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CS5300 Advanced Algorithms HW # 3

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1. Consider the following recurrence equation, defining $T(n)$, as

$$T(n) = \begin{cases} 2 & \text{if } n=1 \\ T(n-1)+2 & \text{Otherwise} \end{cases}$$

Show, by induction, that $T(n) = 2n$

Sol:

1. Basis:

if $n=1$, then $T(1) = 2n = 2(1) = 2$

2. Inductive Hypothenis: Assume $T(K) = 2K \ \forall \ K \leq n$

In particular, Let $K = n-1$

i.e., $\boxed{T(n-1) = 2(n-1)}$

3. Show $T(n) = 2n \ \forall \ n$

$$\begin{aligned} T(n) &= T(n-1)+2 \\ &= 2(n-1)+2 \\ &= 2n-2+2 \\ &= 2n \end{aligned}$$

Hence, $T(n) = 2n \ \forall \ n$

2. Consider the following recurrence equation, defining $T(n)$, as

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-1)+n^2 & \text{otherwise} \end{cases}$$

Show, by induction, that $T(n) = \frac{n(n+1)(2n+1)}{6}$

Sol: 1. Basis:

If $n=1$, then $T(1) = \frac{1(1+1)(2(1)+1)}{6}$

$$T(1) = \frac{2(3)}{6} = \frac{6}{6} = 1$$

2. Inductive Hypothesis:

Assume $T(K) = \frac{K(K+1)(2K+1)}{6} \quad \forall \quad K \leq n$

In particular, Let $K=n-1$

i.e, $T(n-1) = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$

$$= \frac{(n-1)n(2n-2+1)}{6}$$

$$\boxed{T(n-1) = \frac{n(n-1)(2n-1)}{6}}$$

3. Show, $T(n) = \frac{n(n+1)(2n+1)}{6}$

$$T(n) = T(n-1) + n^2$$

$$= \frac{n(n-1)(2n-1)}{6} + n^2$$

$$= \frac{n(n-1)(2n-1)}{6} + \frac{6}{6} n^2$$

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$$= \frac{n(n-1)(2n-1) + 6n^2}{6}$$

$$= \frac{(n^2-n)(2n-1) + 6n^2}{6}$$

$$= \frac{2n^3 - 3n^2 + n + 6n^2}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(2n^2+3n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$2n^2+3n+1$

$2n^2+2n+n+1$

$2n(n+1)+1(n+1)$

$(n+1)(2n+1)$

Hence, $T(n) = \dfrac{n(n+1)(2n+1)}{6} \quad \forall\ n$

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3. Show that the Solution for $T(n) = T(\sqrt{n}) + \Theta(1)$ is $T(n) = \Theta(\lg \lg (n))$

$$T(n) = T(\sqrt{n}) + \Theta(1)$$

Assume $n = 2^K \Rightarrow K = \log_2^n$

$$T(2^K) = T(2^{K/2}) + \Theta(1) \text{ ——— } ①$$

Let's Assume $P(K) = T(2^K)$ ② Then $P(\frac{K}{2}) = T(2^{K/2})$ ——— ③

Substituting Equations ② and ③ in ①

$$P(K) = P(\frac{K}{2}) + \Theta(1)$$

The above equation is in the form of

$$T(n) = aT(\frac{n}{b}) + f(n) \text{ Where } a \geq 1, b > 1$$

Master Method:

$a = 1$, $b = 2$, $f(n) = 1$ $\qquad$ $f(K) = 1$

$$n^{\log_b^a} \text{ vs } f(n)$$

$$n^{\log_2^1} \text{ vs } 1$$

$$n^0 \text{ vs } 1$$

$$1 \text{ vs } 1$$

Case 2:

$$T(n) = \Theta(f(n) * \lg n)$$

We have $n = 2^K$ and $K = \log_2^n$

$$T(n) = \Theta(f(K) * \lg K)$$

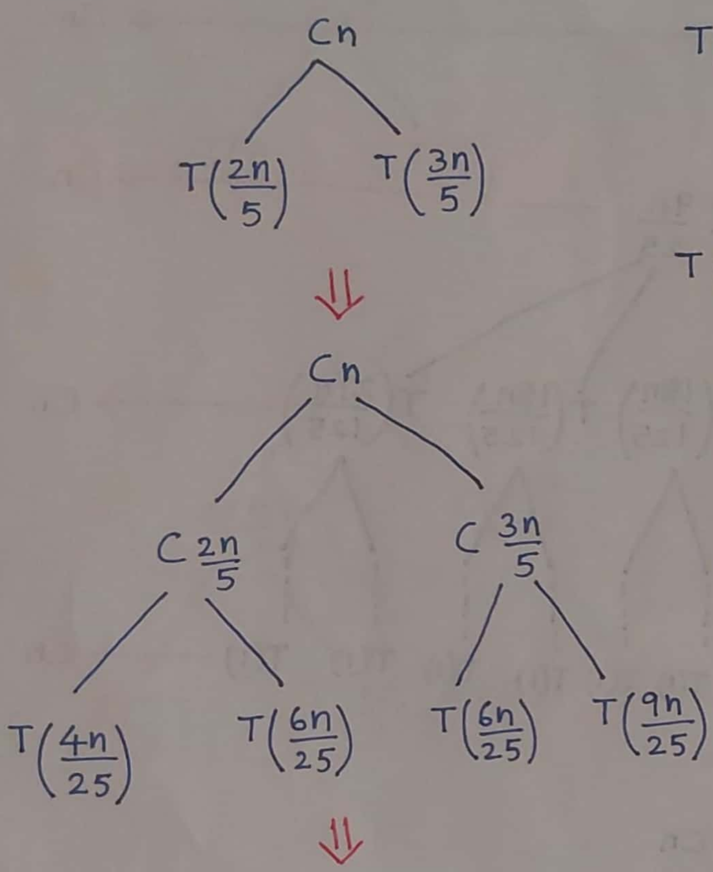
$$T(n) = \Theta(1 * \lg K)$$

$T(n) = \Theta(\lg \log_2^n)$

$\boxed{T(n) = \Theta(\lg \lg(n))}$

Hence, $T(n) = T(\sqrt{n}) + \Theta(1)$ is $T(n) = \Theta(\lg \lg(n))$

4. Draw the recursion tree for $T(n) = T\left(\frac{2n}{5}\right) + T\left(\frac{3n}{5}\right) + \theta(n)$ and find the height of the tree, then generate the guess for the solution.



$$T\left(\frac{2n}{5}\right) = T\left(\frac{2\left(\frac{2n}{5}\right)}{5}\right) + T\left(\frac{3\left(\frac{2n}{5}\right)}{5}\right) + \theta\left(\frac{2n}{5}\right)$$

$$= T\left(\frac{4n}{25}\right) + T\left(\frac{6n}{25}\right) + \theta\left(\frac{2n}{5}\right)$$

$$T\left(\frac{3n}{5}\right) = T\left(\frac{2\left(\frac{3n}{5}\right)}{5}\right) + T\left(\frac{3\left(\frac{3n}{5}\right)}{5}\right) + \theta\left(\frac{3n}{5}\right)$$

$$= T\left(\frac{6n}{25}\right) + T\left(\frac{9n}{25}\right) + \theta\left(\frac{3n}{5}\right)$$

$$T\left(\frac{4n}{25}\right) = T\left(\frac{2\left(\frac{4n}{25}\right)}{5}\right) + T\left(\frac{3\left(\frac{4n}{25}\right)}{5}\right) + \theta\left(\frac{4n}{25}\right)$$

