ON THE COLLATZ CONJECTURE

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Abstract. In this paper a procedure is demonstrated to generate pre-images of the Collatz procedure. As such a path can be traced from the number one to any given odd number using the Collatz procedure in reverse.

1. Introduction

The **digital root** of a natural number **x** in a given base is a procedure by which the digits of x are iteratively summed resulting in a single digit. In this paper, we will restrict ourselves to natural numbers in base 10.

The **digit sum** of n a natural number in base $b > 1, F_b : \mathbb{N} \to \mathbb{N}$ is defined as

$$F_b(n) = \sum_{i=0}^{k} d_i.$$

where $k = \lfloor log_b n \rfloor$ is one less than the number of digits in n and

$$d_i = \frac{n \mod b^{i+1} - n \mod b^i}{b^i}$$

is the value of each digit in n.

Repeatedly applying the **digit sum** yields the **digital root**. Formally; a natural number n is a digital root if it is also a fixed point for F_b, which occurs if $F_b(n) = n$

In the specific case of base 10 natural numbers, the **digital root** $dr : \mathbb{N} \to \mathbb{N}$ can be computed by the following congruence formula

$$F_{10}(n) = dr(n) = \begin{cases} n(\mod 9) & n \not\equiv 0(\mod 9) \\ 9 & n \equiv 0(\mod 9) \end{cases}$$

Simplifying

$$= 1 + [n - 1(\mod 9)]$$

1.1. Operations on and properties of the digital root.

Let $n, m \in \mathbb{N}_{>0}$ in base 10 then

$$dr(n + m) = dr(n) + dr(m)$$
$$dr(n \times m) = dr(n) \times dr(m)$$

The base-10 digital roots of the first few natural numbers $n > 0$ are 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, ...

Let $m \in \mathbb{N}_{>0}$ be a base 10 natural number then $m \equiv 0(\mod 3) \iff dr(m) \in \{3, 6, 9\}$. These are numbers of the form $9k + 3$ or $9k + 6$ or $9k$ where $k \in \mathbb{N}$

Let $m \in \mathbb{N}_{>0}$ be a base 10 natural number $m \equiv 0(\mod 9) \iff dr(m) = 9$. These are numbers of the form $9k$ where $k \in \mathbb{N}$

2. The Collatz Conjecture

The simple statement of the Collatz conjecture is as follows Given a natural number $n \in \mathbb{N}_{>0}$

- If the number is even, divide it by 2.
- If the number is odd, triple it and add one.

We may define this function as follows

$$f(n) = \begin{cases} \frac{n}{2} & n \equiv 0(\mod 2) \\ 3n+1 & n \equiv 1(\mod 2) \end{cases}$$

Perform these operations repeatedly beginning with any natural number and taking the result at each step as the input to the next step. In mathematical notation

$$a_i = \begin{cases} n & i = 0 \\ f(a_{i-1}) & i > 0 \end{cases}$$

In other words a_i is the value of f applied to n recursively for i times: $a_i = f^i(n)$

The Collatz conjecture is: The process outlined above will eventually reach the number 1, regardless of the n chosen. In other words, no matter the number chosen the final state will be $a_i = 1$ for some i. We will refer to this as the Collatz process

3. Pre-images of the Collatz Conjecture sequences

Theorem 3.1. *Let $x \in \mathbb{N}_{>0}$ be a base 10 natural number where $dr(x) \in \{1, 4, 7\}$ then $x \times 2^{2n} = 3 \times \alpha + 1, n = 1, 2, 3...$ and α is odd. If x is odd, then α is a preimage of x under the Collatz process.*

Proof. The digital root $dr(2^{2n}) \in \{1, 4, 7\}$ also the digital root of $3 \times \alpha \in \{3, 6, 9\}$ since $3 \times \alpha$ is a multiple of 3. As such the $dr(3 \times \alpha + 1) \in \{1, 4, 7\}$. Since $dr(x) \in \{1, 4, 7\}$ The product of $x \times 2^{2n} \in \{1, 4, 7\}$ since $dr(a \times b) = dr(a) \times dr(b)$ and any number of the from $y = 3 \times \alpha + 1$ has a $dr(y) \in \{1, 4, 7\}$ since, by the rules of digital root addition $3 + 1 = 4$, $6 + 1 = 7$ and $9 + 1 = 1$. □

Theorem 3.2. *Let $x \in \mathbb{N}_{>0}$ be a base 10 natural number where $dr(x) \in \{2, 5, 8\}$ then $x \times 2^n = 3 \times \alpha + 1, n = 1, 3, 5, 7...$ and α is odd. If x is odd, then α is a preimage of x under the Collatz process.*

Proof. Because the digital root $dr(2^n) \in \{2, 5, 8\}$, and $dr(x) \in \{2, 5, 8\}$ the argument proceeds analogously to the proof above. Since, by the rules of digital root multiplication $2 \times 2 = 4$, $2 \times 5 = 1$, $2 \times 8 = 7$ and $5 \times 2 = 1$, $5 \times 5 = 7$, $5 \times 8 = 4$, finally $8 \times 2 = 7$, $8 \times 5 = 4$ and $8 \times 8 = 1$, generating the set $\{1, 4, 7\}$ in every instance. □

Notice that a number $x \in \mathbb{N}_{>0}$ in base 10 where $dr(x) \in \{3, 6, 9\}$ has no pre-image since by the rules of digital root addition $dr(3 \times \alpha + 1) \in \{\{3, 6, 9\} + 1\} = \{4, 7, 1\}$

It is also clear that the operations above cover all of $\mathbb{N}_{>0}$ and every Collatz preimage is included since the digital roots take all possible values.

Theorem 3.3. *The only base 10 natural number which is its own pre-image is 1.*

Proof. Suppose $y \in \mathbb{N}_{>0}$ is odd and is its own preimage. Then

$$y = \frac{(2^\beta \times y - 1)}{3}$$

where $\beta \in \{1, 2, 3...\}$. This equation places constraints on the value of β and y and is only consistent when $2^\beta = 4$, $\beta = 2$ and $y = 1$. This is consitent with $dr(y) \in \{1, 4, 7\}$.

For example, suppose $y = 5$ instead, choosing $\beta = 3$ would maintain consistency with the pre-image calculation since $dr(y) \in \{2, 5, 8\}$. This would imply that

$$5 = \frac{5 \times (2^3 - 1)}{3}$$

which is impossible.

It is for this reason that the only cycle in any Collatz sequence is 1-4-1. □

Theorem 3.4. *Let $x \in \mathbb{N}_{>0}$ be an odd number. We may write $2^\beta \times x = 3 \times y + 1$ to generate the preimage of x in the Collatz process. Suppose $z = 3 \times y$, then $dr(z) \in \{3, 6, 9\}$. If $dr(z) = 3 \iff dr(y) \in \{1, 4, 7\}$ otherwise if $dr(z) = 6 \iff dr(y) \in \{2, 5, 8\}$ finally if $dr(z) = 9 \iff dr(y) \in \{3, 6, 9\}$. As usual $dr(a)$ is the digital root of a*

Proof. Suppose $dr(z) = 3$ then $z = 9k + 3$ for some $k \in \mathbb{N}$ dividing through by 3 gives $y = 3k + 1$ to find $dr(y)$ we may substitute $dr(k) \in \{1..9\}$.

$$dr(y) = \begin{cases} dr(3 \times 1 + 1) = 4 \\ dr(3 \times 2 + 1) = 7 \\ dr(3 \times 3 + 1) = 1 \\ dr(3 \times 4 + 1) = 4 \\ dr(3 \times 5 + 1) = 7 \\ dr(3 \times 6 + 1) = 1 \\ dr(3 \times 7 + 1) = 4 \\ dr(3 \times 8 + 1) = 7 \\ dr(3 \times 9 + 1) = 1 \end{cases}$$

$\therefore$ $dr(y) \in \{1, 4, 7\} \iff 3 = dr(z) = dr(3y)$ The arguments for $dr(y) \in \{2, 5, 8\} \iff 6 = dr(z) = dr(3y)$ and $dr(y) \in \{3, 6, 9\} \iff 9 = dr(z) = dr(3y)$ proceed by a simillar argument. □

4. Digital roots of the powers of 2

The powers of two have a limited, cyclical and specific set of digital roots.

Theorem 4.1. *Let $n \in \mathbb{N}$ then $dr(2^n) \in \{1, 2, 4, 5, 7, 8\}$*

Proof. The digital root of $dr(2^0) = 1$ similarly $dr(2^1) = 2$, $dr(2^2) = 4$, $dr(2^3) = 8$, $dr(2^4) = 7$, $dr(2^5) = 5$, $dr(2^6) = 1$ after which the cycle restarts. The digital root $dr(x^0) = 1 : x \in \mathbb{N}$, therefore $dr(2^0) = 1$, after that we can use the property that $dr(a \times b) = dr(a) \times dr(b)$ to find the rest □

Corollary 4.2. *The digital root of the even powers of* $2 : dr(2^{2n}) \in \{1, 4, 7\}$ *where* $n \in \{1, 2, 3..\}$

Corollary 4.3. *The digital root of the odd powers of* $2 : dr(2^n) \in \{2, 5, 8\}$ *where* $n \in \{1, 3, 5..\}$

5. Properties of the Collatz procedure

To generate the even preimages of the Collatz procedure, we multiply $x \in \mathbb{N}_{>0}$ where $dr(x) \in \{1, 4, 7\}$ and x is odd, by 2^{2n} where $n \in \{1, 2, 3...\}$. The digital roots of $dr(2^{2n}) \in \{1, 4, 7\}$. We may construct the follwing table of products:

×	1	4	7
1	1	4	7
4	4	7	1
7	7	1	4

Similarly to generate preimages of the Collatz procedure of $x \in \mathbb{N}_{>0}$ where $dr(x) \in \{2, 5, 8\}$ and x is odd, multiply x by 2^n where $n \in \{1, 3, 5...\}$. The digital roots of $dr(2^n) \in \{2, 5, 8\}$. We may construct the table of products:

×	2	5	8
2	4	1	7
5	1	7	4
8	7	4	1

in the case of $x \in \mathbb{N}_{>0}$ where $dr(x) \in \{3, 6, 9\}$ and x is odd we will do the procedure using the usual $3x + 1$ and use the property of the digital root where $dr(a \times b) = dr(a) \times dr(b)$ as follows:

$$dr(3 \times 3 + 1) = 1$$

$$dr(3 \times 6 + 1) = 1$$

$$dr(3 \times 9 + 1) = 1$$

From the above tables, we observe that digital roots in the set $\{3, 6, 9\} \Rightarrow 1$, $\{1, 4, 7\} \Rightarrow 4$ and $\{2, 5, 8\} \Rightarrow 7$

5.1. Properties of even numbers with digital roots in {1,4,7}.

Given any even number y such that $dr(y) \in \{1, 4, 7\} \Rightarrow y = 3\beta + 1$ where β is odd;

We can conclude that when $dr(y) = 1$ then $dr(\beta) \in \{3, 6, 9\}$ and when $dr(y) = 4$ then $dr(\beta) \in \{1, 4, 7\}$ and finally when $dr(y) = 7$ then $dr(\beta) \in \{2, 5, 8\}$.

