➢ Content :-

- Binomial Method
- Poisson Method
- Normal Method
- Gamma Method

<u>BINOMIAL DISTRIBUTION METHOD</u> :-

- The experiment consists of **n identical trials** (simple experiments).
- Each trial results in one of **two outcomes** (success or failure)
- The probability of success on a single trial is equal to π and π remains the same from trial to trial.
- The trials are independent, that is, the outcome of one trial does not influence the outcome of any other trial.
- The random variable y is the number of successes observed during n trials.

$$P(y) = \frac{n!}{y!(n-y)!}\pi^{y}(1-\pi)^{n-y}$$

n!=1x2x3x…x n

$$\mu = n\pi$$

Mean

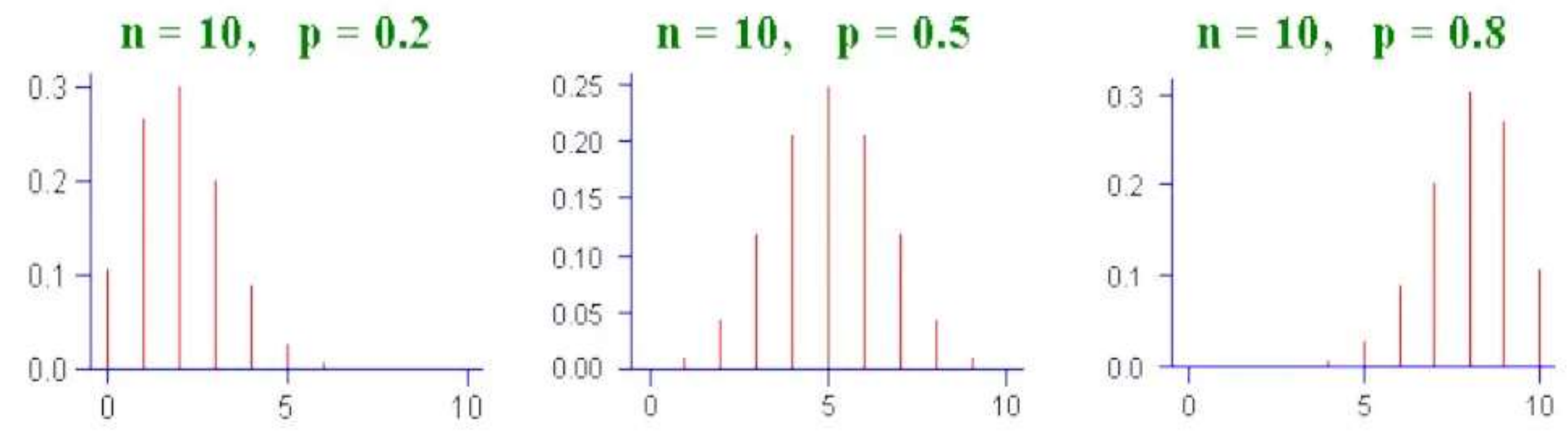
$$\sigma = \sqrt{n\pi(1-\pi)}$$

Standard deviation

➢ Shape Of Binomial Distribution :-

- The shape of the binomial distribution depends on the values of n and p

Fig.1.Binomial distributions for different values of p with n=10



- When p is small (0.2), the binomial distribution is skewed to the right. •When p= 0.5 , the binomial distribution skewed to right.
- When p is larger than 0.5, the distribution is skewed to the left.

➢ Applications for binomial distributions :-

- Binomial distributions describe the possible number of times that a particular event will occur in a sequence of observations.

- They are used when we want to know about the occurrence of an event, not its magnitude.

<u>POISSON DISTRIBUTION METHOD</u> :-

- The Poisson distribution, named after Simeon Denis Poisson (1781-1840). Poisson distribution is a discrete distribution. It describes random events that occurs rarely over a unit of time or space.
- It differs from the binomial distribution in the sense that we count the number of success and number of failures, while in Poisson distribution, the average number of success in given unit of time or space is considered.

