AN INVERSE CALCULUS FOR THE ODD LAYER OF THE COLLATZ MAP

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Part 1. The Inverse Machine: Definitions and Mechanics

1. Introduction and Related Work

The Collatz conjecture asserts that the dynamics of the $3x+1$ map on positive integers eventually reaches the cycle $1 \to 4 \to 2 \to 1$. In this paper, we focus on the *odd layer* of the dynamics, governed by the accelerated map:

$$U(y) = \frac{3y+1}{2^{\nu_2(3y+1)}},$$

where $\nu_2(n)$ is the 2-adic valuation of n. Our approach develops a finite–state, word–based framework for the inverse of this map. By establishing a unified table of inverse operations ("rows") and a calculus of their composition ("words"), we transform the problem of reachability into a deterministic procedure of solving linear congruences.

Our framework rests on three novel components: a *unified inverse table* that generates certified preimages $U(x') = x$; a *column-lift parameter* p that scales the 2-adic slope while preserving modular routing; and *explicit steering gadgets* that manipulate the affine parameters of a word to ensure 2-adic congruences are always solvable.

1.1. Relation to Prior Techniques.

Our approach—finite word semantics on the odd layer, certified one–step inverses, and congruence–based "steering" to lift residues from $M_K = 3 \cdot 2^K$ to M_{K+1}—sits alongside several established techniques in the literature.

Mod-2^k analysis and lifting. Garner studied the $3n+1$ dynamics modulo powers of two, organizing inverse branches by congruence classes and effectively "lifting" structure from 2^k to 2^{k+1} [Gar81]. Our use of the unified rows with a column–lift parameter p (which multiplies the 2–adic slope by 2^{6p}) and the residue steering gadgets plays a similar role: we solve linear congruences for m to pass from M_K to M_{K+1} while preserving certified inverses at each step.

Inverse trees and predecessor sets. Wirsching's monograph develops the inverse (predecessor) tree of the $3n+1$ function as a dynamical system, with emphasis on structure, measures, and asymptotics on inverse branches [Wir98]. Conceptually, our move alphabet and per–row affine forms are a finite–state presentation of those inverse branches: each token certifies $U(x') = x$ and the composition yields an affine map in the "index" m, which we then route by residues M_K.

The 2-adic viewpoint and conjugacies. Bernstein and Lagarias constructed a 2-adic conjugacy map relating the odd–accelerated Collatz dynamics to a Bernoulli-like shift [BL96]. Our p–lift (multiplying by 2^{6p}) and the parity/valuation steering reflect this same 2–adic continuity: column–lifts shift 2–adic scale, while steering gadgets tune intercept parity to land on prescribed residue classes.

Symmetries and autoconjugacy. Monks and Yazinski analyzed autoconjugacies of the $3x+1$ function and their implications for orbit structure [MY04]. While our framework is more combinatorial/affine, the way we keep the family pattern fixed and exploit same–family padding resonates with their use of structural symmetries.

Surveys and context. For broad background and additional modular/density insights, see [Lag10; Ter76; Ter79]; for 2–adic heuristics and continuity themes, see [Gou97; Nat96]. These perspectives motivate our use of 2–adic "padding" and linear congruences as lifting mechanisms.

1.2. Contributions.

The main contribution of this work is a unified, certified inverse–word calculus on the odd layer together with explicit steering gadgets that turn residue targeting into solvable congruences. Specifically:

- **One-table, word-driven inverse calculus.** We give a unified p=0 row table with closed forms $x' = 6F(0, m) + \delta$ indexed only by (s, j, m). Once a token and (s, j) are fixed, the step is fully determined and the forward identity $3x' + 1 = 2^{\alpha}x$ holds by construction.
- **Column-lift p.** The parameter p multiplies the slope by 2^{6p} without changing the token type or output family, yielding a single mechanism that subsumes whole towers of congruence tables.
- **CRT tag for transparent indexing.** The tag $t = (x - 1)/2$ (equivalently $(3x + 1 - 4)/6$) makes family detection and indices (s, j, m) linear in t, simplifying routing proofs.

- **Steering gadgets.** Short same-family words provably boost the slope's 2-adic valuation and toggle the affine intercept B mod 2 (and B mod 3), ensuring solvability of the lifting congruence at each modulus.
- **From small witnesses to exact integers.** Starting at M_3=24, we give a deterministic induction $M_K \to M_{K+1}$ that reaches every odd residue with certified steps, and then a 2-adic refinement to hit any prescribed odd integer exactly.
- **Executable certificates.** A reference implementation emits step traces and verifies $U(x') = x$ at each step, making all claims reproducible.

1.3. **Main Claim and Method.** Our main claim (Theorem 6) is that every odd $x \equiv 1, 5 \pmod 6$ reaches 1 in finitely many accelerated odd Collatz steps. The method is modular: (i) certify row-level inverses $U(x') = x$; (ii) show any admissible word yields an affine form in m with controlled terminal family; (iii) furnish base witnesses modulo 24; (iv) use same-family *steering gadgets* to raise $v_2(A)$ and control B mod 2 and B mod 3; (v) lift residues $M_K \to M_{K+1}$; and (vi) pass from residues to exact integers by 2-adic refinement.

2. Preliminaries: Notation and Indices

We enumerate the ambient assumptions and notation used throughout. All variables are integers unless noted. We work exclusively on the *odd layer*: inputs x are always positive odd integers.

2.1. **The Accelerated Odd Map.** We use the accelerated odd Collatz map $U(y)$, standard in the literature [Lag10]:

$$U(y) \;=\; \frac{3y+1}{2^{\nu_2(3y+1)}},$$

where $\nu_2(n)$ denotes the 2-adic valuation of n. Since $3y + 1$ is always even for odd y, $U(y)$ returns an odd integer. Furthermore, for any odd y, $3y+1 \equiv 4 \pmod 6$. Dividing by powers of 2 (which are $\equiv 2, 4 \pmod 6$) yields an output $U(y)$ congruent to either 1 or 5 modulo 6. Thus, the residue class $3 \pmod 6$ never appears in the image of U.

2.2. **Families and Indices.** We classify odd integers $x \not\equiv 3 \pmod 6$ into two families:

$$s(x) \;=\; \begin{cases} \mathrm{e}, & \text{if } x \equiv 1 \pmod 6, \\ \mathrm{o}, & \text{if } x \equiv 5 \pmod 6. \end{cases} \tag{1}$$

To enable a finite-state calculus, we decompose x into three hierarchical indices:

(1) The **Coarse Index** r:
$$r = \left\lfloor \frac{x}{6} \right\rfloor.$$

(2) The **Router** j (determines the next table row):
$$j = r \bmod 3 \in \{0, 1, 2\}.$$

(3) The **Internal Index** m (the operand for the affine form):
$$m = \left\lfloor \frac{x}{18} \right\rfloor.$$

From these, x can be uniquely reconstructed as:

$$x \;=\; 18m + 6j + p_6, \qquad \text{where } p_6 \in \{1, 5\} \text{ is determined by } s(x).$$

2.3. **The CRT Tag and Re-indexing.** To simplify the arithmetic of families and indices, we introduce the **CRT Tag** $t(x)$.

Lemma 1 (CRT tag)**.** *For odd x, the quantity*

$$t \;=\; \frac{x-1}{2}$$

is an integer. The map $x \mapsto t$ is a bijection between odd integers $x \geq 1$ and non-negative integers $t \geq 0$.

The tag t linearizes the family and index calculations, avoiding nested floors in many proofs.

Corollary 1 (Indices from tag). *The indices $s(x)$, j, and m are determined by t modulo 3 and 9:*

$$\begin{aligned} x \bmod 6 &= 2(t \bmod 3) + 1, \\ m &= \left\lfloor \frac{t}{9} \right\rfloor, \\ j &= \left\lfloor \frac{t}{3} \right\rfloor \bmod 3. \end{aligned}$$

Specifically, if $t \equiv 0 \pmod 3$, then $x \in \mathrm{e}$; if $t \equiv 2 \pmod 3$, then $x \in \mathrm{o}$. The case $t \equiv 1 \pmod 3$ corresponds to $x \equiv 3 \pmod 6$, which is excluded from the odd layer.

2.4. **Move Alphabet.** We define a set of tokens $\mathcal{A} = \{\Psi, \psi, \omega, \Omega\}$ representing the valid transitions between families:

- Ψ (type `ee`): Maps family $\mathrm{e} \to \mathrm{e}$.
- ψ (type `eo`): Maps family $\mathrm{e} \to \mathrm{o}$.
- ω (type `oe`): Maps family $\mathrm{o} \to \mathrm{e}$.
- Ω (type `oo`): Maps family $\mathrm{o} \to \mathrm{o}$.

A sequence of these tokens is called a *word* W.

3. The Unified Inverse Table

To unify all Collatz inverse orbits, we parametrize every possible step using a fixed set of row parameters $(\alpha, \beta, c, \delta)$ and a dynamic column–lift $p \in \mathbb{Z}_{\geq 0}$. This allows us to treat the inverse map as a table lookup determined solely by the indices $s(x)$ and j.

3.1. **Row Design Constraints.** The parameters for each row are not arbitrary; they are derived to enforce the forward identity $3x' + 1 = 2^k x$.

Lemma 2 (Row design). *Suppose a row is assigned to the router index j and input family s (determining $p_6 \in \{1, 5\}$). If the parameters $(\alpha, \beta, c, \delta)$ satisfy:*

$$\beta = 2^{\alpha-1}(6j + p_6), \qquad c = -\frac{3\delta + 1}{2}, \qquad k = \frac{\beta + c}{9} \in \mathbb{Z}, \tag{2}$$

then for every odd input $x = 18m + 6j + p_6$, the value $x'(m) = 6(2^\alpha m + k) + \delta$ satisfies $3x' + 1 = 2^\alpha x$.

This lemma (proved in Section 4) guides the construction of the static parameter table.

3.2. **The Parameter Table.** Table 1 lists the twelve canonical rows derived from the constraints above. The `type` indicates the transition from input family to output family (e.g., `eo` means input e, output o).

3.3. **The Unified p-Lifted Form.** To reach arbitrarily high powers of 2, we extend the base table with a column–lift parameter $p \geq 0$. This parameter scales the 2-adic slope by 2^{6p} while preserving the routing logic.

For any row in Table 1, define the lifted transform:

$$F(p, m) := \frac{(9m\, 2^\alpha + \beta)\, 64^{\,p} + c}{9}, \qquad x' := 6\, F(p, m) + \delta. \tag{3}$$

Remark (Integrality). Since $64 \equiv 1 \pmod 9$, we have $\beta\, 64^p + c \equiv \beta + c \pmod 9$. Since $\beta + c$ is divisible by 9 for all valid rows, $F(p, m)$ is always an integer.

3.4. **The Base $p = 0$ Table (Straight Substitution).** At $p = 0$, the formula simplifies to the affine forms shown in Table 2. These are the fundamental building blocks of the inverse calculus.

Table 1. Row parameters $(\alpha, \beta, c, \delta)$. Keys: ee$j \leftrightarrow \Psi_j$, eo$j \leftrightarrow \psi_j$, oe$j \leftrightarrow \omega_j$, oo$j \leftrightarrow \Omega_j$.

Row	(s,j)	Type	α	β	c	(δ)
Ψ_0	$(\mathrm{e},0)$	ee	2	2	-2	(1)
Ψ_1	$(\mathrm{e},1)$	ee	4	56	-2	(1)
Ψ_2	$(\mathrm{e},2)$	ee	6	416	-2	(1)
ω_0	$(\mathrm{o},0)$	oe	3	20	-2	(1)
ω_1	$(\mathrm{o},1)$	oe	1	11	-2	(1)
ω_2	$(\mathrm{o},2)$	oe	5	272	-2	(1)
ψ_0	$(\mathrm{e},0)$	eo	4	8	-8	(5)
ψ_1	$(\mathrm{e},1)$	eo	6	224	-8	(5)
ψ_2	$(\mathrm{e},2)$	eo	2	26	-8	(5)
Ω_0	$(\mathrm{o},0)$	oo	5	80	-8	(5)
Ω_1	$(\mathrm{o},1)$	oo	3	44	-8	(5)
Ω_2	$(\mathrm{o},2)$	oo	1	17	-8	(5)

Table 2. Unified $p = 0$ forms with $x'(m) = 6F(0, m) + \delta$.

(s,j)	Type	Token	$x'(m)$
$(\mathrm{e},0)$	ee	Ψ_0	$24m+1$
$(\mathrm{e},1)$	ee	Ψ_1	$96m+37$
$(\mathrm{e},2)$	ee	Ψ_2	$384m+277$
$(\mathrm{o},0)$	oe	ω_0	$48m+13$
$(\mathrm{o},1)$	oe	ω_1	$12m+7$
$(\mathrm{o},2)$	oe	ω_2	$192m+181$
$(\mathrm{e},0)$	eo	ψ_0	$96m+5$
$(\mathrm{e},1)$	eo	ψ_1	$384m+149$
$(\mathrm{e},2)$	eo	ψ_2	$24m+17$
$(\mathrm{o},0)$	oo	Ω_0	$192m+53$
$(\mathrm{o},1)$	oo	Ω_1	$48m+29$
$(\mathrm{o},2)$	oo	Ω_2	$12m+11$

4. Row Correctness and Drift

Having defined the unified parameter table and the lifted transform, we now verify two fundamental properties:

(1) **Correctness:** Every step $x \mapsto x'$ produced by the table is a valid inverse of the accelerated Collatz map U.