For example $dr(31) = 4 \in \{1, 4, 7\}$ and $31 \times 2^2 = 124$ and $dr(124) = 7 = 6 + 1$ and $124 = 123 + 1 = 3 \times 41 + 1$ where $dr(123) = dr(3 \times 41) = 6$ as such $dr(41) = 5 \in \{2, 5, 8\}$

Supposing $dr(y) = dr(x2^\alpha) = dr(3\beta + 1) \in \{1, 4, 7\}$ where $x, \beta \in \mathbb{N}_{>0}$ and $\alpha \in \{1, 2, 3...\}$ then

$$dr(y) = \begin{cases} 1 \Rightarrow dr(\beta) \in \{3, 6, 9\} \\ 4 \Rightarrow dr(\beta) \in \{1, 4, 7\} \\ 7 \Rightarrow dr(\beta) \in \{2, 5, 8\} \end{cases}$$

For an even number y to have $dr(y) \in \{1,4,7\}$ of the form $9k+a$ where $a \in \{1,4,7\}$ and $k \in \{0,1,2,3...\}$. Specifically k is even when $a=0$ or $a=4$ and odd otherwise. As such the smallest of these even numbers is $4 = 9 \times 0 + 4$.

Given $dr(y) = dr(3\beta+1) \in \{1,4,7\}$ where y is even and $\beta \in \{1,3,5...\}$ then $dr(3\beta) \in \{3,6,9\}$. Take the case when $dr(3\beta)=3$ these are numbers of the form $9k+3$ where $k = \{0,1,2,3...\}$ we may observe the following:

$$dr(\tfrac{9\times0+3}{3}) = 1 \quad dr(\tfrac{9\times1+3}{3}) = 4 \quad dr(\tfrac{9\times2+3}{3}) = 7$$

$$dr(\tfrac{9\times3+3}{3}) = 1 \quad dr(\tfrac{9\times4+3}{3}) = 4 \quad dr(\tfrac{9\times5+3}{3}) = 7$$

$$dr(\tfrac{9\times6+3}{3}) = 1 \quad dr(\tfrac{9\times7+3}{3}) = 4 \quad dr(\tfrac{9\times8+3}{3}) = 7$$

$$... \qquad ... \qquad ...$$

More generally a number $dr(\frac{9\times k+3}{3}) = 1 \iff k \equiv (0 \mod 3)$ simplifying $dr(3\times k+1) = 1 \iff k \equiv (0 \mod 3)$ generally

$$dr(3\times k+1) = 1 \iff k \equiv (0 \mod 3)$$

$$dr(3\times k+1) = 4 \iff k \equiv (1 \mod 3)$$

$$dr(3\times k+1) = 7 \iff k \equiv (2 \mod 3)$$

Table 1. Group 1 Values.

Equivalently $dr(\frac{9\times k+6}{3}) = 2 \iff k \equiv (0 \mod 3)$

$$dr(3\times k+2) = 2 \iff k \equiv (0 \mod 3)$$

$$dr(3\times k+2) = 5 \iff k \equiv (1 \mod 3)$$

$$dr(3\times k+2) = 8 \iff k \equiv (2 \mod 3)$$

Table 2. Group 2 Values.

Similarly $dr(\frac{9\times k+9}{3}) = 3 \iff k \equiv (0 \mod 3)$

$$dr(3\times k+3) = 3 \iff k \equiv (0 \mod 3)$$

$$dr(3\times k+3) = 6 \iff k \equiv (1 \mod 3)$$

$$dr(3\times k+3) = 9 \iff k \equiv (2 \mod 3)$$

Table 3. Group 3 Values.

It is instances of digital roots $dr(y) \in \{1,4,7\}$ that are produced by the operation $3\times k+1$ where k is odd, of the Collatz procedure, in Table 1(Group 1) above. However we shall see that the other values from the other tables becom important.

The operation $(3 \times k + 1)$ where $k \equiv (x \mod 3)$, $k \in \{1, 3, 5, 7...\}$ yields even numbers.

5.2. **Division by 2.** The only possible digital roots of base 10 natural numbers are $dr(a) \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Consider multiplcation by 2 of the digital roots: $2 \times 5 = 1$, $2 \times 1 = 2$, $2 \times 6 = 3$, $2 \times 2 = 4$, $2 \times 7 = 5$, $2 \times 3 = 6$, $2 \times 8 = 7$, $2 \times 4 = 8$ and $2 \times 9 = 9$. By this means we can establish the results of division by 2 as follows $\frac{1}{2} = 5$, $\frac{2}{2} = 1$, $\frac{3}{2} = 6$, $\frac{4}{2} = 2$, $\frac{5}{2} = 7$, $\frac{6}{2} = 3$, $\frac{7}{2} = 8$, $\frac{8}{2} = 4$ and $\frac{9}{2} = 9$.

We may now investigate how digital roots $dr(y) \in \{1, 4, 7\}$ behave assuming division by 2 of even numbers.

For the number 1:	$1 \to 5 \to 7 \to 8 \to 4 \to 2 \to 1$
For the number 4:	$4 \to 2 \to 1 \to 5 \to 7 \to 8 \to 4$
For the number 7:	$7 \to 8 \to 4 \to 2 \to 1 \to 5 \to 7$

All of which have a cycle of length 6.

6. The sequence of Collatz odd numbers

6.1. **Enumerating odd numbers.** We will use a unique technique to enumerate the odd numbers. In this system and in general an odd number x can be reprere-sented as $x = 2^k + \omega$. Where $\omega \in \mathbb{N}$ is an odd natural number, including zero, such that $\omega \leq 2^k$. The following table provides a few examples:

$k = 0$	$\omega = 0$	$\therefore x = 2^k + \omega$	$= 2^0 + 0 =$	**1**
$k = 1$	$\omega = 1$	$\therefore x = 2^k + \omega$	$= 2^1 + 1 =$	**3**
$k = 2$	$\omega = 1$	...	$= 2^2 + 1 =$	**5**
$k = 2$	$\omega = 3$	...	$= 2^2 + 3 =$	**7**
$k = 3$	$\omega = 1$	...	$= 2^3 + 1 =$	**9**
$k = 3$	$\omega = 3$	...	$= 2^3 + 3 =$	**11**
$k = 3$	$\omega = 5$	...	...	**13**
$k = 3$	$\omega = 7$	...	...	**15**
$k = 4$	$\omega = 1$	...	$= 2^4 + 1 =$	**17**
...	...	...	...	...

This way of generating odd numbers lends itself to analysis of the even z_{prev}. In general we can analyze specific instances of these odd numbers x, for example in the case of $\omega = 5$, $x = 2^k + 5$. Recall that $z_{prev} = \lfloor \frac{x_{prev}}{2} \rfloor$ or equivalently $z_{prev} = \frac{x_{prev}-1}{2}$. in this case $x_{prev} = x = 2^k + 5$ and $z_{prev} = \frac{2^k+5-1}{2} = \frac{2^k+4}{2} = 2^{k-1} + 2$ which is clearly even.

More generally we may set $\omega = 2n + 1$. We notice at once that z_{prev} is only even if n is even. Using this notation we can also see that $\frac{z_{prev}}{2} = \lfloor \frac{x_{prev}}{4} \rfloor = \frac{2^k+2n+1-1}{4} = \frac{2^k+2n}{4}$

6.2. **Definition ${\omega_n}^x$.**

Definition 6.1. Given an odd number x under the Collatz procedure $\frac{3x+1}{2}$ is considered the first step. Every subsequent division by 2 is another step. The step count terminates when $\frac{3x+1}{2^n} = x_{next}$ is odd, where $n \in \{1, 2, 3, 4...\}$. The number of steps $s = n$ where the number of steps $s \in \mathbb{N}_{>0}$.

Definition 6.2. Let $x \in \mathbb{N}_{>0}$ be a base 10 natural number where $dr(x) \in \{1, 4, 7\}$ then $x \times 2^n = 3 \times \alpha_n + 1$ where $n = 2, 4, 6, 8...$ is even and α_n is odd. If x is odd, then α_n is a preimage of x under the Collatz process.

Define ${\omega_n}^x = \alpha_n$ for x. We call this an even ${\omega_n}^x$ for $dr(x) \in \{1, 4, 7\}$. When ${\omega_n}^x$ is used in this way it will produce an even number of steps s equal to n.

Definition 6.3. Let $x \in \mathbb{N}_{>0}$ be a base 10 natural number where $dr(x) \in \{2, 5, 8\}$ then $x \times 2^n = 3 \times \alpha + 1, n = 1, 3, 5, 7...$ and α is odd. If x is odd, then α is a preimage of x under the Collatz process.

Define ${\omega_n}^x = \alpha_n$ for x. We call this an odd ${\omega_n}^x$ for $dr(x) \in \{2, 5, 8\}$. When ${\omega_n}^x$ is used in this way it will produce an odd number of steps s equal to n.

Below are some examples where the ω is fixed. You will notice that the number of steps is also fixed.

Below is a table of $2^n + 1$ odd numbers and their working, in this case $\omega = 1$ which is a preimage of 1 or $1 = \frac{(1 \times 2^2 - 1)}{3}$. The odd numbers in the column next x are the subsequent odd numbers:

Even powers of two 2^{2n}

n	2^n	${\omega_2}^1$	$x = 2^n + \omega$	$z_0 = \lfloor \frac{x}{2} \rfloor$	$z_1 = \lfloor \frac{z_0}{2} \rfloor$	next z	next x	Steps
0	4	1	5	2	1			2
1	16	1	17	8	4	6	13	2
2	64	1	65	32	16	24	49	2
3	256	1	257	128	64	96	193	2
4	1024	1	1025	512	256	384	769	2
5	4096	1	4097	2048	1024	1536	3073	2
6	16384	1	16385	8192	4096	6144	12289	2
7	65536	1	65537	32768	16384	24576	49153	2
8	262144	1	262145	131072	65536	98304	196609	2
9	1048576	1	1048577	524288	262144	393216	786433	2

Odd powers of two 2^n

n	2^n	${\omega_2}^1$	$x = 2^n + \omega$	$z_0 = \lfloor \frac{x}{2} \rfloor$	$z_1 = \lfloor \frac{z_0}{2} \rfloor$	next z	next x	Steps
0	2	1	3	1	2	5	5	2
1	8	1	9	4	2	3	7	2
2	32	1	33	16	8	12	25	2
3	128	1	129	64	32	48	97	2
4	512	1	513	256	128	192	385	2
5	2048	1	2049	1024	512	768	1537	2
6	8192	1	8193	4096	2048	3072	6145	2
7	32768	1	32769	16384	8192	12288	24577	2
8	131072	1	131073	65536	32768	49152	98305	2
9	524288	1	524289	262144	131072	196608	393217	2
10	2097152	1	2097153	1048576	524288	786432	1572865	2

Below is a table of $2^n + 5$ odd numbers and their working , in this case $\omega = 5$ which is a preimage of 1 or $5 = \frac{(1 \times 2^4 - 1)}{3}$. The odd numbers in the column next x are the subsequent odd numbers:

Even powers of two 2^{2n}

n	2^n	$\omega_4{}^1$	$x = 2^n + \omega$	$z_0 = \lfloor \frac{x}{2} \rfloor$	$z_1 = \lfloor \frac{z_0}{2} \rfloor$	z_2	z_3	next z	next x	Steps
0	16	5	21	10	5	2	1			4
1	64	5	69	34	17	8	4	6	13	4
2	256	5	261	130	65	32	16	24	49	4
3	1024	5	1029	514	257	128	64	96	193	4
4	4096	5	4101	2050	1025	512	256	384	769	4
5	16384	5	16389	8194	4097	2048	1024	1536	3073	4
6	65536	5	65541	32770	16385	8192	4096	6144	12289	4
7	262144	5	262149	131074	65537	32768	16384	24576	49153	4
8	1048576	5	1048581	524290	262145	131072	65536	98304	196609	4

