AN INVERSE CALCULUS FOR THE ODD LAYER OF THE COLLATZ MAP

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Abstract.
We develop a finite–state, word–based framework for the accelerated odd Collatz map $U(y) = \frac{3y+1}{2^{\nu_2(3y+1)}}$. Every admissible token (one of $\Psi, \psi, \omega, \Omega$) corresponds to a fixed "row" with parameters $(\alpha, \beta, c, \delta)$ such that for inputs $x = 18m + 6j + p_6$ the update $x' = 6F(p, m) + \delta$ with

$$F(p, m) = \frac{(9m\,2^{\alpha} + \beta)\,64^{\,p} + c}{9}$$

satisfies the forward identity $3x' + 1 = 2^{\alpha+6p}x$. Hence $U(x') = x$ at every step, providing a per–step certificate independent of the starting value. We formalize *steering* by same–family padding: short words that (i) strictly increase the 2-adic valuation of the affine slope and (ii) control the intercept modulo 2 and modulo 3. This yields a deterministic lifting procedure that reduces reachability modulo $3 \cdot 2^{K+1}$ to a linear congruence once modulo 3 is aligned; a 2-adic refinement then promotes compatible residue solutions to an exact integer solution for a fixed word. We include a reference implementation that verifies each row identity, the mod–3 steering action, and example witnesses modulo 24.

The main contribution is a unified, certified inverse–word calculus on the odd layer together with explicit steering gadgets that turn residue targeting into solvable congruences. Because the resulting program would imply convergence of the odd Collatz dynamics to 1, we provide machine–checkable tests and artifacts to facilitate scrutiny.

Contents

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1. Related work: inverse trees, 2-adic lifting, and modular routing

Our approach—finite word semantics on the odd layer, certified one–step inverses, and congruence–based "steering" to lift residues from $M_K = 3 \cdot 2^K$ to M_{K+1}—sits alongside several established techniques.

Mod-2^k analysis and lifting. Garner studied the $3n+1$ dynamics modulo powers of two, organizing inverse branches by congruence classes and effectively "lifting" structure from 2^k to 2^{k+1} [Gar81]. Our use of the unified rows with a column–lift parameter p (which multiplies the 2–adic slope by 2^{6p}) and the residue steering gadgets plays a similar role: we solve linear congruences for m to pass from M_K to M_{K+1} while preserving certified inverses at each step.

Inverse trees and predecessor sets. Wirsching's monograph develops the inverse (predecessor) tree of the $3n+1$ function as a dynamical system, with emphasis on structure, measures, and asymptotics on inverse branches [Wir98]. Conceptually, our move alphabet and per–row affine forms are a finite–state presentation of those inverse branches: each token certifies $U(x') = x$ and the composition yields an affine map in the "index" m, which we then route by residues M_K.

The 2-adic viewpoint and conjugacies. Bernstein and Lagarias constructed a 2-adic conjugacy map relating the odd–accelerated Collatz dynamics to a Bernoulli-like shift [BL96]. Our p–lift (multiplying by 2^{6p}) and the parity/valuation steering reflect this same 2–adic continuity: column–lifts shift 2–adic scale, while steering gadgets tune intercept parity to land on prescribed residue classes.

Symmetries and autoconjugacy. Monks and Yazinski analyzed autoconjugacies of the $3x+1$ function and their implications for orbit structure [MY04]. While our framework is more combinatorial/affine, the way we keep the family pattern fixed (Lemma 11) and exploit same–family padding resonates with their use of structural symmetries.

Surveys and context. For broad background and additional modular/density insights, see [Lag10; Ter76; Ter79]; for 2–adic heuristics and continuity themes, see [Gou97; Nat96]. These perspectives motivate our use of 2–adic "padding" and linear congruences as lifting mechanisms.

What is new here. Our contribution is a single unified p=0 inverse table on the odd layer (Table 5) with a per–step column–lift $p \geq 0$ and explicit steering gadgets that (i) raise v_2 of the word's slope and (ii) toggle the intercept parity, ensuring solvability of the lifting congruences modulo M_{K+1} while keeping each step certified by $U(x') = x$.

Contributions.

- **One-table, word-driven inverse calculus on the odd layer.** We give a unified p=0 row table with closed forms $x' = 6F(0, m) + \delta$ indexed only by (s, j, m). Once a token and (s, j) are fixed, the step is fully determined and the forward identity $3x' + 1 = 2^{\alpha}x$ holds by construction (Lemma 10).
- **Column-lift p that preserves routing while scaling the 2-adic slope.** The parameter p multiplies the slope by 2^{6p} without changing the token type or output family, yielding a single mechanism that subsumes whole towers of congruence tables (Lemmas 17–19).
- **CRT tag for transparent indexing.** The tag $t = (x - 1)/2$ (equivalently $(3x + 1 - 4)/6$) makes family detection and indices (s, j, m) linear in t (Corollary 2), simplifying routing proofs.
- **Steering gadgets that control v_2, $B \bmod 2$, and $B \bmod 3$.** Short same-family words provably boost the slope's 2-adic valuation and toggle the affine intercept $B \bmod 2$, ensuring solvability of the lifting congruence at each modulus (Lemmas 20 and 56, App. 36.).
- **From small witnesses to all moduli and exact integers.** Starting at M_3=24, we give a deterministic induction $M_K \to M_{K+1}$ (Lemma 54) that reaches every odd residue with certified steps, and then a 2-adic refinement to hit any prescribed odd integer exactly (Theorem 35).
- **Row-level invariance certificates.** We isolate a mod-54 one-step invariance (Lemma 15) that explains why fixed tokens reselect the same next row across many starts, aiding certification and automation.
- **Executable, per-step certificates.** A reference implementation emits step traces and verifies $U(x') = x$ at each step, making all claims reproducible from the table (App. C).

Relation to prior techniques.

- **Versus classical modular inverse-tree analyses (Terras, Lagarias, *etc.*).** Prior work develops rich residue classifications and stopping-time bounds; our contribution is a *single* finite-state table with a word calculus and an explicit steering mechanism that turns residue reachability into solvable linear congruences with guaranteed 2-adic headroom.
- **Versus 2-adic dynamical viewpoints (Gouvêa, Nathanson).** Earlier 2-adic studies illuminate topology, measures, and cycles. We use the 2-adic setting constructively: the slope/offset control plus 2-adic completeness converts an infinite ladder of congruences into an exact integer solution anchored to a concrete word.
- **Versus "energy"/almost-everywhere results (Tao 2019 and follow-ups).** These show near-monotone behavior for a density-one set via probabilistic/analytic Lyapunov methods. Our approach is entirely combinatorial and constructive: for each target residue (and ultimately each odd integer) we produce a finite word and certify every inverse step by $3x' + 1 = 2^{\alpha+6p}x$.

Scope note. Standard ingredients (accelerated map U, parity splitting, v_2, and modular routing) are classical; the novelty here is the *unified word/table formalism* with a *routing-preserving p-lift* and *steering gadgets* (including mod-3 control) that together enable a fully constructive lifting from mod 24 to exact integers with stepwise certificates.

Main claim and method. Our main claim (Theorem 37) is that every odd $x \equiv 1, 5 \pmod 6$ reaches 1 in finitely many accelerated odd Collatz steps. The method is modular: (i) certify row-level inverses $U(x') = x$ (Lemma 10); (ii) show any admissible word yields an affine form in m with controlled terminal family (Lemma 12 and Lemma 11); (iii) furnish base witnesses modulo 24 (Table 13); (iv) use same-family *steering gadgets* to raise $v_2(A)$ and control B mod 2 and B mod 3 (Lemmas 20, 56); (v) lift residues $M_K \to M_{K+1}$ (Lemma 54, Theorem 30); (vi) pass from residues to exact integers by 2-adic refinement (Theorem 35). For a discussion addressing common misreadings, see Section 36.

2. Notation, indices, and moves

To unify all Collatz inverse odd orbits we work with an affine form indexed by row parameters (α, β, c) and an orbit–type offset $\delta \in \{1, 5\}$. For any nonnegative integer $p = 0, 1, 2, \ldots$ and $m = \lfloor x/18 \rfloor$, define

$$F_{\alpha,\beta,c}(p, m) \;:=\; \frac{(9m\,2^{\alpha} + \beta)\,64^{\,p} + c}{9} \;=\; 2^{\alpha+6p} m + \frac{\beta\,64^{\,p} + c}{9}, \qquad x' \;=\; 6\,F_{\alpha,\beta,c}(p, m) + \delta.$$

Here p is a *column–lift* that preserves routing/type (and δ) while multiplying the 2-adic slope by 2^{6p}; integrality of $F_{\alpha,\beta,c}(p, m)$ follows from $64 \equiv 1 \pmod 9$. The base table is recovered at $p = 0$ (so $F_{\alpha,\beta,c}(0, m) = 2^{\alpha} m + (\beta + c)/9$), and in all cases the forward identity

$$3x' + 1 \;=\; 2^{\alpha+6p}\,x$$

holds, hence $U(x') = x$. In the following we use orbit and step interchangeably.

Let x be odd with $x \not\equiv 3 \pmod 6$ and define

$$s(x) = \begin{cases} \text{e}, & x \equiv 1 \pmod 6, \\ \text{o}, & x \equiv 5 \pmod 6, \end{cases} \qquad r = \left\lfloor \frac{x}{6} \right\rfloor, \quad j = r \bmod 3 \in \{0, 1, 2\}, \quad m = \left\lfloor \frac{x}{18} \right\rfloor.$$

We use the accelerated odd Collatz map $U(y) = \dfrac{3y+1}{2^{\nu_2(3y+1)}}$, standard in the Collatz literature [Lag10].

The move alphabet is $\mathcal{A} = \{\Psi, \psi, \omega, \Omega\}$ with type mapping

$$\Psi \leftrightarrow \texttt{ee}, \qquad \psi \leftrightarrow \texttt{eo}, \qquad \omega \leftrightarrow \texttt{oe}, \qquad \Omega \leftrightarrow \texttt{oo}.$$

Admissibility by family: if $s(x) = \text{e}$ we may use Ψ or ψ; if $s(x) = \text{o}$ we may use ω or Ω.

A CRT tag for odds, and re-indexing by t. Define, for odd x,

$$y := 3x + 1, \qquad t := \frac{y - 4}{6} = \frac{x - 1}{2} \in \mathbb{Z}.$$

3. Glossary and notation

U**:** Accelerated Collatz map $U(n) = (3n+1)/2^{v_2(3n+1)}$ on odd n.

Family e/o**:** Terminal residue class modulo 6: e for $\equiv 1 \pmod 6$, o for $\equiv 5 \pmod 6$.

Router j**:** $j = \lfloor x/6 \rfloor \bmod 3 \in \{0, 1, 2\}$ chooses the table row.

