Unit-1

Propositional Logic

Proposition :-

A proposition is a declarative sentence i.e, either T (or) F but not both.

Ex:- 2+5=7 (T)

2:- KKd is one of the states in India (F)

Negation :-

If 'P' is any Proposition then, negation of P is denoted by '$\neg P$' (or) '$\sim P$' & it is defined as "it is case of not P"

Ex:- If P: This is a book
$\neg P$: This is not a book.

Truth table :-

If P is any proposition, then

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

Disjunction ($\vee$) :- (or) (+)

If P, q are any 2 propositions, then disjunction of P, q is denoted by '$(P \vee q)$'. It is defined as $P \vee q$ is false, if both are false & true otherwise.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction (∧) and :- (x)

If p, q are any 2 propositions then conjunction of p, q is (p∧q).

If both are true, then p∧q is true, otherwise False.

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Conditional statement :- (⇒, →)

If p, q are any 2 propositions then conditional statement p→q is also proposition. It is false, if p is true & q is false and all other cases it is true

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Bi-conditional statement :- ($\Longleftrightarrow$, $\leftrightarrow$)

If p, q are any 2 propositions then bi-conditional statement is $p \leftrightarrow q$ it is $p \rightarrow q$ and $q \rightarrow p$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exclusive statement :- $\oplus$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Tautology :-

A compound proposition which is true in all cases is called tautology.

contradiction :-

A compound proposition which is False in all cases is called contradiction.

Logically equivalent :-

2 statements are said to be logically equivalent, if they have identical truth tables

Ex :- short

Construct truth table for

$(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Q. Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The truth values of $p \rightarrow q$ and $\neg p \vee q$ identical

$\therefore$ $p \rightarrow q$ and $\neg p \vee q$ are logically equ

De Morgan's Laws :-

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

Imp

Q) Prove that demorgan laws in logic i.e.,

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p) \wedge (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\therefore$ $\neg(p \vee q)$ and $\neg p \wedge \neg q$

Truth values are identical

$$\therefore \neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

2 $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p) \vee (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\therefore$ $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$ are identical

$\therefore$ They are logically Equivalent

$$\therefore \neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

Q. Prove that

i, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

$\therefore p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ are identical

$\therefore$ Truth table of $p \wedge (q \vee r)$, $(p \wedge q) \vee (p \wedge r)$ are identical

$\therefore p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

ii. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

∴ Truth table of $p \vee (q \wedge r)$, $(p \vee q) \wedge (p \vee r)$ are identical

$\therefore p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Q. Prove that

$\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

∴ Truth table of $\neg(p \to q)$ and $p \wedge \neg q$ are identical

$\therefore \neg(p \to q) \equiv p \wedge \neg q$

Logically Equivalences :-

1, **Identity law :-**

$p \wedge T \equiv p$

$p \vee F \equiv p$

2, **Dominent law :-**

$p \vee T \equiv T$

$p \wedge F \equiv F$

3, **Idempotent law :-**

$p \wedge p \equiv p$

$p \vee p \equiv p$

4, **Double negation law :-**

$\neg(\neg p) \equiv p$

5, **Distributive law :-**

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

6, **Associative law :-**

$(p \vee q) \vee r \equiv p \vee (q \vee r)$, $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

7, **Absorption law :-**

$p \wedge (p \vee q) \equiv p$

$p \vee (p \wedge q) \equiv p$

8, **Negation law :-**

$p \wedge \neg p \equiv F$

$p \vee \neg p \equiv T$

Universal and Existential Quantifers :-

1. **Universal quantifiers :-**

If any statement is true for all the values in the domain then that value is called universal Quantifier.

Consider the example,

Every apple is red

this can be written as

For all x, if x is an apple then x is Red

Denote $A(x)$: x is an apple.

$R(x)$: x is Red

Now, given statement can be written as

$$(\forall x)\left[A(x) \rightarrow R(x)\right]$$

In this Quantifier x is called universal quantifier.

Existential Quantifier :-

If any statement is true for some of the values in that domain in this case those values are called existential Quantifier.

$\exists \rightarrow$ there Exists

$\forall \rightarrow$ For all

Consider the statement

Some men are clever.

this can be written as

$(\exists x)$ such that x is man and x is clever

Denote $M(x)$: x is man

$C(x)$: x is clever

Now, Given statement can be written as

$$(\exists x)\left[M(x) \wedge C(x)\right]$$

In this Quantifier x is called Existential Quantifier.

* show that the following conditional statements are tautology using truth tables.

1. $[\neg p \wedge (p \vee q)] \rightarrow q$

2. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

3. $[p \wedge (p \rightarrow q)] \rightarrow q$

1)

p	q	$\neg p$	$(p \vee q)$	$\neg p \wedge (p \vee q)$	$\neg p \wedge (p \vee q) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

this statement is true in all cases

∴ Given statement is a tautology.

∴ $\neg p \wedge (p \vee q) \rightarrow q$.

~~2)~~ $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	$p \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

his statement is true in all cases

Given statement is a tautology

$\therefore [(p \to q) \wedge (q \to r)] \to (p \to r)$

3. $[p \wedge [p \to q)] \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Given statement is true in all cases

this statement is a tautology

$\therefore [p \wedge (p \to q)] \to q$

4. $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

p	q	r	$p \vee q$ (a)	$p \to r$ (b)	$q \to r$ (c)	$a \wedge b \wedge c$	$(a \wedge b \wedge c) \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Given statement is true in all cases

given statement is tautology

$\therefore [(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

Logic Rules :-

1. $p \to q \equiv \neg p \vee q$ (conditional equivalence)
2. $p \to q, q \to r \Rightarrow p \to r$ (chain Rule)
3. $p \to q, p \Rightarrow q$ (Detachment)
4. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow pq + p'q'$

Prove that the following statements are tautology without using truth table.

1. $[\neg p \wedge (p \vee q)] \to q$

sol: $[(\neg p) \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$

$\Leftrightarrow [F \vee (\neg p \wedge q)] \rightarrow q \quad (x \wedge \neg x \equiv F)$

$\Leftrightarrow (\neg p \wedge q) \rightarrow q \quad (x \vee F \equiv x)$

$\Leftrightarrow \neg(\neg p \wedge q) \vee q \quad (\because x \rightarrow y \equiv \neg x \vee y)$

$\Leftrightarrow (\neg(\neg p) \vee \neg q) \vee q$

$\Leftrightarrow p \vee \neg q \vee q \quad (\because x \vee \neg x \Leftrightarrow T)$

$\Leftrightarrow p \vee T$

$\Leftrightarrow T \quad (\because x \vee T \equiv T)$

Mid

2. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

$\Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

$\Leftrightarrow [(\neg p \vee q) \wedge (\neg q \vee r)] \rightarrow (\neg p \vee r) \quad (\because x \rightarrow y \equiv \neg x \vee y)$

$\Leftrightarrow \neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r)$

$\Leftrightarrow [\neg(\neg p \vee q) \vee \neg(\neg q \vee r)] \vee (\neg p \vee r)$

$\Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg r)] \vee (\neg p \vee r)$

$\Leftrightarrow \{[(p \wedge \neg q) \vee q] \wedge [(p \wedge \neg q) \vee \neg r]\} \vee (\neg p \vee r)$

$\Leftrightarrow [(p \vee q) \wedge (\neg q \vee q)] \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)] \vee (\neg p \vee r)$

$\Leftrightarrow [(p \vee q) \wedge T \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)] \vee (\neg p \vee r)$

$\Leftrightarrow [(p \vee q) \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)] \vee (\neg p \vee r) \quad \because (x \wedge T = x)$

$\Leftrightarrow (p \vee q \vee \neg p \vee r) \wedge (p \vee \neg r \vee \neg p \vee r) \wedge (\neg q \vee \neg r \vee \neg p \vee r)$

$\Leftrightarrow T \wedge T \wedge T \quad (\because x \vee \neg x \equiv T, \; x \vee T \equiv T)$

$\Leftrightarrow T$

3. $[p \wedge (p \to q)] \to q$

Sol: $[p \wedge (p \to q)] \to q$

$\Leftrightarrow [p \wedge (\neg p \vee q)] \to q \quad (x \to y \Leftrightarrow \neg x \vee y)$

$\Leftrightarrow [(p \wedge \neg p) \vee (p \wedge q)] \to q$

$\Leftrightarrow [F \vee (p \wedge q)] \to q$

$\Leftrightarrow (p \wedge q) \to q \quad (\because F \vee x \equiv x$

$\Leftrightarrow \neg (p \wedge q) \vee q \quad (\because x \to y \Leftrightarrow \neg x \vee y)$

$\Leftrightarrow \neg p \vee \neg q \vee q$

$\Leftrightarrow \neg p \vee T$

$\Leftrightarrow T$

Rules

Rules of Inference :-

1. $\dfrac{\forall x\, P(x)}{\therefore P(c)}$	Universal Instantiation
2. $P(c)$: c any arbitary c $\therefore \forall x\, P(x)$	Universal Generalization
3. $\dfrac{\exists x\, P(x)}{\therefore P(c)}$ for some c	Existential Instantiation
4. $\dfrac{P(c)}{\therefore \exists x\, P(x)}$ for some element c	Existential Generalization.

Q: show that the premises "everyone in this discrete Mathematical class has taken a course in computer science" and "Maala is a student in this class". implies the conclusion "Maala has taken a course in computer science".

Ans :- $D(x)$: x is in this Discrete mathematics class

$C(x)$: x has taken a course in computer science

Now, given statement can be written as

1) For all x, if x is in D.M class
Then x has taken a course in computer science class.

2) D(Mazala) ~~A C(Mazala)~~

Conclusion :

Mazala has taken a computer science class

Mathematical form of above premisis is

1. $\forall x\,(D(x) \rightarrow C(x))$
2. D(Mazala)

$\therefore$ C(Mazala)

Step		Reason
1. $\forall x\,(D(x) \rightarrow C(x))$	}	Premises
2. D(Mazala		
3. D(Mazala) $\rightarrow$ C(mazala)		Universal Instantiation from ①
4. C(mazala)		from ② & ③ True

Prove that the following argument is valid

$\forall x\,(p(x) \rightarrow q(x))$
$\forall x\,(q(x) \rightarrow r(x))$

$\therefore \forall x\,(p(x) \rightarrow r(x))$

Sol:- St

1. $\forall x$ (
2. $\forall x$
3. P(
4. q(
5. P(
6. $\forall x$

Q) Show tha
not read
passed th
Implie

" some a
read the

Sol:- Let
C(x) :
B(x)
P(x)
$\therefore$ Given

$\exists$
$\forall$

$\therefore$

Sol:- Step | Reason

1. $\forall x\,(p(x) \rightarrow q(x))$ } Premises
2. $\forall x\,(q(x) \rightarrow r(x))$ }
3. $p(a) \rightarrow q(a)$ Universal Instantation from ①
4. $q(a) \rightarrow r(a)$ Universal instantation from ②
5. $p(a) \rightarrow r(a)$ from ③ & ④
6. $\forall x\,(p(x) \rightarrow r(x))$ Universal generalization from ⑤

Q) Show that the premises "A student in this class has not read the book" and "every one in this class passed the first exam" Implies the conclusion "some are who passes the first Exam has not read the Book".

Sol: Let

$C(x)$: x is in this class

$B(x)$: x has read the book

$P(x)$: x passed the first exam

∴ Given Premises are

$$\exists x\,(C(x) \wedge \neg B(x))$$
$$\forall x\,(C(x) \rightarrow P(x))$$
$$\therefore \exists x\,(P(x) \wedge \neg B(x))$$

Step	Reason.
1. $\exists x(C(x) \wedge \neg B(x))$	premises
2. $\forall x(C(x) \rightarrow P(x))$	
3. $C(a) \wedge \neg B(a)$	Existential ~~universal~~ instantation from ①
4. $C(a) \rightarrow P(a)$	universal instantation from ②
5. $C(a)$	from ③
6. $\neg B(a)$	
7. $P(a)$	from ④ ⑤
8. $P(a) \wedge \neg B(a)$	⑥ & ⑦ conjunction
9. $(\exists x)[P(x) \wedge \neg B(x)]$	Existential Generalization.

SETS

1. Union :- $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

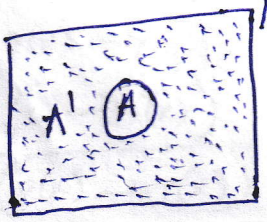
2. Intersection :- $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
(and)

3. Universal set :- Every set is subset of universal set ('μ')

4. complement :- $x \in A' \Leftrightarrow x \in \mu - A$

*5. $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

6.



$A \cap A' = \phi$

$A \cup A' = \mu$

$\mu \cap A = A$

$\mu \cup A = \mu$

7. $A - B = A \cap B'$

(2)

* Prove that the following

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$ } [commutative law]

Sol:- a. $A \cup B = B \cup A$

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow x \in B$ or $x \in A$

$\Leftrightarrow x \in B \cup A$

$\therefore A \cup B = B \cup A$

b. $A \cap B = B \cap A$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow x \in B$ and $x \in A$

$\Leftrightarrow x \in B \cap A$

$\therefore A \cap B = B \cap A$

2. a. $A \cup (B \cup C) = (A \cup B) \cup C$,
b. $A \cap (B \cap C) = (A \cap B) \cap C$ } Associative Law

a. $A \cup (B \cup C) = (A \cup B) \cup C$

$x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C$

$\Leftrightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Leftrightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Leftrightarrow x \in (A \cup B) \text{ or } x \in C$

$\Leftrightarrow x \in (A \cup B) \cup C$

$\therefore A \cup (B \cup C) = (A \cup B) \cup C$

b. $A \cap (B \cap C) = (A \cap B) \cap C$

$x \in A \cap (\underline{B \cap C}) \Leftrightarrow x \in A \text{ and } x \in B \cap C$

$\Leftrightarrow x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Leftrightarrow x \in (A \cap B) \text{ and } x \in C$

$\Leftrightarrow x \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) = (A \cap B) \cap C.$

mid

3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ } Distributive law

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

sol: a. $x \in A \cap (\underline{B \cup C}) \Leftrightarrow x \in A \text{ and } x \in B \cup C$

$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Leftrightarrow x \in A \cap B \text{ or } x \in A \cap C$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B \cap C$

$\Leftrightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

* show that $A - (A \cap B) = A - B$

Note: $(A - B = A \cap B')$

$A - (A \cap B) = A \cap (A \cap B)'$ $\boxed{x - y = x \cap y'}$

$= A \cap (A' \cup B')$

$= (A \cap A') \cup (A \cap B')$

$= \phi \cup (A \cap B')$

$= A \cap B'$

$A - (A \cap B) = A - B$

* show that $A - (B \cup C) = (A - B) \cap (A - C)$

$L.H.S = A - (B \cup C)$

$= A \cap (B \cup C)'$

$= A \cap (B' \cap C')$

$= (A \cap B') \cap (A \cap C')$

$= (A - B) \cap (A - C)$

$\therefore LHS = RHS$

* Show that $A-(B\cap C) = (A-B)\cup(A-C)$

$$LHS = A - B\cap C$$
$$= A\cap(B\cap C)'$$
$$= A\cap(B'\cup C')$$
$$= (A\cap B')\cup(A\cap C')$$
$$= (A-B)\cup(A-C)$$

L.H.S = R.H.S.

Relations :-

If $A=\{1,2,3\}$, $B=\{x,y\}$ then

$$A\times B = \{(1,x)(1,y),(2,x)(2,y),(3,x)(3,y)\}$$

then,

$$R_1 = \{(1,x),(2,x)\}$$
$$R_2 = \{(1,x),(3,y),(2,y)\}$$
$$R_3 = \{(1,x),(3,x),(3,y)\}$$

this are subset of $A\times B$.

Definition :-

If A, B are any two sets then any subset of $A\times B$ is called Relation from A to B.

→ If A is any non empty set then any subset of $A\times A$ is called relation on A.

Ex :- $A=\{1,2,3\}$

Now $A\times A = \{(1,1)(1,2)(1,3),(2,1)(2,2)(2,3),(3,1)(3,2)(3,3)\}$

Let,

$$\left.\begin{aligned} R_1 &= \{(1,1)\ (2,2)\ (3,3),\ (1,3)\} \\ R_2 &= \{(1,1)\ (2,2)\ (3,3)\ (1,3),\ (3,1)\} \\ R_3 &= \{(1,1)\ (2,2)\ (1,3),\ (3,1)\} \end{aligned}\right\} \subseteq A \times A$$

Now R_1, R_2, R_3 are relations on A.

Types of Relations:-

1. Reflexive Relation:-

'A' is any non empty set then a relation R on A is said to be Reflexive.

If $(a,a) \in R \ \forall a \in A$

Note:- In the above example R_1, R_2 are reflexive but not R_3. since, $(3,3)$ doesnot belongs to R_3.

2. Symmetric Relation:-

A relation R on a ~~exp~~ non empty set 'A' is said to be symmetric. If $(a,b) \in R \Rightarrow (b,a) \in R$

(or) $aRb \Rightarrow bRa$

Note:- In the above example R_1 is not symmetric since, $(1,3) \in R_1$, but ~~not~~ $(3,1) \notin R_1$.

R_2, R_3 are symmetric.

3. Transitive Relation:-

A relation R on a ~~set~~ 'A' is said to be transitive,

$(a,b)\ (b,c) \in R \Rightarrow (a,c) \in R$

Equivalence Relation :-

A relation R on a set A which is reflexive, symmetric and transitive is called an Equivalence relation.

Q. $A = \{1, 2, 3, 4\}$

R is relation of A $\ni$

$R = \{(a,b) / a, b \in A, a < b\}$

then write R as

1, Order pairs Notation

2, Matrix of R

3, Directed graph of R

4, Examine R is reflexive, symmetric, transitive (or) not.

Ans:-

Given

$A = \{1, 2, 3, 4\}$

$R = \{(a,b) / a, b \in A, a < b\}$

1, $= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

This is ordered pair Notation of 'R'.

4, R is not Reflexive since, $(1,1) \notin R$

R is not symmetric, since $(1,2) \in R$ But $(2,1) \notin R$.

R is transitive :-

Let $(a,b), (b,c) \in R$

$\Rightarrow a<b \;,\; b<c$

$\Rightarrow a<c$

$\Rightarrow (a,c) \in R$.

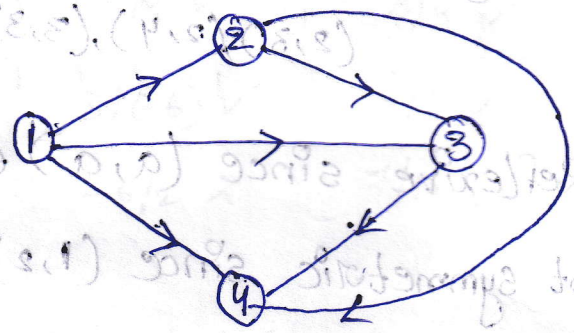
$\because$ R is transitive

This is not an equivalence Relation

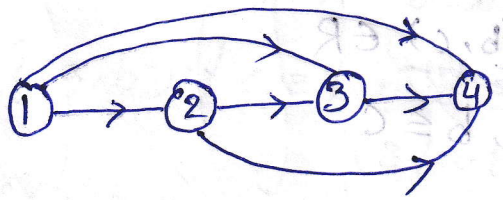
2. <u>Matrix of R</u> :-

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3. <u>Directed graph Representation of R</u> :-



(OR)



2. $A = \{1,2,3,4\}$

R is relation of A $\ni$

$R = \{(a,b) / a,b \in A, a \le b\}$

then, write R as

1) order pairs Notation

2) Matrix of R

3) Directed graph of R

4) Examine R is reflexive, symmetric, transitive (or) not.