➢ DEFINATION :-

The probability that exactly x events will occur in a given time is as follows

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \; x=0,1,2\ldots$$

called as probability mass function of Poisson distribution.
where λ is the average number of occurrences per unit of time

$$\lambda = np$$

➢ CONDITIONS :-

Poisson distribution is the limiting case of binomial distribution under the following assumptions.

- The number of trials n should be indefinitely large i.e.., n->∞
- The probability of success p for each trial is indefinitely small.
- np= λ, should be finite where λ is constant.

➢ APPLICATIONS :-

- In biology, to count the number of bacteria.
- In determining the number of deaths in a district in a given period, by rare disease.
- The number of error per page in typed material.
- The number of plants infected with a particular disease in a plot of field.
- Number of weeds in particular species in different plots of a field.
- It is used in quality control statistics to count the number of defects of an item

➢ EXAMPLES :-

Example 1: Suppose on an average 1 house in 1000 in a certain district has a fire during a year. If there are 2000 houses in that district, what is the probability that exactly 5 houses will have a fire during the year? [given that $e^{-2} = 0.13534$]

Mean, $\bar{x} = np$, $n = 2000$ and $p = \frac{1}{1000}$

$$= 2000 \times \frac{1}{1000}$$

$\lambda=2$

The Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(X = 5) = \frac{e^{-2}2^{5}}{5!}$$

$$= \frac{(0.13534)\times 32}{120}$$

$$= \mathbf{0.036}$$

Example 2: If 2% of electric bulbs manufactured by a certain company are defective. Find the probability that in a sample of 200 bulbs i) less than 2 bulbs ii) more than 3 bulbs are defective.[e-4 = 0.0183]

The probability of a defective bulb $= p = \frac{2}{100} = 0.02$

Given that n = 200 since p is small and n is large We use the

Poisson distribution mean, $m = np = 200 \times 0.02 = 4$

Now, Poisson Probability function, $P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

i) Probability of less than 2 bulbs are defective

$= P(X<2)$

$= P(x = 0) + P(x = 1)$

$= e^{-4} + e^{-4}(4)$

$= e- 4\,(1 + 4) = 0.0183 \times 5$

$= 0.0915$

ii) Probability of getting more than 3 defective bulbs

$P(x > 3) = 1 - P(x \leq 3)$

$= 1 - \{P(x = 0) + P(x = 1) + P(x=2) + P(x=3)\}$

$$= 1 - e^{-4}\left\{1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!}\right\}$$

$= 1 - \{0.0183 \times (1 + 4 + 8 + 10.67)\}$

$= 0.567$

NORMAL DISTRIBUTION METHOD :-

- Introduction

A Probability Distribution will give us a Value of P(x) = P(X=x) to each possible outcome of x. For the values to make a Probability Distribution, we needed two things to happen:

1. $\sum P(x) = P(X = x)$
2. $0 \leq P(x) \leq 1$

- For a Continuous Random Variable, a Probability Distribution must be what is called a Density Curve. This means:

  1. The Area under the Curve is 1.
  2. $0 \le P(x)$ for all outcomes

➢Normal Distribution Definition :-:

1) A continuous variable X having the symmetrical, bell shaped distribution is called a Normal Random Variable.

2) The normal probability distribution (Gaussian distribution) is a continuous distribution which is regarded by many as the most significant probability distribution in statistics particularly in the field of statistical inference.

➢ Characteristics of Normal Distribution:-

1) It is “Bell-Shaped” and has a single peak at the center of the distribution.

2) The arithmetic Mean, Median and Mode are equal.