Internal index u**:** $u = \lfloor x/18 \rfloor$ feeds the next token.

(A, B, δ)**:** Linear surrogate: $x = 6(Am + B) + \delta$ at each step.

M_K**:** Modulus $M_K = 3 \cdot 2^K$.

Pinning: A last row with exponent $\alpha_p \geq K$ forces $x \equiv 6k^{(p)} + \delta \pmod{M_K}$ independently of m.

Table 1. Notation used throughout. Families e, o are $1, 5 \pmod 6$. Indices j, m come from $x = 18m + 6j + p_6$ with $p_6 \in \{1, 5\}$.

Symbol	**Meaning**
$U(y) = \dfrac{3y+1}{2^{\nu_2(3y+1)}}$	Accelerated odd Collatz map (odd layer).
x	Current odd, always $x \equiv 1, 5 \pmod 6$ on the odd layer.
$s(x) \in \{\mathrm{e}, \mathrm{o}\}$	Family of x: e if $x \equiv 1 \pmod 6$, o if $x \equiv 5 \pmod 6$.
$j = \lfloor \frac{x}{6} \rfloor \bmod 3$	Row index (next-row selector), $j \in \{0, 1, 2\}$.
$m = \lfloor \frac{x}{18} \rfloor$	Coarse index used in the closed forms $x'(m)$.
$p \in \mathbb{Z}_{\geq 0}$	Column-lift parameter; each step multiplies the forward power by 2^{6p}.
α, β, c, δ	Row parameters; $\delta \in \{1, 5\}$ is the output family offset.
$k = \dfrac{\beta + c}{9}$	One-step constant at p=0; integrality since $\beta + c \equiv 0 \pmod 9$.
$F(p, m) = \dfrac{(9m\, 2^\alpha + \beta)\, 64^p + c}{9}$	Lifted per-row form; integral since $64 \equiv 1 \pmod 9$.
$x' = 6F(p, m) + \delta$	One-step preimage; satisfies $3x' + 1 = 2^{\alpha + 6p} x$.
$\nu_2(n)$	2-adic valuation of n.
$t = \dfrac{x-1}{2}$	CRT tag (reindexing); bijection $x = 2t + 1$.
$\mathcal{A} = \{\Psi, \psi, \omega, \Omega\}$	Token alphabet; types `ee`, `eo`, `oe`, `oo` respectively.
$W \in \mathcal{A}^*$	A word (sequence of tokens).
$x_W(m) = 6(A_W m + B_W) + \delta_W$	Affine form after a word; $A_W = 3 \cdot 2^{\alpha(W)}$.
$M_K = 3 \cdot 2^K$	Working modulus for lifting; odd residues split into $\mathcal{E}_K, \mathcal{O}_K$.
$\mathcal{E}_K, \mathcal{O}_K$	$\mathcal{E}_K = \{1 + 6t \bmod M_K\}$, $\mathcal{O}_K = \{5 + 6t \bmod M_K\}$.

4. Standing assumptions and conventions

We enumerate the ambient assumptions used throughout. None of these are Collatz–specific hypotheses; they are standard arithmetic facts and explicitly verified table properties.

(A1) **Universe and variables.** All variables are integers unless noted. We work on the *odd layer*: inputs x are odd with $x \geq 1$. The column parameter $p \in \mathbb{Z}_{\geq 0}$, and the step indices are

$$m = \left\lfloor \frac{x}{18} \right\rfloor, \qquad j = \left\lfloor \frac{x}{6} \right\rfloor \bmod 3, \qquad x = 18m + 6j + p_6, \quad p_6 \in \{1, 5\}.$$

(A2) **Accelerated odd map.** We use $U(y) = \dfrac{3y+1}{2^{\nu_2(3y+1)}}$. For odd y, one has $3y + 1 \equiv 4 \pmod 6$, hence $U(y) \equiv 1$ or $5 \pmod 6$.

(A3) **CRT tag.** For odd x, the tag $t = \dfrac{3x + 1 - 4}{6} = \dfrac{x-1}{2} \in \mathbb{Z}$ is used only as a reindexing device; it is bijective via $x = 2t + 1$.

(A4) **Row parameter table is integral/consistent.** For every row $(\alpha, \beta, c, \delta)$ in Table 4:

$$k = \frac{\beta + c}{9} \in \mathbb{Z}, \qquad \delta = \begin{cases} 1, & \texttt{*e} \\ 5, & \texttt{*o} \end{cases}$$

so that $F(0, m) = 2^\alpha m + k$ and $x' = 6F(0, m) + \delta$ are integer-valued.

(A5) **Column lifts are integral.** For $p \geq 0$,

$$F_{\alpha,\beta,c}(p, m) = \frac{(9m\, 2^\alpha + \beta)\, 64^p + c}{9} \in \mathbb{Z} \quad \text{since } 64 \equiv 1 \pmod 9.$$

(A6) **Per–row odd–forward identity.** For every admissible row and $p \geq 0$,

$$3x' + 1 = 2^{\alpha + 6p}\, x,$$

hence $U(x') = x$. (Proved in the text; used as a stepwise certificate.)

(A7) **Word affinity and routing.** Composing admissible rows yields an affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$ with $A_W = 3 \cdot 2^{\alpha(W)}$. Family routing (e/o) depends only on the token's type (`ee`, `eo`, `oe`, `oo`), not on m or p.

(A8) **Steering gadgets exist and are explicit.** There are short same–family composites that (i) raise $v_2(A_W)$ arbitrarily (by repetition) and (ii) provide a parity toggle $B_W \mapsto B_W + 1 \pmod 2$ at $p = 0$. (Concrete tokens are listed in Appendix A; e.g. rows ω_1 and Ω_2 have odd $k = (\beta + c)/9$, enabling the toggle.)

(A9) **Lifting over powers of two uses only standard facts.** We use: solvability of linear congruences $Am \equiv b \pmod{2^K}$; nested lifting to 2^{K+1} (choose a solution compatible modulo 2^{K-1}); and completeness of $\mathbb{Z}_2$ to pass from compatible residues to an integer m. No heuristic or distributional assumptions are used.

(A10) **Base witnesses are explicit (no hidden computation).** The eight mod 24 classes $\{1, 5, 7, 11, 13, 17, 19, 23\}$ are each accompanied by a specific finite word W_r (Table 13), verified stepwise via $U(x') = x$. The proof does not rely on unverifiable large-scale searches.

(A11) **Scope relative to the classical map.** All statements are on the odd layer for U. For the classical Collatz map, even runs are removed by dividing out powers of two between odd iterates; the conclusions then transfer verbatim.

Non-assumptions. We do not assume (i) the Collatz conjecture itself, (ii) any stochastic/heuristic model for the orbit, or (iii) density or randomness properties of residue classes. All steps are constructive and finitely checkable.

5. CRT-tag calculus and odd-step drift

We collect here the normalization fact for the odd layer via the Chinese Remainder Theorem (*CRT tag*), and the resulting closed-form expressions for the single-step drift on the odd layer. The parameters (α, p) below match those used in the row/lift transform $F_{\alpha,\beta,c}(p, m)$.

Lemma 1 (CRT tag for odd inputs)**.** *For odd x one has $3x + 1 \equiv 4 \pmod 6$ and the tag $t = \dfrac{3x+1-4}{6} = \dfrac{x-1}{2}$ is an integer. Moreover, the map $x \mapsto t$ is a bijection between odd integers and all integers via $x = 2t + 1$.*

Proposition 1 (Explicit difference of CRT tags along the odd Collatz map)**.** *Let x be odd and let $x' = \boldsymbol{U}(x) = \dfrac{3x+1}{2^k}$ be the odd Collatz image with $2^k \| (3x + 1)$ and $k \geq 1$. For odd y, write the CRT tag as $t(y) = \dfrac{3y+1-4}{6} = \dfrac{y-1}{2}$ (Lemma 1), and set*

$$d \;:=\; t(x') - t(x), \qquad r := \left\lfloor \frac{x}{6} \right\rfloor.$$

Then d is given by

$$\textbf{(e)} \quad \text{If } x \equiv 1 \pmod 6, \qquad d \;=\; r\big(2^{\alpha+6p} - 3\big)\, 4^p \;+\; 2\, q_p,$$

$$\textbf{(o)} \quad \text{If } x \equiv 5 \pmod 6, \qquad d \;=\; r\big(2^{\alpha+6p} - 3\big)\, 4^p \;+\; 5\, q_p \;-\; 1,$$

where

$$p \in \mathbb{Z}_{\geq 0}, \qquad \alpha \text{ and } p \text{ are the same parameters as in } F_{\alpha,\beta,c}(p, m) \text{ (see its definition in this paper)},$$

and $(q_p)_{p\geq 0}$ is given by $q_0 = 0$, $q_{p+1} = 4q_p + 1$, equivalently $q_p = \dfrac{4^p - 1}{3}$. Moreover,

$$x' - x \;=\; 2\, d.$$

Proof sketch. By Lemma 1, $t(y) = \frac{y-1}{2}$ for odd y, hence $d = t(x') - t(x) = \frac{x'-x}{2}$. Writing $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$ and expanding $x' = \frac{3x+1}{2^k}$ along the odd layer yields the two residue-conditioned expressions. The parameters α and p coincide with those in the transform $F_{\alpha,\beta,c}(p, m)$ used in the row/lift framework; q_p captures the cumulative geometric contribution $1 + 4 + \cdots + 4^{p-1}$. □

Corollary 1 (Explicit x'–x relation)**.** *Let x be odd and let $x' = \boldsymbol{U}(x)$ denote its odd Collatz image. Write $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$ and $r = \lfloor x/6 \rfloor$. Let $p \in \mathbb{Z}_{\geq 0}$ and α be the same parameters as in $F_{\alpha,\beta,c}(p, m)$, and set*

$$K := \left(2^{\alpha+6p} - 3\right) 4^p \quad \text{and} \quad q_p := \frac{4^p - 1}{3}.$$

Then the following hold:

$$\textit{(e) If } x \equiv 1 \pmod 6 \ (\varepsilon = 1), \qquad x' \ = \ x \ + \ 2r\,K \ + \ 4\,q_p,$$

$$\textit{(o) If } x \equiv 5 \pmod 6 \ (\varepsilon = 5), \qquad x' \ = \ x \ + \ 2r\,K \ + \ 10\,q_p \ - \ 2.$$

Equivalently, eliminating $r = \frac{x-\varepsilon}{6}$ gives the affine-in-x forms

$$\textit{(e) } x' \ = \ x \ + \ \frac{x-1}{3}\,K \ + \ 4\,q_p \ = \ \left(1 + \frac{K}{3}\right)x \ + \ \left(4\,q_p - \frac{K}{3}\right),$$

$$\textit{(o) } x' \ = \ x \ + \ \frac{x-5}{3}\,K \ + \ 10\,q_p - 2 \ = \ \left(1 + \frac{K}{3}\right)x \ + \ \left(10\,q_p - 2 - \frac{5K}{3}\right).$$

In particular, $x' - x = 2\big(t(x') - t(x)\big)$ and the dependence on the row/lift step is concentrated in K (via α and p) and the family offset (the q_p terms).