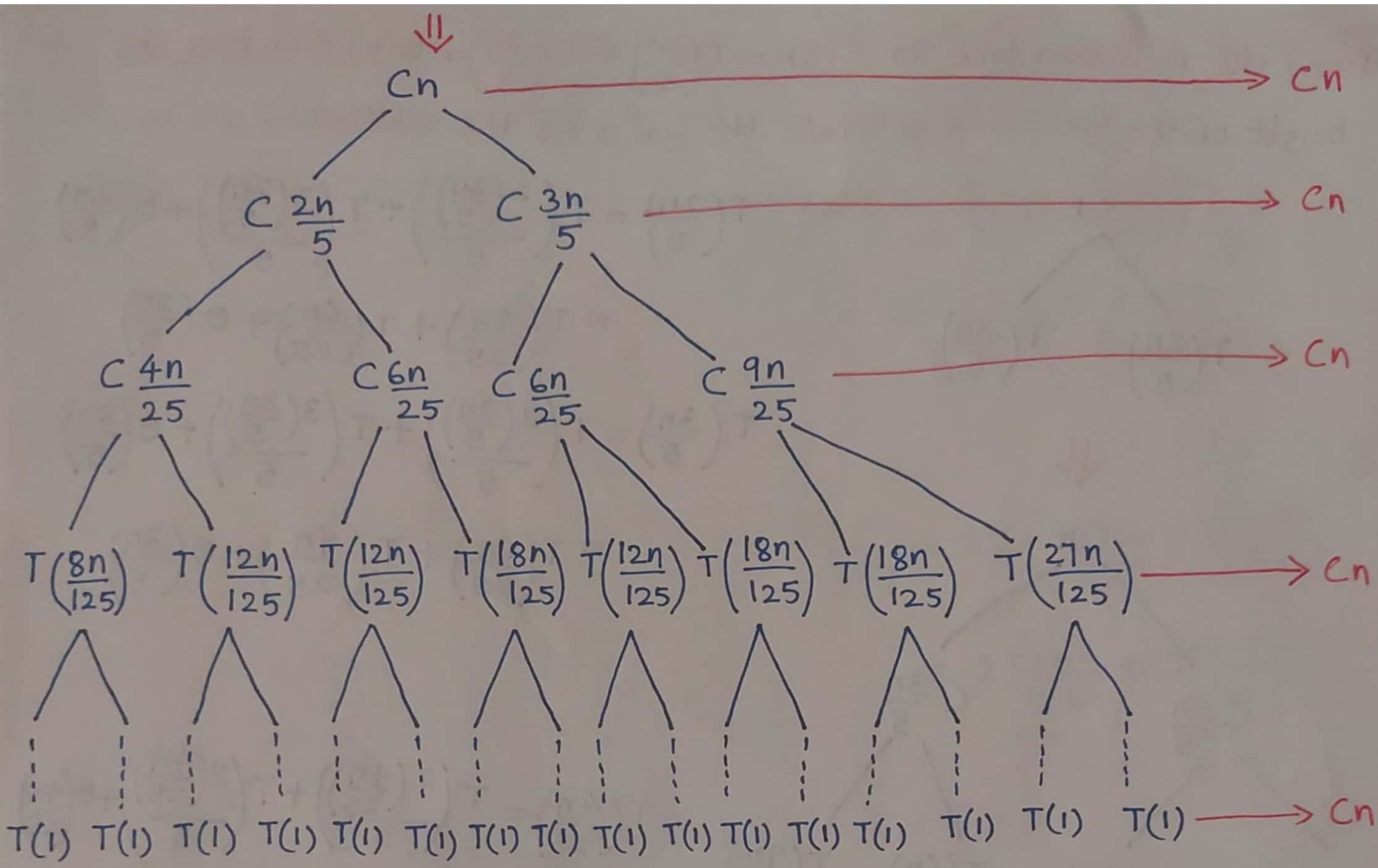
$$= T\left(\frac{8n}{125}\right) + T\left(\frac{12n}{125}\right) + \theta\left(\frac{4n}{25}\right)$$

$$T\left(\frac{6n}{25}\right) = T\left(\frac{2\left(\frac{6n}{25}\right)}{5}\right) + T\left(\frac{3\left(\frac{6n}{25}\right)}{5}\right) + \theta\left(\frac{6n}{25}\right)$$

$$= T\left(\frac{12n}{125}\right) + T\left(\frac{18n}{125}\right) + \theta\left(\frac{6n}{25}\right)$$

$$T\left(\frac{9n}{25}\right) = T\left(\frac{2\left(\frac{9n}{25}\right)}{5}\right) + T\left(\frac{3\left(\frac{9n}{25}\right)}{5}\right) + \theta\left(\frac{9n}{25}\right)$$

$$= T\left(\frac{18n}{125}\right) + T\left(\frac{27n}{125}\right) + \theta\left(\frac{9n}{25}\right)$$



