CS5300 Advanced Algorithms

HW # 3

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1) Consider the following reccurence equation, defining $T(n)$, as

$$T(n) = \begin{cases} 3 & \text{if } n=1 \\ T(n-1)+3 & \text{otherwise} \end{cases}$$

Show by induction, that $T(n) = 3n$

Proof: Step 1! Basis if $n=1 \Rightarrow T(1) = 3(1) = 3$

Step2: Inductive hyp: Assume $T(k) = 3k \ \forall \ k \leq n$.

In Particular let $k=n-1 \Rightarrow \boxed{T(n-1) = 3(n-1)}$

Step3:- Solve $T(n) = 3n \ \forall n$

$$\begin{aligned} T(n) &= T(n-1)+3 \\ &= 3(n-1)+3 \\ &= 3n-3+3 \\ &= 3n \end{aligned}$$

$\therefore$ Hence $T(n) = 3n \ \forall \ n$

2) Consider the following reccurence equation, defining $T(n)$, as.

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-1)+n^2 & \text{otherwise} \end{cases}$$

Show by Induction, that $T(n) = \frac{n(n+1)(2n+1)}{6}$.

Sol: Step 1: Basis if $n=1 \Rightarrow T(1) = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$

Step2: Inductive hyp:

Assume $T(k) = \frac{k(k+1)(2k+1)}{6}. \ \forall \ k \leq n$.

In Particular let $k=n-1$

$$T(n-1) = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6.} = \frac{n(n-1)(2n-1)}{6}$$

$$\Rightarrow \boxed{T(n-1) = \frac{n(n-1)(2n-1)}{6}}$$

<u>Step3:</u> show $T(n) = \frac{n(n+1)(2n+1)}{6} \ \forall n$.

$$T(n) = T(n-1) + n^2$$

$$= \frac{n(n-1)(2n-1)}{6} + n^2 \Rightarrow \frac{(n^2-n)(2n-1)+6n^2}{6}$$

$$\Rightarrow \frac{2n^3-n^2-2n^2+n+6n^2}{6}$$

$$\Rightarrow \frac{2n^3+3n^2+n}{6}$$

$$\Rightarrow \frac{n(2n^2+3n+1)}{6}$$

$$\Rightarrow \frac{n(n+1)(2n+1)}{6}.$$

$\therefore$ Hence $T(n) = \frac{n(n+1)(2n+1)}{6} \ \forall n$

3) Show that the Solution for $T(n) = T(\sqrt{n}) + \Theta(1)$ is.

$T(n) = \Theta(lg(lg(n))$

Sol: Given, $T(n) = T(\sqrt{n}) + \Theta(1)$

Let $n = 2^k$

Apply log on Both sides.

$\Rightarrow \log_0 n = k \Rightarrow \sqrt{n} = 2^{k/2}$

$\Rightarrow T(2^k) = T(\sqrt{2k}) + \Theta(1)$

let $T(n) = T(2^k) = S(k)$

if $k = \frac{k}{2}$

$\Rightarrow S(\frac{k}{2}) = T(2^{k/2})$.

$\Rightarrow S(k) = S(k/2) + \Theta(1)$

Now Applying Master theorem,

$a=1,\ b=2,\ f(k) = 1$

$\log_2 2 = 1$

$k^{\log_b a}$ Vs $f(k)$.

$k^{\log_2 1}$ Vs 1

k^0 Vs $1 \Rightarrow 1$ Vs 1

Case 2 Solution: $T(k) = \Theta(f(k) * lg k)$

$= \Theta(1 * lg k)$

$= \Theta(lg k)$

$\Rightarrow k = lg n$

$\Rightarrow T(n) = \Theta(lg(lg n))$

4) Draw the recursion tree for $T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{4n}{5}\right) + \theta(n)$ and find the height of the tree

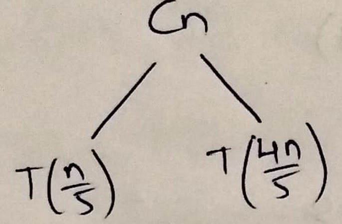
Sol:

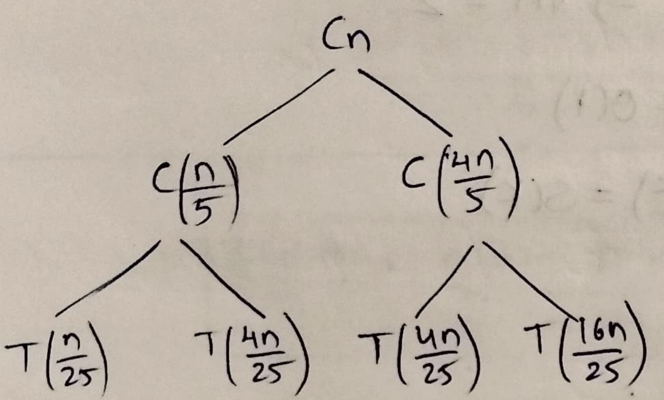
$$T\left(\frac{n}{5}\right) = T\left(\frac{\frac{n}{5}}{5}\right) + T\left(\frac{4\left(\frac{n}{5}\right)}{5}\right) + \theta\left(\frac{n}{5}\right)$$

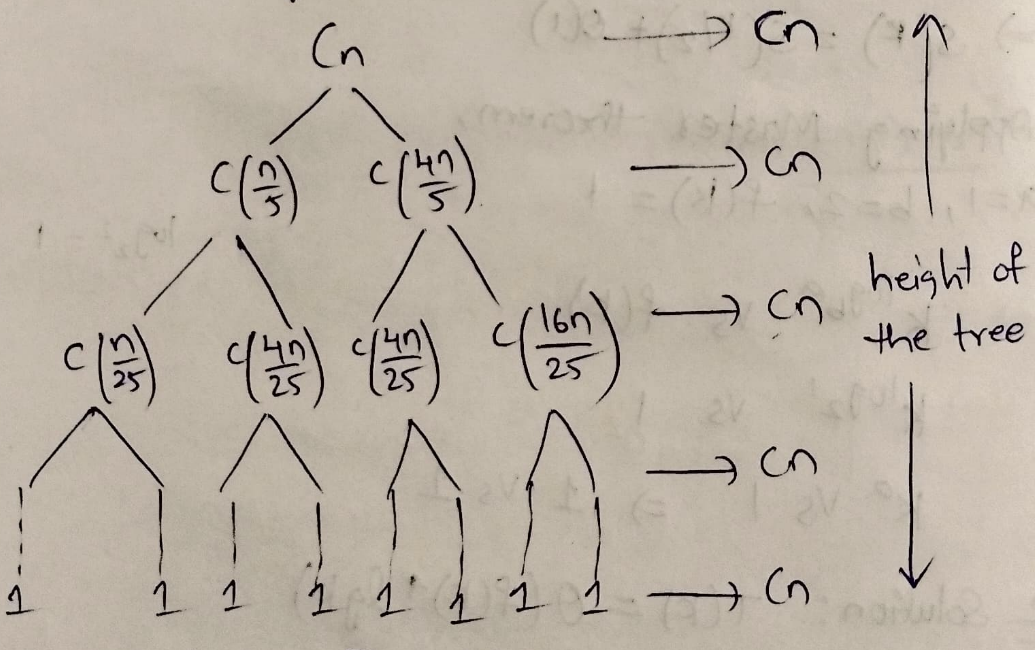
$$= T\left(\frac{n}{25}\right) + T\left(\frac{4n}{25}\right) + \theta\left(\frac{n}{5}\right)$$

$$T\left(\frac{4n}{5}\right) = T\left(\frac{\frac{4n}{5}}{5}\right) + T\left(\frac{4\left(\frac{4n}{5}\right)}{5}\right) + \theta\left(\frac{4n}{5}\right)$$

$$= T\left(\frac{4n}{25}\right) + T\left(\frac{16n}{25}\right) + \theta\left(\frac{4n}{5}\right)$$







Rec. Tree

Guess:-

$$T(n) = Cn + Cn + Cn + \cdots + Cn$$

$$= Cn * \text{height}$$

Height:- $n \rightarrow \left(\frac{4}{5}\right)n \rightarrow \left(\frac{4}{5}\right)^2 n \rightarrow \cdots \rightarrow \left(\frac{4}{5}\right)^k \cdot n = 1$

where k is height

Solve $\left(\frac{4}{5}\right)^k \cdot n = 1$

$$\Rightarrow \left(\frac{4}{5}\right)^k = \frac{1}{n} \Rightarrow \left(\frac{5}{4}\right)^k = n.$$

