Unit-02

Date

Distvributions

(1) Discrete Distributions

→ Bernoulli
→ Binomial
→ Uniform
→ Geometric
→ Hypergeometric
→ Negative Binomial
→ Poisson Distribution

(2) Continuous Distribution

→ Uniform
→ Beta
→ Gamma
→ Normal
→ standard Normal
→ t-Distribution
→ F-Distribution
→ chi-square Distribution
→ Pareto Distribution
→ Weibull Distribution
→ Double exponential
→ Lozy Normal Distribution

Binomial Distuributions

Binomial Distributions is a discrete probability distributions which is obtained when the probability p of the happening of an event is same in all the trials, and there are only two events in each trials.

For the binomial random variable X, where X is the number of success and numbers of Bernoulli trials are n then. Binomial distribution is given by,

$$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

Properties of a Binomial Distribution.

1. It is a discrete distribution which gives the theoretical probabilities.
2. It depends on the parameters p or q, the probability of success or failure and n (the number of trials). The parameter n is always a positive integer.
3. The distribution will be symmetrical if $p=q$
4. The statistics of the Binomial distribution one mean $= np$, Variance $= npq$ and standard deviation $= \sqrt{npq}$.

(5) The mode of the binomial distribution is equal to that value x which has the largest frequency.

(6) The shape of and location of a binomial distribution changes as p changes for a given n or n changes for a given p.

Mean of Binomial Distribution :-

For a binomial distribution the probability function is

$$P(X=r) = {}^nC_r \, p^r q^{n-r}$$

The discrete probability distribution for the binomial distribution can be displayed as follows:

X	0	1	2	...	r	...		n
P(x)	${}^nC_0 q^n$	${}^nC_1 p q^{n-1}$	${}^nC_2 p^2 q^{n-2}$	...	${}^nC_r p^r q^{n-r}$	...		${}^nC_n p^n$

$$\therefore \text{Mean } (\mu) = E(X) = \sum_{r=0}^{n} r \, P(X=r)$$

$$= {}^nC_0 q^n \times 0 + {}^nC_1 q^{n-1} p \times 1 + {}^nC_2 p^2 q^{n-2} \times 2 + \cdots + {}^nC_n p^n \times n$$

$$= 0 + npq^{n-1} + n(n-1) p^2 q^{n-2} + \cdots + np^n$$

$$\text{Mean}(\mu) = np\left[q^{n-1} + (n-1)pq^{n-2} + \cdots + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1} = np \quad (\because p+q = 1)$$

$$\therefore \boxed{\text{Mean}(\mu) = E(X) = np}.$$

Variance of Binomial Distribution

$$\text{Variance } \sigma^2 = \sum_{x=0}^{n} (x-\mu)^2\, p(x)$$

$$\sigma^2 = E(X^2) - \{E(X)\}^2$$

Consider

$$\{E(X^2)\} = \sum_{r=0}^{n} r^2 P(X=r)$$

$$= \sum_{r=0}^{n} \left[r^2 - r + r\right] P(X=r)$$

$$= \sum_{r=0}^{n} \left[r(r-1)\right] P(X=r) + \sum_{r=0}^{n} r\, P(X=r)$$

$$= \sum_{r=0}^{n} r(r-1)\cdot {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\, P(X=r)$$

$$= \sum_{r=0}^{n} r(r-1)\cdot \frac{n!}{r!\,(n-r)!}\, p^r q^{n-r} + np \quad \left[\begin{array}{l}\text{for}\\ \text{Binomial}\end{array}\right.$$

$$\left\{\begin{array}{l}\text{for Binomial distribution}\\ \mu = E(X) = \sum_{r=0}^{n} r\, P(X=r) = np\end{array}\right\}$$

$$E(x^2) = \sum_{r=0}^{n} \frac{\cancel{r(r-1)}n(n-1)(n-2)!}{\cancel{r(r-1)}(r-2)!\{(n-2)-(r-2)\}!} p^r q^{n-r} + np$$

$$= n(n-1)p^2 \sum_{r=0}^{n} \frac{(n-2)!\, p^{r-2} q^{(n-2)-(r-2)}}{(r-2)!\{(n-2)-(r-2)\}!} + np$$

$$= n(n-1)p^2(p+q)^{n-2} + np$$

$$= (n^2-n)p^2 \cdot 1^{n-2} + np \qquad [\because p+q=1]$$

$$E(x^2) = (n^2-n)p^2 + np$$

We know that

$$\sigma^2 = E(x^2) - \{E(x)\}^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p) \qquad \left(\begin{array}{l} p+q=1 \\ q=1-p \end{array} \right)$$

$$\sigma^2 = npq$$

Hence variance $\boxed{\sigma^2 = npq}$

standard deviation $\boxed{\sigma = \sqrt{npq}}$.