Odd powers of two 2^n

n	2^n	$\omega_4{}^1$	$x = 2^n + \omega$	$z_0 = \lfloor \frac{x}{2} \rfloor$	$z_1 = \lfloor \frac{z_0}{2} \rfloor$	z_2	z_3	next z	next x	Steps
0	8	5	13	6	3	1	2	5	5	4
1	32	5	37	18	9	4	2	3	7	4
2	128	5	133	66	33	16	8	12	25	4
3	512	5	517	258	129	64	32	48	97	4
4	2048	5	2053	1026	513	256	128	192	385	4
5	8192	5	8197	4098	2049	1024	512	768	1537	4
6	32768	5	32773	16386	8193	4096	2048	3072	6145	4
7	131072	5	131077	65538	32769	16384	8192	12288	24577	4
8	524288	5	524293	262146	131073	65536	32768	49152	98305	4
9	2097152	5	2097157	1048578	524289	262144	131072	196608	393217	4

6.3. A study of $\omega_n{}^x$.

Below is a table of x where $dr(x) \in \{1, 4, 7\}$ odd numbers and the corresponding $\omega_n{}^x$. In the table below $k \in \{0, 1, 2, 3...\}$. Note that $x = 6k + 1$.

x	$8k+1$	$32k+5$	$128k+21$	$512k+85$	...
1	1	5	21	85	...
7	9	37	149	597	...
13	17	69	277	1109	...
19	25	101	405	1621	...
25	33	133	533	2133	...
31	41	165	661	2645	...
37	49	197	789	3157	...
...	...	...	...	...	...

Below is a table of x where $dr(x) \in \{2, 5, 8\}$ odd numbers and the corresponding $\omega_n{}^x$. In the table below $k \in \{0, 1, 2, 3...\}$. Note that $x = 6k + 5$.

x	$4k+3$	$16k+13$	$64k+53$	$256k+213$	...
5	3	13	53	213	...
11	7	29	117	469	...
17	11	45	181	725	...
23	15	61	245	981	...
29	19	77	309	1237	...
35	23	93	373	1493	...
41	27	109	437	1749	...
...	...	...	...	...	...

In each row in the above tables are preimages of the x column.

The general form of the formulae for preimage columns of the tables above is $2^{n+1} + {\omega_n}^x$.

There is a recurrence relation for the preimages in both the above tables given by $a_n = 4a_{n-1} + 1$ where a_0 is in the first preimage column, a_0 can be obtained by $a_0 = \frac{4x-1}{3}$ for the $8k+1$ column in the first table above, while for the $4k+3$ column in the second table $a_0 = \frac{2x-1}{3}$. A solution to the $a_n = 4a_{n-1} + 1$ recurrence relation is $a_n = \frac{1}{3}(3 \times 4^n \times a_{n-1} + 4^n - 1)$

7. A proof of the Collatz conjecture

7.1. An exploration of $3x + 1$.

Theorem 7.1. *Every even number given by* $3x + 1$ *is of the form* $2(3z + 2)$ *where* x *is a positive odd natural number.*

Proof. Suppose x is a positive odd natural number then $3x + 1 = 6(\frac{x-1}{2}) + 4$

Because x is odd $x - 1$ is even. we may set $x - 1 = 2z$, therefore $3x + 1 = 6z + 4 = 2(3z + 2)$. □

A table of $3z + 2$:

z	3z+2	2(3z+2) = 3x+1	x = 2z+1
0	2	4	1
1	5	10	3
2	8	16	5
3	11	22	7
4	14	28	9
5	17	34	11
6	20	40	13
7	23	46	15
8	26	52	17
9	29	58	19
10	32	64	21
11	35	70	23
12	38	76	25
13	41	82	27
14	44	88	29
15	47	94	31
16	50	100	33
17	53	106	35
18	56	112	37
19	59	118	39
20	62	124	41
21	65	130	43
22	68	136	45
23	71	142	47
24	74	148	49
25	77	154	51
26	80	160	53
27	83	166	55
28	86	172	57
29	89	178	59
30	92	184	61
31	95	190	63
32	98	196	65
33	101	202	67
34	104	208	69
35	107	214	71
36	110	220	73
37	113	226	75
38	116	232	77
39	119	238	79
40	122	244	81

Corollary 7.2. *By the same token* $3z_0 + 2 = 2(3z_1 + 1)$ *where* $z_1 = \lfloor \frac{z_0}{2} \rfloor$ *since* $3z_0 + 2 = 2(\lfloor \frac{3z_0}{2} \rfloor + 1)$

7.1.1. *Collatz even numbers.* Given $y = 3x + 1$ where $x \in \{1, 3, 5...\}$, then y is an even numbers such that its digital root $dr(y) \in \{1, 4, 7\}$ as we have seen before.

Consider $y_0 = 3x + 1 = 2(3z_0 + 2)$ then

$$\begin{array}{r|llll} y_1 = & \frac{y_0}{2} = \frac{3x+1}{2} & = 3z_0 + 2 & \text{if } y_0 \text{ is even} \\ y_2 = & \frac{y_0}{4} = \frac{3x+1}{4}r & = 3z_1 + 1 & \text{if } y_1 \text{ is even} \\ y_3 = & \frac{y_0}{8} = \frac{3x+1}{8} & = 3z_2 + 2 & \text{if } y_2 \text{ is even} \\ y_4 = & \frac{y_0}{16} = \frac{3x+1}{16} & = 3z_3 + 1 & \text{if } y_3 \text{ is even} \\ ... & \quad ... & & \\ y_n = & \frac{y_0}{2^n} = \frac{3x+1}{2^n} & = 3z_n + \{2 \text{ or } 1\} & \text{if } y_{n-1} \text{ is even} \end{array}$$

Where $z_1 = \lfloor \frac{z_0}{2} \rfloor$ and $z_2 = \lfloor \frac{z_1}{2} \rfloor$ and $z_n = \lfloor \frac{z_{n-1}}{2} \rfloor$ and so on.

The procedure terminates when y_n is odd or $z_n = 1$, consequently we may ignore the products $3z_n + 2$ when z_n is even or $3z_n + 1$ when z_n is odd. This is because $3z_n + 2$ is only odd when z_n is odd and $3z_n + 1$ is odd when z_n is even.

This procedure can be further simplified. In fact the procedure depends only on the odd/even parity of z_n and the termination criteria are equivalent to; terminate if z_n is odd and its index n is zero or even, otherwise terminate when z_n is even and its index n is odd. At which point the output is either $3z_n + 2$ in the former case or $3z_n + 1$ in the latter.

Theorem 7.3. $\frac{3x+1}{2}$ *is odd for all natural numbers of the form* $x = 4n + 3$ *where* $n \in \{0, 1, 2, 3...\}$

Proof. Suppose $x = 4n + 3$ and $z = \frac{x-1}{2} = 2n + 1$. z is clearly an odd number. It follows that $3z + 2$ will be an odd number. □

Theorem 7.4. $\frac{3x+1}{4}$ *is odd for all natural numbers of the form* $x = 8n + 1$ *where* $n \in \{0, 1, 2, 3...\}$

Proof. Suppose $x = 8n + 1$ and $z = \frac{x-1}{2} = 4n$. z is clearly an even number. As such $3z + 2$ will be an even number. However the subsequent $3z_1 + 1$ will be odd since $z_1 = 2n$. □

Writing the odd numbers as some $2^n + \omega$, for example, writing all the odd numbers as $x = 8n+1$,$x = 8n+3$,$x = 8n+5$ and $x = 8n+7$ where $n \in \{0, 1, 2, 3...\}$ and in this case $\omega \in \{1, 3, 5, 7\}$ and using the procedure outlined above gives some insight into the process of obtaining the next odd number from some previous odd number. While the $8n + \omega$ where ω is evey odd number less than 8 gives some clarity, $32+\omega$, where ω is every odd number less than 32, would also work. See the appendices for some sample calculations and code for more details.

An alternative method would involve working with the binary representation of z_n. This reduces the problem to seeking occurences of zeros and ones.

A table of $32n + 3$ odd numbers and their working. The odd numbers in the last column are the subsequent odd numbers:

n	x = 32n+3	z = (x - 1)/2	3z+2
0	3	1	5
1	35	17	53
2	67	33	101
3	99	49	149
4	131	65	197
5	163	81	245
6	195	97	293
7	227	113	341
8	259	129	389
9	291	145	437
10	323	161	485
11	355	177	533
12	387	193	581
13	419	209	629
14	451	225	677
15	483	241	725
16	515	257	773
17	547	273	821
18	579	289	869
19	611	305	917
20	643	321	965
21	675	337	1013
22	707	353	1061
23	739	369	1109
24	771	385	1157
25	803	401	1205
26	835	417	1253
27	867	433	1301
28	899	449	1349
29	931	465	1397
30	963	481	1445
31	995	497	1493
32	1027	513	1541
33	1059	529	1589
34	1091	545	1637
35	1123	561	1685

A table of $32n + 5$ odd numbers and their working. The odd numbers in the last column are the subsequent odd numbers:

n	x = 32n+5	z=(x - 1)/2	3z+2	3z+1	3z+2_1	3z+1_2
0	5	2	8	4		
1	37	18	56	28	14	7
2	69	34	104	52	26	13
3	101	50	152	76	38	19
4	133	66	200	100	50	25
5	165	82	248	124	62	31
6	197	98	296	148	74	37
7	229	114	344	172	86	43
8	261	130	392	196	98	49
9	293	146	440	220	110	55
10	325	162	488	244	122	61
11	357	178	536	268	134	67
12	389	194	584	292	146	73
13	421	210	632	316	158	79
14	453	226	680	340	170	85
15	485	242	728	364	182	91
16	517	258	776	388	194	97
17	549	274	824	412	206	103
18	581	290	872	436	218	109
19	613	306	920	460	230	115
20	645	322	968	484	242	121
21	677	338	1016	508	254	127
22	709	354	1064	532	266	133
23	741	370	1112	556	278	139
24	773	386	1160	580	290	145
25	805	402	1208	604	302	151
26	837	418	1256	628	314	157
27	869	434	1304	652	326	163
28	901	450	1352	676	338	169
29	933	466	1400	700	350	175
30	965	482	1448	724	362	181
31	997	498	1496	748	374	187
32	1029	514	1544	772	386	193
33	1061	530	1592	796	398	199
34	1093	546	1640	820	410	205
35	1125	562	1688	844	422	211

7.1.2. *Collatz odd numbers.* One can think of the Collatz procedure as a sequence of odd numbers, the conjecture being that this sequence always terminates at 1. Here we analyze and give an “algebraic” method for finding the next odd number form the previous odd number. We are working with two odd numbers x_{prev} and x_{next}. We assume that $\frac{3x_{prev}+1}{2^n} = x_{next}$ where $n \in \{1, 2, 3...\}$. In other words x_{prev} is the Collatz preimage of x_{next}. z_{prev} and z_{next} are both generalizations of z encountered in earlier discussions. In general the $n^{\text{th}} = z^n_{prev} = \lfloor \frac{x_{prev}}{2^n} \rfloor$ and the first $z^0_{next} = \lfloor \frac{x_{next}}{2} \rfloor$ in relation to x_{next}.

