CS5300 - Advanced Algorithms

HW # 3

Manoj Suvvala

700744733

Feb, 6, 2023

① Find a theta notation for the number of times the statement $x = x+1$ is executed.

```
i=2
while (i<n){
    i= i*i
    x = x+1
}
```

sol) value of i in the loop →

The value of i keeps changing after & before iteration.

The value of i after & before iteration are as follows,

Before 1st iteration → $i = 2^1$

After 1st iteration → $i = 4 = 2^2$

2nd → $i = 16 = 2^4$

3rd → $i = 256 = 2^8$

4th → $i = 65536 = 2^{16}$

we can see that after 'k' iterations

$i = 2^{2^k}$

∴ Number of iterations corresponds to the lowest value of k

$\Rightarrow \boxed{2^k = n}$

lets apply log on both sides

$2^k \log_2^2 = \log_2 n$

$k = \log_2 \log_2 n$

Now the theta notation for the given statement is as follows

$\Rightarrow \Theta(\log_2 \log_2 n)$

---

(2) How can we modify almost any algorithm to have a good best case running time?

---

sol)

1) Incase of a recursive problem, it can be solved by dynamic programming to reduce the time complexity

2) we can use binary search instead of linear search to convert $O(n)$ to $O(\log n)$

3) Sometimes, Hashmap can also be used to get the constant time complexity $O(1)$.

---

(3) Consider the following recurrence equation, defining $T(n)$, as

$$T(n) = \begin{cases} 2 & \text{if } n=1 \\ T(n-1)+2 & \text{otherwise} \end{cases}$$

show by induction $T(n) = 2n$.

step:1 Basis if $n=1$, then $T(n) = 2n$

$T(1) = 2(1) = 2$

step-2 Inductive hypothesis:

let $T(k) = 2k \quad \forall\, k<n$

in particular let $k = (n-1)$

$\Rightarrow T(n-1) = 2(n-1) = 2n-2$

step-3 show $T(n) = 2n$

$T(n) = T(n-1) + 2$

$= 2n - \cancel{2} + \cancel{2}$

$= 2n$

Hence, $T(n) = 2n \quad \forall n$

---

(4) consider the following equation of reccurence defining $T(n)$, as

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-1)+n^2 & \text{otherwise} \end{cases}$$

show by induction that $T(n) = \dfrac{n(n+1)(2n+1)}{6}$

---

sol) Step 1: Basis if $n=1$, then

$$T(1) = \frac{1(1+1)(2(1)+1)}{6} = 6/6 = 1$$

Step:2 Inductive hypothesis:

Assume $T(k) = \frac{k(k+1)(2k+1)}{6}$

In particular $k = n-1$

$$T(n-1) = \frac{(n-1)(n-1+1)(2(n-1)+1}{6}$$

$$T(n-1) = \frac{(n-1)(n)(2n-1)}{6}$$

Step: 3 show that $T(n) = \frac{n(n+1)(2n+1)}{6}$

$$T(n) = T(n-1) + n^2$$

$$= \frac{(n-1)(n)(2n-1)}{6} + n^2$$

$$= \frac{2n^3 - 2n^2 - n^2 + n}{6} + n^2$$

$$= 2n^3 - 3n^2 + n + 6n^2/6$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Hence,

$$T(n) = \frac{n(n+1)(2n+1)}{6} \quad \forall n$$

(5) prove by induction that the solution to $T(n) = T(\frac{n}{2}) + 1$ is given by $T(n) = O(\lg n)$

---

sol)

proof by induction:

As it is given in asymptotic notation upper bound is being used such as,

$$T(n) \leq c[\lg n]$$

Now, substitute $n/2$

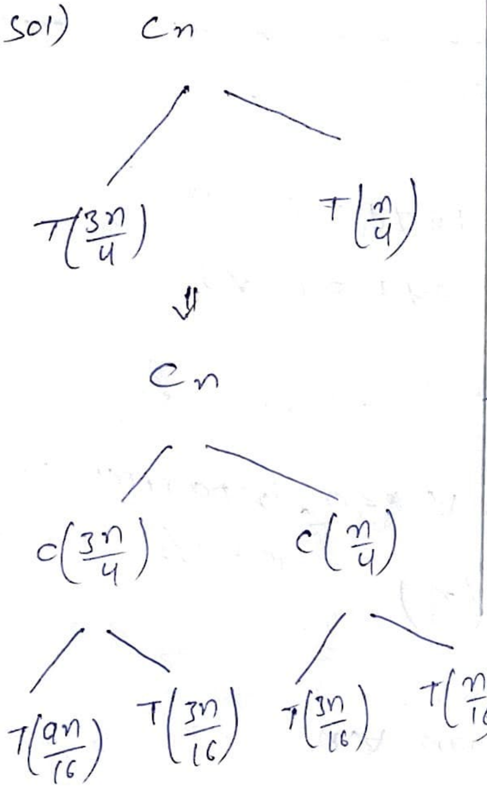
$$T(n/2) \leq c\left(\lg \frac{n}{2}\right)$$
$$\leq c(\lg n - \lg 2)$$
$$\leq c(\lg n)$$

$\therefore$ $\boxed{T(n) = O(\lg n)}$

⑥ a) Draw the reccursion tree for

$$T(n) = T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + \Theta(n)$$

Sol)