Ex. (1) Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution When one coin is thrown

The probability of getting a head $= \frac{1}{2} = p$

$\therefore$ The probability of not getting a head $= 1 - \frac{1}{2} = \frac{1}{2} = q$

Then $P(\text{at least } 7 \text{ heads})$

$$= P(7 \text{ heads}) + P(8 \text{ heads}) + P(9 \text{ heads}) + P(10 \text{ heads})$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9}$$

$$+ {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$$

$$= \frac{1}{2^{10}} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \frac{1}{2^{10}} \left[\frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4} \cdot \cancel{7!}}{\cancel{7!} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} + \frac{\cancel{10}^{5} \cdot 9 \cdot \cancel{8!}}{\cancel{8!} \cdot 2 \cdot 1} + \frac{10 \cdot \cancel{9!}}{\cancel{9!} \cdot 1!} + \frac{\cancel{10!}}{\cancel{10!}\, 0!} \right]$$

$$= \frac{1}{2^{10}} \left[120 + 45 + 10 + 1 \right]$$

$$= \frac{176}{1024} = \frac{11}{64} \quad \left[\text{Divided by } 16 \right]$$

Prove that Variance of the Binomial distribution is $\sigma^2 = \sum_{x=0}^{n} (x-\mu)^2 p(x) = \{E(x^2)\} - \{E(x)\}^2 = npq$

$$\therefore \sigma^2 = \sum_{x=0}^{n} (x-\mu)^2 p(x) = \sum_{x=0}^{n} (x^2 + \mu^2 - 2x\mu)\, p(x)$$

$$= \sum_{x=0}^{n} x^2 p(x) + \mu^2 \sum_{x=0}^{n} p(x) - 2\mu \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x^2 p(x) + \mu^2 \cdot 1 - 2\mu \cdot \mu$$

$$= E(x^2) + \mu^2 - 2\mu^2$$

$$= E(x^2) + (-\mu^2)$$

$$= E(x^2) - \{E(x)\}^2$$

also we know that $\sigma^2 = npq$

$$\Rightarrow \boxed{\sigma^2 = npq} \boxed{= E(x^2) - \{E(x)\}^2 = \sum_{x=0}^{n} (x-\mu)^2 p(x)}$$

Ex. ② In a lot of 200 articles 10 are defective, find the probability of: (i) no defective article (ii) one defective article, (iii) at least one defective article, in a random sample of 20 articles.

Soln:- The probability of defective articles is $\frac{10}{200} = \frac{1}{20}$

$\therefore p = \frac{1}{20}$

The probability of non-defective article

$$= 1 - \frac{1}{20} = \frac{20-1}{20} = \frac{19}{20} = q.$$

(i) The probability of no-defective article out of 20

$$= {}^{20}C_0 (p)^0 q^{20} = 1 \cdot 1 \cdot \left(\frac{19}{20}\right)^{20}.$$

(ii) The probability of exactly one defective articles

$$= {}^{20}C_1 \, p^1 \, q^{19} = 20 \times \frac{1}{20} \times \left(\frac{19}{20}\right)^{19}$$

$$= \left(\frac{19}{20}\right)^{19}$$

(iii) The probability of at least one will be defective

= 1 − [probability of that none will be defective]

$$= 1 - {}^{20}C_{20} \left(\frac{19}{20}\right)^{20} = 1 - \left(\frac{19}{20}\right)^{20}$$

Ex. 3 If on an average, one ship out of 10 is wrecked, find the probability that out of 5 ships expected to arrive the port, at least four will arrive safely.

Sol^n:- p be the probability of a ship arriving safely

$= 1 - \frac{1}{10} = \frac{9}{10}$; $q = 1 - \frac{9}{10} = \frac{1}{10}$

Binomial Distribution is $\left(\frac{1}{10} + \frac{9}{10}\right)^5$.

