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CS5300 Advanced Algorithms

HW #4

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1. Given the recurrence relation

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ 7T\left(\frac{n}{5}\right)+10n & \text{Otherwise} \end{cases}$$

Find $T(625)$

$T(625) = 7T\left(\frac{625}{5}\right)+10(625) \Rightarrow 7T(125)+6250$ —— ①

$T(125) = 7T\left(\frac{125}{5}\right)+10(125) \Rightarrow 7T(25)+1250$ —— ②

$T(25) = 7T\left(\frac{25}{5}\right)+10(25) \Rightarrow 7T(5)+250$ —— ③

$T(5) = 7T\left(\frac{5}{5}\right)+10(5) \Rightarrow 7T(1)+50$ —— ④

Here $T(1)=1$

Replace $T(1)$ in Equation 4

$T(5) = 7(1)+50 = 57$

Replace $T(5)$ in Equation 3

$T(25) = 7(57)+250 = 649$

Replace $T(25)$ value in Equation 2

$T(125) = 7(649)+1250 = 4543+1250 = 5793$

Replace $T(125)$ in Equation 1

$T(625) = 7(5793)+6250 = 40551+6250 = 46801$

Hence, $T(625) = 46801$

2. Solve the following recurrence equation

$$T(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ 4T(n-1)-3T(n-2) & \text{otherwise} \end{cases}$$

$T(n)-4T(n-1)+3T(n-2)=0 \quad n>1$

$\downarrow$

Degree of Polynomia

Step 1: Characteristic Equation

$$r^2-4r+3=0$$
$$r^2-3r-r+3=0$$
$$r(r-3)-1(r-3)=0$$
$$(r-1)(r-3)=0$$
$$r=1 \text{ or } r=3$$

Step 2: Apply the Theorem

$$T(n)=c_1r_1^n+c_2r_2^n$$
$$=c_1(3)^n+c_2(1)^n$$

Step 3: Find c_1 and c_2

$T(0)=c_1(3)^0+c_2(1)^0=0 \Rightarrow c_1+c_2=0$ —— ①

$T(1)=c_1(3)^1+c_2(1)^1=1 \Rightarrow 3c_1+c_2=1$ —— ②

$c_1+c_2=0 \Rightarrow c_1=-c_2$

Solve $3c_1+c_2=1$

$$3(-c_2)+c_2=1$$
$$-3c_2+c_2=1$$
$$-2c_2=1$$
$$c_2=-\frac{1}{2}$$

$c_1=-c_2=-\left(\frac{-1}{2}\right)=\frac{1}{2} \Rightarrow c_1=\frac{1}{2},\ c_2=\frac{-1}{2}$

Hence, $T(n)=\frac{1}{2}(3)^n-\frac{1}{2}(1)^n$

3. Solve the following recurrence equation

$$T(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ 2 & \text{if } n=2 \\ -6T(n-1)-11T(n-2)-6T(n-3) & \text{otherwise} \end{cases}$$

$$T(n)+6T(n-1)+11T(n-2)+6T(n-3)=0 \quad \text{if } n>2$$

Step 1: Characteristic Equation : $r^3+6r^2+11r+6=0$

Factors of 6 : $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of 1 : ± 1

Partial Rationalize Zero (PRZ) : $\pm 1, \pm 2, \pm 3, \pm 6$

Synthetic Division

-1	1	6	11	6
		-1	-5	-6
	1	5	6	0

$$r^3+6r^2+11r+6 = (r+1)(r^2+5r+6)=0$$
$$= (r+1)(r^2+3r+2r+6)=0$$
$$(r+1)(r(r+3)+2(r+3))=0$$
$$(r+1)(r+2)(r+3)=0$$
$$r=-1,\ r=-2,\ r=-3$$