Following from the above discussion if $x_{next} = 3z_{prev} + 2$ is odd then z_{prev} is also odd. In genral z_{next} is defined as $z_{next} = \frac{x_{next}-1}{2}$ therefore $x_{next} = 2z_{next} + 1$ it follows that $x_{next} = 2z_{next} + 1 = 3z_{prev} + 2$ and $z_{next} = \frac{3z_{prev}+1}{2}$.

On the other hand, if $x_{next} = 3z_{prev} + 1$ is odd, then z_{prev} is even. As before z_{next} is defined as $z_{next} = \frac{m-1}{2}$ and $x_{next} = 2z_{next} + 1$. It follows that $x_{next} = 2z_{next} + 1 = 3z_{prev} + 1$ and $z_{next} = \frac{3z_{prev}}{2}$.

In this way, we can relate z_{prev} to the next z_{next} of the next odd number in the Collatz sequence. z_{next} is either $z_{next} = \frac{3z_{prev}+1}{2}$ where z_{prev} is odd or $z_{next} = \frac{3z_{prev}}{2}$ where z_{prev} is even.

Furthermore x_{next} the next odd number is $x_{next} = 2z_{next} + 1 = 3z_{prev} + 2$ if z_{prev} is odd and $x_{next} = 2z_{next} + 1 = 3z_{prev} + 1$ otherwise.

Theorem 7.5. *An odd number x_{next} is strictly less than its odd preimage x_{prev} in the Collatz process if $\lfloor \frac{x_{prev}}{2} \rfloor$ is even.*

Proof. We note that every odd number can be written as $2n + 1$. Furthermore $\lfloor \frac{2n+1}{2} \rfloor = n$. As a result we are concerned with odd numbers $2n + 1$ where $n \in \{2, 4, 6, 8...\}$ Recall the procedure outlined above in which $z_0 = \lfloor \frac{x_{prev}}{2} \rfloor$. In this case z_0 is even. The first step in the procedure is to check if z_0 is odd and if not divide by 2 again producing $z_1 = \lfloor \frac{z_0}{2} \rfloor$. If at that point z_1 is even then the next odd number $x_{next} = 3z_1 + 1$.

As such $z_1 = \lfloor \frac{z_0}{2} \rfloor = \lfloor \frac{x_{prev}}{4} \rfloor$. In other words, any subsequent z_n produced in the same way must obey $z_n < z_1 = \lfloor \frac{x_{prev}}{4} \rfloor$ and at least $3z_1 + 1 < x_{prev}$ □

Theorem 7.6. *An odd number x_{next} is strictly greater than its odd preimage x_{prev} in the Collatz process if $\lfloor \frac{x_{prev}}{2} \rfloor$ is odd.*

Proof. We note that every odd number can be written as $2n + 1$. Furthermore $\lfloor \frac{2n+1}{2} \rfloor = n$. As a result we are concerned here with odd numbers $2n + 1$ where $n \in \{1, 3, 5, 7...\}$ Recall the procedure outlined above in which $z_0 = \lfloor \frac{x_{prev}}{2} \rfloor$. In this case z_0 is odd. The first step in the procedure is to check if z_0 is odd and if it is multiply by 3 and add 1 producing $x_{next} = 3z_0 + 1$.

As such $z_0 = \lfloor \frac{x_{prev}}{2} \rfloor$ and $3z_0 + 2 > x_{prev}$

Furthermore when $z_0 = z_{prev}$ is odd, $\frac{3 \times x_{prev}+1}{2}$ is also odd and $\frac{3 \times x_{prev}+1}{2} = x_{next}$ □

Let us examine the evolution of z_{prev} and z_{next} since these control the changes to successive x_{next} throughout the calculation.

Recall that x_{next} is the imediate next odd number after x_{prev} and as such $x_{next} = 2z_{next} + 1 = 3z_{prev} + 2$ if z_{prev} is odd and $x_{next} = 2z_{next} + 1 = 3z_{prev} + 1$ if z_{prev} is even.

We can model the relationships between on z_{prev} to another z_{next} as a recurrence relation, one in which $z_{next} = 3(\lfloor \frac{z_{prev}}{2^n} \rfloor) + 1$ or $z_{next} = 3(\lfloor \frac{z_{prev}}{2^n} \rfloor)$ where n is the number of steps. Solving this reccurence relation resullts in $z_{next} = \lfloor (\frac{3}{2^n}) \times (z_{prev} + \frac{1}{2}) \rfloor$ which is itself a recurrence relation. A similar recurrence realtion exists for x_{prev} and x_{next} as follows $x_{next} = \lfloor (\frac{3}{2^n}) \times (x_{prev} + \frac{1}{2}) \rfloor$

It is possible to execute the function $x_{next} = \lfloor (\frac{3}{2^n}) \times (x_{prev} + \frac{1}{2}) \rfloor$ even if n, the nubmer of steps, is unknown by gradually incrimenting n from $n = 1$ until the function produces the first odd number. The following Python code demonstrates how this might work

```
def nextX(x):
```

```
    n = 1
    xnext = 2
    while xnext % 2 == 0:
        xnext = math.floor((3/2**n)*(x + 0.5))
        n += 1
    return xnext
```

7.1.3. *Digital roots in the Collatz procedure.* Digital roots and the number of steps are related. Given a pair of natural numbers which are Collatz orbitals, xprev and xnext then the table below relates a given input digital root $dr(xprev)$, the number of steps and the output digital root $dr(xnext)$.

Below is an example of a table of where $xprev_n = (2^n \times 3^q) + xprev$ odd numbers and the corresponding $xnext_n = (2^{n-s} \times 3^{q+1}) + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$ and $xprev = 3$, $xnext = 5$ and the number of steps $s = 1$.

xprev digital root	xnext digital root						
	1 step	2 steps	3 steps	4 steps	5 steps	6 steps	7 steps
1	2	1	5	7	8	4	2
2	8	4	2	1	5	7	8
3	5	7	8	4	2	1	5
4	2	1	5	7	8	4	2
5	8	4	2	1	5	7	8
6	5	7	8	4	2	1	5
7	2	1	5	7	8	4	2
8	8	4	2	1	5	7	8
9	5	7	8	4	2	1	5

The following Python code illustrates this procedure.

```
def getOutputDigitalRoot(n, steps):
        if n in [1, 4,7]:
            x = 5
        elif n in [2, 5, 8]:
            x =3
        elif n in [3,6,9]:
            x = 1
        mod6 = ((steps - 1) + x) % 6
        return self.digitSum(5**mod6)
```

This relationship between the digital roots is given by the following formulae, there are three cases:

If $dr(xprev) \in \{1,4,7\}$ and the number of steps is s then first obtain

$$r = (5 + (s-1)) \mod 6$$

then

$$dr(xnext) = dr(5^r)$$

If $dr(xprev) \in \{2,5,8\}$ and the number of steps is s then first obtain

$$r = (3 + (s-1)) \mod 6$$

then

$$dr(xnext) = dr(5^r)$$

Finally if $dr(xprev) \in \{3,6,9\}$ and the number of steps is s then first obtain

$$r = (1 + (s-1)) \mod 6$$

then

$$dr(xnext) = dr(5^r)$$

7.1.4. *Type One Collatz sequences.* For a pair of natural numbers which are Collatz orbitals, xprev and xnext, for example $xprev = 3$ and $xnext = 5$ where $3 \times 3 + 1 = 10, \frac{10}{2} = 5$, in this case the number of steps $s = 1$ we deduce two different types of Collatz sequences. For Type One; one may create a table of values in which $xprev_n = (2^p \times 3^q)n + xprev$ and $xnext_n = (2^{p-s} \times 3^{q+1})n + xnext$, below is an example:

Below is a table where $xprev_n = (2^p \times 3^q)n + xprev$ odd numbers and the corresponding and $xnext_n = (2^{p-s} \times 3^{q+1})n + xnext$. In the table below $n \in \{0,1,2,3...\}$, $p = 2$, $q = 1$, $xprev = 3$, $xnext = 5$ and the number of steps $s = 1$.

xprev digital root	*xprev*	*steps*	xnext digital root	*xnext*
3	3	1	5	5
6	15	1	5	23
9	27	1	5	41
3	39	1	5	59
6	51	1	5	77
9	63	1	5	95
3	75	1	5	113
6	87	1	5	131
9	99	1	5	149
3	111	1	5	167
6	123	1	5	185
9	135	1	5	203
3	147	1	5	221
6	159	1	5	239
9	171	1	5	257
3	183	1	5	275
6	195	1	5	293
9	207	1	5	311
3	219	1	5	329
6	231	1	5	347
9	243	1	5	365
3	255	1	5	383
6	267	1	5	401
9	279	1	5	419
3	291	1	5	437
6	303	1	5	455
9	315	1	5	473
3	327	1	5	491
6	339	1	5	509
9	351	1	5	527

The table below demonstrates another Type One table with more than one orbit of the Collatz procedure.

Below is an example of a table of where $xprev_n = (2^p \times 3^q)n + xprev$ odd numbers and the corresponding and $xnext_n = (2^{p-s} \times 3^{q+1})n + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$, $p = 2$, $q = 1$, $xprev = 3$, $xnext = 5$ and the number of steps $s = 1$.

$xprev_n = (2^5 \times 3^1)n + 1$	steps	$xnext_n = (2^3 \times 3^2)n + 1$	steps	$xnext_n = (2^1 \times 3^2)n + 1$
1	2	1	2	1
97	2	73	2	55
193	2	145	2	109
289	2	217	2	163
385	2	289	2	217
481	2	361	2	271
577	2	433	2	325
673	2	505	2	379
769	2	577	2	433
865	2	649	2	487
961	2	721	2	541
1057	2	793	2	595
1153	2	865	2	649
1249	2	937	2	703
1345	2	1009	2	757
1441	2	1081	2	811
1537	2	1153	2	865
1633	2	1225	2	919
1729	2	1297	2	973
1825	2	1369	2	1027
1921	2	1441	2	1081
2017	2	1513	2	1135
2113	2	1585	2	1189
2209	2	1657	2	1243
2305	2	1729	2	1297
2401	2	1801	2	1351
2497	2	1873	2	1405
2593	2	1945	2	1459
2689	2	2017	2	1513
2785	2	2089	2	1567

7.1.5. *Type Two Collatz sequences.* Alternatively we may vary n as follows $xprev_n = (2^n \times 3^q) + xprev$ and $xnext_n = (2^{n-s} \times 3^{q+1}) + xnext$, this is the Type Two calculation which yields a table like this:

Below is a table of where $xprev_n = (2^n \times 3^q) + xprev$ odd numbers and the corresponding $xnext_n = (2^{n-s} \times 3^{q+1}) + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$ and $xprev = 3$, $xnext = 5$ and the number of steps $s = 1$.

xprev digital root	xprev	steps	xnext digital root	xnext
6	15	1	5	23
9	27	1	5	41
6	51	1	5	77
9	99	1	5	149
6	195	1	5	293
9	387	1	5	581
6	771	1	5	1157
9	1539	1	5	2309
6	3075	1	5	4613
9	6147	1	5	9221
6	12291	1	5	18437
9	24579	1	5	36869
6	49155	1	5	73733
9	98307	1	5	147461
6	196611	1	5	294917
9	393219	1	5	589829
6	786435	1	5	1179653
9	1572867	1	5	2359301
6	3145731	1	5	4718597
9	6291459	1	5	9437189
6	12582915	1	5	18874373
9	25165827	1	5	37748741
6	50331651	1	5	75497477
9	100663299	1	5	150994949
6	201326595	1	5	301989893
9	402653187	1	5	603979781
6	805306371	1	5	1207959557
9	1610612739	1	5	2415919109

Below is an example of a table of where $xprev_n = 2^n \times 3^q + xprev$ odd numbers and the corresponding $xnext_n = 2^{n-s} \times 3^{q+1} + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$ and $xprev = 3$, $xnext = 5$ and the number of steps $s = 1$.

