Linear Programming Problems Summary Report

MIT 4203 – Assignment 1

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ABSTRACT

This report summarizes the Simplex Method, Transportation Problem and Assignment Problems. All three topics are touched over without diving into complete examples, but rather highlighting the key points and concepts used in each. Mathematical models are presented for all topics. Examples are discussed where necessary.

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1. Simplex Method

The simplex method is used to solve Linear Programming problems where there are more than 2 decision variables. It produces feasible solutions within the feasible range, which may or may not be optimal. The algorithm for the Simplex Method is as follows.

1. Locate an extreme point of the feasible region.

2. Examine each edge for values which increase / decrease the objective function (depending on whether the problem is maximization or minimization).

3. If a value is found to yield desirable outcomes of the objective function, move along the edge to find an extreme point yielding an optimal solution.

4. Repeat this process until no other desirable outcomes for the objective function are found by 'walking' across edges.

The algorithm above is expressed in terms of a Mathematical model, where the edges represent gradients between extreme points and the extreme points themselves are represented as vertices of a polygon. In practice, the simplex method (simplex algorithm) is carried out by using a table notation named, 'Simplex Tableau' (matrix notation).

1.1. Simplex Tableau

The column headings for the Simplex Tableau in its tabular form is shown below in Table 1.

		Coefficients							
Equation Number	Basic Variable	Z	x1	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable

Table 1 - Simplex Tableau Structure

Definitions of the columns in Table 1 are provided below.

- Equation Number – The ID number of the LP Equation populating this row.
- Basic Variable – Variable with a coefficient of 1.
- Coefficients – Coefficients of each variable in the equation.
- Right Hand Side – Equality or inequality value of the equation.
- Upper bound on Entering Variable – Used for sensitivity analysis and is calculated in Tableau iterations as follows.

$$Upper\ Bound = \frac{(R.H.S.)}{Smallest\ Coefficient\ for\ that\ Row}$$

- Entering Variable – Variable of the column, containing the most negative Coefficient in the entire table.
- Leaving Variable –Variable of the row, containing the smallest upper bound on Entering Variable.

1.1.1. Process Overview

The process of solving a Simplex Problem using the Simplex Tableau method can be summarized as follows.

1. Develop a Linear Programming Model for the problem.
2. Ensure that the LP equations are in Standard form.
3. Assign the initial coefficients.

Iteratively,

4. Identify the Basic Variable for each Row.
    - The R.H.S. for each Basic Variable (except Z) yields a solution for the LP problem.
    - Continue to Step 5, if any of the coefficients in the Objective Function are negative.
5. Identify the column with the Entering Variable.
6. Identify the row with the Leaving Variable.
7. Perform row operations to generate the Tableau iteration.
8. Repeat from Step 4 until all coefficients in the Objective Function are positive or, until a further optimal solution cannot be obtained.

1.1.2. Row Operations

The Simplex Method's process outline mentions the task of "Row Operations" in Step 7. The Row operations allow us to generate the next Tableau iteration from the current set of values.

To capture the row operations in a clear manner, an example will be used. Afterwards, the Row Operations involved are summarized as steps.

Initial Tableau

		Coefficients							
Equation Number	Basic Variable	Z	**x1**	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable
A1	Z	1	-13	-11	0	0	0	0	N/A
B1	S1	0	4	5	1	0	0	1500	375
C1	**S2**	0	**5**	3	0	1	0	1575	315
D1	S3	0	1	2	0	0	1	420	420

Table 2 - Row Operations Initial Tableau

In the above example, the Entering and Leaving variables are as follows.

- Entering Variable – x1 (red)
- Leaving Variable – S2 (blue)

Operation 1

Find the intersection of the Entering Variable Column and Leaving Variable Row. This is marked in purple color (in Table 2). Divide all coefficients and the RHS in the Leaving Variable Row by the highlighted value. This will yield the table values depicted in Table 3. The affected rows are highlighted in pale blue color.

		Coefficients							
Equation Number	Basic Variable	Z	**x1**	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable
A1	Z	1	-13	-11	0	0	0	0	N/A
B1	S1	0	4	5	1	0	0	1500	375
C2	x1	0	1	3/5	0	1/5	0	315	
D1	S3	0	1	2	0	0	1	420	420

Table 3 – Row Operations First Intermediate

Operation 2

The next set of operations are obtained using the following method.