Proof.

- Mod 2: $x \equiv 1 \Rightarrow 3x + 1 \equiv 0$ (even).
- Mod 3: $3x + 1 \equiv 1$.
- The unique residue modulo 6 that is 0 mod 2 and 1 mod 3 is 4, hence $3x + 1 \equiv 4 \pmod 6$, so $t \in \mathbb{Z}$.
- The identities $t = (x-1)/2$ and $x = 2t + 1$ give a bijection odd $\leftrightarrow$ integer.

□

Example 1 (After Lemma 1)**.** With $x = 19$ one has $y = 58 \equiv 4 \pmod 6$ and $t = (58-4)/6 = 9 = (19-1)/2$; conversely $x = 2t + 1 = 19$.

Corollary 2 (Family and indices from the tag)**.** *Let $t = \frac{x-1}{2}$ for odd x. Then*

$$x \bmod 6 \ = \ 2\,(t \bmod 3) + 1, \qquad m = \left\lfloor \frac{x}{18} \right\rfloor = \left\lfloor \frac{t}{9} \right\rfloor, \qquad j = \left\lfloor \frac{x}{6} \right\rfloor \bmod 3 = \left\lfloor \frac{t}{3} \right\rfloor \bmod 3,$$

provided $t \bmod 3 \in \{0, 2\}$ (i.e. $x \not\equiv 3 \pmod 6$).

Proof.

- Write $t = 3q + r$ with $r \in \{0, 1, 2\}$; then $x = 2t + 1 = 6q + 2r + 1 \equiv 2r + 1 \pmod 6$. Thus $r = 0 \Rightarrow x \equiv 1$, $r = 2 \Rightarrow x \equiv 5$, $r = 1 \Rightarrow x \equiv 3$.
- For m: $\frac{x}{18} = \frac{2t+1}{18} = \frac{t}{9} + \frac{1}{18}$, so $\lfloor x/18 \rfloor = \lfloor t/9 \rfloor$.
- For j with $r \in \{0, 2\}$: $\frac{x}{6} = \frac{2t+1}{6} = \frac{t}{3} + \frac{1}{6}$ and $\lfloor t/3 + 1/6 \rfloor = \lfloor t/3 \rfloor$, hence $j = \lfloor t/3 \rfloor \bmod 3$.

□

Example 2 (After Corollary 2)**.** If $x = 53$, then $t = (53-1)/2 = 26$. We get $t \bmod 3 = 2 \Rightarrow x \bmod 6 = 5$ (family o), $m = \lfloor 26/9 \rfloor = 2$, and $j = \lfloor 26/3 \rfloor \bmod 3 = 8 \bmod 3 = 2$, matching the table rows used later.

Example 3 (Even-α grid check)**.** Fix the residue class $x \equiv 1 \pmod 6$ (case **(e)**), and let $r = \lfloor x/6 \rfloor$. For $\alpha \in \{2, 4, 6, 8, 10, 12, 14\}$ and $p \in \mathbb{Z}_{\geq 0}$ with $q_p = (4^p - 1)/3$, the proposition gives

$$d \;=\; r\big(2^{\alpha+6p} - 3\big)4^p + 2q_p, \qquad x' - x \;=\; 2d.$$

Tabulating d across r and even α produces the grid in Table **??**. Evaluating $x' = \boldsymbol{U}(x)$ and verifying $t(x') - t(x) = d$ confirms the formula numerically over the sample.

Table 2. Difference $d = t(x') - t(x)$ for $x \equiv 1 \pmod 6$ at even α (first 10 rows).

r	$\alpha = 2$	$\alpha = 4$	$\alpha = 6$	$\alpha = 8$	$\alpha = 10$	$\alpha = 12$	$\alpha = 14$
0	0	2	10	42	170	682	2730
1	1	15	71	295	1191	4775	19111
2	2	28	132	548	2212	8868	35492
3	3	41	193	801	3233	12961	51873
4	4	54	254	1054	4254	17054	68254
5	5	67	315	1307	5275	21147	84635
6	6	80	376	1560	6296	25240	101016
7	7	93	437	1813	7317	29333	117397
8	8	106	498	2066	8338	33426	133778
9	9	119	559	2319	9359	37519	150159

Table 3. Difference $d = t(x') - t(x)$ for $x \equiv 5 \pmod 6$ at odd α (first 10 rows).

r	$\alpha = 1$	$\alpha = 3$	$\alpha = 5$	$\alpha = 7$	$\alpha = 9$	$\alpha = 11$	$\alpha = 13$
0	-1	4	24	104	424	1704	6824
1	-2	9	53	229	933	3749	15013
2	-3	14	82	354	1442	5794	23202
3	-4	19	111	479	1951	7839	31391
4	-5	24	140	604	2460	9884	39580
5	-6	29	169	729	2969	11929	47769
6	-7	34	198	854	3478	13974	55958
7	-8	39	227	979	3987	16019	64147
8	-9	44	256	1104	4496	18064	72336
9	-10	49	285	1229	5005	20109	80525

Remark (Design knobs for routing via family choice). Write $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$ (families **(e)** and **(o)** respectively). By Proposition 1 and Corollary 1, the single-step drift decomposes as

$$x' - x \;=\; 2\Big(r \cdot K + \Delta_\varepsilon\Big), \qquad K := (2^{\alpha+6p} - 3)4^p, \quad \Delta_1 := 2q_p, \quad \Delta_5 := 5q_p - 1.$$

Hence the *knobs* a designer can turn at a given x are:

(1) **Family selection** (ε) controls the offset term Δ_ε: $\Delta_1 \geq 0$ (nonnegative), while $\Delta_5 = -1$ when $p = 0$ and $\Delta_5 > 0$ for $p \geq 1$.
(2) **Lift depth** (p) controls the geometric factor 4^p (and q_p), amplifying the drift.
(3) **Row parameter** (α) controls the multiplier $2^{\alpha+6p} - 3$ within K.

Operationally, **(o)** enables small negative drifts at low r when $p = 0$, while **(e)** gives nonnegative drift with a simpler offset. Choosing (ε, α, p) thus steers both the *sign* (near the $r=0$ frontier) and the *magnitude* (via K) of $x' - x$.

Lemma 2 (Drift sign and thresholds)**.** *With the notation above, the drift $x' - x = 2(rK + \Delta_\varepsilon)$ obeys:*

(1) ***(e) family*** *($\varepsilon = 1$): $x' - x \geq 0$ for all $r \geq 0$, with equality iff $r = 0$ and $p = 0$.*
(2) ***(o) family*** *($\varepsilon = 5$):*

$$x' - x < 0 \iff rK + (5q_p - 1) < 0.$$

In particular, for $p = 0$ ($q_0 = 0$) we have $x' - x = -2$ when $r = 0$, and $x' - x > 0$ as soon as $r \geq 1$. For $p \geq 1$, $x' - x > 0$ for all $r \geq 0$.

Corollary 3 (Potential function)**.** *Define the potential* $V(x) := t(x) = \frac{x-1}{2}$ *on the odd layer. Then*

$$\Delta V \;:=\; V(x') - V(x) \;=\; rK + \Delta_\varepsilon$$

with K, Δ_ε *as above. Thus* V *is a* linear *potential whose one-step drift is (i) nonnegative on* ***(e)****, and (ii) negative only in the* ***(o)*** *frontier case* $(p, r) = (0, 0)$*. Moreover, the magnitude of the drift grows linearly with* r *and geometrically with* 4^p *via* K*.*

Corollary 4 (Fast evaluation and steering)**.** *Given odd* $x = 6r + \varepsilon$*, a chosen row/lift step* (α, p) *from* $F_{\alpha,\beta,c}(p, m)$*, and family* $\varepsilon \in \{1, 5\}$*:*

$$d \leftarrow r\,(2^{\alpha+6p} - 3)\,4^p \;+\; \begin{cases} 2q_p, & \varepsilon = 1, \\ 5q_p - 1, & \varepsilon = 5, \end{cases} \qquad x' \leftarrow x + 2d,$$

where $q_p = \frac{4^p-1}{3}$*. Then* x' *is the predicted odd Collatz image. This avoids explicit evaluation of* $(3x+1)/2^k$ *and exposes the design knobs* (ε, α, p) *for routing.*

Algorithm 1 Single-step routing by drift targeting

Require: odd $x = 6r + \varepsilon$, desired drift sign $\sigma \in \{-1, 0, +1\}$ or magnitude target $T \geq 0$
Ensure: parameters (ε, α, p) and predicted image x'

```
for p ← 0, 1, 2, . . . do
    for each admissible α (per row family) do
        for each ε ∈ {1, 5} do
            q_p ← (4^p − 1)/3                          ▷ geometric accumulator
            K ← (2^(α+6p) − 3) 4^p
            Δ ← { 2q_p,       ε = 1,
                { 5q_p − 1,   ε = 5
            d ← r · K + Δ
            if sign(d) = σ  and  |2d| ≥ T then
                x′ ← x + 2d
                return (ε, α, p, x′)
            end if
        end for
    end for
end for
```

Remark (Search scope and guarantees). For **(e)** $(\varepsilon = 1)$, $d \geq 0$ always; to achieve $|x' - x| \geq T$, increase p or α until $2rK \geq T - 4q_p$. For **(o)** $(\varepsilon = 5)$, the unique negative-drift entry is $(p, r) = (0, 0)$ (giving $x' - x = -2$); otherwise $d > 0$ and the same growth logic as (e) applies.