Step 2: Apply the Theorem

$$T(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$$

$$T(n) = c_1(-1)^n + c_2(-2)^n + c_3(-3)^n$$

$T(0) = c_1(-1)^0 + c_2(-2)^0 + c_3(-3)^0$

$T(0) = c_1 + c_2 + c_3 = 0$ ——— ①

$T(1) = c_1(-1)^1 + c_2(-2)^1 + c_3(-3)^1$

$T(1) = -c_1 - 2c_2 - 3c_3 = 1$ ——— ②

$T(2) = c_1(-1)^2 + c_2(-2)^2 + c_3(-3)^2$

$T(2) = c_1 + 4c_2 + 9c_3 = 2$ ——— ③

Solve Equation 1 and 2

$$\begin{array}{r} \cancel{c_1} + c_2 + c_3 = 0 \\ -\cancel{c_1} - 2c_2 - 3c_3 = 1 \\ \hline -c_2 - 2c_3 = 1 \end{array}$$ ——— ④

Solve Equation 2 and 3

$$\begin{array}{r} -\cancel{c_1} - 2c_2 - 3c_3 = 1 \\ \cancel{c_1} + 4c_2 + 9c_3 = 2 \\ \hline 2c_2 + 6c_3 = 3 \end{array}$$ ——— ⑤

Solve Equation 4 and 5

$2(-c_2 - 2c_3) = 1 \times 2 \Rightarrow$

$$\begin{array}{r} -2\cancel{c_2} - 4c_3 = 2 \\ \cancel{2c_2} + 6c_3 = 3 \\ \hline 2c_3 = 5 \end{array} \Rightarrow c_3 = 5/2$$

Substitute C_3 Value in Equation 4

$$-C_2 - 2C_3 = 1$$

$$-C_2 - 2\left(\frac{5}{2}\right) = 1 \Rightarrow -C_2 - \frac{10}{2} = 1 \Rightarrow -C_2 - 5 = 1$$

$$-C_2 = 6$$

$$C_2 = -6$$

Substitute C_2 and C_3 value in Eq 1

$$C_1 + C_2 + C_3 = 0$$

$$C_1 - 6 + \frac{5}{2} = 0$$

$$C_1 = 6 - \frac{5}{2}$$

$$C_1 = \frac{7}{2}$$

Hence $C_1 = \frac{7}{2}$, $C_2 = -6$, $C_3 = \frac{5}{2}$

Hence, the solution is

$$T(n) = \frac{7}{2}(-1)^n + (-6)(-2)^n + \frac{5}{2}(-3)^n$$

4. Use the binary Search algorithm to Search for $x = 6$, if

$S = [3, 5, 7, 8, 10, 11, 17, 18, 19, 31]$ Show all the steps.

$n = 10$

$x = 6$

	1	2	3	4	5	6	7	8	9	10
S	3	5	7	8	10	11	17	18	19	31
	low				Mid					High

Low	High	Location	Mid
~~1~~	~~10~~	0	~~5~~
3	~~4~~		~~2~~
	2		3

$1 <= 10$ && $0 == 0$

$Mid = \left\lfloor \frac{1+10}{2} \right\rfloor = 5$

$6 == 10$ # Fail

$6 < 10$ # Pass

$high = Mid - 1 = 5 - 1 = 4$

1	2	3	4
3	5	7	8
Low	Mid		High

$1 <= 4$ && $0 == 0$

$Mid = \left\lfloor \frac{1+4}{2} \right\rfloor = 2$

$6 == 5$ # Fail

$6 < 5$ # Fail

$Low = Mid + 1 = 2 + 1 = 3$

3	4
7	8
Low	High

$3 <= 4$ && $0 == 0$

$$Mid = \left\lfloor \frac{3+4}{2} \right\rfloor = 3$$

$6 == 7$ # Fail

$6 < 7$

$high = Mid - 1 = 3 - 1 = 2$

3
7

↑ High ↑ Low

$3 <= 2$ # Fail

Print Location

Output : 0

5. Use the binary Search algorithm to search for $X = 19$, if $S = [3, 5, 7, 8, 10, 11, 17, 18, 19, 31]$. Show all the steps.

