CS5300-Advanced Algorithms

Homework # 4

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1) Show that the solution for $T(n) = T(\sqrt{n}) + \Theta(1)$ is $T(n) = \Theta(lg(lg(n)))$.

sol) Given that $T(n) = T(\sqrt{n}) + \Theta(1)$

let $n = 2^R$

Now, lets apply the log on both sides, we get,

$R = lg n$

$T(2^R) = T(\sqrt{2^R}) + \Theta(1)$

$T(2^R) = T(2^{R/2}) + \Theta(1)$

let $T(2^R) = h(R) = T(n)$

$\Rightarrow h(R) = h\left(\frac{R}{2}\right) + \Theta(1)$

Now we apply masters theorem on the above equation

$a = 1, b = 2, f(R) = 1$

$R^{\log_b a}$ (vs) $f(R)$

$R^{\log_2 1}$ (vs) 1

R^0 vs 1

1 vs 1

since $R^{\log_b a} = f(R)$

then it belongs to case: 2

<u>case: 2</u>

$$h(R) = \Theta(f(R) * \lg R)$$

$$\Rightarrow h(R) = \Theta(1 * \lg R)$$

$$\Rightarrow T(2^R) = \Theta(\lg R)$$

since $R = \lg n \cdot \varepsilon_1\, 2^R = n$

$$\boxed{\Rightarrow T(n) = \Theta(\lg \lg n)}$$

Hence proved.

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(2) solve the following recurrence equation

$T(n) = 4T(\frac{n}{2}) + \Theta(n^2)$

<u>Sol</u> From the given recurrence equation

$$T(n) = 4T(\tfrac{n}{2}) + \Theta(n^2)$$

$$a = 4, b = 2, f(n) = n^2$$

$$\Rightarrow n^{\log_b a} \text{ vs } f(n)$$

$$\Rightarrow n^{\log_2 4} \text{ vs } n^2$$

$$\Rightarrow n^2 \text{ vs } n^2$$

so this means $n^{\log_b^a} = f(n)$ then it

belongs to

case: 2

$$T(n) = \Theta(f(n).\lg n)$$

$$\boxed{T(n) = \Theta(n^2 \lg n)}$$

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(3) Solve the following recurrence

equation: $T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^3)$

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sol) From the given equation

$$T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^3)$$

$$a = 5, b = 2, f(n) = n^3$$

$$n^{\log_b a} \text{ (vs) } f(n)$$

$$n^{\log_2 5} \text{ vs } n^3$$

$$n^{\log_2 5} < n^3$$

case: 3

$$T(n) = \Theta(f(n))$$

$$T(n) = \Theta(n^3) \text{ with } \varepsilon = 3 - \log_2 5$$

Regularity condition:

$af\left(\frac{n}{b}\right) \leq c(f(n))$ where $0<c<1$

$$a\,f\left(\frac{n}{b}\right) = 5f\left(\frac{n}{2}\right) = 5\left(\frac{n^3}{8}\right)$$

$$\left(\because f\left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^3 = \frac{n^3}{8}\right)$$

$$5\left(\frac{n^3}{8}\right) \leq c(n^3)$$

$$\frac{5}{8}(n^3) \leq c(n^3)$$

while comparing,

$$c = 5/8,\ f(n) = n^3.$$

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4) solve the following recurrence:

$$T(n) = 7T\left(\frac{n}{2}\right) + O(n^2)$$

Sol $a=7,\ b=2\quad f(n) = n^2$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 7} \text{ (vs) } n^2$$

Here, $n^{\log_2 7} > n^2$ Hence, it belongs to

Case: 1

$$T(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^{\log_2 7})$$

so $T(n) = \Theta(n^{\log_2 7})$ with $\varepsilon = \log_2 7 - 2$

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(5) Use Binary search to search for the integer 120 in the following array of integers. show the action step by step.

$A = \{12, 34, 37, 45, 57, 82, 99, 120, 134\}$

sol Binary-Search Algorithm:

~~Write~~

low = 1, high = n, location = 0

while (low ≤ high && location = 0)

mid = floor[(low + high)]/2

if (x == s[mid])

location = mid

else if (x < s[mid])

high = mid - 1

else

low = mid + 1

A =

12	34	37	45	57	82	99	120	134
1	2	3	4	5	6	7	8	9
low				mid				high

Step: 1

low = 1

high = 9

$mid = \left(\frac{1+9}{2}\right) = 5$

$A[5] = 57 < 120$

$\therefore low = mid + 1 = 5 + 1 = 6$

12	34	37	45	57	82	99	120	134
1	2	3	4	5	6 low	7	8	9
						mid		high

Step: 2

low = 6

high = 9

low ≤ high i.e $6 < 9$

$$mid = \left\lfloor \frac{6+9}{2} \right\rfloor = 7$$

$$A[7] = 99 < 120$$

$$\therefore low = mid + 1 = 7 + 1 = 8$$

A =

12	34	37	45	57	82	99	120	134
1	2	3	4	5	6	7	8	9
							low	high

Step: 3

low = 8

high = 9

low ≤ high i.e $8 < 9$

$$mid = \left(\frac{8+9}{2} \right) = 8$$

$$A(8) = 120 \Rightarrow 120$$

$$\therefore A(8) = key$$

$\therefore$ location = 8

$\therefore$ location of 120 is '8'.