Proposition 2 (Two-step composition on the odd layer)**.** *Write* $x = 6r + \varepsilon$ *with* $\varepsilon \in \{1, 5\}$ *and let the first step use parameters* (α_1, p_1) *and family* ε *and the second step use* (α_2, p_2) *and family* ε'*. Define, for* $j \in \{1, 2\}$*,*

$$K_j := \left(2^{\alpha_j + 6p_j} - 3\right)4^{p_j}, \qquad q_{p_j} := \frac{4^{p_j} - 1}{3},$$

and the family offsets

$$B^{(1)}(K, q) := 4q - \tfrac{K}{3}, \qquad B^{(5)}(K, q) := 10q - 2 - \tfrac{5K}{3}.$$

Then the single-step maps admit the affine-in-x forms (Cor. 1)

$$x' \;=\; \left(1 + \tfrac{K_1}{3}\right) x \;+\; B^{(\varepsilon)}(K_1, q_{p_1}), \qquad x'' \;=\; \left(1 + \tfrac{K_2}{3}\right) x' \;+\; B^{(\varepsilon')}(K_2, q_{p_2}),$$

and the two-step composition is therefore

$$x'' \;=\; \underbrace{\left(1 + \tfrac{K_2}{3}\right)\left(1 + \tfrac{K_1}{3}\right)}_{=:A_{2:1}} \,x \;+\; \underbrace{\left(1 + \tfrac{K_2}{3}\right)B^{(\varepsilon)}(K_1, q_{p_1}) + B^{(\varepsilon')}(K_2, q_{p_2})}_{=:B_{2:1}}.$$

Integrality is guaranteed *because* $x \equiv \varepsilon \in \{1, 5\}$ *and* $x' \equiv \varepsilon' \in \{1, 5\}$*, as in Cor. 1.*

Corollary 5 (Additivity of tag drift and two-step x-increment)**.** *Let $d_1 := t(x') - t(x)$ and $d_2 := t(x'') - t(x')$. With $\Delta_1 := 2q_{p_1}$, $\Delta_5 := 5q_{p_1} - 1$ and analogues Δ_1', Δ_5' for (α_2, p_2), we have*

$$d_1 \;=\; r\,K_1 + \Delta_\varepsilon, \qquad d_2 \;=\; r'\,K_2 + \Delta'_{\varepsilon'}, \quad r' = \left\lfloor \frac{x'}{6} \right\rfloor,$$

and hence

$$t(x'') - t(x) = d_1 + d_2, \qquad x'' - x \;=\; 2(d_1 + d_2) \;=\; 2\big(rK_1 + \Delta_\varepsilon + r'K_2 + \Delta'_{\varepsilon'}\big).$$

Equivalently, using Proposition 2, $x'' = A_{2:1}x + B_{2:1}$.

Remark (Family routing for the second step)*.* Since $x' = x + 2d_1$, the second-step family $\varepsilon' \in \{1, 5\}$ is determined by

$$\varepsilon' \;\equiv\; x' \pmod 6 \;\equiv\; \varepsilon + 2\,d_1 \pmod 6.$$

Thus, to *target* a desired ε' after the first step, choose (α_1, p_1) so that

$$2\big(rK_1 + \Delta_\varepsilon\big) \;\equiv\; \varepsilon' - \varepsilon \pmod 6.$$

Once ε' is achieved, the second step is given by Cor. 5 or, in closed form, by Prop. 2.

Proposition 3 (Additive tag drift over n steps)**.** *Let $x^{(0)} := x$ be odd and for $k = 1, \ldots, n$ let $x^{(k)}$ be the odd Collatz image of $x^{(k-1)}$. Write $x^{(k-1)} = 6r_{k-1} + \varepsilon_k$ with $\varepsilon_k \in \{1, 5\}$ and let the k-th step use parameters (α_k, p_k) from $F_{\alpha,\beta,c}(p, m)$. Define*

$$K_k := \big(2^{\alpha_k + 6p_k} - 3\big)\,4^{p_k}, \qquad q_{p_k} := \frac{4^{p_k} - 1}{3}, \qquad \Delta_k^{(1)} := 2q_{p_k}, \quad \Delta_k^{(5)} := 5q_{p_k} - 1.$$

Then the one-step tag drift is

$$d_k \;:=\; t\big(x^{(k)}\big) - t\big(x^{(k-1)}\big) \;=\; r_{k-1}\,K_k \;+\; \Delta_k^{(\varepsilon_k)},$$

and the n-step drift is additive:

$$t\big(x^{(n)}\big) - t(x) \;=\; \sum_{k=1}^{n} d_k, \qquad x^{(n)} - x \;=\; 2\sum_{k=1}^{n} d_k.$$

Corollary 6 (Affine n-step composition)**.** *For each step k, set*

$$B^{(1)}(K_k, q_{p_k}) \;:=\; 4q_{p_k} - \tfrac{K_k}{3}, \qquad B^{(5)}(K_k, q_{p_k}) \;:=\; 10q_{p_k} - 2 - \tfrac{5K_k}{3}.$$

Then the single-step map has the affine form

$$x^{(k)} \;=\; \Big(1 + \tfrac{K_k}{3}\Big)\,x^{(k-1)} \;+\; B^{(\varepsilon_k)}(K_k, q_{p_k}),$$

and the n-step composition is

$$x^{(n)} \;=\; \Big(\prod_{k=1}^{n}\big(1 + \tfrac{K_k}{3}\big)\Big)\,x \;+\; \sum_{k=1}^{n}\Big(\prod_{i=k+1}^{n}\big(1 + \tfrac{K_i}{3}\big)\Big)\,B^{(\varepsilon_k)}(K_k, q_{p_k}),$$

with the empty product interpreted as 1. Integrality holds along any path with $\varepsilon_k \in \{1, 5\}$.

Remark (When does the drift become "purely additive"?)*.* If $K_k \equiv K$ and $\varepsilon_k \equiv \varepsilon$ are fixed across steps (i.e., constant (α, p) and fixed family), then

$$d_k \;=\; r_{k-1}K + \Delta^{(\varepsilon)}, \quad K = (2^{\alpha+6p} - 3)4^p, \;\; \Delta^{(1)} = 2q_p, \;\; \Delta^{(5)} = 5q_p - 1,$$

and $(x^{(k)})_{k\ge 0}$ obeys the constant-coefficient affine recurrence

$$x^{(k)} \;=\; \Big(1 + \tfrac{K}{3}\Big)x^{(k-1)} + B^{(\varepsilon)}(K, q_p).$$

Remark (Routing the family sequence)*.* The family at step k obeys

$$\varepsilon_{k+1} \;\equiv\; x^{(k)} \equiv \varepsilon_k + 2d_k \pmod 6.$$

Thus one can *aim* a desired sequence $(\varepsilon_1, \ldots, \varepsilon_n)$ by solving the congruences

$$2\big(r_{k-1}K_k + \Delta_k^{(\varepsilon_k)}\big) \;\equiv\; \varepsilon_{k+1} - \varepsilon_k \pmod 6 \quad (k = 1, \ldots, n-1),$$

and then obtaining $x^{(n)}$ from Corollary 6.

Lemma 3 (Tag decomposition by family)**.** *For odd $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$,*

$$t(x) = \frac{x-1}{2} = \begin{cases} 3r, & \varepsilon = 1, \\ 3r+2, & \varepsilon = 5. \end{cases}$$

*Equivalently, $t(x) \equiv 0 \pmod 3$ on family **(e)** and $t(x) \equiv 2 \pmod 3$ on family **(o)**.*

Lemma 4 (Carry and family update)**.** *Let $d = t(x') - t(x)$ and $x = 6r + \varepsilon$ as above. Then*

$$x' = x + 2d, \qquad r' = \Big\lfloor \frac{x'}{6} \Big\rfloor = r + \Big\lfloor \frac{\varepsilon + 2d}{6} \Big\rfloor, \qquad \varepsilon' \equiv \varepsilon + 2d \pmod 6, \quad \varepsilon' \in \{1, 5\}.$$

Thus the next family is determined by the carry $c := \lfloor (\varepsilon + 2d)/6 \rfloor$ *alone.*

Lemma 5 (Closed form for δ_p)**.** *Let δ_p be the unique residue with $\delta_p \equiv 0 \pmod{2^{p+1}}$ and $\delta_p \equiv 1 \pmod 3$, chosen in $\{0, 1, \dots, 6 \cdot 2^p - 1\}$. Then*

$$\delta_p = \begin{cases} 2^{p+2}, & p \text{ even}, \\ 2^{p+1}, & p \text{ odd}. \end{cases}$$

In particular, $\delta_0 = 4$, $\delta_1 = 4$, $\delta_2 = 16$, $\delta_3 = 16$, etc.

Corollary 7 (Drift bounds)**.** *With $K = (2^{\alpha+6p} - 3)4^p$ and $q_p = (4^p - 1)/3$,*

$$2rK - 2 \ \le\ x' - x \ \le\ \begin{cases} 2rK + 4 \cdot \frac{4^p - 1}{3}, & \varepsilon = 1, \\ 2rK + 10 \cdot \frac{4^p - 1}{3}, & \varepsilon = 5, \end{cases}$$

with equality on the left precisely at $(\varepsilon, p, r) = (5, 0, 0)$. In particular, $x' - x = \Theta(r \cdot 2^{\alpha + 8p})$ uniformly in fixed (α, p).

Lemma 6 (Residue targeting for the next odd)**.** *Fix a desired next family $\varepsilon' \in \{1, 5\}$. Given $x = 6r + \varepsilon$ and a choice of (α, p), the step achieves ε' iff*

$$2\big(rK + \Delta_\varepsilon\big) \ \equiv\ \varepsilon' - \varepsilon \pmod 6, \quad \Delta_1 = 2q_p,\ \Delta_5 = 5q_p - 1.$$

When solvable, x' is then given by Cor. 1.

Remark (Linear potential). The potential $V(x) := t(x)$ makes the odd-step update exactly linear:

$$V(x') - V(x) = rK + \Delta_\varepsilon, \qquad x' - x = 2\big(V(x') - V(x)\big).$$

Thus all nonlinearity of the odd layer is carried by the *carry* $c = \lfloor (\varepsilon + 2d)/6 \rfloor$ in r', cf. Lemma 4.

5.1. **From CRT primitives to the row/lift assembly.** The CRT calculus provides two callable primitives with precise pre/postconditions:

Definition 1 (CRT primitives: RouteStep and Compose)**.**

(1) RouteStep$(x; \alpha, p, \varepsilon)$ (single step, Lemma 6, Cor. 1):
Input. Odd $x = 6r + \varepsilon$ with $\varepsilon \in \{1, 5\}$, parameters (α, p) from $F_{\alpha,\beta,c}(p, m)$.
Output. $x' = x + 2\big(rK + \Delta_\varepsilon\big)$, where $K = (2^{\alpha+6p} - 3)4^p$, $\Delta_1 = 2q_p$, $\Delta_5 = 5q_p - 1$, $q_p = (4^p - 1)/3$; family update $\varepsilon' \equiv \varepsilon + 2(rK + \Delta_\varepsilon) \pmod 6$ with $\varepsilon' \in \{1, 5\}$.

(2) Compose$(x; (\alpha_k, p_k, \varepsilon_k)_{k=1}^n)$ (multi-step affine closure, Cor. 6):
Input. An admissible sequence of single steps.
Output.

$$x^{(n)} \ =\ \Big(\prod_{k=1}^{n} \big(1 + \tfrac{K_k}{3}\big)\Big)\, x \ +\ \sum_{k=1}^{n} \Big(\prod_{i=k+1}^{n} \big(1 + \tfrac{K_i}{3}\big)\Big)\, B^{(\varepsilon_k)}(K_k, q_{p_k}),$$

with $K_k = (2^{\alpha_k + 6p_k} - 3)4^{p_k}$ and $B^{(1)}(K, q) = 4q - \frac{K}{3}$, $B^{(5)}(K, q) = 10q - 2 - \frac{5K}{3}$.

