Normal Distribution

Normal distribution is the most popular and commonly used distribution. It was discovered by De Moivre in 1733 after 20 years when Bernoulli gave Binomial distribution.

This distribution is a limiting case of Binomial distribution when neither $p \rightarrow q$ is too small and n, the number of trials becomes infinitely large i.e. $n \rightarrow \infty$.

In fact any quantity whose variation depends on random cause will be distributed according to the normal distribution whereas in Binomial and Poisson distribution X assume value like $0, 1, 2 \ldots$ and thus these distribution are discrete distribution.

The continuous random variable X is said to have a normal distribution, if its probability density function is defined as

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ;} \quad -\infty \leq X \leq \infty \;, \quad \sigma > 0 .$$

Where μ and σ are parameters of normal distribution.

Mean of the Normal Distribution :-

The normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty \leq X \leq \infty$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$ then $\frac{dx}{\sigma} = dz \Rightarrow dx = \sigma dz$

and $x = \mu + \sigma z$ limits are unaltered.

$$\mu = E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{\mu \cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz\right] + \frac{1}{\sigma\sqrt{2\pi}} \sigma^2 \left[\int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz\right]$$

$$= \frac{\mu}{\cancel{\sigma}\sqrt{2\pi}} \cdot 2\cancel{\sigma} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + 0$$

$\left\{ \because z e^{-\frac{1}{2}z^2} \text{ is an odd and } e^{-\frac{1}{2}z^2} \text{ is even funct} \right\}$

$$= \frac{2\mu}{\sqrt{2}\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\sqrt{2}}$$

$\left\{ \int_{0}^{\infty} e^{-\left(\frac{z}{2}\right)^2} dz = \sqrt{\frac{\pi}{2}} \right\}$

$= \mu$. $\therefore$ The mean of the normal distribution is μ.

Variance of the Normal distribution

since

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$= E(X-\mu)^2$$

Now

$$E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma dz$ limits are unaltered

var(X)

$$= E(x-\mu)^2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

put $\frac{z^2}{2} = t \Rightarrow z dz = dt \Rightarrow dz = \frac{dt}{z}$

$$\Rightarrow dz = \frac{dt}{\sqrt{2t}}$$

$$var(x) = \frac{\sigma^2}{\sqrt{2\pi}} \times 2 \int_0^{\infty} 2t \cdot e^{-t} \cdot \frac{dt}{\sqrt{2t}}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \times 2 \times \sqrt{2} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt$$

$$\operatorname{Var}(X) = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\frac{3}{2} = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{\cancel{2}}\cancel{\sqrt{\pi}} = \sigma^2$$

$$\boxed{\operatorname{Var}(X) = \sigma^2}$$

Variance of normal distribution is σ^2.

Median of the normal distribution :-

Suppose 'M' is median of normal distribution

then $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$

Now take, $\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma dz$

Also limits are when $x = -\infty$ then $z = -\infty$

Also when $x = \mu$, then $= z = 0$.

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{0} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} = 1/2.$$

$$\Rightarrow \frac{1}{2} + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{\mu}^{M} f(x)\,dx = 0 \quad \left\{ \text{if } \int_{a}^{b} f(x)\,dx = 0 \text{ then } a = b \right\}$$

$$\Rightarrow \boxed{\mu = M}$$

Hence, the median of the normal distribution is 'μ'.

Mode of the normal distribution :-

Mode of the normal distribution is the value of x at which $f(x)$ has maximum value. Then $f'(x) = 0$ and $f''(x) = -ve$ at that value of x.

Now the probability density function of 'x' is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \le X \le \infty$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left\{ \frac{-1}{\cancel{2}} \cdot \cancel{2}\left(\frac{x-\mu}{\sigma}\right) \right\}$$

$$= -\frac{1}{\sigma^2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} (x-\mu)$$

$$= 0 \quad \text{when at } (x = \mu).$$

and $f''(x) = (x-\mu)\left\{-\frac{1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right\}$

$\left\{-\frac{1}{2}\cdot 2\left(\frac{x-\mu}{\sigma}\right)\right\} + \left\{\frac{-1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right\}$

at $x=\mu$,

$$f''(\mu) = -\frac{1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2}$$

$$= -\frac{1}{\sigma^2\sqrt{2\pi}} e^{0} = -\frac{1}{\sigma^2\sqrt{2\pi}} < 0, \text{ where } \sigma > 0.$$

$\therefore f(x)$ has maximum value, when the value of $x=\mu$,

$\therefore$ The mode of the normal distribution is μ.