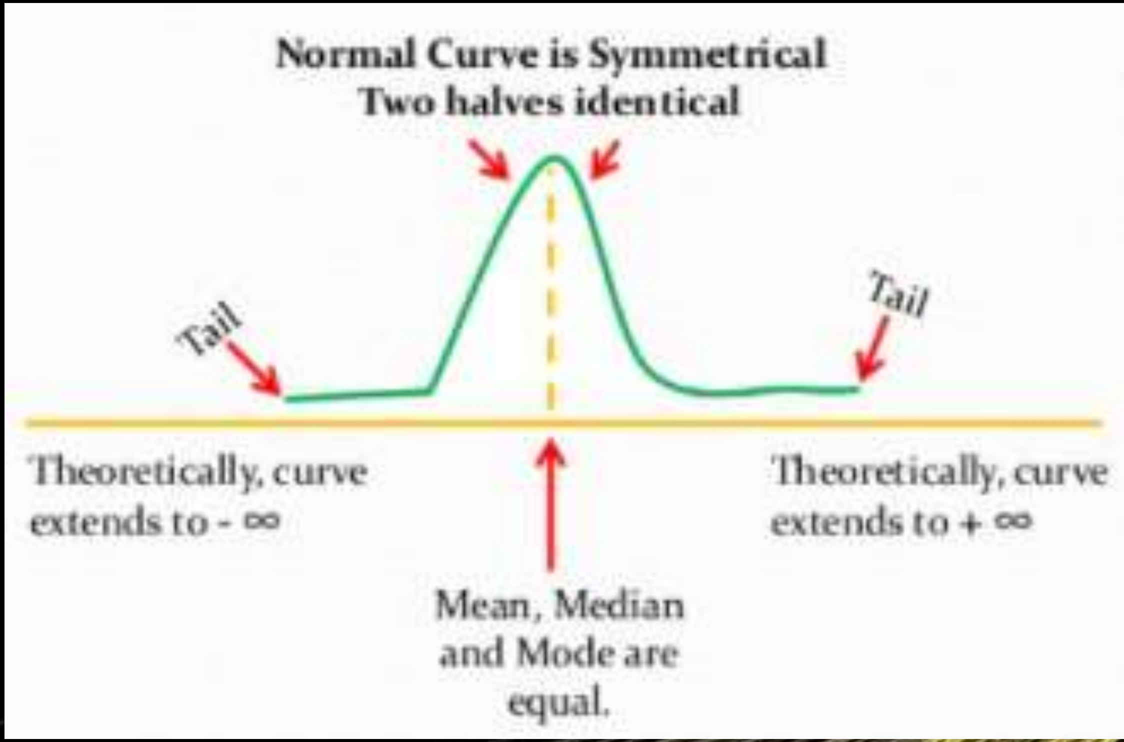
3) The total area under the curve is 1.00; half the area under the normal curve is to the right of this center point and the other half to the left of it.

4) It is Symmetrical about the mean.

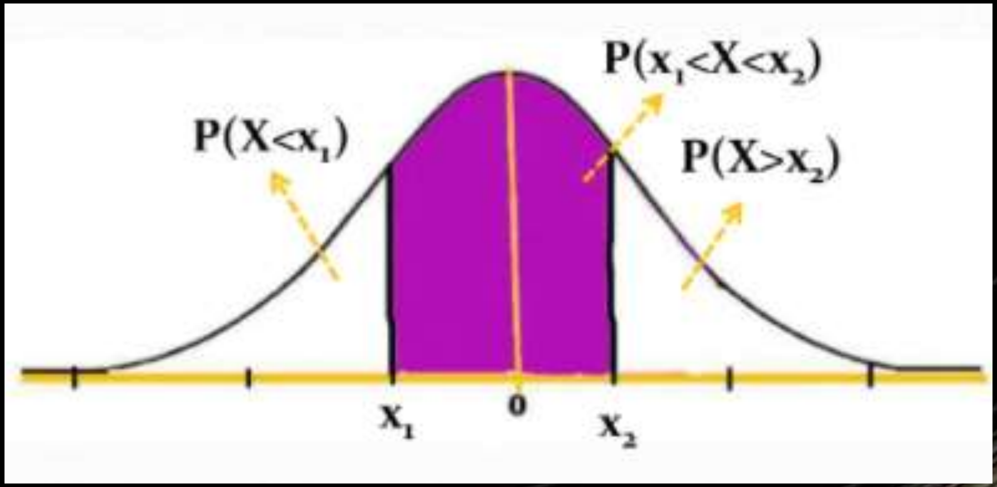
5) It is Asymptotic: The curve gets closer and closer to the X – axis but never actually touches it. To put it another way, the tails of the curve extend indefinitely in both directions.