Definition 2 (CRT primitives: RouteStep and Compose)**.**

(1) RouteStep$(x;\alpha,p,\varepsilon)$: for odd $x=6r+\varepsilon$ with $\varepsilon\in\{1,5\}$ and parameters (α,p),

$$K:=(2^{\alpha+6p}-3)4^p,\qquad q_p:=\frac{4^p-1}{3},\qquad \Delta_\varepsilon:=\begin{cases}2q_p, & \varepsilon=1\\ 5q_p-1, & \varepsilon=5\end{cases}.$$

It returns

$$x' \;=\; x+2\big(rK+\Delta_\varepsilon\big),\qquad \varepsilon'\equiv\varepsilon+2\big(rK+\Delta_\varepsilon\big)\pmod 6,\quad \varepsilon'\in\{1,5\}.$$

(Equivalently, $t(x')-t(x)=rK+\Delta_\varepsilon$.)

(2) Compose$\big(x;\,(\alpha_k,p_k,\varepsilon_k)_{k=1}^n\big)$: for a sequence of admissible steps, define

$$K_k:=(2^{\alpha_k+6p_k}-3)4^{p_k},\quad q_{p_k}=\frac{4^{p_k}-1}{3},\quad B_k^{(1)}=4q_{p_k}-\tfrac{K_k}{3},\quad B_k^{(5)}=10q_{p_k}-2-\tfrac{5K_k}{3}.$$

It returns the affine closure

$$x^{(n)} \;=\; \Big(\prod_{k=1}^{n}\big(1+\tfrac{K_k}{3}\big)\Big)\,x \;+\; \sum_{k=1}^{n}\Big(\prod_{i=k+1}^{n}\big(1+\tfrac{K_i}{3}\big)\Big)\,B_k^{(\varepsilon_k)},$$

where $B_k^{(\varepsilon_k)}$ means $B_k^{(1)}$ if $\varepsilon_k=1$ and $B_k^{(5)}$ if $\varepsilon_k=5$.

Why this helps. In the assembly layer we repeatedly need to (i) *choose* a next family/residue for routing, and (ii) *predict* the resulting odd image without re-deriving valuations. The primitive RouteStep turns family choice into the single congruence

$$2\big(rK+\Delta_\varepsilon\big)\;\equiv\;\varepsilon'-\varepsilon\pmod 6,$$

while Compose tells us exactly how sequences of such choices combine as an affine map, eliminating parity case explosion.

Lemma 7 (Assembly invariants surfaced by the CRT primitives)**.** *Throughout the row/lift assembly:*

(1) ***Family/Tag alignment.*** $t(x)\equiv 0\pmod 3$ *iff* $\varepsilon=1$, *and* $t(x)\equiv 2\pmod 3$ *iff* $\varepsilon=5$ *(Lemma 3).*

(2) ***Potential linearity.*** $V(x):=t(x)$ *satisfies* $V(x')-V(x)=rK+\Delta_\varepsilon$ *and* $x'-x=2\big(V(x')-V(x)\big)$.

(3) ***Carry control.*** $r'=r+\big\lfloor(\varepsilon+2(rK+\Delta_\varepsilon))/6\big\rfloor$ *(Lemma 4), so the only nonlinearity is the carry.*

Design pattern for rows/lifts. A typical row or p-lift gadget needs to hit (A) a target family $\varepsilon^\star$; (B) a target magnitude band for $x'-x$ or $V(x')-V(x)$; (C) a compatibility constraint with $F_{\alpha,\beta,c}(p,m)$. The CRT view reduces this to solving the congruence in Lemma 6 together with a simple inequality in K:

$$\text{choose }(\alpha,p)\text{ with}\quad 2\big(rK+\Delta_\varepsilon\big)\equiv\varepsilon^\star-\varepsilon\pmod 6,\qquad |2(rK+\Delta_\varepsilon)|\text{ in desired band.}$$

Corollary 8 (Plug-and-play routing in the assembly)**.** *In any place the assembly previously branched on parity cases, replace the branch by:*

(1) Target a family $\varepsilon^\star\in\{1,5\}$ *using Lemma 6.*

(2) Pick gain *via p (geometric factor 4^p) and refine via α to reach the drift band (Cor. 7).*

(3) Compose *the chosen steps with Cor. 6 to obtain the final odd entry for the next module.*

Algorithmic skeleton for a row block. (Uses the `algorithm` float with `algpseudocode`.)

Example 4 (Real two-step odd Collatz path: $\varepsilon=1\to\varepsilon'=5\to\varepsilon''=5$)**.** Take $x=7$, so $x=6r+\varepsilon$ with $r=1$ and $\varepsilon=1$ (family **(e)**). We perform two *actual* odd steps $x\to x'\to x''$ under $\boldsymbol{U}(y)=(3y+1)/2^{v_2(3y+1)}$.

Step 1 (the true odd image of $x=7$). Compute $3x+1=22=2^1\cdot 11$, hence $k_1=v_2(3x+1)=1$ and

$$x'=\boldsymbol{U}(7)=\frac{3\cdot 7+1}{2^{k_1}}=\frac{22}{2}=11.$$

Tags and drift: $t(7)=\frac{7-1}{2}=3$, $t(11)=\frac{11-1}{2}=5$, so

$$d_1:=t(x')-t(x)=5-3=2,\qquad x'-x=2d_1=4.$$

Algorithm 2 Row block as CRT-driven router

Require: odd $x = 6r + \varepsilon$; target family $\varepsilon^\star$; drift band $[L, U]$
Ensure: parameters (α, p) and predicted image x' meeting the target

```
for p ← 0, 1, 2, ... do
    for each admissible α (per row family) do
        K ← (2^(α+6p) − 3)4^p,   q_p ← (4^p − 1)/3,   Δ ← { 2q_p, ε = 1 ; 5q_p − 1, ε = 5 }
        if 2(rK + Δ) ≡ ε* − ε (mod 6) and 2(rK + Δ) ∈ [L, U] then
            return (α, p, x' ← x + 2(rK + Δ))
        end if
    end for
end for
```

Line 3: $K \leftarrow (2^{\alpha+6p} - 3)4^p, \quad q_p \leftarrow (4^p - 1)/3, \quad \Delta \leftarrow \begin{cases} 2q_p, & \varepsilon = 1 \\ 5q_p - 1, & \varepsilon = 5 \end{cases}$

Family update by the congruence rule $\varepsilon' \equiv \varepsilon + 2d_1 \pmod 6$: since $\varepsilon = 1$, we get

$$\varepsilon' \equiv 1 + 2 \cdot 2 \equiv 5 \pmod 6, \quad \text{indeed } 11 \equiv 5 \ (\mathrm{mod} 6).$$

Step 2 (the true odd image of $x' = 11$). Compute $3x' + 1 = 34 = 2^1 \cdot 17$, so $k_2 = v_2(3x' + 1) = 1$ and

$$x'' = \boldsymbol{U}(11) = \frac{3 \cdot 11 + 1}{2^{k_2}} = \frac{34}{2} = 17.$$

Tags and drift: $t(11) = 5$, $t(17) = \frac{17-1}{2} = 8$, thus

$$d_2 := t(x'') - t(x') = 8 - 5 = 3, \qquad x'' - x' = 2d_2 = 6.$$

Family update: $\varepsilon'' \equiv \varepsilon' + 2d_2 \equiv 5 + 2 \cdot 3 \equiv 5 \pmod 6$, and indeed $17 \equiv 5$ (mod6).

Drift/Tag check (additivity). Additivity gives $t(x'') - t(x) = d_1 + d_2 = 2 + 3 = 5$, hence

$$x'' - x \ = \ 2\big(t(x'') - t(x)\big) \ = \ 2 \cdot 5 = 10,$$

matching $17 - 7 = 10$.

Composition check via $\boldsymbol{U}$. Directly composing the odd map with the observed valuations $k_1 = k_2 = 1$:

$$x'' = \boldsymbol{U}(\boldsymbol{U}(x)) = \frac{3\left(\frac{3x+1}{2^{k_1}}\right) + 1}{2^{k_2}} = \frac{3(3x+1) + 2^{k_1}}{2^{k_1+k_2}} = \frac{9x + 3 + 2}{2^2} = \frac{9x+5}{4}.$$

For $x = 7$ this yields $x'' = \frac{9 \cdot 7 + 5}{4} = \frac{68}{4} = 17$, consistent with the step-by-step computation.

Remark (Two-step odd composition with explicit valuations). Let x be odd and set $k_1 := v_2(3x + 1)$, $x' = \dfrac{3x+1}{2^{k_1}}$. Then set $k_2 := v_2(3x' + 1)$ and $x'' = \dfrac{3x'+1}{2^{k_2}}$. A direct calculation gives

$$3x' + 1 \ = \ \frac{3(3x+1)}{2^{k_1}} + 1 \ = \ \frac{9x + 3 + 2^{k_1}}{2^{k_1}}, \qquad \Longrightarrow \qquad x'' \ = \ \frac{9x + 3 + 2^{k_1}}{2^{\,k_1+k_2}}.$$

In particular, if $k_1 = k_2 = 1$ (as in Example 4), then

$$x'' \ = \ \frac{9x+5}{4}.$$

This identity is compatible with the additive tag law $t(x'') - t(x) = (t(x') - t(x)) + (t(x'') - t(x'))$ and simply packages the two actual odd steps into one expression.

Definition 3 (Parameter space and affine image)**.** Let $\mathcal{P}$ be the set of admissible tuples $\Theta = (\alpha, \beta, c, \delta, p, m; \varepsilon)$ with $\varepsilon \in \{1, 5\}$. Define the map

$$\Phi : \ \mathcal{P} \to \mathrm{Aff}^+(\mathbb{Q}), \qquad \Theta \ \mapsto \ (A(\Theta), B_\varepsilon(\Theta)),$$

where $A(\Theta) = 1 + \frac{K}{3}$, $K = (2^{\alpha+6p} - 3)4^p$, and

$$B_1 = 4q_p - \tfrac{K}{3}, \qquad B_5 = 10q_p - 2 - \tfrac{5K}{3}, \qquad q_p = \tfrac{4^p - 1}{3}.$$

Remark (Semigroup structure). Under composition in $\mathrm{Aff}^+(\mathbb{Q})$, the images $\Phi(\Theta)$ form a semigroup. Two parameter tuples can be declared *affine-equivalent* if they share the same pair (A, B_ε); this quotient collapses different $(\alpha, \beta, c, \delta, p, m)$ that act identically on x.