<u>Note</u> :- for the normal distribution, the mean, median and mode are equal.

i.e $\boxed{\text{Mean} = \text{Median} = \text{Mode}}$

Properties of the Normal distribution

1. The normal probability curve with mean μ and standard deviation σ is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right] \quad -\infty \le x \le \infty$$

2. The curve is bell-shaped and symmetrical about the line $x = \mu$.

3. Mean, Median and Mode of the normal distribution of coincide the normal distribution is uni modal.

4. $f(x)$ decreases rapidly as x increases

5. X-axis is an asymptote to the curve

6. The maximum probability occurs at the point $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$.

7. Mean deviation about mean $= \frac{4}{5}\sigma$.

8. Since $f(x)$, being the probability, can never the negative, so that no portion of the curve lies below the x-axis.

9. A linear function of independent normal variates is also normal variate.

10. The point of inflexion of the curve at $x = \mu \pm \sigma$

(i) Area of the normal curve between $(\mu - \sigma)$ and $(\mu + \sigma)$ is 0.6826

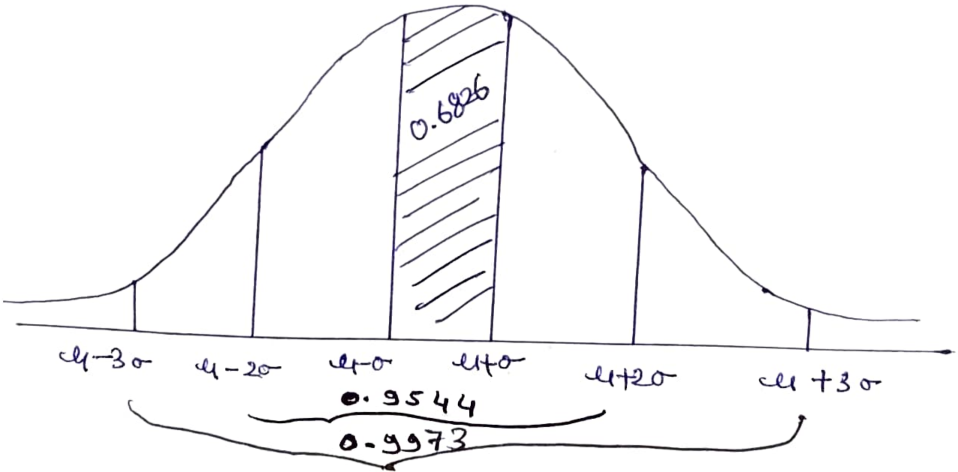
i.e. $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$
$= 68.26\%$

(ii) Area of the normal curve between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 0.9544

i.e. $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$
$= 95.44\%$

(iii) Area of the normal curve between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.9973

i.e. $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$
$= 99.73\%$



Example of Normal distributions

Ex. (1) If $\mu = 50$ and $\sigma = 10$ find : (i) $P(50 \le X \le 80)$, (ii) $P(60 \le X \le 70)$ (iii) $P(30 \le X \le 40)$ (iv) $P(40 \le X \le 60)$.

Use $P(0 \le Z \le 3) = 0.4987$, $P(0 \le Z \le 2) = 0.4772$, $P(0 \le Z \le 1) = 0.1359$.