Sol:- $A = \{1,2,3,4\}$

1) Given $R = \{(a,b) / a,b \in A, a \le b\}$

$= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),$
$(2,3),(2,4),(3,3),(3,4),(4,4)\}$

4) R is Reflexive since $(a,a) \in R \ \forall a \in A$

R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$

R is transitive :-

Let $(a,b),(b,c) \in R$

$\Rightarrow a \le b, b \le c$

$\Rightarrow a \le c$

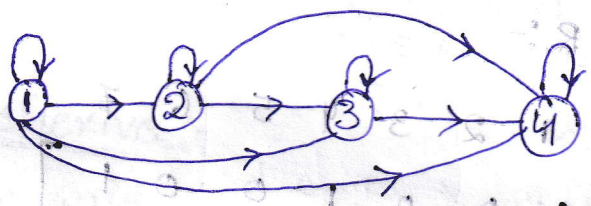
$\Rightarrow (a,c) \in R$

→ This is not an equivalence relation

2. Matrix of R :-

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

3. Directed graph of R :-



Note :-

a 'DIVIDES' b $\Longleftrightarrow a/b$

$\Longleftrightarrow b/a$ = Integer

$2/0$ ✓ $3/9$ ✓ $3/3$ ✓

$4/2$ ✗ $3/(-6)$ ✓ $6/7$ ✗

$2/4$ ✓ $-6/2$ ✗ $7/14$ ✓

Mid

Problem :-

$A = \{1, 2, 3, 4, 5, 6, 7\}$

R is a relation on A such that

$R = \{(a, b) / a \equiv b \bmod 3\}$

$(3 / a-b)\}$

Sol :- $A = \{1, 2, 3, 4, 5, 6, 7\}$

$a \equiv b \bmod m$
$\Longleftrightarrow m / (a-b)$

i) $R = \{(a,b) \,/\, 3 \mid (a-b)\} = \{(a,b) \,/\, a \equiv b \bmod 3\}$

$= \{(1,1)\ (1,4)\ (1,7)\ (2,2)\ (2,5)$
$(3,3)\ (3,6)\ (4,1)\ (4,4)\ (4,7)$
$(5,2)\ (5,5)\ (6,3)\ (6,6)\ (7,1)$
$(7,4)\ (7,7)\}$

$\boxed{\frac{a-b}{3} = \text{integer}}$

$\frac{1-1}{3} = 0$

$\frac{1-4}{3} = -1$

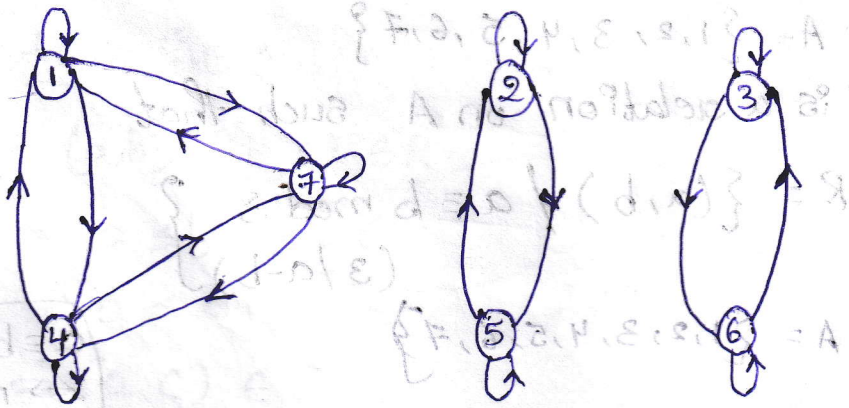
$\frac{1-7}{3} = -6 = -2$

(1,1) (1,4) (1,7)
(2,2) (2,5)
(3,3) (3,6) (3,
(4,4) (4,
(5,5)
(6,6)
(7,7)

ii) Matrix of R :-

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

iii) Graph of R :-



Imp ** mid

Problem :-

R is a Relation on set of Real numbers such that

$aRb \Leftrightarrow a-b$ is an integer $a, b \in \mathbb{R}$

Is it equivalence relation

Sol: Let $\mathbb{R}$ be the set of real numbers

Given that R is a relation $\mathbb{R}$

Such that

$aRb \Leftrightarrow a-b$ is an integer

where $a, b \in \mathbb{R}$

i) R is Reflexive :-

clearly

$aRa \ \forall a \in R$.

since $a-a=0$ is an integer $\forall a \in \mathbb{R}$

$\therefore$ R is Reflexive.

ii) R is symmetric :-

Let aRb

$\Rightarrow a-b$ is an integer

$\Rightarrow -(a-b)$ is an integer

$\Rightarrow b-a$ is an integer

$\Rightarrow bRa$

$\therefore$ R is symmetric

iii) R is transitive :-

Let aRb, bRc

$\Rightarrow a-b, b-c$ are integers

$\Rightarrow (a-b)+(b-c)$ is integers

$\Rightarrow a-c$ is an integer

$\Rightarrow aRc$

$\therefore$ R is transitive

$\therefore$ R is Reflexive, symmetric and transitive

$\therefore$ R is an Equivalence Relation.

Note :-

Congruence Relation :-

Congruence Relation on set of integers ($\mathbb{Z}$)

$m \in \mathbb{Z}, m > 1, a, b \in \mathbb{Z}$

$a \equiv b \mod m \Leftrightarrow m / a-b$

Problem :-

m is an integer, $m > 1$ then the congruence Relation $a \equiv b \mod m$ is an equivalence Relation on set of integers.

Sol :- Let, $\mathbb{Z}$ be the set of all integers and $m \in \mathbb{Z}, m > 1$

Congruence Relation is

$a \equiv b \pmod m \Leftrightarrow m / a-b$ where $a, b \in \mathbb{Z}$

i) $\equiv$ is Reflexive :-

Clearly $m / a-a = 0 \ \forall a \in \mathbb{Z}$

$\Rightarrow m / a-a \ \forall a \in \mathbb{Z}$

$\therefore a \equiv a \pmod m \ \forall a \in \mathbb{Z}$

$\therefore$ '$\equiv$' is Reflexive

ii) '$\equiv$' is symmetric :-

$\boxed{a/b = a/-b}$

Let $a \equiv b \pmod m \Rightarrow m/a-b$

$\Rightarrow m/-(a-b)$

$\Rightarrow m/b-a$

$\Rightarrow b \equiv a \pmod m$

$\therefore$ '$\equiv$' is symmetric

iii) '$\equiv$' is transitive :-

Let $a \equiv b \pmod m$, $b \equiv c \pmod m$

$\Rightarrow m/a-b$, $m/b-c$

$\Rightarrow m/a-\cancel{b}+\cancel{b}-c$

$\boxed{a/b,\ a/c \Rightarrow a/(b \pm c)}$

$\Rightarrow m/a-c$

$\Rightarrow a \equiv c \pmod m$

$\therefore$ '$\equiv$' is transitive

$\therefore$ '$\equiv$' is an equivalence Relation on Z

Note :-

The set of all strings formed by English letters is

$A = \{\Lambda, a, b, \ldots z, ab, aa, aaa, bbaa, az, \ldots\}$

$\Lambda \rightarrow$ Null string

If $x \in A$ then length of x is denoted by $l(x)$

Ex: $l(aabz) = 5$, $l(\Lambda) = 0$

Q. R be the relation on set of strings of English Letters search that $aRb \Leftrightarrow l(a) = l(b)$ where $l(x)$ is length of string (x).

sol: Let A be the set of strings formed by English Letters

Given R is a relation on A $\ni$

$aRb \Leftrightarrow l(a) = l(b)$

i) R is Reflexive :-

clearly $l(x) = l(x)$ $\forall x \in A$

$\Rightarrow xRx$ $\forall x \in A$

$\therefore$ R is Reflexive on A.

ii) R is symmetric :-

Let $x, y \in A \ni$

$xRy \Rightarrow l(x) = l(y)$

$\Rightarrow l(y) = l(x)$

$\Rightarrow yRx$

$\therefore$ R is symmetric

iii) R is transitive :-

Let xRy, yRz

$\Rightarrow l(x) = l(y)$, $l(y) = l(z)$,

$\Rightarrow l(x) = l(y) = l(z)$

$\Rightarrow l(x) = l(z)$

$\Rightarrow xRz$

$\therefore$ R is transitive.

$\therefore$ R is an equivalence Relation on A.

Anti symmetric Relation :-

A relation R on a set A is said to be Anti symmetric if $aRb, bRa \Rightarrow a=b$

Partially ordered relation :-

A Relation R on a non empty set is said to be a partially ordered relation if R is Reflexive Anti symmetric and Transitive.

Note :-

N = Natural numbers $= \{1, 2, 3, \ldots\}$

Z = Integers $= \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$

Note :-

$a, b \in N$

$a/b, b/a \Rightarrow a=b$.

Problem :-

R is a Relation on set of Natural numbers N $\ni$

$aRb \Leftrightarrow a/b$ Then show that R is partially ordered relation on N. i.e. $(N, /)$ is a partially ordered set (PO set)

sol:- Given R be the relation on N $\ni$ $aRb \Leftrightarrow a/b$ where $a, b \in N$

i) R is Reflexive :-

clearly $a/a \quad \forall a \in N$

$\Rightarrow aRa \quad \forall a \in N$

$\therefore$ R is Reflexive

ii) R is Anti Symmetric :-

Let aRb, $bRa \Rightarrow a/b, b/a$

$\Rightarrow a = b$

$\therefore$ R is Anti symmetric

iii) R is Transitive :-

Let $aRb, bRc \Rightarrow a/b, b/c$

$\Rightarrow a/c$

$\Rightarrow aRc$

$\therefore$ R is Reflexive, Anti symmetric and transitive

$\therefore$ R is partially ordered Relation on N.

$(N, |)$ is a po set.

D_{30} (Divisors of 30) $(D_{30}, |)$:-

Power set :-

If 'A' is any non-empty set, then the set of all possible subsets of A is called power set of A.

The power set of A is denoted by '$P(A)$'.

Ex:-

1, if $A = \{a, b\}$ then

$$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

2, If $A = \{1, 2, 3\}$ then

$$P(A) = \{\phi \ \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{2,3\}\}$$

Q, If X is any non-empty set and $\leq$ is a relation on $P(x)$ such that

$A \leq B \Leftrightarrow A \subseteq B$ then $(P(x), \leq)$ poset

Sol:- Given that $\leq$ on $P(x) \ni$

$$A \leq B \Leftrightarrow A \subseteq B$$

i, Reflexive:- ($\leq$ is reflexive)

we know that $A \subseteq A \quad \forall A \in P(x)$

$\Rightarrow A \leq A \quad \forall A \in P(x)$

$\therefore$ '$\leq$' is reflexive.

ii, $\leq$ is anti-symmetric:-

Let $A \leq B$, $B \leq A$

$\Rightarrow A \subseteq B$, $B \subseteq A$

$\Rightarrow A = B$

$\therefore \leq$ is anti-symmetric.

(iii) $\leq$ is Transitive :-

Let $A \leq B$, $B \leq C$

$\Rightarrow A \subseteq B$, $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow A \leq C$

$\therefore$ '$\leq$' is transitive

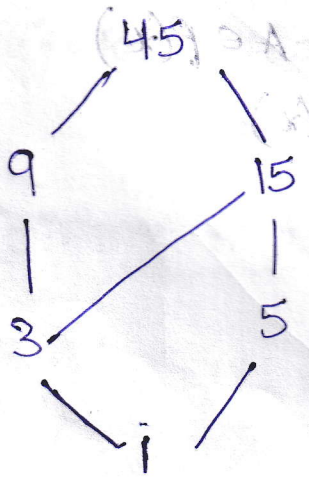
$\therefore$ '$\leq$' is partially ordered relation of $P(x)$

$\therefore (P(x), \leq)$ poset

Hasse Diagrams for posets :-

1. Draw the hasse diagram for $(D_{45}, |)$ where $(D_{45}, |)$ set of all divisors 45

Sol:- $D_{45} = \{1, 3, 5, 9, 15, 45\}$



$\boxed{1}$	$\boxed{3,5}$
$1 \mid \times \mid 3$	$3 \mid \times \mid 9$
$1 \mid \overset{3}{\times} \mid 5$	$3 \mid \overset{9}{\times} \mid 15$
$1 \mid 3 \mid 9$	$3 \mid 9 \mid 45$
$1 \mid 3 \mid 15$	$5 \mid \times \mid 15$
$1 \mid 5 \mid 45$	$5 \mid 15 \mid 45$

2. Draw the Hasse diagram for partial ordering $\{(A, B) / A \subseteq B\}$ on the Power set $P(S)$ where $S = \{a, b, c\}$

Sol: Given $S = \{a, b, c\}$

Power set of S is

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\} \{a,c\} \{b,c\} \{a,b,c\}\}$

Given relation on $P(S)$ such that

$\{(A,B) / A \subseteq B\}$

clearly $\subseteq$ is partially ordered relation on $P(S)$

since ① $A \subseteq A \ \forall \ A \in P(S)$

$\therefore \subseteq$ is Reflexive

2. $A \subseteq B, B \subseteq A \Rightarrow A = B$

$\therefore \subseteq$ is Antisymmetric

3. $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

$\therefore \subseteq$ is transitive

$\therefore (P(S), \subseteq)$ is Poset.

Hasse Diagram for this poset:-

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\} \{a,c\} \{b,c\} \{a,b,c\}\}$

①

$\emptyset \subseteq x \subseteq \{a\}$

$\emptyset \subseteq x \subseteq \{b\}$

$\emptyset \subseteq x \subseteq \{c\}$

$\emptyset \subseteq \{a\} \subseteq \{a,b\}$

$\emptyset \subseteq \{a\} \subseteq \{a,c\}$

$\emptyset \subseteq \{b\} \subseteq \{b,c\}$

$\emptyset \subseteq \{a\} \subseteq \{a,b,c\}$

②

$\{a\} \subseteq x \subseteq \{a,b\}$

$\{a\} \subseteq x \subseteq \{a,c\}$

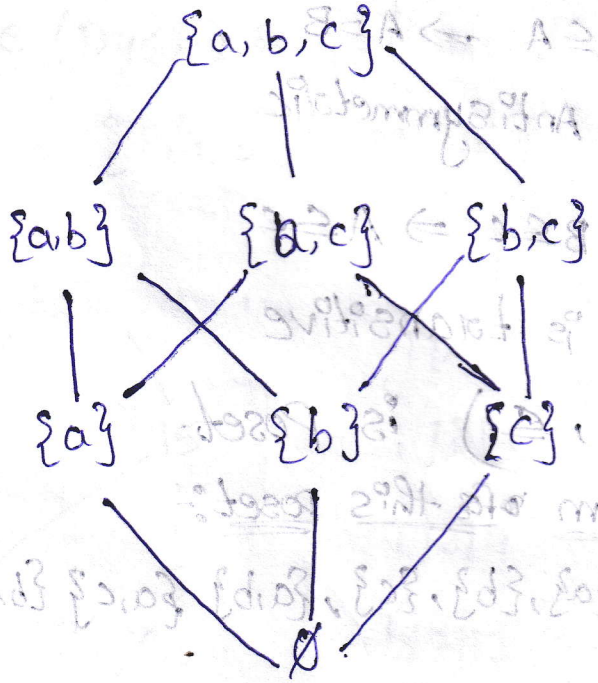
$\{a\} \subseteq \{a,c\} \subseteq \{a,b,c\}$

$\{b\} \subseteq x \subseteq \{a,b\}$

$\{b\} \subseteq x \subseteq \{b,c\}$

$\{c\} \subseteq x \subseteq \{a,c\}$

$\{c\} \subseteq x \subseteq \{b,c\}$



Problem: Hasse diagram for $(P(S), \subseteq)$ where $S = \{a,b\}$

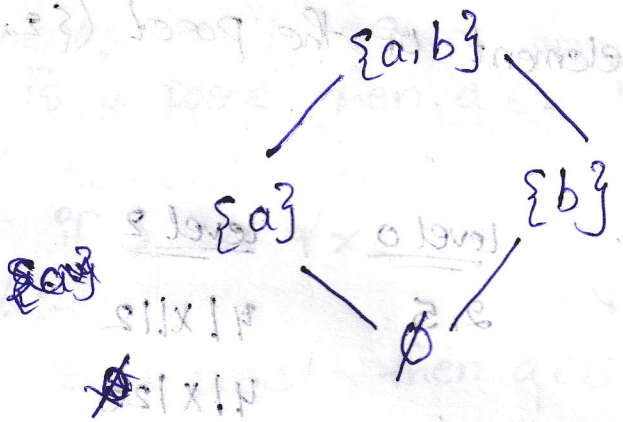
$P(S) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

①

$\{a\} \subseteq x \subseteq \{a,b\}$ $\quad \emptyset \subseteq x \subseteq \{a\}$

$\{b\} \subseteq x \subseteq \{a,b\}$ $\quad \emptyset \subseteq x \subseteq \{b\}$

$\emptyset \subseteq \{a\} \subseteq \{a,b\}$



Mid

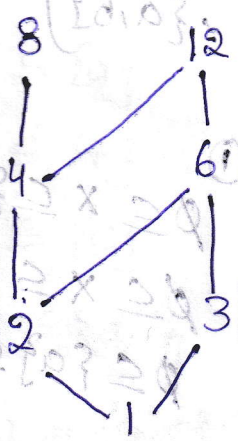
2) Draw the Hasse diagram Representing the partial ordering $\{(a,b) / a$ devides $b\}$ on $\{1,2,3,4,5,6,8,12\}$

Sol: Given partially ordered relation on S is $\{(a,b) / a$ Divides $b\}$

Let $S = \{1,2,3,4,5,6,8,12\}$

Hasse diagram for this poset is

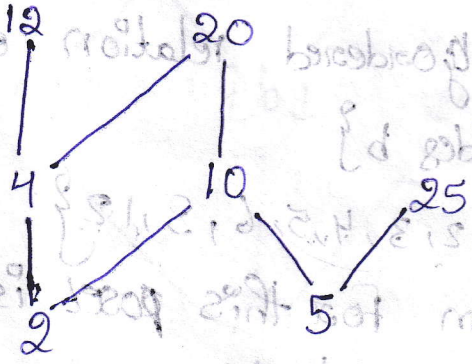
Level 1	Level-2	Level 3
1|x|2	2|x|4	4|x|8
1|x|3	2|x|6	4|x|12
1|2|4	2|4|8	6|x|12
1|3|6	2|4|12	
1|4|8	3|x|6	
1|4|12	3|6|12	



which element of the poset ({2, 4, 5, 10, 12, 20, 25}

1)

Level 1	Level 0	Level 2	Level 3
2 | x | 4	2, 5	4 | x | 12	
~~2 | x | 8~~		4 | x | 20	
2 | x | 10		10 | x | 20	
2 | 4 | 12			
2 | 10 | 20			



Maximal elements are in this poset are 12, 20, 25

Minimal elements are 2, 5

Maximal Element :-

An element 'a' in a poset $(S, \leq)$ is said to be maximal element in S.

If there is no $b \in S \ni a < b$

Minimal Element :-

$(S, \leq)$ is a poset then $a \in S$ is said to be minimal element in S

If there is no $b \in S \ni b < a$

20, 25 }. Greatest Element :-

$(S, \leq)$ is a poset then $a \in S$ is said to be Greatest element in S

if $x \leq a \ \forall x \in S$.

