3/10/24 Thursday

$L\{f(t)\}$ $t>0$ is defined as
Laplace transform of function $f(t)$

$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$ provided that integral exist

i.e $L\{f(t)\} = F(s)$

Laplace transform of some standard function.

① $L\{1\} = 1/s$

Proof
W.K.T

$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

$L\{1\} \therefore \int_0^\infty (1)e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{\cancel{e^{-st}}}{-\infty} - \left(\frac{e^{-s(0)}}{-s}\right) = 1/s$

② $L\{e^{at}\} = \frac{1}{s-a}$ $s>a$ $s-a>0$

W.K.T

$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$

$\therefore f(t) = e^{at}$ $\therefore L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt$

$$\Rightarrow L\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{e^{-(s-a)(\infty)}}{-(s-a)} + \frac{e^{-(s-a)(0)}}{s-a}$$

$$= \frac{1}{s-a} //$$

Similarly

$$L\{e^{-at}\} = \frac{1}{s+a} \qquad s > -a$$

$$s+a > 0,$$

(3) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

consider

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at} - e^{-at}\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2-a^2} //$$

④ $L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

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⑤ $L\{\sin at\}$

Proof

$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$

use formula or ILATE

$= \left[\frac{e^{-st}}{(-s)^2+a^2}\left[-s\sin at - a\cos at\right]\right]_0^\infty$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$

$e^{-s(\infty)} \to 0$

$= 0 - \left[\frac{1}{s^2+a^2}[0-a]\right] = \frac{a}{s^2+a^2}$ //

---

⑥ $L\{\cos at\}$

Proof

$L\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$

$= \left[\frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos at + a\sin at\right]\right]_0^\infty = 0 - \frac{(e^{-0(t)})(-s\cos a(0) + a\sin a(0))}{s^2+a^2}$

$$0 - \frac{e^{-s(0)}}{s^2+a^2}\left[-s\cos a(0) + a\sin a(0)\right] = \frac{s}{s^2+a^2} //$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

⑦ $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

24/10/24 Friday 2 hours

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n+1) = n!$ ∀ +ve integer n

$\Gamma(n+1) = n\Gamma(n)$ ∀ n

Proof :-

$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$

$L\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$

let $st = x$ $\left[\Rightarrow t = x/s \right]$ $t=0$ $x \to 0$

$dt = \frac{1}{s} dx$ $t \to \infty$ $x \to \infty$

$= \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{1}{s}\, dx$

$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$

$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty x^{(n+1)-1} e^{-x}\, dx$

Now compare to gamma $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{\Gamma(n+1)}{s^{n+1}}$ //

note. also $L\{t^n\} = \frac{n!}{s^{n+1}}$ if n is +ve integer

Linearity Property

If $f(t)$ & $g(t)$ are functions of t with Laplace transforms $F(s)$ and $G(s)$ then $L\{c_1 f(t) \pm c_2 g(t)\} = c_1 L\{f(t)\} \pm c_2 L\{g(t)\} = c_1 F(s) \pm c_2 G(s)$

Q). find the L.T of the following.

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 + \cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

① $\sin^2 3t$

Sol:- w.k.t $\sin^2 3t = \frac{1-\cos 6t}{2} = \frac{1}{2}(1-\cos 6t) = f(t)$

$\therefore L\{f(t)\} = \frac{1}{2}L\{1-\cos \cancel{8t}\}$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+6^2}\right] = \frac{1}{2}\left[\frac{s^2-s^2+6^2}{s(s^2+6^2)}\right]$$

$$\frac{36}{2(s)(s^2+6^2)} //$$

---

② $e^{4t} + t^3 + \sin^2 t$

Sol:- ~~Let~~ Let $e^{4t}+t^3+\sin^2 t = f(t)$

$$L\{f(t)\} = L\{e^{4t}\} + L\{t^3\} + L\{\sin^2 t\}$$

$$= \frac{1}{s-4} + \frac{\Gamma(4)}{s^4} + \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$= \frac{1}{s-4} + \frac{3!}{s^4} + \frac{2}{s(s^2+4)}$$

---

③. $\sqrt{t} + \frac{1}{\sqrt{t}}$

Let $f(t) = t^{1/2} + t^{-1/2}$

$$L\{f(t)\} = L\{t^{1/2}\} + L\{t^{-1/2}\}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} + \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{\frac{3}{2}\sqrt{\Gamma}(\frac{1}{2})}{s^{3/2}} + \frac{\sqrt{\Gamma}(\frac{1}{2})}{s^{1/2}}$$ $\Gamma(n+1) = n\Gamma(n)$ $\Gamma(\frac{1}{2}+1) = \frac{1}{2}$

$$L\{f(t)\} = \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{\sqrt{\pi}}{s^{1/2}}$$