Soln:- We know that standard normal variate

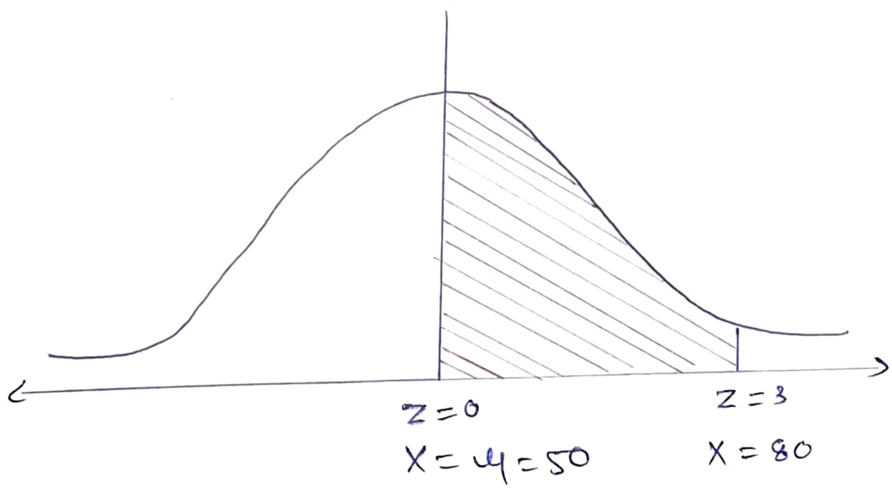
$$Z = \frac{X - \mu}{\sigma} = \frac{X - 50}{10}.$$

(i)

$$Z = \frac{50 - 50}{10} = 0 \quad \text{when } X = 50$$

$$Z = \frac{80 - 50}{10} = \frac{30}{10} = 3 \quad \text{when } X = 80$$

Hence $P(50 \le X \le 80) = P(0 \le Z \le 3) = 0.4987$

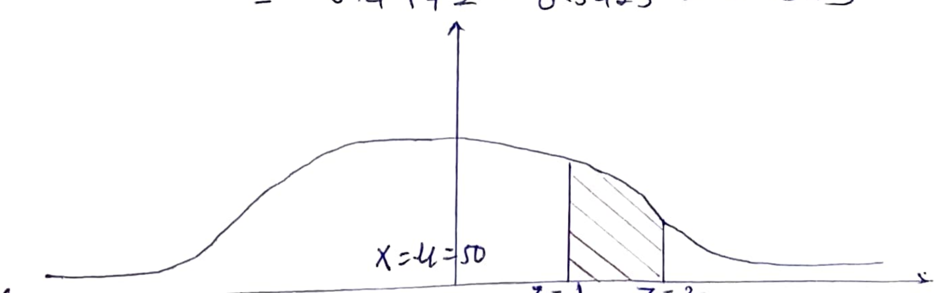


(ii) $P(60 \le X \le 70) = P(1 \le Z \le 2)$

$=$ Area from $Z = 1$ to $Z = 2$.

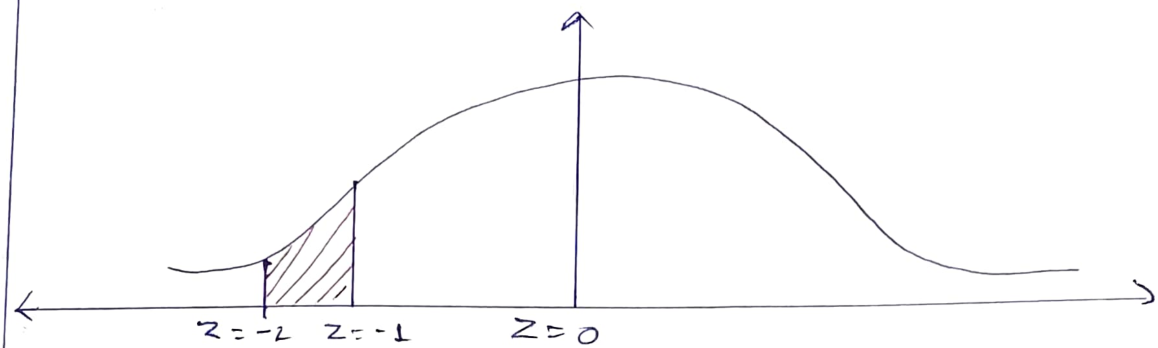
$=$ (Area from $Z = 0$ to $Z = 2$) $-$ (Area from $Z = 0$ to 1)

$= 0.4772 - 0.3413 = 0.1359$



(iii) $P(30 \le X \le 4) = P(-2 \le z \le -1)$

Due to symmetry, area between $z = -1$ to $z = -2$ will be same as between $z = 1$ to $z = 2$, which is the same as (ii), i.e. 0.1359.

