Q1. Given

$$\begin{array}{ccccc} & & & \text{To} & \\ & & C & N & D \\ & C & 0.5 & 0.4 & 0.1 \\ \text{From} & N & 0.3 & 0.4 & 0.3 \\ & D & 0.2 & 0.3 & 0.5 \end{array}$$

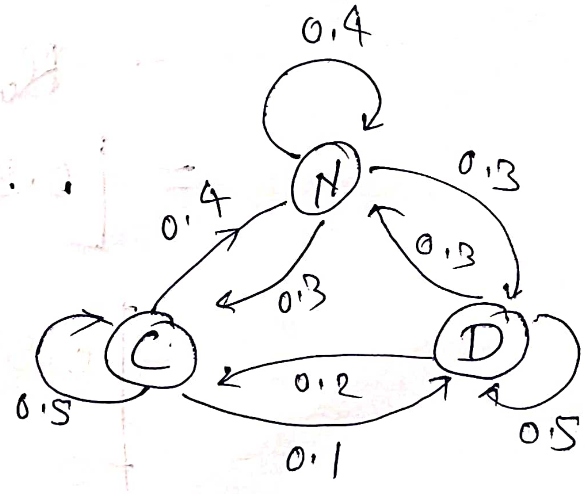
a) Since the probability of the $(n+1)^{th}$ day is Cheerfull or Normal or depressed is depends on n^{th} day, where $n \geq 0$.

So we can say that X_n is a markov chain.

properties: markov chain is a Stochastic model describing a Sequence of possible Events in which the probability of Each event depends only on the State attained in the previous event.

b) $P =$

$$\begin{array}{ccccc} & & & \text{To} & \\ & & C & N & D \\ & C & 0.5 & 0.4 & 0.1 \\ \text{From} & N & 0.3 & 0.4 & 0.3 \\ & D & 0.2 & 0.3 & 0.5 \end{array}$$



Q 2// Given

State Space = {0, 1, 2}

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.75 & 0.25 & 0 \\ 1 & 0.25 & 0.5 & 0.25 \\ 2 & 0 & 0.75 & 0.25 \end{array}$$

i) $p(X_2 = 2, X_1 = 1 \mid X_0 = 2) = ?$

$= \pi \; P_{21} \times P_{12}$

π is the initial distribution which is $[1/3, 1/3, 1/3]$

$= 0.3 \times 0.75 \times 0.25 = 0.061$ Ans.

ii) $p(X_1 = 1) \Rightarrow$ Step 1 States 1

$= R_0 \times P^{(1)}$

$$= \underset{1\times 3}{\begin{bmatrix} 0.33, & 0.33, & 0.33 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.75 & 0.25 \end{bmatrix}}$$

$= \begin{bmatrix} 0.33 & 0.33 \times 1.5 & 0.33 \times 0.5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0.33 & 0.495 & 0.165 \end{bmatrix}$

$p(X_1 = 1) = 0.495$ Ans

Q 3//

Given

State space $S = \{0, 1\}$

Initial distribution $\pi = (0.5, 0.5)$

Transition matrix

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix}$$

Soln

$P(X_2 = 0)$ So State 0

time period = 2

$= \pi \times P^2$

$$P^2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.01 + 0.27 & 0.09 + 0.63 \\ 0.03 + 0.21 & 0.27 + 0.49 \end{bmatrix}$$

$$= \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}$$

Now $\pi \times P^2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}_{1\times2} \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}_{2\times2}$

$$= \begin{bmatrix} 0.5\times0.28 + 0.5\times0.24 & 0.5(0.72 + 0.76) \end{bmatrix}$$

