hence, $u \to w \to v$ is a path between u and v in $\bar{G}$. $\square$

Exercise 2

Proof. **Case:** $n = 3$



Case: $n > 3$



Here is a more general construction for $k-$**regular graphs on** $2n$ **vertices, which contain no odd cycles**.

Let $G = (U \cup V, E)$ where $U = \{u_0, u_2, \ldots u_{n-1}\}$ and $V = \{v_0, v_2, \ldots v_{n-1}\}$ and $E = \{\{u_i, v_j\} \mid 0 \leq i \leq n-1, i \leq j \leq i+k-1 \pmod{n}\}$

This is clearly $k-regular$ and it doesn't have any odd cycles because it is a bipartite graph.

$\square$

Exercise 3

We shall prove the following stronger assertion: If a graph G has exactly two no-cut-vertices, then G is a path.

Proof. Let P be an arbitrary maximal path in G and let a and b be its endpoints.
(Note that all neighbours of a and b lie on P by maximality of P)
We claim that a and b are no-cut vertices. For that let C be the connected component containing a and b. Now consider any $x, y \in C$. We will show that there is a path connecting x, y such that a, b do not lie on this path. So let Q be any path joining x, y. If $a \in Q$, say $Q = x \dots vav' \dots y$ then since $v, v' \in N(a)$, we know from above reasoning that $v, v' \in P$ and so we can eliminate a from Q. Similar argument holds for b. Therefore, a, b are no-cut vertices.
But G has exactly two no-cut vertices, so G must be connected because for $v \in G$, let P_v be a maximal path passing through v. So, a and b must be the end-points of P_v and hence every vertex v lies in C (connected component of a, b).
Now we claim that $G = P$. Suppose there is a vertex $x \notin P$. Hence x is a cut vertex. So, $G \setminus \{x\}$ has atleast two connected components. Clearly, a and b still lie in the same connected component. But choose any y from the other connected component of $G \setminus \{x\}$ and let P_y be maximal path containing y in G. Then $P_y = Q_a Q_b$ where $Q_a = a \dots y$ and $Q_b = y \dots b$. Since y is in a different connected component, $x \in P_y$. If $x \in Q_a$ then path Q_b is still a path in $G \setminus \{x\}$ which means y is the same connected component as b, which is a contradiction. Similarly, if $x \in Q_b$ then y is in the same connected component as a, which is again a contradiction.
Furthermore, let $P = av_1v_2 \dots v_k$ then there cannot be any extra edge like $\{v_i, v_j\}$ because otherwise v_{i+1} will be a cut vertex. Hence $G = P$.

□

Exercise 4

Proof.

$$2|E| = \sum_v d(v) \geq 2n \implies |E| \geq n$$

We know that a tree has exactly $n - 1$ edges.
Since G is connected and has more than $n - 1$ edges, G contains a cycle.
This is not true if the graph G is infinite. For example, consider the two-sided infinite path $G = (V, E)$ where $V = \mathbb{Z}$ and $E = \{\{i, i + 1\} \mid i \in \mathbb{Z}\}$ □