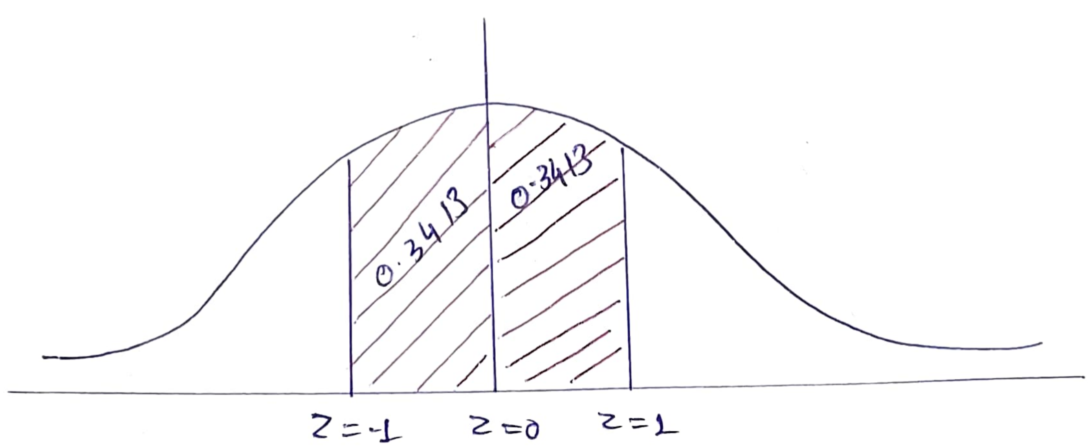


(iv) $P(40 \le X \le 60) = P(-1 \le z \le 1)$

= area between $z = -1$ to $z = 1$

= twice the area between $z = 0$ to $z = 1$.

$= 2 \times 0.3413 = 0.6826$.



Gamma distribution :-

A continuous random variable X is said to be follow gamma distribution with parameter α and d if its p.d.f is

$$f(x) = \begin{cases} \dfrac{\alpha^{d} e^{-\alpha x} x^{d-1}}{\Gamma d}, & x \geq 0, \alpha, d > 0 \\ 0, & \text{otherwise} \end{cases}$$

Remarks :-

1. The function $f(x)$ represents a probability density function since

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} \frac{\alpha^{d} e^{-\alpha x} x^{d-1}}{\Gamma d}\,dx$$

$$= \frac{\alpha^{d}}{\Gamma d} \int_{0}^{\infty} e^{-\alpha x} x^{d-1}\,dx$$

$$= \frac{\alpha^{d}}{\Gamma d} \cdot \frac{\Gamma d}{\alpha^{d}} \qquad \left[\because \int_{0}^{\infty} e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^{n}} \text{ gamma function}\right]$$

$$= 1.$$

Hence $f(x)$ represents a p.d.f.

2. Mean of Gamma distribution :-

As we know,

$$\text{Mean} = \mu_1' = E(X) = \bar{x}$$

i.e. $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$

$$= 0 + \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{x \alpha^{d} e^{-\alpha x}}{\Gamma d} \cdot x^{d-1}\,dx$$

$$= \frac{\alpha^d}{\Gamma d} \cdot \int_0^{\infty} e^{-\alpha x} x^{(d+1)-1}\,dx$$

$$= \frac{\alpha^d}{\Gamma d} \cdot \frac{\Gamma d+1}{\alpha^{d+1}} = \frac{\cancel{\alpha^d}}{\cancel{\Gamma d}} \cdot \frac{d\,\cancel{\Gamma d}}{\cancel{\alpha^d} \cdot \alpha^1}$$

$$\therefore \boxed{E(X) = \frac{d}{\alpha}}.$$

3. Variance of Gamma distribution :-

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$\because E(X^2) = \int_0^{\infty} x^2 \cdot \frac{\alpha^d}{\Gamma d} \cdot x^{d-1} e^{-\alpha x}\,dx$$

$$= \frac{\alpha^d}{\Gamma d} \cdot \int_0^{\infty} e^{-\alpha x} \cdot x^{(d+2)-1}\,dx$$

$$= \frac{\alpha^d}{\Gamma d} \cdot \frac{\Gamma d+2}{\alpha^{d+2}} = \frac{\alpha^d}{\Gamma d} \cdot \frac{(d+1)(d)\Gamma d}{\alpha^d \cdot \alpha^2}$$