(4) $e^{-3/2t} + \sinh\sqrt{2}t + 4$

Sol: let $f(t) = e^{-3/2t} + \sinh\sqrt{2}t + 4$

$$L\{f(t)\} = L\{e^{-3/2t}\} + L\{\sinh\sqrt{2}t\} + L\{4\}$$

$$L\{f(t)\} = \frac{1}{s+3/2} + \frac{\sqrt{2}}{s^2-2} + \frac{4}{s} //$$

(5). $e^{3t}\sinh t$ $\quad f(t) = e^{3t}\sinh t$ ← $\sinh t + \cosh t$

Sol: $~~L\{f(t)\} = \int_0^\infty f(t)\, e^{-st} \sinh t\, dt~~$

$$~~= \int_0^\infty e^{3t-st} \sinh t\, dt~~$$

$$~~= \int_0^\infty e^{t(3-s)} \sinh t\, dt~~$$

: ~~Length~~ ×

$\sinh t = \frac{e^x - e^{-x}}{2}$

$$\sinh t = \frac{e^x - e^{-x}}{2} = \frac{1}{2}(e^x - e^{-x})$$

$$\therefore e^{3t}\sinh t = \frac{1}{2}(e^{3t}e^{t} - e^{3t}e^{-t}) = \frac{1}{2}(e^{4t} - e^{2t})$$

$$f(t) = \frac{1}{2}(e^{4t} - e^{2t})$$

$$\therefore L\{f(t)\} = \frac{1}{2}\left[\frac{1}{s-4} - \frac{1}{s-2}\right] //$$

⑥ $\sin^2 t \cos t$

$\frac{(1-\cos 2t)}{2}\cos t = \cos t$

$(1-\cos^2 t)\cos t = \cos t - \cos^3 t$

$\frac{1}{2}(1-\cos 2t)\cos t$

$\Rightarrow \frac{1}{2}(\cos t - \cos 2t \cos t)$

$\frac{1}{2}\cos t - \frac{1}{2}\,\frac{1}{2}(\cos 3t + \cos t)$

$\frac{1}{2}\cos t - \frac{1}{4}\cos 3t - \frac{1}{4}\cos t = f(t)$

$L\{f(t)\} = \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{4}\frac{s}{s^2+9} - \frac{1}{4}\frac{s}{s^2+1}$ //

$\frac{1}{2}\frac{s}{s^2+1} - \frac{1}{4}\frac{s}{s^2+9}$

---

⑦ $f(t) = \sin(2t+3) = \sin 2t\cos 3 + \cos 2t \sin 3$

$\therefore L\{f(t)\} = \frac{2}{s^2+4}\cos 3 + \sin 3\,\frac{s}{s^2+4}$ //

---

⑧ $\sin\sqrt{t}$

$\sin\sqrt{t} = \sin\sqrt{a} + \frac{1}{1!}(t-a)\frac{\cos\sqrt{a}}{2\sqrt{a}} + \frac{1}{2!}(t-a)^2 + \ldots$

$L\{f(t)\} = \frac{\sin\sqrt{a}}{s} + \frac{\cos\sqrt{a}}{2\sqrt{a}}\frac{\Gamma(t)}{s^2} - \frac{a\cos\sqrt{a}}{2\sqrt{a}(s)}$

(5) or ~~$\sin\sqrt{t}$~~ about 0,

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} + \cdots$

$\sin\sqrt{t} = t^{1/2} - \frac{1}{3!} t^{3/2} + \frac{1}{5!} t^{5/2}$

Apply L on both side

$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!}\{t^{5/2}\}$

$= \frac{\Gamma(1/2+1)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(3/2+1)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(5/2+1)}{s^{7/2}}$

$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\frac{3/2 \cdot 1/2\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!}\frac{5/2\ 3/2\ 1/2\sqrt{\pi}}{s^{7/2}}$

$= \sqrt{\pi}\left[\frac{1}{2s^{3/2}} - \frac{3}{3!\,4\,s^{5/2}} + \frac{15}{5!\,8\,s^{7/2}}\right] + \cdots$ //

---

(9) $\sin^3 t$

w.k.T

$= \sin 3x = 3\sin x - 4\sin^3 x$

$f(t) = 3\sin t - 4\sin^3 t = \sin 3t$

$\therefore f(t) = \sin^3 t = (3\sin t - \sin 3t)\,1/4$

$L\{f(t)\} = \frac{1}{4}\left[\frac{3(1)}{s^2+1} - \frac{3}{s^2+9}\right]$

08/10/2024 Tuesday

10 $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3 = (\sqrt{t})^3 + \left(\frac{1}{\sqrt{t}}\right)^3 + 3(\sqrt{t})^2\left(\frac{1}{\sqrt{t}}\right) + 3\left(\frac{1}{\sqrt{t}}\right)^2(\sqrt{t})$

$f(t) = t^{3/2} + t^{-3/2} + 3t^{1/2} + 3t^{-1/2}$

$L\{f(t)\} = \frac{\Gamma(3/2+1)}{s^{5/2}} + \frac{\Gamma(-3/2+1)}{s^{-3/2+1}} + \frac{\Gamma(1/2+1)}{s^{1/2+1}} + \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$

$\Gamma(n+1) = n\Gamma(n)$

$= \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}} + \frac{\Gamma(-1/2+1)}{-1/2\, s^{-1/2}} + \frac{1/2\sqrt{\pi}}{s^{3/2}} + \frac{\Gamma(1/2)}{s^{1/2}}$

$\Gamma(n) = \frac{\Gamma(n+1)}{n}$ → -ve fraction

$\Gamma(n+1) = n\Gamma(n)$ → +ve fraction

$= \frac{3/2\ 1/2\sqrt{\pi}}{s^{5/2}} + \frac{(-2)\sqrt{\pi}}{s^{-1/2}} + \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{\sqrt{\pi}}{s^{1/2}}$ //

11 $\sin 6t\ \sin 2t$

Sol:- $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$

$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

$\sin 6t \sin 2t = \frac{1}{2}\left[\cos(4t) - \cos(8t)\right] = f(t)$

$L\{f(t)\} = \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{s}{s^2+64}\right]$ //