6) The location of a normal distribution is determined by the Mean, μ, the Dispersion or spread of the distribution is determined by the Standard Deviation, σ.

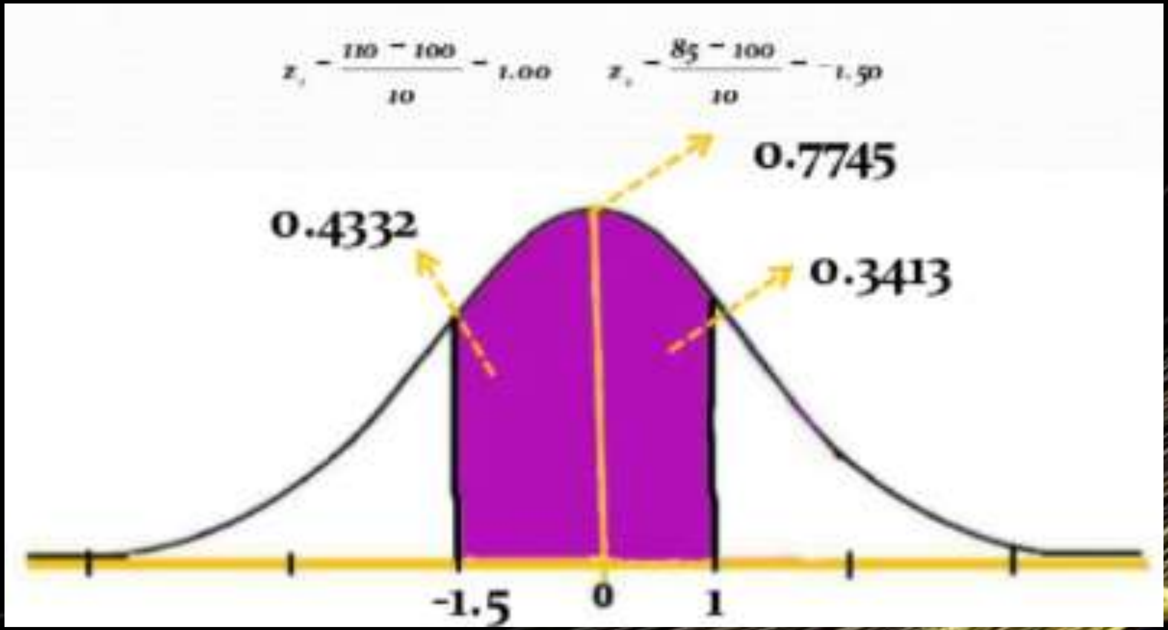
➢ THE NORMAL DISTRIBUTION - GRAPHICALLY :-



- AREA UNDER THE NORMAL CURVE : -



- Let us consider a variable X which is normally distributed with a mean of 100 and a standard deviation of 10. We assume that among the values of this variable are x1= 110 and x2 = 85.



➢ The Standard Normal Probability Distribution :-

- The Standard Normal Distribution is a Normal Distribution with a Mean of 0 and a Standard Deviation of 1.

- It is also called the z distribution .

- A z –value is the distance between a selected value , designated X, and the population Mean μ, divided by the Population Standard Deviation, σ.