1. Find the cell that intersects with the Entering Variable's Column.
2. Obtain the inverse of that cell value.
3. Multiply all coefficients & the RHS in the C2 row by that value
4. Add the corresponding result of each column, to the original coefficients in that row.

Considering Row A1, the cell that intersects with the Entering Variable Column is in the second column. Its value is -13, and the inverse is 13. The corresponding next set of values would be as shown in Table 4 below. The resulting intermediate table is Table 5.

		Z	x1	x2	S1	S2	S3	RHS
Step 1	**C2**	0	1	3/5	0	1/5	0	315
Step 2	**A1' = C2 * 13**	0	13	39/5	0	13/5	0	4095
Step 3	**A1**	1	-13	-11	0	0	0	0
Step 4	**A2 = A1' + A1**	1	0	$\frac{39}{5} + (-11)$	0	13/5	0	4095
	A2	1	0	-16/5	0	13/5	0	4095

Table 4 – Second Intermediate Operation Calculation Steps

		Coefficients							
Equation Number	Basic Variable	Z	**x1**	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable
A2	Z	1	0	-16/5	0	13/5	0	4095	
B1	S1	0	4	5	1	0	0	1500	375
C2	x1	0	1	3/5	0	1/5	0	315	
D1	S3	0	1	2	0	0	1	420	420

Table 5 - Row Operation Second Intermediate

There are 2 remaining operations left that calculate the values for the B1 and D1 rows as shown in Table 5. The steps for these two operations are similar to that of Operation 2. The differences are,

- The row that will be mutated
- The value used for multiplication

Operation 3

Considering Row B1, the cell intersecting with the Entering Variable column is the second column. Its value is 4, and the inverse is -4. Similar to Operation 2, first the values of row C2 must be multiplied with 4. The resulting values must be added to the original values in Row B1. The table consisting of these values are show in **Error! Reference source not found.**.

		Coefficients							
Equation Number	Basic Variable	Z	**x1**	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable
A2	Z	1	0	-16/5	0	13/5	0	4095	
B2	S1	0	0	13/5	1	-4/5	0	240	
C2	x1	0	1	3/5	0	1/5	0	315	
D1	S3	0	1	2	0	0	1	420	420

Table 6 - Row Operations Third Intermediate

Operation 4

Considering Row D1, the cell intersecting with the Entering Variable column is the second column. Its value is 1, and the inverse is -1. Similar to Operation 3, first the values of row C2 must be multiplied by 1. The resulting values must be added to the original values in Row D1. The table consisting of these values are show in Table 7.

		Coefficients							
Equation Number	Basic Variable	Z	**x1**	x2	S1	S2	S3	Right Hand Side	Upper bound on Entering Variable
A2	Z	1	0	-16/5	0	13/5	0	4095	
B2	S1	0	0	13/5	1	-4/5	0	240	
C2	x1	0	1	3/5	0	1/5	0	315	
D2	S3	0	0	7/5	0	-1/5	1	105	

Table 7 - Row Operation Fourth Intermediate

2. Transportation Problem

The Transportation Problem describes a Minimizing problem where commodities must be physically transported from its manufacturing facilities to the consumer outlets. Notice the plurals in the previous sentence.

In the Transportation Problem, there are no pre-existing constraints as to which outlet sells commodities from which facility. The goal of the problem is to provide this information, with the additional requirement of minimizing the cost of transportation.

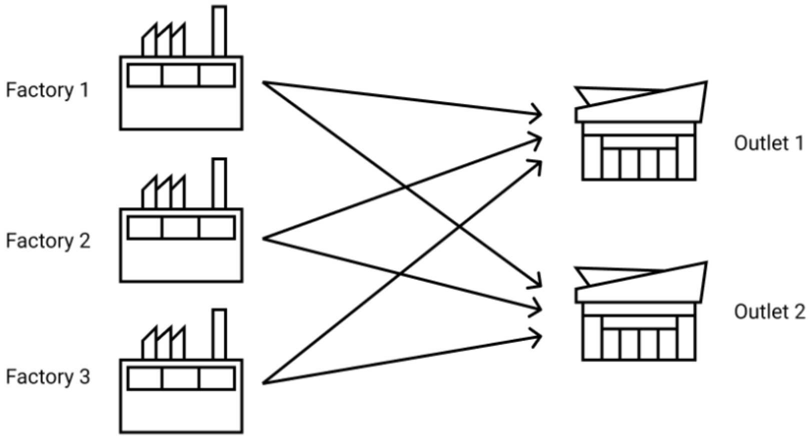

Figure 1 - Transportation Problem Example

Consider the above example diagram shown in Figure 1. On the left side, there are 3 factories which produce the required commodity (repr. by m). On the right, there are 2 outlets which will be selling the commodity (repr. by n).

