Optiver - Problem 1(a) Solution

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Problem: Three players A, B, C play the following game. First, A picks a real number between 0 and 1 (both inclusive), then B picks a number in the same range (different from A's choice) and finally C picks a number, also in the same range, (different from the two chosen numbers). We then pick a number in the range uniformly randomly. Whoever's number is closest to this random number wins the game. Assume that A, B and C all play optimally and their sole goal is to maximise their chances of winning. Also assume that if one of them has several optimal choices, then that player will randomly pick one of the optimal choices.
If A chooses 0, then what is the best choice for B?

Answer: Best choice for B is $\boxed{\frac{2}{3}}$

Proof. We begin by noting that $P(\text{tie}) = 0$ because the final number is chosen from a continuous probability distribution, so we need not worry about this corner case in our following analysis.
Say B chooses $x > 0$ then C has two options, either they can choose between 0 and x or between x and 1. Let's see which incentivizes C more.

Case 1: C chooses y such that $x < y < 1$.
Now we choose a number uniformly at random from $[0, 1]$ and the winner is whoever is closest to this number. That is,

$$P(\text{B wins}) = \frac{x+y}{2} - \frac{x}{2} = \frac{y}{2}$$

$$P(\text{C wins}) = 1 - \frac{x+y}{2} < 1 - x \text{ (as } y > x)$$

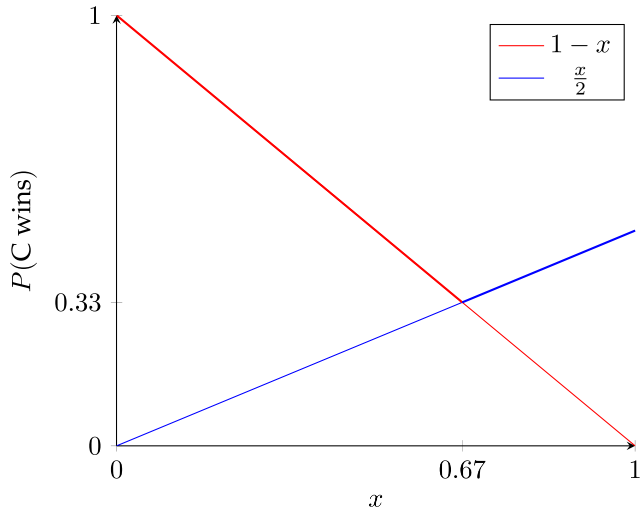
Since C need to maximize their chances of winning, we note that y can be chosen arbitrarily close to x and so $P(\text{C wins})$ can be made arbitrarily close to $1 - x$

Case 2: C chooses y such that $0 < y < x$.
Now we choose a number uniformly at random from $[0, 1]$ and the winner is whoever is closest to this number. That is,

$$P(\text{B wins}) = 1 - \frac{x+y}{2}$$

$$P(\text{C wins}) = \frac{x+y}{2} - \frac{y}{2} = \frac{x}{2}$$

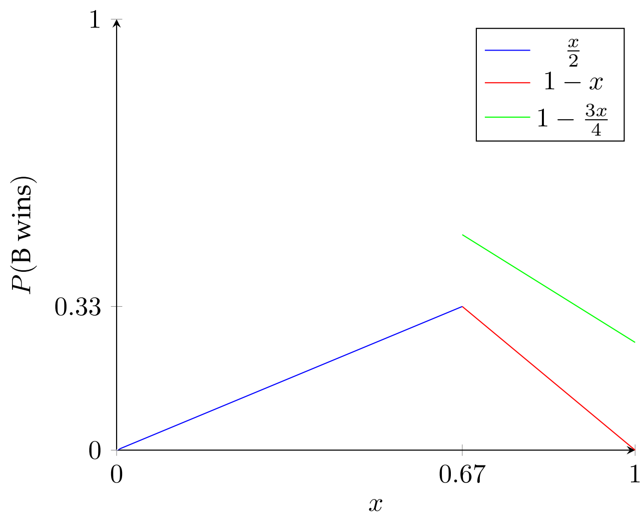
So in this case, no matter what C chooses their chances of winning remain constant (and so choice of C is random in this case)



From the graph, it is clear that if $x < \frac{2}{3}$, then C chooses Case 1, otherwise if $x \geq \frac{2}{3}$ then C chooses Case 2.
Now that we exactly know the strategy for C, we can use it to optimize for B.

If B chooses $x < \frac{2}{3}$ then $P(\text{B wins}) = \frac{y}{2}$ and since in this case C's optimal choice was to choose arbitrarily close to x, we get that $P(\text{B wins})$ is arbitrarily close to $\frac{x}{2}$.

If B chooses $x \geq \frac{2}{3}$ then $P(\text{B wins}) = 1 - \frac{x+y}{2}$ and since in this case C's optimal choice was to choose a random number in $(0, x)$, meaning y is a uniform random variable with expectation $\frac{x}{2}$ (assuming C chooses it uniformly at random).
So in expectation, $P(\text{B wins}) = 1 - \frac{3x}{4}$.



Hence we see that $P(\text{B wins})$ is maximum at $x = \frac{2}{3}$ giving us that B should choose $\frac{2}{3}$ to maximize their chances of winning.

Even if we did a worst-case analysis for B, that is C can choose a number arbitrarily close to x (so that chances of B winning is minimized), then $P(\text{B wins})$ is almost equal to $1 - x$ in which case yet $x = \frac{2}{3}$ maximizes the chances of winning of B.

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