Least Element :-

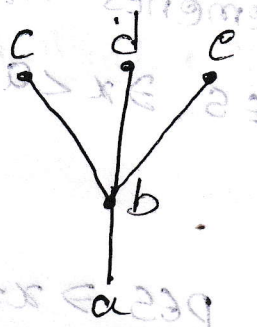
$(S, \leq)$ is a poset then $a \in S$ is said to be least element in S

if $a \leq x \ \forall x \in S$.

Problem :- Mid.

Find maximal, minimal, Greatest & least elements in the following poset

1.



Sol:- $\because$ c is maximal element in given poset $(S, \leq)$ since there is no $x \in S \ni c < x$

Simillarly d, e are also maximal element in S.

$\therefore$ Maximal elements in $(S, \leq)$ are $\boxed{c, d, e}$

ii) $a \in S$ is only minimal element in S since, there is no $x \in S \ni x < a$.

Greatest element :-

iii) There is no element $p \in S \ni x \leq p \ \forall x \in S$

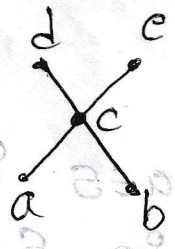
$\therefore$ S has no greatest element.

iv) Least element :-

Clearly $a \leq x \ \forall x \in S$

$\therefore$ 'a' is only least element in S

2.



i) d, e is maximal elements in given poset $(S, \leq)$. since, there is no $x \in S \ni d, e < x$.

$\therefore$ Maximal elements in $(S, \leq)$ are $\boxed{d, e}$

ii) $a, b \in S$ are minimal elements in S since, there is no $x \in S \ni x < a, b$

iii) Greatest element :-

There is no element $p \in S \ni x \leq p \ \forall x \in S$

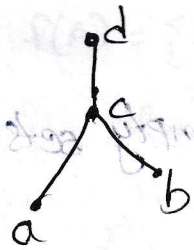
$\therefore$ S has no greatest element

iv) Least element :-

There is no element $p \in S \ni p \leq x \ \forall x \in S$

$\therefore$ There is no least element in S.

3.



Function :-

If A and B are any two Non-empty sets then

$$A \times B = \left\{ (a,b) \,/\, \begin{matrix} a \in A \\ b \in B \end{matrix} \right\}$$

then any subset $A \times B$ is called relation from A to B.

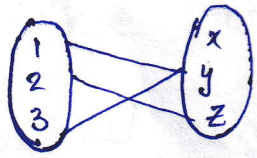
A relation $f : A \rightarrow B$ is said to be function if every element in A is Associated with unique element in B.

Example :-

If $A = \{1, 2, 3\}$

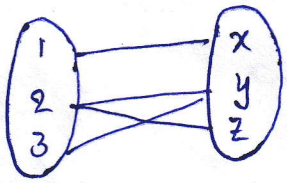
$B = \{x, y, z\}$ then

$f_1 : A \longrightarrow B$



f_1 is a function

$f_2 : A \longrightarrow B$



f_2 is not a function.

Domain :- Codomain , Range :-

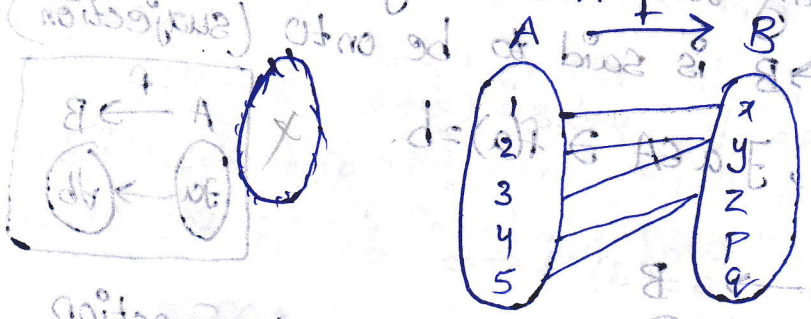
If A , B are any two non-empty sets and $f : A \rightarrow B$ is a function where

A : Domain of f

B : Codomain of f

$f(A) = \{f(x) \mid x \in A\}$ is Range of f

Consider the Function.



Domain $A = \{1, 2, 3, 4, 5\}$

Codomain $B = \{x, y, z, p, q\}$

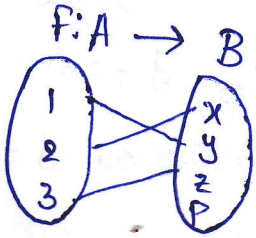
Range $f(A) = \{x, y, z\}$

One one Function :- [one to one function (or) Injection]

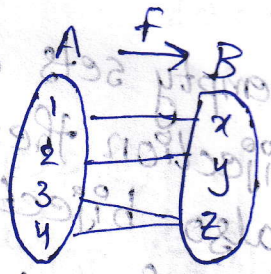
If A, B are any 2 non-empty sets then A Function $f: A \to B$ is said to be one one, if $a, b \in A$ $a \neq b \Rightarrow f(a) \neq f(b)$

(or) $f(a) = f(b) \Rightarrow a = b$

Ex:-



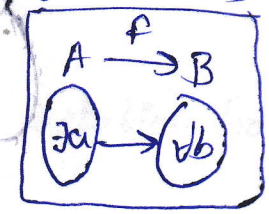
This is one one Function



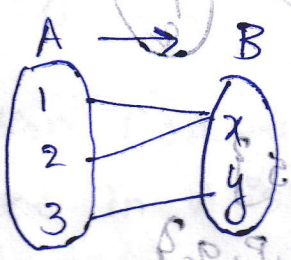
This not one one function.

Onto function :-

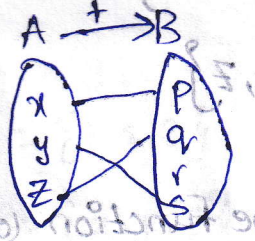
A, B are any two non-empty sets then, A function $f: A \to B$ is said to be onto (surjection) if $\forall\, b \in B$, $\exists\, a \in A \ni f(a) = b$.



Ex:-



This is onto function but not one-one.

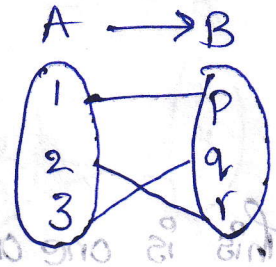


Not onto function, this is one-one.

Bijection :-

A, B are any 2 non-empty sets then, A function $f: A \to B$ is said to be bijection, if f is both one-one, onto

Ex:-



Inverse of a function :-

A, B are any 2 non-empty sets and, $f: A \to B$ is a bijection then $f^{-1}: B \to A$ is also bijection

If $a \in A$ and $f(a) = b$ then we write $f^{-1}(b) = a$.

Problem :-

1. $F: R \to R \ni f(x) = 3x+4 \ \forall x \in R$ then find F^{-1}

Sol:- Given

$F: R \to R \ni f(x) = 3x+4 \quad \forall x \in R$

Let $a \in R$

Let $f(a) = b \iff f^{-1}(b) = a \longrightarrow (1)$

$f(a) = b \Rightarrow 3a+4 = b$

$\Rightarrow 3a = b-4$

$\Rightarrow a = \frac{b-4}{3} \longrightarrow (2)$

from (1) & (2)

$f^{-1}(b) = \frac{b-4}{3}$

$f^{-1}(x) = \frac{x-4}{3}$

2. $F: R \to R \ni f(x) = 5x-3 \ \forall x \in R$ then find F^{-1}

Sol:- Given

$F: R \to R \ni f(x) = 5x-3 \ \forall x \in R$

Let $a \in R$

Let $f(a) = b \iff f^{-1}(b) = a$ — (1)

$f(a) = b \Rightarrow 5a-3 = b$

$5a = b+3$

$a = \frac{b+3}{5}$ — (2)

From (1) & (2)

$f^{-1}(b) = \frac{b+3}{5}$

$f^{-1}(x) = \frac{x+3}{5}$

Growth Function (Big Oh Notation (or) Big O Notation):-

The growth function is often describe using special Notation i.e. Big O Notation.

f, g are Functions from [the set of all integers or the set of all Real Numbers] to the set of all Real numbers.

That is $f, g : (\mathbb{Z} \text{ or } \mathbb{R}) \to \mathbb{R}$ then f is said to be $O(g(x))$.

if, $\exists$ constants $c, k \ni$

$$|f(x)| \leq c|g(x)| \quad \forall x > k$$

In this case we Read it as $f(x)$ is big oh of $g(x)$

Problem:-

1. Show that $f(x) = x^2 + x + 1$ is $O(x^2)$

Sol:- $f(x) = x^2 + x + 1$

Let $x > 1 \Rightarrow x^2 > 1$

$x > 1 \Rightarrow x^2 > x$

$2x^2 > x + 1$

$\Rightarrow 2x^2 + x^2 > x^2 + x + 1$

$\Rightarrow 3x^2 > f(x)$

$f(x) < 3x^2$

$\Rightarrow |f(x)| < 3|x^2|$

$\Rightarrow f(x) < c|g(x)|$ where $c = 3$

$g(x) = x^2$

$\forall x > K = 1$

$f(x)$ is $O(g(x))$

$\therefore f(x)$ is $O(x^2)$

Problem:-

Show that $7x^2$ is $O(x^3)$

Sol:- Let $f(x) = 7x^2$

$x > 7 \Rightarrow 7 < x$

$\Rightarrow 7x^2 < x^3$

$\Rightarrow f(x) < x^3$

$\Rightarrow |f(x)| < 1\,|x^3|$

$\Rightarrow |f(x)| < c\,|g(x)|$, where $c = 1$ and $g(x) = x^3$ and $k = 7$

$\therefore f(x)$ is $O(g(x))$

$\therefore 7x^2$ is $O(x^3)$

Big oh (O) Notation of Sum of first N natural numbers:-

$f(x)$ is $O(g(x))$ $\Leftrightarrow |f(x)| \le c\,|g(x)|$

$f(n)$ is $O(g(n))$ $\Leftrightarrow |f(n)| \le c\,|g(n)|$

Sol:- $1+2+3+\dots+n$ is sum of first n natural numbers

Let $f(n) = 1+2+3+\dots+n$

Now $f(n) = 1+2+3+\dots+n$

$< n+n+n+\dots n$ times

$= nn$

$= n^2$

$f(n) < n^2$

$1 < n$
$2 < n$
$3 < n$
$\vdots$
$n-1 < n$
$n = n$ add

$\Rightarrow f(n) < n^2$

$\Rightarrow |f(n)| < 1|n^2|$

$\Rightarrow |f(n)| < c|g(n)|$ where $g(n) = n^2$ ∴ $c = 1$

∴ $f(n)$ is $O(g(n))$

$\Rightarrow 1+2+3+\dots+n$ is $O(n^2)$

Q. Big O Notation of Factorial n that is n!

(or) prove that $n!$ is $O(n^n)$

Sol:- Let $f(n) = n!$

$= 1.2.3.4.\dots n$

$< n.n.n \dots n$ times

$= n^n$

$f(n) < n^n$

$|f(n)| < 1|n^n|$

$\Rightarrow |f(n)| < c|g(n)|$ where $c = 1$

$g(n) = n^n$

$n > 1 = k$

∴ $f(n)$ is $O(g(n))$

∴ $n!$ is $O(n^n)$

$1 < n$
$2 < n$
$3 < n$
$\vdots \quad \vdots$
$n-1 < n$
$n = n$ multiplication

$1.2.3.n < n.n.\dots n$ times $= n^n$

Note :-

→ $\log a + \log b = \log ab$

→ $\log a + \log b + \log c = \log abc$

$\rightarrow \log_x n = n \log x$

Q. Prove that Big O Notation of $\log n!$

Sol:- $f(n) = \log n!$

$= \log(1.2.3 \ldots n)$

$\log 1 < \log n$

$\log 2 < \log n$

$\vdots$

$\log(n-1) < \log n$

$\log n = \log n$

add

$\log 1 + \log 2 + \ldots + \log n$

$= n \log n$

$= \log 1 + \log 2 + \log 3 \ldots + \log n$

$< \log n + \log n + \log n + \ldots n \text{ times}$

$= n \log n$

$\therefore f(n) < n \log n$

$|f(n)| < 1 |n \log n|$

$|f(n)| < c.|g(n)|$ where $c = 1$

$g(n) = n \log n$

$n > 1 = k$

$\therefore f(n)$ is $O\, g(n)$

$\therefore \log n!$ is $O(n \log n)$

UNIT - III

Graph Theory

Graph :-

A graph $G = (V, E)$ where,

V : set of vertices

E : set of edges

each edge associated with two vertices

Directed graph :-

$G = (V, E)$ be the graph if directions are assigned to the all the edges then graph G is called "directed graph."

Undirected graph :-

$G = (V, E)$ be the graph if directions are not assigned to the all the edges then graph G is called "undirected graph."

Loop :-

If the end points of an edge are same then that edge is called "Loop."

Multiple edges :-

If more than one edge between any two vertices then those edges are called "Multiple edges".

Simple Graph :-

A graph $G = (V, E)$ is said to be simple if it has no loops & multiple edges.

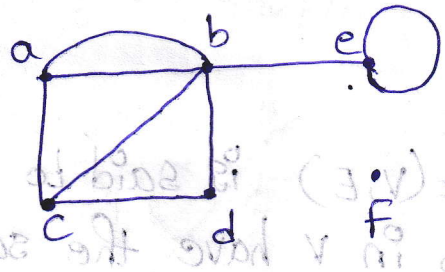
Degree of the vertex :-

A degree of the vertex in an undirected graph is the no. of edges incident to it. (the number of connections at that vertex). In case of Loop we have to count two times

Isolated vertex :-

Any vertex of degree '0' in given graph G is called "Isolated vertex"

Example :- $G = (V, E)$ is given by



$\deg(a) = 3$

$\deg(b) = 5$

$\deg(c) = 3$

$\deg(d) = 2$

$\deg(e) = 3$

$\deg(f) = 0$

where f is Isolated vertex.

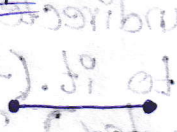
Complete graph :-

A graph $G = (V, E)$ is said to be complete graph if ① G is simple graph

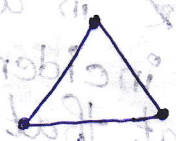
② It contains Exactly one edge b/w any two vertices (or) each pair of distinct vertices.

The complete graph of n vertices is denoted by K_n

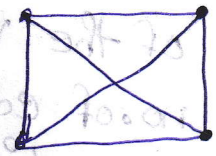
Example :-



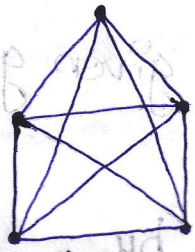
K_2



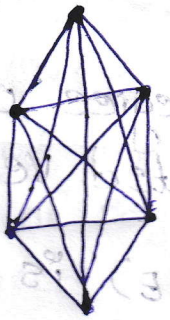
K_3



K_4



K_5

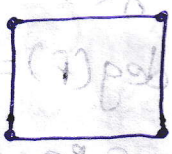


K_6

Regular graph :-

A graph $G=(V,E)$ is said to be regular all the vertices in V have the same degree

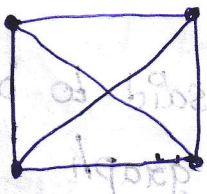
Eg:- ①



This is regular graph but not complete

2-regular

②



This is regular and complete graph.

3-regular

In degree and out degree :-

A graph $G=(V,E)$ be the directed graph then the in degree of the vertex (v) is denoted by $deg^-(v)$ and it is defined as Number of

edges with v as their terminal vertex.

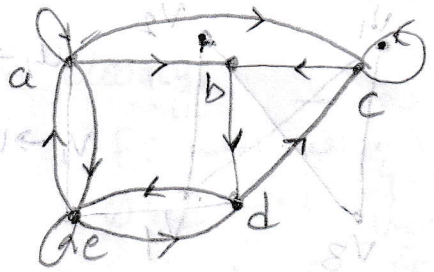
Outdegree :-

The outdegree of vertex v is denoted by $deg^+(v)$ is the number of edges with v as their initial vertex.

Note that a loop at a vertex contributes 1 to both the indegree and outdegree of the vertex

Problem :-

Find the indegree and outdegree of the every vertex in the given graph.



Sol:- Indegree of vertices

$deg^-(a) = 2$

$deg^-(b) = 2$

$deg^-(c) = 3$

$deg^-(d) = 2$

$deg^-(e) = 3$

Out degree of vertices

$deg^+(a) = 4$

$deg^+(b) = 1$

$deg^+(c) = 2$

$deg^+(d) = 2$

$deg^+(e) = 3$

Graph Isomorphism :-

Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists one to one and onto function

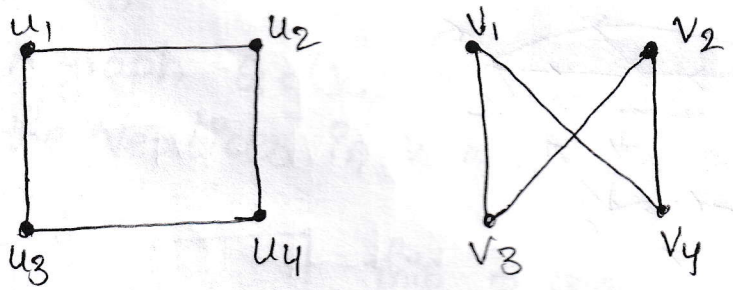
$f : V_1 \to V_2$ and $\phi : E_1 \to E_2 \Rightarrow$

$\phi(\overline{ab}) = \overline{f(a) f(b)}$ where $\overline{ab}$ is edge in E_1 and $\overline{f(a) f(b)}$ is edge in E_2.

** Easy

Problems:-

Examine the following graphs are isomorphic are not.



$G(V, E)$ $\qquad$ $H = (W, F)$

$u_1 \to u_2 \to u_4 \to u_3$

$v_1 \to v_3 \to v_2 \to v_4$

Sol:- Given Graphs are

$G = (V, E)$, $H = (W, F)$

where $V = \{u_1, u_2, u_3, u_4\}$

$E = \{\overline{u_1u_2}, \overline{u_2u_4}, \overline{u_4v_3}, \overline{u_3u_1}\}$

$W = \{v_1, v_2, v_3, v_4\}$

$F = \{\overline{v_1v_3}, \overline{v_3v_2}, \overline{v_2v_4}, \overline{v_4v_1}\}$

$|V| = |W| = 4$

$|E| = |F| = 4$

define $f: V \to W \ni$

$$f(u_1) = v_1$$
$$f(u_2) = v_3$$
$$f(u_4) = v_2$$
$$f(u_3) = v_4$$

$$\boxed{\begin{array}{l} u_1 \to u_2 \to u_4 \to u_3 \\ v_1 \to v_3 \to v_2 \to v_4 \end{array}}$$

$\phi : E \to F \ni$

$\phi(\overline{u_1u_2}) = \overline{f(u_1)f(u_2)} = \overline{v_1v_3}$ edge in F

$\phi(\overline{u_2u_4}) = \overline{f(u_2)f(u_4)} = \overline{v_3v_2}$ Edge in F

$\phi(\overline{u_4u_3}) = \overline{f(u_4)f(u_3)} = \overline{v_2v_4}$ edge in F

$\phi(\overline{u_3u_1}) = \overline{f(u_3)f(u_1)} = \overline{v_4v_1}$ edge in F

$\therefore$ clearly

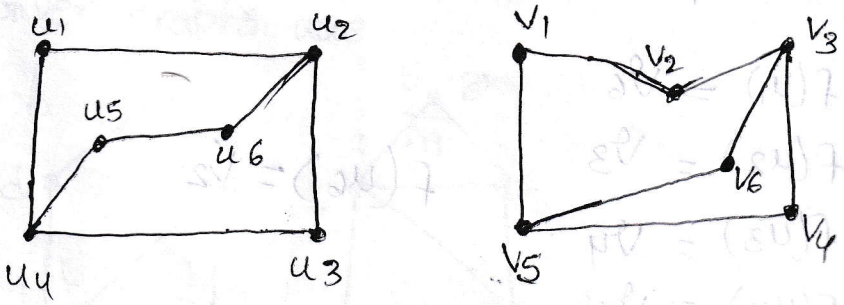
$\left.\begin{array}{l} f: V \to W \\ \phi: E \to F \end{array}\right\}$ are one one

and onto Fuction such that

$$\phi(\overline{ab}) = \overline{f(a)f(b)}$$
$$\forall \overline{ab} \text{ in } E$$

$\therefore$ Given graphs are Isomorphic

Show that the following graphs are Isomorphic



Sol:- Given Graphs are $G_1 = (V_1, E_1)$

$H = (V_2, E_2)$

where $V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$

$E_1 = \{\overline{u_1u_4}, \overline{u_4u_3}, \overline{u_3u_2}, \overline{u_2u_1}, \overline{u_2u_5}, \overline{u_6u_5}, \overline{u_5u_4}\}$

$V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E_2 = \{\overline{v_1v_2}, \overline{v_2v_3}, \overline{v_3v_4}, \overline{v_4v_5}, \overline{v_5v_6}, \overline{v_6v_3}\}$

Clearly, $|V_1| = |V_2| = 6$

$|E_1| = |E_2| = 7$

degree:	3	2	3	2	2	3	2	3
V_1:	u_4	u_3	u_2	u_6	u_5	u_4	u_1	u_2
V_2:	v_5	v_4	v_3	v_2	v_1	v_5	v_6	v_3

Define $f: V_1 \to V_2$?