08/01/2024 Tuesday

Shifting Rule

$$\boxed{\begin{aligned}&\text{If } L\{f(t)\} = F(s) \text{ then} \\ &L\{e^{at}f(t)\} = F(s-a)\end{aligned}}$$ left shifting

$\int_0^\infty e^{-st} f(t)\,dt$

Proof:- We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

In place of s there is s-a

$$\Rightarrow L\{e^{at}f(t)\} = F(s-a)$$

Similarly

$$L\{e^{-at}f(t)\} = F(s+a) \rightarrow \text{right shifting}$$

ex:

$$L\{\cos 3t\} = \frac{s}{s^2+9}$$

$$L\{e^{2t}\cos 3t\} = \left(\frac{s}{s^2+9}\right)_{s \to s-2} = \frac{s-2}{(s-2)^2+9}$$

s changes to s-2

---

(1) $e^{-3t}[2\cos 6t - 3\sin t]$

Sol.

$$L\{e^{-3t}[2\cos 6t - 3\sin t]\} = \frac{2(s+3)}{(s+3)^2+(6)^2} - \frac{3(1)}{(s+3)^2+(6)^2}$$

$$= \left[\frac{2s}{s^2+6^2} - \frac{3}{s^2+6^2}\right]_{(s \to s+3)}$$

$\cos 2x = 2\cos^2 x - 1$

$\frac{1}{2}(\cos 2x + 1)$

② $e^{2t}\cos^2 t$

$\frac{e^{2t}}{2}[\cos 2x + 1]$

$L\{\frac{e^{2t}}{2}[\cos 2x + 1]\} = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{1}{s-2}\right]$ //

③ $t e^{-2t}\cosh 3t$

Sol $L\{t e^{2t}\left[\frac{e^{3t}+e^{-3t}}{2}\right]\} = L\{\frac{t}{2}\left[e^{t} + e^{-5t}\right]\}$

$= L\{\frac{e^{t}}{2}[t]\} + L\{\frac{e^{-5t}}{2}[t]\}$

first find $L\{f(t)\}$ then replace s by $s-a$ after

$L\{te^{-2t}\left(\frac{e^{3t}+e^{-3t}}{2}\right)\} = \frac{1}{2}\left(\frac{1!}{(s-1)^2} + \frac{1!}{(s+5)^2}\right]$

④ $L\{t^8\cosh 2t\} = L\{\frac{e^{2t}}{2}t^8 + \frac{e^{2t}t^8}{2}\}$

$= \frac{1}{2}\left[\frac{8!}{(s-2)^9} + \frac{8!}{(s+2)^9}\right]$ //

If
$$\boxed{\begin{aligned}&L\{f(t)\} = F(s) \text{ then}\\ &L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))\end{aligned}}$$

Proof: we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Now $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ — Leibnitz rule of differentiation under integration

diff w.r.t s on both sides

$$\frac{d\,F(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

change differ & integration when constant limits are there

using leibnitz rule of diff^n under integral sign

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \underset{\text{f act as const}}{\frac{\partial}{\partial s}\left(e^{-st} f(t)\right)dt} = \int_0^\infty e^{-st}(-t) f(t)\,dt$$

$$-\frac{d}{ds}(F(s)) = \int_0^\infty e^{-st}\, t\, f(t)\,dt = \int_0^\infty e^{-st}\left(t f(t)\right)dt \quad - *$$

$$(-1)\frac{d}{ds}(F(s)) = L\{t f(t)\}$$

Induction proof can be done

differentiate $*$ w.r.t s again

$$(-1)\frac{d^2}{ds^2}(F(s)) = \int_0^\infty \frac{\partial}{\partial s}\left(t e^{-st} f(t)\right)dt = \int_0^\infty -t\, e^{-st}(t) f(t)\,dt$$

$$\therefore (-1)^2\frac{d^2}{ds^2}(F(s)) = \int_0^\infty e^{-st}\left(t^2 f(t)\right)dt$$

$$(-1)^2\frac{d^2(F(s))}{ds^2} = L\{t^2 f(t)\}$$

Similarly

$$(-1)^3\frac{d^3}{ds^3}(F(s)) = L\{t^3 f(t)\}$$

$$\vdots$$

In general

$$(-1)^n\frac{d^n}{ds^n}(F(s)) = L\{t^n f(t)\} \quad //$$

14/10/24 Monday

$$\boxed{\text{If } L\{f(t)\} = F(s) \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_{s=s}^{\infty} F(s)\,ds}$$

Proof :

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)\,dt = F(s)$$

$$F(s) = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

Integrating w.r.t s between the limits $s=s$ to $s=\infty$

$$\int_{s=s}^{\infty} F(s)\,ds = \int_{s=s}^{\infty}\left(\int_{t=0}^{\infty} e^{-st} f(t)\,dt\right) ds$$

write variable limit first

$$\int_{s=s}^{\infty} F(s)\,ds = \int_{t=0}^{\infty}\int_{s=s}^{\infty} e^{-st} f(t)\,ds\,dt$$

$$= \int_{t=0}^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_{s=s}^{\infty} dt$$

$$= \int_{t=0}^{\infty} \frac{f(t)}{-t}\left[-e^{-st} + \cancel{e^{-\infty(t)}}^{\,0}\right] dt$$

$$= \int_{t=0}^{\infty} \frac{f(t)\, e^{-st}}{+t}\,dt$$

$$\int_{s=s}^{\infty} F(s)\,ds = \int_{t=0}^{\infty} e^{-st}\frac{f(t)}{t}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

Similarly

$$L\left\{\frac{f(t)}{t^2}\right\} = \int_{s=s}^{\infty}\int_{s=s}^{\infty} F(s)\,ds\,ds$$

$$L\left\{\frac{f(t)}{t^3}\right\} = \int_{s=s}^{\infty}\int_{s=s}^{\infty}\int_{s=s}^{\infty} F(s)\,ds\,ds\,ds$$

Find the Laplace transform of the following.