- The formula is :

$$z = \frac{X - \mu}{\sigma}$$

➢ EXAMPLES :-

Ques 1) A random variable X has a normal distribution with mean 5 and variance 16.

a.) Find an interval (b,c) so that the probability of X lying in the interval is 0.95.

b.) Find d so that the probability that X ≥ d is 0.05. 1

1 $z = \frac{X - \mu}{\sigma} = \frac{X - 5}{4}$

2 $\frac{0.95}{2} = 0.475$

Solution: A.

$P(b \leq X \leq c) = P(Z_b \leq Z \leq Z_c)$

$= P(-1.96 \leq \frac{X-5}{4} \leq 1.96)$

$= P(-1.96(4) \leq X - 5 \leq 1.96(4))$

$= P(-7.84 + 5 \leq X \leq 7.84 + 5)$

$P(b \leq X \leq c) = P(-2.84 \leq X \leq 12.84)$

thus: b = -2.84 and c = 12.84

Solution B:

$P(X \geq d) = P(Z \geq Z_d) = 0.05 \geq$

$= P\left(\frac{X-5}{4}\right) \geq 1.64$

$= P[X - 5 \geq (1.64)(4)]$

$= P(X - 5 \geq 6.56)$

$= P(X \geq 6.56 + 5)$

$P(X \geq d) = P(X \geq 11.56)$

thus: d = 11.56

Ques 2) A certain type of storage battery last on the average 3.0 years, with a standard deviation σ of 0.5 year. Assuming that the battery are normally distributed, find the probability that a given battery will last less than 2.3 years.

Solution:

$$z = \frac{X - \mu}{\sigma} = \frac{2.3 - 3}{0.5} = \frac{-0.7}{0.5} = -1.4$$

$$P(X < 2.3) = P(Z < -1.4)$$

$$= 0.5 - \Phi(-1.4)$$

$$= 0.5 - 0.4192$$

$$P(X < 2.3) = 0.0808$$

GAMMA DISTRIBUTIONS METHOD :-

The gamma distribution is another widely used distribution. Its importance is largely due to its relation to exponential and normal distributions. Before introducing the gamma random variable, we need to introduce the gamma function.

***Gamma function*:** The gamma function , shown by $\Gamma(x)$, is an extension of the factorial function to real (and complex) numbers. Specifically, if $n \in\{1,2,3,\ldots\}$; then