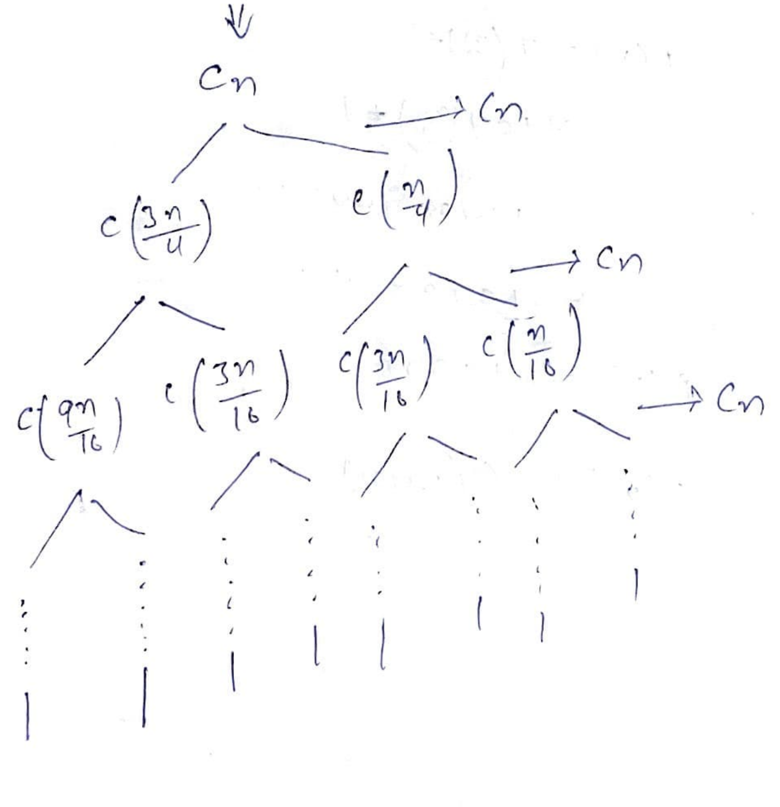
$$T\left(\frac{3n}{4}\right) = T\left(\frac{9n}{16}\right) + T\left(\frac{3n}{16}\right) + \Theta\left(\frac{3n}{4}\right)$$

$$T\left(\frac{n}{4}\right) = T\left(\frac{3n}{16}\right) + T\left(\frac{n}{16}\right) + \Theta\left(\frac{n}{4}\right)$$

$$T\left(\frac{9n}{16}\right) = T\left(\frac{27n}{64}\right) + T\left(\frac{9n}{64}\right) + \Theta\left(\frac{9n}{16}\right)$$

$$T\left(\frac{3n}{16}\right) = T\left(\frac{9n}{64}\right) + T\left(\frac{3n}{64}\right) + \Theta\left(\frac{3n}{16}\right)$$

$$T\left(\frac{n}{16}\right) = T\left(\frac{3n}{64}\right) + T\left(\frac{n}{64}\right) + \Theta\left(\frac{n}{16}\right)$$



b) find the height of the tree

$Cn + Cn + \dots + Cn = Cn * (\text{height of the tree})$

$n \rightarrow \frac{3n}{4} \rightarrow \frac{9n}{16} \dots \left(\frac{3}{4}\right)^{kn} = 1$ where k is the

height

$$\left(\frac{3}{4}\right)^{kn} = 1$$

$$n = \left(\frac{4}{3}\right)^{k}$$

$$\boxed{k = \log\left(\frac{4}{3}\right)^{n}} \rightarrow \text{height}$$

---

c) Guess the solution

$$T(n) = Cn + Cn + Cn + \dots + Cn$$

$$= Cn * \text{height}$$

$$= Cn * \log_{4/3} n$$

$$T(n) = \Theta(n \log n)$$

---

7) Draw the recurssion tree for $T(n) = T\left(\frac{n}{3}\right) + 1$

1 $\vert$ $T\left(\frac{n}{3}\right)$ $\Downarrow$ 1 $\vert$ $c\left(\frac{n}{3}\right)$ $\vert$ $T\left(\frac{n}{9}\right)$	$T\left(\frac{n}{3}\right) = T\left(\frac{n/3}{3}\right) + 1 = T\left(\frac{n}{9}\right)$ $T\left(\frac{n}{9}\right) = T\left(\frac{n/9}{3}\right) + 1 = T\left(\frac{n}{27}\right)$ $T\left(\frac{n}{27}\right) = T\left(\frac{\frac{n}{27}}{3}\right) + 1 = T\left(\frac{n}{81}\right)$

$\Downarrow$

$$1$$

$$\Big| \rightarrow Cn$$

$$c\left(\frac{n}{3}\right)$$

$$\Big| \rightarrow Cn$$

$$c\left(\frac{n}{27}\right)$$

$$\Big| \rightarrow Cn$$

$$T\left(\frac{n}{81}\right)$$

$$\Big|$$

$$\vdots$$

$$1$$

b) find the height of the tree

$Cn + Cn + Cn + \ldots + Cn = 1$ * height of the tree

$$1 \rightarrow \frac{n}{3} \rightarrow \frac{n}{9} \rightarrow \frac{n}{27} \rightarrow \ldots \frac{n}{3^k} = 1,$$ where k is height

$$\frac{n}{3^k} = 1$$

$$n = 3^k$$

Take log on both sides

$$\log n = \log 3^k$$

$$\log n = k \log 3$$

$$\frac{\log n}{\log 3} = k$$

$$\boxed{k = \log n}$$

---

c) Guess the solution

$T(n) = Cn + Cn + Cn + \dots + Cn$

$= Cn * \text{weight}$

$= 1 * \text{weight}$

$= 1 * \log n$

$T(n) = \Theta(\log n)$