$$= \begin{bmatrix} 0.26 & 0.74 \end{bmatrix}$$

So $P(X_2 = 0) = 0.26$ Ans//

Q4

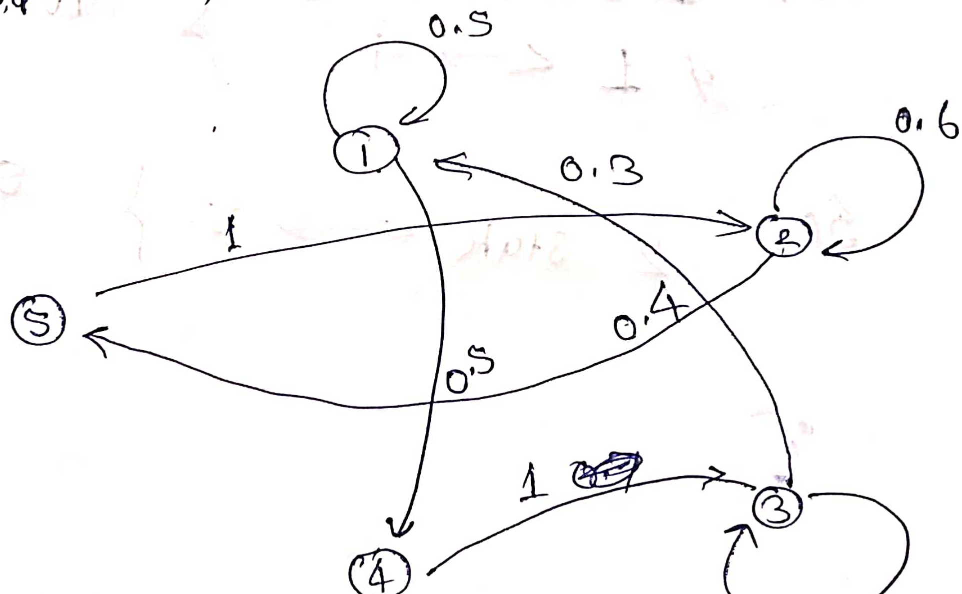
Given

$$S = \{1, 2, 3, 4, 5\}$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.6 & 0 & 0 & 0.4 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Soln:

$$\cancel{t_{0,0} = t^{(1)}_{0,0} + t_{0,}}$$



$$t_{1,1} = t^{(1)}_{1,1} + t^{(2)}_{1,1} + t^{(3)}_{1,1}$$

$$= 0.5 + 0 + 0.5 \times 1 \times 0.3 = 0.65$$

$$t_{2,2} = t^{(1)}_{2,2} + t^{(2)}_{2,2}$$

$$= 0.6 + 0.4 \times 1 = 1$$

$t_{11} = 0.65$

which is less then 1.

So 1 State is transient.

$t_{22} = 1$

which is equal to 1

So State 2 is recurrent.

Theorem:

$i \leftrightarrow j$, i is recurrent $\Rightarrow$ j recurrent

$i \leftrightarrow j$, i is transient $\Rightarrow$ j transient

So, 1 State $\leftrightarrow$ 3
& 1 $\leftrightarrow$ 4 } 3 and 4 State transient

So; 2 State $\leftrightarrow$ 5 } 5 is recurrent.

Q5.

Given

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Show that

$$p^{(n)} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{pmatrix}$$

Soln :

In the two state markov chain p has first row $[p \;\; 1-p]$ and second row is reverse of this. Then $p^{(n)}$ also have entries in the same form with first row as $[a_n \;\; b_n]$ and second row $[b_n \;\; a_n]$.

Lets do the calculation for 1st row.

$$p^{(n)} = \left[\frac{1}{2} + (2p-1)^n/2 \quad \frac{1}{2} - (2p-1)^n/2\right]$$

$$p^{(1)} = \left[\frac{1}{2} + p - \frac{1}{2} \quad \frac{1}{2} - p + \frac{1}{2}\right]$$

$$= [p \quad 1-p]$$

which is true,

$$P^{(n+1)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Considering only first row, because of symmetry.

$$P^{(n+1)} = \begin{bmatrix} a_n p + b_n (1-p) & a_n (1-p) + b_n p \\ a_n (1-p) + b_n p & a_n p + b_n (1-p) \end{bmatrix}$$

Now

$$a_n p + b_n (1-p)$$

$$= \frac{1}{2} p + \frac{1}{2} p (2p-1)^n + \frac{1}{2}(1-p)$$

$$- \frac{1}{2}(1-p)(2p-1)^n$$

$$= \frac{1}{2}(p+1-p) + \frac{1}{2}(2p-1)^n \{p-1+p\}$$

$$= \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} = a_{n+1}$$

Similarly

$$a_n(1-p) + b_n p$$

$$= \frac{1}{2}(1-p) + \frac{1}{2}(1-p)(2p-1)^n + \frac{1}{2}p - \frac{1}{2}p(2p-1)^n$$

$$= \frac{1}{2}(1-p+p) + \frac{1}{2}(2p-1)^n(1-p-p)$$

$$= \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} = b_{n+1}$$

So, if $p(n)$ is true than $p(n+1)$ is also true. So by mathematical induction the formula is true proved.