$xprev_n = 2^n \times 3^1 + 1$	steps	$xnext_n = 2^{n-2} \times 3^2 + 1$	steps	$xnext_n = 2^{n-4} \times 3^3 + 1$
97	2	73	2	55
193	2	145	2	109
385	2	289	2	217
769	2	577	2	433
1537	2	1153	2	865
3073	2	2305	2	1729
6145	2	4609	2	3457
12289	2	9217	2	6913
24577	2	18433	2	13825
49153	2	36865	2	27649
98305	2	73729	2	55297
196609	2	147457	2	110593
393217	2	294913	2	221185
786433	2	589825	2	442369
1572865	2	1179649	2	884737
3145729	2	2359297	2	1769473
6291457	2	4718593	2	3538945
12582913	2	9437185	2	7077889
25165825	2	18874369	2	14155777
50331649	2	37748737	2	28311553
100663297	2	75497473	2	56623105
201326593	2	150994945	2	113246209
402653185	2	301989889	2	226492417
805306369	2	603979777	2	452984833
1610612737	2	1207959553	2	905969665
3221225473	2	2415919105	2	1811939329
6442450945	2	4831838209	2	3623878657
12884901889	2	9663676417	2	7247757313
25769803777	2	19327352833	2	14495514625
51539607553	2	38654705665	2	28991029249

7.1.6. *The Fundamental Collatz sequences.* In the examples below we demonstrate what we call the Fundamental sequence because this formulation produces all odd numbers and the corresponding xnext orbital. Recall that in the Type One tables $xprev_n = (2^p \times 3^q)n + xprev$ where $q \geq 1$. In the Fundamental formulation initially $q = 0$ for the $xprev_n$ calculation, effectively this formulation means that $xprev_n = 2^p + xprev$ or $xprev_n = (2^p)n + xprev$ where $p \geq s + 1$ where s is the number of steps and $n \in \{0, 1, 2, 3...\}$. This means that $xnext_n = 2^{n-s} \times 3^1 + xnext$

As discussed above, in the table below we set $q = 0$ intially given that $xprev_n = (2^n \times 3^q) + xprev$ odd numbers and the corresponding $xnext_n = (2^{n-s} \times 3^{q+1}) + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$ and $xprev = 1$, $xnext = 1$, initially $q = 0$ and the number of steps $s = 2$.

xprev digital root	$xprev_n = (2^3)n + 1$	steps	xnext digital root	$xnext_n = (2^1 \times 3^1)n + 1$
1	1	2	1	1
9	9	2	7	7
8	17	2	4	13
7	25	2	1	19
6	33	2	7	25
5	41	2	4	31
4	49	2	1	37
3	57	2	7	43
2	65	2	4	49
1	73	2	1	55
9	81	2	7	61
8	89	2	4	67
7	97	2	1	73
6	105	2	7	79
5	113	2	4	85
4	121	2	1	91
3	129	2	7	97
2	137	2	4	103
1	145	2	1	109
9	153	2	7	115
8	161	2	4	121
7	169	2	1	127
6	177	2	7	133
5	185	2	4	139
4	193	2	1	145
3	201	2	7	151
2	209	2	4	157
1	217	2	1	163
9	225	2	7	169
8	233	2	4	175

Below is an example of a sequence of where $xprev_n = (2^n \times 3^q) + xprev$ odd numbers and the corresponding $xnext_n = (2^{n-s} \times 3^{q+1}) + xnext$. In the table below $n \in \{0, 1, 2, 3...\}$ and $xprev = 3$, $xnext = 5$, initially $q = 0$ and the number of steps $s = 1$.

xprev digital root	$xprev_n = (2^2)n + 3$	steps	xnext digital root	$xnext_n = (2^1 \times 3^1)n + 1$
3	3	1	5	5
7	7	1	2	11
2	11	1	8	17
6	15	1	5	23
1	19	1	2	29
5	23	1	8	35
9	27	1	5	41
4	31	1	2	47
8	35	1	8	53
3	39	1	5	59
7	43	1	2	65
2	47	1	8	71
6	51	1	5	77
1	55	1	2	83
5	59	1	8	89
9	63	1	5	95
4	67	1	2	101
8	71	1	8	107
3	75	1	5	113
7	79	1	2	119
2	83	1	8	125
6	87	1	5	131
1	91	1	2	137
5	95	1	8	143
9	99	1	5	149
4	103	1	2	155
8	107	1	8	161
3	111	1	5	167
7	115	1	2	173
2	119	1	8	179

The folloing pairs of xprev and xnext produce all the Collatz odd number pairs for the given number of steps using the Fundamental formulation.

For even numbered steps

2 steps	$xprev = 1$	$xnext = 1$
4 steps	$xprev = 5$	$xnext = 1$
6 steps	$xprev = 21$	$xnext = 1$
8 steps	$xprev = 85$	$xnext = 1$
...	...	
n even steps	$xprev = \frac{(2^n - 1)}{3}$ or $4 \times xprev + 1$	$xnext = 1$

For odd numbered steps

1 step	$xprev = 3$	$xnext = 5$
3 steps	$xprev = 13$	$xnext = 5$
5 steps	$xprev = 53$	$xnext = 5$
7 steps	$xprev = 213$	$xnext = 5$
...	...	
n odd steps	$xprev = \frac{(2^n \times 5 - 1)}{3}$ or $4 \times xprev + 1$	$xnext = 5$

We can obtain the xprev values using the preimage procedure descrivbed earlier in this paper.

7.1.7. *Re-examination of the Collatz orbitals.* Take the following Collatz products:

Below a table of the Collatz procedure for 17, notice the cumulative difference column in particular. The cumulative difference is calculated by taking the previous product minus the next one and adding them up, for example in this case $17 - 13 = 4$ for the first difference. Notice that the final value 16 is one less than the starting value 17

orbits	no. steps	$xnext$	cumulative difference
0	start	17	0
1	2	13	4
2	3	5	12
3	4	1	16

Here is anothe example starting with 31:

Below a table of the Collatz procedure for 31, as above notice the cumulative difference column in particular.

orbits	no. steps	*xnext*	cumulative difference
0	start	31	0
1	1	47	-16
2	1	71	-40
3	1	107	-76
4	1	161	-130
5	2	121	-90
6	2	91	-60
7	1	137	-106
8	2	103	-72
9	1	155	-124
10	1	233	-202
11	2	175	-144
12	1	263	-232
13	1	395	-364
14	1	593	-562
15	2	445	-414
16	3	167	-136
17	1	251	-220
18	1	377	-346
19	2	283	-252
20	1	425	-394
21	2	319	-288
22	1	479	-448
23	1	719	-688
24	1	1079	-1048
25	1	1619	-1588
26	1	2429	-2398
27	3	911	-880
28	1	1367	-1336
29	1	2051	-2020
30	1	3077	-3046
31	4	577	-546
32	2	433	-402
33	2	325	-294
34	4	61	-30
35	3	23	8
36	1	35	-4
37	1	53	-22
38	5	5	26
39	4	1	30

Since we can use the procedure to calculate *xprev* and *xnext* to calculate the difference, we may be able to boil the Collatz procedure to the calculation of a single number, the individual differences between *xprev* and *xnext*.

7.1.8. *Difference Formula.* The output of the equation below (7.1) is interpreted as half of the difference between the image and the preimage of a Collatz orbital number. In this context, the preimage is restricted to integers congruent to 1 (mod 6).

(7.1) quantifies the difference between the image and the preimage between two Collatz orbitals.

Here, r denotes the row, s the number of steps in the orbit, and $m \in \{0, 1, 5, 21, 85, \dots\}$ given by $m_n = 4m_{n-1} + 1$ is a parameter whose value depends on s, varying with the number of steps.

$$de = r\big(2^s - 3\big)4^q + 2m_s, \qquad \text{for preimages } \equiv 1 \pmod 6, \tag{7.1}$$

$$do = r\big(2^s - 3\big)4^q + 5m_s - 1, \qquad \text{for preimages } \equiv 5 \pmod 6. \tag{7.2}$$

The output of equation (7.2) represents the half of the difference between the image and the preimage of a Collatz orbital number in the complementary case to equation (7.1). In this setting, the preimage is restricted to integers congruent to $5 \pmod 6$, in contrast with the $1 \pmod 6$ condition used for equation (7.1). Here, r denotes the row, s the number of steps. Finally $m \in \{0, 1, 5, 21, 85, \dots\}$ given by $m_n = 4m_{n-1} + 1$ is a parameter whose value depends on s, varying with the number of steps.

Together, equations (7.1) and (7.2) characterize the difference between image and preimage values along Collatz orbital paths. The first case corresponds to preimages congruent to $1 \pmod 6$, while the second describes preimages congruent to $5 \pmod 6$. These two conditions exhaust the admissible congruence classes for odd preimages within the orbit, thereby capturing the structural transitions governing the Collatz trajectory.

The exponent q is included for completeness. When the preimages are congruent to $1 \pmod 6$ or $5 \pmod 6$, $q = 0$. The value of q changes only if finer congruence classes of preimages are considered. For example, if the preimages are taken modulo 24, for example preimages congruent to $1 \pmod{24}$ or $5 \pmod{24}$ would require $q = 1$, while classes such as $1 \pmod{24}$ versus $5 \pmod{24}$ could require $q = 1$. Thus, q encodes deeper structural distinctions that emerge at higher modulus levels of the Collatz dynamics.

To illustrate the role of the parameter q, we present examples for both $q = 0$ and $q = 1$. When $q = 0$, the preimages are taken modulo 6, corresponding to the two admissible congruence classes $1 \pmod 6$ and $5 \pmod 6$. In this setting, q does not contribute additional factors of 4, and the differences de and do follow directly from equations (7.1) and (7.2).

By contrast, when $q = 1$, the preimages are refined to lie in congruence classes modulo 24. In this case, the term 4^q enters nontrivially, scaling the contribution of $(2^s - 3)$ and thereby modifying the values of de and do. The following tables give sample values of these quantities for small s and corresponding $m_s \in \{0, 1, 5, 21, 85, \dots\}$. In general we take the preimages to be $2^n \times 3$ where $n \in \{1, 3, 5, 7, \dots\}$. In this paper we primarily focus on $1 \pmod 6$, $3 \pmod 6$ and $5 \pmod 6$ which are equivalence classes for all the odd numbers.