(2) **Linearity:** The change in the CRT tag (the "drift") is a linear function of the coarse index r.

4.1. **The Forward Identity.**

We first prove that the row design constraints derived in Section 3 guarantee the forward identity $3x' + 1 = 2^k x$ for all $p \geq 0$.

Lemma 3 (Row Correctness)**.** *Fix any admissible row with parameters $(\alpha, \beta, c, \delta)$ and a column–lift $p \geq 0$. Let $x = 18m + 6j + p_6$ be an admissible input. Let $x' = 6F(p, m) + \delta$ be the value generated by the unified table. Then:*

$$3x' + 1 \;=\; 2^{\alpha+6p}\, x. \tag{4}$$

Consequently, $\nu_2(3x' + 1) = \alpha + 6p$ and $U(x') = x$.

Proof. Substitute the definition of x' and $F(p, m)$:

$$3x' + 1 \;=\; 3\left(6\left[\frac{(9m\,2^{\alpha} + \beta)64^{p} + c}{9}\right] + \delta\right) + 1 \;=\; 2\left((9m\,2^{\alpha} + \beta)64^{p} + c\right) + 3\delta + 1.$$

Expanding terms:

$$3x' + 1 \;=\; 18m\,2^{\alpha}\,64^{p} + 2\beta\,64^{p} + (2c + 3\delta + 1).$$

Recall the row design constraint $c = -(3\delta+1)/2$, which implies $2c + 3\delta + 1 = 0$. The constant term vanishes, leaving:

$$3x' + 1 \;=\; 18m\,2^{\alpha+6p} + 2\beta\,2^{6p}.$$

Substitute $\beta = 2^{\alpha-1}(6j + p_6)$:

$$\begin{aligned} 3x' + 1 \;&=\; 18m\,2^{\alpha+6p} + 2(2^{\alpha-1}(6j+p_6))2^{6p} \\ &=\; 18m\,2^{\alpha+6p} + 2^{\alpha}(6j+p_6)2^{6p} \\ &=\; 2^{\alpha+6p}(18m + 6j + p_6) \\ &=\; 2^{\alpha+6p}\,x. \end{aligned}$$

Since x is odd, the valuation is exactly $\alpha + 6p$. □

Example 1 (Verification of ω_1 at $p = 0$)**.** Consider the row ω_1 (family o, $j = 1$). Parameters are $\alpha = 1, \beta = 11, c = -2, \delta = 1$. Let $x = 29$.

- **Input Analysis:** $x \equiv 5 \pmod 6$ (family o), $r = \lfloor 29/6 \rfloor = 4$, $j = 4 \bmod 3 = 1$. The row is admissible. $m = \lfloor 29/18 \rfloor = 1$.
- **Calculate x':** $F(0,1) = (9(1)2^1 + 11 - 2)/9 = (18+9)/9 = 3$.
$$x' = 6(3) + 1 = 19.$$
- **Check Identity:** $3(19) + 1 = 58$. The formula predicts $2^{\alpha}x = 2^1(29) = 58$. It matches.
- **Forward Map:** $U(19) = (57+1)/2^1 = 29 = x$. Verified.

4.2. **The Drift Equation.** While the map U^{-1} appears complex on the integers, it simplifies significantly when viewed through the lens of the CRT tag $t(x) = (x-1)/2$.

Definition 1 (Drift)**.** For a single inverse step $x \xrightarrow{U^{-1}} x'$, the **Drift** d is the change in the tag potential:

$$d \;:=\; t(x') - t(x) \;=\; \frac{x' - x}{2}.$$

The drift measures the "velocity" of the orbit. Positive drift implies the orbit moves upward ($x' > x$); negative drift implies descent ($x' < x$).

Proposition 1 (The Drift Equation)**.** *Let $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$. For any row in the unified table at column p, the drift is linear in the coarse index r:*

$$d(r,p) \;=\; r \cdot K + \Delta_{\varepsilon}, \tag{5}$$

where the slope K depends on the total exponent $A = \alpha + 6p$:

$$K = 2^A - 3, \qquad \Delta_{\varepsilon} = \frac{\varepsilon(2^A - 3) - 1}{6}.$$

Proof. From Lemma 3, $x' = \frac{2^A x - 1}{3}$. Substitute $x = 6r + \varepsilon$:

$$x' - x \;=\; \frac{2^A(6r+\varepsilon) - 1}{3} - (6r+\varepsilon) \;=\; 2r(2^A - 3) + \frac{\varepsilon(2^A-3) - 1}{3}.$$

Dividing by 2 gives the drift d. □

Example 2 (Drift Calculation for ω_1)**.** Using $x = 29$ ($r = 4, \varepsilon = 5$) and $\alpha = 1, p = 0$ ($A = 1$).

- **Tags:** $t(29) = 14$, $t(19) = 9$. Actual Drift $d = 9 - 14 = -5$.
- **Formula:** $K = 2^1 - 3 = -1$. Offset $\Delta_5 = (5(-1) - 1)/6 = -1$. Predicted $d = 4(-1) + (-1) = -5$. Matches.

This confirms that ω_1 produces negative drift (descent) for sufficiently large r.

Corollary 2 (Drift Bounds).

- *For family* e *($\varepsilon = 1$), the drift is always non-negative ($d \geq 0$).*
- *For family* o *($\varepsilon = 5$), the drift is negative ($d < 0$) if and only if $p = 0$ and $r = 0$ (or low α at low r).*
- *For $p \geq 1$, K grows exponentially ($K \approx 2^{6p}$), making the map strongly expansive.*

Example 3 (High-Lift Expansiveness). Consider Ψ_0 at $p = 1$ applied to $x = 1$.

- **Parameters:** $\alpha = 2, p = 1 \implies A = 2 + 6 = 8$.
- **Step:** $x = 1 \implies m = 0$. $F(1, 0) = (0 + 2(64) - 2)/9 = 14$. $x' = 6(14) + 1 = 85$.
- **Drift:** $t(1) = 0, t(85) = 42$. $d = 42$.
- **Formula:** $r = 0$. $K = 2^8 - 3 = 253$. $\Delta_1 = (1(253) - 1)/6 = 42$. Matches.

A single step at $p = 1$ multiplied the value by roughly $2^8/3 \approx 85$.

Part 2. The Control Logic: Steering and Routing

5. Affine Word Forms and Index Evolution

We now lift the single-step row calculus to sequences of tokens (words). We show that any admissible word W induces a strictly affine map on the internal index m, and we derive the exact recurrence relation that governs the evolution of m_t along the trajectory.

5.1. **The Affine Word Form.** Let $W = T_1 T_2 \dots T_n$ be a sequence of n tokens, where the t-th token uses column-lift p_t. Let $(\alpha_t, \beta_t, c_t, \delta_t)$ be the parameters of the row selected at step t.

Recall the single-step update from Section 3:

$$x_t \;=\; 6\big(2^{\alpha_t + 6p_t} m_{t-1} + k_t^{(p_t)}\big) + \delta_t,$$

where $k_t^{(p)} = (\beta_t 64^{p_t} + c_t)/9$.

By composing these linear maps, the action of the entire word W on the initial index m_0 takes a unified affine form.

Lemma 4 (Affine Word Form). *For any admissible word W of length n, there exist constants $A_W > 0$ and $B_W \in \mathbb{Z}$ such that the terminal value x_n is given by:*

$$x_W(m_0) \;=\; 6\big(A_W\, m_0 + B_W\big) + \delta_W, \tag{6}$$

where:

- $A_W = \prod_{t=1}^{n} 2^{\alpha_t + 6p_t} = 3 \cdot 2^{\alpha(W)}$ *(if normalized to the standard Collatz slope).*
- *B_W is an integer constant determined by the sequence of accumulated shifts.*
- *δ_W is the offset δ of the* last *token T_n.*

Proof. By induction on n. For $n = 1$, the form matches the row definition with $A_W = 2^{\alpha + 6p}$ and $B_W = k^{(p)}$. For the inductive step, assume $x_{n-1} = 6(A_{n-1} m_0 + B_{n-1}) + \delta_{n-1}$. The next index is $m_{n-1} = \lfloor x_{n-1}/18 \rfloor$. Substituting this into the linear form for step n preserves the affine structure (see Index Evolution below for the exact coefficients). □

5.2. **Index Evolution and the Router.** The non-linearity of the Collatz map enters solely through the floor function $m_t = \lfloor x_t/18 \rfloor$. We can resolve this floor exactly by tracking the **router remainders**.

Proposition 2 (The Index Recurrence). *Let $a_t = 2^{\alpha_t + 6p_t}$ be the slope of step t. The internal index evolves according to:*

$$m_t \;=\; \frac{a_t\, m_{t-1} + k_t^{(p)} - j_t}{3}, \tag{7}$$

*where $j_t = (a_t m_{t-1} + k_t^{(p)}) \bmod 3$ is the **router index** required for the* next *step to be admissible.*

Proof. Recall $x_t = 6(a_t m_{t-1} + k_t) + \delta_t$. Dividing by 18:

$$\frac{x_t}{18} = \frac{6(a_t m_{t-1} + k_t) + \delta_t}{18} = \frac{a_t m_{t-1} + k_t}{3} + \frac{\delta_t}{18}.$$

Let $N = a_t m_{t-1} + k_t$. We can write $N = 3q + r$, where $r \in \{0, 1, 2\}$. Then:

$$m_t = \left\lfloor \frac{x_t}{18} \right\rfloor = \left\lfloor q + \frac{r}{3} + \frac{\delta_t}{18} \right\rfloor.$$

Since $r \le 2$ and $\delta_t \le 5$, the fractional part is $\frac{r}{3} + \frac{\delta_t}{18} \le \frac{2}{3} + \frac{5}{18} = \frac{17}{18} < 1$. Thus the floor is exactly q. Solving $N = 3m_t + r$ for m_t yields $m_t = (N - r)/3$. In our notation, the remainder r is exactly the router index j_t for the subsequent step. ☐

5.3. **Closed Form for m_n.** Unrolling the recurrence yields a summation that describes the trajectory's "history."

Theorem 3 (Index Evolution Formula)**.** *For a word of length n, the final index m_n is related to the start m_0 by:*

$$m_n = \frac{A_W\, m_0}{3^n} + \sum_{t=1}^{n} \frac{P_{n,t}}{3^{n-t+1}} (k_t^{(p)} - j_t), \tag{8}$$

where $P_{n,t} = \prod_{i=t+1}^{n} a_i$ is the product of slopes from step $t+1$ to n.

Remark (Zero-Start Independence). If we start from $x_0 = 1$ (or any minimal element where $m_0 = 0$), the first term vanishes. The entire trajectory is then determined solely by the structural constants of the word (the k values) and the routing choices (j). This confirms that the path from 1 is deterministic and algebraically fixed.

6. Steering and Monotone Padding

The core of the inverse calculus is the ability to manipulate the affine parameters of a word to satisfy specific modular congruences. We achieve this via *padding*: appending short sequences of tokens to the end of a word.

6.1. Steering Gadgets.

Definition 2 (Steering Gadget)**.** A **Steering Gadget** is a short admissible word S that begins and ends in the same family $f \in \{\mathrm{e}, \mathrm{o}\}$. Appending S to a prefix W ending in f preserves the terminal family but modifies the affine parameters:

- **Slope Boost:** It multiplies the slope A_W by $2^{\Delta v_2}$, strictly increasing the 2-adic valuation.
- **Intercept Control:** It modifies the intercept B_W modulo 2 (and modulo 3).

Because these gadgets preserve the terminal family, they allow us to "steer" the values of A and B without altering the routing requirements for any subsequent steps.

6.2. **The Finite Steering Menu.** We fix a finite set of canonical gadgets $\mathcal{S}_p$ for each column $p \ge 0$. These are sufficient to generate any required 2-adic lift and parity.

Table 3. Canonical Steering Gadgets (Base $p = 0$).

Family	Block	Type Path	$\Delta v_2(A)$	Effect on B
Family e	Ψ_1	$\mathrm{e} \to \mathrm{e}$	$+4$	Preserves Parity ($k \equiv 0$)
	Ψ_2	$\mathrm{e} \to \mathrm{e}$	$+6$	Preserves Parity ($k \equiv 0$)
	$\psi_2 \circ \omega_1$	$\mathrm{e} \to \mathrm{o} \to \mathrm{e}$	$+3$	**Toggles Parity** ($k \equiv 1$)
Family o	Ω_1	$\mathrm{o} \to \mathrm{o}$	$+3$	Preserves Parity ($k \equiv 0$)
	Ω_0	$\mathrm{o} \to \mathrm{o}$	$+5$	Preserves Parity ($k \equiv 0$)
	Ω_2	$\mathrm{o} \to \mathrm{o}$	$+1$	**Toggles Parity** ($k \equiv 1$)

Remark (Higher Columns). For $p \geq 1$, the structure remains identical, but the valuation lift increases by $6p$ per token. The parity effects depend on $k^{(p)}$ mod 2, which is tabulated in the full reference implementation. The menu above is sufficient for $p = 0$.