Ans

Q 6,, Given

$$R_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P = \begin{array}{c|cc} & R & B \\ \hline R & 0.8 & 0.2 \\ B & 0.2 & 0.8 \end{array}$$

Soln :

State at time 1

$$R_1 = R_0 P = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8+0 & 0.2+0 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$$

State at time 2

$$R_2 = R_1 P$$

$$= \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64+0.04 & 0.16+0.16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.68 & 0.32 \end{bmatrix}$$

State at time 3

$$R_3 = R_2 P = \begin{bmatrix} 0.68 & 0.32 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.608 & 0.392 \end{bmatrix}$$

State at time 4

$$R_4 = R_3 P = \begin{bmatrix} 0.608 & 0.392 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.56 & 0.435 \end{bmatrix}$$

State at time 5

$$R_5 = R_4 P = \begin{bmatrix} 0.56 & 0.435 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.53 & 0.46 \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 0.53 & 0.46 \end{bmatrix}$$

The probability that the fifth ball selected is red

$= 0.53$

Ans

Q 7. Given:

Balls are successively distributed among 8 urns.
probability that there will be exactly 3 nonempty urs after 9 balls have been distributed.

Soln: If we let X_n be the number of nonempty urns after n balls have been distributed, then $X_n, n \geq 0$ is a Markov Chain with states $0, 1, \ldots 8$ and transition probabilities

$$P_{i,i} = i/8 = 1 - P_{i,i+1}, \quad i = 0, 1 \ldots 8.$$

The desired probability is $P^9_{0,3} = P^8_{1,3}$, where the equality follows because $P_{0,1} = 1$ ~~Now~~.

We need to consider the transition probability matrix with states $1, 2, \ldots 8$. However, because we ~~also~~ only need probability, starting with single occupied urn, that there are 3 occupied urns after an additional 8 balls have been distributed we can make use of the fact that the state of the markov chain can't decrease to collapse all states $4, 5, \ldots 8$ into single state 4 with the interpretation that state is 4 whenever four or more of the urns are occupied.

Consequently, we need only determine the eight-step transition probability $p^8_{1,3}$ of the markov chain states 1, 2, 3, 4 having transition probability matrix P given by.

$$P = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Raising the preceding matrix to the power 4 yields the matrix P^4 given

$$P^4 = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0 & 0.0039 & 0.0952 & 0.9009 \\ 0 & 0 & 0.0198 & 0.9802 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$p^8_{1,3} = 0.0002 \times 0.2563 + 0.0256 * 0.0952 + 0.2563 \times 0.0198 + 0.07178 \times 0$$

$$= 0.00756$$

~~Ans~~

$$P^9_{0,3} = P^8_{1,3} = 0.00756$$

Ans

Q 8 Given:

$\{N(t), t \geq 0\}$ be a $PP(\lambda)$.

S_K be the time of occurrence of the Kth event in this poisson process.

Show that $N(t) = 1$, S_1 is uniformly distributed over $[0, t]$

Soln:

$$P(\{S_1 < s \mid N(t) = 1\})$$

$$= \frac{P(\{S_1 < s, N(t) = 1\})}{P(N(t) = 1)}$$

$$= \frac{P(\{1 \text{ event in } [0, s], 0 \text{ events in } [s, t]\})}{P(\{N(t) = 1\})}$$

$$= \frac{P(\{1 \text{ event in } [0, s]\}) \cdot P(\{0 \text{ events in } [s, t]\})}{P(\{N(t) = 1\})}$$

$$= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}}$$

$$F_{S_1}(s) = P(\{S_1 < s \mid N(t) = 1\}) = \frac{s}{t} \quad (1)$$

Since c.d.f signifies uniquely to particular distribution & from ①
$F_{S_1}(s)$ is c.d.f of uniform random variable ones $[0, t]$.

Q 9, Given

$\{N(t),\ t \geq 0\}$ be a $PP(\lambda)$, $s, t \geq 0$

joint distribution.