Probability that at least four ship out of five arrive safely $= P(4) + P(5)$

$$= {}^5C_4\, p^4 q^{5-4} + {}^5C_5\, p^5 q^{5-5}$$

$$= 5 \cdot \left(\frac{9}{10}\right)^4 \left(\frac{1}{10}\right)^1 + 1 \cdot \left(\frac{9}{10}\right)^5 \cdot \left(\frac{1}{10}\right)^0$$

$$= \left(\frac{9}{10}\right)^4 \left[\frac{5+9}{10}\right] = \left(\frac{9}{10}\right)^4 \cdot \frac{14}{10}$$

$$= \left(\frac{9}{10}\right)^4 \cdot \frac{7}{5} = 0.91854.$$

Ex.4 The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men aged 60 now, at least 7 would live to be 70?

Sol^n:- Probability of survival upto the age of 70

$= p = 0.65$

Probability of non-survival upto the age of 70

$= q = 1 - p = 1 - 0.65 = 0.35$

Probability that out of 10 such men at least 7 would survive as desired

$= P(7) + P(8) + P(9) + P(10)$

$= {}^{10}C_7\, p^7 q^3 + {}^{10}C_8\, p^8 q^2 + {}^{10}C_9\, p^9 q^1 + {}^{10}C_{10}\, p^{10}$

$$= \frac{10\times 9\times 8\times 7!}{7!\times 3\times 2\times 1} p^7 q^3 + \frac{10\times 9\times 8!}{8!\,2\times 1} p^8 q^2 + 10 p^9 q + p^{10}$$

$$= 120 p^7 q^3 + 45 p^8 q^2 + 10 p^9 q + p^{10}$$

$$= p^7 \left(120 q^3 + 45 p q^2 + 10 p^2 q + p^3 \right)$$

$$= (0.65)^7 \left[120 \times (0.35)^3 + 45 (0.65)(0.35)^2 + 10 (0.65)^2 (0.35) + (0.65)^3 \right]$$

$= 0.514$, the required result.

Ex (5) Six dice are thrown together at a time, the process is repeated 729 times. How many times do you expect at least 3 dice to have 4 or 6 ?

Soln :- The chance of getting 4 or 6 with one dice is $\frac{1}{6}+\frac{1}{6}$

i.e. $p = \frac{2}{6} = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$.

In one throw of six dice together, we have probability of getting at least 3 dice to have 4 or 6,

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3 p^3 q^3 + {}^6C_4 p^4 q^2 + {}^6C_5 p^5 q + {}^6C_6 p^6$$

$$= \frac{6\times 5\times 4\times 3!}{3!\,3\cdot 2\cdot 1}\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^3 + \frac{6\times 5\times 4!}{4!\,2\times 1}\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^2 + \frac{6\times 5!}{5!\,1!}\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right)^1 + \left(\tfrac{1}{3}\right)^6$$

$$= \frac{1}{3^6}\left[20\times 8 + 15\times 4 + 12 + 1\right] = \frac{233}{3^6}$$

Now the process is repeated 729 times

$\therefore$ Required number of times at least 3 dice have 4 or 6

$$= \cancel{729} \times \frac{233}{\cancel{3^6}} = 233,$$ the required result.

Note. In the above case the binomial distribution is $N(p+q)^n$ where $N = 729$, $n = 6$

Ex. 6 If the sum of the mean and the variance of binomial distribution of 5 trials is 4.8, find the distribution.

Soln: Let the required binomial distribution be ${}^nC_r p^r q^{n-r}$, where n = number of trials = 5, Mean of the distribution = np and the variance of distribution = npq

By the given condition $np + npq = 4.8$

$$5p + 5pq = 4.8$$

$$50\,p(1+q) = 48$$

or $50(1-q)(1+q) = 48$

$$50(1-q^2) = 48$$

$$50 - 50q^2 = 48$$

$$50q^2 = 50 - 48$$

$$q^2 = \frac{2}{50} = \frac{1}{25}$$

$$\boxed{q = \frac{1}{5}}$$

$$50p(1+1-p) = 48$$

$$50(2p-p^2) = 48$$

$$50p^2 - 100p + 48 = 0$$

and $p = 1 - q = 1 - \frac{1}{5}$

$$= \frac{4}{5}.$$

Hence the required binomial distribution is ${}^5C_r \left(\frac{4}{5}\right)^r \left(\frac{1}{5}\right)^{5-r}$.