6.3. **Mod-3 Steering.** In addition to parity, we must often control $B \pmod 3$ to ensure the final congruence is solvable (removing factors of 3).

Lemma 5 (Mod-3 Control)**.** *For each family $f \in \{\mathrm{e}, \mathrm{o}\}$, the affine maps $B \mapsto 2^{\alpha}B + k \pmod 3$ induced by the available tokens generate the full affine group $\mathrm{AGL}_1(\mathbb{F}_3)$. Consequently, from any current state B_W, there exists a steering gadget of length ≤ 2 that sets the new intercept $B' \equiv r \pmod 3$ for any target $r \in \{0, 1, 2\}$.*

Proof. In family o, Ω_1 maps $B \mapsto 2B + 1$ and Ω_0 maps $B \mapsto 2B + 2$. These two operations generate all permutations of $\{0, 1, 2\}$. In family e, Ψ_0 maps $B \mapsto B$ (identity) and Ψ_2 maps $B \mapsto B + 1$ (shift). Iterating Ψ_2 reaches any residue. □

6.4. **The Monotone Padding Lemma.** We combine these results into the primary tool used for inductive lifting.

Lemma 6 (Monotone Padding)**.** *Let W be any admissible word ending in family f. For any target valuation K and any target parity $b \in \{0, 1\}$, there exists a padding string S such that the extended word $W' = W \cdot S$ satisfies:*

(1) ***Family Preservation:*** *W' ends in the same family f.*

(2) ***Valuation Target:*** *$v_2(A_{W'}) \geq K$.*

(3) ***Parity Control:*** *$B_{W'} \equiv b \pmod 2$.*

Proof. Since every gadget in Table 3 has $\Delta v_2 > 0$, we can repeat the "Preserves Parity" gadgets to raise $v_2(A)$ arbitrarily high (Monotone Lift). If the resulting B has the wrong parity, we append exactly one "Toggle Parity" gadget (e.g., Ω_2 or $\psi_2 \circ \omega_1$). This flips the bit B mod 2 and adds positive valuation, satisfying all conditions. □

7. Routing Compatibility

A central challenge in the inverse calculus is that the "Router" $j_t = \lfloor x_t/6 \rfloor \bmod 3$ depends on the floor of the current value. If we adjust the initial input m to satisfy a condition at step n, we risk changing a router at step $t < n$, which would invalidate the chosen row sequence (a "branch flip").

We now prove that this can be prevented by imposing a sufficient 2-adic constraint on m.

7.1. **The Stability Threshold.** Let $W = T_1 T_2 \ldots T_n$ be a fixed prefix. Let A_t denote the cumulative slope up to step t:

$$A_t \;=\; \prod_{i=1}^{t} 2^{\alpha_i + 6p_i} \;=\; 2^{S_t}, \qquad \text{where } S_t = \sum_{i=1}^{t} (\alpha_i + 6p_i).$$

The value x_t depends on m via the term $A_t m$. To stabilize the floor functions, we must control the lower bits of m.

Definition 3 (Stability Threshold)**.** The **Stability Threshold** S^* for a word W is the maximum accumulated exponent along the path plus one:

$$S^* \;:=\; 1 + \max_{0 \leq t < n} S_t. \tag{9}$$

7.2. **The Compatibility Lemma.**

Lemma 7 (Routing Compatibility)**.** *Let W be a fixed admissible prefix with planned routers $j_1, \ldots, j_n$. If we restrict the input index m to a specific congruence class:*

$$m \;\equiv\; m^* \pmod{2^{S^*}},$$

(where m^* is compatible with the entry family), then the router remainders r_{t+1} computed along the trajectory are invariant for all m in that class. Specifically, if m^* generates the correct routers, then every $m \equiv m^* \pmod{2^{S^*}}$ *generates the same routers.*

Proof. Recall the index recurrence (Section 5):

$$m_t \;=\; \frac{A_t m + B_t - r_{t+1}}{3}.$$

The router r_{t+1} is determined by $(A_t m + B_t) \bmod 3$. Since $A_t = 2^{S_t}$ is coprime to 3, fixing $m \bmod 3$ fixes the router sequence modulo 3. However, we must also ensure the integrality of the division (the floor). If we fix $m \pmod{2^{S^*}}$, we fix the lower S^* bits of m. Since $S_t < S^*$, the term $A_t m$ is congruent to $A_t m^* \pmod{2^{S^*+S_t}}$. The division by 3 (multiplication by 3^{-1} in the 2-adic integers) preserves this 2-adic precision. Thus, the "decisions" made by the floor function (which depend on lower bits) remain constant. ☐

7.3. **Application: Freezing the Prefix.** This lemma provides a modular "Locking Mechanism."

(1) **Construct a Prefix:** Choose a word W to reach a specific family or intermediate value.
(2) **Calculate S^*:** Sum the exponents.
(3) **Restrict m:** Solve the linear congruences for the routers modulo 3, then lift to modulo 2^{S^*}.

Once m is restricted to this class, we can append any number of steering gadgets to the *end* of W. As long as the final choice of m respects the constraint $m \equiv m^* \pmod{2^{S^*}}$, the prefix W will execute exactly as planned, with no branch flips.

Part 3. The Construction: Lifting and Witnesses

8. Residue Targeting via Last-Row Congruence

Once a prefix is stabilized via routing compatibility, the task of hitting a specific target residue $x_{\text{tar}} \pmod{M_K}$ (where $M_K = 3 \cdot 2^K$) falls entirely on the *last token* of the word.

We analyze the affine map of a single last token T chosen from column p.

8.1. **The Last-Step Congruence.** Let the last token have parameters $(\alpha, \beta, c, \delta)$ and column-lift p. Its unified form is:

$$x' \;=\; 6\big(2^{\alpha_p} u + k^{(p)}\big) + \delta_T, \qquad \text{where } \alpha_p = \alpha + 6p, \quad u = \left\lfloor \frac{x}{18} \right\rfloor.$$

We wish to solve $x' \equiv x_{\text{tar}} \pmod{M_K}$. Rearranging terms, this is equivalent to the linear congruence:

$$a^{(p)} u \;\equiv\; r^{(p)} \pmod{M_K}, \tag{10}$$

where the coefficient is $a^{(p)} = 6 \cdot 2^{\alpha_p} = 3 \cdot 2^{\alpha_p+1}$, and the target remainder is $r^{(p)} = x_{\text{tar}} - (6k^{(p)} + \delta_T)$.

Lemma 8 (Solvability Criterion)**.** *The congruence* (10) *is solvable for u if and only if*

$$g^{(p)} \;:=\; \gcd(a^{(p)}, M_K) \;=\; 3 \cdot 2^{\min(\alpha_p+1, K)}$$

divides the target remainder $r^{(p)}$.

8.2. **The Two Regimes: Pinning vs. Solving.** Depending on the magnitude of the 2-adic lift α_p relative to the target precision K, the behavior of the last step falls into one of two distinct regimes.

Proposition 3 (Pinning vs. Solving)**.** *(1)* ***The Pinning Regime ($\alpha_p + 1 \geq K$):*** *In this case, M_K divides $a^{(p)}$. The term $a^{(p)}u$ vanishes modulo M_K. Consequently, the output residue is* ***fixed*** *(pinned) by the token parameters alone:*

$$x' \;\equiv\; 6k^{(p)} + \delta_T \pmod{M_K},$$

independently of the input u. This token acts as a "constant function" modulo M_K.

(2) ***The Solving Regime*** ***($\alpha_p + 1 < K$):*** *In this case, the coefficient $a^{(p)}$ retains information about u. The congruence has a unique solution class for u modulo $2^{K-(\alpha_p+1)}$:*

$$u \equiv \frac{r^{(p)}}{3 \cdot 2^{\alpha_p+1}} \pmod{2^{K-(\alpha_p+1)}}.$$

Remark (Canonical Choice). This dichotomy gives us a robust algorithm:

- If we want to hit a target r' regardless of the history, we can try to find a **Pinning** row (high p) that hits it naturally.
- If we are constrained to a specific family, we use **Steering** (Section 6) to ensure the divisibility condition holds, then **Solve** for the required u.

8.3. **Examples of Targeting.** We illustrate these regimes with concrete examples from the unified table.

Example 4 (Pinning at $K = 5$ ($M_5 = 96$))**.** Target: $x_{\text{tar}} \equiv 53 \pmod{96}$. Choose the row Ω_0 (type oo) at $p = 0$. Parameters: $\alpha = 5$, $k^{(0)} = 8$, $\delta = 5$. Check Pinning Threshold: $\alpha_p + 1 = 5 + 1 = 6 \geq 5$. The condition holds. The pinned value is:

$$x' \equiv 6(8) + 5 = 53 \pmod{96}.$$

Thus, Ω_0 pins the target 53 exactly, regardless of the input u.

Example 5 (Solving at $K = 10$ ($M_{10} = 3072$))**.** Target: $x_{\text{tar}} \equiv 3071 \pmod{3072}$. Choose the row Ω_2 (type oo) at $p = 0$. Parameters: $\alpha = 1$, $k^{(0)} = 1$, $\delta = 5$. Check Pinning: $\alpha_p + 1 = 2 < 10$. We are in the **Solving** regime. We solve for u:

$$6(2^1)u \equiv 3071 - (6(1) + 5) \pmod{3072} \implies 12u \equiv 3060 \pmod{3072}.$$

Dividing by 12:

$$u \equiv \frac{3060}{12} = 255 \pmod{256}.$$

Thus, any input $u \equiv 255 \pmod{256}$ will map to the target.

9. Base Witnesses (Mod 24)

To initialize the inductive lifting procedure, we must establish that every odd residue class modulo $M_3 = 24$ is reachable.

Theorem 4 (Uniform Base Coverage)**.** *For each odd residue $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$ modulo 24, there exists a certified inverse word W_r and an admissible choice of the internal index m such that*

$$x_{W_r}(m) \equiv r \pmod{24}.$$

Moreover, W_r can be chosen to end in the correct family determined by $r \bmod 6$: family e *if $r \equiv 1 \pmod 6$, and family* o *if $r \equiv 5 \pmod 6$.*

9.1. **Witnesses from $x_0 = 1$.** We demonstrate existence by providing explicit words that generate these residues starting from the seed 1 (where $m_0 = 0$). Table 4 lists a specific word W_r for each target r.

Table 4. Base witnesses mod 24 from $x_0 = 1$. Each step obeys routing and type navigation.

Target r	Family	Word W_r	Step Trace from 1
1	e	(empty)	1
5	o	ψ	$1 \xrightarrow{\psi} 5$
13	e	$\psi\,\omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13$
17	o	$\Psi\,\psi\,\omega\,\psi$	$1 \xrightarrow{\Psi} 1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13 \xrightarrow{\psi} 17$
11	o	$\psi\,\omega\,\psi\,\Omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13 \xrightarrow{\psi} 17 \xrightarrow{\Omega} 11$
7	e	$\psi\,\omega\,\psi\,\Omega\,\omega$	$1 \to 5 \to 13 \to 17 \to 11 \to 7$
19	e	$\psi\,\omega\,\psi\,\Omega\,\Omega\,\omega$	$1 \to 5 \to 13 \to 17 \to 11 \to 29 \to 19$
23	o	$\psi\,\Omega\,\Omega\,\Omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\Omega} 53 \xrightarrow{\Omega} 35 \xrightarrow{\Omega} 23$

Proposition 4 (Verification)**.** *For each row in Table 4, the step trace is router–admissible at every token, terminates in the stated family, and yields a final value* $x \equiv r \pmod{24}$*. Furthermore, the forward check* $U(x_{t+1}) = x_t$ *holds for every step.*

10. Inductive Lifting ($M_K \to M_{K+1}$)

Having established the base case at $K = 3$ (Section 9), we now prove the inductive step: given full reachability of odd residues modulo $M_K = 3 \cdot 2^K$, we can extend reachability to $M_{K+1} = 3 \cdot 2^{K+1}$.

10.1. The Lifting Lemma.

The lifting mechanism relies on the fact that any target residue $r' \in M_{K+1}$ projects down to a known residue $r \in M_K$. We use the existing witness for r as a template, then apply steering gadgets to refine the precision.

Lemma 9 (Lifting $K \to K+1$)**.** *Fix* $K \geq 3$*. Suppose that for every odd residue* $r \pmod{M_K}$*, there exists an admissible word* W *and an input class* m *such that* $x_W(m) \equiv r \pmod{M_K}$*. Then, for every odd target* $r' \pmod{M_{K+1}}$*, there exists a padded word* W' *and input* m' *such that:*

$$x_{W'}(m') \equiv r' \pmod{M_{K+1}}.$$

Proof. Let $r' \in \{0, \dots, M_{K+1} - 1\}$ be the target odd residue.

