Sample Space

A set S that consists of all possible outcomes of a random experiment is called a sample space, and each outcome is called a sample point.

EXAMPLE 1. If we toss a die, one sample space, or set of all possible outcomes, is given by {1, 2, 3, 4, 5, 6} while another is {odd, even}.

Events

An event is a subset A of the sample space S, i.e., it is a set of possible outcomes. If the outcome of an experiment is an element of A, we say that the event A has occurred. An event consisting of a single point of S is often called a simple or elementary event.

EXAMPLE 1.8 If we toss a coin twice, the event that only one head comes up is the subset of the sample space that consists of points (0, 1) and (1, 0), as indicated in Fig. 1-2.

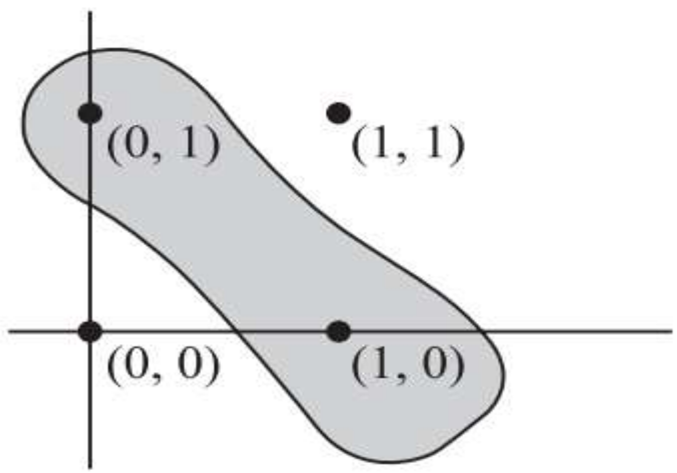
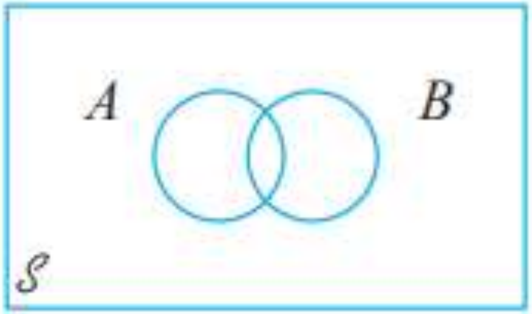


Fig. 1-2

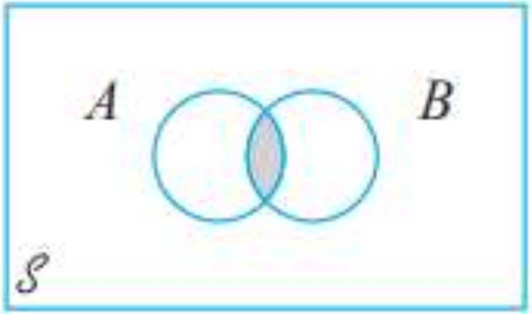
Some Relations from Set Theory

1. The **complement** of an event A, denoted by A', is the set of all outcomes in $\mathcal{S}$ that are not contained in A.
2. The **union** of two events A and B, denoted by $A \cup B$ and read "*A or B*," is the event consisting of all outcomes that are *either in A or in B or in both events* (so that the union includes outcomes for which both A and B occur as well as outcomes for which exactly one occurs)—that is, all outcomes in at least one of the events.
3. The **intersection** of two events A and B, denoted by $A \cap B$ and read "*A and B*," is the event consisting of all outcomes that are in *both A and B*.