$f(u_1) = v_6$

$f(u_2) = v_3$

$f(u_3) = v_4$

$f(u_4) = v_5$

$f(u_5) = v_1$

$f(u_6) = v_2$

and $\phi : E_1 \to E_2 \ni$

$\phi(\overline{u_1u_4}) = f(u_1)f(u_4) = \overline{v_6v_5}$ Edge in E_2

$\phi(\overline{u_4u_3}) = f(u_4)f(u_3) = \overline{v_5v_4}$ is edge in E_2

$\phi(\overline{u_3u_2}) = f(u_3)f(u_2) = \overline{v_4v_3}$ is edge in E_2

$\overline{u_2u_6}$

$\phi(\overline{u_2u_1}) = f(u_2)f(u_1) = \overline{v_3v_6}$ is edge in E_2

$\phi(\overline{u_2u_6}) = f(u_2)f(u_6) = \overline{v_3v_2}$ is edge in E_2

$\phi(\overline{u_6u_5}) = f(u_6)f(u_5) = \overline{v_2v_1}$ is edge in E_2

$\phi(\overline{u_5u_4}) = f(u_5)f(u_4) = \overline{v_1v_5}$ is edge in E_2

$\overline{v_5v_1}$

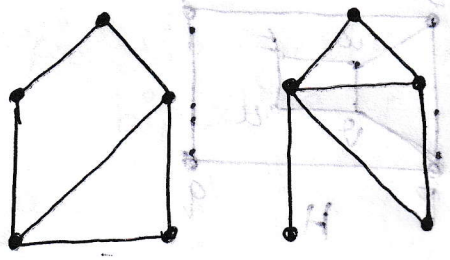
clearly $f : V_1 \to V_2$

$\phi : E_1 \to E_2$ are one to one

and onto mapping

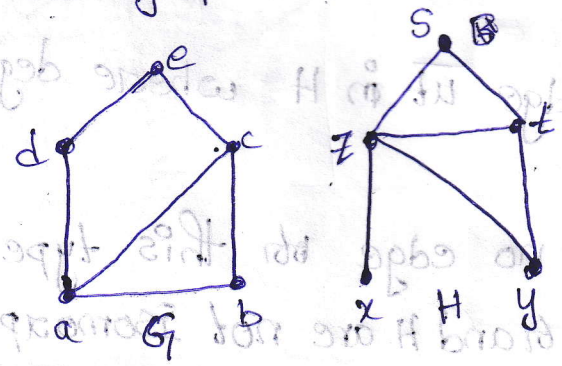
$\therefore$ Given graphs are Isomorphic.

*Examine the following graphs are Isomorphic or not.



G H

Sol:- Given graphs are

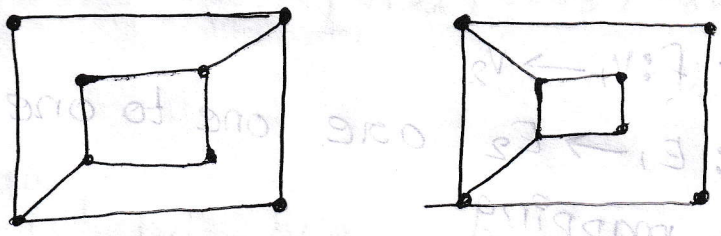


Given two graphs G and H have ~~by~~ 5 vertices & 6 edges.

However graph ~~edge~~ H as a ~~degree~~ vertex of degree 1 namely vertex x where as G has no vertex of degree 1.

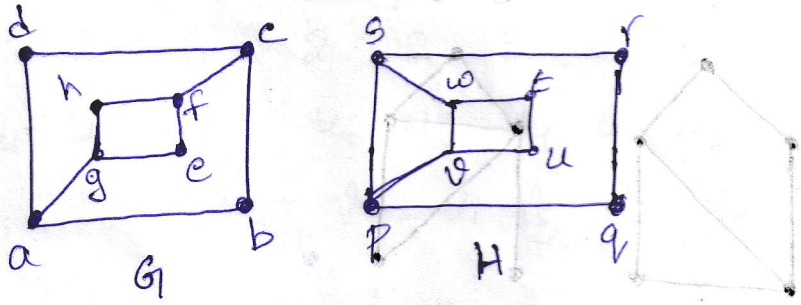
Therefore, given graphs G and H are not Isomorphic.

* Examine the following graph is isomorphic or not.



Sol:

Given graphs are



Both the graphs G & H have 8 vertices and 10 edges.

Consider the edge $\overline{ut}$ in H where $\deg(u)=2$, $\deg(t)=2$.

But there is no edge of this type in G.

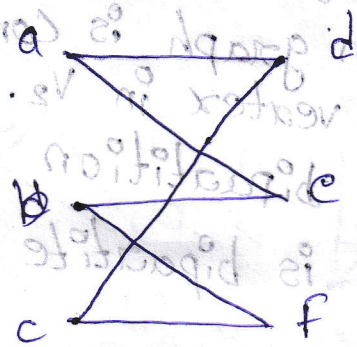
∴ Given graphs G and H are not Isomorphic.

Bipartite graph :-

A simple graph $G=\{V,E\}$, is said to b bipartite graph if 'V' is partitioned into two disjoint sets V_1 & V_2. such that every ed in the graph connects a vertex V_1 and a vert in V_2.

Where V_1, V_2 are called bipartition of V

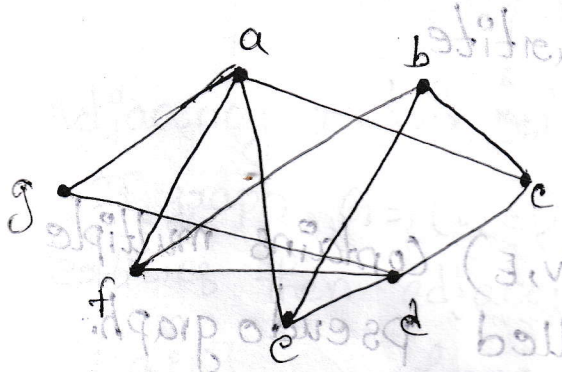
Example:-



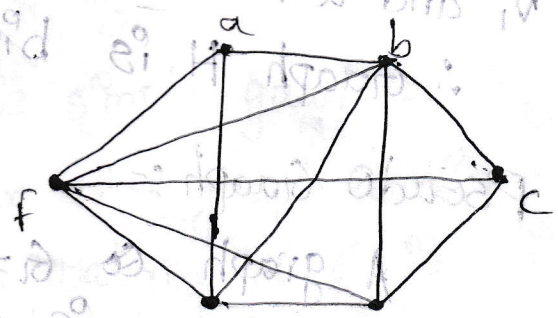
$V_1 = \{a, b, c\}$, $V_2 = \{d, e, f\}$

$\therefore$ It is bipartite graph where $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$ are Bipartition of V.

Problem:- Are the graphs G and H are bipartite.



G



H

Sol:- Graph $G=(V,E)$

$V=\{a,b,c,d,e,f,g\}$

$V_1=\{a,b,d\}$

$V_2=\{c,e,f,g\}$

we observed that V is partitioned into 2 disjoint sets V_1 & V_2 such that every edge in given graph is connects a vertex in V_1 and a vertex in V_2.

$\therefore$ V_1, V_2 are bipartition of V.

$\therefore$ Given graph is bipartite graph.

2) Consider the graph $H=(V,E)$

where $V=\{a,b,c,d,e,f,g\}$

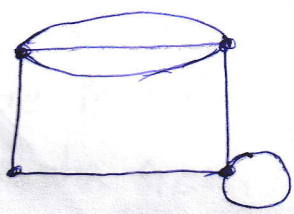
where V cannot be partition into two disjoint sets V_1 and V_2. such that every edge in graph H is connects a vertex in V_1 and a vertex in V_2.

$\therefore$ Graph H is bipartite

Pseudo Graph :-

A graph $G=(V,E)$ contains multiple edges and loops is called pseudo graph.

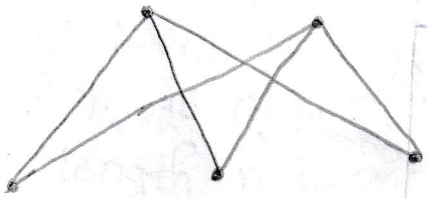
Eg:-



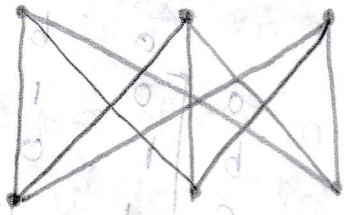
Complete bipartite graph :-

A graph $G = (V, E)$ is said to be complete bipartite if V is partitioned into two disjoint sets V_1, V_2 so that every vertex in V_1 is connect with all the vertices in V_2.

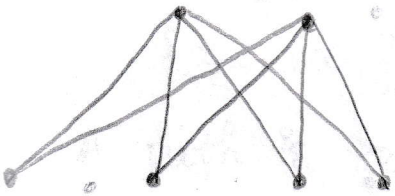
if $|V_1| = m$, $|V_2| = n$ then we denote that graph as $K_{m,n}$ bipartite.



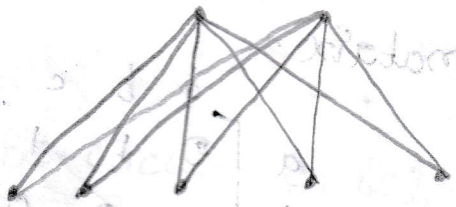
$K_{2,3}$.



$K_{3,3}$



$K_{2,4}$



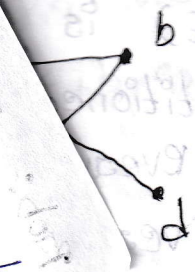
$K_{2,5}$

Adjacency matrix of the simple graph :-

If graph $G = (V, E)$ be the graph with n vertices then adjacency matrix of G is

$$[a_{ij}]_{n \times n} \text{ where } a_{ij} = \begin{cases} 1 & \text{if there is an edge } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

adjacency matrix of the Graph



Sol: 1) Given

Let V_1 is a complete bipartite graph where V_1 and V_2 are disjoint

Let the vertices of the vertices as a, b, c, d.

Then adjacency matrix of given graph is

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array}$$

Q. Draw the Graph with the adjacency matrix.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

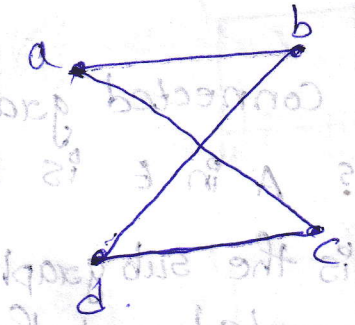
Sol: Given adjacency matrix is of order 4.

∴ graph has 4 vertices.

clearly, given graph is symmetric.

∴ Graph is undirected graph.

Let vertices in graph G are a, b, c, d.



Path, simple path, circuit (circle) :-

If $G=(V,E)$ is undirected graph

$$v = x_0 \overset{e_1}{\text{———}} x_1 \overset{e_2}{\text{———}} x_n \cdots x_{n-1} \overset{e_n}{\text{———}} x_n = V$$

If 'n' is a positive integer then, a path of length n from vertex 'v' to 'v' in G is a sequence of n edges of G.

such that e_1 is associated with $\{x_0, x_1\}$ ... and e_n associated with $\{x_{n-1}, x_n\}$

A path is said to be circuit (circle), if it begins and ends at the same vertex.

A path (or) a circuit is said to be simple if it does not contains the same edge more than ones.

Connected Graph :-

A graph $G=(V,E)$ is said to be connected if there is a path between any 2 different vertices in G.

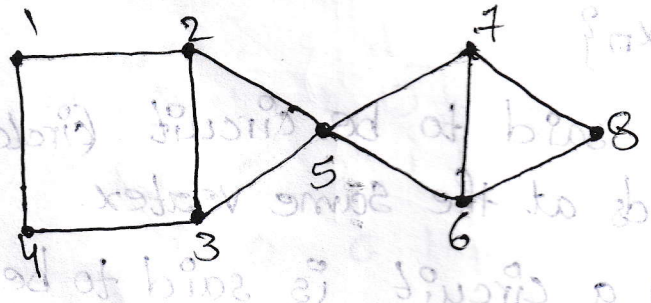
Cut set :-

$G = (V, E)$ is any connected graph then the set of edges A in E is said to be cut set if $G - A$ (that is the sub graph of G with $E-A$ edges) is disconnected and there is no $B \subset A \ni G-B$ is disconnected.

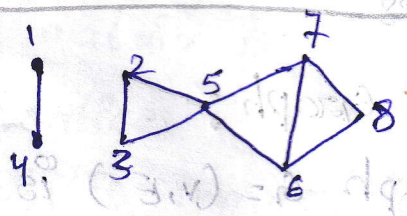
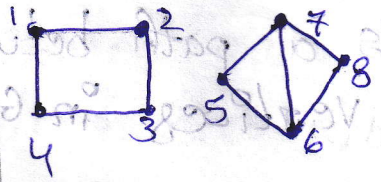
Cut vertex :-

If $G = (V, E)$ be the connected graph then a vertex v in G is said to be cut vertex if $G - \{v\}$ (that is subgraph of G obtained by deleating vertix v and all edges incident to v) is disconnected.

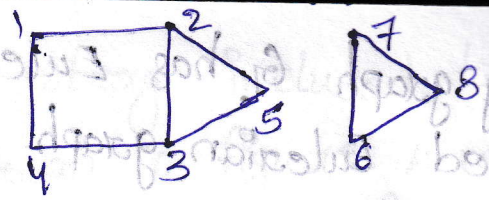
Q) Find cut sets and cut vertices of the following graph.



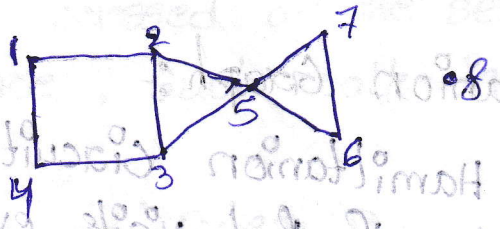
Sol:

Cut sets :-	disconnected sub graph.
$\{\{1,2\},\{3,4\}\}$	
$\{\{2,5\},\{3,5\}\}$	

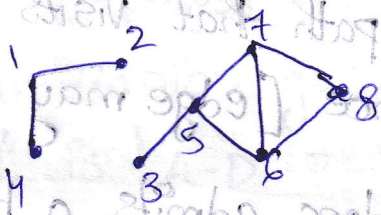
$\{\{5,7\},\{5,6\}\}$



$\{\{7,8\},\{6,8\}\}$

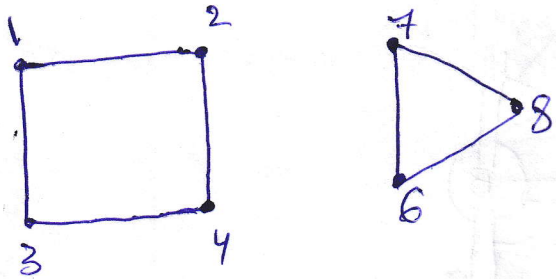


$\{\{4,3\},\{3,2\}\{2,5\}$



Cut vertices :-

Vertex '5' is only cut vertex. Since the graph $G-\{5\}$ is disconnected.



Q. write about HAMiltonian and Eulerian Graphs.

* Eulerian Graph :-

An Eulerian circuit in a graph G is a simple circuit containing all the edges of G.

An Eulerian path in G is a simple path containing every edge of G. [where edge cannot be repeat] ~~but may~~ vertex may repeat].

If any graph G has Eulerian circuit then G is called Eulerian graph.

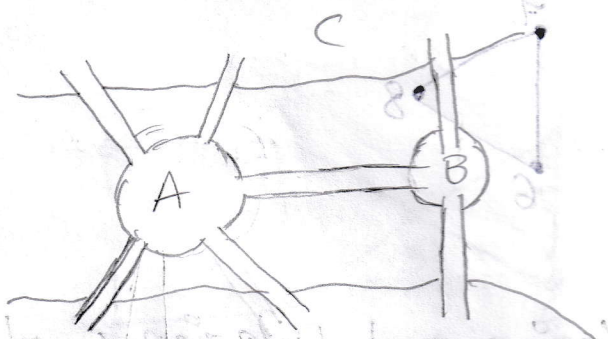
Hamiltanion Graph :-

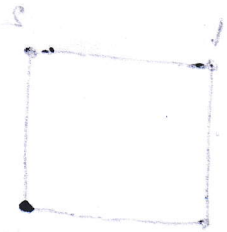
A Hamiltanion circuit in a graph G is a closed path that visits every vertex in G Exactly once [edge may repeat]

If G does admits a hamiltanion circuit then graph G is called Hamiltanion graph.

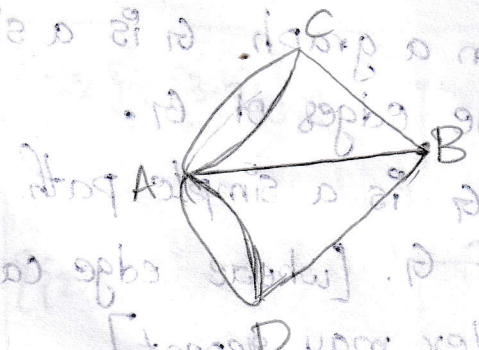
Example :- 1

Bridges in konigsberg :-





Multigraph of Bridges in konigsberg :-



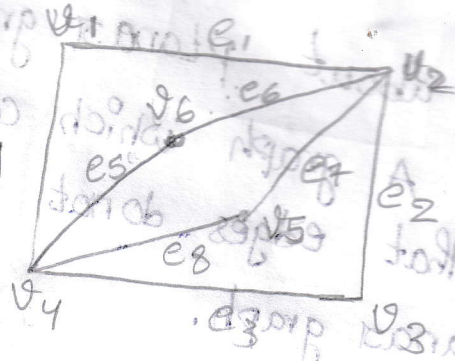
Town in a konigsberg included and 7 bridges shown in figure. A anywhere and he crossed all the bridges exactly once, such walk Possible.