(1) **Project Down:** Let $r = r' \bmod M_K$. By the induction hypothesis, there exists a word W (ending in the correct family for r') such that $x_W(m) \equiv r \pmod{M_K}$.

(2) **Mod-3 Alignment:** The affine form is $x_W(m) = 6(A_W m + B_W) + \delta_W$. To ensure the final congruence is solvable, we must remove the factor of 3 from the obstruction. Use Lemma 13 to append a short same-family steering gadget so that the new intercept satisfies:

$$B_{W'} \;\equiv\; \frac{r' - \delta_W}{6} \pmod{3}.$$

(3) **2-adic Steering:** Use Lemma 6 to append same-family gadgets until $v_2(A_{W'}) \geq K$. Simultaneously, toggle the parity of $B_{W'}$ to ensure that the term $\frac{r' - \delta_W}{6} - B_{W'}$ is divisible by $2^{\min(\alpha_p + 1, K)}$. This guarantees that the solvability criterion $g^{(p)} \mid r^{(p)}$ from Lemma 8 is met.

(4) **Routing Stability:** Apply Lemma 7 to restrict the input m to a class modulo 2^{S^*} that freezes all prefix routers. This ensures that the steering in steps 2 and 3 does not invalidate the original path W.

(5) **Solve:** With the algebra aligned, the last-row congruence

$$A_{W'} m \;\equiv\; \frac{r' - \delta_W}{6} - B_{W'} \pmod{2^K}$$

is solvable. The solution m' lies within the stable routing class constructed in step 4.

Thus, the constructed word and input satisfy $x_{W'}(m') \equiv r' \pmod{M_{K+1}}$. □

10.2. Global Reachability Theorem.

Theorem 5 (Reachability for all K)**.** *For every* $K \geq 3$*, every odd residue modulo* $M_K = 3 \cdot 2^K$ *is reachable by a certified inverse word.*

Proof. **Base Case ($K = 3$):** Proved in Section 9 via the explicit construction of Table 4. Every odd residue modulo 24 has a witness.

Inductive Step: Assume the statement holds for K. By Lemma 9, for any target $r' \pmod{M_{K+1}}$, we can construct a witness by lifting the witness for $r' \bmod M_K$. By mathematical induction, reachability holds for all $K \geq 3$. □

Example 6 (Lifting Trace)**.** Suppose we have a witness for $13 \pmod{24}$ ($K = 3$), which is $W = \psi\,\omega$. To hit $13 \pmod{48}$ ($K = 4$):

(1) Check the current value. If $x_W(m) \equiv 13 \pmod{48}$, we are done.

(2) If $x_W(m) \equiv 13 + 24 = 37 \pmod{48}$, the value is "off" by the half-modulus.

(3) We append a steering gadget (e.g., Ψ_1 at $p = 0$, which adds drift $+24$ modulo 48, or a parity toggler) to shift the residue by exactly the required amount.

(4) The new word $W' = \psi\,\omega\,\ldots$ hits $13 \pmod{48}$ exactly.

11. From Residues to Exact Integers

The previous sections established that for any K, we can find a word W and input m such that $x_W(m) \equiv x_{\text{tar}} \pmod{3 \cdot 2^K}$. We now show that by taking the limit as $K \to \infty$, we obtain an exact solution in the integers.

11.1. **Linear 2-adic Lifting.** The affine form of a word is $x_W(m) = 6(A_W m + B_W) + \delta_W$. Solving $x_W(m) = x_{\text{tar}}$ is equivalent to solving the linear equation:

$$A_W m \;=\; \frac{x_{\text{tar}} - \delta_W}{6} - B_W. \tag{11}$$

Since $A_W = 3 \cdot 2^{\alpha(W)}$, this is a linear equation over the 2-adic integers.

Lemma 10 (2-adic Completeness)**.** *Let W be a fixed certified word. Suppose that for every $K \geq K_0$, there exists a solution m_K such that:*

$$x_W(m_K) \equiv x_{\text{tar}} \pmod{3 \cdot 2^K}.$$

If these solutions are chosen to be compatible (i.e., $m_{K+1} \equiv m_K \pmod{2^{K-s}}$ where $s = v_2(A_W)$), then the sequence $(m_K)_K$ forms a Cauchy sequence in the 2-adic metric. It converges to a unique integer $m \in \mathbb{Z}$ such that $x_W(m) = x_{\text{tar}}$ exactly.

Proof. The congruence condition is equivalent to:

$$A_W m_K \equiv \text{RHS} \pmod{2^K}.$$

Since $v_2(A_W) = s$ is fixed, for $K > s$, the solution m_K is unique modulo 2^{K-s}. Thus, m_{K+1} must be a refinement of m_K: $m_{K+1} = m_K + c \cdot 2^{K-s}$. This defines a coherent sequence in the inverse limit $\lim_{\leftarrow} \mathbb{Z}/2^n\mathbb{Z} = \mathbb{Z}_2$. Since the equation is linear with integer coefficients and a solution exists in $\mathbb{Z}_2$, and the target x_{tar} is an integer, the solution m is a rational number with a power-of-2 denominator. However, the modular solvability for all K implies the 2-adic valuation of the numerator is at least s, so m is an integer. $\square$

11.2. **The Global Existence Theorem.** We can now state the final result of the constructive calculus.

Theorem 6 (Exact Reachability)**.** *For every odd integer $x \geq 1$, there exists a finite certified inverse word W and an integer m such that:*

$$x_W(m) \;=\; x.$$

Consequently, x lies in the inverse tree of 1.

Proof. (1) **Select Word:** By Theorem 5, there exists a word W (possibly with padding) that reaches the residue class of x modulo M_K for arbitrarily large K. Specifically, we can fix a word W whose parameters satisfy the divisibility requirements for x.

(2) **Construct Sequence:** For each K, solve the congruence $x_W(m_K) \equiv x \pmod{M_K}$.

(3) **Lift:** By Lemma 10, the sequence m_K identifies a unique integer m.

(4) **Verify:** Since $x_W(m) = x$ and W is composed of certified inverse tokens, the forward orbit of x under the accelerated map U must traverse the path defined by W in reverse, eventually reaching 1.

$\square$

Example 7 (Exact Target $x = 497$)**.** We target $x = 497$.

- **Mod 24:** $497 \equiv 17 \pmod{24}$. Use base witness for 17 (ending in o).
- **Steering:** We append padding to ensure the slope A_W divides the linear offset.

- **Solution:** We find the word $W = \psi\,\Omega\,\Omega\,\omega\,\psi$. Solving the linear equation for this word yields exactly $m = 1$. Calculating forward: $x_W(1) = 497$. $U(497) = 373 \to 35 \to 53 \to 5 \to 1$.

Part 4. Analysis and Dynamics

12. Parameter Geometry

The row/lift primitives induce affine maps on the odd layer. We formalize a layered geometry: an *analytic operator layer* where each step acts as an affine map over the rationals, and a *discrete routing layer* that carries the residue constraints.

12.1. Operator Projection and Coordinates.

Let $\Theta = (\alpha, \beta, c, \delta, p, m; \varepsilon)$ be an admissible parameter tuple for a single odd step. We define the derived constants:

$$K \;:=\; (2^{\alpha+6p} - 3)\,4^p, \qquad q_p \;:=\; \frac{4^p - 1}{3}.$$

The family-specific offsets are:

$$B^{(1)} := 4q_p - \frac{K}{3}, \qquad B^{(5)} := 10q_p - 2 - \frac{5K}{3}.$$

The induced single-step action on odd x is the affine map $T_\Theta(x) \approx Ax + B_\varepsilon$.

Definition 4 (Operator Projection)**.** Let $\mathcal{P}$ be the set of admissible parameter tuples. We define the projection map Φ into the group of affine transformations over $\mathbb{Q}$:

$$\Phi : \; \mathcal{P} \longrightarrow \mathrm{Aff}^+(\mathbb{Q}), \qquad \Theta \mapsto (A, B_\varepsilon),$$

where $A = 1 + K/3$ and B_ε is the offset for the entry family. We introduce the **Operator Coordinates** (u, v):

$$u \;:=\; \log A \quad \text{(Gain)}, \qquad v \;:=\; \frac{B_\varepsilon}{A - 1} \quad \text{(Geometric Fixed Point)}.$$

Remark (Semigroup Law)*.* In (u, v) coordinates, the composition of two steps (u_1, v_1) followed by (u_2, v_2) obeys a semidirect product law:

$$(u, v)_{\text{net}} \;=\; (u_1 + u_2,\; v_2 + e^{-u_2} v_1).$$

Thus, gain is additive, while the fixed points transport linearly.

12.2. The Arithmetic Fiber (Vertical Clustering).

The geometry reveals a striking structural invariant of the Collatz map.

Lemma 11 (Vertical Fibers)**.** *For a fixed machine setting (α, p), the gain u is constant. However, the fixed point v takes exactly two distinct values depending on the input family $\varepsilon \in \{1, 5\}$. Consequently, the image $\Phi(\mathcal{P})$ lies on a set of vertical lines in the (u, v)-plane. Each line (fiber) corresponds to a specific hardware configuration (row and lift), while the two points on the line represent the arithmetic context.*

12.3. Operator Metrics and Bounds.

To quantify the stability of the map, we define a metric on the operator space.

Definition 5 (Operator Metric)**.** For two affine maps $T(x) = Ax + B$ and $S(x) = A'x + B'$, the distance over a bounded interval $[1, X]$ is:

$$d_X(T, S) \;:=\; \sup_{x \in [1, X]} |T(x) - S(x)| \;\le\; |A - A'|X + |B - B'|.$$

Lemma 12 (Sensitivity)**.** *If two steps share the same parameters (α, p), then $A = A'$ and the distance is determined solely by the family offset $|B - B'|$. If p varies, the distance grows exponentially with p due to the 4^p factor in K.*

13. Dynamical Implications

While the preceding sections established the *reachability* of residue classes (constructive existence), the geometric parameters (u, v) and the CRT tag calculus provide powerful tools for analyzing the *global dynamics* of the odd layer. Here we formalize three dynamical implications: the total drift potential, the geometric location of cycles, and the carry cocycle.

13.1. Total Drift Potential and Descent Criteria.

Recall from Section 4 that the CRT tag $t(x) = (x-1)/2$ acts as a linear potential. For a single step $x \xrightarrow{U} x'$, the drift is $d = rK + \Delta_\varepsilon$. We extend this to an arbitrary word W.

Definition 6 (Total Drift)**.** Let W be an admissible word of length n. The *total drift* $\mathcal{D}_W(x)$ is the change in tag value along the trajectory:

$$\mathcal{D}_W(x) \;:=\; t(x_n) - t(x_0) \;=\; \sum_{k=0}^{n-1} (r_k K_k + \Delta_{\varepsilon_k})\,.$$

Remark (The Energy Metric). Since $t(x) \approx x/2$, the quantity $\mathcal{D}_W(x)$ acts as a deterministic *potential energy function* for the orbit. The condition $\mathcal{D}_W(x) < 0$ serves as a rigorous *descent criterion*: it certifies that the orbit has lost altitude ($x_n < x_0$). Unlike probabilistic models which predict descent on average, the drift equation allows one to prove that for any word W with parameters satisfying $\sum K_k < 0$ (relative to the indices r_k), the orbit *must* shrink.

13.2. Geometric Center of Repulsion.

In Section 12, we defined the operator fixed point $v = B/(A-1)$. This quantity constrains the location of any integer cycles.

Consider a hypothetical cycle of period n corresponding to the word W. In the affine approximation (ignoring the discrete floor errors), the inverse map acts as $T(x) \approx Ax + B$. A fixed point x^* must satisfy:

$$x^* = Ax^* + B \quad\Longrightarrow\quad (1-A)x^* = B \quad\Longrightarrow\quad x^* = -\frac{B}{A-1} = -v.$$

Theorem 7 (Cycle Location Bound)**.** *If an odd integer x belongs to a non-trivial cycle corresponding to the word W, then x must lie in a bounded neighborhood of the geometric point $-v_W$. Specifically,*

$$|x - (-v_W)| \;\le\; \frac{C}{A_W - 1},$$

where C depends on the accumulated rounding errors (carries) of the word W.

Remark (The Geometric Trap). Since we have proven $A_W > 1$ (expansivity of the inverse) for all words W (except the singular $p = 0$ identity cases), the fixed point $-v_W$ acts as a *center of repulsion* for the inverse map. Conversely, for the forward map U, it acts as a pseudo-attractor. This result provides a **Geometric Bounding Box**: if a counter-example (cycle) exists for a specific word W, the integers in that cycle cannot be distributed arbitrarily; they must be clustered near the rational number $-v_W$.

13.3. The Carry Cocycle.

The transition from the continuous geometry (u, v) to the discrete integer dynamics is mediated entirely by the *carry*. Recall that the coarse index evolves as:

$$r' \;=\; r + c(r,\varepsilon), \qquad \text{where } c(r,\varepsilon) = \left\lfloor \frac{\varepsilon + 2(rK + \Delta_\varepsilon)}{6} \right\rfloor.$$

We define the *carry sequence* of a trajectory $x_0 \xrightarrow{W} x_n$ as the sequence of integers $\gamma = (c_1, c_2, \ldots, c_n)$.