Ex. 7 Ten coins are tossed 1024 times and the following frequencies are observed. Compare these frequencies with the expected frequencies :-

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Frequencies	2	10	38	106	188	257	226	128	59	7	3

Solⁿ :- Here $n = 10$, $N = 1024$

p = The change of getting a head in one toss

$= \frac{1}{2}$

$\therefore q = 1 - p = \frac{1}{2}$

The expected frequencies are the respective terms of the binomial $1024\left(\frac{1}{2}+\frac{1}{2}\right)^{10}$

The frequency of r head $(0 \le r \le 10)$ is

$$= 1024 \cdot {}^{10}C_r \left(\frac{1}{2}\right)^{10-r}\left(\frac{1}{2}\right)^{r}$$

$$= 1024 \cdot {}^{10}C_r \cdot \frac{1}{2^{10}} = \cancel{1024} \cdot {}^{10}C_r \cdot \frac{1}{\cancel{1024}}$$

$$= {}^{10}C_r$$

Hence, we have the following comparison

Number of heads	0	1	2	3	4	5	6	7	8	9	10
Observed frequency	2	10	38	106	188	257	226	128	59	7	3
Expected frequency	1	10	45	120	210	252	210	120	45	10	1

Objective Questions

(a) The probability that a man hit a target is given as $\frac{1}{5}$. Then his probability of atleast one hit in 10 shots is

(i) $1-\left(\frac{4}{5}\right)^{10}$ (ii) $\left(\frac{1}{5}\right)^{10}$ (iii) $1-\left(\frac{1}{5}\right)^{10}$ (iv) None.

Soln: (a) (i) $1-\left(\frac{4}{5}\right)^{10}$

$p=\frac{1}{5}$

$q=\frac{4}{5}$ → The probability of no hit a target out of 10 shots $= {}^{10}C_0 \, p^0 q^{10}$

$= \left(\frac{4}{5}\right)^{10}$

→ The probability of atleast one hit in 10 shots is

$= 1-\{$ The probability of no hit a target out of 10 shots $\}$

$= 1-\left(\frac{4}{5}\right)^{10}$.

(b) 8 coins are tossed simultaneously. The probability of getting at least 6 head is

(i) $\frac{57}{64}$ (ii) $\frac{229}{256}$ (iii) $\frac{7}{64}$ (iv) $\frac{37}{256}$

Soln:-

$$P(6)+P(7)+P(8) = {}^{8}C_{6}\,p^{6}q^{2} + {}^{8}C_{7}\,p^{7}q^{1} + {}^{8}C_{8}\,p^{8}q^{0}$$

$$= \frac{\overset{4}{\cancel{8}}\cdot 7\cdot \cancel{6!}}{\cancel{6!}\cdot \cancel{2}}\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{2} + \frac{8\cdot \cancel{7!}}{\cancel{7!}\cdot 1}\left(\frac{1}{2}\right)^{7}\left(\frac{1}{2}\right) + 1\cdot\left(\frac{1}{2}\right)^{8}\cdot 1$$

$$= \frac{1}{2^{8}}\left[28+8+1\right]$$

$$= \frac{37}{2^{8}} = \frac{37}{256}.$$ Ans.

Poisson Distribution

It is due to the French Mathematician S.D. Poisson (1781 – 1840) who published its derivation in 1837. It is a discrete probability distribution which has the following characteristic :-

(i) It is the limiting form of the Binomial distribution as n becomes infinitely large i.e. $n \to \infty$ and p, the constant probability of success for each trial becomes indefinitely small i.e. $p \to 0$ in such a manner that $np = m$ remains a finite number.

(ii) It consists of a single parameter m only. The entire distribution can be obtained once m is known.

It has wide applications in physical, engineering and manggement sciences as well as economics, operations research and reliability technology.

Poisson distribution occurs when there are events which do no occur as outcomes of definite number of trials of an experiment but which occurs at random point of time and space where in our interest lies only in the number of occurence of the events.

Where the probability of the event remains very small and the Poisson distribution is applicable, are :

(i) the occurence of accidents in a factory in a given period,

(ii) the number of defective articles produced by some factory in a fixed time period,

(iii) the number of twins births per year in some hospital,

(iv) and the number of deaths due to snake bite in a city per year,

(v) the number of printing mistakes which page of the book.