$$\Gamma(n)=(n-1)!$$

More generally, for any positive real number α, Γ(α) is defined as

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \text{for } \alpha > 0.$$

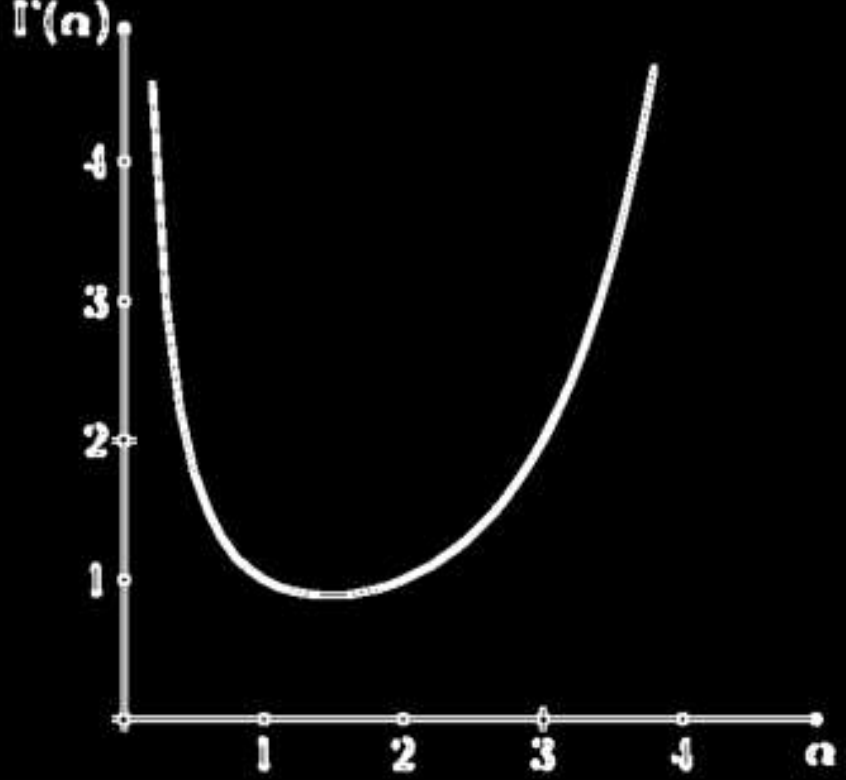


Fig. shows the gamma function for some Positive real values of α

Note that for $\alpha = 1$, we can write

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx$$
$$= 1.$$

Using the change of variables $x = \lambda y$, we can show the following equation that is
often useful when working with the gamma distribution:

$$\Gamma(\alpha) = \lambda^\alpha \int_0^\infty y^{\alpha - 1} e^{-\lambda y}\, dy \qquad \text{for } \alpha, \lambda > 0.$$

Also, using integration by parts it can be shown that

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \qquad \text{for } \alpha > 0.$$

Note that if $\alpha = n$, where n is a positive integer, the above equation reduces to

$$n! = n.(n-1)!$$

Properties of the gamma function:

For any real number α :

1. $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1} e^{-x} \, dx$
2. $\int_0^\infty x^{\alpha-1} e^{-x} \, dx = \Gamma(\alpha)/\lambda^\alpha$, for $\lambda > 0$;
3. $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$;
4. n, $\Gamma(n)=(n-1)!$; for n= 1,2,3,……;
5. $\Gamma(1/2) = \sqrt{\pi}$.
6. For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1).\Gamma(\alpha - 1)$

(via integration by parts)

Example :-

1. Find $\Gamma(7/2)$.
2. Find the value of the following integral:

$$I = \int_0^\infty x^6 \; e^{-5x} \; dx.$$

Solution:

1. To find $\Gamma(7/2) = 5/2.\,\Gamma(5/2)$ (using property 3)

$= 5/2.3/2.\,\Gamma(3/2)$ (using property 3)

$= 5/2.3/2.1/2\,\Gamma(1/2)$ (using property 3)

$= 5/2.3/2.1/2.\,\sqrt{\pi}.$ **(using property 5)**

$= \mathbf{15/8.\,\sqrt{\pi}.}$

2. Using property 2 with $\alpha = 7$ and $\lambda = 5$, we obtain

$$I = \int_0^\infty x^6 \; e^{-5x} \; dx.$$

$= \Gamma(7)/5^7$

$= 6!/5^7$ (using property 4)

≈ 0.0092

Random Variables:

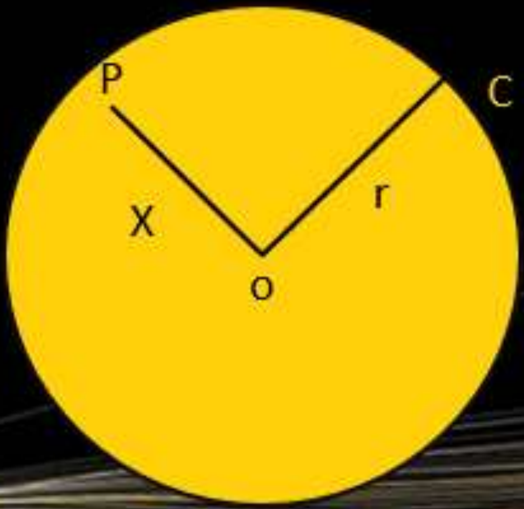
Definition: A random variable X on a sample space S is a rule that assigns a numerical value to each outcome of S or in other words a function from S into the set R of real numbers. These are random responses corresponding to subjects randomly selected from a population.

$$X: S \rightarrow R$$

x : value of random variable

R_x : The set of numbers assigned by random variable X, i.e. range space.