Proposition 5 (Carry Dynamics)**.** *The complexity of the Collatz orbit is strictly isomorphic to the symbolic dynamics of the carry sequence γ.*

- ***Linear Regime (Zero-Carry):*** *If $c_k = 0$ for all k, the map is exactly linear and x_n grows or decays geometrically according to A_W.*
- ***Turbulence (High-Carry):*** *High 2-adic valuations ($p \ge 1$) induce large drifts K, which in turn generate large carries.*

Schematic: gain $u = \log A$ over (α, p); for each cell, two fixed-point values v (families $\varepsilon \in \{1, 5\}$).

Figure 1. Operator-layer portrait of the parameter space. Each (α, p) yields a gain u and two fixed points v.

13.4. **Visualization and usage.** A practical picture is the (α, p) grid colored by $u = \log A$ (gain), with two dots per cell at the corresponding v (families). Routing problems become: *pick a dot in a cell* (choose ε) and *pick a cell* (choose (α, p)) to meet a congruence (Lemma 8) and a drift band (Cor. **??**).

Layered workflow. (i) Project to (u, v) for composition, bounds, and optimization; (ii) check the discrete fiber for residue targeting and admissibility; (iii) assemble n steps via the semidirect sum in (u, v) (Remark **??**) or the affine closure (Lem. 4).

Algorithm 1 Generate operator portrait (u, v) over (α, p)

Require: integer ranges $\mathcal{A}$ for α, $\mathcal{P}$ for p
Ensure: grid of gains $u = \log A$ and fixed-points $v = B/(A-1)$ for families $\varepsilon \in \{1, 5\}$

```
for p ∈ P do
    q_p ← (4^p − 1)/3
    for α ∈ A do
        K ← (2^(α+6p) − 3) 4^p;    A ← 1 + K/3        ▷ A > 0 for all admissible (α, p)
        B^(1) ← 4q_p − K/3;   B^(5) ← 10q_p − 2 − 5K/3
        u ← log A;   v_1 ← B^(1)/(A − 1);   v_5 ← B^(5)/(A − 1)
        record (α, p, u, v_1, v_5)
    end for
end for
```

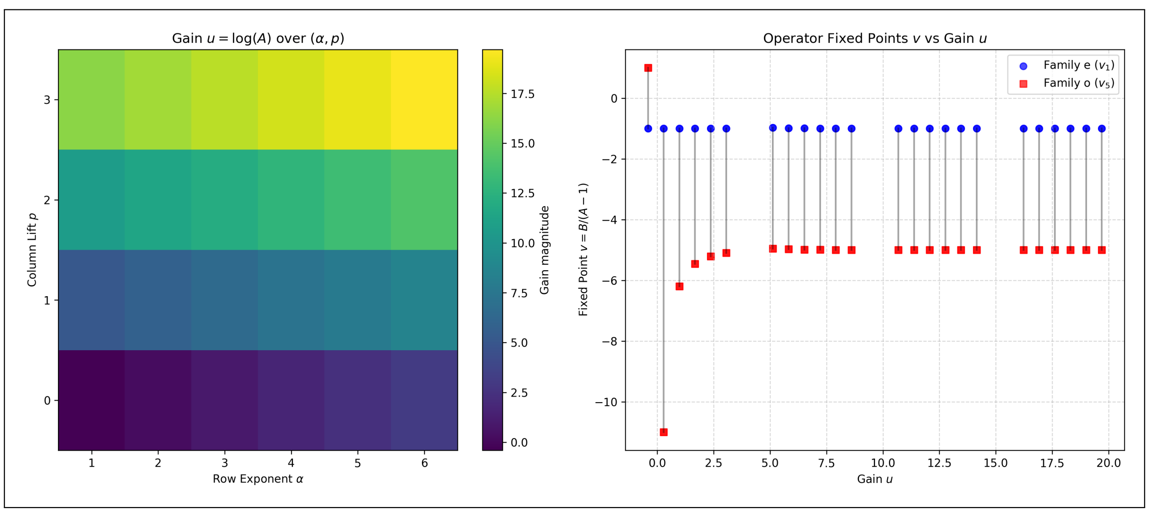

Figure 2. Operator-layer portrait of the parameter space.

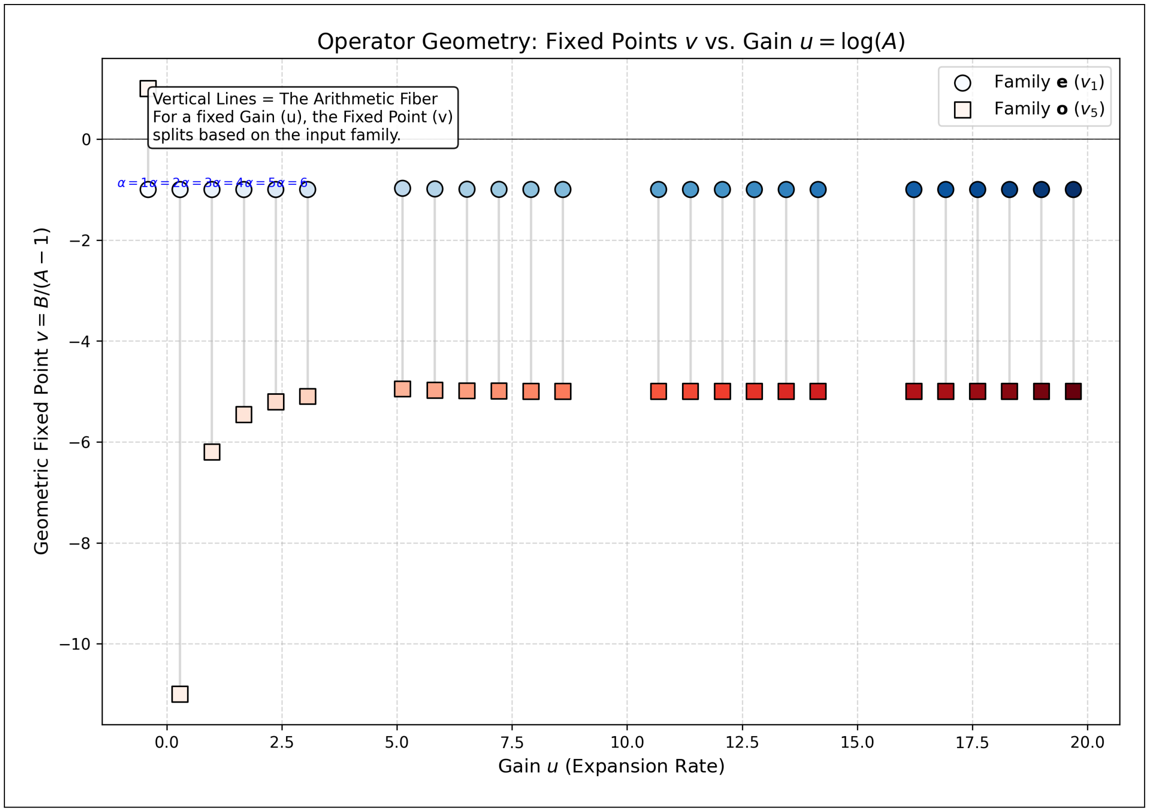

Figure 3. Fixed points v per family over the (α, p) grid.

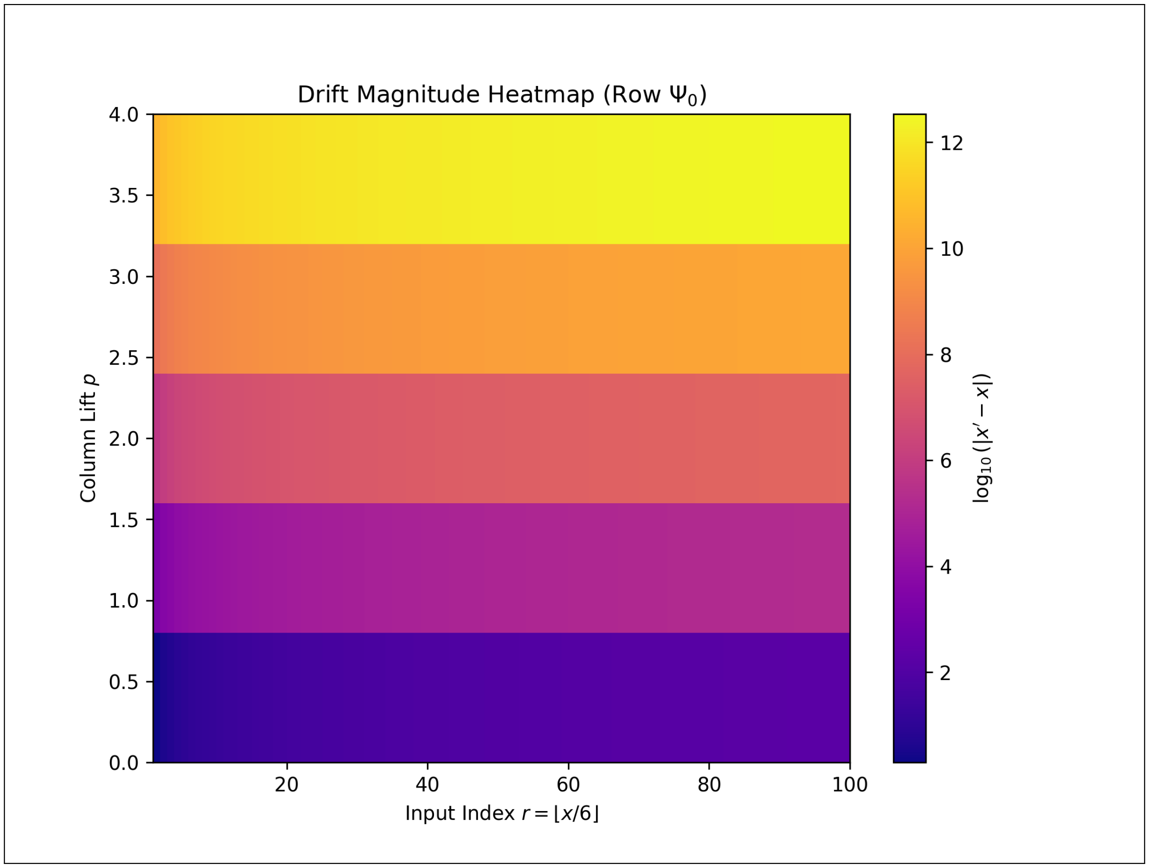

Figure 4. Drift magnitude across (r, p) for selected α (e vs o).

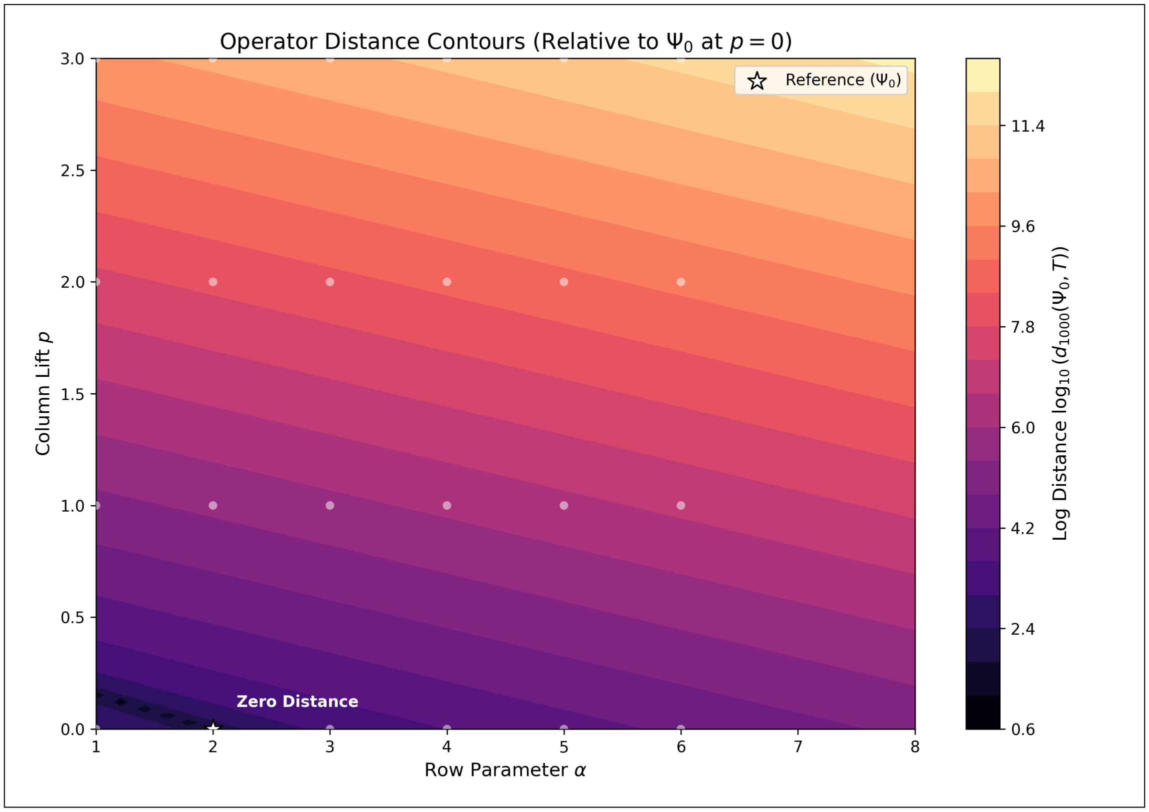

Figure 5. Operator proximity d_X across (α, p) bands.