	1	2	3	4	5	6	7	8	9	10
S	3	5	7	8	10	11	17	18	19	31
	low				Mid					High

$n = 10$

$X = 19$

Low	High	Location	Mid
~~1~~	10	~~0~~	~~5~~
~~6~~		9	~~8~~
9			9

$1 <= 10 \; \&\& \; 0 == 0$

$$Mid = \left\lfloor \frac{1+10}{2} \right\rfloor = 5$$

$19 == 10$ # Fail

$19 < 10$ # Fail

$Low = Mid + 1 = 5 + 1 = 6$

6	7	8	9	10
11	17	18	19	31
low		Mid		High

$6 <= 10 \; \&\& \; 0 == 0$

$$Mid = \left\lfloor \frac{6+10}{2} \right\rfloor = 8$$

$19 == 18$ # Fail

$19 < 18$ # Fail

$Low = Mid + 1 = 8 + 1 = 9$

9	10
19	31
low Mid	High

$9 <= 10$ && $0 == 0$

$$Mid = \left\lfloor \frac{9+10}{2} \right\rfloor = 9$$

$19 == 19$

Location = Mid = 9

$9 <= 10$ && $9 == 0$ # Fail

Print Location

Output: 9

6. Use the Sequential Search algorithm to search for $X=6$ if $S=[3,5,7,8,10,11,17,18,19,31]$. Show all the steps

$n=10$

$X=6$

S

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
1

$1<=10$ && $3!=6$ # Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
~~1~~
2

$2<=10$ && $5!=6$ # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
~~1~~
~~2~~
3

$3<=10$ && $7!=6$ # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
~~1~~
~~2~~
~~3~~
4

$4<=10$ && $8!=6$ # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
~~1~~
~~2~~
~~3~~
~~4~~
5

$5<=10$ && $10!=6$ # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

location

Location
~~1~~
~~2~~
~~3~~
~~4~~
~~5~~
6

6<=10 && 11!=6 # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ 7

7<=10 && 17!=6

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ 8

8<=10 && 18!=6 # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ ~~8~~ 9

9<=10 && 19!=6 # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ ~~8~~ ~~9~~ 10

10<=10 && 31!=6 # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ ~~8~~ ~~9~~ ~~10~~ 11

11<=10 # Fail

11 > 10 # Pass

Print Location 0

Output = 0

Location

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ ~~8~~ ~~9~~ ~~10~~
~~11~~ 0

7. Use the sequential search algorithm to search for $x=19$, if $S=[3,5,7,8,10,11,17,18,19,31]$. Show all the steps.

S

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ Location

Location
1

$n=10$

$X=19$

1<=10 && 3!=19 # Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
2

2<=10 && 5!=19 # Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
~~2~~
3

3<=10 && 7!=19

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
~~2~~
~~3~~
4

4<=10 && 8!=19 # pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
~~2~~
~~3~~
~~4~~
5

5<=10 && 10!=19 # Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
~~2~~
~~3~~
~~4~~
~~5~~
6

6<=10 && 11!=19 #Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~
~~2~~
~~3~~
~~4~~
~~5~~
~~6~~
7

7<=10 && 17!=19 #Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ ~~7~~ 8

8<=10 && 18!=19 #Pass

1	2	3	4	5	6	7	8	9	10
3	5	7	8	10	11	17	18	19	31

↑ location

Location
~~1~~ ~~2~~ ~~3~~ 4 ~~5~~
~~6~~ ~~7~~ ~~8~~ 9

9<=10 && 19!=19 # Fail

9>10 # Fail

Print Location

Output : 9

8. How can we modify almost any algorithm to have a good best case running time?

When designing any algorithm to treat its best-case Scenario as a special case and return a predetermined Solution.

For Example, For any sorting algorithm, design a precheck Condition to check whether the input array is already Sorted and if it is, we can return without doing anything, Which saves time in a different scenarios if the algorithm is already sorted partially, an algorithm like Insertion sort can run with the best case running time of $\Theta(n)$. This is Linear.

Design different algorithms for Specific types of Input data. For example, Quick sort can be modified to choose a pivot element that minimizes the number of swaps for sorted arrays.