Recursion Tree

Guess: $T(n) = \underbrace{Cn + Cn + Cn + \cdots + Cn}_{\text{Height of the Tree}}$

$$= Cn * \text{Height} = Cn * \log_{5/3}^{n} = \Theta(n \lg n)$$

Height: $n \rightarrow \frac{3}{5}n \rightarrow \left(\frac{3}{5}\right)^2 n \rightarrow \cdots\cdots \rightarrow \left(\frac{3}{5}\right)^K n = 1$, Where K is the height

Solve $\left(\frac{3}{5}\right)^K n = 1$ for $K \Rightarrow \left(\frac{3}{5}\right)^K = \frac{1}{n}$

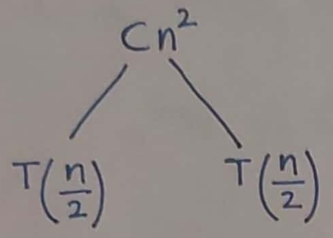
$$\Rightarrow \left(\frac{5}{3}\right)^K = n$$

$$\Rightarrow \log_{\frac{5}{3}}^{\left(\frac{5}{3}\right)^K} = \log_{\frac{5}{3}}^{n}$$

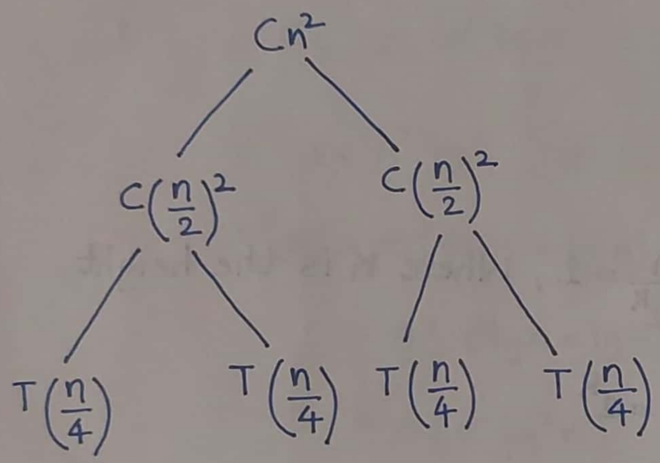
$$\boxed{K = \log_{\frac{5}{3}}^{n}}$$

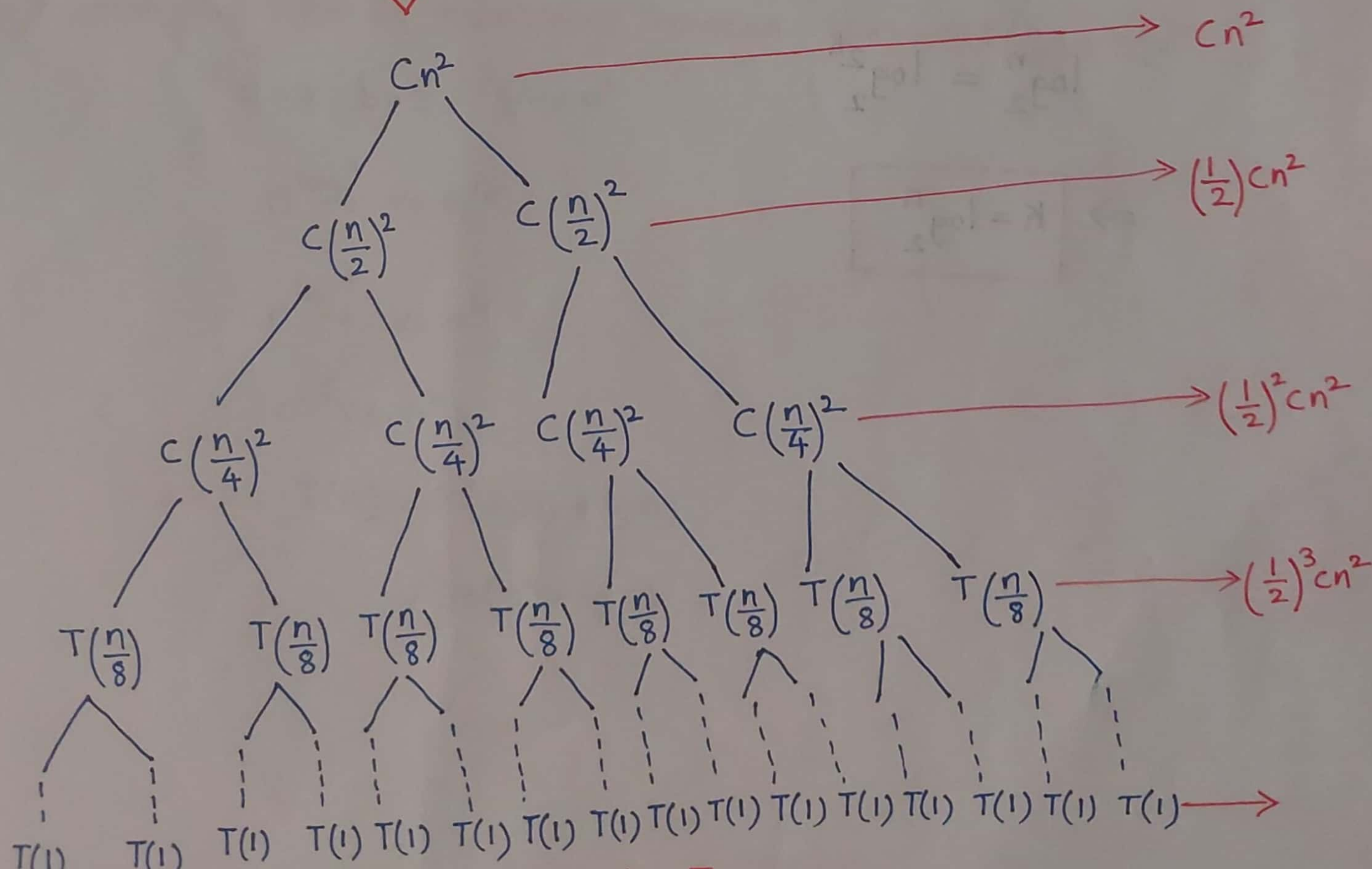
5. Draw the recursion tree for $T(n) = 2T\left(\frac{n}{2}\right) + \theta(n^2)$ and find the height of the tree, then generate the guess for the solution.

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n/2}{2}\right) + \theta\left(\left(\frac{n}{2}\right)^2\right)$$
$$= 2T\left(\frac{n}{4}\right) + \theta\left(\left(\frac{n}{2}\right)^2\right)$$



$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n/4}{2}\right) + \theta\left(\left(\frac{n}{4}\right)^2\right)$$
$$= 2T\left(\frac{n}{8}\right) + \theta\left(\left(\frac{n}{4}\right)^2\right)$$





Recursion Tree

Guess: $T(n) = \left(\frac{1}{2}\right)^0 cn^2 + \left(\frac{1}{2}\right)^1 cn^2 + \left(\frac{1}{2}\right)^2 cn^2 + \dots + ?$

$$= \sum_{K=0}^{\overset{\text{Height of the tree}}{?}} \left(\frac{1}{2}\right)^K cn^2$$

$$= \sum_{K=0}^{\log_2^n} \left(\frac{1}{2}\right)^K cn^2$$

$$= \Theta(n^2)$$

Height of the Tree:

$n \to \frac{n}{2} \to \frac{n}{2^2} \to \dots\dots \to \frac{n}{2^K} = 1$, Where K is the height