$p(N(s) = i,\ N(s+t) = j),\quad 0 \leq i \leq j < \infty$

Soln:

let $\{N(t),\ t \geq 0\}$ be a $PP(\lambda)$

the joint probability distribution of $p(N(s)) = i,\quad N(s+t) = j$ is as follows

$$f_{x,y}(x, y) = f_x(x/y)\, f_y(y)$$

$$f_{N(s),\,N(s+1)}(N(s), N(s+t)) = f_{N(s)}(n(s) = i / n(s+t) = j)\; f_{N(s+t)}(n(s+t) = j)$$

$$= \frac{\exp(-(n(s+t) = j))}{(n(s+t) = j)},\quad 0 \leq i \leq j < \infty$$

then

$$P_{N(s),\,N(s+t)}\left(n(s)=i,\; n(s+t)=j\right)$$

$$= \frac{\exp\left(-\left(n(s+t)=j\right)\right)}{\left(n(s+t)=j\right)}, \quad 0 \le i \le j \le \infty$$

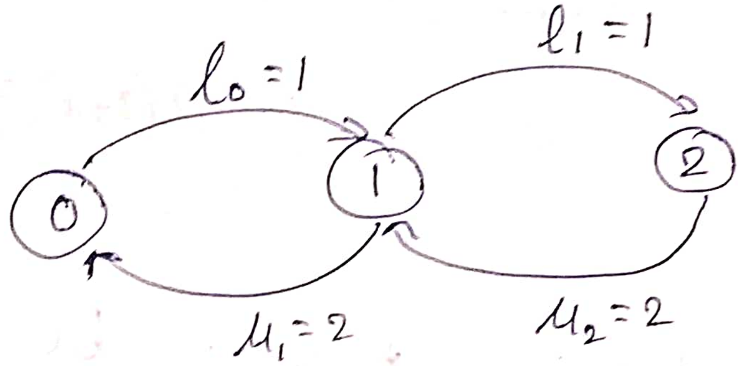
Q 10 Given:

States → (0, 1, 2)

birth rate → 1

~~deth~~ death rate → 2

Ans a)



Ans b)

$$P_1 = \frac{l_0}{\mu_1} P_0 = \frac{1}{2} P_0$$

$$P_2 = \frac{l_0 l_1}{\mu_1 \mu_2} P_0 = \frac{1}{4} P_0$$

$$P_0 + P_1 + P_2 = 1$$

Ans c) :

$$P_0 + P_1 + P_2 = 1$$

$$P_0 + \frac{1}{2} P_0 + \frac{1}{4} P_0 = 1$$

$$7 P_0 = 4 \Rightarrow P_0 = 4/7$$

$$P_1 = \frac{1}{2} P_0 = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$$

$$P_2 = \frac{1}{4} \times P_0 = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$$

Ans

Q1. Given

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{ccc} C & N & D \end{array} \\ \text{From} & \begin{array}{c} C \\ N \\ D \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

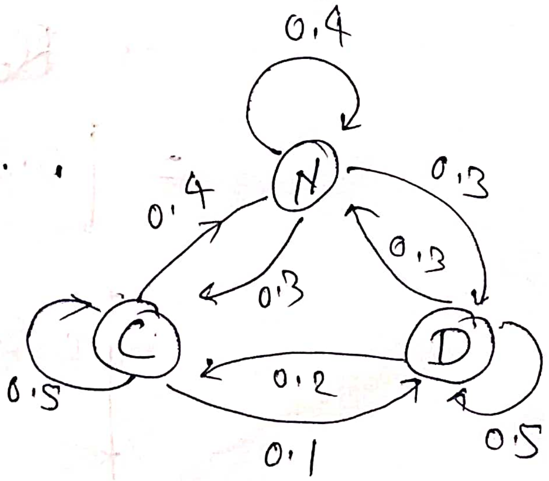
a) Since the probability of the $(n+1)^{th}$ day is Cheerfull or Normal or depressed is depends on n^{th} day, where $n \geq 0$.