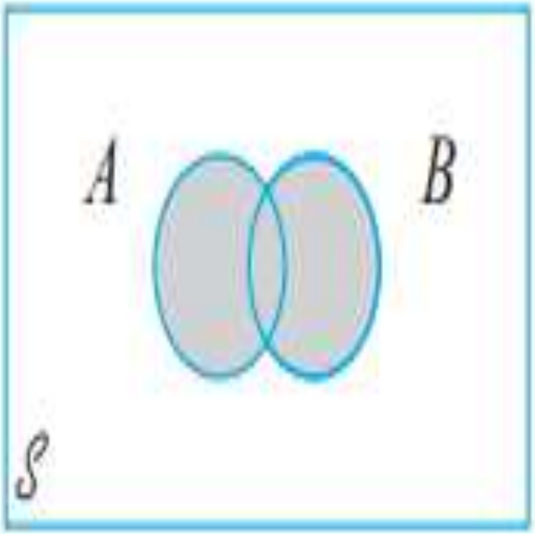
Let $\varnothing$ denote the *null event* (the event consisting of no outcomes whatsoever). When $A \cap B = \varnothing$, A and B are said to be **mutually exclusive** or **disjoint** events.



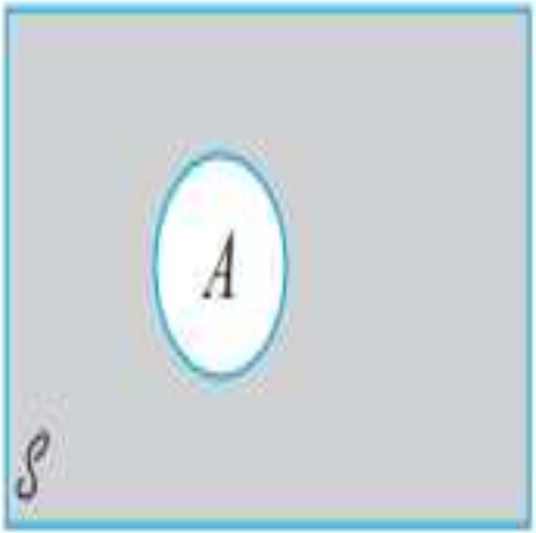
(a) Venn diagram of events A and B



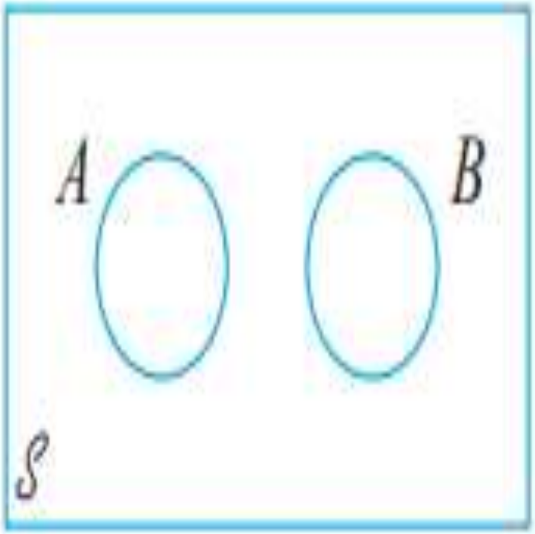
(b) Shaded region is $A \cap B$



(c) Shaded region is $A \cup B$



(d) Shaded region is A'



(e) Mutually exclusive events

Axioms, Interpretations, and Properties of Probability

AXIOM 1 For any event A, $P(A) \geq 0$.

AXIOM 2 $P(\mathcal{S}) = 1$.

AXIOM 3 If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

$P(\varnothing) = 0$ where $\varnothing$ is the null event

More Probability Properties

For any event A, $P(A) + P(A') = 1$, from which $P(A) = 1 - P(A')$.

Proof In Axiom 3, let $k = 2$, $A_1 = A$, and $A_2 = A'$. Since by definition of A', $A \cup A' = \mathcal{S}$ while A and A' are disjoint, $1 = P(\mathcal{S}) = P(A \cup A') = P(A) + P(A')$. ■

PROPOSITION

For any event A, $P(A) \leq 1$.

This is because $1 = P(A) + P(A') \geq P(A)$ since $P(A') \geq 0$.

When events A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$. For events that are not mutually exclusive, adding $P(A)$ and $P(B)$ results in "double-counting" outcomes in the intersection. The next result shows how to correct for this.