9. Use the recursive version of the Fibonacci number to calculate Fib(6)

n	output
0	0
1	1

6<=1 # False

Fib(6-1) + Fib(6-2) = Fib(5) + Fib(4)

5<=1 # False

Fib(5-1) + Fib(5-2) = Fib(4) + Fib(3)

4<=1 # False

Fib(4-1) + Fib(4-2) = Fib(3) + Fib(2)

3<=1 # False

Fib(3-1) + Fib(3-2) = Fib(2) + Fib(1)

2<=1 # False

Fib(2-1) + Fib(2-2) = Fib(1) + Fib(0)

1<=1 # True

1

0<=1 # True

0

Fib(6) = Fib(5) + Fib(4)

= Fib(4) + Fib(3) + Fib(3) + Fib(2)

= Fib(3) + Fib(2) + Fib(2) + Fib(1) + Fib(2) + Fib(1) + Fib(1) + Fib(0)

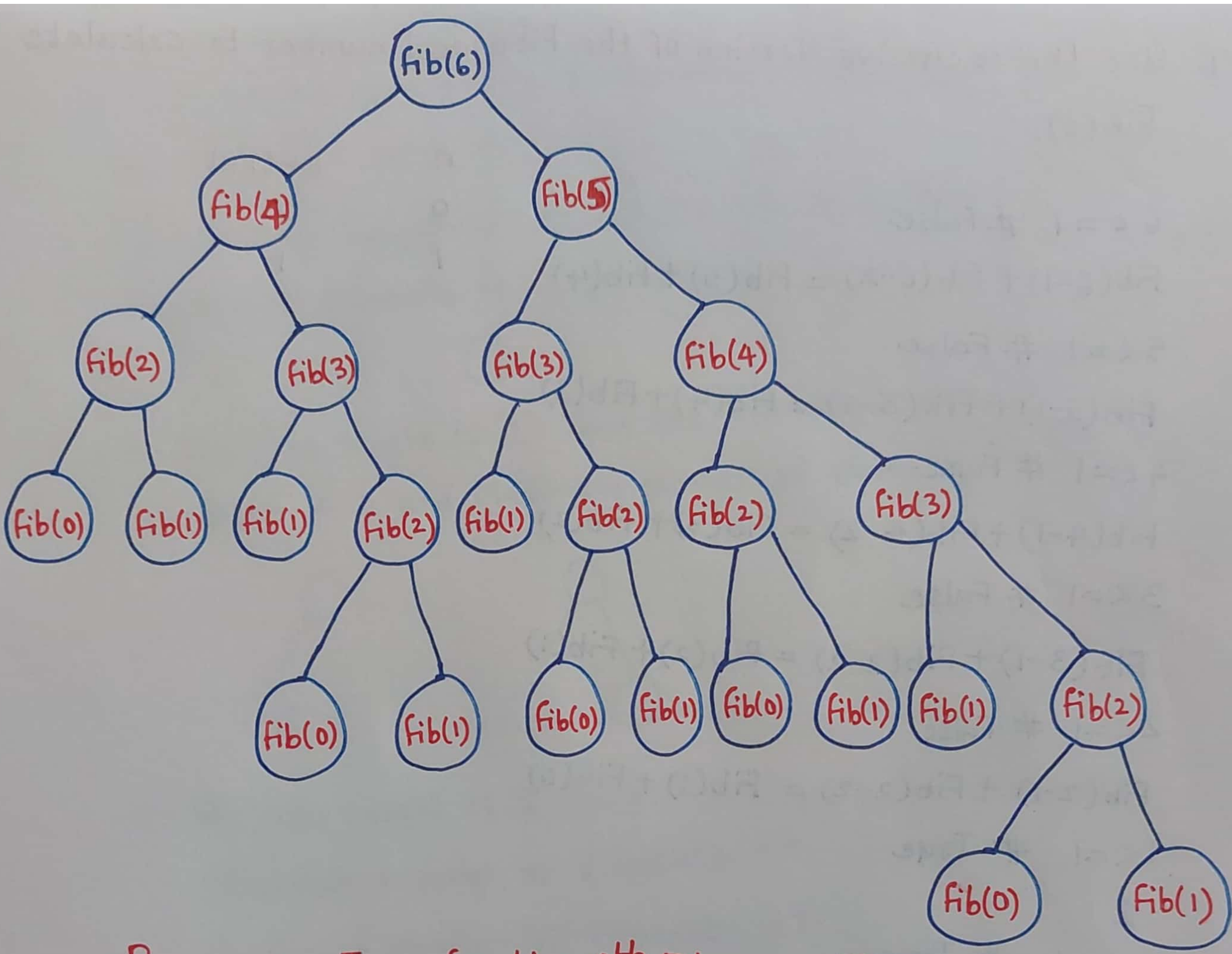
= Fib(2) + Fib(1) + Fib(1) + Fib(0) + Fib(1) + Fib(0) + Fib(1) + Fib(1) + Fib(0) + Fib(1) + Fib(1) + Fib(0)