Apply lg on Both sides

$$\lg_{\left(\frac{5}{4}\right)} \left(\frac{5}{4}\right)^k = \lg_{\left(\frac{5}{4}\right)} n. \Rightarrow k \cdot \underbrace{\lg_{\left(\frac{5}{4}\right)} \left(\frac{5}{4}\right)}_{1} = \lg_{\left(\frac{5}{4}\right)} n.$$

$$\Rightarrow \boxed{k = \lg_{\left(\frac{5}{4}\right)} n.}$$

Guess:

$$\Rightarrow T(n) = cn * \text{height}$$

$$= cn * \lg_{\left(\frac{5}{4}\right)} n \Rightarrow T(n) = \Theta(n \lg n)$$

5) Solve the following recurrence:

$$T(n) = 16T\left(\frac{n}{2}\right) + \Theta(n^4 \lg^2 n)$$

Sol: $a = 16,\quad b = 2,\quad f(n) = n^4 \lg^2 n.$

$$n^{\log_b a} \text{ vs } f(n) \qquad (\because \log_2 16 = 4)$$

$$n^{\log_2 16} \text{ vs } n^4 \lg^2 n$$

Case 2: $T(n) = \Theta(f(n) * \lg n)$

$$= \Theta(n^4 \lg^2 n * \lg n)$$

$$= \Theta(n^4 \lg^3 n^2)$$

6) Solve the following recurrence equation

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n^2 \lg n)$$

Sol:- $a=3,\ b=2,\ f(n) = n^2 \lg n.$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 3} \text{ vs } n^2 \lg n.$$

Case 3:- $T(n) = \Theta(f(n)) = \Theta(n^2 \lg n)$

with $\varepsilon = 2 - \log_2 3$

Reg. Condition: $a \cdot f\left(\frac{n}{b}\right) = 3 \cdot f\left(\frac{n}{2}\right)$

$$= 3 \cdot \left(\frac{n}{2}\right)^2 \lg\left(\frac{n}{2}\right)$$

$$= \frac{3n^2}{4} \lg \frac{n}{2}$$

$$\leq \frac{3}{4} n^2 \lg \frac{n}{2} \leq c \cdot f(n)$$

$$\boxed{c = \frac{3}{4}}$$

where $0 < \frac{3}{4} < 1$

7) Solve the following recurrence

$$T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Sol:- $a=7,\ b=2,\ f(n) = n^2$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 7} \text{ vs } n^2$$

Case 1: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 7})$

$$= \Theta(n^{\log_2 7})$$

$$\Rightarrow \boxed{T(n) = \Theta(n^{\log_2 7})}$$

where $\varepsilon = \log_2 7 - 2$

8) Solve the following recurrence equation.

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Sol:- $a=4,\ b=2,\ f(n)=n^2$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 4} \text{ vs } n^2$$

$$n^2 \text{ vs } n^2$$

Case 2: $T(n) = \Theta(f(n) * \lg(n))$

$$= \Theta(n^2 \lg n)$$

9) Solve the following recurrence:

$$T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^3).$$

Sol:- $a=5,\ b=2,\ f(n)=n^3$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 5} \text{ vs } n^3$$

Case 3: $T(n) = \Theta(f(n)) = \Theta(n^3)$

where $\varepsilon = 3 - \log_2 5$

Reg. Condition: $a f\left(\frac{n}{b}\right) = 5 \cdot f\left(\frac{n}{2}\right)$

$$= 5 \cdot \left(\frac{n}{2}\right)^3$$

$$a \cdot f\left(\frac{n}{b}\right) = 5\left(\frac{n^3}{8}\right)$$

$$= \frac{5}{8} n^3 \leq \frac{5}{8} n^3$$

$$\therefore \boxed{c = \frac{5}{8}}$$

where $0 < \frac{5}{8} < 1$

10) $T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^2)$

Solve the following recurrence:

<u>Sol:-</u> $a=5,\ b=2,\ f(n) = n^2$

$$n^{\log_b a} \text{ vs } f(n)$$

$$n^{\log_2 5} \text{ vs } n^2$$

<u>Case 1:</u> $T(n) = \Theta\left(n^{\log_b a}\right)$

$$= \Theta\left(n^{\log_2 5}\right)$$

with $\varepsilon = \log_2 5 - 2$