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof Note first that $A \cup B$ can be decomposed into two *disjoint* events, A and $B \cap A'$; the latter is the part of B that lies outside A (see Figure 2.4). Furthermore, B itself is the union of the two disjoint events $A \cap B$ and $A' \cap B$, so $P(B) = P(A \cap B) + P(A' \cap B)$. Thus

$$\begin{aligned} P(A \cup B) = P(A) + P(B \cap A') &= P(A) + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

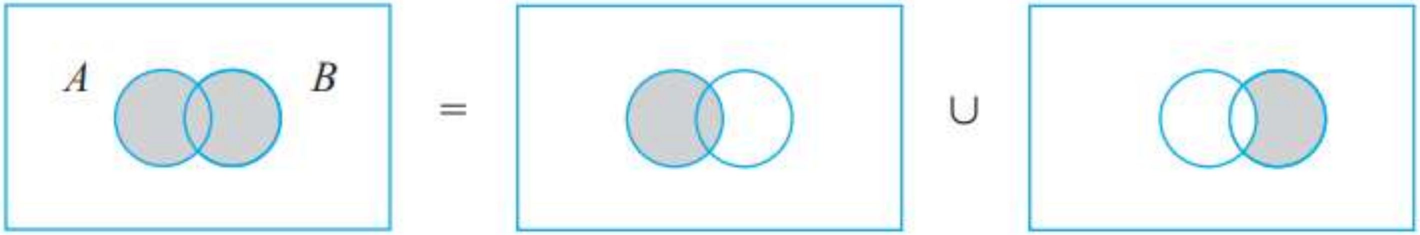


Figure 2.4 Representing A ∪ B as a union of disjoint events ■

Example

In a certain residential suburb, 60% of all households get Internet service from the local cable company, 80% get television service from that company, and 50% get both services from that company. If a household is randomly selected, what is the probability that it gets at least one of these two services from the company, and what is the probability that it gets exactly one of these services from the company?

With $A = \{\text{gets Internet service}\}$ and $B = \{\text{gets TV service}\}$, the given information implies that $P(A) = .6$, $P(B) = .8$, and $P(A \cap B) = .5$. The foregoing proposition now yields

P(subscribes to at least one of the two services)

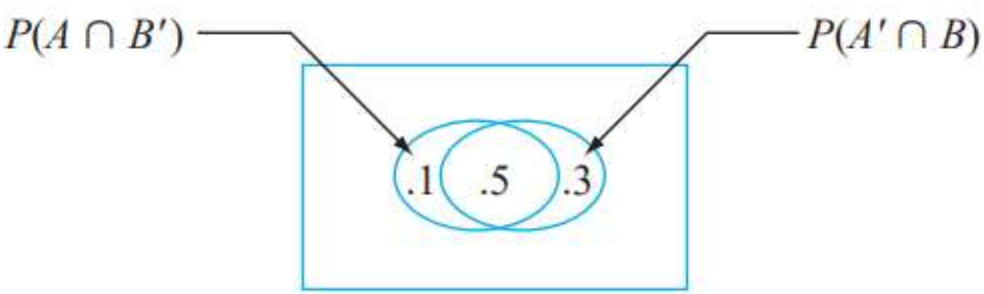
$$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .8 - .5 = .9$$

The event that a household subscribes only to tv service can be written as $A' \cap B$ [(not Internet) and TV]. Now Figure 2.4 implies that

$$.9 = P(A \cup B) = P(A) + P(A' \cap B) = .6 + P(A' \cap B)$$

from which $P(A' \cap B) = .3$. Similarly, $P(A \cap B') = P(A \cup B) - P(B) = .1$. This is all illustrated in Figure 2.5, from which we see that

$$P(\text{exactly one}) = P(A \cap B') + P(A' \cap B) = .1 + .3 = .4$$



For any three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ - P(B \cap C) + P(A \cap B \cap C)$$

1. **If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and high selectivity is 0.18, what is probability that a system with high fidelity will also have high selectivity?**

<u>Solution</u> : If A is the event that a communication system has high selectivity and B is the event that it has high fidelity, we have $P(B) = 0.81$ and $P(A \cap B) = 0.18$, and substitution into the formula yields

$$\boxed{P(A \mid B) = \frac{0.18}{0.81} = \frac{2}{9}}$$

General multiplication rule of probability

Theorem 3.8 If A and B are any events in $\mathcal{S}$, then

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B \mid A) \qquad \text{if} \quad P(A) \neq 0 \\ &= P(B) \cdot P(A \mid B) \qquad \text{if} \quad P(B) \neq 0 \end{aligned}$$

2.The supervisor of a group of 20 construction workers wants to get the opinion of 2 of them (to be selected at random) about certain new safety regulations. If 12 workers favor the new regulations and the other 8 are against them, what is the probability that both of the workers chosen by the supervisor will be against the new safety regulations?