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⑥ solve the following recurrence equation

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 4T(n-1) - 3T(n-2) & \text{otherwise.} \end{cases}$$

Sol Given that $T(0) = 0$, $T(1) = 1$, $T(n) = 4T(n-1) - 3T(n-2)$

$\Rightarrow Tn - 4T(n-1) + 3T(n-2) = 0$

Step: 1 Characteristic equation:

$T(n) - 4T(n-1) + 3T(n-2) = 0$

$r^2 - 4r + 3 = 0$

Step: 2 solve the equation to find zeroes

$r^2 - 4r + 3 = 0$ $\quad 3r^2$

$r^2 - 3r - r + 3 = 0$ $\quad -3r \quad -r$

$r(r-3) - 1(r-3) = 0$

$(r-1) \cdot (r-3) = 0$

$r = 1, r = 3.$

i.e $r_1 = 1, r_2 = 3$

Step: 3 write the solution:

$T(n) = C_1 r_1^n + C_2 r_2^n$

$\Rightarrow T(n) = C_1 (1)^n + C_2 (3)^n$

Step: 4 find C_1 & C_2

Given that,

$T(0) = 0$

$\Rightarrow C_1 (1)^0 + C_2 (3)^0 = 0$

$\Rightarrow C_1 + C_2 = 0$ — (1)

Also given $T(1) = 1$

$\Rightarrow c_1(1)^1 + c_2(3)^1 = 1$

$\Rightarrow c_1 + 3c_2 = 1$ —— (2)

On solving equation (1) & (2), we get

$2c_2 = 1 \rightarrow c_2 = \frac{1}{2}$

Now, $c_1 = -\frac{1}{2}$

So the solution is $\boxed{T(n) = \left(-\frac{1}{2}\right)1^n + \left(\frac{1}{2}\right)3^n}$

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(7) Solve the following recurrence equation

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ -6T(n-1) - 11T(n-2) - 6T(n-3) & \text{otherwise} \end{cases}$$

Sol Given that

$T(0) = 0$, $T(1) = 1$, $T(2) = 2$, $T(n) = -6T(n-1) - 11T(n-2) - 6T(n-3)$

$\Rightarrow T(n) + 6T(n-1) + 11T(n-2) + 6T(n-3) = 0$

Step:1 Characteristic equation:

$r^3 + 6r^2 + 11r + 6 = 0$

Step:2 Solve above equation to find the zeros

By trail & error method, let $r = -1$

$\Rightarrow (-1)^3 + 6(-1)^2 + 11(-1) + 6$

$= -1 + 6 - 11 + 6 = 0$

Hence, r can be -1 so $r=-1$ is

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

so, the quadratic equation will be

$$r^2+5r+6=0$$

$$\Rightarrow r^2+3r+2r+6=0$$

$$r(r+3)+2(r+3)=0$$

$$(r+2)(r+3)=0$$

$$r=-2,\ r=-3$$

$$\therefore r_1=-1,\ r_2=-2,\ r_3=-3$$

Step:3 write the solution

$$T(n)=C_1 r_1^n+C_2 r_2^n+C_3 r_3^n$$

$$\Rightarrow T(n)=C_1(-1)^n+C_2(-2)^n+C_3(-3)^n$$

Step:4 Find $C_1, C_2, \& C_3$

Given $T(0)=0$

$$\Rightarrow C_1+C_2+C_3=0 \quad (1)$$

Also given $T(1)=1$

$$\Rightarrow C_1(-1)^1+C_2(-2)^1+C_3(-3)^1=1$$

$$\Rightarrow -C_1-2C_2-3C_3=1 \quad (2)$$

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$\Rightarrow c_1(-1)^2 + c_2(-2)^2 + c_3(-3)^2 = 2$

$\Rightarrow c_1 + 4c_2 + 9c_3 = 2$ — (3)

solving equations (1), (2) & (3), we get

$$c_1 = -c_2 - c_3$$

$\Rightarrow$ from (2), $-c_2 - 2c_3 = 1$ — (4)

from (3), $3c_2 + 8c_3 = 5$ — (5)

from (4) & (5)

$$-3c_2 - 6c_3 = 3$$
$$3c_2 + 8c_3 = 2$$
$$2c_3 = 5$$
$$c_3 = 5/2$$

$\Rightarrow$ from (4) $\quad -c_2 - \frac{\cancel{10}^{5}}{\cancel{2}} = 1$

$$\Rightarrow c_2 = -6$$

$\Rightarrow$ from (1), $c_1 - \frac{7}{2} = 0$

$$c_1 = 7/2$$

$\therefore c_1 = 7/2,\ c_2 = -6,\ c_3 = 5/2$

$\therefore$ The recurrence equation after solving will be,

$$\boxed{T(n) = \frac{7}{2}(-1)^n - 6(-2)^n + \frac{5}{2}(-3)^n}$$

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