Example: a point P is chosen at random in a circle with radius r. Let X be the distance of the point from the centre of the circle. Then X is a (continuous) random variables with R_x = [0, r].



Probability distributions:-

A listing of the possible outcomes and their probabilities (discrete random variables) or their densities (continuous random variables).

Density curve – Gamma Distribution :

The density curve of the normal distribution is symmetric and bell-shaped. There are many practical situations where the variable of interest is skewed (not symmetric). This property is shown by gamma distributions.

Gamma Distribution:

We now define the gamma distribution by providing its probability distribution/ density function PDF :

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

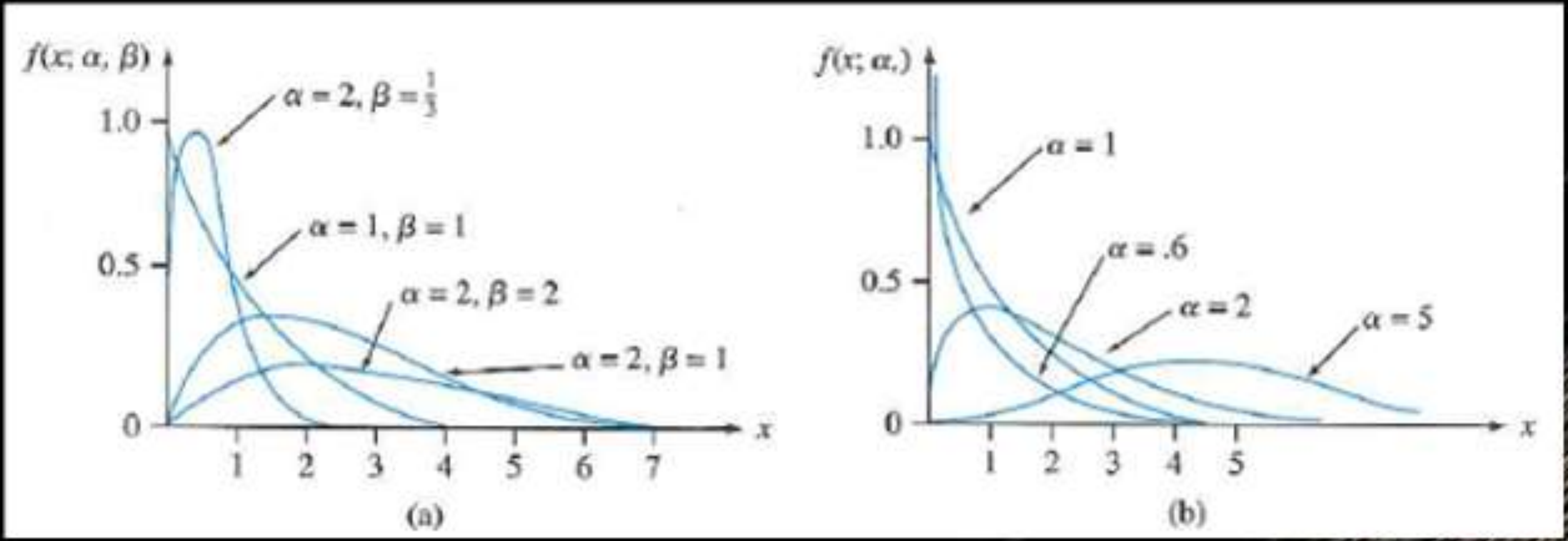
where the parameters satisfy $\alpha>0,\beta>0$.

- The standard gamma distribution has $\beta=1$. β is called a scale parameter (values other than $\beta=1$ stretch or compress the distribution in the x direction.)
- The exponential distribution results from taking $\alpha=1$ and $\beta=1/\lambda$.