$$= \frac{d(d+1)}{\alpha^2}.$$

$$Var(X) = E(X^2) - \{E(X)\}^2 = \frac{d(d+1)}{\alpha^2} - \left(\frac{d}{\alpha}\right)^2$$

$$= \frac{d}{\alpha^2}[\cancel{d}+1-\cancel{d}] = \frac{d}{\alpha^2}[1]$$

$$= \frac{d}{\alpha^2}.$$

$$\therefore \quad Var(X) = \frac{d}{\alpha^2}.$$

4. **M.G.F. of Gamma Distribution :-**

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}.f(x)\,dx$$

$$= \int_0^\infty e^{tx}.\frac{\alpha^d}{\Gamma d} e^{-\alpha x} x^{d-1}\,dx$$

$$= \frac{\alpha^d}{\Gamma d}\int_0^\infty e^{-(t-\alpha)x} x^{d-1}\,dx$$

put $y = (\alpha - t)x \Rightarrow dy = (\alpha - t)\,dx$

$\frac{1}{\alpha - t}dy = dx$, limit If $x \to 0 \Rightarrow y \to 0$

If $x \to \infty \Rightarrow y \to \infty$

$$M_x(t) = \frac{\alpha^d}{\Gamma d}\int_0^\infty e^{-y}.\left(\frac{y}{\alpha - t}\right)^{d-1}.\frac{1}{(\alpha - t)}\,dy$$

$$= \frac{\alpha^d}{\Gamma d}\frac{1}{(\alpha - t)^{d-1+1}}\int_0^\infty e^{-y}.y^{d-1}\,dy$$

$$= \frac{\alpha^d}{\cancel{\Gamma d}}.\frac{1}{(\alpha - t)^d}.\cancel{\Gamma d} = \frac{\alpha^d}{(\alpha - t)^d} ; (t < \alpha)$$

$$M_x(t) = \left(1-\frac{t}{\alpha}\right)^{-d} ; \quad t<d.$$

Cumulative distribution function of Gamma distribution :-

The cdf of $X \sim \text{gamma}(\alpha, d)$ is

$$F(x) = \int_0^x \frac{\alpha^d e^{-\alpha x} x^{d-1}}{\Gamma d} dx$$

$$= \frac{\alpha^d}{\Gamma d} \int_0^x x^{d-1} e^{-\alpha x} dx \qquad \text{put } \alpha x = y \Rightarrow dx = \frac{dy}{\alpha}$$

$$= \frac{\alpha^d}{\Gamma d} \int_0^{\alpha x} \left(\frac{y}{\alpha}\right)^{d-1} . e^{-y} . \frac{dy}{\alpha}$$

$$= \frac{\alpha^d}{\Gamma d} . \frac{1}{\alpha^d} \int_0^{\alpha x} y^{d-1} e^{-y} dy$$

This is called incomplete gamma function.

Additive Property of Gamma distribution :-

If X_1 and X_2 be independent random variables following gamma distribution with parameters d_1 and d_2 respectively. Then the moment generating function of the sum of two gamma distribution will have parameter d_1+d_2.

Since $M_{x_i}(t) = \left(1-t/\alpha\right)^{-d_i}$

$$M_{X_1+X_2+\cdots+X_k}(t) = M_{X_1}(t) \cdots M_{X_k}(t) = \left(1-\frac{t}{\alpha}\right)^{-(d_1+d_2+\cdots+d_k)}$$

$=$

Exponential Distribution :-

A random variable X is said to have an exponential distribution with parameter $\alpha > 0$; if its probability density function is given by :

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

The cumulative distribution function $F(x)$ is given by $F(x) = P(X \leq x) = \int_{-\infty}^{x} \alpha e^{-\alpha x}\, dx$

$$\Rightarrow F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\alpha x} & , x \geq 0 \\ 1 & , x = \infty \end{cases}$$

Remark :- Sometimes exponential distribution is defined by the p.d.f.

$$\boxed{f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{1}{\beta}x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}}$$