Solution Assuming equal probabilities for each selection (which is what we mean by the selections being random), the probability that the first worker selected will be against the new safety regulations is $\frac{8}{20}$, and the probability that the second worker selected will be against the new safety regulations given that the first one is against them is $\frac{7}{19}$. Thus, the desired probability is

$$\frac{8}{20} \cdot \frac{7}{19} = \frac{14}{95}$$

Special product rule of probability

Theorem 3.9 Two events A and B are independent events if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

3.What is the probability of getting two heads in two flips of a balanced coin?
Solution Since the probability of heads is 1/2 for each flip and the two flips are not physically connected, we treat them as independent. The probability is 1/2 . 1/2 = 1/4

Independence and selection with and without replacement

4.Two cards are drawn at random from an ordinary deck of 52 playing cards. What is the probability of getting two aces if
(a) the first card is replaced before the second card is drawn;
(b) the first card is not replaced before the second card is drawn?

Solution **(a)** Since there are four aces among the 52 cards, we get

$$\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$$

(b) Since there are only three aces among the 51 cards that remain after one ace has been removed from the deck, we get

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

Note that

$$\frac{1}{221} \neq \frac{4}{52} \cdot \frac{4}{52}$$

so independence is violated when the sampling is without replacement. ■

You are given the following information about 25 members of a sports club. Choose a person at random.

	Heavy Drinker	Moderate Drinker	Non-Drinker	
Women	1	7	2	**10**
Men	4	8	3	**15**
Total	**5**	**15**	**5**	**25**

a) What is the probability the chosen person is a man and a moderate drinker?

b) If the chosen person is a man, what is the probability he is a heavy drinker?

c) Are the events women and heavy drinker, mutually exclusive?

d) Find two independent events.

You are given the following information about 25 members of a sports club. Choose a person at random.

	Heavy Drinker	Moderate Drinker	Non-Drinker	
Women	1	7	2	**10**
Men	4	8	3	**15**
Total	**5**	**15**	**5**	**25**

a) What is the probability the chosen person is a man and a moderate drinker?

8/25=.32

b) If the chosen person is a man, what is the probability he is a heavy drinker?

4/15=.267

You are given the following information about 25 members of a sports club. Choose a person at random.

	Heavy Drinker	Moderate Drinker	Non-Drinker	
Women	1	7	2	**10**
Men	4	8	3	**15**
Total	**5**	**15**	**5**	**25**

c) Are the events women and heavy drinker, mutually exclusive?

NO

d) Find two independent events.

Non-drinker and Women

Non-drinker and Man

Reverend Thomas Bayes (1701-1761),
studied logic and theology as an undergraduate student
at the University of Edinburgh from 1719-1722.

Bayes' Theorem

Bayes Theorem

Let A and B be two events from a (countable) sample space Ω, and $P : \Omega \to [0, 1]$ a probability distribution on Ω, such that $0 < P(A) < 1$, and $P(B) > 0$. Then

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A})}$$

Proof of Bayes' Theorem:

Let A and B be events such that $0 < P(A) < 1$ and $P(B) > 0$.

By definition, $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. So: $P(A \cap B) = P(A \mid B)P(B)$.

Likewise, $P(B \cap A) = P(B \mid A)P(A)$.

Likewise, $P(B \cap \overline{A}) = P(B \mid \overline{A})P(\overline{A})$. (Note that $P(\overline{A}) > 0$.)