= Fib(1) + Fib(0) + 1 + 1 + 0 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0

= 1 + 0 + 1 + 1 + 0 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0

= 8

Fib(6) = 8

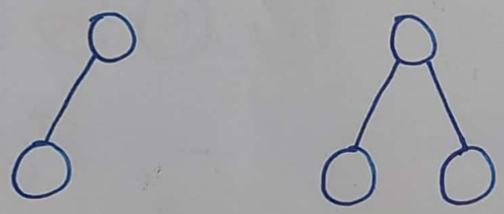


Recursion Tree for the 6th Fibonacci Term.

10. What are the minimum and maximum number of elements in a heap of height h ?

If the tree height is '0' Then the minimum and maximum number of elements is 1

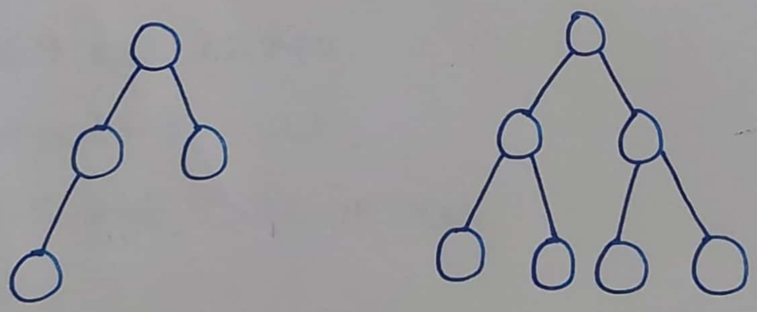
If the tree height is '1'. Then the Minimum number of elements is 2 and Maximum number of elements is 3



If the tree height is 2. Then

Minimum number of Elements = 4

Maximum number of Elements = 7



Minimum Number of Nodes happen in a heap in which the last level Contains only one node

Minimum no of Nodes $= 2^0 + 2^1 + 2^2 + \dots + 2^{h-1} + 1 = 2^h$

Maximum no of Nodes happens in a heap in which the last level is Full.

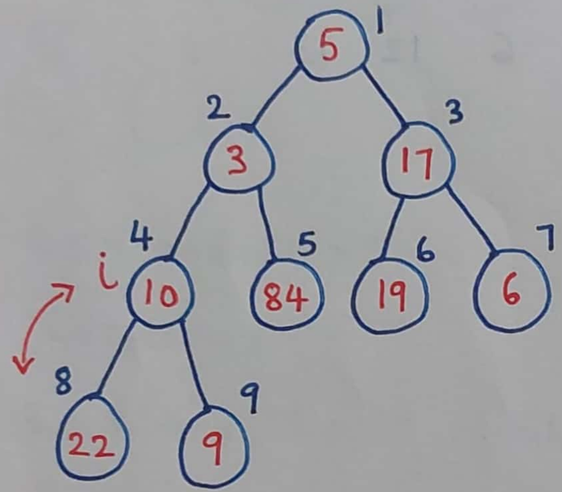
Maximum number of Nodes $= 2^0 + 2^1 + 2^2 + \dots + 2^h = 2^{h+1} - 1$