Solve $\frac{n}{2^K} = 1$ for $K \Rightarrow n = 2^K$

$$\log_2^n = \log_2^{2^K}$$

$$\Rightarrow \boxed{K = \log_2^n}$$

6. Solve the following recurrence : $T(n) = 16\,T\left(\frac{n}{4}\right) + \Theta(n^2 \lg n)$

$a = 16,\ b = 4,\ f(n) = n^2 \lg n$

$n^{\log_b^a}$ vs $f(n)$

$n^{\log_4^{16}}$ vs $n^2 \lg n$

$n^{\log_4^{4^2}}$ vs $n^2 \lg n$

n^2 vs $n^2 \lg n$ // Compare polynomials Ignore lg

Case 2: $T(n) = \Theta(f(n) * \lg n)$

$= \Theta(n^2 \lg n \times \lg n)$

$= \Theta(n^2 \lg^2 n)$

7. Solve the following reccurrence Equation : $T(n) = 4\,T\left(\frac{n}{2}\right) + \Theta(n^2)$

$a = 4,\ b = 2,\ f(n) = n^2$

$n^{\log_b^a}$ vs $f(n)$

$n^{\log_2^4}$ vs n^2

n^2 vs n^2

Case 2: $T(n) = \Theta(f(n) * \lg n)$

$= \Theta(n^2 * \lg n)$

8. Solve the following recurrence: $T(n) = 7T(\frac{n}{2}) + \Theta(n^3)$

$a=7,\ b=2,\ f(n)=n^3$, $f(\frac{n}{2}) = (\frac{n}{2})^3$

$n^{\log_b^a}$ Vs $f(n)$

$n^{\log_2^7}$ Vs n^3

$\log_2^7 = 2.8$

Case 3: $T(n) = \Theta(f(n))$

$= \Theta(n^3)$

With ε (Espilon) $= 3 - \log_2^7$

Regularity Condition : $af(\frac{n}{b}) \leq cf(n)$ where $0 < c < 1$

$af(\frac{n}{b}) = 7f(\frac{n}{2})$

$= 7(\frac{n}{2})^3$

$= \frac{7}{8}n^3$

$\leq \frac{7}{8}\underbrace{n^3}_{f(n)}$

Let $c = \frac{7}{8}$

9. Solve the following recurrence : $T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n^2 \lg n)$

$a=3$, $b=2$, $f(n) = n^2 \lg n$, $f\left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^2 \lg \frac{n}{2}$

$n^{\log_b^a}$ vs $f(n) \Rightarrow n^{\log_2^3}$ vs $n^2 \lg n$

Case 3: $T(n) = \Theta f(n)$ with $\varepsilon = ?$

$= \Theta(n^2 \lg n)$

With $\varepsilon = 2 - \log_2^3$

Regularity Condition: $af\left(\frac{n}{b}\right) \leq cf(n)$ Where $0<c<1$

$$
\begin{aligned}
af\left(\frac{n}{b}\right) &= 3f\left(\frac{n}{2}\right) \\
&= 3\left(\frac{n}{2}\right)^2 \lg \frac{n}{2} \\
&= \frac{3}{4} n^2 [\lg n - \lg 2] \\
&= \frac{3}{4} n^2 \lg n - \frac{3}{4} n^2 \lg 2 \qquad \lg 2 = 1 \\
&= \frac{3}{4} n^2 \lg n - \frac{3}{4} n^2 \\
&\leq \underbrace{\frac{3}{4} n^2 \lg n}_{f(n)}
\end{aligned}
$$

Let $c = \frac{3}{4}$

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10. Solve the following recurrence : $T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^2)$

$a=5, b=2, f(n) = n^2$

$n^{\log_b^a}$ vs $f(n) \Rightarrow n^{\log_2^5}$ vs n^2

Case 1: $T(n) = \Theta\left(n^{\log_b^a}\right)$ with $\varepsilon = ?$

$= \Theta\left(n^{\log_2^5}\right)$

$\varepsilon = \log_2^5 - 2$

11. Solve the following recurrence : $T(n) = 11T\left(\frac{n}{3}\right) + \Theta(n^2)$

$a=11, b=3, f(n) = n^2$

$n^{\log_b^a}$ vs $f(n) \Rightarrow n^{\log_3^{11}}$ vs n^2

Case 1: $T(n) = \Theta\left(n^{\log_b^a}\right)$ with $\varepsilon = ?$

$= \Theta\left(n^{\log_3^{11}}\right)$

With $\varepsilon = \log_3^{11} - 2$