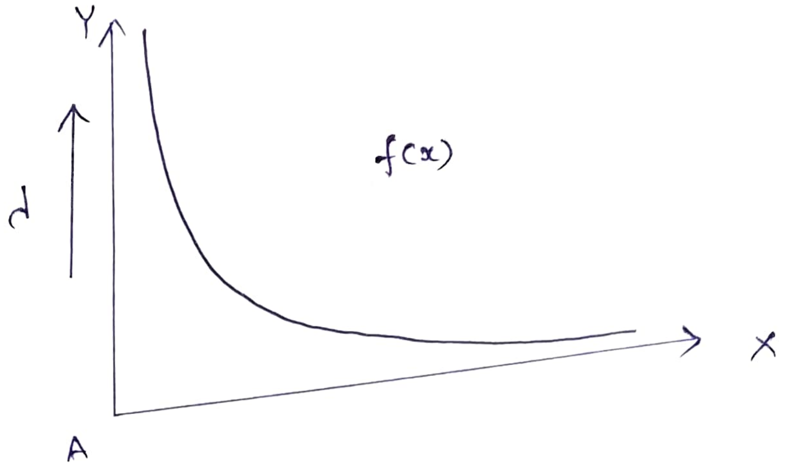
Graph of Exponential Probability density function -

For $f(x) = \alpha e^{-\alpha x}$

x	0	1	2	3	4	5	...	∞
$f(x)$	α	$\alpha e^{-\alpha}$	$\alpha e^{-2\alpha}$	$\alpha e^{-3\alpha}$	$\alpha e^{-4\alpha}$	$\alpha e^{-5\alpha}$	...	0

When $\alpha = 1$, $f(x) = 1, e^{-1}, e^{-2}, \dots 0$

$\alpha = 2$, $f(x) = 2, 2e^{-2}, 2e^{-4}, \dots 0$



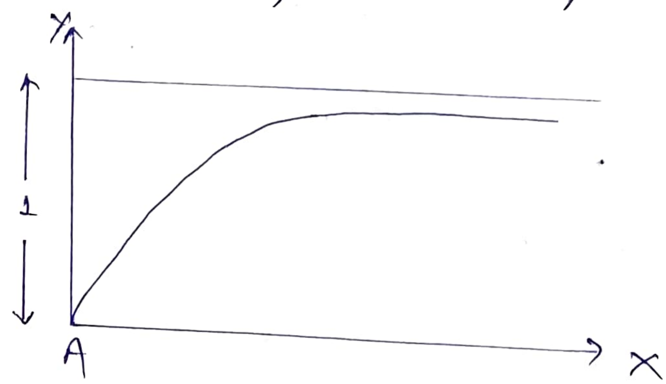
Graph of Distribution Function :-

For $F(x) = 1 - e^{-\alpha x}$

x	0	1	2	...	∞
$F(x)$	0	$1-e^{-\alpha}$	$1-e^{-2\alpha}$	...	1

When $\alpha = 1$, $F(x) = 0, 1-e^{-1}, 1-e^{-2}, \dots 1$

$\alpha = 2$, $F(x) = 0, 1-e^{-2}, 1-e^{-4}, \dots 1$



5. Moment Generating Function :-

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \lambda e^{-\lambda x} \cdot e^{tx}\, dx = \int_0^\infty \lambda e^{-(\lambda - t)x}\, dx$$

$$= \frac{\lambda}{\lambda - t}\left[-e^{-(\lambda - t)x}\right]_0^\infty = \frac{\lambda}{\lambda - t}\left[-e^{-\infty} + e^{0}\right] = \frac{\lambda}{\lambda - t}$$

$$= \left(\frac{\lambda - t}{\lambda}\right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \frac{t^4}{\lambda^4} + \cdots$$

$$\mu_r' = \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_x(t)$$

$$= \frac{r!}{\lambda^r}.$$

In particular, $\mu_1' = \frac{1}{\lambda}$, $\mu_2' = \frac{2!}{\lambda^2}$, $\mu_3' = \frac{3!}{\lambda^3}$,

$\mu_4' = \frac{4!}{\lambda^4}$ and so on.