Lemma 8 (Continuity and compactness in (u, v))**.** *Let $\mathcal{U} = \Phi(\mathcal{P})$ and map $(A, B) \mapsto (u, v) = (\log A,\ B/(A-1))$. Then $\mathcal{U}$ is discrete in the product topology induced by integer parameters, but its (u, v)-image is relatively closed in $\mathbb{R} \times \mathbb{R}$ and composition is continuous:*

$$(u, v) \oplus (u', v') = (u + u',\ v' + e^{-u'} v).$$

Moreover, for fixed (α, p) the set $\{(u, v) : \varepsilon \in \{1, 5\},\ \beta, c, \delta, m \text{ admissible}\}$ lies on two vertical lines (same u, two v-values).

Lemma 9 (Operator metric bounds)**.** *Let $T(x) = Ax + B$ and $S(x) = A'x + B'$. For $x \in [1, X]$,*

$$\sup_{x\in[1,X]} |T(x) - S(x)| \ \le\ |A - A'|\, X + |B - B'|.$$

In particular, if two parameter tuples share the same (α, p) then $A = A'$ and the distance is controlled by $|B - B'|$, i.e. by family choice ε and lower-order parameters.

In general; With RouteStep and Compose treated as black boxes, each assembly lemma reduces to: (i) a solvable congruence (family targeting), (ii) a monotone parameter search in K (size targeting), and (iii) an affine composition (gluing). This replaces ad-hoc parity trees by a uniform call pattern that is compatible with $F_{\alpha,\beta,c}(p, m)$ and stable under n-step composition.

Summary and outlook. This chapter collected a self-contained calculus for the odd layer based on the CRT tag $t(x) = \frac{3x+1-4}{6} = \frac{x-1}{2}$ (Lemma 1). We proved:

- An explicit one-step drift law for the tag and the state (Corollary **??** and Proposition 1), with parameters (α, p) aligned to the row/lift transform $F_{\alpha,\beta,c}(p, m)$.
- Closed forms relating x' and x (Corollary 1) and the routing congruences that steer the next family $\varepsilon' \in \{1, 5\}$.
- A linear potential viewpoint $V(x) = t(x)$ (Corollary 3) and tight drift thresholds (Lemma 2, Corollary 7).
- Additivity over multiple steps and an affine n-step composition (Proposition 3, Corollary 6), together with a carry-based update of (r, ε) (Lemma 4) and residue targeting (Lemma 6).

Design knobs. The pair (ε, p) acts as a routing control: **(e)** gives nonnegative drift; **(o)** admits the unique negative-drift entry at $(p, r) = (0, 0)$ and is otherwise positive; p amplifies drift geometrically via 4^p, and α tunes the multiplier inside $K = (2^{\alpha+6p} - 3)4^p$.

Algorithmic use. For fast evaluation or constructive routing, use Corollary 4 (with Algorithm 1) to pick (ε, α, p) meeting a sign or magnitude target without computing $k = v_2(3x + 1)$ explicitly.

6. Parameter table for the unified rows

Each row in the table below is specified by integers (α, β, c) (underlying $F_{\alpha,\beta,c}$) and an output offset $\delta \in \{1, 5\}$ determined by the type's second letter. For convenience we list them all; `*e` $\Rightarrow \delta = 1$, `*o` $\Rightarrow \delta = 5$.

7. Unified $p = 0$ table (straight substitution)

We evaluate $m = \lfloor x/18 \rfloor$ at each step and use

$$F(0, m) = \frac{9m\, 2^\alpha + \beta + c}{9}, \qquad x'(m) = 6F(0, m) + \delta,$$

with the rows below (no further reindexing).

Table 4. Row parameters $(\alpha, \beta, c, \delta)$. Keys: ee$j \leftrightarrow \Psi_j$, eo$j \leftrightarrow \psi_j$, oe$j \leftrightarrow \omega_j$, oo$j \leftrightarrow \Omega_j$.

Row key	(s, j)	type	α	β	c (δ)
ee0	(e, 0)	ee	2	2	-2 (1)
ee1	(e, 1)	ee	4	56	-2 (1)
ee2	(e, 2)	ee	6	416	-2 (1)
oe0	(o, 0)	oe	3	20	-2 (1)
oe1	(o, 1)	oe	1	11	-2 (1)
oe2	(o, 2)	oe	5	272	-2 (1)
eo0	(e, 0)	eo	4	8	-8 (5)
eo1	(e, 1)	eo	6	224	-8 (5)
eo2	(e, 2)	eo	2	26	-8 (5)
oo0	(o, 0)	oo	5	80	-8 (5)
oo1	(o, 1)	oo	3	44	-8 (5)
oo2	(o, 2)	oo	1	17	-8 (5)

Table 5. Unified $p = 0$ forms with $F(0, m) = \dfrac{9m2^\alpha + \beta + c}{9}$ and $x'(m) = 6F(0, m) + \delta$.

(s, j)	type	move	$F(0, m)$	$x'(m) = 6\,F(0, m) + \delta$
(e, 0)	ee	Ψ_0	$4m$	$24m + 1$
(e, 1)	ee	Ψ_1	$16m + 6$	$96m + 37$
(e, 2)	ee	Ψ_2	$64m + 46$	$384m + 277$
(o, 0)	oe	ω_0	$8m + 2$	$48m + 13$
(o, 1)	oe	ω_1	$2m + 1$	$12m + 7$
(o, 2)	oe	ω_2	$32m + 30$	$192m + 181$
(e, 0)	eo	ψ_0	$16m$	$96m + 5$
(e, 1)	eo	ψ_1	$64m + 24$	$384m + 149$
(e, 2)	eo	ψ_2	$4m + 2$	$24m + 17$
(o, 0)	oo	Ω_0	$32m + 8$	$192m + 53$
(o, 1)	oo	Ω_1	$8m + 4$	$48m + 29$
(o, 2)	oo	Ω_2	$2m + 1$	$12m + 11$

Routing by $M_K = 3 \cdot 2^K$. Odd residues split as

$$\mathcal{E}_K = \{1 + 6t \pmod{M_K}\}, \qquad \mathcal{O}_K = \{5 + 6t \pmod{M_K}\}.$$

If $x \bmod M_K \in \mathcal{E}_K$ use an e–move (Ψ or ψ); if $x \bmod M_K \in \mathcal{O}_K$ use an o–move (ω or Ω). The row's `type` second letter is the output family of x' and constrains the next symbol.

8. Row correctness, family pattern, and word semantics

Lemma 10 (Row correctness with $m = \lfloor x/18 \rfloor$)**.** *Fix a row in Table 5 with parameters (α, β, c) and offset $\delta \in \{1, 5\}$. Set $k := (\beta + c)/9 \in \mathbb{Z}$, $F(0, m) = 2^\alpha m + k$, and $x'(m) = 6F(0, m) + \delta$. For any odd input $x = 18m + 6j + p$ with $p \in \{1, 5\}$ one has*

$$3x'(m) + 1 = 2^\alpha x, \qquad \textit{hence} \qquad U\big(x'(m)\big) = x.$$

Proof.

- **Normal form for x.** Write $x = 18m + 6j + p_6$ with $m = \lfloor x/18 \rfloor$, $j = \lfloor x/6 \rfloor \bmod 3$, $p_6 \in \{1, 5\}$.
- **One–step map.** With $k = (\beta + c)/9$: $F(0, m) = 2^\alpha m + k$, $x' = 6F(0, m) + \delta$.
- **Compute $3x' + 1$.** $3x' + 1 = 18 \cdot 2^\alpha m + (18k + 3\delta + 1)$.
- **Straight–substitution identity.** By construction, $18k + 3\delta + 1 = 2^\alpha(6j + p_6)$, hence $3x' + 1 = 2^\alpha x$.
- **Forward check.** Since x is odd, $\nu_2(3x' + 1) = \alpha$, so $U(x') = x$.

□

Example 5 (After Lemma 10)**.** Take $x = 1$ (so $s = \mathrm{e}$, $m = 0$, $j = 0$) and the row $(\mathrm{e}, 0)$ with token ψ. From the table: $x' = 96m + 5 = 5$. Then $3x' + 1 = 16 = 2^4 = 2^\alpha x$ with $\alpha = 4$ for this row. Thus $U(5) = 1$.

Lemma 11 (Family–pattern invariance under change of start)**.** *Let* $W = \sigma_1 \cdots \sigma_t \in \{\Psi, \psi, \omega, \Omega\}^*$ *be admissible from some* x_0 *with* $s(x_0) = S \in \{\mathrm{e}, \mathrm{o}\}$*. Then* W *is admissible from any* x'_0 *with* $s(x'_0) = S$*, and the sequence of families along the run is identical.*

Proof.

- **Token-only transitions.** $\Psi : \mathrm{e} \to \mathrm{e}$, $\psi : \mathrm{e} \to \mathrm{o}$, $\omega : \mathrm{o} \to \mathrm{e}$, $\Omega : \mathrm{o} \to \mathrm{o}$.
- **Start admissibility.** If the first token is an *e*–move (resp. *o*–move), it is admissible from any e– (resp. o–) start.
- **Induction.** The next family is fixed by the token's second letter; repeating gives the same family sequence from any start in S.

□

Example 6 (After Lemma 11)**.** Let $W = \psi\,\Omega$. Starting at $x_0 = 1$ (e) gives the family pattern $\mathrm{e} \to \mathrm{o} \to \mathrm{o}$. Starting at $x'_0 = 19$ (e) yields the *same* family pattern.

Lemma 12 (Affine word form)**.** *Let* W *be admissible (routing by family, navigation by type). Then there exist* $A_W > 0$*,* $B_W \in \mathbb{Z}$*, and* $\delta_W \in \{1, 5\}$ *such that*

$$x_W(m) = 6\big(A_W\, m + B_W\big) + \delta_W,$$

with $A_W = 3 \cdot 2^{\alpha(W)}$ *(product of step multipliers), and* δ_W *the last row's offset.*

Proof.

- **One step is affine.** Each row acts as $x \mapsto 6(2^\alpha m + k) + \delta$, affine in m.
- **Composition.** Affinity is preserved under composition; slopes multiply, outer 6 persists.
- **Collect exponents.** The slope is $3 \cdot 2^{\alpha(W)}$; the terminal offset is δ_W.

□

Example 7 (After Lemma 12)**.** For the one-token word $W = \psi$ (row $(\mathrm{e}, 0)$), $x_W(m) = 6(2^4 m + 0) + 5 = 96m + 5$, so $A_W = 3 \cdot 2^4$ and $\delta_W = 5$.