11. Illustrate the Operation of BUILD-MAX-HEAP on the array

$A = \langle 5, 3, 17, 10, 84, 19, 6, 22, 9 \rangle$

A									
A	5	3	17	10	84	19	6	22	9

Heap size = 9



$i = \left\lfloor \frac{A.length}{2} \right\rfloor = \left\lfloor \frac{9}{2} \right\rfloor = 4$

MAX-HEAPIFY(A, i) ⇒ MAX-HEAPIFY(9, 4)

i	L	r	Largest
4	8	9	8
8	16	17	

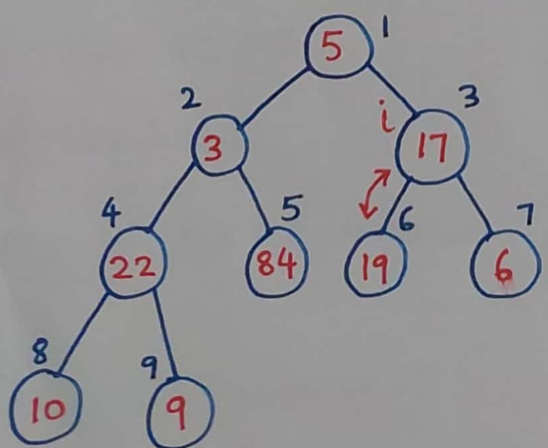
$8 \leq 9$ and $22 > 10$

largest = 8

$9 \leq 9$ and $9 > 22$ # False

$8 \neq 4$

Exchange A[4] with A[8]



MAX-HEAPIFY (A, largest) $\Rightarrow$ MAX-HEAPIFY (A, 8)

$16 \leq 9$ # False

$6 \leq 9$ and $19 > 17$

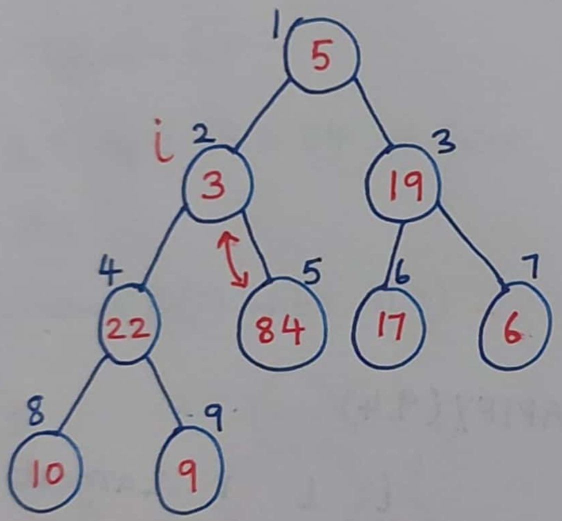
Largest = l = 6

$7 \leq 9$ and $6 > 19$ # False

$6 \neq 3$

Exchange A[3] with A[6]

i	l	r	largest
3	6	7	6
6	12	13	



$4 \leq 9$ and $22 > 3$

largest = l = 4

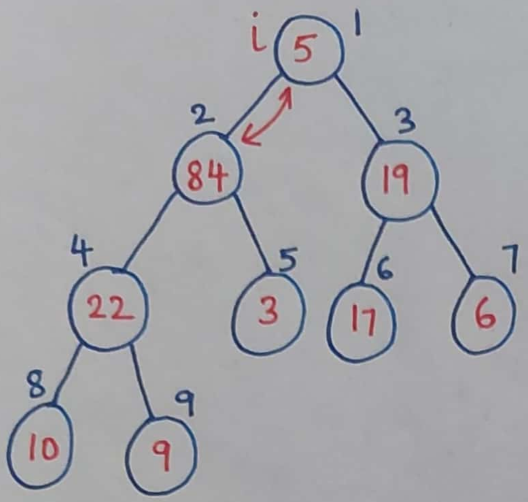
$5 \leq 9$ and $84 > 22$

largest = r = 5

$5 \neq 2$

Exchange A[2] with A[5]