∴ This graph is not Eulerian.

But this is Hamiltanion graph

Hamiltanion path is A–D–B–C–

Example 2:



This graph is Eulerian but not hamiltanion graph.

Eulerian path is

$$v_4 \xrightarrow{e_3} v_3 \xrightarrow{e_2} v_2 \xrightarrow{e_1} v_1 \xrightarrow{e_4} v_4 \xrightarrow{e_8} v_5 \xrightarrow{e_7} v_2 \xrightarrow{e_6} v_6 \xrightarrow{e_5} v_4$$

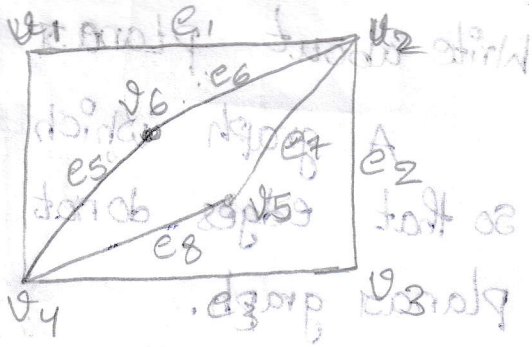
Town in a konigsberg included two islands and 7 bridges shown in figure. A man began anywhere and he crossed all the seven bridges exactly once, such walk is not Possible.

∴ This graph is not Eulerian.

But this is Hamiltanion graph.

Hamiltanion path is A–D–B–C–A

Example: 2:-

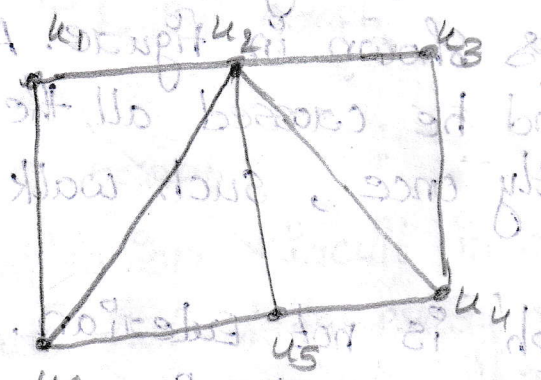


This graph is Eulerian but not hamiltanion graph.

Eulerian path is

$$v_4 \xrightarrow{e_3} v_3 \xrightarrow{e_2} v_2 \xrightarrow{e_1} v_1 \xrightarrow{e_4} v_4 \xrightarrow{e_8} v_5 \xrightarrow{e_7} v_2 \xrightarrow{e_6} v_6 \xrightarrow{e_5} v_4$$

Example 3:-



This graph is hamiltanion but not Eulerian

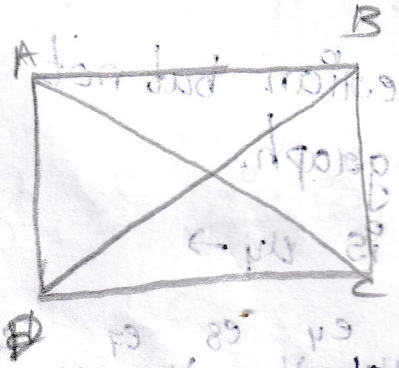
Hamiltakion path is

$$u_1 - u_2 - u_3 - u_4 - u_5 - u_6$$

Write about planar graphs

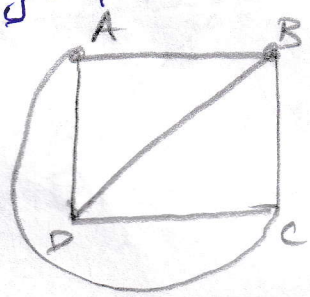
A graph which can be drawn in a plane so that edges donot intersects is said to be planar graph.

Example:-

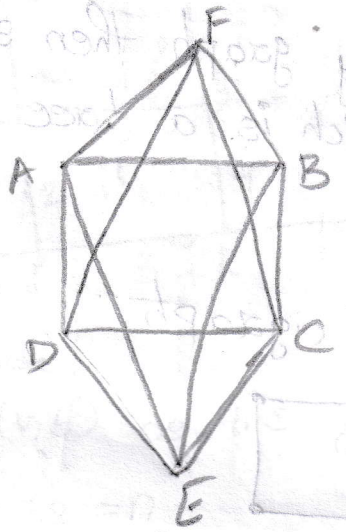


is a planar graph

planar Representation is

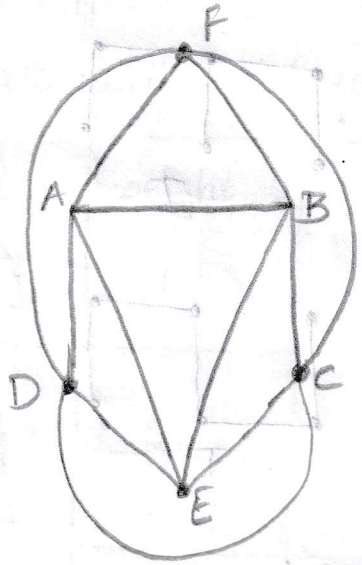


Example 2 :-



Is a planar graph

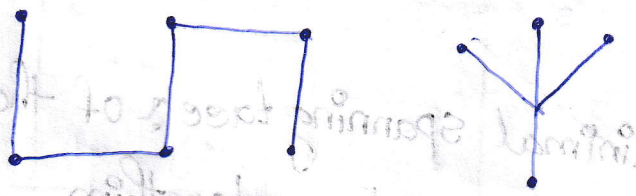
Planar Representation is.



Tree :-

A connected graph with no cycles is called Tree.

Ex :-

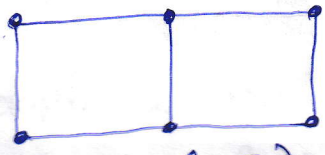


Spanning Tree :-

$G = (V, E)$ is any graph then spanning tree is subgraph T which is a tree with all vertices in G.

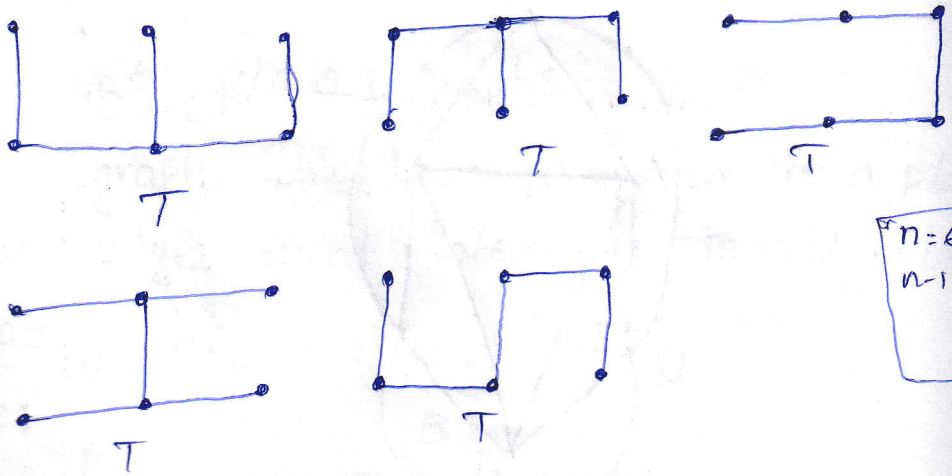
Ex :-

Consider the graph



$G = (V, E)$

Then spanning trees of G are



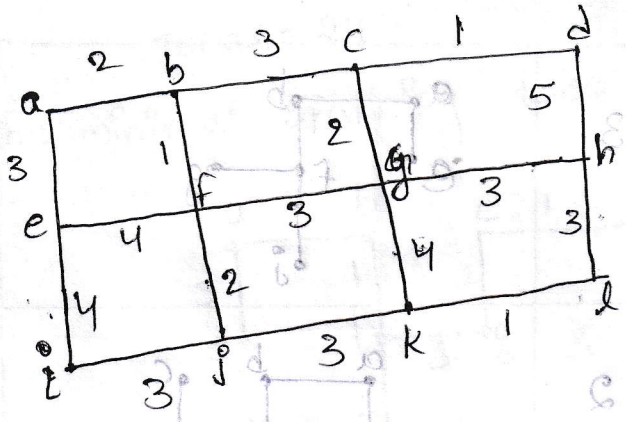
$n = 6$
$n - 1 = 6 - 1$
$= 5$
edges

Minimal Spanning tree :-

$G = (V, E)$ is a graph then minimal spanning tree of G is the spanning tree which as the sum of all weights assigned to edges is least.

Problem :- 1

Find the Minimal spanning tree of the following graph using prims algrothim



[For 15 Marks]

Sol:-

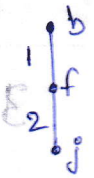
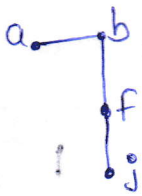
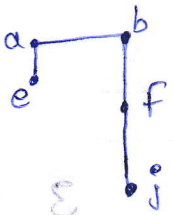
Let $G = (V, E)$ be the given graph

$|V| = 12 = n$

$\therefore$ The minimal spanning tree contains

$n-1 = 11$ edges

Let us starts with minimum weight edge

choice	edge	weight	diagram.
1	$\{b, f\}$	1	
2	$\{f, j\}$	2	
3.	$\{a, b\}$	2	
4	$\{a, e\}$	3	

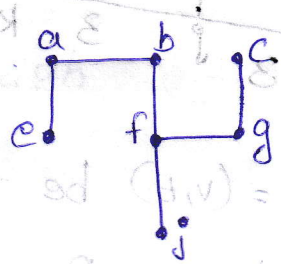
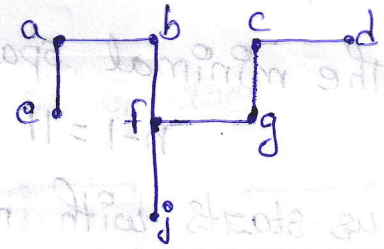
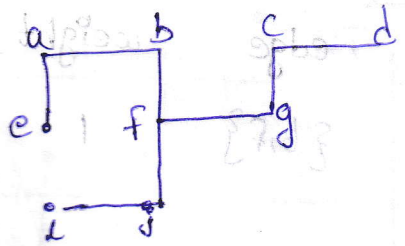
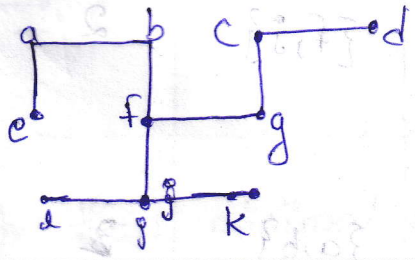
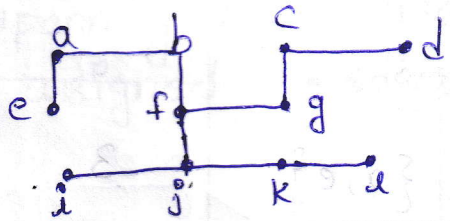
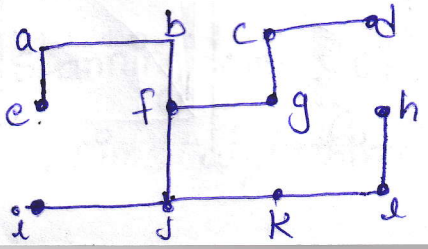
Spanning Tree :

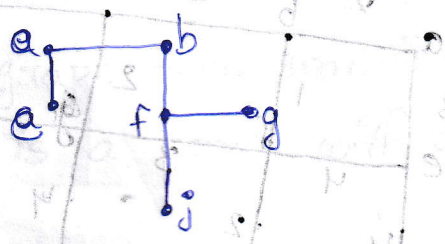
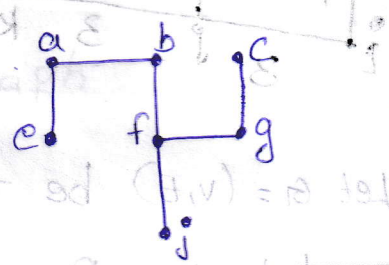
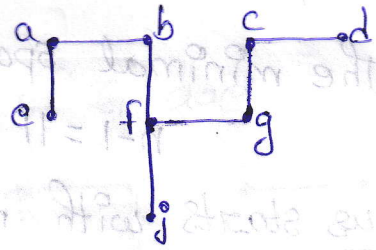
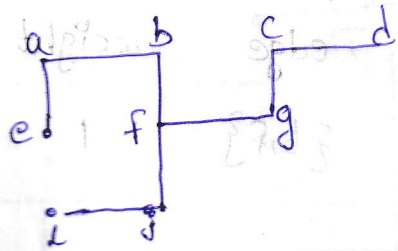
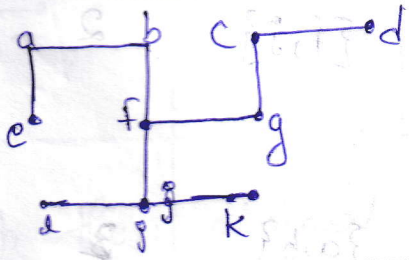
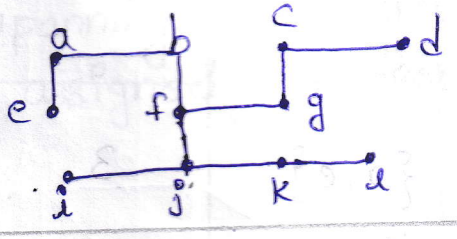
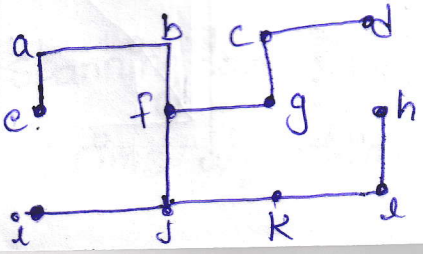
$G = (V, E)$

is Suba

vertic

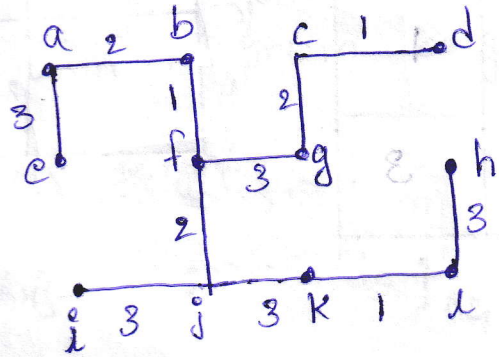
[for 15 marks]

6		2	
7.	$\{c, d\}$	1	
8	$\{j, k\}$	3	
9.	$\{j, k\}$	3	
10.	$\{k, l\}$	1	
11	$\{l, h\}$	3	

5	{f, g}	3	
6	{g, c}	2	
7.	{c, d}	1	
8	{j, i}	3	
9.	{j, k}	3	
10.	{k, l}	1	
11	{l, h}	3	

Total = 24

The minimal spanning tree is



Another model :- [for 8 Marks]

Given graph is $G = (V, E)$

where $|V| = 12$.

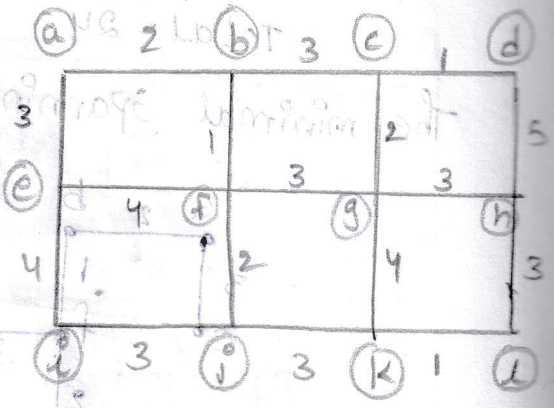
$\therefore$ Minimal spanning tree contains

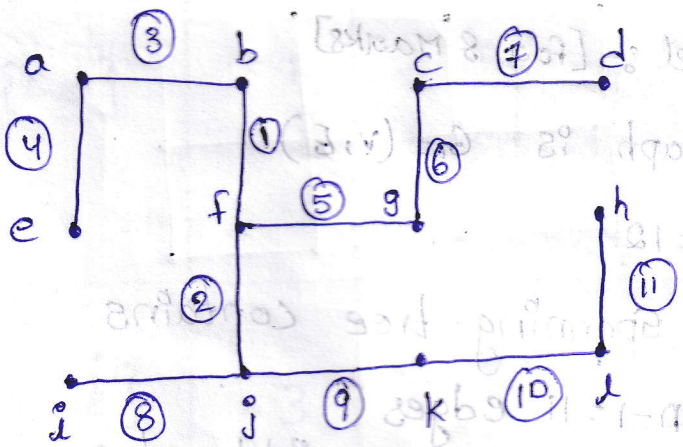
$n-1 = 11$ edges

Starts with minimum weight edge

choice	edge	weight
1	{b, f}	1
2	{f, j}	2
3	{a, b}	2
4	{a, e}	3
5	{f, g}	3
6	{g, c}	2
7	{c, d}	1

8	{j, l}	3
9	{j, k}	3
10	{k, l}	1
11	{l, h}	3

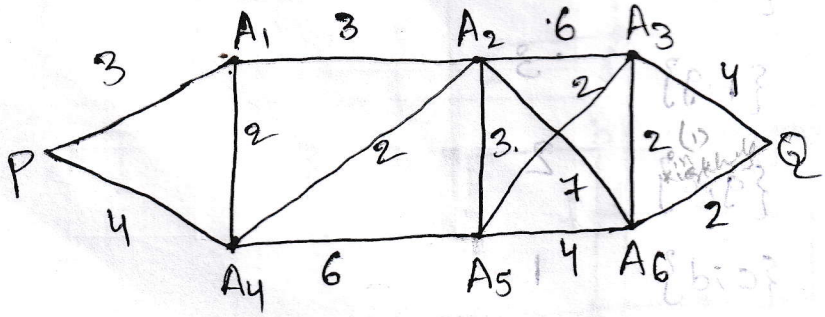




∴ This is the minimal spanning tree with cost 24.

Problem-2 :-

Find the minimal spanning tree of the following graph using prims algorithm.

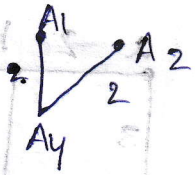
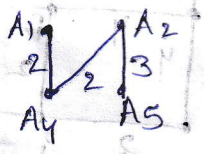
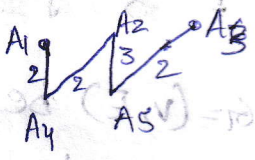
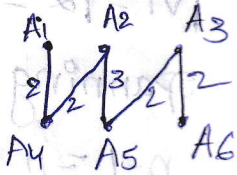
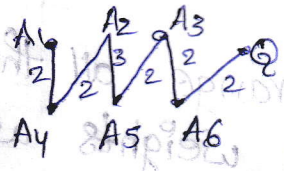
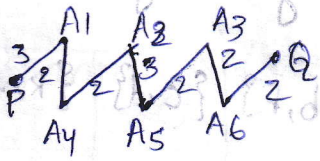


Sol: Let $G = (V, E)$ be the given graph

$|V| = 8 = n$

$\therefore$ The minimal spanning tree contains $n-1 = 7$ edge

let us starts with minimum weight edge

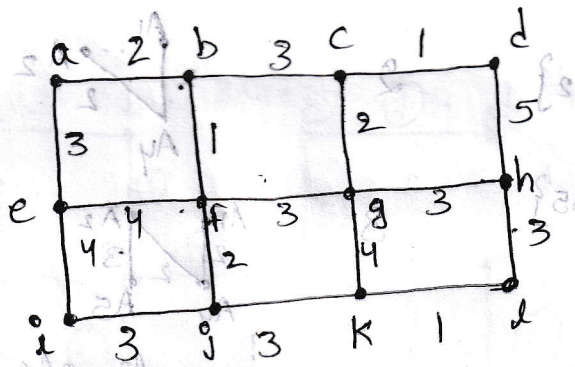
choice	edge	weight	
1	$\{A_1, A_4\}$	2	
2	$\{A_4, A_2\}$	2	
3.	$\{A_2, A_5\}$	3	
4.	$\{A_5, A_3\}$	2	
5.	$\{A_3, A_6\}$	2	
6.	$\{A_6, Q\}$	2	
7.	$\{A_1, P\}$	3	
		16	

Krauskal's Algorithm :-

Krauskal's Algorithm to find minimal spaning tree.