(vi) the number of persons blind in a large city every year etc.

Binomial Approximation To Poisson Distribution

Claim

Poisson distribution as a limiting case of Binomial distribution when $p \to 0$, $n \to \infty$ such that $np = m$ (a finite quantity).

Proof:- We know that in a Binomial distribution, the probability of r successes is given by,

$$P(r) = {}^nC_r p^r q^{n-r} \quad \text{where } p = \frac{m}{n};\ q = 1 - \frac{m}{n}$$

$$= \frac{n(n-1)(n-2)\dots(n-(r-1))\cancel{(n-r)!}}{r!\,\cancel{(n-r)!}} \left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r}$$

$$P(r) = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!\ n^r} \cdot m^r \left(1-\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{-r}$$

$$P(r) = \frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\dots\left(\frac{n-r+1}{n}\right) \cdot \frac{m^r}{r!} \left(1-\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{-r}$$

$$P(r) = 1 \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{(r-1)}{n}\right) \cdot \frac{m^r}{r!} \left(1-\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{-r}$$

when $n \to \infty$, we have

$$\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\right] = 1$$

$$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^n = e^{-m};\quad \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-r} = 1.$$

As such in the limiting form,

$$\boxed{P(r) = \frac{e^{-m} \cdot m^r}{r!}}$$

This is the probability of r successes for Poisson Distribution.

For $r = 0, 1, 2, 3, \ldots$, we get the probabilities of $0, 1, 2, 3, \ldots$ successes as

$$P(0) = e^{-m}, \quad P(1) = m e^{-m}, \quad P(2) = \frac{m^2}{2!} e^{-m},$$

$$P(3) = \frac{m^3}{3!} e^{-m} \ldots \text{ and so on}$$

Note: 1. The sum of the probabilities $P(r)$ for $r = 0, 1, 2, 3, \ldots$ is 1

i.e. $$\Sigma P(r) = P(0) + P(1) + P(2) + P(3) + \ldots$$

$$= e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\right]$$

$$= e^{-m} \times e^{m}$$

$$= e^{0}$$

$$= 1.$$

Note (2). Poisson distribution possesses only one parameter 'm'. If we consider the length of interval as 'd' instead of unit length, the average number of occurence in 'd' length of interval is 'md' Thus, the probability function in this situation is,

$$P(r) = \frac{e^{-md}(md)^r}{r!}.$$

Condition Under which Poisson Distribution is Used

1. The random variable x should be discrete.
2. A dichotomy exists, i.e. the happening of the event must be of two alternatives such as success and failure, occurence and non-occurences etc.
3. It is applicable in those case where the number of trials n is very large and the probabilities of success p is very small but the mean $np = m$ is finite.
4. p should be very small (close of zero). If $p \to 0$, then the distribution is J-shaped and unimodal.
5. Statistical independence is assumed.

Mean of Poisson Distribution :-

Poisson distribution is $P(r) = \frac{e^{-m} \cdot m^r}{r!}$, $r = 0, 1, 2, \ldots$

$$\text{Mean } (\mu) = E(x) = \sum_{r=0}^{\infty} r \cdot \frac{e^{-m} m^r}{r!} = \sum_{r=0}^{\infty} r \cdot p(r), \quad r = 0, 1, 2, \ldots$$

$$= 0 \cdot e^{-m} + 1 \cdot e^{-m} m + 2 \cdot \frac{m^2 e^{-m}}{2!} + 3 \cdot \frac{m^3 e^{-m}}{3!} + \cdots$$

$$= e^{-m} \left[m + m^2 + \frac{m^3}{2!} + \frac{m^4}{3!} + \cdots \right]$$

$$= m e^{-m} \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right]$$

$$= m e^{-m} \cdot e^{m} \quad \left\{ e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right\}$$

$$\boxed{\mu = m}.$$

Variance of Poisson Distribution :-

$$\text{Variance } (\sigma^2) = \sum_{r=0}^{\infty} r^2 p(r) - \mu^2 = E(x^2) - \{E(x)\}^2, \quad \text{---(1)}$$

