ML Exam

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Problem 1

For $r = 1, 2, 3, 4, 5$ - Find the best rank-r approximation of

0.7820	0.7190	0.5210	0.6280	0.8640	0.5630
1.0750	0.8780	0.6030	0.9290	1.0630	0.8520
1.1250	0.7250	0.4340	1.0860	0.8930	1.0280
0.6600	0.7770	0.6130	0.4290	0.9210	0.3540
0.3600	0.3080	0.2160	0.3020	0.3710	0.2750
1.2270	0.8830	0.5680	1.1300	1.0790	1.0560
0.5520	0.3850	0.2430	0.5160	0.4710	0.4840

Find the $L^2 - norm$ error between the approximation and the original matrix. If we were to design an efficient value of r, which r will we choose and why?

Solution

We know the Singular Value Decomposition for a Matrix A is given by

$$A = U\Sigma V^T$$

To get the values U, Σ and V we use the following

$$A^T A = VU^T UV^T = V\Sigma^2 V^T (U.U^T = I)$$

$$AA^T = UV^T VU^T = U\Sigma^2 U^T (V.V^T = I)$$

$$A^T A = \begin{bmatrix} 5.408 & 4.241 & 2.857 & 4.775 & 5.148 & 4.408 \\ 4.241 & 3.44 & 2.357 & 3.677 & 4.166 & 3.377 \\ 2.857 & 2.357 & 1.628 & 2.454 & 2.851 & 2.247 \\ 4.775 & 3.677 & 2.454 & 4.255 & 4.469 & 3.939 \\ 5.148 & 4.166 & 2.851 & 4.469 & 5.046 & 4.106 \\ 4.408 & 3.377 & 2.247 & 3.939 & 4.106 & 3.65 \end{bmatrix}$$

Characteristic Equation of A^TA is given as $|A^TA - \lambda I| = 0$

$$\begin{vmatrix} 5.408-\lambda & 4.241 & 2.857 & 4.775 & 5.148 & 4.408 \\ 4.241 & 3.44-\lambda & 2.357 & 3.677 & 4.166 & 3.377 \\ 2.857 & 2.357 & 1.628-\lambda & 2.454 & 2.851 & 2.247 \\ 4.775 & 3.677 & 2.454 & 4.255-\lambda & 4.469 & 3.939 \\ 5.148 & 4.166 & 2.851 & 4.469 & 5.046-\lambda & 4.106 \\ 4.408 & 3.377 & 2.247 & 3.939 & 4.106 & 3.65-\lambda \end{vmatrix} = 0$$

The Eigen Values of the above system are

$$[22.99959, 0.427182, 0.00000079133, 0.0000003815, 0.0000001551, 0.0000001101855]$$

The corresponding eigen vectors for the above eigen values for the U matrix.

$$U = \begin{bmatrix} -0.35 & 0.298 & 0.24 & -0.177 & -0.077 & -0.734 & -0.395 \\ -0.467 & 0.082 & 0.421 & 0.676 & 0.288 & 0.077 & 0.229 \\ -0.462 & -0.488 & -0.362 & 0.139 & 0.018 & 0.192 & -0.601 \\ -0.318 & 0.756 & -0.26 & -0.17 & 0.004 & 0.469 & -0.102 \\ -0.158 & 0.069 & -0.711 & 0.297 & -0.162 & -0.412 & 0.425 \\ -0.517 & -0.258 & 0.215 & -0.391 & -0.525 & 0.138 & 0.415 \\ -0.231 & -0.153 & -0.126 & -0.471 & 0.78 & -0.101 & 0.259 \end{bmatrix}$$

The Σ Matrix is a diagonal matrix, with the values having square root of the eigen values.

$$\Sigma = \begin{bmatrix} 4.79578 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.65359 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00088 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.00061 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00039 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.00033 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now for AA^T,

$$AA^T = \begin{bmatrix} 2.857 & 3.767 & 3.659 & 2.658 & 1.280 & 4.126 & 1.838 \\ 3.767 & 5.009 & 4.941 & 3.440 & 1.696 & 5.533 & 2.470 \\ 3.659 & 4.941 & 5.013 & 3.224 & 1.664 & 5.543 & 2.484 \\ 2.658 & 3.440 & 3.224 & 2.572 & 1.177 & 3.696 & 1.638 \\ 1.280 & 1.696 & 1.664 & 1.177 & 0.575 & 1.868 & 0.833 \\ 4.126 & 5.533 & 5.543 & 3.696 & 1.868 & 6.164 & 2.757 \\ 1.838 & 2.470 & 2.484 & 1.638 & 0.833 & 2.757 & 1.234 \end{bmatrix}$$

Characteristic Equation of AA^T is given as $|AA^T - \lambda I| = 0$

The Eigen Values are same as we got in teh above case, and the corresponding Eigen Vector. The Matrix is transposed to get the final V^T

$$V^T = \begin{bmatrix} -0.484 & -0.383 & -0.259 & -0.426 & -0.465 & -0.393 \\ -0.161 & 0.389 & 0.44 & -0.447 & 0.429 & -0.496 \\ -0.626 & -0.384 & -0.016 & 0.305 & 0.593 & 0.126 \\ -0.039 & 0.432 & -0.839 & -0.147 & 0.293 & -0.007 \\ -0.498 & 0.575 & 0.096 & 0.501 & -0.39 & -0.091 \\ 0.312 & -0.192 & -0.159 & 0.504 & 0.102 & -0.759 \end{bmatrix}$$

For Rank 1 Approximation we take only the first eigen value in the Σ matrix and get the new A_1 which is Rank 1 approximation of A

$$\Sigma = \begin{bmatrix} 4.79578 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = U \Sigma V^T$$

$$A_1 = \begin{bmatrix} 0.813 & 0.643 & 0.435 & 0.715 & 0.78 & 0.659 \\ 1.084 & 0.857 & 0.58 & 0.953 & 1.04 & 0.878 \\ 1.074 & 0.849 & 0.574 & 0.944 & 1.03 & 0.87 \\ 0.739 & 0.585 & 0.395 & 0.65 & 0.709 & 0.599 \\ 0.367 & 0.29 & 0.196 & 0.322 & 0.352 & 0.297 \\ 1.2 & 0.949 & 0.642 & 1.055 & 1.151 & 0.972 \\ 0.536 & 0.424 & 0.287 & 0.471 & 0.514 & 0.434 \end{bmatrix}$$

$L^2 - norm$ between original A and Rank-1 approximation A_1 is given by $\sqrt{\sum_i n \sum_j n |a_{i,j}|^2}$

$$|A - A_1|_2 = 0.6535$$

Similarly Rank-2 Approximation is:

$$\Sigma = \begin{bmatrix} 4.79578 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.65359 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = U \Sigma V^T$$

$$A_2 = \begin{bmatrix} 0.782 & 0.719 & 0.521 & 0.628 & 0.864 & 0.563 \\ 1.075 & 0.878 & 0.603 & 0.929 & 1.063 & 0.852 \\ 1.125 & 0.725 & 0.434 & 1.086 & 0.893 & 1.028 \\ 0.66 & 0.777 & 0.613 & 0.429 & 0.921 & 0.354 \\ 0.36 & 0.308 & 0.216 & 0.302 & 0.371 & 0.275 \\ 1.227 & 0.883 & 0.568 & 1.13 & 1.079 & 1.056 \\ 0.552 & 0.385 & 0.243 & 0.516 & 0.471 & 0.484 \end{bmatrix}$$

$$|A - A_2|_2 = 0.001199$$