1) $\frac{\cos at - \cos bt}{t}$

let $f(t) = \cos at - \cos bt$

$$\cancel{F(s)} = L\{f(t)\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Now

$$L\left\{\frac{f(t)}{t}\right\} = L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_{s=s}^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$s^2+a^2 = K$

$2s\,ds = dK \Rightarrow s\,ds = \frac{dK}{2}$

$s^2+b^2 = K_1$

$2s\,ds = dK_1$

$s\,ds = \frac{dK_1}{2}$

$$\int_{s^2+a^2}^{\infty}\frac{dK}{2(K)} - \int_{s^2+b^2}^{\infty}\frac{dK_1}{2K}$$

$$\frac{1}{2}\Big[\ln K\Big]_{s^2+a^2}^{\infty} - \frac{1}{2}\Big[\ln K_1\Big]_{s^2+b^2}^{\infty}$$

$$= \cancel{\frac{1}{2}\left[\left[\ln K\right]_{s^2+a^2}^{\infty} - \left[\ln K_1\right]_{s^2+b^2}^{\infty}\right]}$$

$$\frac{1}{2}\left[\left[\ln s^2+a^2\right]_s^{\infty} - \left(\ln s^2+b^2\right]_s^{\infty}\right]$$

$$\frac{1}{2}\left[\ln\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$\frac{1}{2}\lim_{s\to\infty}\ln\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\lim_{s\to s}\ln\frac{s^2+a^2}{s^2+b^2}$$

$$\frac{1}{2}\ln\left[\lim_{s\to\infty}\frac{s^2+a^2}{s^2+b^2}\right] - \frac{1}{2}\ln\left[\lim_{s\to s}\frac{s^2+a^2}{s^2+b^2}\right]$$

$$\frac{1}{2}\ln\left(\cancel{\frac{s^2}{s^2}}\right)^{0} - \frac{1}{2}\ln\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\ln\frac{s^2+b^2}{s^2+a^2}$$

Note

$$e^{3t}\left(\frac{\cos at-\cos bt}{t}\right) \rightarrow \frac{1}{2}\ln\frac{(s-3)^2+b^2}{(s-3)^2+a^2}$$

(2) $\frac{t-\sinh at}{t}$

let

$f(t) = t - \sinh at$

$$L\{f(t)\} = F(s) = \frac{1!}{s^2} - \frac{a}{s^2-a^2} = \frac{1}{s^2} - \frac{a}{s^2-a^2}$$

$$\therefore \text{Now } L\left\{\frac{f(t)}{t}\right\} = L\left\{\frac{t-\sinh at}{t}\right\} = \int_{s=s}^{\infty}\left[\frac{1}{s^2} - \frac{a}{s^2-a^2}\right]ds$$

$$= \cancel{\left[\frac{-1}{s} - \frac{a\sin^{-1}\frac{s}{a}}{a}\right]_s^\infty}$$

$$= \left[\frac{-1}{s} - \frac{a}{2a}\log\left(\frac{s-a}{s+a}\right)\right]_s^\infty$$

$$= \left[\frac{-1}{s} - \frac{1}{2}\log\frac{s-a}{s+a}\right]_s^{\infty}$$

$$= \frac{-1}{\infty} - 0 + \frac{1}{s} + \frac{1}{2}\log\frac{s-a}{s+a}$$

$$L\left\{\frac{t-\sinh at}{t}\right\} = \frac{1}{s} + \log\sqrt{\frac{s-a}{s+a}}$$

$\frac{1}{2}(\cos(a-b) - \cos(a+b))$

$\frac{1}{2}(\cos a\cos b + \sin a\sin b - \cos a\cos b + \sin a\sin b)$

3. $\frac{\sin t\sin 3t}{t}$

$$L\{\sin t\sin 3t\} = \frac{1}{2}L\{\cos 2t - \cos 4t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+16}\right]$$

$$L\left\{\frac{\sin t\sin 3t}{t}\right\} = \frac{1}{2}\int_{s=s}^{\infty}\left[\frac{s}{s^2+4} - \frac{s}{s^2+16}\right]ds$$

$$= \cancel{\tfrac{1}{2}}\ \frac{1}{4}\int_{s=s}^{\infty}\left(\frac{2s}{s^2+4} - \frac{2s}{s^2+16}\right)ds$$

$$= \frac{1}{4}\left[\ln\left(\frac{s^2+4}{s^2+16}\right)\right]_s^{\infty}$$

$$= \frac{1}{4}[0] + \cancel{\ln}\ \frac{1}{4}\ln\frac{s^2+16}{s^2+4}$$

$$= \frac{1}{4}\ln\left[\frac{s^2+16}{s^2+4}\right]$$

Note:

$$L\left\{e^{-\sqrt{3}t}\frac{\sin t\sin 3t}{t}\right\} = \frac{1}{4}\ln\left[\frac{(s+\sqrt{3})^2+16}{(s+\sqrt{3})^2+4}\right]$$

(Q) find Laplace transform.