Problem :-

Find the minimal spanning tree of the following graph using krauskal's algorithm.



Sol:- Let $G=(V,E)$ be the given graph

where $|V| = |2| = n$

Minimal spanning tree contains

$n-1 = 11$ edges.

First arrange all the edges in the order of increasing weights assigned to edges

edge	{b,f}	{c,d}	{k,l}	{a,b}	{c,g}	{f,j}	{a,e}	{i,j}	{b,c}
weight	1	1	1	2	2	2	3	3	3
	{j,k}	{f,g}	{g,h}	{l,h}	{e,f}	{e,i}	{g,k}	{d,h}	
	3	3	3	3	4	4	4	5	

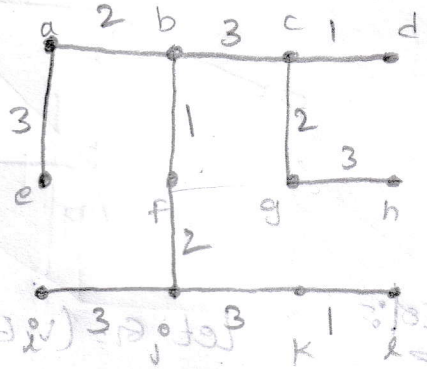
{g,k}

S.No	edge
1.	{b,f}
2	{c,d}
3.	{k,l}
4.	{a,b}
5	{f,j}
6.	{c,g}
7.	{a,e}
8.	{i,j}
9.	{b,c}
10.	{j,k}
11	{f,g}
12.	{g,h}

∴ cost

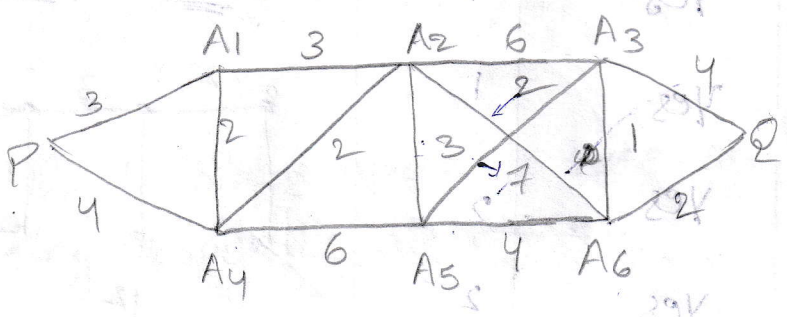
S.No	edge	do not form cycle	weight
1.	{b, f}	yes	1
2	{c, d}	yes	1
3.	{k, l}	yes	1
4.	{a, b}	yes	2
5	{f, j}	yes	2
6.	{c, g}	yes	2
7.	{a, e}	yes	3
8.	{i, j}	yes	3
9.	{b, c}	yes	3
10.	{j, k}	yes	3
11	{f, g}	no	–
12.	{g, h}	yes	3
		Total	24

Minimal spanning tree:-



∴ cost of the minimal spanning tree is 24

2. Find the minimal Spanning tree of the following graph using Kruskal's Algorithm.



Sol:- Let $G = (V, E)$ be the given graph

where $|V| = 8 = n$

Minimal spanning tree contains

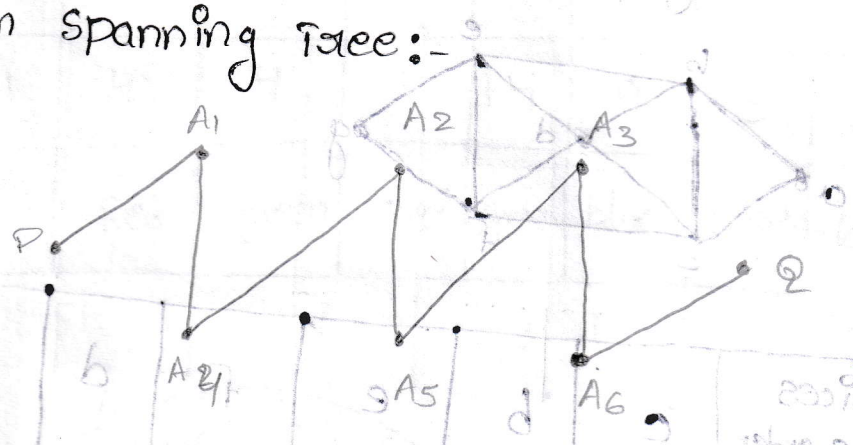
$n-1 = 7$ edges

First arrange all the edges in the order of increasing weights assigned to edges.

edge	$\{A_3, A_6\}$	$\{A_1, A_4\}$	$\{A_2, A_4\}$	$\{A_3, A_5\}$	$\{A_6, Q\}$
weight	1	2	2	2	2
	$\{A_1, A_2\}$	$\{P, A_1\}$	$\{A_2, A_5\}$	$\{P, A_4\}$	$\{A_5, A_6\}$
	3	3	3	4	4
	$\{A_3, Q\}$	$\{A_4, A_5\}$	$\{A_2, A_3\}$	A_2, A_6	
	4	6	6	7	

S.NO	edge	do not from cycle	weight
1	$\{A_3, A_6\}$	yes	1
2	$\{A_1, A_4\}$	yes	2
3	$\{A_2, A_4\}$	yes	2
4	$\{A_3, A_5\}$	yes	2
5	$\{A_6, Q\}$	yes	2
6	$\{A_1, A_2\}$	yes	3
7	$\{A_2, A_5\}$	yes	3
		Total	15

Minimum spanning Tree:-



∴ cost of Minimal spanning tree is 15

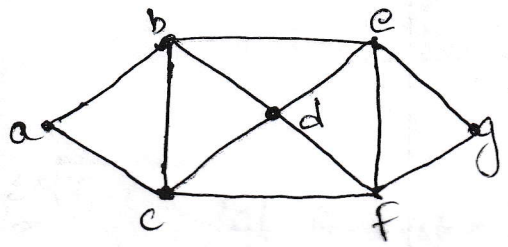
Graph colouring and chromatic number :-

A colouring of a simple graph is an assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

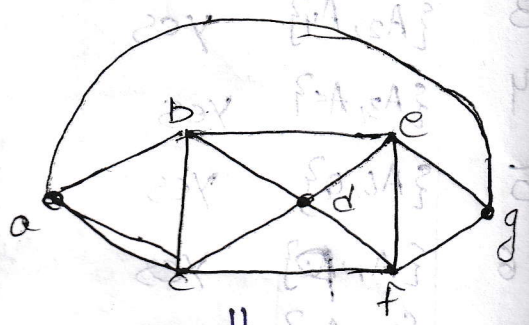
A chromatic number of a graph G is least number of colours needed to colouring of the graph(G).

chromatic number of graph G is denoted by '$\chi'(G)$ (chi).

Example:- Find the chromatic number of the following graph G and H.

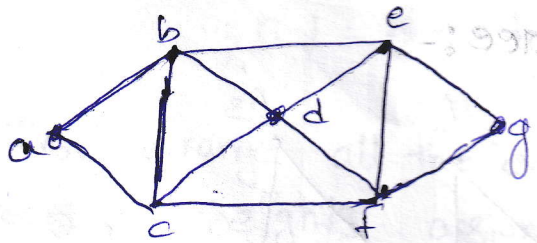


G

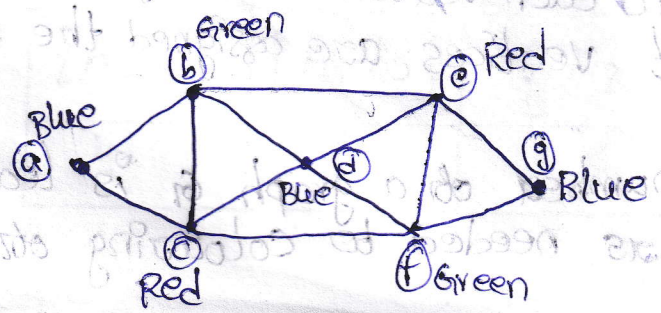


H

Sol:- Given graph G is



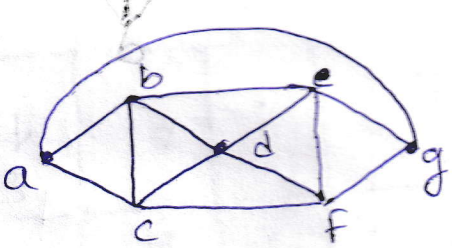
vertices in the order of ↓ degree	c	b	e	f	d	a	g
degree	4	4	4	4	4	2	2
Assigned colour	Red	green	Red	Green	Blue	Blue	Blue



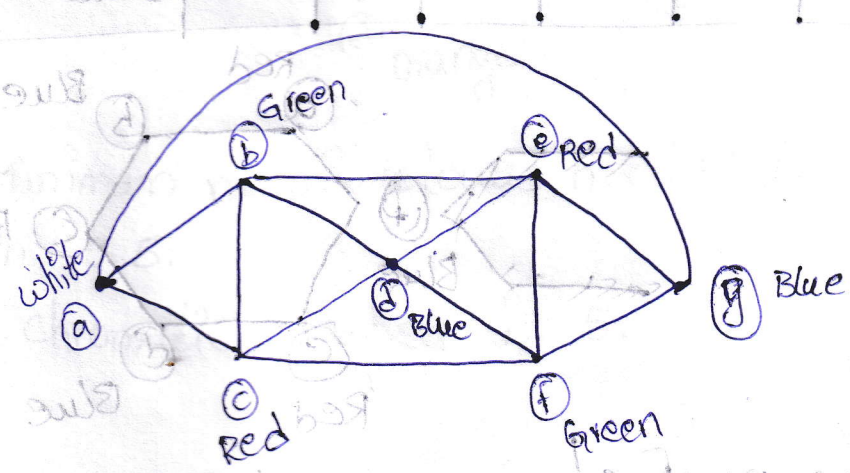
∴ Minimum number of colours needed to colour the graph is 3.

∴ chromatic Number of G is $\chi(G) = 3$.

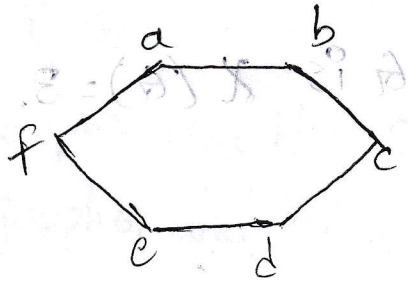
ii, Given graph H is

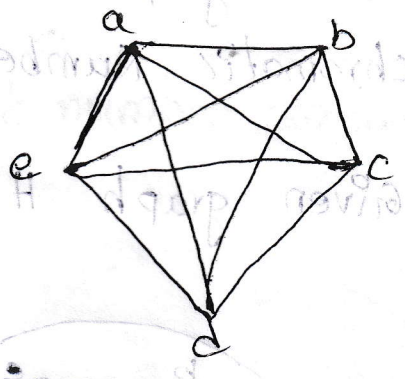


vertices in order of ↓ degree	e	b	e	f	d	a	g
degree	4	4	4	4	4	3	3
Assigned colour	Red	green	Red	green	blue	white	Blue

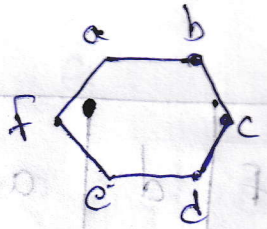


Q, Find the chromatic no. of the following graph.

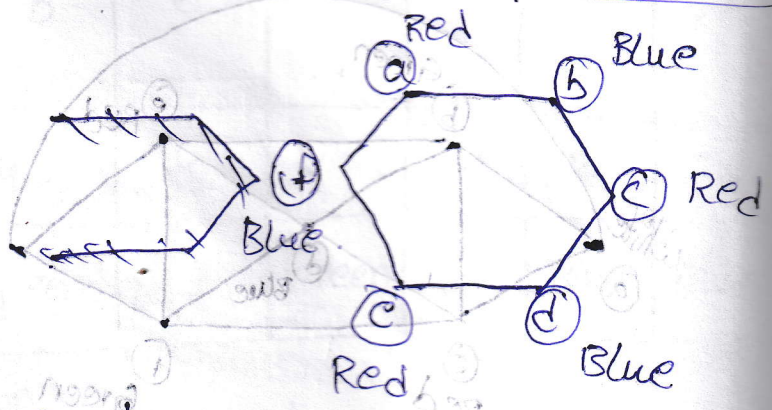




Sol:- Given graph is



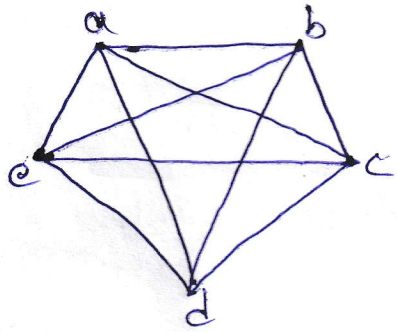
Vertex in the order of ↓ weight	a	b	c	d	e	f
weights	2	2	2	2	2	2
colour Assigned	Red	Blue	Red	Blue	Red	Blue



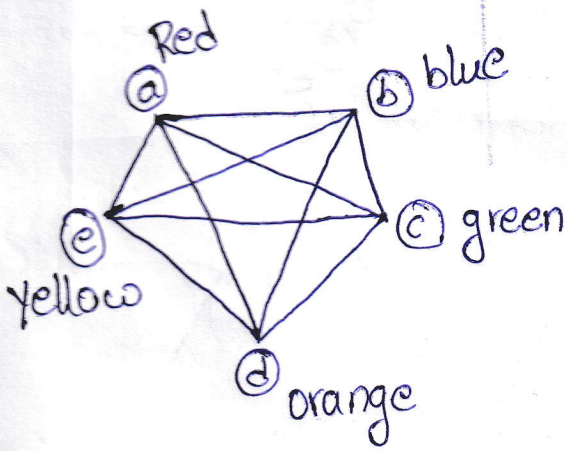
Minimum no. of colours needed to colour the graph is 2

∴ chromatic number is 2

b) Given graph is



vertex in order of ↓ weights	a	b	c	d	e
weights	4	4	4	4	4
colour Assigned	Red	blue	green	orange	yellow



Minimum no. of colours needed to colour the graph is 5.

∴ Chromatic number is 5.

Unit-II

Recurence Relation :-

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ is Recurence relation of degree k where $c_1, c_2, \dots c_k$ are real numbers.

$\boxed{r_1 = r_2 = r}$ → equal roots

→ $a_n = (\alpha_1 + \alpha_2 n) r^n$

Solution of Recurence Relation :-

$a_n = r^n$

* $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(r+1)^3 = r^3 + 3r^2 + 3r + 1$

Recurence Relation :-

A linear homogeneous recurence relation of degree k with constant coefficients is a recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ where

$c_1, c_2, \dots c_k$ are real numbers and $c_k \neq 0$.

Method to solve recurence relation :-

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \rightarrow (1)$

$(c_k \neq 0)$ is a recurence relation of degree k.

Let $a_n = r^n$ be the solution then

from (1) $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$

Divide with r^{n-k}

$$\frac{r^n}{r^{n-k}} = c_1 \frac{r^{n-1}}{r^{n-k}} + c_2 \frac{r^{n-2}}{r^{n-k}} + \dots + c_k$$

$\Rightarrow r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$

$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$

This is called characteristic equation of given recurence relation.

Let are $r_1, r_2, \ldots r_k$

Note:- If Reference Relation is second degree

Case-i:- If r_1, r_2 are distinct roots then

$$a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n$$

If r_1, r_2, r_3 are distinct roots of 3rd degree R.R

Then $a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n + \alpha_3 (r_3)^n$

Case-ii:- If $r_1 = r_2$ then let $(r_1 = r_2 = r)$ then

$$a_n = (\alpha_1 + \alpha_2 n) r^n$$

If $r_1 = r_2 = r_3 = r$ (Three roots are equal) then

$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2) r^n$$

Problem-1:

What is the soln of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$, $a_1 = 7$

Sol:- Given R.R is $a_n - a_{n-1} - 2a_{n-2} = 0$

Let $a_n = r^n$ be the solution

Now characteristic equation is

$$r^2 - r - 2 = 0$$

$\Rightarrow (r-2)(r+1) = 0 \Rightarrow r = 2, -1$ are distinct roots

General solution $a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n = \alpha_1 (2)^n + \alpha_2$

Given that $a_0 = 2 \Rightarrow \alpha_1 + \alpha_2 = 2$ —— ②

$a_1 = 7 \Rightarrow 2\alpha_1 - \alpha_2 = 7$ —— ③

solving ② ③ : ② + ③ $\Rightarrow 3\alpha_1 = 9 \Rightarrow \alpha_1 = 3$

from ② $\alpha_2 = -1$

$\therefore$ from ① solution is $a_n = 3(2)^n - 1(-1)^n$.

2. What is the soln of the Recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} = 0$ with initial condition $a_0 = 1, a_1 = 6$

Sol:- Given R.R is $a_n - 6a_{n-1} + 9a_{n-2} = 0$

let $a_n = r^n$ be the solution

now characteristic equation is

$r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0$

$\Rightarrow r = 3, 3$ (two roots are equal)

General solution is

$a_n = (\alpha_1 + \alpha_2 n) r^n$

$\Rightarrow a_n = (\alpha_1 + \alpha_2 n) 3^n \longrightarrow ①$

given that $a_0 = 1 \Rightarrow (\alpha_1 + \alpha_2 (0)) 3^0 = 1$

$\Rightarrow \alpha_1 = 1$

$a_1 = 6 \Rightarrow (\alpha_1 + \alpha_2 (1)) 3^1 = 6$.

$\Rightarrow (1 + \alpha_2) = 2 \Rightarrow \alpha_2 = 1$

$\therefore \alpha_1 = 1, \ \alpha_2 = 1$

from ① solution is

$$a_n = (1+n)\,3^n$$
$$= 3^n + n3^n$$

Note :- FIBONACCI NUMBERS :-

$f_n = f_{n-1} + f_{n-2}$ & $f_0 = 0, f_1 = 1$

f_0,	f_1,	f_2,	f_3,	f_4,	f_5,	f_6 ...
0	1	1	2	3	5	8 ...

Mid Sem

3) Find an explicit formula for the Fibonacci numbers.

Sol:- Fibonacci Recurrence relation is ...