Note that $P(A \mid B)P(B) = P(A \cap B) = P(B \mid A)P(A)$. So,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Furthermore,

$$\begin{aligned} P(B) &= P((B \cap A) \cup (B \cap \overline{A})) = P(B \cap A) + P(B \cap \overline{A}) \\ &= P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A}) \end{aligned}$$

So: $$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A})}. \quad \square$$

Generalized Bayes' Theorem

Suppose that $E, F_1, \ldots, F_n$ are events from sample space Ω, and that $P : \Omega \rightarrow [0, 1]$ is a probability distribution on Ω. Suppose that $\cup_{i=1}^{n} F_j = \Omega$, and that $F_i \cap F_j = \emptyset$ for all $i \neq j$.
Suppose $P(E) > 0$, and $P(F_j) > 0$ for all j. Then for all j:

$$P(F_j \mid E) = \frac{P(E \mid F_j)P(F_j)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)}$$

Proof of Generalized Bayes' Theorem: Very similar to the proof of Bayes' Theorem. Observe that:

$$P(F_j \mid E) = \frac{P(F_j \cap E)}{P(E)} = \frac{P(E \mid F_j)P(F_j)}{P(E)}$$

So, we only need to show that $P(E) = \sum_{i=1}^{n} P(E \mid F_i)P(F_i)$.
But since $\bigcup_i F_i = \Omega$, and since $F_i \cap F_j = \emptyset$ for all $i \neq j$:

$$\begin{aligned} P(E) &= P(\bigcup_i (E \cap F_i)) \\ &= \sum^{n} P(E \cap F_i) \quad \text{(because } F_i\text{'s are disjoint)} \end{aligned}$$

$$= \sum_{i=1}^{n} P(E \mid F_i) P(F_i). \quad \square$$

Random Variable Definition

- **A random variable is a rule that assigns a numerical value to each outcome in a sample space.**
- **Random variables may be either discrete or continuous. A random variable is said to be discrete if it assumes only specified values in an interval.**
- **Otherwise, it is continuous.**
- **We generally denote the random variables with capital letters such as X and Y.**
- **When X takes values 1, 2, 3, …, it is said to have a discrete random variable.**

Variate

- ❑A variate can be defined as a generalization of the random variable.
- ❑It has the same properties as that of the random variables without stressing to any particular type of probabilistic experiment.
- ❑It always obeys a particular probabilistic law.
- ❑A variate is called discrete variate when that variate is not capable of assuming all the values in the provided range.
- ❑If the variate is able to assume all the numerical values provided in the whole range, then it is called continuous variate.

Types of Random Variable

As discussed in the introduction, there are two random variables, such as:

- ✓ **Discrete Random Variable**
- ✓ **Continuous Random Variable**

➢ Discrete Random Variable

A discrete random variable can take only a finite number of distinct values such as 0, 1, 2, 3, 4, … and so on. The probability distribution of a random variable has a list of probabilities compared with each of its possible values known as probability mass function.

Continuous Random Variable

- **A numerically valued variable is said to be continuous if, in any unit of measurement, whenever it can take on the values a and b.**
- **If the random variable X can assume an infinite and uncountable set of values, it is said to be a continuous random variable.**
- **When X takes any value in a given interval (a, b), it is said to be a continuous random variable in that interval.**

Random Variable Formula

For a given set of data the mean and variance random variable is calculated by the formula. So, here we will define two major formulas:

- Mean of random variable
- Variance of random variable

Mean of random variable:

If X is the random variable and P is the respective probabilities, the mean of a random variable is defined by:

$$\textbf{Mean } (\mu) = \sum XP$$

where variable X consists of all possible values and P consist of respective probabilities.

Variance of Random Variable:

The variance tells how much is the spread of random variable X around the mean value. The formula for the variance of a random variable is given by;

$Var(X) = \sigma^2 = E(X^2) - [E(X)]^2$

where $E(X^2) = \sum X^2 P$ and $E(X) = \sum XP$

Random Variable Example

Question: Find the mean value for the continuous random variable, f(x) = x, 0 ≤ x ≤ 2.

Solution:

Given: f(x) = x, 0 ≤ x ≤ 2.