i	l	r	largest
~~2~~	~~4~~	~~5~~	~~4~~
5	10	11	5



$2 \le 9$ and $84 > 5$

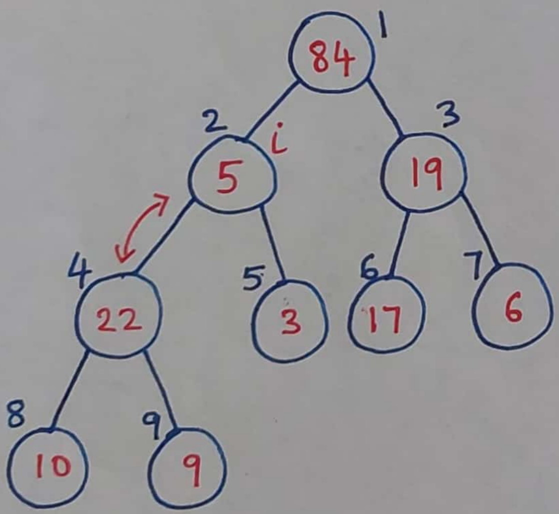
Largest $= l = 2$

$3 \le 9$ and $19 > 84$ # False

$2 \ne l$

Exchange A[1] with A[2]

i	l	r	largest
~~1~~	~~2~~	~~3~~	~~2~~
~~2~~	~~4~~	~~5~~	~~4~~
4	8	9	8
8	16	17	



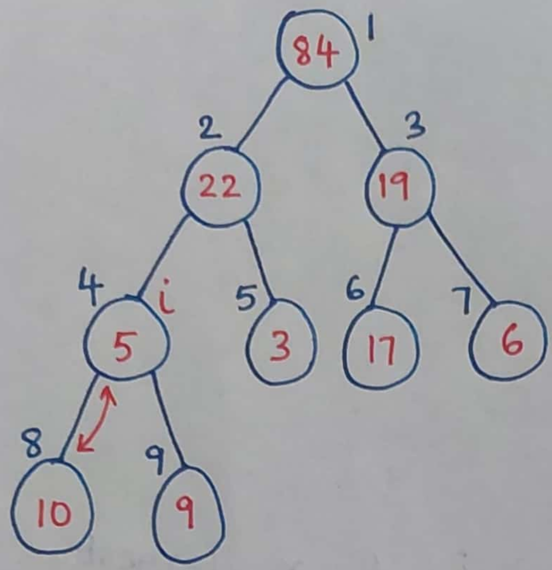
$4 \le 9$ and $22 > 5$

largest $= l = 4$

$5 \le 9$ and $3 > 22$ # False

$4 \ne 2$

Exchange A[2] with A[4]



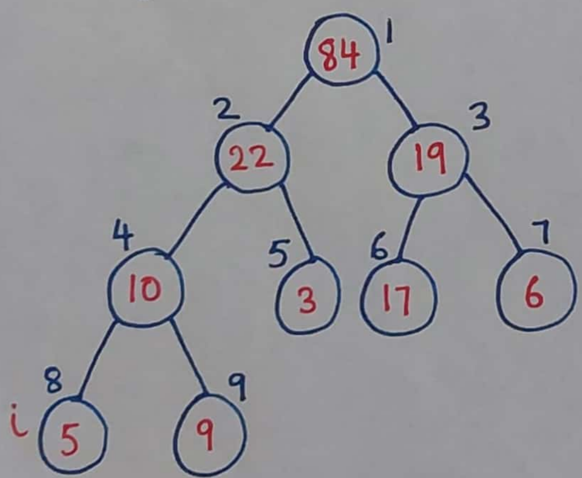
$8 \leq 9$ and $10 > 5$

largest $= l = 8$

$9 \leq 9$ and $9 > 10$ # False

$8 \neq 4$

Exchange A[4] with A[8]



$16 \leq 9$ # False

$17 \leq 9$ # False

$8 \neq 8$ # False STOP MAX-HEAPIFY

$i = 0$

Output: A

84	22	19	10	3	17	6	5	9