6. Cumulant Generating Function :-

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$K_x(t) = \log(M_x(t)) = -\log\left(1 - \frac{t}{\lambda}\right)$$

$$= +\left\{\frac{t}{\lambda} + \frac{1}{2}\left(\frac{t^2}{\lambda^2}\right) + \frac{1}{3}\left(\frac{t^3}{\lambda^3}\right) + \cdots + \frac{1}{r}\left(\frac{t^r}{\lambda^r}\right) + \cdots\right\}$$

$$\therefore\ k_1 = \frac{1}{\lambda},\ k_2 = \frac{1}{\lambda^2},\ k_3 = \frac{2}{\lambda^3},\ \ldots,\ k_r = \frac{(r-1)!}{\lambda^r}.$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\left(\frac{9}{d^4}\right)}{\left(\frac{1}{d^2}\right)^2} = 9$$

$$\gamma_2 = \beta_2 - 3 = 9 - 3 = 6.$$

4. **Mean Deviation About Mean :-**

$$f(x) = \begin{cases} d e^{-dx} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad E(X) = \text{Mean} = \frac{1}{d}$$

Hence, mean deviation about mean,

$$\begin{aligned}
\text{M.D.} &= E|X - E(X)| \\
&= \int_0^\infty \left|x - \frac{1}{d}\right| d e^{-dx} \, dx \\
&= \int_0^\infty |dx - 1| e^{-dx} \, dx \\
&= \frac{1}{d} \int_0^\infty |t-1| e^{-t} \, dt \qquad \text{where } t = dx \\
&= \frac{1}{d} \left[\int_0^1 (1-t) e^{-t} \, dt + \int_1^\infty (t-1) e^{-t} \, dt \right] \\
&= \frac{1}{d} \left[e^{-1} + e^{-1} \right] = \frac{2}{d} \cdot e^{-1}.
\end{aligned}$$

Constants of Exponential Distribution :-

1. Moments About Origin :-

$$\mu_1' \text{ or mean} = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = 0 + \int_{0}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \alpha e^{-\alpha x}\, dx$$

$$= \alpha \int_{0}^{\infty} x e^{-\alpha x}\, dx$$

$$= \alpha \cdot \frac{\Gamma 2}{\alpha^2} \quad \left[\because \int_0^{\infty} x^{n-1} e^{-ax}\, dx = \frac{\Gamma n}{a^n} \right]$$

$$\boxed{\text{mean} = \frac{1}{\alpha}}$$

$$\mu_2' = E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = 0 + \int_0^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \alpha e^{-\alpha x} = \alpha \cdot \frac{\Gamma 3}{\alpha^3} = \frac{2}{\alpha^2}$$

Similarly, $\mu_3' = E(x^3) = \dfrac{6}{\alpha^3}$

and $\mu_4' = E(x^4) = \dfrac{24}{\alpha^4}$.

2. Moments About Mean :-

first Central Moment $\rightarrow \mu_1 = 0$ (Always)

Second Central Moment $\rightarrow \mu_2 = \mu_2' - \mu_1'^2 = \dfrac{2}{\alpha^2} - \left(\dfrac{1}{\alpha}\right)^2 = \dfrac{1}{\alpha^2}$

$= \text{Variance } (\sigma^2)$.

$\therefore$ Standard deviation $\sigma = \sqrt{\mu_2}$

$= \sqrt{var(x)} = \frac{1}{d}$.

third central moments

$\rightarrow \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= \frac{6}{d^3} - \frac{3\times 2}{d^2}\times\frac{1}{d} + 2\cdot\left(\frac{1}{d}\right)^3$

$= \frac{6}{d^3} - \frac{6}{d^3} + \frac{2}{d^3}$

$$\boxed{\mu_3 = \frac{2}{d^3}}$$

fourth central moment

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1)^2 - 3(\mu_1')^4$

$= \frac{24}{d^4} - \frac{4\cdot 6}{d^3}\cdot\frac{1}{d} + 6\times\frac{2}{d^2}\times\left(\frac{1}{d}\right)^2 - 3\left(\frac{1}{d}\right)^4$

$= \cancel{\frac{24}{d^4}} - \cancel{\frac{24}{d^4}} + \frac{12}{d^4} - \frac{3}{d^4}$

$$\boxed{\mu_4 = \frac{9}{d^4}}.$$

3. Beta and Gamma Coefficients :-

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left(\frac{2}{d^3}\right)^2}{\left(\frac{1}{d^2}\right)^3} = \frac{\frac{4}{d^6}}{\frac{1}{d^6}} = 4.$

$\gamma_1 = \sqrt{\beta_1} = 2.$