MA2223: SOLUTIONS TO ASSIGNMENT 4

1. Prove directly that the following three norms on $\mathbb{R}^2$ are equivalent.
   (a) $\|\mathbf{x}\|_1 = |x_1| + |x_2|$
   (b) $\|\mathbf{x}\|_2 = \sqrt{(x_1)^2 + (x_2)^2}$
   (c) $\|\mathbf{x}\|_\infty = \max_{i=1,2} |x_i|$

   where $\mathbf{x} = (x_1, x_2)$.

   Solution: First we will show that the 1-norm and the maximum norm are equivalent. We claim that for all points $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ we have

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq 2\|\mathbf{x}\|_\infty$$

   To show the first inequality note that

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|\} \leq |x_1| + |x_2| = \|\mathbf{x}\|_1$$

   For the second inequality we have

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| \leq \|\mathbf{x}\|_\infty + \|\mathbf{x}\|_\infty = 2\|\mathbf{x}\|_\infty$$

   Next we will show that the maximum norm and the Euclidean norm are equivalent. We claim that for all points $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ we have

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{2}\|\mathbf{x}\|_\infty$$

   To show the first inequality note that

$$|x_1|^2 = x_1^2 \leq x_1^2 + x_2^2 \implies |x_1| \leq \sqrt{x_1^2 + x_2^2}$$

$$|x_2|^2 = x_2^2 \leq x_1^2 + x_2^2 \implies |x_2| \leq \sqrt{x_1^2 + x_2^2}$$

   Combining these two observations gives

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|\} \leq \sqrt{x_1^2 + x_2^2} = \|\mathbf{x}\|_2$$

For the second inequality note that

$$(x_1)^2 = |x_1|^2 \leq \|\mathbf{x}\|_\infty^2$$

$$(x_2)^2 = |x_2|^2 \leq \|\mathbf{x}\|_\infty^2$$

Combining these two observations gives

$$(x_1)^2 + (x_2)^2 \leq \|\mathbf{x}\|_\infty^2 + \|\mathbf{x}\|_\infty^2 = 2\|\mathbf{x}\|_\infty^2$$

We have

$$\|\mathbf{x}\|_2 = \sqrt{(x_1)^2 + (x_2)^2} \leq \sqrt{2}\|\mathbf{x}\|_\infty$$

Note that equivalence of norms is transitive so now we automatically have the Euclidean norm is equivalent to the 1-norm. To see this directly our claim is that

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq 2\|\mathbf{x}\|_2$$

To show the first inequality note that

$$(x_1)^2 + (x_2)^2 = |x_1|^2 + |x_2|^2 \leq |x_1|^2 + 2|x_1||x_2| + |x_2|^2 = (|x_1| + |x_2|)^2$$

$$\implies \|\mathbf{x}\|_2 = \sqrt{(x_1)^2 + (x_2)^2} \leq |x_1| + |x_2| = \|\mathbf{x}\|_1$$

For the second inequality note that

$$|x_1|^2 = (x_1)^2 \leq (x_1)^2 + (x_2)^2 \implies |x_1| \leq \sqrt{(x_1)^2 + (x_2)^2} = \|\mathbf{x}\|_2$$

$$|x_2|^2 = (x_2)^2 \leq (x_1)^2 + (x_2)^2 \implies |x_2| \leq \sqrt{(x_1)^2 + (x_2)^2} = \|\mathbf{x}\|_2$$

Combining these two observations gives

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| \leq \|\mathbf{x}\|_2 + \|\mathbf{x}\|_2 = 2\|\mathbf{x}\|_2$$

2. Compute the Frobenius norm, 1-norm, maximum norm and spectral norm of the following matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Solution:

(a)

$$A^tA = \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 13 & -10 \\ -10 & 17 \end{pmatrix}$$

The Frobenius norm is

$$\|A\|_F = \sqrt{trace(A^tA)} = \sqrt{30}$$

The 1-norm is

$$\|A\|_1 = \max\{3, 7\} = 7$$

The maximum norm is

$$\|A\|_\infty = \max\{5, 5\} = 5$$

The characteristic polynomial of A^tA is

$$(13 - \lambda)(17 - \lambda) - 100 = 0$$

$$\implies \lambda^2 - 30\lambda + 121 = 0$$

$$\implies \lambda = \frac{30 \pm \sqrt{900 - 484}}{2} = 15 \pm 2\sqrt{26}$$

The spectral norm is

$$\begin{aligned}\|A\|_2 &= \sqrt{\text{max eigenvalue of } A^tA} \\ &= \sqrt{15+2\sqrt{26}} \\ &\simeq 5.02\end{aligned}$$

(b)

$$B^tB = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

The Frobenius norm is

$$\|B\|_F = \sqrt{trace(B^tB)} = \sqrt{13}$$

The 1-norm is

$$\|B\|_1 = \max\{2,3,0\} = 3$$

The maximum norm is

$$\|B\|_\infty = \max\{0,2,3\} = 3$$

The spectral norm is

$$\begin{aligned}\|B\|_2 &= \sqrt{\text{max eigenvalue of } B^tB} \\ &= 3\end{aligned}$$

(c)

$$
\begin{aligned}
C^tC &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

The Frobenius norm is

$$\|C\|_F = \sqrt{trace(C^tC)} = 4$$

The 1-norm is

$$\|C\|_1 = \max\{4, 4, 4, 4\} = 4$$

The maximum norm is

$$\|C\|_\infty = \max\{4, 4, 4, 4\} = 4$$

The spectral norm is

$$
\begin{aligned}
\|C\|_2 &= \sqrt{\text{max eigenvalue of } C^tC} \\
&= 2
\end{aligned}
$$