The formula to find the mean value is:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(X) = \int_0^2 x f(x) dx$$

$$E(X) \int_0^2 x . x dx$$

$$E(X) \int_0^2 x^2 dx$$

$$E(X) = \left(\frac{x^3}{3}\right)_0^2$$

$$E(X) = \left(\frac{2^3}{3}\right) - \left(\frac{0^3}{3}\right)$$

$$E(X) = \left(\frac{8}{3}\right) - (0)$$

$$E(X) = \frac{8}{3}$$

Therefore, the mean of the continuous random variable, E(X) = 8/3.

What is meant by a random variable?

Solution:-

A random variable is a rule that assigns a numerical value to each outcome in a sample space, or it can be defined as a variable whose value is unknown or a function that gives numerical values to each of an experiment's outcomes.

What is a random variable and its types?

Solution:-

As we know, a random variable is a rule or function that assigns a numerical value to each outcome of the experiment in a sample space.

There are two types of random variables, i.e. discrete and continuous random variables.

How do you identify a random variable?

Solution:-

In general, random variables are represented by capital letters for example, X and Y.

How do you know whether a random variable is continuous or discrete?

A discrete variable is a variable whose value can be obtained by counting since it contains a possible number of values that we can count. In contrast, a continuous variable is a variable whose value is obtained by measuring.

What are the examples of a discrete random variable?

Solution:-

The probability of any event in an experiment is a number between 0 and 1, and the sum of all the probabilities of the experiment is equal to 1. Examples of discrete random variables include the number of outcomes in a rolling die, the number of outcomes in drawing a jack of spades from a deck of cards and so on.

Moment Generating Functions

Moment generating functions are useful for several reasons, one of which is their application to analysis of sums of random variables. Before discussing MGFs, let's define moments.

Definition The *n*th moment of a random variable X is defined to be

is defined to be $E[X^n]$. The *n*th central moment of X is defined to be $E[(X-EX)^n]$.

For example, the first moment is the expected value E[X]. The second central moment is the variance of X. Similar to mean and variance, other moments give useful information about random variables.

The moment generating function (MGF) of a random variable X is a function $M_X(s)$ defined as

$$M_X(s) = E\left[e^{sX}\right].$$

We say that MGF of X exists, if there exists a positive constant a such that $M_X(s)$ is finite for all $s \in [-a, a]$.

Example For each of the following random variables, find the MGF.

a. X is a discrete random variable, with PMF

$$P_X(k) = \begin{cases} \frac{1}{3} & k = 1 \\ \\ \frac{2}{3} & k = 2 \end{cases}$$

b. Y is a $Uniform(0,1)$ random variable.

Solution

a. For X, we have

$$\begin{aligned} M_X(s) &= E\left[e^{sX}\right] \\ &= \frac{1}{3}e^{s} + \frac{2}{3}e^{2s}. \end{aligned}$$

which is well-defined for all $s \in \mathbb{R}$.

b. For Y, we can write

$$\begin{aligned} M_Y(s) &= E\left[e^{sY}\right] \\ &= \int_0^1 e^{sy}\, dy \\ &= \frac{e^{s} - 1}{s}. \end{aligned}$$

Note that we always have $M_Y(0) = E[e^{0 \cdot Y}] = 1$, thus $M_Y(s)$ is also well-defined for all $s \in \mathbb{R}$.

Random variables

A **random variable** is any function that assigns a numerical value to each possible outcome.

Probability distributions

The **probability distribution** of a discrete random variable X is a list of the possible values of X together with their probabilities

$$f(x) = P[X = x]$$

The probability distribution always satisfies the conditions

$$f(x) \geq 0 \quad \text{and} \quad \sum_{\text{all } x} f(x) = 1$$

EXAMPLE 1 **Checking for nonnegativity and total probability equals one**

Check whether the following can serve as probability distributions:

(a) $f(x) = \dfrac{x-2}{2}$ for $x = 1, 2, 3, 4$

(b) $h(x) = \dfrac{x^2}{25}$ for $x = 0, 1, 2, 3, 4$

Solution (a) This function cannot serve as a probability distribution because $f(1)$ is negative.

(b) The function cannot serve as a probability distribution because the sum of the five probabilities is $\frac{6}{5}$ and not 1. ■