So we can say that X_n is a markov chain.

properties: markov chain is a Stochastic model describing a sequence of possible Events in which the probability of Each event depends only on the State attained in the previous event.

b)

$$P = \begin{array}{ccc} & & \text{To} \\ & & \begin{array}{ccc} C & N & D \end{array} \\ \text{From} & \begin{array}{c} C \\ N \\ D \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$



Q 2// Given

State Space = {0, 1, 2}

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.75 & 0.25 \end{pmatrix} \end{array}$$

i) $p(X_2 = 2, X_1 = 1 \mid X_0 = 2) = ?$

$= \pi \; P_{21} \times P_{12}$

π is the initial distribution which is $[1/3, 1/3, 1/3]$

$= 0.3 \times 0.75 \times 0.25 = 0.061$ Ans.

ii) $p(X_1 = 1) \Rightarrow$ Step 1 States 1

$= \pi_0 \times p^{(1)}$

$$= \underset{1\times3}{\begin{bmatrix} 0.33 , & 0.33 , & 0.33 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.75 & 0.25 \end{bmatrix}}$$

$= \begin{bmatrix} 0.33 & 0.33 \times 1.5 & 0.33 \times 0.5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0.33 & 0.495 & 0.165 \end{bmatrix}$

$p(X_1 = 1) = 0.495$ Ans//

Q 3//

Given

State space $S = \{0, 1\}$

Initial distribution $\pi = (0.5, 0.5)$

Transition matrix

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix}$$

Soln

$P(X_2 = 0)$ So State 0
time period = 2

$$= \pi \times P^2$$

$$P^2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.01 + 0.27 & 0.09 + 0.63 \\ 0.03 + 0.21 & 0.27 + 0.49 \end{bmatrix}$$

$$= \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}$$

Now $\pi \times P^2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}_{1\times2} \begin{bmatrix} 0.28 & 0.72 \\ 0.24 & 0.76 \end{bmatrix}_{2\times2}$

$$= \begin{bmatrix} 0.5\times0.28 + 0.5\times0.24 & 0.5(0.72 + 0.76) \end{bmatrix}$$

$$= \begin{bmatrix} 0.26 & 0.74 \end{bmatrix}$$

So $P(X_2 = 0) = 0.26$ Ans//

Q4

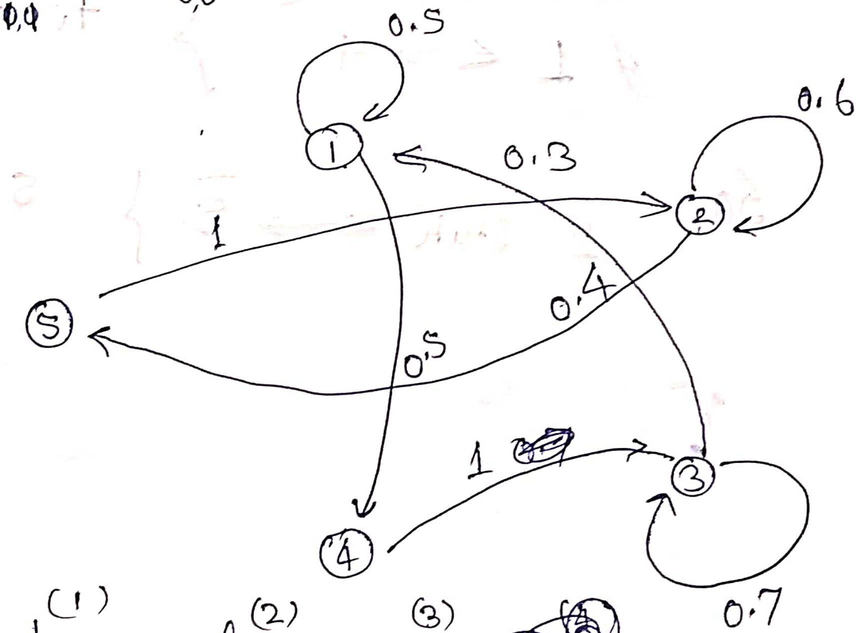
Given

$$S = \{1, 2, 3, 4, 5\}$$

$$P = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0.5 & 0 & 0 & 0.5 & 0 \\ 2 & 0 & 0.6 & 0 & 0 & 0.4 \\ 3 & 0.3 & 0 & 0.7 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Soln:

$\cancel{t_{0,0} = t_{0,0}^{(1)} + t_{0,}}$



$$t_{1,1} = t_{1,1}^{(1)} + t_{1,1}^{(2)} + t_{1,1}^{(3)}$$

$$= 0.5 + 0 + 0.5 \times 1 \times 0.3 = 0.65$$

$$t_{2,2} = t_{2,2}^{(1)} + t_{2,2}^{(2)}$$

$$= 0.6 + 0.4 \times 1 = 1$$

Q5.