Each factory i has a maximum production capacity, a_i. Similarly, each outlet j has a minimum demand b_j that must be met. We are also provided with the transportation costs between each unique factory & outlet combination as, c_{ij}.

Using this information, we must calculate all product (commodity) quantities where each factory i produces and outlet j consumes x_{ij} commodities.

The above description can be modelled mathematically as follows.

$$Total\ Cost = Z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}x_{ij}$$

Find $Min\ Z$ subject to,

$$a_i \geq \sum_{j=1}^{m} x_{ij} \quad , \quad b_j \leq \sum_{i=1}^{n} x_{ij} \quad , \quad x_{ij} \geq 0$$

2.1. Assumptions

The Transportation Problem attempts to solve a recurring issue with efficient commodity logistics. However, it does not represent real world scenarios. This is due to the following ideal situation assumptions made when attempting a solution.

1. Only one type of product (commodity) is being distributed amongst all Factories & Outlets.
2. There are no slack or surplus quantities of items left. Total factory supply meets total outlet demand.

$$Supply \equiv Demand\,;\ \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$$

3. Transport costs between factories & outlets are known in advance. They must be fixed values and not probable ranges.

2.3 Cost Matrix

Before attempting to solve the Transportation Problem, the details are summarized in a Cost Matrix (table). An example Cost Matrix is shown below in Table 8.

		Destinations		Maximum Supplies
		O1	O2	
Sources	S1	3	4	7
	S2	6	5	11
	S3	7	8	15
Minimum Demand		16	17	33

Table 8 - Example Transportation Problem Cost Matrix

It is possible that the Total Supply / Demand is not equal in which case the problem is classified as an Un-Balanced Transportation Problem (UBTP). As necessary, dummy sources / destinations must be added to the problem to provide / absorb the unbalanced item quantities.

2.4. Solving Methods

Transportation problems are solved in two phases. In Phase 1, an Initial Feasible Solution is obtained by one of the following three methods.

1. North-West corner Method
2. Least cost cell Method
3. Vogel's Approximation Method

It is possible that the three methods yield different initial solutions. Any solution is accepted as it is the baseline requirement for Phase 2. Once a feasible solution is obtained, this solution is further optimized in Phase 2. The result of this phase is the Optimal Solution. The Optimal Solution can be obtained by one of the following two methods.

1. UV Method (U – Rows, V – Columns)
2. Stepping Stone Method

Phase 1 methods are different with respect to the steps involved. However, the principle is the same. All methods present a cell selection method, that is executed repeatedly until the supply from suppliers are spread across the demand requirement in such a way, that all supplies are used (equal to 0) and all demands are *completely* met.

Phase 2 methods use the principle of least costly assignment. The UV Method and Stepping Stone method are substantially different in how they are executed. However, both methods iteratively check whether each current assignment can be further improved by swapping / reassigning them among other cells in a way that reduces the objective function's value.

3. Assignment Problem

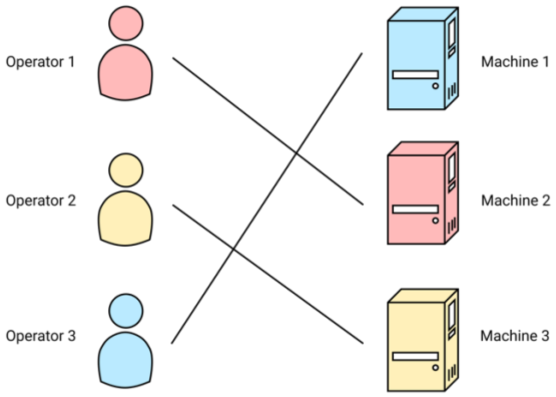

Figure 2 - Assignment Problem Example Diagram

Assignment Problem is the Linear Programming problem that deals with building 1-to-1 relationships between the source and destination items. This is in contrast to the Transportation Problem which deals with many-to-many associations.