Now,

$$\sum_{r=0}^{\infty} r^2 p(r) = E(x^2) = \sum_{r=0}^{\infty} r^2 . \frac{e^{-m} m^r}{r!}$$

$$= 0 + 1^2 . \frac{e^{-m} . m}{1!} + 2^2 . \frac{e^{-m} . m^2}{2!} + 3^2 . \frac{e^{-m} m^3}{3!} + 4^2 . \frac{e^{-m} m^4}{4!} + \cdots$$

$$= e^{-m} m \left[1 + 2m + \frac{3}{2} m^2 + \frac{4}{3!} m^3 + \cdots \right]$$

$$= e^{-m} . m \left[1 + m + m + \frac{1}{2} m^2 + m^2 + \frac{1}{3!} m^3 + \frac{3}{3!} m^3 + \cdots \right]$$

$$= e^{-m} . m \left[\left(1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right) + \left(m + m^2 + \frac{3}{3!} m^3 + \cdots \right) \right]$$

$$= e^{-m} . m \left[\left(1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots \right) + m \left(1 + m + \frac{m^2}{2!} + \cdots \right) \right]$$

$$= e^{-m} . m \left[e^m + m e^m \right]$$

$$= e^{-m} e^m . m [1 + m]$$

$$= e^0 . m (1 + m)$$

and $\mu = m$ for poisson distri.

$E(x^2) = m(1+m)$

Hence $\sigma^2 = m + \cancel{m^2} - \cancel{m^2} \Rightarrow \boxed{\sigma^2 = m}$.

Standard deviation $= \sqrt{\text{Variance}} = \sqrt{m}$.

Examples of Poisson distribution

(1) The number of telephone calls arriving on an internal switch board of an office is 90 per hour. Find the probability that at ~~that~~ the most 1 to 3 calls in a minute on the board arrive. (Use $e^{-1.5} = 0.223$)

Solⁿ:- $m = \text{mean} = \frac{90}{60} = 1.5$, obviously, x will follow Poisson distribution,

Now the probability that at the most 1 to 3 calls in one minute

$$= P(1) + P(2) + P(3)$$

$$= \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!} + \frac{e^{-1.5}(1.5)^3}{3!}$$

$$= e^{-1.5}\left[1.5 + \frac{2.25}{2} + \frac{3.375}{6}\right]$$

$$= 0.223\,[1.5 + 1.125 + 0.562]$$

$$= 0.223\,[1.5 + 1.125 + 0.562]$$

$$= 0.223\,[3.187]$$

$$= 0.710701$$

$$= 0.711.$$

Ex. 2. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomely distributed throught the book, what is the probability that 10 pages, selected at random, will be free from errors?

(Use $e^{-0.735} = 0.4795$)

Soln:- Here $p = \frac{43}{585} = 0.0735$ and $10 = n$

$m = np = 10 \times 0.0735 = 0.735$

Cleary, p is very small and n is large

So, it is a case of Poisson distribution.

Let X denote the number of errors in 10 pages,

Then $P(X=r) = \frac{e^{-m} \cdot m^{r}}{r!} = \frac{e^{-0.735} \times (0.735)^{r}}{r!}$

$\therefore P(\text{no error}) = P(X=0)$

$$= \frac{e^{-0.735} \times (0.735)^{0}}{0!}$$

$$= e^{-0.735} = 0.4795$$

Hence, the required probability is 0.4795.

Ex. 3 Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents are:

(i) at least one

(ii) at most one. (Given $e^{-1.8} = 0.16529$)

Soln :- The probability function of the Poisson distribution is

$$P(X=r) = \frac{e^{-m} \cdot m^r}{r!} \quad (r=0,1,2,\ldots)$$

Given that $m = 1.8$

(i) $P(X \geq 1)$, $P(X \geq 1) = 1 - p(r<1)$

$$= 1 - p(r=0)$$

$$= 1 - \frac{e^{-m} m^0}{0!}$$

$$= 1 - e^{-1.8}$$

$$= 1 - 0.16529$$

$$= 0.83471$$

∴ The probability the number of accidents at least one is 0.83471.

(ii) $P(X \leq 1)$, $P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{e^{-m} m^0}{0!} + \frac{e^{-m} \cdot m^1}{1!} = e^{-1.8} + e^{-1.8} \cdot (1.8)$$

$$= 0.16529 + (0.16529) \times (1.8)$$

$P(X \leq 1) = e^{-1.8}(1+1.8) = (0.16529)(2.8)$

$= 0.4628$

Ex. ④ If a random variable X follows a poisson distribution such that $P(X=2) = 9 \cdot P(X=4) + 90\, P(X=6)$, find the mean and variance of X.