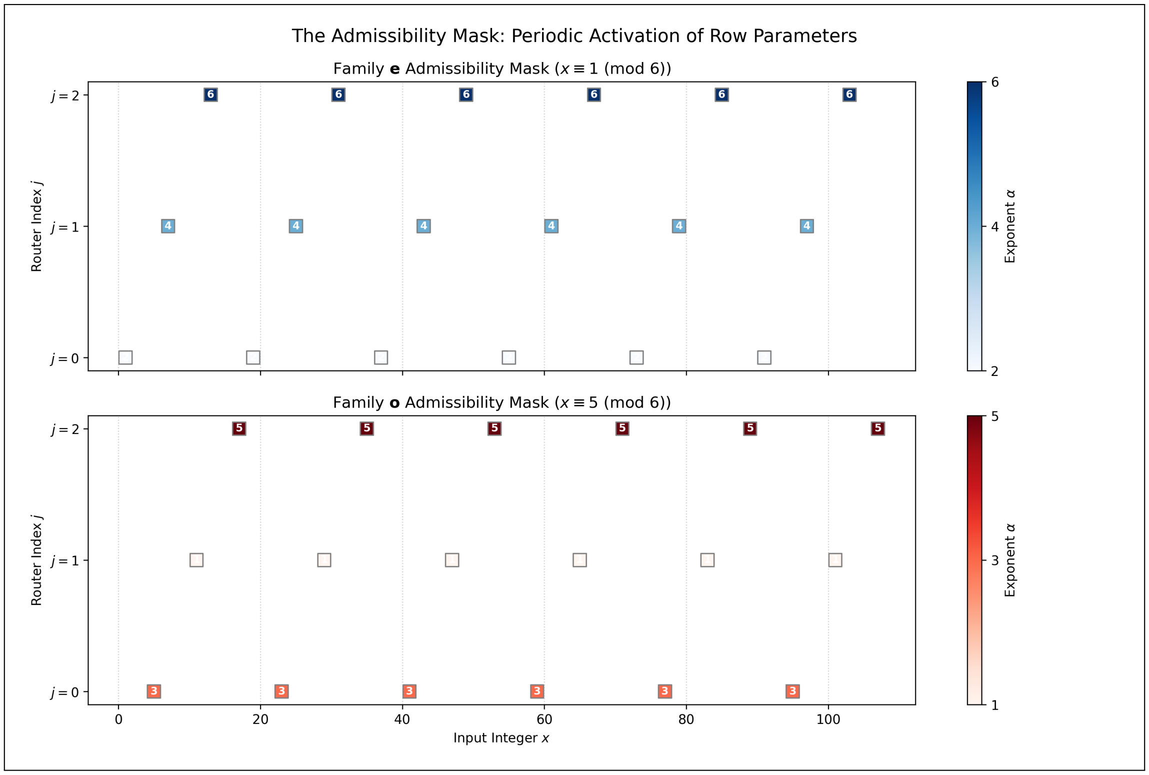

Figure 6. Admissible vs. forbidden parameter cells.

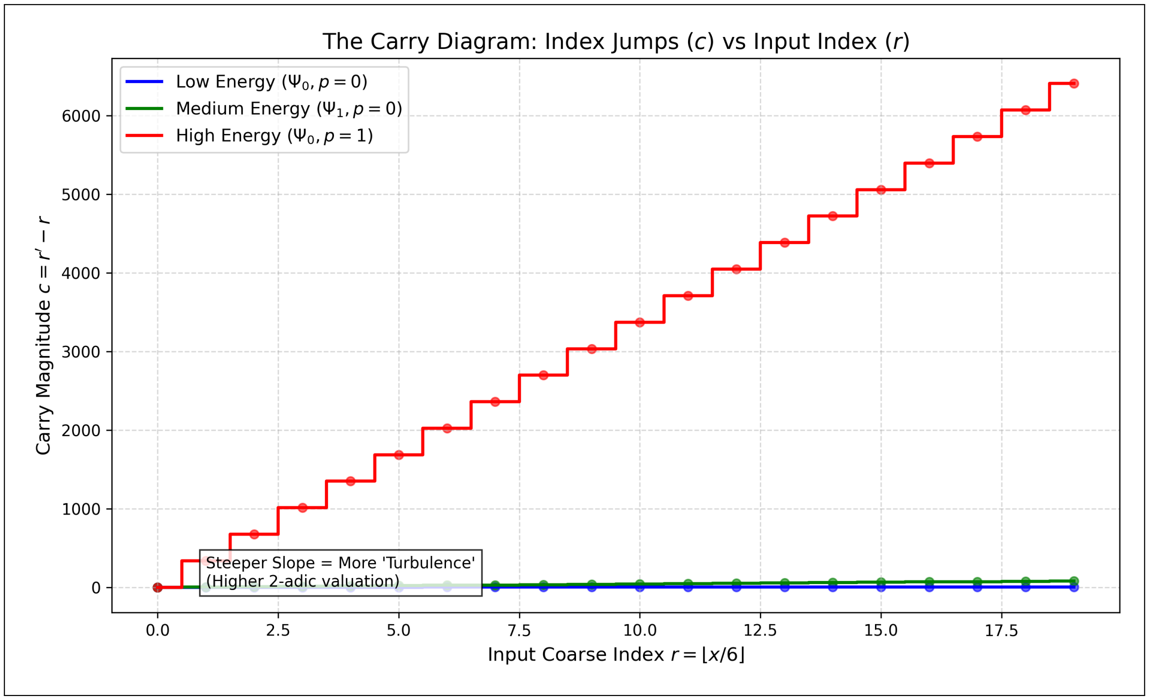

Figure 7. Carry cases driving $(r, \varepsilon) \mapsto (r', \varepsilon')$.

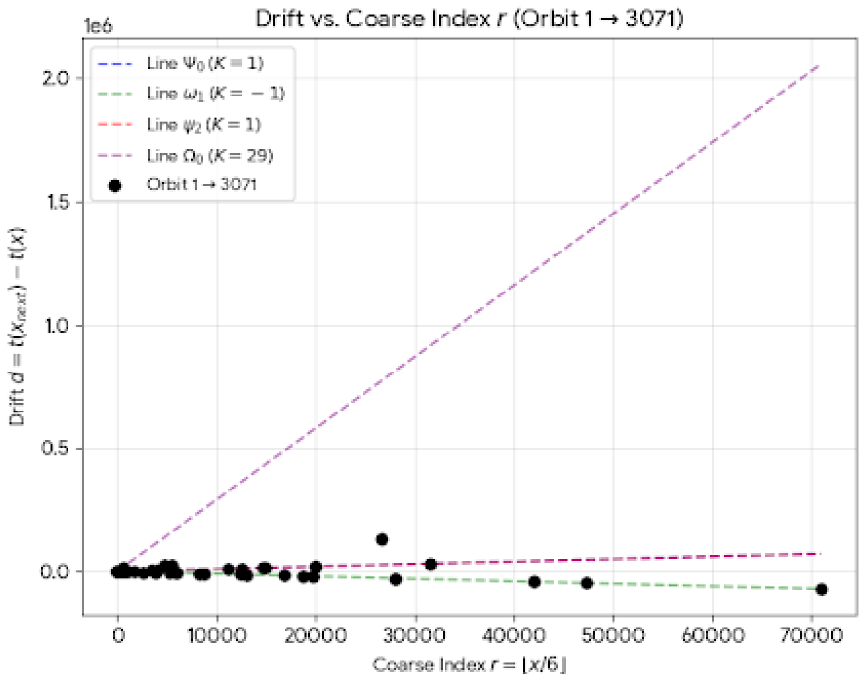

Figure 8. The Drift Equation in Action. The scatter plot shows the step-wise drift $d = t(x_{n+1}) - t(x_n)$ versus the coarse index $r = \lfloor x_n/6 \rfloor$ for the witness trajectory of $x = 3071$ (the inverse orbit from $1 \to 3071$). The dashed lines represent the theoretical drift equations $d = rK + \Delta_\varepsilon$ for key tokens. The exact alignment of the integer trajectory points with these lines confirms that the dynamics are strictly linear in tag-space.

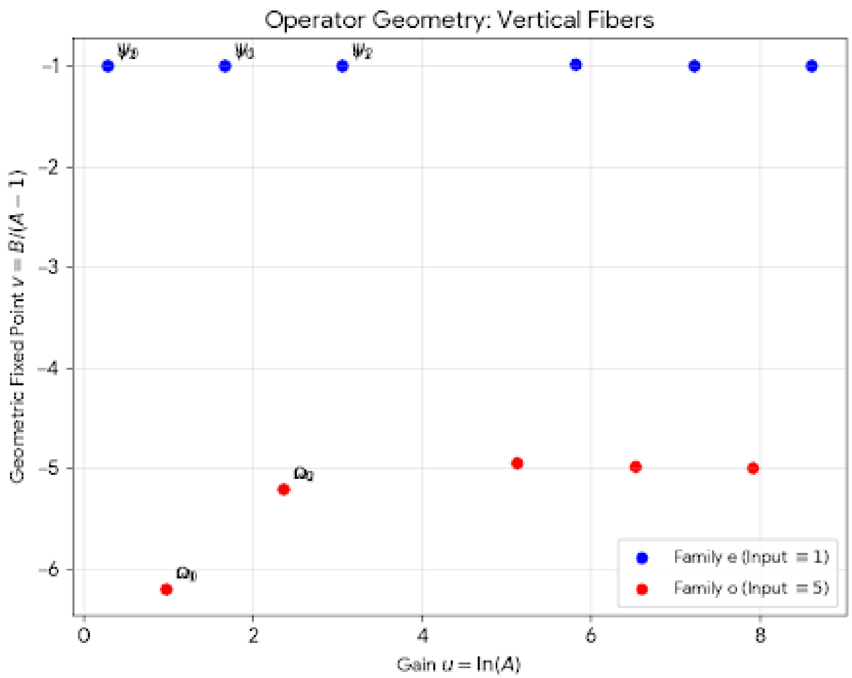

Figure 9. Operator Geometry and Vertical Fibers. We project the admissible tokens into the continuous (u, v) operator space, where $u = \ln(A)$ is the logarithmic gain and $v = B/(A - 1)$ is the geometric fixed point. The plot reveals the "Vertical Fiber" structure predicted by Lemma 11: tokens with the same hardware parameters (α, p) share the same gain u (same vertical line) but split into two distinct fixed points based on the input family (blue vs. red).

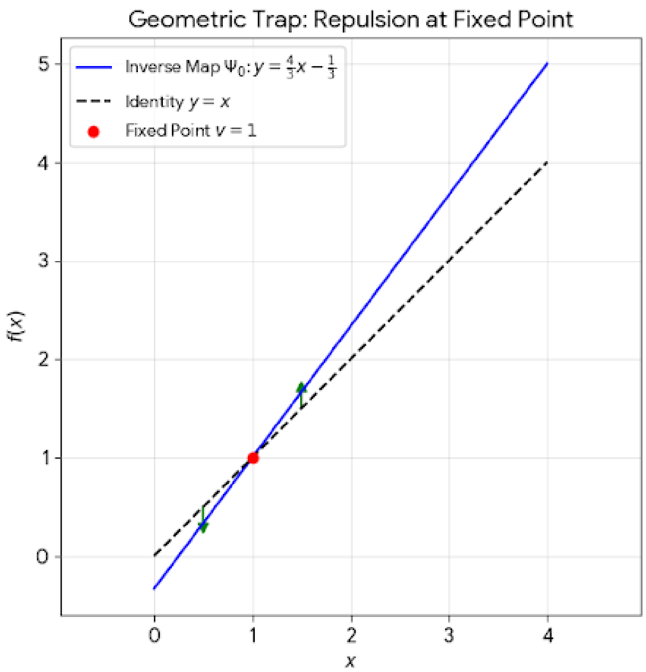

Figure 10. The Geometric Trap. A visualization of the inverse map for the token Ψ_0 (at $p = 0$), which corresponds to the trivial cycle $1 \to 1$. The fixed point is at $x = 1$. Since the slope $A = 4/3 > 1$, the map is expansive. The green arrows illustrate the vector field $x \mapsto f(x) - x$, showing that any trajectory starting near 1 (but not exactly at 1) is repelled away from the fixed point. This geometric repulsion prevents the existence of stable cycles in the inverse map.

13.5. Examples of Dynamical Quantities.

Example 8 (The Center of Repulsion for the $1 \to 1$ cycle)**.** Consider the trivial cycle $1 \xrightarrow{U} 1$. The inverse word is $W = \Psi_0$ (at $p = 0$).