$t^4 e^{-3/2 t}$

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$t e^{4t} \sin 3t$

$t^2 \sin 3t \, e^{-2t}$

15/10/24 Tuesday

(1) $t \sin^2 3t$

$t\left(\frac{1-\cos 6t}{2}\right)$

$f(t) = \frac{1-\cos 6t}{2}$

$f(t) = \frac{1}{2}\left(1-\cos 6t\right)$

$L\{f(t)\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$

$L\{t f(t)\} = \frac{1}{2}(-1)^1\left(\frac{-1}{s^2} - \frac{(s^2+36)(1)-(2s)(s)}{(s^2+36)^2}\right)$

$= -\frac{1}{2}\left[\frac{-(s^2+36)^2 - s^2+36+2s^2}{s^2(s^2+36)^2}\right]$

$= -\frac{1}{2}\left[\frac{-(s+36)^2+s^2+36}{s^2(s^2+36)^2}\right]$

(2) $t^4 e^{-3/2 t}$

sol $\cancel{L\{e^{-3/2 t}\} = \frac{1}{s+3/2}}$

$e^{-3/2 t}[$

$L\{e^{at} f(t)\} = F(s-a)$

$t^4 = f(t)$

$\therefore F(s) = \frac{4!}{s^5} = \frac{24}{s^5}$

$L\{e^{-3/2 t} t^4\} = \frac{24}{(s+3/2)^5}$

(3) sol

(4) $f(t$

③ $te^{-4t}\sin 3t$

Sol $f(t) = t\sin 3t$ $\qquad F(s) = \frac{3}{s^2+9}$ $\qquad (s^2+9)^{-1}$

$F(s) = (-1)^1 \, 3 \cdot \frac{(2s)(-1)}{(s^2+9)^2}$

$L\{te^{-4t}\sin 3t\} = F(s+4) = +\frac{6(s+4)}{((s+4)^2+9)^2}$ //

④ $t^2e^{-2t}\sin 3t$

$f(t) = e^{-2t}(t^2\sin 3t)$

let $f_1(t) = t^2\sin 3t$ $\qquad \sin 3t = f_2(t)$

$F_2(s) = \frac{3}{s^2+9}$

$F_1(s) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$ $\qquad (s^2+9)^{-1}$

$-\frac{2s}{(s^2+9)^2}$

$= 3\frac{((s^2+9)^2(-2) + 4s^2(2)(s^2+9))}{(s^2+9)^4}$ $\qquad \frac{(s^2+9)^2(-2) - (-2s)(2(s^2+9)2s)}{(s^2+9)^4}$

$= 3\frac{(s^2+9)(-2(s^2+9) + 8s^2)}{(s^2+9)^4}$

$= 3\frac{(s^2+9)(6s^2-18)}{(s^2+9)^4}$

$F_1(s) = 3\frac{(6s^2-18)}{(s^2+9)^3}$ //

Now $L\{e^{-2t}t^2\sin 3t\} = F_1(s+2) = \frac{3(6(s+2)^2-18)}{((s+2)^2+9)^3}$ //

Note:

Evaluate $\int_0^\infty e^{-2t} t^2 \sin 3t\, dt$ =

$$\int_0^\infty e^{-st} e^{-2t} t^2 \sin 3t\, dt = \frac{18(s+2)^2 - 54}{[(s+2)^2+9]^3}$$

Put $s=0$,

$$\int_0^\infty e^{-2t} t^2 \sin 3t\, dt = \frac{18(2)^2-54}{((2)^2+9)^3}$$

$$= \frac{18(4)-54}{(4+9)^3} = \frac{72-54}{(13)^3}$$

$$= \frac{18}{(13)^3}$$

Evaluate $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sin 3t}{t} dt$ using Laplace transform

Sol: w.k.T

$$L\left\{\frac{\sin t \sin 3t}{t}\right\} = \frac{1}{2} L\left\{\frac{\cos 2t - \cos 4t}{t}\right\}$$

$$\int_0^\infty \frac{e^{-st} \sin t \sin 3t}{t} = \frac{1}{4} \log\left[\frac{s^2+16}{s^2+4}\right]$$

Put $s=\sqrt{2}$

$$\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sin 3t}{t} \quad \frac{1}{4}\log\left[\frac{18}{6}\right] = \frac{1}{4}\log 3$$

Laplace transform of derivatives

If $f(t)$ & its $1^{st}(n-1)^{th}$ derivatives be continuous then

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) + \cdots - f^{n-1}(0)$$

Laplace transform of nth derivative.