Solⁿ:- By poisson distribution, we have

$$P(X=r) = \frac{e^{-m}\, m^r}{r!}$$

$$P(X=2) = 9 \cdot P(X=4) + 90\, P(X=6)$$

$$\frac{e^{-m} \cdot m^2}{2!} = 9 \times \frac{e^{-m} \cdot m^4}{4!} + 90 \cdot \frac{e^{-m} \cdot m^6}{6!}$$

$$\frac{\cancel{e^{-m}}\, \cancel{m^2}}{\cancel{2!}} = \frac{\cancel{e^{-m}} \cdot \cancel{m^2}}{\cancel{2!}} \left[\frac{\overset{3}{\cancel{9}}}{4 \times \cancel{3}} m^2 + \frac{\overset{15}{\cancel{90}} \times m^4}{\cancel{6} \times 5 \times 4 \times 3} \right]$$

$$1 = \frac{3}{4} m^2 + \frac{m^4}{4}$$

$\Rightarrow m^4 + 3m^2 = 4$

$\Rightarrow m^4 + 3m^2 - 4 = 0$

$\Rightarrow m^4 + 4m^2 - m^2 - 4 = 0$

$\Rightarrow m^2(m^2+4) - 1(m^2+4) = 0$

$\Rightarrow m^2 - 1 = 0, \quad m^2 + 4 = 0$

$\Rightarrow m = 1 \quad [\because m > 0 \text{ and } m^2+4 \neq 0]$

Hence, mean $= 1$ and variance $= 1$.

Ex. 5 Find a Poisson distribution on the following:

x	0	1	2	3	4	
f	192	100	24	3	1	

(Given that $e^{-0.5} = 0.6065$)

Soln:

$P(r) = \frac{e^{-m} m^r}{r!}$; m = mean of the distribution

$m = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{0\times192 + 1\times100 + 2\times24 + 3\times3 + 4\times1}{192+100+24+3+1}$

$= \frac{100+48+9+4}{320} = \frac{161}{320} = 0.50$ (approx).

$P(0) = \frac{e^{-0.5}(0.5)^0}{0!} = 0.6065$ i.e. $f = 320\times0.6065$
$= 194$ (approx)

$P(1) = \frac{e^{-0.5}(0.5)^1}{1!} = \frac{0.6065 \times 0.5}{1!} = 0.30325$

i.e. $f = 320 \times 0.30325 = 97$ (approx).

$P(2) = \frac{e^{-0.5}(0.5)^2}{2!} = \frac{0.6065\times0.25}{2} = \frac{0.151625}{2} = 0.0758125$

ie. f = 24 (approx) = 320×0.0758125.

$P(3) = \frac{e^{-0.5}(0.5)^3}{3!} = 0.0126$ i.e. $f = 320\times0.0126$
= 4 (Approx)

$$p(4) = \frac{e^{-0.5}(0.5)^4}{4!} = 0.0016$$

i.e. $f = 320 \times 0.0016$
$= 0.512$ or 1 (Approx)

As total number of trials = 320.
We have the approximate value as obtained by poisson distribution as:

x	0	1	2	3	4
f	194	97	24	4	1

Ex.6 If a random variable has a poisson distrib. such that $p(1) = p(2)$. find:

(i) mean of the distribution

(ii) $P(4)$

Soln.- $p(1) = p(2) \Rightarrow \frac{e^{-m} m^1}{1!} = \frac{e^{-m} \cdot m^2}{2!}$

$\Rightarrow 2m = m^2$

$\Rightarrow m^2 - 2m = 0$

$\Rightarrow m(m-2) = 0$

$\Rightarrow m = 2 \quad , m > 0$

Hence mean of distribution is 2 and variance of distribution is also 2.

$$p(r) = \frac{e^{-m} \cdot m^r}{r!} \Rightarrow p(4) = \frac{e^{-2} \cdot 2^4}{4!}$$

$$= \frac{\cancel{16}\ 2}{\cancel{4} \times 3 \times \cancel{2} \times 1 \times e^2} = \frac{2}{3e^2} \quad \underline{\text{Ans.}}$$