Below are sample values of de for small s and corresponding $m_s \in \{0, 1, 5, 21, 85, \dots\}$ when $q = 0$. Here the preimages are confined to the congruence class 1 modulo 6, so the factor 4^q does not contribute. In this case $s = 2$ for 2 steps and $m_s = 0$, $s = 4$ for 4 steps and $m_s = 1$, $s = 6$ for 6 steps and $m_s = 5$ etc. Take row 1, where $r = 1$, the number is 7, then the preimages of 7 are

$7+\big(2\times\mathbf{1}\big)$, or $7+\big(2\times\mathbf{15}\big)$, or $7+\big(2\times\mathbf{71}\big)$, or $7+\big(2\times\mathbf{295}\big)$, or $7+\big(2\times\mathbf{1191}\big)$, or $7+\big(2\times\mathbf{4775}\big)$...etc.

r	num	2 steps	4 steps	6 steps	8 steps	10 steps	12 steps	14 steps
0	1	0	2	10	42	170	682	2730
1	7	1	15	71	295	1191	4775	19111
2	13	2	28	132	548	2212	8868	35492
3	19	3	41	193	801	3233	12961	51873
4	25	4	54	254	1054	4254	17054	68254
5	31	5	67	315	1307	5275	21147	84635
6	37	6	80	376	1560	6296	25240	101016
7	43	7	93	437	1813	7317	29333	117397
8	49	8	106	498	2066	8338	33426	133778
9	55	9	119	559	2319	9359	37519	150159
10	61	10	132	620	2572	10380	41612	166540
11	67	11	145	681	2825	11401	45705	182921
12	73	12	158	742	3078	12422	49798	199302
13	79	13	171	803	3331	13443	53891	215683
14	85	14	184	864	3584	14464	57984	232064
15	91	15	197	925	3837	15485	62077	248445
16	97	16	210	986	4090	16506	66170	264826
17	103	17	223	1047	4343	17527	70263	281207
18	109	18	236	1108	4596	18548	74356	297588
19	115	19	249	1169	4849	19569	78449	313969
20	121	20	262	1230	5102	20590	82542	330350
21	127	21	275	1291	5355	21611	86635	346731
22	133	22	288	1352	5608	22632	90728	363112
23	139	23	301	1413	5861	23653	94821	379493
24	145	24	314	1474	6114	24674	98914	395874
25	151	25	327	1535	6367	25695	103007	412255
26	157	26	340	1596	6620	26716	107100	428636
27	163	27	353	1657	6873	27737	111193	445017
28	169	28	366	1718	7126	28758	115286	461398
29	175	29	379	1779	7379	29779	119379	477779
30	181	30	392	1840	7632	30800	123472	494160
31	187	31	405	1901	7885	31821	127565	510541
32	193	32	418	1962	8138	32842	131658	526922
33	199	33	431	2023	8391	33863	135751	543303
34	205	34	444	2084	8644	34884	139844	559684
35	211	35	457	2145	8897	35905	143937	576065
36	217	36	470	2206	9150	36926	148030	592446
37	223	37	483	2267	9403	37947	152123	608827
38	229	38	496	2328	9656	38968	156216	625208
39	235	39	509	2389	9909	39989	160309	641589
40	241	40	522	2450	10162	41010	164402	657970

Below are sample values of do for small s and corresponding $m_s \in \{0, 1, 5, 21, 85, \dots\}$ when $q = 0$. Here the preimages are confined to the congruence class 5 modulo 6, so the factor 4^q does not contribute. In this case $s = 1$ for 1 steps and $m_s = 0$, $s = 3$ for 3 steps and $m_s = 1$, $s = 5$ for 5 steps and

$m_s = 5$ etc. Take row 1, where $r = 1$, the number is 11, then the preimages of 11 are $11 + \big(2 \times \mathbf{-2}\big)$, or $11 + \big(2 \times \mathbf{9}\big)$, or $11 + \big(2 \times \mathbf{53}\big)$, or $11 + \big(2 \times \mathbf{229}\big)$, or $11 + \big(2 \times \mathbf{933}\big)$, or $11 + \big(2 \times \mathbf{3749}\big)$...etc.

r	num	1 steps	3 steps	5 steps	7 steps	9 steps	11 steps	13 steps
0	5	-1	4	24	104	424	1704	6824
1	11	-2	9	53	229	933	3749	15013
2	17	-3	14	82	354	1442	5794	23202
3	23	-4	19	111	479	1951	7839	31391
4	29	-5	24	140	604	2460	9884	39580
5	35	-6	29	169	729	2969	11929	47769
6	41	-7	34	198	854	3478	13974	55958
7	47	-8	39	227	979	3987	16019	64147
8	53	-9	44	256	1104	4496	18064	72336
9	59	-10	49	285	1229	5005	20109	80525
10	65	-11	54	314	1354	5514	22154	88714
11	71	-12	59	343	1479	6023	24199	96903
12	77	-13	64	372	1604	6532	26244	105092
13	83	-14	69	401	1729	7041	28289	113281
14	89	-15	74	430	1854	7550	30334	121470
15	95	-16	79	459	1979	8059	32379	129659
16	101	-17	84	488	2104	8568	34424	137848
17	107	-18	89	517	2229	9077	36469	146037
18	113	-19	94	546	2354	9586	38514	154226
19	119	-20	99	575	2479	10095	40559	162415
20	125	-21	104	604	2604	10604	42604	170604
21	131	-22	109	633	2729	11113	44649	178793
22	137	-23	114	662	2854	11622	46694	186982
23	143	-24	119	691	2979	12131	48739	195171
24	149	-25	124	720	3104	12640	50784	203360
25	155	-26	129	749	3229	13149	52829	211549
26	161	-27	134	778	3354	13658	54874	219738
27	167	-28	139	807	3479	14167	56919	227927
28	173	-29	144	836	3604	14676	58964	236116
29	179	-30	149	865	3729	15185	61009	244305
30	185	-31	154	894	3854	15694	63054	252494
31	191	-32	159	923	3979	16203	65099	260683
32	197	-33	164	952	4104	16712	67144	268872
33	203	-34	169	981	4229	17221	69189	277061
34	209	-35	174	1010	4354	17730	71234	285250
35	215	-36	179	1039	4479	18239	73279	293439
36	221	-37	184	1068	4604	18748	75324	301628
37	227	-38	189	1097	4729	19257	77369	309817
38	233	-39	194	1126	4854	19766	79414	318006
39	239	-40	199	1155	4979	20275	81459	326195
40	245	-41	204	1184	5104	20784	83504	334384

By contrast below are sample values of de where $q = 1$ for small s and corresponding $m_s \in \{0, 1, 5, 21, 85, \dots\}$. The preimages are confined to the congruence class 1 modulo 24. In this case $s = 2$ for 2 steps and $m_s = 0$, $s = 4$ for

4 steps and $m_s = 1$, $s = 6$ for 6 steps and $m_s = 5$ etc. Take row 1, where $r = 1$, the number is 25, then the preimages of 25 are $25 + \left(2 \times \mathbf{4}\right)$, or $25 + \left(2 \times \mathbf{54}\right)$, or $25 + \left(2 \times \mathbf{254}\right)$, or $25 + \left(2 \times \mathbf{1054}\right)$, or $25 + \left(2 \times \mathbf{4254}\right)$, or $25 + \left(2 \times \mathbf{17054}\right)$...etc.

r	num	2 steps	4 steps	6 steps	8 steps	10 steps	12 steps	14 steps
0	1	0	2	10	42	170	682	2730
1	25	4	54	254	1054	4254	17054	68254
2	49	8	106	498	2066	8338	33426	133778
3	73	12	158	742	3078	12422	49798	199302
4	97	16	210	986	4090	16506	66170	264826
5	121	20	262	1230	5102	20590	82542	330350
6	145	24	314	1474	6114	24674	98914	395874
7	169	28	366	1718	7126	28758	115286	461398
8	193	32	418	1962	8138	32842	131658	526922
9	217	36	470	2206	9150	36926	148030	592446
10	241	40	522	2450	10162	41010	164402	657970
11	265	44	574	2694	11174	45094	180774	723494
12	289	48	626	2938	12186	49178	197146	789018
13	313	52	678	3182	13198	53262	213518	854542
14	337	56	730	3426	14210	57346	229890	920066
15	361	60	782	3670	15222	61430	246262	985590
16	385	64	834	3914	16234	65514	262634	1051114
17	409	68	886	4158	17246	69598	279006	1116638
18	433	72	938	4402	18258	73682	295378	1182162
19	457	76	990	4646	19270	77766	311750	1247686
20	481	80	1042	4890	20282	81850	328122	1313210
21	505	84	1094	5134	21294	85934	344494	1378734
22	529	88	1146	5378	22306	90018	360866	1444258
23	553	92	1198	5622	23318	94102	377238	1509782
24	577	96	1250	5866	24330	98186	393610	1575306
25	601	100	1302	6110	25342	102270	409982	1640830
26	625	104	1354	6354	26354	106354	426354	1706354
27	649	108	1406	6598	27366	110438	442726	1771878
28	673	112	1458	6842	28378	114522	459098	1837402
29	697	116	1510	7086	29390	118606	475470	1902926
30	721	120	1562	7330	30402	122690	491842	1968450
31	745	124	1614	7574	31414	126774	508214	2033974
32	769	128	1666	7818	32426	130858	524586	2099498
33	793	132	1718	8062	33438	134942	540958	2165022
34	817	136	1770	8306	34450	139026	557330	2230546
35	841	140	1822	8550	35462	143110	573702	2296070
36	865	144	1874	8794	36474	147194	590074	2361594
37	889	148	1926	9038	37486	151278	606446	2427118
38	913	152	1978	9282	38498	155362	622818	2492642
39	937	156	2030	9526	39510	159446	639190	2558166
40	961	160	2082	9770	40522	163530	655562	2623690

7.1.9. *Succession Formulae*

.

Collatz preimage arrays (odd classes 1 and 5 mod 6).

We work with two array–like structures, each indexed by *rows* and *columns*. The first array collects preimages of numbers congruent to 1 (mod 6); the second does the same for numbers congruent to 5 (mod 6). In both arrays, the *row number* $r \in \mathbb{N}_0$ is an index into the array (it labels the row), and the *column number* $j \in \mathbb{N}_0$ records the j^{th} preimage.

Let G be the Collatz map and G^j its j^{th} preimage. For $p \in \{1, 5\}$ and row index $r \in \mathbb{N}_0$, define the *row image* (not displayed in the array)

$$n_r^{(p)} := 6r + p.$$

The array itself contains *only preimages.* We index columns by $j \in \mathbb{N}_0$ so that *column* 0 *lists the immediate (preimage-depth* 0*) preimages*, column 1 lists preimages of depth 1, etc.:

$$\mathcal{A}_{r,j}^{(p)} \;:=\; \left\{\, x \in \mathbb{N} \;\middle|\; G^{\,j}(x) = n_r^{(p)}, \;\; x \equiv p \pmod 6 \,\right\}, \qquad r, j \in \mathbb{N}_0.$$

Thus, given the row index r and which array ($p = 1$ or $p = 5$), one recovers the image as $6r + p$, while the table cells $\mathcal{A}_{r,j}^{(p)}$ list only the corresponding preimages.