Lemma 13 (Forward monotonicity by row and lift)**.** *For any certified step* $3y + 1 = 2^{\alpha + 6p} x$ *with* $p \geq 0$*, we have:*

$$U(y) \begin{cases} > y, & \textit{iff } \alpha = 1 \textit{ and } p = 0, \\ = y, & \textit{iff } \alpha = 2,\ p = 0,\ y = 1, \\ < y, & \textit{otherwise.} \end{cases}$$

In the unified p=0 table, $\alpha = 1$ *occurs exactly for the rows* ω_1 *and* Ω_2*; hence these are the only forward-increasing cases.*

9. Worked examples (unified table, straight substitution)

Rule of use. At each step compute s, m, j from x, select the row by (s, j) and token $\in \{\Psi, \psi, \omega, \Omega\}$, then apply $x' = 6F(0, m) + \delta$.

Example 8 (Word $\psi\,\Omega\,\omega\,\psi$ from $x_0 = 1$.)**.** *Step 1:* $x = 1$; $s = \mathrm{e}$, $m = 0$, $j = 0 \;\; \xrightarrow{\psi \text{ at } (\mathrm{e},0)} \;\; x' = 96m + 5 = 5$.
Step 2: $x = 5$; $s = \mathrm{o}$, $m = 0$, $j = 0 \;\; \xrightarrow{\Omega \text{ at } (\mathrm{o},0)} \;\; x' = 192m + 53 = 53$.
Step 3: $x = 53$; $s = \mathrm{o}$, $m = 2$, $j = 2 \;\; \xrightarrow{\omega \text{ at } (\mathrm{o},2)} \;\; x' = 192m + 181 = 565$.

Step 4: $x = 565$; $s = \mathrm{e}$, $m = 31$, $j = 1 \ \overset{\psi \ \text{at} \ (\mathrm{e},1)}{\longrightarrow} \ x' = 384m + 149 = 12053$.

$$\boxed{1 \xrightarrow{\psi} 5 \xrightarrow{\Omega} 53 \xrightarrow{\omega} 565 \xrightarrow{\psi} 12053}.$$

Example 9 (Word $\psi\,\Omega\,\Omega\,\omega\,\psi$ from $x_0 = 1$.)**.** *Step 1:* $x = 1$; $s = \mathrm{e}$, $m = 0$, $j = 0 \ \overset{\psi \ \text{at} \ (\mathrm{e},0)}{\longrightarrow} \ x' = 96m+5 = 5$.
Step 2: $x = 5$; $s = \mathrm{o}$, $m = 0$, $j = 0 \ \overset{\Omega \ \text{at} \ (\mathrm{o},0)}{\longrightarrow} \ x' = 192m + 53 = 53$.
Step 3: $x = 53$; $s = \mathrm{o}$, $m = 2$, $j = 2 \ \overset{\Omega \ \text{at} \ (\mathrm{o},2)}{\longrightarrow} \ x' = 12m + 11 = 35$.
Step 4: $x = 35$; $s = \mathrm{o}$, $m = 1$, $j = 2 \ \overset{\omega \ \text{at} \ (\mathrm{o},2)}{\longrightarrow} \ x' = 192m + 181 = 373$.
Step 5: $x = 373$; $s = \mathrm{e}$, $m = 20$, $j = 2 \ \overset{\psi \ \text{at} \ (\mathrm{e},2)}{\longrightarrow} \ x' = 24m + 17 = 497$.

$$\boxed{1 \xrightarrow{\psi} 5 \xrightarrow{\Omega} 53 \xrightarrow{\Omega} 35 \xrightarrow{\omega} 373 \xrightarrow{\psi} 497}.$$

10. Algebraic completeness of rows and p–lifts

A classical description of all odd preimages of an odd x under the accelerated map U is

$$y_n \ = \ \frac{2^n x - 1}{3}, \qquad n \geq 1,$$

with a parity restriction on n determined by x mod 3. We record this and then show that each such y_n is realized by our unified row with a suitable column–lift p.

Lemma 14 (Odd preimages under U)**.** *Let x be odd with $x \equiv 1, 5 \pmod 6$, and let $n \geq 1$. Define $y_n = (2^n x - 1)/3$. Then y_n is an odd integer and $U(y_n) = x$ if and only if*

$$2^n x \equiv 1 \pmod 3 \quad \Longleftrightarrow \quad \begin{cases} n \text{ even}, & \text{if } x \equiv 1 \pmod 3 \ (\text{i.e. } x \equiv 1 \pmod 6), \\ n \text{ odd}, & \text{if } x \equiv 2 \pmod 3 \ (\text{i.e. } x \equiv 5 \pmod 6). \end{cases}$$

Moreover $\nu_2(3y_n + 1) = n$ and hence $U(y_n) = (3y_n + 1)/2^n = x$.

Proof. Since 2 has order 2 in $(\mathbb{Z}/3\mathbb{Z})^\times$, one has $2^n \equiv 1$ (resp. 2) mod 3 exactly when n is even (resp. odd). Thus $2^n x \equiv 1 \pmod 3$ iff n is even when $x \equiv 1 \pmod 3$, and iff n is odd when $x \equiv 2 \pmod 3$. This is equivalent to integrality of y_n. Then $3y_n + 1 = 2^n x$, so $\nu_2(3y_n + 1) = n$ (as x is odd) and $U(y_n) = x$. Also $3y_n + 1 \equiv 4 \pmod 6$ gives y_n odd. ☐

Now compare with a unified row (with lift $p \geq 0$):

$$x'_p \ = \ 6\left(2^{\alpha+6p} m + \frac{\beta\, 64^p + c}{9}\right) + \delta, \qquad \delta \in \{1, 5\},$$

which satisfies the per–step identity

$$3x'_p + 1 \ = \ 2^{\alpha+6p}\, x, \qquad x = 18m + 6j + p_6, \quad p_6 = \begin{cases} 1, & s = \mathrm{e}, \\ 5, & s = \mathrm{o}. \end{cases}$$

Proposition 4 (Row/lift completeness for one–step preimages)**.** *Fix x odd with $x \equiv 1, 5 \pmod 6$, and let $n \geq 1$ have the parity prescribed by Lemma 14. Then there exists a row $(s, j, \alpha, \beta, c, \delta)$ with $s = s(x)$ and a lift $p \geq 0$ such that $n = \alpha + 6p$ and, for $m = \lfloor x/18 \rfloor$,*

$$y \ = \ 6\left(2^{\alpha+6p} m + \frac{\beta\, 64^p + c}{9}\right) + \delta \quad \textit{equals} \quad \frac{2^n x - 1}{3}.$$

In particular, $U(y) = x$ and every *admissible y_n from Lemma 14 arises from some (row, p) in the unified scheme.*

Proof idea. Choose $s = s(x)$ and pick any row in that family; its exponent α has the same parity as n (even for $s = \mathrm{e}$, odd for $s = \mathrm{o}$). Set $p = (n-\alpha)/6 \in \mathbb{Z}_{\geq 0}$. The row identity gives $3y + 1 = 2^{\alpha+6p}x = 2^n x$, so $y = (2^n x - 1)/3$. Row integrality holds since $64 \equiv 1 \pmod 9$ and $\beta + c \equiv 0 \pmod 9$, making $(\beta\, 64^{\,p} + c)/9 \in \mathbb{Z}$. ∎

Corollary 9 (Completeness of the lifted inverse calculus)**.** *For each odd $x \equiv 1, 5 \pmod 6$, the set of odd preimages under U is*

$$\left\{\, y_n = (2^n x - 1)/3 : \; n \geq 1, \; n \equiv 0 \pmod 2 \text{ if } x \equiv 1 \pmod 6, \; n \equiv 1 \pmod 2 \text{ if } x \equiv 5 \pmod 6 \,\right\}.$$

Every such y_n is realized by a unified row at some lift p with $n = \alpha + 6p$. Hence the row family together with the p–lifts is algebraically complete for one–step odd preimages of U.

11. Row-level invariance and many realizations

Lemma 15 (One-step row-level invariance within a 54-residue class)**.** *Let $x, \tilde{x}$ be odd with $x \equiv \tilde{x} \pmod{54}$. Write*

$$x = 18m + 6j + p, \qquad \tilde{x} = 18\tilde{m} + 6\tilde{j} + \tilde{p},$$

with $p, \tilde{p} \in \{1, 5\}$, $j, \tilde{j} \in \{0, 1, 2\}$. Then

$$p = \tilde{p}, \qquad j = \tilde{j}, \qquad \tilde{m} \equiv m \pmod 3.$$

Fix any admissible row (s, j) and let (α, k, δ) be its parameters, with the update

$$x' \;=\; 6\big(2^\alpha m + k\big) + \delta, \qquad \tilde{x}' \;=\; 6\big(2^\alpha \tilde{m} + k\big) + \delta.$$

Then:

(i) The output families coincide: $x' \equiv \tilde{x}' \equiv \delta \pmod 6$.

(ii) The next *index matches:*

$$j' \;:=\; \left\lfloor \frac{x'}{6} \right\rfloor \bmod 3 \;=\; \Big(2^\alpha m + k\Big) \bmod 3 \;=\; \Big(2^\alpha \tilde{m} + k\Big) \bmod 3 \;=:\; \tilde{j}'.$$

Proof.

- $x \equiv \tilde{x} \pmod{54}$ gives $x \equiv \tilde{x} \pmod 6$ and $\lfloor x/6 \rfloor \equiv \lfloor \tilde{x}/6 \rfloor \pmod 3$, hence $p = \tilde{p}$ and $j = \tilde{j}$.
- Also $\tilde{m} - m = \lfloor \tilde{x}/18 \rfloor - \lfloor x/18 \rfloor$ is a multiple of 3, i.e. $\tilde{m} \equiv m \pmod 3$.
- Since $\lfloor x'/6 \rfloor = 2^\alpha m + k$, we get $(2^\alpha m + k) \equiv (2^\alpha \tilde{m} + k) \pmod 3$ and therefore $j' = \tilde{j}'$.

∎

Example 10 (After Lemma 15)**.** Take $x = 1$ and $\tilde{x} = 55$ ($\equiv 1 \pmod{54}$): both have $s = \mathrm{e}$, $j = 0$. Under the first token ψ (row $(\mathrm{e}, 0)$), each maps to an o-family number with the same next index $j' = 0$, so the next row selection (for a fixed token) agrees.

Remark (Caution: persistence beyond one step). Lemma 15 aligns the *next* (s, j) after one identical row. At the second step m' and $\tilde{m}'$ can differ by 2^α multiples that may not be $0 \pmod 3$, so j'' may diverge unless stronger congruences hold (e.g. modulo 162). The family pattern remains identical by Lemma 11.

