Math 171B Homework Assignment 1 Solutions

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1. (5 points) Let A be the following symmetric matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find an orthogonal matrix Q such that $Q^\top AQ = \Lambda$ is diagonal, what is Λ?

Solution: Λ is an diagonal matrix of the eigenvalues of A. Since the rank A is 1, A has two 0 eigenvalues. $\det(\lambda I_3 - A) = 0 = \lambda^2(\lambda - 3)$, so $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3$. We now calculate the eigenvectors, x, such that $Ax = \lambda x$. For $\lambda_1,\ \lambda_2 = 0$, $Ax = 0$, get two eigenvector $v_1 = \frac{1}{\sqrt{2}}[-1, 1, 0]^\top,\ v_2 = \frac{1}{\sqrt{6}}[-1, -1, 2]^\top$, and $v_3 = \frac{1}{\sqrt{3}}[1, 1, 1]^\top$. Let

$$Q = \begin{bmatrix} v_1 \ v_2 \ v_3 \end{bmatrix}$$

$Q^\top Q = I_3$, and $\Lambda = Diag(0, 0, 3)$.

2. (5 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $Q \in \mathbb{R}^{n\times n}$ be nonsingular. Show that if A is positive definite, then $Q^\top AQ$ is also positive definite.

Proof. To show that $Q^\top AQ$ is positive definite, we show that for all $x \neq 0$, $x^\top Q^\top AQx > 0$. We first note that if $x \neq 0$, then $y = Qx \neq 0$ since the null space of Q is 0, and so $x \notin Null(Q)$. So we have:

$$\begin{aligned} x^\top Q^\top AQx &= (Qx)^\top A(Qx) \\ &= y^\top Ay \\ &> 0 \end{aligned}$$

Since A is positive definite.

(Another method) A is positive definite, if and only if all the eigenvalues of A are positive. But $Q^\top AQ$ has the same eigenvalues as A, so all the eigenvalues of $Q^\top AQ$ are positive, thus $Q^\top AQ$ is positive definite. □

3. (10 points) Let $x, y \in \mathbb{R}^n$, show that

$$\|xy^\top\|_F = \|yx^\top\|_F = \|x\|_2\|y\|_2.$$

Proof. By the definition of Frobenius norm of matrix A,

$$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2} = \sqrt{\text{Trace}(A^\top A)}.$$

Using the conclusion Trace(AB) = Trace(BA), we have that

$$\|xy^\top\|_F = \sqrt{\text{Trace}(yx^\top xy^\top)} = \sqrt{\|x\|^2\|y\|^2} = \|x\|_2\|y\|_2.$$

Similarly,

$$\|yx^\top\|_F = \|x\|_2\|y\|_2.$$

□

4. (10 points) Let $A \in \mathbb{R}^{n\times n}$, show that

$$n^{-1/2}\|A\|_2 \leq \|A\|_\infty \leq n^{1/2}\|A\|_2$$

Proof. The definition of the induced matrix p-norm is

$$\|A\|_p = \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}. \quad p = 1, 2, \infty.$$

The vector p-norms have the property:

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

Here is a proof for $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$ (as this is all we will need here), Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $k = \arg\max_{1\leq i\leq n} |x_i|$

$$\begin{aligned}
\|x\|_2^2 &= \sum_{i=1}^{n} |x_i|^2 \\
&\leq \sum_{i=1}^{n} |x_k|^2 \\
&= n|x_k|^2 \\
&= n\|x\|_\infty^2
\end{aligned}$$

So $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$. The other inequality is easy:

$$\begin{aligned}
\|x\|_\infty^2 &= |x_k|^2 \\
&\leq \sum_{i=1}^{n} |x_i|^2 \\
&= n\|x\|_2^2
\end{aligned}$$

Using the above inequality, and the definition of the induced matrix norm, we obtain:

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x\neq 0} \frac{\sqrt{n}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty.$$

and:

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{x\neq 0} \frac{\sqrt{n}\|Ax\|_2}{\|x\|_2} = \sqrt{n}\|A\|_2.$$

The result follows.