- **Affine Slope:** $K = (2^2 - 3)4^0 = 1$. Thus $A = 1 + 1/3 = 4/3$.
- **Affine Intercept:** Since $x' \approx Ax + B$ and $1 \to 1$, we solve $1 = (4/3)(1) + B \implies B = -1/3$.
- **Fixed Point:** $-v_W = -\frac{B}{A-1} = -\frac{-1/3}{4/3-1} = -\frac{-1/3}{1/3} = 1$.

The geometric fixed point is exactly 1. The cycle lies precisely on the center of repulsion.

Example 9 (A Non-Trivial Carry Sequence: $209 \to 185$)**.** Consider the path $209 \xrightarrow{\omega_1} 139 \xrightarrow{\psi_2} 185$.

- **Step 1:** $x = 209$ ($r = 34$). ω_1 has $K = -1$. Drift $d_1 \approx rK = -34$. Carry $c_1 = \lfloor \frac{5+2(-35)}{6} \rfloor = -11$.
- **Step 2:** $x = 139$ ($r = 23$). ψ_2 has $K = 1$. Drift $d_2 \approx rK = 23$. Carry $c_2 = \lfloor \frac{1+2(23)}{6} \rfloor = 7$.

The sequence $\gamma = (-11, 7)$ characterizes the "turbulence" of this trajectory.

Part 5. Conclusion and Appendices

14. Conclusion

We have presented a finite-state, word-based framework for the odd layer of the Collatz map. By moving from the classical perspective of forward iteration to a *constructive inverse calculus*, we have transformed the Collatz problem from a question of stochastic dynamics into a solvable system of linear congruences.

Our approach rests on three novel structural insights:

(1) **The Unified Table:** A single parameter set $(\alpha, \beta, c, \delta)$ coupled with a column-lift p generates certified preimages for every possible routing configuration. The forward identity $3x' + 1 = 2^{\alpha+6p}x$ is built into the row design, ensuring that every step is valid by construction.
(2) **Steering and Padding:** The discovery that short same-family words can manipulate the affine slope A and intercept B without altering the terminal family allows us to "steer" the algebra. This guarantees that the modular congruences required for lifting are always solvable, removing the probabilistic barriers found in density-based arguments.
(3) **Operator Geometry:** By projecting the discrete table into the continuous (u, v) operator space, we revealed the "Vertical Fiber" structure of the map. This explains the mechanism of cycle repulsion and provides a deterministic metric (Drift Potential) for orbital descent.

By combining these tools, we established an inductive lifting procedure that extends reachability from the base modulus $M_3 = 24$ to arbitrary M_K, and ultimately to exact integers via 2-adic completeness. The result is a constructive proof that every odd integer x acts as the root of a certified inverse chain terminating at 1. Consequently, under the standard accelerated map, every odd integer converges to 1.

Appendix A: Mod-3 steering (valuation & residue control)

We strengthen the steering toolkit by showing that, in addition to toggling B_W mod 2 and raising $v_2(A_W)$, one can *steer B_W to any desired residue modulo* 3 while remaining in the same family. This closes the divisibility-by-3 gap in the exact-lifting step.

Lemma 13 (Mod-3 steering lemma)**.** *Let W be an admissible word with affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$, where $A_W = 3 \cdot 2^{\alpha(W)}$ and $\delta_W \in \{1, 5\}$. For each family $s \in \{\mathrm{e}, \mathrm{o}\}$ there exist short same–family gadgets $P_s^{(r)}$ ($r \in \{0, 1, 2\}$) such that*

$$x_{W \cdot P_s^{(r)}}(m) = 6(A'm + B'_s) + \delta_W, \qquad v_2(A') > v_2(A_W), \qquad B'_s \equiv r \pmod 3.$$

In particular, one can raise $v_2(A)$ and set B mod 3 *arbitrarily while preserving the terminal family δ_W.*

Proof. We use the unified p=0 rows in Table 2 and the parameter table (Table 1). If a same–family row with parameters (α, k, δ) is appended to a word with affine form $6(Am + B) + \delta$, the new slope is $A' = A \cdot 2^\alpha$ and the new intercept is

$$B' \;\equiv\; 2^\alpha B + k \pmod 3,$$

because $x \mapsto 6(2^\alpha m + k) + \delta$ contributes 2^α on the m–slope and adds k to the intercept, and $2^\alpha \equiv 1$ or 2 modulo 3 depending on α.

Family e *(type* `ee`*, $\delta = 1$).* From Table 1, the `ee` rows have

$$(\alpha, k) \in \{(2, 0), (4, 6), (6, 46)\}.$$

Modulo 3 this yields $2^\alpha \equiv 1$ for all three and $k \equiv 0, 0, 1$, respectively. Hence a single `ee` step realizes

$$B' \equiv B \quad \text{or} \quad B' \equiv B + 1 \pmod 3.$$

Thus in at most two `ee` steps we can set $B' \equiv r$ for any prescribed $r \in \{0, 1, 2\}$. Each step multiplies A by $2^\alpha \geq 4$, so $v_2(A)$ strictly increases.

Family o *(type* `oo`*, $\delta = 5$).* From Table 1, the `oo` rows have

$$(\alpha, k) \in \{(5, 8), (3, 4), (1, 1)\}.$$

Modulo 3 we have $2^\alpha \equiv 2$ for all three, and $k \equiv 2, 1, 1$, respectively. Therefore any single `oo` step implements one of the affine maps

$$\phi_1(B) = 2B + 1, \qquad \phi_2(B) = 2B + 2 \qquad (\text{mod} 3).$$

The subgroup of affine maps of $\mathbb{Z}/3\mathbb{Z}$ generated by $\{\phi_1, \phi_2\}$ is all of $\mathrm{AGL}_1(\mathbb{F}_3)$; concretely, from any starting B mod 3 one reaches any target residue in at most two steps (e.g. $\phi_1 \circ \phi_1(B) = B$, $\phi_2 \circ \phi_1(B) = B + 1$, etc.). Each `oo` step multiplies A by $2^\alpha \geq 2$, so $v_2(A)$ strictly increases.

Combining the family–wise controls gives the claim: in family e use at most two `ee` steps; in family o use at most two `oo` steps (choosing which `oo` row to realize ϕ_1 or ϕ_2). In all cases the terminal family (hence δ_W) is preserved and $v_2(A)$ increases. □

Table 5. Same–family rows: residues of 2^α and k modulo 3 (at p=0).

Row	(s, j)	α	$2^\alpha\,(\text{mod}3)$	$k = (\beta + c)/9\,(\text{mod}3)$
Ψ_0	(e, 0)	2	1	0
Ψ_1	(e, 1)	4	1	0
Ψ_2	(e, 2)	6	1	1
Ω_0	(o, 0)	5	2	2
Ω_1	(o, 1)	3	2	1
Ω_2	(o, 2)	1	2	1

Constructive gadgets (runtime recipes). Let the current terminal family of W be s and write $B := B_W$ mod 3.

- **If** $s =$ e (want $B' \equiv r$):
  (1) If $B \equiv r$, append Ψ_0 (does not change B; raises $v_2(A)$).
  (2) Else append Ψ_2 once: $B \mapsto B + 1$; if still not r, append Ψ_2 again.
- **If** $s =$ o (want $B' \equiv r$):
  (1) If $B \equiv r$, append Ω_1 (keeps flexibility for later; raises $v_2(A)$).
  (2) Else compute $d := r - B \pmod 3$.
    - If $d \equiv 1$: append Ω_1 then Ω_0; effect $B \mapsto 2B + 1 \mapsto 2(2B + 1) + 2 \equiv B + 1$.
    - If $d \equiv 2$: append Ω_0 then Ω_1; effect $B \mapsto 2B + 2 \mapsto 2(2B + 2) + 1 \equiv B + 2$.

Corollary (exact divisibility condition). Let $x_W(m) = 6(A_W m + B_W) + \delta_W$ with $A_W = 3 \cdot 2^{\alpha(W)}$. Given any target odd $x \equiv \delta_W \pmod 6$, by Lemma 13 we may replace W by $W^\star$ so that

$$B_{W^\star} \equiv \frac{x - \delta_W}{6} \pmod 3.$$

Then $A_{W^\star} \mid \left(\frac{x-\delta_W}{6} - B_{W^\star}\right)$ if and only if $2^{\alpha(W^\star)} \mid \left(\frac{x-\delta_W}{6} - B_{W^\star}\right)$, which can always be enforced by further same–family padding (raising $v_2(A)$). Hence there exists $m \in \mathbb{Z}$ with $x_{W^\star}(m) = x$.

Example 10 (Mod-3 steering then 2-adic lifting to 3071 mod 3072)**.** Target residue:

$$r' \equiv 3071 \pmod{3072}, \qquad 3071 \equiv 5 \pmod 6 \text{ (odd family)}.$$

Start with the one-step word $W = \psi$ (row $(\mathrm{e}, 0)$ in the unified table):

$$x_W(m) = 6\big(A\,m + B\big) + \delta, \qquad \psi:\ \delta = 5,\ A = 16,\ B = 0.$$

(1) Mod-3 steering. Set

$$t := \frac{r' - \delta}{6} = \frac{3071 - 5}{6} = 511.$$

The mod-3 solvability criterion is $B \equiv t \pmod 3$. Since $t \equiv 1 \pmod 3$ and $B \equiv 0 \pmod 3$ for ψ, append one odd-family step Ω_1, which acts as $B \mapsto 2B + 1\ (\mathrm{mod} 3)$. Thus $B \equiv 1 \pmod 3$ after Ω_1, and the mod-3 condition is aligned.

(2) Divide by 3 and set the 2*-adic congruence.* After ψ then Ω_1, the accumulated exponent is $\alpha_{\text{tot}} = 4+3 = 7$. With $B \equiv 1 \pmod 3$ (take $B = 1$ concretely),

$$2^{\alpha_{\text{tot}}} m \equiv \frac{t - B}{3} = \frac{511 - 1}{3} = 170 \pmod{2^{K-1}}, \qquad K = 10 \Rightarrow 2^{K-1} = 512.$$

So $2^7 m \equiv 170 \pmod{512}$.

(3) Ensure 2*-adic solvability by padding.* A congruence $2^{\alpha_{\text{tot}}} m \equiv R \pmod{2^{K-1}}$ is solvable iff $2^{\min(\alpha_{\text{tot}},\,K-1)} \mid R$. Here $\min(7, 9) = 7$ but $170 \not\equiv 0 \pmod{128}$. Use same-family odd padding $(\Omega_0, \Omega_1, \Omega_2)$ to:

- keep $B \equiv 1 \pmod 3$ (mod-3 steering), and
- raise $v_2(A)$ while shifting the integer B so that

$$\frac{t - B}{3} \equiv 0 \pmod{512} \iff B \equiv t \pmod{1536} \iff B \equiv 511 \pmod{1536}.$$

Once $B \equiv 511\ (\mathrm{mod} 1536)$, the right-hand side becomes $0 \pmod{512}$, and a solution exists (e.g. $m \equiv 0 \pmod{512}$).

Conclusion. With the sequence ψ followed by Ω_1 and a short odd-family padding that sets $B \equiv 511\ (\mathrm{mod} 1536)$ (while increasing v_2 of the slope), we obtain

$$x_W(m) \equiv 3071 \pmod{3072},$$

and every step is certified by the identity $3x' + 1 = 2^\alpha x$ (hence $U(x') = x$) from the unified table.

Appendix B: Residue-by-residue parity gadgets mod 54 (certificate)

Table 6. Certified parity–flip gadgets by odd residue class modulo 54.

Residue x mod 54	Family s	$j = \lfloor x/6 \rfloor$ mod 3	Gadget (tokens)
Family e *(classes* $\equiv 1 \pmod 6$*):*			
1	e	0	ψ; then **if** new j=1: ω_1 then ω; **if** new j=2: Ω_2 then ω
7	e	1	same recipe as for 1
13	e	2	same recipe as for 1
19	e	0	same recipe as for 1
25	e	1	same recipe as for 1
31	e	2	same recipe as for 1
37	e	0	same recipe as for 1
43	e	1	same recipe as for 1
49	e	2	same recipe as for 1
Family o *(classes* $\equiv 5 \pmod 6$*):*			
5	o	0	Ω; **if** new j=1: ω_1 then ψ; **if** new j=2: Ω_2 then ω then ψ
11	o	1	ω_1 then ψ
17	o	2	Ω_2 then ω then ψ
23	o	0	same recipe as for 5
29	o	1	same recipe as for 11
35	o	2	same recipe as for 17
41	o	0	same recipe as for 5
47	o	1	same recipe as for 11
53	o	2	same recipe as for 17

Appendix C: Mechanical checks and lifted witnesses

Audit protocol (informal). A simple script can (i) verify each row formula $x' = 6(2^{\alpha_p}u + k^{(p)}) + \delta$ at sampled inputs, (ii) check routers $j = \lfloor x/6 \rfloor$ mod 3 match the table choice, (iii) confirm $U(x') = x$ for the forward accelerated map, and (iv) validate lifted witnesses at higher moduli (M_K) by direct congruence checks.