Array for $p = 1 \pmod 6$ (preimages only).

row r	$j = 0$	$j = 1$	$j = 2$	$\cdots$
0	$\mathcal{A}_{0,0}^{(1)}$	$\mathcal{A}_{0,1}^{(1)}$	$\mathcal{A}_{0,2}^{(1)}$	$\cdots$
1	$\mathcal{A}_{1,0}^{(1)}$	$\mathcal{A}_{1,1}^{(1)}$	$\mathcal{A}_{1,2}^{(1)}$	$\cdots$
2	$\mathcal{A}_{2,0}^{(1)}$	$\mathcal{A}_{2,1}^{(1)}$	$\mathcal{A}_{2,2}^{(1)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Array for $p = 5 \pmod 6$ (preimages only).

row r	$j = 0$	$j = 1$	$j = 2$	$\cdots$
0	$\mathcal{A}_{0,0}^{(5)}$	$\mathcal{A}_{0,1}^{(5)}$	$\mathcal{A}_{0,2}^{(5)}$	$\cdots$
1	$\mathcal{A}_{1,0}^{(5)}$	$\mathcal{A}_{1,1}^{(5)}$	$\mathcal{A}_{1,2}^{(5)}$	$\cdots$
2	$\mathcal{A}_{2,0}^{(5)}$	$\mathcal{A}_{2,1}^{(5)}$	$\mathcal{A}_{2,2}^{(5)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Indexing convention. Column j corresponds to preimage depth j under G. The values $n_r^{(p)} = 6r + p$ are *not* printed in the arrays; they are determined from the row index r and the chosen array ($p = 1$ or $p = 5$).

Below is an example with $p = 1$. Incidentally the column $j = 0$ is composed of numbers which are congruent to $1 \equiv \pmod 8$, all the other columns contain numbers which are congruent to $5 \equiv \pmod 8$. This provides an convenient algorithm to search this table for the row and column of an arbitrary odd number.

r	n	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
0	1	1	5	21	85	341	1365	5461
1	7	9	37	149	597	2389	9557	38229
2	13	17	69	277	1109	4437	17749	70997
3	19	25	101	405	1621	6485	25941	103765
4	25	33	133	533	2133	8533	34133	136533
5	31	41	165	661	2645	10581	42325	169301
6	37	49	197	789	3157	12629	50517	202069
7	43	57	229	917	3669	14677	58709	234837
8	49	65	261	1045	4181	16725	66901	267605
9	55	73	293	1173	4693	18773	75093	300373
10	61	81	325	1301	5205	20821	83285	333141
11	67	89	357	1429	5717	22869	91477	365909
12	73	97	389	1557	6229	24917	99669	398677
13	79	105	421	1685	6741	26965	107861	431445
14	85	113	453	1813	7253	29013	116053	464213
15	91	121	485	1941	7765	31061	124245	496981
16	97	129	517	2069	8277	33109	132437	529749
17	103	137	549	2197	8789	35157	140629	562517
18	109	145	581	2325	9301	37205	148821	595285
19	115	153	613	2453	9813	39253	157013	628053
20	121	161	645	2581	10325	41301	165205	660821

Below is an example with $p = 5$. Here the column $j = 0$ is composed of numbers which are congruent to $3 \equiv \pmod 4$, all the other columns contain numbers which are congruent to $5 \equiv \pmod 8$. This provides an convenient algorithm to search this table for the row and column of an arbitrary odd number.

r	n	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
0	5	3	13	53	213	853	3413	13653
1	11	7	29	117	469	1877	7509	30037
2	17	11	45	181	725	2901	11605	46421
3	23	15	61	245	981	3925	15701	62805
4	29	19	77	309	1237	4949	19797	79189
5	35	23	93	373	1493	5973	23893	95573
6	41	27	109	437	1749	6997	27989	111957
7	47	31	125	501	2005	8021	32085	128341
8	53	35	141	565	2261	9045	36181	144725
9	59	39	157	629	2517	10069	40277	161109
10	65	43	173	693	2773	11093	44373	177493
11	71	47	189	757	3029	12117	48469	193877
12	77	51	205	821	3285	13141	52565	210261
13	83	55	221	885	3541	14165	56661	226645
14	89	59	237	949	3797	15189	60757	243029
15	95	63	253	1013	4053	16213	64853	259413
16	101	67	269	1077	4309	17237	68949	275797
17	107	71	285	1141	4565	18261	73045	292181
18	113	75	301	1205	4821	19285	77141	308565
19	119	79	317	1269	5077	20309	81237	324949
20	125	83	333	1333	5333	21333	85333	341333

Succesive Collatz preimages.

Below are two related tables, related by the Type column in each table. They are the starting point for a method for finding successive preimages. Later we find a general formula for T_n and then a general formula for the row index r_x of the preimage of a number x.

Type	image (mod 6)	row (mod 3)	preimage (mod 6)
ee0	1	0	1
ee1	1	1	1
ee2	1	2	1
eo0	1	0	5
eo1	1	1	5
eo2	1	2	5
oe0	5	0	1
oe1	5	1	1
oe2	5	2	1
oo0	5	0	5
oo1	5	1	5
oo2	5	2	5

Type	T_0	T_1	T_2	T_3
ee0	0	$2^1 \cdot 7$	$2^7 \cdot 7 + 2^1 \cdot 7$	$2^{13} \cdot 7 + 2^7 \cdot 7 + 2^1 \cdot 7$
ee1	6	$2^3 \cdot 49 + 6$	$2^9 \cdot 49 + 2^3 \cdot 49 + 6$	$2^{15} \cdot 49 + 2^9 \cdot 49 + 2^3 \cdot 49 + 6$
ee2	46	$2^5 \cdot 91 + 46$	$2^{11} \cdot 91 + 2^5 \cdot 91 + 46$	$2^{17} \cdot 91 + 2^{11} \cdot 91 + 2^5 \cdot 91 + 46$
eo0	0	$2^3 \cdot 7$	$2^9 \cdot 7 + 2^3 \cdot 7$	$2^{15} \cdot 7 + 2^9 \cdot 7 + 2^3 \cdot 7$
eo1	24	$2^5 \cdot 49 + 24$	$2^{11} \cdot 49 + 2^5 \cdot 49 + 24$	$2^{17} \cdot 49 + 2^{11} \cdot 49 + 2^5 \cdot 49 + 24$
eo2	2	$2^1 \cdot 91 + 2$	$2^7 \cdot 91 + 2^1 \cdot 91 + 2$	$2^{13} \cdot 91 + 2^7 \cdot 91 + 2^1 \cdot 91 + 2$
oe0	2	$2^2 \cdot 35 + 2$	$2^8 \cdot 35 + 2^2 \cdot 35 + 2$	$2^{14} \cdot 35 + 2^8 \cdot 35 + 2^2 \cdot 35 + 2$
oe1	1	$2^0 \cdot 77 + 1$	$2^6 \cdot 77 + 2^0 \cdot 77 + 1$	$2^{12} \cdot 77 + 2^6 \cdot 77 + 2^0 \cdot 77 + 1$
oe2	30	$2^4 \cdot 119 + 30$	$2^{10} \cdot 119 + 2^4 \cdot 119 + 30$	$2^{16} \cdot 119 + 2^{10} \cdot 119 + 2^4 \cdot 119 + 30$
oo0	8	$2^4 \cdot 35 + 8$	$2^{10} \cdot 35 + 2^4 \cdot 35 + 8$	$2^{16} \cdot 35 + 2^{10} \cdot 35 + 2^4 \cdot 35 + 8$
oo1	4	$2^2 \cdot 77 + 4$	$2^8 \cdot 77 + 2^2 \cdot 77 + 4$	$2^{14} \cdot 77 + 2^8 \cdot 77 + 2^2 \cdot 77 + 4$
oo2	1	$2^0 \cdot 119 + 1$	$2^6 \cdot 119 + 2^0 \cdot 119 + 1$	$2^{12} \cdot 119 + 2^6 \cdot 119 + 2^0 \cdot 119 + 1$

General Case:

. All the succession formulae have the following template:

If $T_n = A + C\sum_{j=0}^{n-2} 64^j$ (so $T_1 = A$ and the first added block is C), then $T_n = A + \frac{C}{63}\big(64^{n-1} - 1\big)$, $\qquad T_n - 64\,T_{n-1}$

Or more generally:

If $T_n = A + C\sum_{j=0}^{n-2} r^j$, then $T_n = A + \frac{C}{r-1}\big(r^{n-1} - 1\big)$, $\qquad T_n - r\,T_{n-1} = C - (r-1)A$.

ee0:

. We require $T_1 = 0$

$$T_n \coloneqq \sum_{j=0}^{n-2} 7 \cdot 2^{\,1+6j} \qquad (n \geq 1).$$

Thus

$$T_n = 7 \cdot 2 \sum_{j=0}^{n-2} \bigl(2^6\bigr)^j = 14 \sum_{j=0}^{n-2} 64^{\,j} = 14\, \frac{64^{\,n-1} - 1}{64 - 1} = \frac{2}{9}\bigl(64^{\,n-1} - 1\bigr).$$

ee1:

. Define the sequence by

$$T_n \coloneqq 6 \;+\; 49 \sum_{j=0}^{n-2} 2^{\,3+6j} \;=\; 6 \;+\; 392 \sum_{j=0}^{n-2} 64^{\,j} \qquad (n \geq 1).$$

Summing the geometric series yields the closed form

$$T_n \;=\; 6 \;+\; \frac{392}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{56}{9}\, 64^{\,n-1} \;-\; \frac{2}{9}, \qquad n \geq 1,$$

which is equivalent to the recurrence

$$T_1 = 6, \qquad T_n = 64\, T_{n-1} + 14 \quad (n \geq 2).$$

ee2:

. Define, for $n \geq 1$,

$$T_n \coloneqq 46 \;+\; 91 \sum_{j=0}^{n-2} 2^{\,5+6j} \;=\; 46 \;+\; 2912 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives

$$T_n \;=\; 46 \;+\; \frac{2912}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{416}{9}\, 64^{\,n-1} \;-\; \frac{2}{9},$$

which is equivalent to the recurrence

$$T_1 = 46, \qquad T_n = 64\, T_{n-1} + 14 \quad (n \geq 2).$$

eo0:

. Define, for $n \geq 1$,

$$T_n \coloneqq 7 \sum_{j=0}^{n-2} 2^{\,3+6j} \;=\; 56 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives the closed form

$$T_n \;=\; \frac{56}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{8}{9}\,\bigl(64^{\,n-1} - 1\bigr).$$