Given

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Show that

$$P^{(n)} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{pmatrix}$$

Soln:

In the two state markov chain P has first row $[p \;\; 1-p]$ and second row is reverse of this. Then $P^{(n)}$ also have entries in the same form with first row as $[a_n \;\; b_n]$ and second row $[b_n \;\; a_n]$.

Lets do the calculation for 1st row.

$$p^{(n)} = \left[\frac{1}{2} + (2p-1)^n/2 \quad \frac{1}{2} - \frac{(2p-1)^n}{2}\right]$$

$$p^{(1)} = \left[\frac{1}{2} + p - \frac{1}{2} \quad \frac{1}{2} - p + \frac{1}{2}\right]$$

$$= [p \quad 1-p]$$

which is true,

$$P^{(n+1)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Considering only first row, because of Symmetry.

$$P^{(n+1)} = \begin{bmatrix} a_n p + b_n (1-p) & a_n (1-p) + b_n p \\ a_n (1-p) + b_n p & a_n p + b_n (1-p) \end{bmatrix}$$

Now

$$a_n p + b_n (1-p)$$

$$= \frac{1}{2}p + \frac{1}{2}p(2p-1)^n + \frac{1}{2}(1-p) - \frac{1}{2}(1-p)(2p-1)^n$$

$$= \frac{1}{2}(p+1-p) + \frac{1}{2}(2p-1)^n \{p-1+p\}$$

$$= \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} = a_{n+1}$$

Similarly

$$a_n(1-p) + b_n p$$

$$= \frac{1}{2}(1-p) + \frac{1}{2}(1-p)(2p-1)^n$$
$$+ \frac{1}{2}p - \frac{1}{2}p(2p-1)^n$$

$$= \frac{1}{2}(1-p+p) + \frac{1}{2}(2p-1)^n(1-p-p)$$

$$= \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} = b_{n+1}$$

So, if $p(n)$ is true than $p(n+1)$ is also true. So by mathematical induction the formula is true proved.

Ans

Q 6,, Given

$$R_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P = \begin{array}{c} \\ R \\ B \end{array} \begin{array}{c} \begin{array}{cc} R & B \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

Soln:

State at time 1

$$R_1 = R_0 P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8+0 & 0.2+0 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$$

State at time 2

$$R_2 = R_1 P$$

$$= \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64+0.04 & 0.16+0.16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.68 & 0.32 \end{bmatrix}$$

State at time 3

$$R_3 = R_2 P = \begin{bmatrix} 0.68 & 0.32 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.608 & 0.392 \end{bmatrix}$$

State at time 4

$$R_4 = R_3 P = \begin{bmatrix} 0.608 & 0.392 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.56 & 0.435 \end{bmatrix}$$

State at time 5

$$R_5 = R_4 P = \begin{bmatrix} 0.56 & 0.435 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.53 & 0.46 \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 0.53 & 0.46 \end{bmatrix}$$

The probability that the fifth ball selected is red

$= 0.53$

Ans//

Q 7. Given:

Balls are successively distributed among 8 urns.

probability that there will be exactly 3 nonempty urns after 9 balls have been distributed.

Soln: If we let X_n be the number of nonempty urns after n balls have been distributed, then $X_n, n \geq 0$ is a Markov Chain with states 0, 1, ... 8 and transition probabilities

$$P_{i,i} = i/8 = 1 - P_{i,i+1}, \quad i = 0, 1 \ldots 8.$$

The desired probability is, $P^9_{0,3} = P^8_{1,3}$. where the equality follows because $P_{0,1} = 1$
~~~~

We need to consider the transition probability matrix with states 1, 2, ..8 However, because we ~~also~~ only need probability starting with single occupied urn, that there are 3 occupied urns, after an additional 8 balls have been distributed we can make use of the fact that the state of the Markov Chain can't decrease to collapse all states 4, 5, ... 8 into single state 4 with the interpretation
~~~~

that state is 4 whenever four or more of the urns are occupied.

Consequently, we need only determine the eight-step transition probability $P^8_{1,3}$ of the markov chain states 1, 2, 3, 4 having transition probability matrix P given by.