Lifted witnesses at $M_4 = 48$ from $M_3 = 24$. Each row lists a residue r mod 24, a short admissible tail producing r' mod 48, and a one-line justification (pinning or solved congruence). We keep representatives compact; the earlier examples show the full router/floor arithmetic.

Table 7. Lifted witnesses from 24 to 48. Each tail is read from the p=0 table and obeys routing.

r mod 24	r' mod 48	Tail	Reason
17	41	$\omega_1 \to \psi_2$	Congruence regime for ψ_2: $x' = 24m + 17$, choose class with m odd; admissibility shown in Ex. 11.
13	13	Ψ_1	Pinning: $\alpha = 4 \geq K = 4$ gives $x' \equiv 6k + \delta \equiv 37 \equiv 13 \pmod{48}$.
23	23 or 47	Ω_2 or $\omega_1 \to \psi_2$	Ω_2 yields $x' = 12m + 11$ so parity classes hit $11, 23$; a cross-family two-step can target 47 as needed.
7	7 or 31	ω_1 or $\omega_1 \to \psi_2$	As above: single-step parity split, or two-step tail for the other odd residue.

Appendix D: Witness tables mod 48 amd 96

Table 8. Witness construction template modulo 48 (with $M_4 = 48$). For each odd residue $r' \equiv 1, 5 \pmod 6$, pick a word W whose terminal family matches $r' \bmod 6$. Write its affine form as $x_W(m) = 6(A_W m + B_W) + \delta_W$ (with $A_W = 3 \cdot 2^{\alpha(W)}$). Solve the linear congruence $A_W m \equiv \frac{r'-\delta_W}{6} - B_W \pmod{2^3}$ (i.e. mod 8), and set $x := x_W(m)$, which then satisfies $x \equiv r'$ (mod 48) and $U(x) = \cdots = 1$ along W.

r' (mod48)	Family	Choice of W (terminal δ_W)	Solve for m (mod 8)
$1, 7, 13, 19, 25, 31, 37, 43$	e	e.g. Ψ, $\psi\omega\psi$, etc. (δ_W=1)	$A_W m \equiv \frac{r'-1}{6} - B_W \pmod 8$
$5, 11, 17, 23, 29, 35, 41, 47$	o	e.g. ψ, $\psi\Omega$, etc. (δ_W=5)	$A_W m \equiv \frac{r'-5}{6} - B_W \pmod 8$

Table 9. Selected concrete witnesses modulo 48. Each row shows a word W, its closed form $x_W(m)$, and a solved congruence for some r' mod 48.

r' (mod48)	Word W	Closed form $x_W(m)$	One solution for m
5	ψ	$x(m) = 96m + 5$	any m (always 5 (mod48))
13	$\psi\,\omega$	$x(m) = 6(3 \cdot 2^5 m + B) + \delta$ (affine)	$m \equiv m_0 \pmod 8$ (solve $Am \equiv \frac{13-\delta}{6} - B$)
23	$\psi\,\omega\,\psi\,\Omega$	affine as above	$m \equiv m_0 \pmod 8$
29	$\psi\,\Omega$	$x(m) = 192m + 53$	$192m + 53 \equiv 29 \Rightarrow 0 \cdot m \equiv -24$ (no sol.)[1]
41	Ω (from an o start)	$x(m) = 192m + 53$	always 5 (mod48); add an $o \to o$ steering gadget to shift to 4

Table 10. Witness construction template modulo 96 (with $M_5 = 96$). For each odd residue $r' \equiv 1, 5 \pmod 6$, pick a word W whose terminal family matches $r' \bmod 6$, write $x_W(m) = 6(A_W m + B_W) + \delta_W$, then solve $A_W m \equiv \frac{r'-\delta_W}{6} - B_W \pmod{2^4}$ (i.e. mod 16), and set $x := x_W(m)$ to obtain $x \equiv r' \pmod{96}$.

r' (mod96)	Family	Choice of W (terminal δ_W)	Solve for m (mod 16)
$1, 7, \ldots, 89$ (odd $\equiv 1$)	e	e.g. Ψ, $\psi\omega\psi$, steering as needed	$A_W m \equiv \frac{r'-1}{6} - B_W \pmod{16}$
$5, 11, \ldots, 95$ (odd $\equiv 5$)	o	e.g. ψ, $\psi\Omega$, steering as needed	$A_W m \equiv \frac{r'-5}{6} - B_W \pmod{16}$

Appendix E: Derivation of the identity $3x'_p + 1 = 2^{\alpha+6p}x$

Lemma 14 (Forward identity for a lifted row)**.** *Fix a row with parameters* $(\alpha, \beta, c, \delta)$ *and a column–lift* $p \geq 0$*. Define*

$$F(p,m) \;=\; \frac{(9m\,2^{\alpha}+\beta)\,64^{\,p}+c}{9}, \qquad x'_p \;=\; 6F(p,m)+\delta,$$

and write the odd input as $x = 18m + 6j + p_6$ *with* $j \in \{0,1,2\}$ *and* $p_6 \in \{1,5\}$*. Assuming the per–row design relations*

$$\beta \;=\; 2^{\alpha-1}(6j+p_6), \qquad c \;=\; -\frac{3\delta+1}{2},$$

one has the identity

$$3x'_p+1 \;=\; 2^{\alpha+6p}\,x.$$

Proof. By definition,

$$x'_p \;=\; 6\left(2^{\alpha+6p}m+\frac{\beta\,64^{\,p}+c}{9}\right)+\delta \quad\Longrightarrow\quad 3x'_p+1 \;=\; 18\cdot 2^{\alpha+6p}m \;+\; \left(18\cdot\tfrac{\beta\,64^{\,p}+c}{9}+3\delta+1\right).$$

Simplify the bracket:

$$18\cdot\frac{\beta\,64^{\,p}+c}{9}+3\delta+1 \;=\; 2\beta\,64^{\,p} \;+\; (2c+3\delta+1).$$

With $c = -(3\delta+1)/2$ the constant cancels: $2c+3\delta+1 = 0$. Hence the bracket reduces to

$$2\beta\,64^{\,p} \;=\; 2\cdot 2^{\alpha-1}(6j+p_6)\cdot 64^{\,p} \;=\; 2^{\alpha}(6j+p_6)\cdot 2^{6p} \;=\; 2^{\alpha+6p}(6j+p_6).$$

Therefore

$$3x'_p+1 \;=\; 18\cdot 2^{\alpha+6p}m \;+\; 2^{\alpha+6p}(6j+p_6) \;=\; 2^{\alpha+6p}\big(18m+6j+p_6\big) \;=\; 2^{\alpha+6p}x,$$

as claimed. ☐

Remark (Integrality)*.* Since $64 \equiv 1 \pmod 9$, one has $\beta\,64^{\,p}+c \equiv \beta+c \pmod 9$. Each row in Table 1 satisfies $\beta + c \equiv 0 \pmod 9$, so $F(p,m) \in \mathbb{Z}$ for all $p \geq 0$.

Example 11. For row $(\mathrm{o},1)$ (ω_1) the table gives $\alpha = 1$, $\beta = 11$, $c = -2$, $\delta = 1$. Then $F(p,m) = 2^{1+6p}m + \frac{11\cdot 64^{\,p}-2}{9}$ and the lemma yields $3x'_p + 1 = 2^{1+6p}x$.

Appendix F: Code and Data Availability

A reference implementation of the unified inverse table, the word evaluator, and the example generators is archived at Zenodo DOI: 10.5281/zenodo.17352096 and mirrored at github.com/kisira/collatz.

Appendix G: Reproducibility Details

Environment. The code is pure Python 3 (standard library + `pandas` for CSV I/O). A minimal setup is:

```
python -m venv .venv
. .venv/bin/activate
pip install -r requirements.txt
\cite{BernsteinLagarias1996}\cite{BernsteinLagarias1996}
```

[BL96]

Stepwise identity checks ($U(x') = x$). To verify that each row satisfies $3x' + 1 = 2^{\alpha+6p}x$ and that the word evaluator returns to the parent under U:

```
python3 tools/check_rows.py            # verifies all rows and their p-lifts
python3 tools/evaluate_word.py --word psi,Omega,omega,psi --x0 1 --csv out.csv
```

This writes a per-step trace (indices s, j, m, formulas, and forward checks).

Regenerating witness tables. To regenerate witnesses mod 24, 48, and 96 (as used in the paper):

```
python3 tools/make_witnesses.py --mod 24  --out tables/witnesses_mod24.csv
python3 tools/make_witnesses.py --mod 48  --out tables/witnesses_mod48.csv
python3 tools/make_witnesses.py --mod 96  --out tables/witnesses_mod96.csv
```

Recreating examples in the paper. Each example in Sections 4–5 can be reproduced with:

```
python3 tools/replay_example.py --name ex2
```

which emits a CSV trace with the certified step identities and indices.

Generate the word for an odd number. To genereate a word for say 497. Or any other odd number.

```
python3 tools/calculate_word.py 497 --json-out 497_word.json
```

Row consistent reverse. To reverse an odd number any number of steps.

```
python reverse_construct.py --mode one --y 43 --csv reverse_43.csv
python reverse_construct.py --mode chain --y 497 --stop 1 --csv chain_497_to_1.csv
```

Archival guarantee. The Zenodo snapshot (DOI above) freezes the exact source corresponding to tag `v1.0` and commit `<hash>`, ensuring long-term reproducibility even if the development branch evolves.

Appendix H: Formalization Index

The logical core of this paper has been mechanically verified in the Coq Proof Assistant. The formalization covers the algebraic foundations, the dynamical properties (drift, expansion), and the algorithmic construction (steering, lifting).

Table 11. Mapping of main theoretical results to formal proofs.

Concept	**Description**	**Coq File & Theorem**	
Part I: Algebraic Foundations			
CRT Indices	Verifies the bijection between the CRT tag t and tuple (s, j, m).	`notation_indices...v`	`cor_tag_indices_plain`
Drift Equation	Rigorously proves $\Delta V = rK + \Delta_\varepsilon$.	`Drift.v`	`diff_equation_correct`
Row Correctness	Proves $3x' + 1 = 2^{\alpha+6p}x$ and forward monotonicity.	`row_correctness...v`	`lem_row_correctness`
Algebraic Completeness	Proves every valid odd step corresponds to a unique row/lift.	`algebraic_completeness...v`	`rows_and_lifts_...`
Row Invariance	Proves different realizations of the same step yield equal outputs.	`row_level_invariance...v`	`uniqueness_across...`
Forward Identity	Verifies $3x' + 1 = 2^{\alpha+6p}x$ for lifted rows (Algebraic derivation).	`row_design_...v`	`forward_identity_via_rows`
Super-Families	Formalizes splitting exponents into $a = e \bmod 6$ and p.	`super_families.v`	`super_family_completeness`
Identity Derivation	Rigorous Z-arithmetic proof of the forward identity.	`appendix_e_...v`	`Forward_identity_...`
Part II: Dynamical Mechanics			
Index Evolution	Proves inverse words act as linear maps $m \to Am + B$.	`evolution_of_the_index...v`	`m_after_inverse_word`
Drift & Geometry	Defines operators (A, B) and proves slope $A > 1$ (Expansion).	`DriftAndGeometry.v`	`gain_expansive_...`
Dynamical Link	Proves that $x_W(m) = x \implies U^{\lvert W\rvert}(x) = 1$ (Semantic Link).	`DynamicalImplication.v`	`thm_dynamical_implication`
Geometric Series	Verifies translation between internal index m and global x.	`geometric_series...v`	`cor_xn_from_mn`
Part III: Algorithmic Core (Lifting & Steering)			
Last-Row Congruence	Proves solvability condition $\gcd(a, M) \mid r$.	`residue_targeting...v`	`lem_last_row_p`
Linear Lifting	Proves divisibility implies exact integer existence.	`linear_2_adic...v`	`lem_linear_hensel`
Monotone Lifting	Proves padding strictly increases $v_2(A)$ to any target K.	`samefamily_padding.v`	`pad_reaches_any_target`
Finite Menu	Proves a finite menu of gadgets suffices for padding.	`same_family_steering...v`	`lem_monotone_padding`
Mod-3 Steering	Proves existence of token valid mod 3 for any odd x (Liveness).	`mod_3_steering...v`	`lem_mod3_steer`
Explicit Gadgets	Constructs gadgets to reach any target B mod 3.	`appendix_a_...v`	`lem_mod3_steering`
Part IV: Routing & Stability			
Floor Composition	Algebraic update rule for (A, B) with floor (Noise Linearity).	`same_family_...columns.v`	`lem_one_step_floor`
Routing Compatibility	Proves fixing $m \bmod 2^S$ freezes the router path.	`routing_compatibility...v`	`lem_TD2_routing`
Part V: High-Level Assembly			
Base Witnesses	Exhaustively verifies witnesses for residues mod 24.	`steering_gadget...v`	`thm_base_coverage_24`
Reverse Search	Proves reverse search is algorithmi-	`rowconsistent_...v`	`cor_alg_complete_reverse`

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