The decision variables state whether an item is assigned or not, so it will be either 0 or 1. The decision variables are multiplied with their associated costs which make up the total cost (value of the objective function).

The "cost" in the Assignment Problem can have meanings such as Required Time, Labor Cost, Exam Pass Rate, etc. The cost is a direct result of choosing to assign two items together. Optimizing this assignment where the total cost is minimized.

3.1. Mathematical Model

Unlike the Transportation Problem where the number of Factors & Outlets may be independent, the Assignment Problem requires the number of items on either side to be exactly equal. Further, one time may be assigned with only one other item. This is also depicted in Figure 2 above.

There are m operators and machines. The labor cost of each Operator i using Machine j per hour is denoted as $c_{ij.}$ Whether or not the Operator i is assigned to Machine j is denoted as $x_{ij.}$ The Objective Function Z can be written as follows.

$$Z = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\, x_{ij} \qquad \text{find } Min\ Z \text{ subject to,} \qquad \sum_{j=1}^{m} x_{ij} = 1 \quad , \quad \sum_{i=1}^{m} x_{ij} = 1 \quad , \quad x_{ij} \geq 0$$

3.2. The Hungarian Method

The Hungarian Method consists of 2 Phases.

- Phase 1 – Find Initial Basic Feasible Solution
- Phase 2 – Optimize the Basic Solution iteratively

Phase 1 is based on the principle of Row & Column Reduction. Further, Phase 1 uses a matrix where intersections provide Cost (c_{ij}) values and not Assignment (x_{ij}), values. Reduction happens by subtracting the smallest value among the row / column by all values of that row / column, as shown in Figure 3.

		Operator			
		O1	O2	O3	Min
Machine	M1	3	4	5	3
	M2	8	2	6	2
	M3	4	2	3	2

➔

		Operator		
		O1	O2	O3
Machine	M1	0	1	2
	M2	6	0	4
	M3	2	0	1
	Min	0	0	1

➔

		Operator		
		O1	O2	O3
Machine	M1	0	1	1
	M2	6	0	2
	M3	2	0	0

Figure 3 - Hungarian Method Phase 1 Example

Once the initial feasible solution has been obtained, Phase 2 can commence. Phase 2 mainly comprises of the following steps.

1. Row scanning – Find rows with only 1 zero, mark the "0" in that row, and cross off the column perpendicular to that cell (don't consider already crossed cells).
2. Column scanning – Find columns with only 1 zero, mark the "0" in that row, and cross off the row perpendicular to that cell (don't consider already crossed cells).
3. If the number of marked "zeroes" is equal to number of rows, the marked zeroes indicate the optimal assignment of Operators and Machines yielding the minimum cost.
4. Otherwise, create a new matrix from the current cost matrix using the following rules.
    a. Find the minimum value that is not "crossed-off".
    b. Copy values that are crossed but are not at cross intersections.
    c. Copy values on cross intersections, by adding min value in step 4a.
    d. Copy all other values, by subtracting min value in step 4a.
5. Repeat from step 1.

An example of the Matrix Copy process mentioned in Step 4 is shown below (Figure 4).

Step A

		Operator				
		O1	O2	O3	O4	O5
Machine	M1	2	4	7	2	0
	M2	0	9	7	5	4
	M3	6	7	0	0	2
	M4	2	0	1	1	0
	M5	0	5	7	1	7

Red	Column Scanning
Blue	Row Scanning
Green	Intersections
Purple	Min. value

Step B

		Operator				
		O1	O2	O3	O4	O5
Machine	M1	2	4			0
	M2	0	9			4
	M3			0	0	
	M4	2	0			0
	M5	0	5			7

Copying crossed, but non-intersection values

Step C

		Operator				
		O1	O2	O3	O4	O5
Machine	M1	2	4			0
	M2	0	9			4
	M3	7	8	0	0	3
	M4	2	0			0
	M5	0	5			7

Copying intersections and adding

Step D

		Operator				
		O1	O2	O3	O4	O5
Machine	M1	2	4	6	1	0
	M2	0	9	6	4	4
	M3	7	8	0	0	3
	M4	2	0	0	0	0
	M5	0	5	6	0	7

Copying remaining values and subtracting

Figure 4 - Hungarian Method Matrix Formation Steps Example