$f_n = f_{n-1} + f_{n-2} \longrightarrow ①$ $f_0 = 0, f_1 = 1$

Let $f_n = r^n$ be the solution of ①

characteristic equation is

$r^2 - r - 1 = 0$

$$\boxed{\begin{array}{l} ar^2 + br + c = 0 \\ a = 1, \ b = -1, \ c = -1 \end{array}}$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore r_1 = \frac{1+\sqrt{5}}{2}, \ r_2 = \frac{1-\sqrt{5}}{2}$$

General solution is $f_n = \alpha_1 . r_1^n + \alpha_2 . r_2^n$

$\Rightarrow f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow (2)$

$f_0 = 0 \Rightarrow \alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_1 \rightarrow (3)$

$f_1 = 1 \Rightarrow \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\Rightarrow \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) - \alpha_1 \left(\frac{1-\sqrt{5}}{2}\right) = 1$ from (3)

$\alpha_1 \left[\frac{1+\sqrt{5} - 1 + \sqrt{5}}{2}\right] = 1$

$\alpha_1 \sqrt{5} = 1 \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}$

from (3) $\alpha_2 = -\frac{1}{\sqrt{5}}$

from (2) solution.

$\therefore f_n = \frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}} \left[\frac{1-\sqrt{5}}{2}\right]^n$

4. Imp. Find the solution to the recurrence relation.

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial condition $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

Given Recurrence relation is

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

Let $a_n = r^k$ be the solution of this eqn

$\therefore$ characteristic equation is

$r^3 = 6r^2 - 11r + 6$

$\Rightarrow r^3 - 6r^2 + 11r - 6 = 0$

$r=1 \Rightarrow 1-6+11-6=0$ $\boxed{r=1}$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$6 = 2\times 3$

$r^2 - 5r + 6 = 0$

$r^2 - 2r - 3r + 6 = 0$

$\Rightarrow r(r-2) - 3(r-2) = 0$

$(r-2)(r-3) = 0$

$\boxed{r = 2, 3}$

$r = 1, 2, 3$ are distinct roots

General solution:-

$a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n + \alpha_3 (r_3)^n$

$= \alpha_1 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n \rightarrow (1)$

where $\alpha_1, \alpha_2, \alpha_3$ are real numbers.

Particular solution:-

Given that $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

from (1)

$a_0 = 2 \Rightarrow 2 = \alpha_1 + \alpha_2 + \alpha_3 \rightarrow (2)$

$a_1 = 5 \Rightarrow 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \rightarrow (3)$

$a_2 = 15 \Rightarrow 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3 \rightarrow (4)$

solving (2) (3) & (4).

③ - ② $\Rightarrow \alpha_2 + 2\alpha_3 = 3 \rightarrow$ ⑤

④ - ③ $\Rightarrow 2\alpha_2 + 6\alpha_3 = 10$

$\Rightarrow \alpha_2 + 3\alpha_3 = 5 \rightarrow$ ⑥

⑥ - ⑤ $\Rightarrow \boxed{\alpha_3 = 2}$

from ⑥ $\alpha_2 = 5 - 3\alpha_3 = 5 - 6 = -1$

$\boxed{\alpha_2 = -1}$

from ② $\alpha_1 - 1 + 2 = 2 \Rightarrow \boxed{\alpha_1 = 1}$

from ① particular solution is

$a_n = 1.(1)^n + (-1).2^n + 2.3^n$

$= 1 - 2^n + 2.3^n$

5. Find the solution of the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial condition $a_0 = 1, a_1 = -2, a_2 = -1$

Sol:- Given R.R is

$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

Let $a_n = r^n$ be the solution then characteristic equation is

$r^3 + 3r^2 + 3r + 1 = 0$

$\Rightarrow (r+1)^3 = 0$ $\boxed{(r+1)^3 = r^3 + 3r^2 + 3r + 1}$

$\Rightarrow r = -1, -1, -1.$

GS is

$$a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3)\, r^n$$

$$= (\alpha_1 + n\alpha_2 + n^2\alpha_3)(-1)^n \rightarrow (1)$$

$a_0 = 1$: put $n=0$ in (1)

$$(\alpha_1 + 0\alpha_2 + 0\alpha_3)(-1)^0 = 1$$

$$\Rightarrow \boxed{\alpha_1 = 1}$$

$a_1 = -2$: put $n=1$ in (1)

$$(\alpha_2 + 1\alpha_2 + 1^2\alpha_3)(-1)^1 = -2$$

$$\Rightarrow (1 + \alpha_2 + \alpha_3)(-1) = -2$$

$$1 + \alpha_2 + \alpha_3 = 2$$

$$\alpha_2 + \alpha_3 = 1 \rightarrow (2)$$

$a_2 = -1$: put $n=2$ in (1)

$$\Rightarrow (\alpha_1 + 2\alpha_2 + 4\alpha_3)(-1)^2 = -1$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + 4\alpha_3 = -1$$

$$\Rightarrow 1 + 2\alpha_2 + 4\alpha_3 = -1$$

$$\Rightarrow 2\alpha_2 + 4\alpha_3 = -2$$

$$\Rightarrow \alpha_2 + 2\alpha_3 = -1 \rightarrow (3)$$

solving (2) & (3) :

$$(3) - (2) \Rightarrow \alpha_3 = -2$$

from (2) $\alpha_2 - 2 = 1$

$$\Rightarrow \boxed{\alpha_2 = 3}$$

from ① Particular solution is

$$a_n = (1+3n-2n^2)(-1)^n$$

Definition :-

Mid

The principle of Inclusion and Exclusion

If A and B are any two sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(this means number of elements in $A \cup B$)

= (Number of elements in A) + (Number of elements in B)

−(Number of elements in $A \cap B$)

Ex:- If $|A| = 6$ $|B| = 4$, $|A \cap B| = 1$

Then find Number of elements in $A \cup B$.

$$|A \cup B| = |A| + |B| - |A \cap B| = 6+4-1 = 9$$

If A, B, C are three sets then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Problem:-

In a discrete mathematics class every student is a major in computer science or Mathematics or both. The Number of students having computer science as a major (Possibly along with mathematics) is 25. The Number of students having mathematics as a major (Possibly along with computer science)

is 13 and the Number of students in both computer science and mathematics is 8. How many students are in this class.

Sol:- Let A : The set of students in the class in computer science.

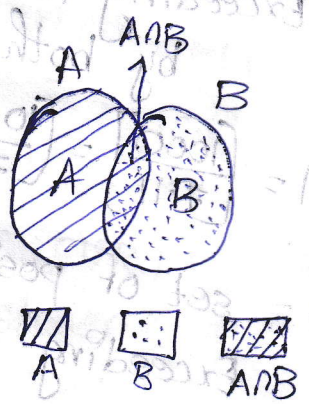
B : The set of students in the class who in mathematics.

$A \cap B$: The set of students in the class who are joint mathematics and computer science

Given $|A| = 25$ $|B| = 13$

and $|A \cap B| = 8$

Number of elements in class is $|A \cup B|$



By Inclusion and Exclusion principle

$\therefore |A \cup B| = |A| + |B| - |A \cap B|$

$= 25 + 13 - 8$

$= 38 - 8$

$= 30$

$\therefore$ There are 30 students in their class.

Long Problem-2 :-

How many positive integers not exceeding 100 are divisible by 7 or 11.

sol:- Let A = set of positive integers divisible by 7 (not exceeding, 1000)

$$|A| = \left[\frac{1000}{7}\right] = [142.\ldots] = 142$$

B : set of positive integers not Exceeding 1000 and divisble by 11

$$|B| = \left[\frac{1000}{11}\right] = [90.9] = 90$$

$A \cap B$ = set of integrs (+ve) and not Exceeding 1000 and divisble by both 7, 11

$$|A \cap B| = \left[\frac{1000}{7 \cdot 11}\right] = \left[\frac{1000}{77}\right] = [12.9] = 12$$

$A \cup B$ = set of positive integers not Exceeding 7 or 11

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 142 + 90 - 12$$
$$= 232 - 12$$
$$= 220.$$

7) 1000 (142.
7
30
28
20
14
6

11) 1000 (90.9
99
10
0
100
99
1

77) 1000 (12.9
77
230
154
76

Mid

3. A Total ~~130~~ 1232 students have taken a course in spanish, 879 have taken a course in ~~Russian~~. ~~Further~~ French and 114 have taken course in Russian. Further 103 have taken courses in both spanish & French, 23 have taken course in both spanish & Russian and 14 have

taken course in both french & Russian. If 2092 students have taken at least one of spanish, french and russian. How many student have taken a course in all three language.

sol:- Let S: set of all students who have taken a course in spanish

Let F: set of all students who have a taken a course in french

R: set all students who have taken a cause in russian

Given that

$|S| = 1232$ $|F| = 879$ $|R| = 114$ $|S \cap F| = 103$

$|S \cap R| = 23$ $|F \cap R| = 14$ and $|S \cup F \cup R| = 2092$

we have to find number of students in the set $(S \cap F \cap R)$ that is $(S \cap F \cap R)$

we know that.

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |F \cap R| - |R \cap S| + |S \cap F \cap R|$$

$$\Rightarrow 2092 = 1232 + 879 + 114 - 103 - 14 - 23 + |S \cap F \cap R|$$

$$= 2225 - 140 + |S \cap F \cap R|$$

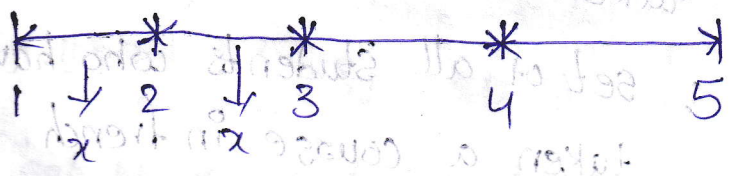
$$= 2085 + |S \cap F \cap R|$$

$$|S \cap F \cap R| = 2092 - 2085$$

$$= 7$$

∴ Number of students who have taken a course in all three languages is 7

Note :- Integral part of a real number



$[1] = 1$ $[x] = 1$ $[x] = 2$

$[1.1] = 1$ $[2.5] = 2$

$[1.2] = 1$ $[2.85] = 2$

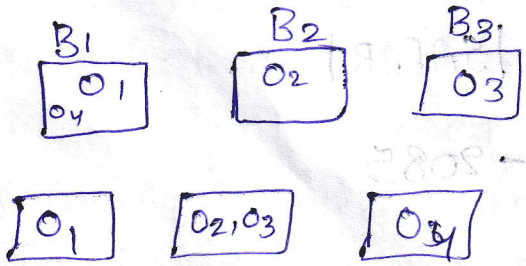
$[1.999] = 1$ $[2.999] = 2$

short answer Mid

The PIGEON HOLE Principle :-

If k is any positive integer then if $k+1$ are more objects are placed into k boxes then there is atleast one box containing two or more objects.

Example-I :-



$k = 3$ boxes

O_1, O_2, O_3, O_4 are

$k+1 = 4$ or more objects.

clearly atleast one Box have more than one object.

Ex:- Among any group of 367 people there must be atleast two with the same Birthday, because there are only 366 possible bithdays.

Non Homogeneous Recursive Relation :-

A Recurrence relation is of the form

$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + F(n)$ is called non homogeneous Reccurence relation where solution of this sequence is

$$a_n = a_n^{[h]} + a_n^{[P]}$$

where $a_n^{[h]}$ is solution of

$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ and

$a_n^{[P]}$ is particular solution.

Note :-

	$a_n^{[P]}$
$F(n) = 7^n$	$C.7^n$
$F(n) = 8^n$	$C.8^n$
$F(n) = 2n, 3n$	$Cn = Cn + d$

Problem :- Mid long.

Solve $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Sol:- Given Recurrence Relation is

$$a_n - 5a_{n-1} + 6a_{n-2} = 7^n \rightarrow (1)$$

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$$^nP_3 = n(n-1)(n-2)$$

→ The number of permutations of 'n' dissimilar things taken 'r' at a time is ^nP_r i.e. the no. of arrangements of 'r' things from given 'n' things is ^nP_r.

Ex:- Find the no. of 4 letter words that can be formed using the letters of the word Equation.

i) How many of these words began with E

ii) How Many words begin with E and Ends with N

Sol:- the word equation has 8 Letters

No. of ways to fill box $\underset{8}{\square}\ \underset{7}{\square}\ \underset{6}{\square}\ \underset{5}{\square}$

the number of 4 Letter words from the Letters in the word equation is $^8P_4 = 8.7.6.5 = 1680$

i) words begin with E

$\boxed{E}\ \square\ \square\ \square$

First put E in 1st place

Now, we can arrange the remaining 7 letters in 3 places in $^7P_3$ ways ⇒ $^7P_3 = 7.6.5 = 210$

ii, First put E in 1st place and N in last place

E			N

Now, we can arrange the remaining 6 Letters in 2 places in $6P_2$ ways.

$\Rightarrow 6P_2 = 6.5 = 30$ places.

Pg: 158: Find the no. of four Letter words that can be formed using the word MIXTURE with

i, contains Letter X

ii, Do not contain the letter x.

sol: The word mixture has 7 letters is $7P_4$

$\Rightarrow 7.6.5.4 = 840$

1, contains 'x'.

1st we put x in 4 places this can be done in 4 ways Now, we can fill the 3 places with 6 letters in $6P_3 = 6.5.4 = 120$ ways.

∴ Total no. of 4 letter words contains 'x' is

$4 \times 6P_3 = 4 \times 120 = 480$

ii, without the letter 'x' we have to fill the four places with remaining 6 Letters

This is in $6P_4 = 6.5.4.3 = 360$ ways

∴ The no. of 4 letter words does not contain 'x' are 360.

Find the no. of ways of arranging 5 boys and 4 girls in a row that the row i, begin and ends with boys. ii, begin with a boy and ends with a girl

Sol:- i, total no. of persons are 9 (5 boys + 4 girls)

B								B

First fill first and last place with boys this can be done in $5P_2$ ways

Now we have to fill the remaining 7 Places with remaining 7 person (3 boys + 4 girls) in 7 ways

Total number of arrangements are $5P_2 \times 7!$

$= 5 \cdot 4 \times 5040$

$= 100800$

ii,

B								G

First place will fill with boys in 5 ways and last place fill with girl in 4 ways and the remaining 7 Places fill with remaining 7 persons in 7 ways.

The no. of required arrangements is

$5 \times 4 \times 7! = 100800$

Pg: 159, Eg-1:-

If $nP_4 = 1680$ find n.

Sol:- $nP_4 = 1680$

$n(n-1)(n-2)(n-3) = 1680$

$8 \cdot 7 \cdot 6 \cdot 5 = 1680$

$n = 8$.

Eg-5: Find the no. of ways of arranging 6 boys and 6 girls in a row. In how many of arrangments i, all the girls are together. ii, No two girls are together iii, Boys and girls come alternately.

Sol: Total no. of persons is 12 (6 boys + 6 girls)

we can arrange the 12 persons in 12! ways

i, Assume 6 girls as 1 unit.

Now, 7 units can be arranged in 7! ways

Now in that unit 6 girls themselves arrange in 6! ways.

∴ No. of arrangements in which all 6 girls together is 7! 6!

ii, First we can arrange 6 boys in 6! ways. Now girls can be arranged in 7 places between the boys and beginning and ending in $^7P_6$ ways

∴ Total no. of required arrangements is $6!\,^7P_6 = 6!\,^7P_6$

$\times \underset{G}{\boxed{B}} \times \underset{G}{\boxed{B}} \times \underset{G}{\boxed{B}} \times \underset{G}{\boxed{B}} \times \underset{G}{\boxed{B}} \times \underset{G}{\boxed{B}} \times$ 6! 6! ways

iii, Begin with Boy

B	G	B	G	B	G	B	G	B	G	B	G
1	2	3	4	5	6	7	8	9	10	11	12

6! 6! ways

Begin with girls:

G	B	G	B	G	B	G	B	G	B	G	B
1	2	3	4	5	6	7	8	9	10	11	12

6! 6! ways

A row begin with a boy then, boys can arrange 1, 3, 5, 7, 9, 11th places in 6! ways and girls can be arranged in 2, 4, 6, 8, 10, 12th places in 6! ways

∴ Total number of required arrangement is 6! 6!

Similarly, no begin with girl arrangements are 6! 6!

Total no. of arrangements is 2 × 6! 6!

Ex-7: Find the number of 4 letter words that can be formed during the letters of the word MIRACLE how many of them,

i, begin with an vowel ii, begin and ends with vowels iii, end with a consonant.

Sol: In the word "miracle" vowels are A, E, I - 3 vowels, consonants are: M, R, C, L - 4 consonants

The no. of 4 letter words using 7 letters in the word miracle is $^7P_4 = \frac{7!}{3!} = 7 \times 6 \times 5 \times 4 = 840$

i,

we can fill the 1st place with 3 vowels in $^3P_1 = 3$ ways

Now, we can fill the remaining three places with remaining 6 letters in $^6P_3$ ways

Required no. of words begin with vowels is $3 \times {}^6P_3 = 3 \times 6 \cdot 5 \cdot 4 = 360$.

ii, | x | | | x | First we fill the first and last places with vowels in $3P_2$ ways.

Now, we can refill the remaining 2 places with remaining 5 letters in $5P_2$ ways

∴ the number of words, starts and ends with vowels is

$$3P_2 \times 5P_2 = (3 \cdot 2)(5 \cdot 4) = 120.$$

iii, | | | | x | we can fill the last place with one of the four consonants in $4P_1 = 4$ ways

Remaining 3 places can be filled with remaining 6 letters in $6P_3$ ways

∴ No. of 4 letter words ends with consonant is

$$4 \times 6P_3 = 4 \times 6 \cdot 5 \cdot 4 = 480.$$

Ex-8:-

Find the number of ways of permutting the letters of the word PICTURE so that

i, All the vowels come together

ii, No two vowels come together

iii, The relative positions of vowels and consonants are not disturbed.

Sol: In the word PICTURE

Vowels are I, U, E (3 vowels)

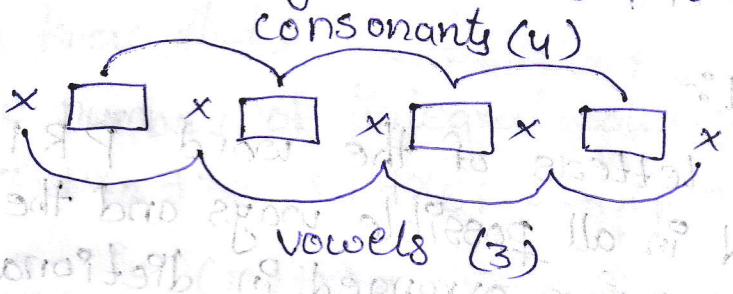
Consonants are P, C, T, R (4 consonants)

i) Let all the vowels are treat an unit then we can arrange 5 letters.

(4 consonants + unit) in 5! ways and In the unit 3 vowels can arrange then also in 3! ways. Number of permutations in which 3 vowels come together is $(5!)3! = 120 \times 6 = 720$

or

ii)



First arrange four consonants in four factorial ways.

Then, we have to arrange vowels in five places (Beginning, ending and between the consonants) in $^5P_3$ ways.

∴ Total Number of permutation in which no two vowels come together is

$4!\,^5P_3 = 24 \cdot \frac{6!}{2!} = 12(120) = 1440$.

	P	I	C	T	U	R	E
Vowels		I			U		E
Consonants	P		C	T		R	

3 vowels can be arranged in their relative positions in 3! ways.

4 consonants can be arranged in their relative Position in 4! ways.

Required number of required permutations is $3!\,4!$

$= 6 \times 24$

$= 144.$

Example-9:-

If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order Find the Rank of the word PRISON

Sol:- the letters of the given word in dictionary order is I N O P R S

In the dictionary order first we write all the words begin with capital I.