$\frac{d^n f(t)}{dt^n}$

provided $\lim_{t \to \infty} e^{-st} f^m(t) = 0$, where $m = 0, 1, 2 \ldots n-1$

$n=1$, 1^{st} derivative

Proof:- $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

ILATE

ILATE

Integrate by parts

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\int_0^\infty e^{-st} f(t) - f(0)$$

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0) \quad \text{—(1)}$$

Hence proved for $n=1$

Consider

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

Integrate by parts

$$\left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f'(t)\,dt$$

$$0 - f'(0) + s\left[s\,L\{f(t)\} - f(0)\right]$$

substitute (1)

$$L\{f''(t)\} = s^2 \underbrace{L\{f(t)\}}_{F(s)} - s f(0) - f'(0)$$

Similarly

$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$

for n

$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \dots - f^{n-1}(0)$

18/10/24 Friday

Laplace transform of Integrals

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

let $g(t) = \int_0^t f(t)dt \longrightarrow g'(t) = f(t)$

also $g(0) = 0$

Consider

$L\{g(t)\} = \int_0^\infty \underset{II}{e^{-st}} \underset{I}{g(t)} dt$

Integrals by parts

$= [g(t)]\left[\frac{e^{-st}}{-s}\right]_0^\infty - \int \frac{g'(t)\, e^{-st}}{-s} dt$

$= \cancel{g(0)\left[\frac{e^{-s(\infty)}}{-s} + \frac{1}{s}\right] + \frac{1}{s}}$

$= \frac{g(0)\, e^{-s(0)}}{-s} - \frac{e^{-s(\infty)}\, g(\infty)}{-s} + \frac{1}{s}\int g'(t) e^{-st} dt$

$L\{g(t)\} = \frac{1}{s}\int g'(t) e^{-st} dt = \frac{1}{s}\int f(t) e^{-st} dt$

$L\left\{\int_0^\infty f(t)dt\right\} = \frac{1}{s} L\{f(t)\}$

$L\left\{\int_0^{\infty} f(t)\,dt\right\} = \frac{1}{s}F(s)$

Problems :-

Find the Laplace transform of the following

① $\int_0^t e^{-t}\cos t\,dt$

sol:- $L\{\cos t\} = \frac{s}{s^2+1}$

$L\{e^{-st}\cos t\} = \frac{s+1}{(s+1)^2+1}$

$\therefore L\left\{\int_0^{\infty} e^{-st}\cos t\,dt\right\} = \frac{1}{s}\left[\frac{s+1}{(s+1)^2+1}\right]$ //

② $\int_0^t \frac{e^t \sin t}{t}\,dt$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\left\{\frac{f(t)}{t}\right\} = \int_{s=s}^{\infty} F(s)\,ds$ $\quad f(t) = F(s)$

$L\left\{\frac{\sin t}{t}\right\} = \int_{s=s}^{\infty} \frac{1}{s^2+1}\,ds$

$y = \tan^{-1}x$

$= \left[\tan^{-1}s\right]_s^{\infty}$

$\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$

$= \tan^{-1}\infty - \tan^{-1}s$

$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$L\left\{\frac{\sin t}{t}\right\} = \pi/2 - \tan^{-1}s$ $\quad L\{e^{\pm at}f(t)\} = F(s \mp a)$

$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$ //

$L\left\{\frac{e^t \sin t}{t}\right\} = \pi/2 - \tan^{-1}(s-1)$

$L\left\{\int_0^{\infty} f(t)\,dt\right\} = \frac{1}{s}F(s)$

$\therefore L\left\{\int_0^t \frac{e^t \sin t}{t}\right\} = \frac{1}{s}\left[\pi/2 - \tan^{-1}(s-1)\right]$ // $= \frac{1}{s}\left[\cot^{-1}(s-1)\right]$ //

Evaluate following integral

(i) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$

Sol:- $L\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = L\left\{\frac{e^{-t}}{t} - \frac{e^{-3t}}{t}\right\}$

$= L\left\{\frac{e^{-t}}{t}\right\} - L\left\{\frac{e^{-3t}}{t}\right\}$

$= \int_{s=s}^{\infty}\frac{1}{s+1}ds - \int_{s=s}^{\infty}\frac{1}{s+3}ds$

$= \left[\ln[s+1] - \ln[s+3]\right]_s^\infty$

$= \left[\ln\left[\frac{s+1}{s+3}\right]\right]_s^\infty$ $\quad \left(\log\left(\frac{s(1+1/s)}{s(1+3/s)}\right)\right.$

$\log(1/1) = 0$

$= \ln(1) + \ln\left(\frac{(s+3)}{(s+1)}\right)$

$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = \ln\left[\frac{s+3}{s+1}\right]$

Put $s=0$

$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t} = \ln 3$

(ii) $\int_0^\infty t\,e^{-3t}\sin t\,dt$

$\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$ $\quad (s^2+1)^{-1}$ $\quad \frac{-2s}{(s^2+1)^2}$

Sol:- $L\{e^{-3t}t\sin t\} = \cancel{\left[\int_{s=s}^{\infty}\frac{1}{s^2+1}ds\right]_{s\to s+3}} = (-1)^1\left[\frac{-2s}{(s^2+1)^2}\right]_{s\to s+3}$

$L\{e^{-3t}t\sin t\} = \frac{2(s+3)}{((s+3)^2+1)^2}$

or put $s=3$

$\frac{+6}{10^2}$

$$\int_0^\infty e^{-st} e^{3t} t \sin t = \frac{2(s+3)}{((s+3)^2+1)^2}$$

$s = 0$

$$\therefore \int_0^\infty e^{3t} t \sin t = \frac{2(3)}{100} = \frac{3}{50} = \frac{3}{50}$$

Note

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$$

Now

$L\{f(t)\} = F(s)$

$L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

$L\{f''''(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

L.T of Factor

Laplace transform of derivative

Example 1:-

If $L\left\{\frac{2\sqrt{t}}{\sqrt{\pi}}\right\} = \frac{1}{s^{3/2}}$ show that

S.T $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Sol:

$f(t) = \frac{2\sqrt{t}}{\sqrt{\pi}}$

$f'(t) = \frac{2 \cdot 1}{2\sqrt{t\pi}} = \frac{1}{\sqrt{\pi t}}$

$$\therefore L\{f'(t)\} = L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s\left[\frac{1}{s^{3/2}}\right] - f(0)$$

$$= \frac{1}{\sqrt{s}} - 2\frac{\sqrt{0}}{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{s}} \quad //$$

Q) Given

$$L\{2\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

then S.T

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$$

Sol:- $f(t) = 2\sin\sqrt{t}$

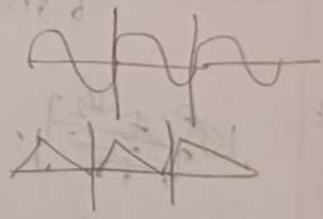
$$f'(t) = \frac{2\cos\sqrt{t}}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$\therefore L\{f'(t)\} = sF(s) - f(0)$$

$$= s\left[\frac{\sqrt{\pi}}{2s^{3/2}}\right]e^{-1/4s} - 2\sin\sqrt{0}$$

$$L\{f'(t)\} = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-1/4s} \quad //$$

Periodic function



Periodic function: $f(t)$ $T > 0$ if $f(t+T) = f(t)$

$$f(T + nT) = f(t)$$

$\sin at \rightarrow T = \frac{2\pi}{a}$ $\qquad a\sin(at+b) + k$

$\tan t \rightarrow \pi$ // $\qquad T\downarrow\ F\uparrow$

A function $f(t)$ is said to be a periodic function with period T (>0 positive) if $f(t+nT) = f(t)$

examples

$\sin t, \cos t$ are periodic function with period 2π

$\sin nt, \cos nt$ are periodic functions with period $\frac{2\pi}{n}$

theorem:

If $f(t)$ is a periodic function with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Periodic function

Proof:

we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

since $f(t)$ is periodic function with period T

$$\therefore L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

put $t = u+T$

$\frac{dt}{du} = 1 + 0$

$dt = du$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du$$

if $f(t)$ is perio $f(u)$ should be period

$f(u+T) = f(u)$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-sT} e^{-su} f(u)\,du$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

laplace integral with variable u

$L\{f(u)\}$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT} L\{f(t)\}$$

By Property $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

$$L\{f(t)\} - e^{-sT}L\{f(t+T)\} = \int_0^T e^{-st} f(t)\,dt$$

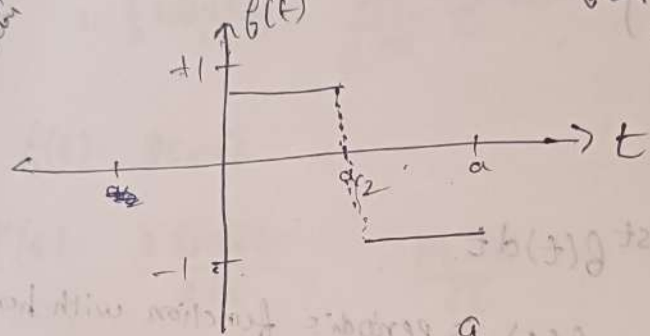
$$L\{f(t)\}(1 - e^{-sT}) = \int_0^T e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \quad // \text{Hence proved}$$

where T is Period.

Q) 17/10/24 Find the Laplace of square wave function of period a. defined by $f(t) = \begin{cases} 1 & ; 0 \le t \le a/2 \\ -1 & ; a/2 < t \le a. \end{cases}$

Sol:-



period $= a - 0 = a$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\,dt$$

$$F(s) = \frac{1}{1-e^{-sa}} \left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-sa}} \left[\int_0^{a/2} e^{-st}(1)\,dt + \int_{a/2}^{a} (-1) e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-sa}} \left[\left[\frac{e^{-st}}{-s} \right]_0^{a/2} - \left[\frac{e^{-st}}{-s} \right]_{a/2}^{a} \right]$$

$$= \frac{1}{1-e^{-sa}} \left[\frac{e^{-sa/2}}{-s} + \frac{1}{s} - \frac{e^{-sa/2}}{s} + \frac{e^{-sa}}{s} \right]$$

$$= \frac{(1-e^{-sa})}{(1-e^{-sa})} \cdot \frac{1}{s^{2}}$$

$\cancel{F(s)} = \frac{1}{s}$ $\qquad f(t)=1.$

$$\frac{1}{s(1-e^{-sa})}\left[-2e^{-sa/2}+e^{-as}+1\right]$$

$$\frac{1}{s(1-e^{-sa})}\left[\left(1-e^{-\frac{sa}{2}}\right)^{2}\right]$$

$$F(s) = \frac{\left(1-e^{-\frac{sa}{2}}\right)^{2}}{s(1-e^{-sa})} = \frac{1}{s}\frac{(1-e^{-sa/2})^{2}}{(1+e^{-sa/2})(1-e^{-sa/2})}$$

$$= \frac{1}{s}\frac{(1-e^{-sa/2})}{(1+e^{-sa/2})} \times \frac{e^{+sa/4}}{e^{sa/4}}$$