P =

1/8	7/8	0	0
0	2/8	6/8	0
0	0	3/8	5/8
0	0	0	1

Raising the preceding matrix to the power 4 yields the matrix P^4 given

P^4 =

0.0002	0.0256	0.2563	0.7178
0	0.0039	0.0952	0.9009
0	0	0.0198	0.9802
0	0	0	1

Hence

$$P^8_{1,3} = 0.0002 \times 0.2563 + 0.0256 * 0.0952 + 0.2563 \times 0.0198 + 0.07178 \times 0$$

$$= 0.00756$$ ~~Ans~~

$$P^9_{0,3} = P^8_{1,3} = 0.00756$$

Ans

Q 8

Given:

$\{N(t), t \geq 0\}$ be a $PP(\lambda)$.

S_k be the time of occurrence of the kth event in this poisson process.

Show that $N(t) = 1$, S_1 is uniformly distributed over $[0, t]$

Soln:

$$P(\{S_1 < s \mid N(t) = 1\})$$

$$= \frac{P(\{S_1 < s, N(t) = 1\})}{P(N(t) = 1)}$$

$$= \frac{P(\{1 \text{ event in } [0, s], 0 \text{ events in } [s, t]\})}{P(\{N(t) = 1\})}$$

$$= \frac{P(\{1 \text{ event in } [0, s]\}) \cdot P(\{0 \text{ events in } [s, t]\})}{P(\{N(t) = 1\})}$$

$$= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}}$$

$$F_{S_1}(s) = P(\{S_1 < s \mid N(t) = 1\}) = \frac{s}{t} \quad \text{— (1)}$$

Since c.d.f signifies uniquely to particular distribution & From ①
$F_{S_1}(s)$ is c.d.f of uniform random variable ones $[0, t]$.

Q 9, Given

$\{N(t), t \geq 0\}$ be a $PP(\lambda)$, $s, t \geq 0$

joint distribution.

$p(N(s) = i, N(s+t) = j)$, $0 \leq i \leq j < \infty$

Soln:

let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$

the joint probability distribution of $p(N(s)) = i$, $N(s+t) = j$ is as follows

$$f_{x,y}(x, y) = f_x(x/y) \, f_y(y)$$

$$f_{N(s), N(s+t)}(n(s), n(s+t)) = f_{N(s)}(n(s) = i / n(s+t) = j) \, f_{N(s+t)}(n(s+t) = j)$$

$$= \frac{\exp(-(n(s+t) = j))}{(n(s+t) = j)}, \quad 0 \leq i \leq j < \infty$$

then

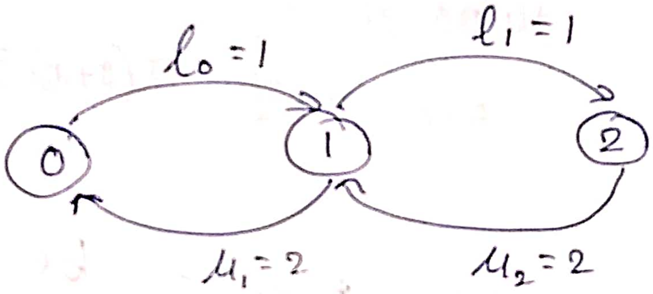
$$P_{N(s),N(s+t)}\left(n(s)=i,\ n(s+t)=j)\right)$$

$$= \frac{\exp\left(-(n(s+t)=j)\right)}{\left(n(s+t)=j\right)}, \quad 0 \le i \le j \le \infty$$

Q 10, Given:

States → (0, 1, 2)
birth state → ~~0~~ 1
~~deth~~ death state → 2

Ans a)



Ans b)

$$P_1 = \frac{l_0}{\mu_1} P_0 = \frac{1}{2} P_0$$

$$P_2 = \frac{l_0 l_1}{\mu_1 \mu_2} P_0 = \frac{1}{4} P_0$$

$$P_0 + P_1 + P_2 = 1$$

Ans c):

$$P_0 + P_1 + P_2 = 1$$

$$P_0 + \frac{1}{2} P_0 + \frac{1}{4} P_0 = 1$$

$$7 P_0 = 4 \Rightarrow P_0 = 4/7$$

$$P_1 = \frac{1}{2} P_0 = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$$

$$P_2 = \frac{1}{4} \times P_0 = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$$

Ans