If we put I in first place the remainning five places can fill in 5! ways.

similarly we have to write the words begin with N the following table shows the Rank of the word prison.

*Note :-

$$\boxed{x_1 + x_2 + \cdots + x_n = r, \quad x_i \geq 0}$$

The no. of non negative integral solutions of the equation $x_1 + x_2 + x_3 + \cdots + x_n = r$ is

$c(n-1+r, r)$

where $c(n-1+r, r)$

$= (n-1+r)C_r$

Q How many integral solutions are there to

$x_1+x_2+x_3+x_4+x_5 = 12$, $x_i \geq 0$

Sol: we know that,

The number of integral solutions of the eqn $x_1+x_2+x_3+x_4+x_5+\dots+x_n = r$. each $x_i \geq 0$ is $(n-1+r)C_r$

The number of integral solution of $x_1+x_2+x_3+x_4+x_5 = 12$ is

$(n-r+1)C_r$ where $n=5, r=12$

$= (5-1+12)C_{12}$

$= {}^{16}C_{12}$ $\left[\because {}^nC_r = \frac{n!}{(n-r)!\,r!}\right]$

$= \frac{16!}{(16-12)!\,12!}$

$= \frac{16 \times 15 \cdot 14 \cdot 13 \cdot \cancel{(12 \cdots 1)}}{4!\,\cancel{(12 \cdot 11 \cdots 1)}}$

$= \frac{\overset{2}{\cancel{16}} \cdot \overset{5}{\cancel{15}} \cdot 14 \cdot 13}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$

$= 1820$

Q. How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where each $x_i \geq 2$

Sol: we have to find the no. of integral solutions of

$$x_1+x_2+x_3+x_4+x_5=20, \quad x_i \geq 2$$

This is same as the no. of integral solutions of $y_1+y_2+y_3+y_4+y_5=10$, each $y_i \geq 0$.

$\therefore$ No. of required integral solutions are

$C(n-1+r, r)$ where $n=5, r=10$

$= C(5-1+10, 10)$

$= C(14, 10) = {}^{14}C_{10}$ $\qquad {}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$

$$= \frac{14!}{(14-10)!\,10!} = \frac{14!}{4!\,10!}$$

$$= \frac{\cancel{14}^{7}\cdot 13\cdot \cancel{12}^{3}\cdot 11\cdot \cancel{(10)!}}{\cancel{4}\cdot 3\cdot 2\cdot 1\ \cancel{(10)!}}$$

$$= 7\cdot 13\cdot 11$$

$$= 1001.$$

Q. How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where

$x_1 \geq 3,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 6,\ x_5 \geq 0$

Sol: we have to find no. of integral solutions $x_1+x_2+x_3+x_4+x_5=20$ where

$x_1 \geq 3,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 6,\ x_5 \geq 0$

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 \geq 3 + 2 + 4 + 6 + 0$

$= 15$

This is same as the number of integral solution of

$y_1 + y_2 + y_3 + y_4 + y_5 = 20 - 15$

$= 5$

where each $y_i \geq 0$.

$\therefore$ The no. of required solutions are

$C(n-1+r, r)$ where $n = 5, r = 5$

$= C(5-1+5, 5)$

$= C(9, 5)$

$= {}^9C_5$

$= \dfrac{9!}{(9-5)!\,5!} = \dfrac{9!}{4!\,5!}$

$= \dfrac{9 \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6}\,(\cancel{5!})}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1\,(\cancel{5!})}$

$= 9 \cdot 2 \cdot 7$

$= 126.$

(2) How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where $x_1 \geq -3,\ x_2 \geq 0,\ x_3 \geq 4,\ x_4 \geq 2$, and $x_5 \geq 2$

Sol: We have to find no. of intergal solutions $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where

$x_1 \geq -3,\ x_2 \geq 0,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 2$

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 \geq -3 + 0 + 4 + 2 + 2 = 5$

This is same as

The no. of integral solutions of

$y_1 + y_2 + y_3 + y_4 + y_5 = 20 - 5$

$= 15$ where each $y_i \geq 0$

$\therefore$ The no. of required solutions are

$C(n-1+r, r)$ where $n = 5, r = 15$

$= C(5-1+15, 15)$

$= C(19, 15)$

$= {}^{19}C_{15}$

$= \dfrac{19!}{(19-15)!\,15!} = \dfrac{19!}{4!\,15!}$

$= \dfrac{19 \cdot \cancel{18}^{3} \cdot 17 \cdot \cancel{16}^{4}\,\cancel{(15)!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1\,\cancel{(15)!}}$

$= 19 \cdot 3 \cdot 17 \cdot 4$

$= 3876.$

Transitive closure :-

If R is any relation on a set A and M_R be the relation matrix then relation matrix of transitive closure of R is

$$M_R^* = M_R \vee M_R^{(2)} \vee M_R^{(3)} \vee \ldots \vee M_R^{(n)}$$

when $|A| = n$.

Q. Find the transitive closure of the relation represented by the matrix $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Sol: Given

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Transitive closure of R is R^* where

$$M_R^* = M_R \vee M_R^{(2)} \vee M_R^{(3)} \quad \text{—— (1)}$$

$$M_R^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R^{(3)} = M_R^{(2)} \cdot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

From ①

$$M_R^* = M_R \vee M_R^{(2)} \vee M_R^{(3)}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R^* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

UNIT-IV

Boolean Algebra

x	y	$f(x,y)=x\cdot y$	$f(x,y)=x+y$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Problem:-

Find the values of the Boolean function represented by $F(x,y) = x\bar{y}$.

Sol:-

x	y	$\bar{y}$	$F(x,y) = x\bar{y}$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

Problem 2:-

Find the value of the Boolean function represented by $F(x,y,z) = xy + \bar{z}$

Sol:- $F(x,y,z) = xy + \bar{z}$ is given Boolean function.

x	y	z	xy	$\bar{z}$	$f(x,y,z)=xy+\bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Q. Prove that the distributive law in Boolean algebra

$\therefore x(y+z) = xy+xz$

$\therefore$ ~~$x(y+z) = xy+xz$~~ $x+yz = (x+y)(x+z)$

Sol: Given

$x(y+z) = xy + xz$

x	y	z	xy	xz	$y+z$	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	0	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	0	0	0	0
0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0

~~Sum~~ From the table $x(y+z) = (xy)+(xz)$ have same value

$\therefore x(y+z) = xy+xz$

ii, Given

$x+yz = (x+y)(x+z)$

x	y	z	yz	$x+y$	$x+z$	$x+yz$	$(x+y)(x+z)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	1	0	0	0
0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0

From the table $x+yz$, $(x+y)\cdot(x+z)$ have same values

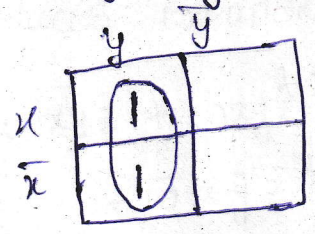
$\therefore x+yz = (x+y)(x+z)$

K-maps

Q, Simplify the sum of products (SOP) Expansions.

a, $xy+\bar{x}y$ b, $x\bar{y}+\bar{x}y$ c, $x\bar{y}+\bar{x}y+\bar{x}\bar{y}$

Sol: a, $xy+\bar{x}y$



$\therefore xy+\bar{x}y = y$

b, $x\bar{y} + \bar{x}y$

	y	$\bar{y}$
x		1
$\bar{x}$	1	

$\therefore \bar{x}y + x\bar{y}$ can't simplify.

c, $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$

	y	$\bar{y}$
x		1
$\bar{x}$	1	1

$\therefore x\bar{y} + \bar{x}y + \bar{x}\bar{y} = \bar{x} + \bar{y}$

Use K-map to minimise these sum of product expansion.

a, $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$

b, $x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

c, $xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

d, $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

sol: a, $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$	1		1	

$\therefore xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{y}\bar{z} + \bar{x}yz$

b) $x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x			1	1
$\bar{x}$	1		1	1

$\therefore x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = \bar{y} + \bar{x}z$

c) $xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	1		1	1

$\therefore xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = x + \bar{y} + yz$

d) $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$			1	1

$\therefore xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = x\bar{z} + \bar{x}\bar{y}$

Q) Use k-map to minimize the following ~~the~~ sum of products expansions.

a. $wxyz + wxy\bar{z} + wx\bar{y}\bar{z} + w\bar{x}yz + w\bar{x}\bar{y}z + w\bar{x}\bar{y}\bar{z}$
$+ \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}\bar{x}y\bar{z}$

Sol:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1	1	1	
$w\bar{x}$	1		1	1
$\bar{w}\bar{x}$	1	1		
$\bar{w}x$			1	

$\therefore$ Minimal expression is.

$wyz + \bar{w}\bar{x}y + wx\bar{z} + w\bar{x}\bar{y} + x\bar{y}\bar{z}$

b. $wx\bar{y}\bar{z} + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z}$

Sol:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx			1	
$w\bar{x}$	1	1	1	
$\bar{w}\bar{x}$		1	1	
$\bar{w}x$			1	

$\therefore$ Minimal Expression is

$\bar{x}\bar{z} + \bar{y}\bar{z} + w\bar{x}y$

c. $wxy\bar{z} + wx\bar{y}\bar{z} + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}xyz$
$+ \bar{w}xy\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}x\bar{y}z + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z}$

Sol:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx		1	1	
$w\bar{x}$	1	1	1	
$\bar{w}\bar{x}$		1	1	
$\bar{w}x$	1	1	1	1

$\therefore$ Minimal Expression is

$$w\bar{x}y + \bar{w}x + \bar{z}$$

Finite state Machine with outputs :-

A Finite state Machine

$M = (S, I, O, f, g, S_0)$ where

S: Finite set of states

I: Finite set of Input Alphabet

O: Finite set of output Alphabet

$f: S \times I \rightarrow S$ is transition function.

$g: S \times I \rightarrow O$ is output Function

S_0: Initial state.

Q, Construct the state diagram for the finite state machine with the following state table.

state	f Input 0	f Input 1	g Input 0	g Input 1
S_0	S_1	S_0	1	0
S_1	S_3	S_0	0	1
S_2	S_1	S_2	0	1
S_3	S_2	S_1	0	0

Sol:- Given finite state machine is

$$M = (S, I, O, f, g, S_0)$$

where, $S = \{S_0, S_1, S_2, S_3\}$ in set of states

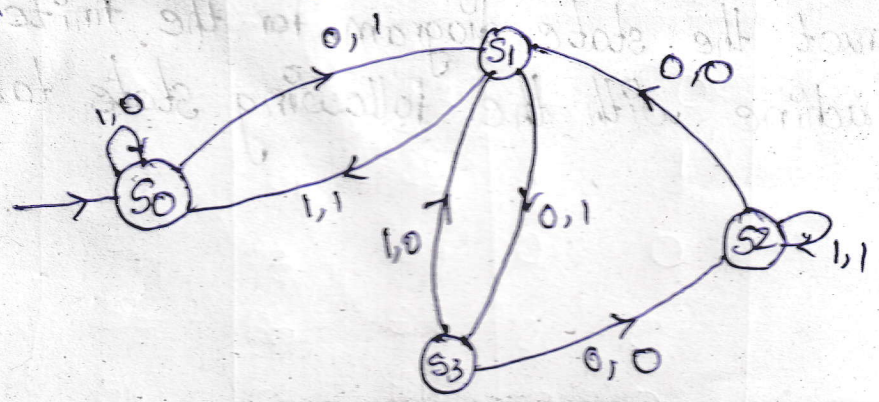
$I = \{0,1\}$ in set of input alphabet

$O = \{0,1\}$ in set of output alphabet

S_0 : Initial state

and $\left.\begin{array}{l} f: S \times I \rightarrow S \\ S \times I \rightarrow O \end{array}\right\}$ are given in the table.

$\therefore$ state diagram for given finite state machine is



Construct the state diagram for the finite state machine with following state table.

state	f Input 0	f Input 1	g Input 0	g Input 1
S_0	S_0	S_4	1	1
S_1	S_0	S_3	0	1
S_2	S_0	S_2	0	0
S_3	S_1	S_1	1	1
S_4	S_1	S_0	1	0

Given finite state machine is

$M = (S, I, O, f, g, S_0)$

where, $S = \{S_0, S_1, S_2, S_3, S_4\}$ in set of states

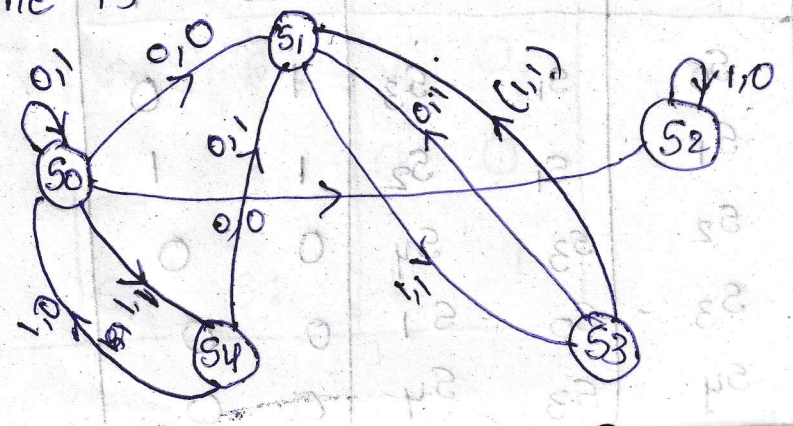
$I = \{0, 1\}$ in set of input alphabet

$O = \{0, 1\}$ in set of output alphabet

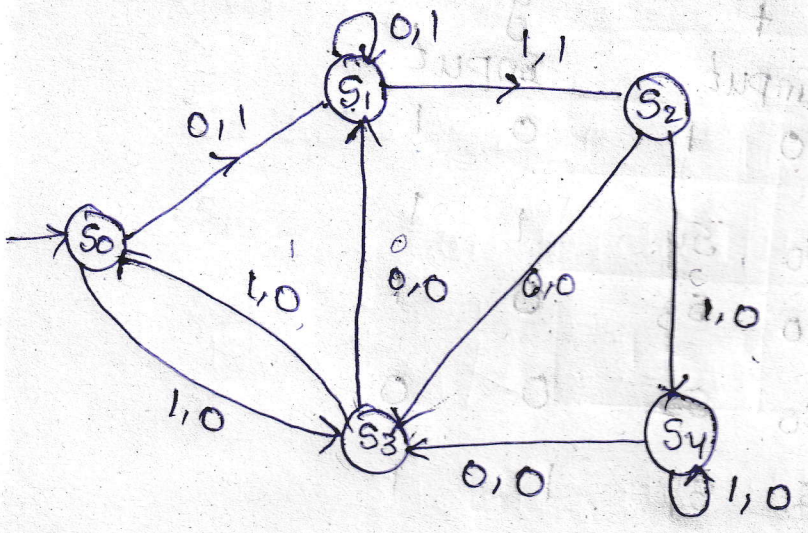
S_0 : Initial state

and $f : S \times I \rightarrow S$, $g : S \times I \rightarrow O$ are given in the table

∴ state diagram for given finite state machine is



Q, Construct the state table for the finite state machine with the state diagram.



Sol: Given finite state machine is

$$M = (S, I, O, f, g, S_0)$$

where $S = \{S_0, S_1, S_2, S_3, S_4\}$

$I = \{0, 1\}$

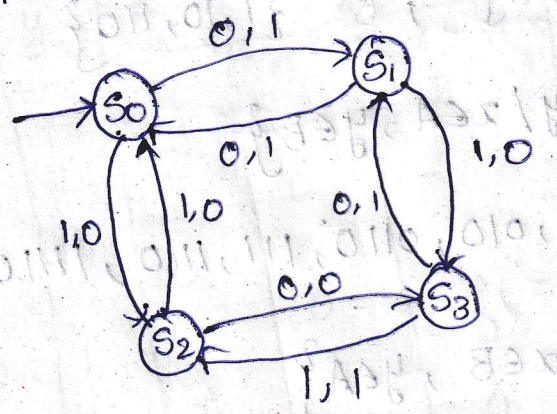
$O = \{0, 1\}$

$f : S \times I \rightarrow S$
$g : S \times I \rightarrow O$ } are in the following state table.

state table for M is

State	f		g	
	Input 0	Input 1	Input 0	Input 1
S_0	S_1	S_3	1	0
S_1	S_1	S_2	1	1
S_2	S_3	S_4	0	0
S_3	S_0	S_1	0	0
S_4	S_3	S_4	0	0

Q, Construct the state table for the finite state machine with the state diagram.



Sol:- Given finite state machine is

$M = \{S, I, O, f, g, S_0\}$

where $S = \{S_0, S_1, S_2, S_3\}$

$I = \{0, 1\}$

$O = \{0, 1\}$

$f: S \times I \rightarrow S$
$g: S \times I \rightarrow O$ } are in the following state table.

state table for M is

state	f Input 0	f Input 1	g Input 0	g Input 1
S_0	S_1	S_2	1	0
S_1	S_0	S_3	1	0
S_2	S_0	S_3	0	0
S_3	S_1	S_2	1	1

If $A = \{0, 11\}$ $B = \{1, 10, 110\}$ then find AB, BA

Given $A = \{0, 11\}$, $B = \{1, 10, 110\}$

$$AB = \{xy / x \in A, y \in B\}$$

$$= \{01, 010, 0110, 111, 1110, 11110\}$$

$$BA = \{xy / x \in B, y \in A\}$$

$$= \{10, 100, 1100, 111, 1011, 11011\}$$

Find the sum of product Expansion of

i) $F(x, y, z) = (x+y)\bar{z}$

Sol: Method - I :- $F(x, y, z) = (x+y)\bar{z}$

$$F(x,y,z) = (x+y)\bar{z}$$
$$= x\bar{z} + y\bar{z}$$
$$= x1\bar{z} + 1y\bar{z}$$
$$= x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z}$$
$$= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z}$$
$$= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}.$$

Method - II :-

x	y	z
1	1	1
1	1	0
1	0	1
1	0	0
0	1	
0	1	
0	0	
0	0	

From the
Expansi
F (x,y

ii) $F(x, y, z)$

Sol:

x	y
1	1
1	1
1	0
1	0
0	1
0	1
0	0
0	0

Method-II :- $F(x,y,z) = (x+y)\bar{z}$

x	y	z	$\bar{z}$	$x+y$	$F(x,y,z) = (x+y)\bar{z}$
1	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	1	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	1	0	0

From the Truth table sum of product Expansion of $F(x,y,z)$ is

$$F(x,y,z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}.$$

ii) $F(x,y,z) = x+y+z$.

Sol:-

x	y	z	$x+y$	$F(x,y,z) = x+y+z$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	1
0	0	1	0	1
0	0	0	0	0

From the truth table, the sum of product from of $F(x,y,z) = x+y+z$ is

$$F(x,y,z) = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z.$$

$\therefore$ $F(x,y,z) = (x+z)y.$

Prove that

$$xy + x'y' = (x'+y)(x+y')$$

Sol:-

$$xy + x'y' = (x + x'y')(y + x'y')$$

$$= (x+x')(x+y')(y+x')(y+y')$$

$$= 1(x+y')(x'+y)(1)$$

$$= (x+y')(x'+y)$$

$$xy + x'y' = (x'+y)(x+y')$$

Method-II

x	y	x'	y'	xy	$x'y'$	$x'+y$	$x+y'$	$xy+x'y'$	$(x'+y)(x+y')$
1	1	0	0	1	0	1	1	1	1
1	0	0	1	0	0	0	1	0	0
0	1	1	0	0	0	1	0	0	0
0	0	1	1	0	1	1	1	1	1