A continuous random variable X is said to have a gamma distribution with parameters $\alpha>0$ and $\lambda>0$, shown as $X \sim Gamma(\alpha,\lambda)$, if its PDF is given by

$$f_X(x)=\begin{cases}\frac{\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} & x>0\\ 0 & \text{otherwise}\end{cases}$$

a. Gamma density function and b. standard gamma density function

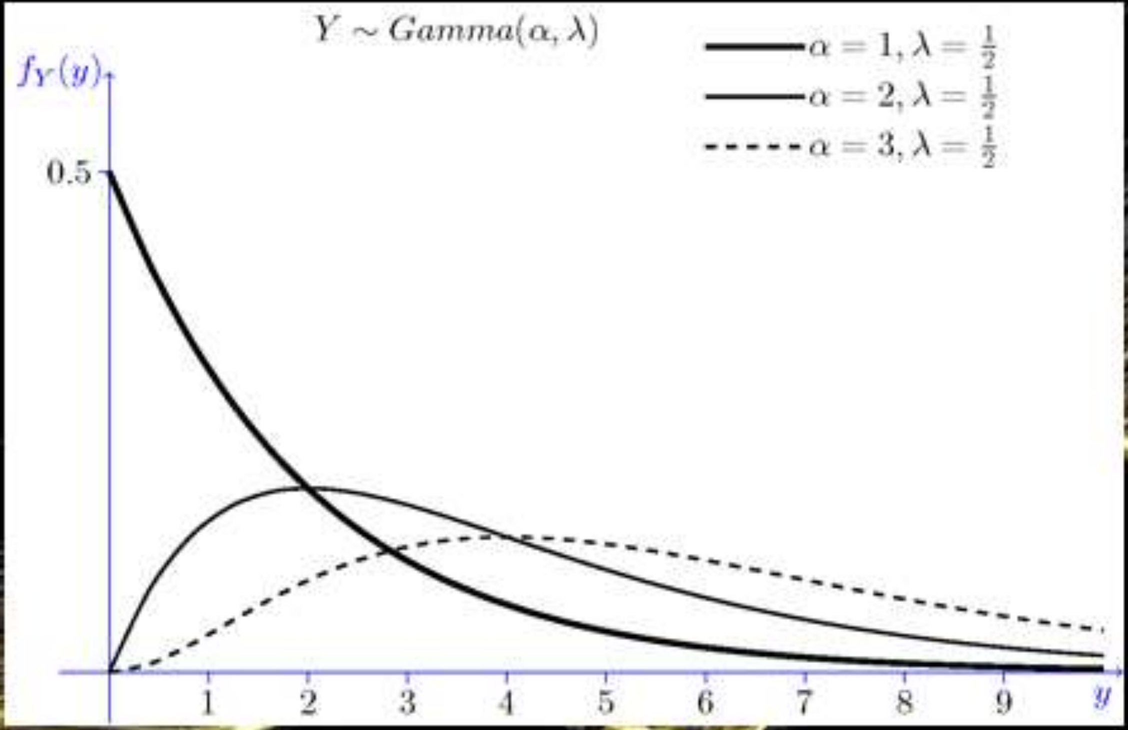


If we let $\alpha = 1$, we obtain

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude ***Gamma(1,λ)=Exponential(λ).*** More generally, if you sum ***n*** independent *Exponential(λ)* random variables, then you will get a *Gamma(n, λ)* random variable. Figure below shows the PDF of the gamma distribution for several values of α.

Fig. PDF of the Gamma distri- - bution for some values of α and λ.



Example :

Using the properties of the gamma function, show that the gamma PDF integrates to 1 , i.e. , show that α, λ >0 , we have

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1.$$

We can write

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha} \qquad \text{(using Property 2 of the gamma function)}$$

$$= 1.$$

In the Solved Problems section, we calculate the mean and variance for the gamma distribution. In particular, we find out that if $X \sim Gamma(\alpha,\lambda)$, then

$$E(X) = \alpha / \lambda \text{ or } \alpha\beta, \quad Var\ (X) = \alpha / \lambda^2 \text{ or } \alpha\beta^2.$$

Thank You