Theorem
Let the homogeneous linear equation with constant coefficients $a_0T(n) + a_1T(n-1) + \cdots + a_kT(n-k) = 0$ be given. If its characteristic equation $a_0r^k + a_1r^{k-1} + \cdots + a_kr^0 = 0$ has k distinct solutions $r_1, r_2, \dots, r_k$, then the solution is given by $T(n) = c_1r_1^n + c_2r_2^n + \cdots + c_kr_k^n$ where c_i are constants.

Example1:
Solve $T(0) = 0, T(1) = 1, T(n) - 3T(n-1) - 4T(n-2) = 0 \; if \; n > 1$

Solution
Step1: Find the characteristic equation: $r^2 - 3r - 4 = 0, (r-4)(r+1) = 0, implies \; r = 4 \; or \; r = -1$

Step2: Apply the theorem: $T(n) = c_1 4^n + c_2(-1)^n$

Step 3: Find $\boldsymbol{c_1 and\; c_2}$
$T(0) = c_1 4^0 + c_2(-1)^0 = 0$ implies $c_1 + c_2 = 0 \dots (1)$
$T(1) = c_1 4^1 + c_2(-1)^1 = 1$ implies $4c_1 - c_2 = 1 \dots (2)$
Solve equations (1) & (2) to get $c_1 = \frac{1}{5}, c_2 = -\frac{1}{5}$. Hence, the solution is: $T(n) = \frac{1}{5}4^n - \frac{1}{5}(-1)^n$

Example2:
Solve $T(0) = 0, T(1) = 1, T(n) - T(n-1) - T(n-2) = 0 \; if \; n > 1$

Solution
Step1: Find the characteristic equation: $r^2 - r - 1 = 0, r = \frac{1 \pm \sqrt{5}}{2}$

Step2: Apply the theorem: $T(n) = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$
Step 3: Find $\boldsymbol{c_1 and\; c_2}$
$T(0) = c_1(\frac{1+\sqrt{5}}{2})^0 + c_2(\frac{1-\sqrt{5}}{2})^0 = 0$ implies $c_1 + c_2 = 0 \dots (1)$
$T(1) = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1 = 1$ implies $c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1 \dots (2)$
Solve equations (1) & (2) to get $c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$. Hence, the solution is: $T(n) = ((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)/\sqrt{5}$

Theorem

Let r be a root of multiplicity m, then $T(n) = r^n, T(n) = nr^n, T(n) = n^2r^n, \dots, T(n) = n^{m-1}r^n$ are all solutions.

Example:

Solve $T(0) = 0, T(1) = 1, T(2) = 2,\ T(n) - 7T(n-1) + 15T(n-2) - 9T(n-3) = 0\ if\ n > 2$

Solution

Step1: Find the characteristic equation: $r^3 - 7r^2 + 15r - 9 = 0\ implies\ (r-1)(r-3)^2 = 0, implis\ r = 1\ or\ r = 3\ of\ multipicity\ 2$

Step2: Apply the theorem: $T(n) = c_1 1^n + c_2 3^n + c_3 n 3^n$

Step 3: Find $\boldsymbol{c_1, c_2, and\ c_3}$

$$c_1 = -1, c_2 = 1, c_3 = -\frac{1}{3}$$

Hence the solution is: $T(n) = (-1)1^n + (1)3^n + \left(-\frac{1}{3}\right)n3^n = -1 + 3^n - n3^{n-1}$