$$\frac{1}{s}\frac{(e^{sa/4}-e^{-sa/4})/2}{e^{sa/4}-e^{-sa/4}/2}$$

$$= \frac{1}{s}\frac{\sinh sa/4}{\cosh sa/4}$$

$$F(s) = \frac{1}{s}\tanh\left(\frac{sa}{4}\right)$$

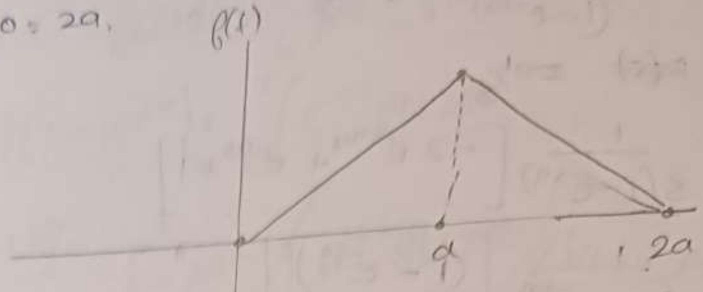
Note:-

$\sinh x = \frac{e^{x}-e^{-x}}{2}$ $\quad \cosh x = \frac{e^{x}+e^{-x}}{2}$ $\quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

Q). Find the laplace transform of the triangular wave function.

$$f(t) = \begin{cases} t & ; 0 \le t < a \\ 2a-t & ; a < t < 2a \end{cases} \qquad T = 2a-0 = 2a$$

$T = 2a - 0 = 2a.$



$\therefore L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)$

$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(t)\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$

$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(t)\,dt - \int_a^{2a} e^{-st}(t)\,dt + \int_a^{2a} e^{-st}(2a)\,dt\right]$

I (II) I (I)

$t\frac{e^{-st}}{-s} + \frac{1}{s}\int e^{-st}$

$\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}$

$= \frac{1}{1-e^{-2as}}\left[\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a + \left[\frac{te^{-st}}{s} + \frac{e^{-st}}{s^2}\right]_a^{2a} - \left[\frac{2ae^{-st}}{s}\right]_a^{2a}\right]$

$= \frac{1}{1-e^{-2as}}\left[\cancel{\frac{-ae^{-sa}}{s}} - \frac{e^{-sa}}{s^2} + 0 + \frac{1}{s^2} + \cancel{\frac{2ae^{-2as}}{s}} + \frac{e^{-2as}}{s^2} \cancel{- \frac{ae^{-sa}}{s}} - \frac{e^{-sa}}{s^2} + \cancel{\frac{-2ae^{-2as}}{s}} + \cancel{\frac{2ae^{-as}}{s}}\right]$

$[f(x) - g(x)]_b^a$

$f(a) - g(a) - f(b) + g(b)$

$f(a) - f(b) - (g(a) - g(b))$

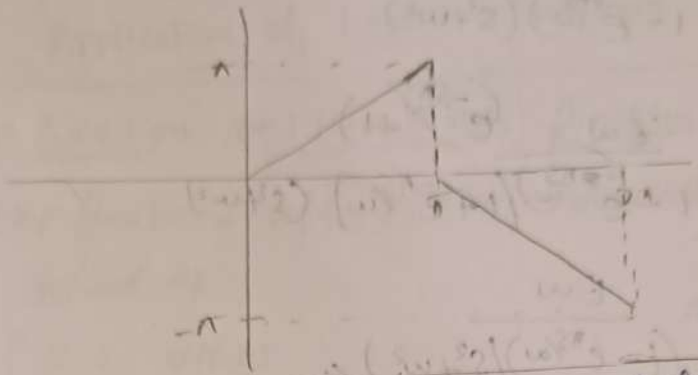
$= \frac{1}{1-e^{-2as}(s^2)}\left[1 - 2e^{-sa} + e^{-2as}\right] = \frac{1 - e^{-sa}}{s^2(1-e^{-as})(1+e^{-as})}$

$= \frac{(1-e^{-sa})^2}{(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{1+e^{-as}} \times \frac{e^{as/2}}{e^{as/2}}$

$= \frac{1}{s^2}\,\frac{e^{sa/2} - e^{-sa/2}}{e^{sa/2} + e^{-sa/2}} = \frac{1}{s^2}\tanh$

① $f(t) = \begin{cases} t & : 0<t<\pi \\ \pi - t & : \pi<t<2\pi \end{cases}$ Draw graph

sol:



21/10/24 Monday

Q③. Show that the ~~*laplace*~~ transform of the periodic

function $f(t) = \begin{cases} E\sin\omega t & : 0<t<\pi/\omega \\ 0 & : \pi/\omega < t < 2\pi/\omega. \end{cases}$

when E and ω are constants, is $\dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

sol: $T = 2\pi/\omega - 0 = 2\pi/\omega$

$$L\{f(t)\} = \frac{E}{1-e^{-\frac{2\pi}{\omega}s}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt + \cancel{\int_{\pi/2}^{2\pi/\omega} 0\, e^{-st}\,dt}\right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$ ILATE

$$= \frac{E}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}(0+\omega) - \frac{e^{-0}}{s^2+\omega^2}(0-\omega)\right]$$

$$= \frac{E\omega\left(e^{-\frac{s\pi}{\omega}} + 1\right)}{\left(1 - e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}$$

$$= \frac{E\omega \cancel{\left(e^{-\frac{s\pi}{\omega}} + 1\right)}}{\left(1 - e^{-\frac{\pi s}{\omega}}\right)\cancel{\left(1 + e^{-\pi s/\omega}\right)}(s^2+\omega^2)}$$

$$F(s) = \frac{E\omega}{\left(1 - e^{-\pi s/\omega}\right)(s^2+\omega^2)}$$