Equivalently, the recurrence is

$$T_1 = 0, \qquad T_n = 64\, T_{n-1} + 56 \quad (n \geq 2).$$

eo1:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 24$),

$$T_n := 24 + 49 \sum_{j=0}^{n-2} 2^{5+6j} = 24 + 1568 \sum_{j=0}^{n-2} 64^j.$$

Summing the geometric series gives the closed form

$$T_n = 24 + \frac{1568}{63}\bigl(64^{n-1} - 1\bigr) = \frac{224}{9} 64^{n-1} - \frac{8}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 24, \qquad T_n = 64\, T_{n-1} + 56 \quad (n \geq 2).$$

eo2:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 2$),

$$T_n := 2 + 91 \sum_{j=0}^{n-2} 2^{1+6j} = 2 + 182 \sum_{j=0}^{n-2} 64^j.$$

Summing the geometric series yields the closed form

$$T_n = 2 + \frac{182}{63}\bigl(64^{n-1} - 1\bigr) = \frac{26}{9} 64^{n-1} - \frac{8}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 2, \qquad T_n = 64\, T_{n-1} + 56 \quad (n \geq 2).$$

oe0:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 2$),

$$T_n := 2 + 35 \sum_{j=0}^{n-2} 2^{2+6j} = 2 + 140 \sum_{j=0}^{n-2} 64^j.$$

Summing the geometric series gives the closed form

$$T_n = 2 + \frac{140}{63}\bigl(64^{n-1} - 1\bigr) = \frac{20}{9} 64^{n-1} - \frac{2}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 2, \qquad T_n = 64\, T_{n-1} + 14 \quad (n \geq 2).$$

oe1:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 1$),

$$T_n := 1 + 77 \sum_{j=0}^{n-2} 2^{6j} = 1 + 77 \sum_{j=0}^{n-2} 64^j.$$

Summing the geometric series gives the closed form

$$T_n = 1 + \frac{77}{63}\bigl(64^{n-1} - 1\bigr) = \frac{11}{9} 64^{n-1} - \frac{2}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 1, \qquad T_n = 64\,T_{n-1} + 14 \quad (n \geq 2).$$

oe2:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 30$),

$$T_n \coloneqq 30 \;+\; 119 \sum_{j=0}^{n-2} 2^{\,4+6j} \;=\; 30 \;+\; 1904 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives the closed form

$$T_n \;=\; 30 \;+\; \frac{1904}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{272}{9}\,64^{\,n-1} \;-\; \frac{2}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 30, \qquad T_n = 64\,T_{n-1} + 14 \quad (n \geq 2).$$

oo0:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 8$),

$$T_n \coloneqq 8 \;+\; 35 \sum_{j=0}^{n-2} 2^{\,4+6j} \;=\; 8 \;+\; 560 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives the closed form

$$T_n \;=\; 8 \;+\; \frac{560}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{80}{9}\,64^{\,n-1} \;-\; \frac{8}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 8, \qquad T_n = 64\,T_{n-1} + 56 \quad (n \geq 2).$$

oo1:

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 4$),

$$T_n \coloneqq 4 \;+\; 77 \sum_{j=0}^{n-2} 2^{\,2+6j} \;=\; 4 \;+\; 308 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives the closed form

$$T_n \;=\; 4 \;+\; \frac{308}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{44}{9}\,64^{\,n-1} \;-\; \frac{8}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 4, \qquad T_n = 64\,T_{n-1} + 56 \quad (n \geq 2).$$

<u>oo2:</u>

. Define, for $n \geq 1$ (empty sum $= 0$ so $T_1 = 1$),

$$T_n \coloneqq 1 \;+\; 119 \sum_{j=0}^{n-2} 2^{\,6j} \;=\; 1 \;+\; 119 \sum_{j=0}^{n-2} 64^{\,j}.$$

Summing the geometric series gives the closed form

$$T_n \;=\; 1 \;+\; \frac{119}{63}\bigl(64^{\,n-1} - 1\bigr) \;=\; \frac{17}{9}\,64^{\,n-1} \;-\; \frac{8}{9}, \qquad n \geq 1.$$

Equivalently, the recurrence is

$$T_1 = 1, \qquad T_n = 64\,T_{n-1} + 56 \quad (n \geq 2).$$

<u>Applications</u>

Table 4. Functions, 1 (mod 6)

Type	Augend F_m	Addend $T_n, n = c+1$
$ee0$	2^{2+6m}	$\frac{2}{9}\,64^{\,n-1} - \frac{2}{9}$
$ee1$	2^{4+6m}	$\frac{56}{9}\,64^{\,n-1} - \frac{2}{9}$
$ee2$	2^{6+6m}	$\frac{416}{9}\,64^{\,n-1} - \frac{2}{9}$
$eo0$	2^{4+6m}	$\frac{8}{9}\,64^{\,n-1} - \frac{8}{9}$
$eo1$	2^{6+6m}	$\frac{224}{9}\,64^{\,n-1} - \frac{8}{9}$
$eo2$	2^{2+6m}	$\frac{26}{9}\,64^{\,n-1} - \frac{8}{9}$

Table 5. Functions, 5 (mod 6)

Type	Augend F_m	Addend $T_n, n = c+1$
$oe0$	2^{3+6m}	$\frac{20}{9}\,64^{\,n-1} - \frac{2}{9}$
$oe1$	2^{1+6m}	$\frac{11}{9}\,64^{\,n-1} - \frac{2}{9}$
$oe2$	2^{5+6m}	$\frac{272}{9}\,64^{\,n-1} - \frac{2}{9}$
$oo0$	2^{5+6m}	$\frac{80}{9}\,64^{\,n-1} - \frac{8}{9}$
$oo1$	2^{3+6m}	$\frac{44}{9}\,64^{\,n-1} - \frac{8}{9}$
$oo2$	2^{1+6m}	$\frac{17}{9}\,64^{\,n-1} - \frac{8}{9}$

Given an odd integer $x_{image} \geq 0$, we define:

$$r_{x_{image}} = x_{image_6} := \left\lfloor \frac{x_{image}}{6} \right\rfloor.$$

$$c_{image} := \left\lfloor \frac{\lfloor x_{image}/6 \rfloor}{3} \right\rfloor = \left\lfloor \frac{x_{image_6}}{3} \right\rfloor = \left\lfloor \frac{x_{image}}{18} \right\rfloor.$$

Using $r_{x_{image}}$ and c_{image} we identify the type of x from among $type \in \{ee0, ee1, ee2, eo0, eo1, eo2, oe0, oe1, oe2, oo0, oo1, oo2\}$. Since we use two input variables $r_{x_{image}}$ and c_{image} there are two options for the output $r_{x_{preimage}}$. For example if $r_{x_{image}} = 1$ and $c_{image} = 0$ the preimage would be of both $type \in \{ee0, eo0\}$.

Finally to get the actual preimage row $r_{x_{preimage}} = F_m \cdot c_{image} + T_n, m \in \mathbb{N}_0, n = m + 1$, the actual $x_{preimage} = 6 \cdot r_{x_{preimage}} + p$ where p is 1 when $x_{preimage_6} \equiv x_{preimage} \in 1 \equiv \pmod 6$ and 5 when $x_{preimage_6} \equiv x_{preimage} \in 5 \equiv \pmod 6$, the $x_{preimage_6}$ is available in the key table above(column: *preimage* (mod 6)). The equations above can further be consolidated into a single equation as below,

ee0.

$$A_(m, n) \;:=\; \frac{\left(9m \cdot 2^2 + 2\right) 64^n + (0 - 2)}{9}$$

ee1.

$$A_(m, n) \;:=\; \frac{\left(9m \cdot 2^4 + 56\right) 64^n + (54 - 56)}{9}$$

ee2.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^6 + 416\right) 64^n + (414 - 416)}{9}$$

eo0.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^4 + 8\right) 64^n + (0 - 8)}{9}$$

eo1.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^6 + 224\right) 64^n + (216 - 224)}{9}$$

eo2.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^2 + 26\right) 64^n + (18 - 26)}{9}$$

oe0.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^3 + 20\right) 64^n + (18 - 20)}{9}$$

oe1.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^1 + 11\right) 64^n + (9 - 11)}{9}$$

oe2.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^5 + 272\right) 64^n + (270 - 272)}{9}$$

oo0.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^5 + 80\right) 64^n + (72 - 80)}{9}$$

oo1.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^3 + 44\right) 64^n + (36 - 44)}{9}$$

oo2.

$$A_(m,n) \;\coloneqq\; \frac{\left(9m \cdot 2^1 + 17\right) 64^n + (9 - 17)}{9}$$

In practice the parameters m, n also restrict the columns from which the preimage could be taken.

Below is an example with $x_{image} \equiv 1 \pmod 6$. Here the column is represented by j. Each value in each column of any row represents the preimage of the previous value in the row. All the row values are $v \equiv 1 \pmod 6$.

r_{image}	x_{image}	$j = 0$ $m = 0$ $n = 1$	$j = 1$ $m = 0$ $n = 1$	$j = 2$ $m = 0$ $n = 1$	$j = 3$ $m = 0$ $n = 1$
0	1	1	1	1	1
1	7	37	49	1045	1393
2	13	277	1477	1969	10501
3	19	25	133	709	3781
4	25	133	709	3781	5041
5	31	661	14101	75205	100273
6	37	49	1045	1393	7429
7	43	229	4885	26053	138949
8	49	1045	1393	7429	158485
9	55	73	97	517	11029
10	61	325	433	577	769
11	67	1429	7621	40645	54193
12	73	97	517	11029	235285
13	79	421	2245	47893	1021717
14	85	1813	38677	825109	4400581
15	91	121	2581	13765	293653
16	97	517	11029	235285	1254853
17	103	2197	2929	62485	333253
18	109	145	193	4117	87829
19	115	613	817	4357	5809
20	121	2581	13765	293653	391537

Below is an example with $x_{image} \equiv 5 \pmod 6$. Here the column is represented by j. Each value in each column of any row represents the preimage of the previous value in the row. All the row values are $v \equiv 5 \pmod 6$.

r_{image}	x_{image}	$j = 0$ $m = 0$ $n = 1$	$j = 1$ $m = 0$ $n = 1$	$j = 2$ $m = 0$ $n = 1$	$j = 3$ $m = 0$ $n = 1$
0	5	53	35	23	245
1	11	29	77	821	2189
2	17	11	29	77	821
3	23	245	653	6965	4643
4	29	77	821	2189	5837
5	35	23	245	653	6965
6	41	437	4661	3107	8285
7	47	125	83	221	2357
8	53	35	23	245	653
9	59	629	419	4469	47669
10	65	173	461	1229	13109
11	71	47	125	83	221
12	77	821	2189	5837	62261
13	83	221	2357	1571	16757
14	89	59	629	419	4469
15	95	1013	10805	115253	76835
16	101	269	179	119	317
17	107	71	47	125	83
18	113	1205	803	2141	1427
19	119	317	845	563	6005
20	125	83	221	2357	1571

In general.
For integers $n \geq 0$ and $m \in \mathbb{Z}$, and parameters $\alpha \in \{1,2,3,4,5,6\}$, $c \in \{-2,-8\}$, and $\beta \in \mathbb{Z}$ satisfying $\beta \equiv -c \pmod 9$, define

$$F_{\alpha,\beta,c}(n,m) \;:=\; \frac{\bigl(9\,m\,2^{\alpha} + \beta\bigr)\,64^{\,n} + c}{9}.$$

Integrality. Since $64^n \equiv 1 \pmod 9$,

$$(9m2^{\alpha} + \beta)64^n + c \equiv \beta + c \equiv 0 \pmod 9,$$

so $F_{\alpha,\beta,c}(n,m) \in \mathbb{Z}$.

Initial value and recurrence in n.

$$F_{\alpha,\beta,c}(0,m) = 2^{\alpha}m + \frac{\beta + c}{9}, \qquad F_{\alpha,\beta,c}(n+1,m) = 64\,F_{\alpha,\beta,c}(n,m) \;-\; 7c.$$

The series as instances.

$$(+14 \text{ family}, c = -2): \quad (\alpha,\beta) \in \{(2,2),(4,56),(6,416),(3,20),(1,11),(5,272)\},$$
$$(+56 \text{ family}, c = -8): \quad (\alpha,\beta) \in \{(4,8),(6,224),(2,26),(5,80),(3,44),(1,17)\}.$$

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