□

5. (10 points) Let $C \in \mathbb{R}^{n\times n}$ be symmetric positive definite, define a function $\|\cdot\|_C$ on $\mathbb{R}^n$ as $\|x\|_C = \sqrt{x^\top Cx}$ for all $x \in \mathbb{R}^n$. Show that

(1) The function $\|\cdot\|_C$ is indeed a vector norm on $\mathbb{R}^n$;

Proof. C has Cholesky decomposition, i.e. there exists R upper-triangular matrix with $C = R^\top R$, so $\|x\|_C = \sqrt{x^\top Cx} = \sqrt{x^\top R^\top Rx} = \sqrt{(Rx)^\top (Rx)} = \|Rx\|_2$.

It's known that $2-norm$ is a norm on R^n, so it is easy to check $\|\cdot\|_C$ is a vector norm.

(I) $\|x\|_C = \|Rx\|_2 \geq 0$, and $\|x\|_C = 0$ if and only if $\|Rx\|_2 = 0$, so $Rx = 0$, as R is nonsingular, so $x = 0$, i.e. $\|x\|_C = 0$ if and only if $x = 0$;

(II) for any $\alpha \in \mathbb{R}$, $\|\alpha x\|_C = \|R(\alpha x)\|_2 = |\alpha|\|Rx\|_2 = |\alpha|\|x\|_C$.

(III) for any $x, y \in \mathbb{R}^n$, $\|x+y\|_C = \|R(x+y)\|_2 \leq \|Rx\|_2 + \|Ry\|_2 = \|x\|_C + \|y\|_C$. □

(2) The dual norm of $\|\cdot\|_C$ is $\|\cdot\|_{C^{-1}}$, that is $\|x\|_{C_*} = \|x\|_{C^{-1}}$.

Proof. It is known that p norm, the dual norm is q norm. So vector 2 norm, the dual is still 2 norm.

$\|x\|_C = \|Rx\|_2$. $C = R^T R$, $C^{-1} = (R^{-T})^T R^{-T}$. $\|x\|_{C^{-1}} = \|R^{-T}x\|_2$.

$\|x\|_{C_*} = \max_{v\neq 0} \frac{|x^\top v|}{\|v\|_C} = \max_{v\neq 0} \frac{|x^\top v|}{\|Rv\|_2}$. Let $z = Rv$,

$\|x\|_{C_*} = \max_{z\neq 0} \frac{|x^\top R^{-1}z|}{\|z\|_2} = \|R^{-T}x\|_{2*} = \|R^{-T}x\|_2 = \|x\|_{C^{-1}}$.

□

6. (10 points) For any $X = (X_{ij}) \in \mathbb{R}^{n\times n}$, define

$$\|X\|_{\max} = \max_{1\leq i,j\leq n} |X_{ij}|.$$

Show that $\|X\|_{\max}$ defined above is a matrix norm on $\mathbb{R}^{n\times n}$. Is it also an operator norm? If yes, give proof, if no, give a counterexample.

Proof. Check by definition.

(1) $\|X\|_{\max} \geq 0$ for any $X \in \mathbb{R}^{n\times n}$;

(2) $\|X\|_{\max} = 0$ iff $X_{ij} = 0$ for any i, j, i.e. $X = 0$;

(3) $\|\alpha X\|_{\max} = \max\limits_{1\leq i,j\leq n} |\alpha X_{ij}| = |\alpha|\|X\|_{\max}$.

$$\begin{aligned}
(4)\|X+Y\|_{\max} &= \max_{i,j} |X_{ij} + Y_{ij}| \\
&\leq \max_{i,j} |X_{ij}| + |Y_{ij}| \\
&\leq \max_{i,j} |X_{ij}| + \max_{i,j} |Y_{ij}| \\
&= \|X\|_{\max} + \|Y\|_{\max}.
\end{aligned}$$

An operator norm satisfies $\|XY\|_{\max} \leq \|X\|_{\max}\|Y\|_{\max}$.

This is not true. For example, $X = Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $\|X\|_{\max} = \|Y\|_{\max} = 1$, $\|XY\|_{\max} = 2$, and $2 > 1 \times 1 = 1$. So this matrix norm does not satisfy the last condition, hence, it is not an operator norm. $\square$