Inverse Pendulum Dynamic and Control

December 2022

1 Equations of Motion

$$x_p = x - \ell\sin(\theta),\ \dot{x}_p = \dot{x} - \ell\dot{\theta}\cos(\theta) \tag{1}$$

$$y_p = \ell\cos(\theta),\ \dot{y}_p = -\ell\dot{\theta}\sin(\theta) \tag{2}$$

2 Potential Energy

$$V = m_g y_p \to V = mg\ell\cos(\theta) \tag{3}$$

3 Kinetic Energy

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2) \tag{4}$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}[\dot{x}^2 - 2\ell\dot{\theta}\dot{x}\cos(\theta) + \ell^2\theta^2(\cos^2(\theta) + \sin^2(\theta)] \tag{5}$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 - m\ell\dot{\theta}\dot{x}\cos(\theta) \tag{6}$$

4 Lagrange Equations

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{7}$$

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} \tag{8}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = Q_i \tag{9}$$

$$L = T - V \tag{10}$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 - m\ell\dot{\theta}\dot{x}\cos(\theta) - m\cos(\theta) \tag{11}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \begin{bmatrix} \partial L/\partial \dot{x} \\ \partial L/\partial \dot{\theta} \end{bmatrix} = \begin{bmatrix} (M+m)\dot{x} - m\ell\dot{\theta}\cos(\theta) \\ m\ell^2\dot{\theta} - m\ell\dot{x}\cos(\theta) \end{bmatrix} \tag{12}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] = \begin{bmatrix} (M+m)\ddot{x} - (m\ell\ddot{\theta}\cos(\theta) - m\ell\ddot{\theta}^2\sin(\theta)) \\ m\ell^2\ddot{\theta} - (m\ell\ddot{x}\cos(\theta) - m\ell\dot{x}\dot{\theta}\sin(\theta)) \end{bmatrix} \tag{13}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} 0 \\ m\ell\dot{x}\dot{\theta}\sin(\theta) + mg\ell\sin(\theta) \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} F(t) \\ 0 \end{bmatrix} = \begin{bmatrix} (M+m)\ddot{x} - m\ell\ddot{\theta}\cos(\theta) + m\ell\ddot{\theta}^2\sin(\theta) \\ m\ell^2\ddot{\theta} + m\ell\dot{x}\dot{\theta}\sin(\theta) - m\ell\dot{x}\cos(\theta) - m\ell\dot{x}\dot{\theta}\sin(\theta) - mg\ell\sin(\theta) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} F(t) \\ 0 \end{bmatrix} = \begin{bmatrix} (M+m)\ddot{x} - m\ell\ddot{\theta}\cos(\theta) + m\ell\ddot{\theta}^2\sin(\theta) \\ m\ell\ddot{x} - m\ell\ddot{x}\cos(\theta) - mg\ell\sin(\theta) \end{bmatrix} \tag{16}$$

5 State Space Equations

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{17}$$

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \tag{18}$$

$$\textbf{Small angle approximation...} \tag{19}$$

$$\begin{bmatrix} F(t) \\ 0 \end{bmatrix} = \begin{bmatrix} (M+m)\ddot{x} - m\ell\ddot{\theta} \\ m\ell\ddot{\theta} - m\ell\ddot{x} - mg\ell\theta \end{bmatrix} \tag{20}$$

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & (M+m)g/M\ell & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/M\ell \end{bmatrix} F \tag{21}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} F \tag{22}$$

$$\ddot{x} = \frac{F + m\ell\ddot{\theta}}{M+m} \tag{23}$$

$$\ddot{\theta} = \frac{m\ell\ddot{x} + mg\ell\theta}{m\ell^2} = \frac{\ddot{x} + g\theta}{\ell} \tag{24}$$

$$\ddot{x} = \frac{F + m\ddot{x} + mg\theta}{M+m} = \frac{F}{M} + \frac{mg\theta}{M} = \frac{F}{M} + \frac{m}{M} g\theta \tag{25}$$

$$\ddot{\theta} = \frac{F/M + m/M(g\theta) + g(\theta)}{\ell} = \frac{F + (M+m)g\theta}{M\ell} \tag{26}$$