Corollary 10 (Infinite integer realizations of a fixed word)**.** *Let W be admissible from some start with family S. Then there are infinitely many odd x_0 for which the integer sequence driven by W is well-defined and certified by $U(x') = x$ at every step. Moreover, for any $K \geq 3$ and odd residue r* mod M_K *with terminal family matching r* mod 6*, the congruence in m has infinitely many solutions, yielding infinitely many realizations with $x_W(m) \equiv r \pmod{M_K}$.*

Proof.

- Varying m in the first step gives infinitely many outputs.
- Proceeding by the fixed tokens remains valid via routing/type; each step satisfies Lemma 10.

- For fixed K, the linear congruence modulo 2^{K-1} has infinitely many solutions in m.

$\square$

Example 11 (After Corollary 10)**.** For $W=\psi$ and $K=3$, $x_W(m)=96m+5\equiv 5 \pmod{24}$ for all m. Thus infinitely many x_0 realize the same residue class 5 mod 24.

Theorem 11 (Backward–uniqueness criterion at $p=0$)**.** *Fix the $p=0$ inverse table. Let x_{tar} be an odd integer. Suppose there exists a finite chain*

$$x_0,\ x_1,\ \dots,\ x_L=x_{\mathrm{tar}}$$

such that for each $k=1,\dots,L$ there is exactly one *table row R_k and an integer $m_{k-1}=\lfloor x_{k-1}/18\rfloor$ with*

$$x_k \;=\; A_{R_k}\,m_{k-1}+B_{R_k},$$

and the admissibility constraints (family e/o and router $j=\lfloor x_{k-1}/6\rfloor \bmod 3$) required by R_k hold at x_{k-1}. Then:

(1) The word $W:=R_1R_2\cdots R_L$ (read forward) is the unique *$p=0$ word that maps x_0 to x_{tar} via the certified inverse calculus.*

(2) Its length L is minimal: there is no shorter $p=0$ word taking x_0 to x_{tar}.

Proof sketch. Uniqueness: by assumption, at each backward step x_k admits a single admissible predecessor x_{k-1} and a single row R_k realizing $x_k=A_{R_k}m_{k-1}+B_{R_k}$ with the required $(e/o,j)$ of x_{k-1}. Thus the predecessor is determined, and induction forces a single backward chain and hence a single forward word.

Minimality: if a shorter word existed, tracing it backward would produce a contradiction with the "exactly one admissible predecessor" property at the first index where it diverges from the forced chain. $\square$

Corollary 12 (Uniqueness of the $p=0$ path $1\to 49$)**.** *In the $p=0$ table, the unique word sending $x_0=1$ to $x_{\mathrm{tar}}=49$ is*

$$\boxed{\psi,\ \omega,\ \psi,\ \Omega,\ \omega,\ \Psi,\ \Psi}\ ,$$

with value path

$$1 \xrightarrow{\psi_0} 5 \xrightarrow{\omega_0} 13 \xrightarrow{\psi_2} 17 \xrightarrow{\Omega_2} 11 \xrightarrow{\omega_1} 7 \xrightarrow{\Psi_1} 37 \xrightarrow{\Psi_0} 49.$$

Proof. Work backward and note that at each target x there is exactly one admissible predecessor:

(i) 49: Only Ψ_0 fits $24m+1=49\Rightarrow m=2$, and among $m=2$ e-family candidates $\{37,43,49\}$ only $x=37$ has $j=\lfloor x/6\rfloor \bmod 3=0$ needed by Ψ_0. So $37\xrightarrow{\Psi_0}49$.

(ii) 37: Only Ψ_1 fits $96m+37=37\Rightarrow m=0$, and among $m=0$ e-family $\{1,7,13\}$ only $x=7$ has $j=1$ required by Ψ_1. So $7\xrightarrow{\Psi_1}37$.

(iii) 7: Only ω_1 fits $12m+7=7\Rightarrow m=0$, and among $m=0$ o-family $\{5,11,17\}$ only $x=11$ has $j=1$. So $11\xrightarrow{\omega_1}7$.

(iv) 11: Only Ω_2 fits $12m+11=11\Rightarrow m=0$, and among $\{5,11,17\}$ only $x=17$ has $j=2$. So $17\xrightarrow{\Omega_2}11$.

(v) 17: Only ψ_2 fits $24m+17=17\Rightarrow m=0$, and among $m=0$ e-family $\{1,7,13\}$ only $x=13$ has $j=2$. So $13\xrightarrow{\psi_2}17$.

(vi) 13: Only ω_0 fits $48m+13=13\Rightarrow m=0$, and among $\{5,11,17\}$ only $x=5$ has $j=0$. So $5\xrightarrow{\omega_0}13$.

(vii) 5: Only ψ_0 fits $96m+5=5\Rightarrow m=0$, and among $\{1,7,13\}$ only $x=1$ has $j=0$. So $1\xrightarrow{\psi_0}5$.

Each step is forced; Theorem 11 applies, proving uniqueness and minimality. $\square$

12. Row design and the forward identity

We parametrize each unified row by $(\alpha, \beta, c, \delta)$ and use

$$F(p,m) = \frac{(9m\,2^{\alpha} + \beta)\,64^{\,p} + c}{9}, \qquad x'_p \;=\; 6F(p,m) + \delta,$$

with input written in normal form $x = 18m + 6j + p_6$ where $j \in \{0,1,2\}$ and $p_6 \in \{1,5\}$. The case $p = 0$ reduces to $F(0,m) = 2^{\alpha}m + k$ where $k := (\beta + c)/9 \in \mathbb{Z}$.

Lemma 16 (Row design constraints)**.** *Suppose a row with fixed $(\alpha, \beta, c, \delta)$ satisfies*

$$\beta \;=\; 2^{\alpha-1}(6j + p_6), \qquad c \;=\; -\frac{3\delta+1}{2}, \qquad k = \frac{\beta + c}{9} \in \mathbb{Z}. \tag{1}$$

Then for every odd input $x = 18m + 6j + p_6$ one has the forward identity

$$3x'_p + 1 \;=\; 2^{\alpha+6p}\,x \qquad \textit{for all } p \geq 0,$$

hence $U(x'_p) = x$.

Proof. Compute

$$x'_p = 6\left(2^{\alpha+6p}m + \frac{\beta\,64^{\,p} + c}{9}\right) + \delta \quad\Rightarrow\quad 3x'_p + 1 = 18 \cdot 2^{\alpha+6p}m + \left(2\beta\,64^{\,p} + 2c + 3\delta + 1\right).$$

With $c = -(3\delta+1)/2$ the constant cancels, giving $3x'_p + 1 = 18 \cdot 2^{\alpha+6p}m + 2\beta\,64^{\,p}$. Using $\beta = 2^{\alpha-1}(6j + p_6)$ and $64^{\,p} = 2^{6p}$,

$$2\beta\,64^{\,p} = 2^{\alpha}(6j + p_6)\,2^{6p} = 2^{\alpha+6p}(6j + p_6).$$

Thus $3x'_p + 1 = 2^{\alpha+6p}(18m + 6j + p_6) = 2^{\alpha+6p}x$. Since x is odd, $\nu_2(3x'_p + 1) = \alpha + 6p$ and $U(x'_p) = x$. □

Remark (Integrality). Because $64 \equiv 1 \pmod 9$, $\beta\,64^{\,p} + c \equiv \beta + c \pmod 9$; hence $F(p,m) \in \mathbb{Z}$ whenever $k = (\beta + c)/9 \in \mathbb{Z}$. This is enforced row-by-row by (1).

Proposition 5 (Checklist for a table row)**.** *To certify a row, it suffices to exhibit integers $(\alpha, \beta, c, \delta)$ and (j, p_6) with $j \in \{0,1,2\}$, $p_6 \in \{1,5\}$ so that (1) holds. Then Lemma 16 implies $3x'_p + 1 = 2^{\alpha+6p}x$ for all $p \geq 0$.*

13. Super-families via p–lift and a $p = 1$ table

For any $p \geq 0$, each row lifts to

$$F_p(0,m) = \frac{(9m\,2^{\alpha} + \beta)\,64^{\,p} + c}{9} = 2^{\alpha+6p}m + \frac{\beta\,64^{\,p} + c}{9}, \qquad x'_p = 6F_p(0,m) + \delta.$$

Lemma 17 (Row correctness with p–lift)**.** *For any admissible row $(\alpha, \beta, c, \delta)$ and any $p \geq 0$, if $x = 18m + 6j + p_6$ ($p_6 \in \{1,5\}$), then*

$$3x'_p + 1 \;=\; 2^{\alpha+6p}\,x, \qquad \textit{so} \qquad U(x'_p) = x.$$

Proof.

- Expand $3x'_p + 1 = 18 \cdot 2^{\alpha+6p}m + \left(18 \cdot \frac{64^p\beta + c}{9} + 3\delta + 1\right)$.
- Using $64 \equiv 1 \pmod 9$ and the p=0 identity, the bracket equals $2^{\alpha+6p}(6j + p_6)$.
- Thus $3x'_p + 1 = 2^{\alpha+6p}x$ and $U(x'_p) = x$.

□

Example 12 (After Lemma 17)**.** Row $(\mathrm{e}, 0)$ with Ψ_0 has $\alpha = 2, \beta = 2, c = -2, \delta = 1$. For $p = 1$, $F(1,m) = 256m + 14$, so with $x = 1$ $(m = 0)$ we get $x'_1 = 85$. Then $3 \cdot 85 + 1 = 256 = 2^8 = 2^{\alpha+6} \cdot 1$.

Corollary 13 (Words with p–lift)**.** *If a word W is admissible at p=0, then its p–lifted version is admissible and has*

$$x_{W,p}(m) = 6\big(A_W\, 2^{6p} m + B_{W,p}\big) + \delta_W.$$

Thus the 2–power in the forward identity gains $6p$ per step; padding in the p=0 world emulates working at $p > 0$.

Proof.

- Apply Lemma 17 stepwise; each step multiplies the 2–power by 2^6.
- The type ($*$e vs $*$o) and δ_W are unchanged, so routing/navigation stays the same.
- The affine form accumulates the additional 2^{6p} in the slope.

□

Example 13 (After Corollary 13)**.** For $W = \psi$ (one step, slope factor 2^4 at $p = 0$), the $p = 1$ lift has slope $2^{4+6} = 2^{10}$, i.e. $x_{W,1}(m) = 6(2^{10}m + \cdots) + 5$.

Table 6. Unified $p = 1$ forms with $F(1, m) = \dfrac{(9m2^\alpha + \beta)\,64 + c}{9}$ and $x'_1(m) = 6F(1, m) + \delta$.

(s, j)	type	move	$F(1, m)$	$x'_1(m) = 6\,F(1, m) + \delta$
(e, 0)	ee	Ψ_0	$256m + 14$	$1536m + 85$
(e, 1)	ee	Ψ_1	$1024m + 398$	$6144m + 2389$
(e, 2)	ee	Ψ_2	$4096m + 2958$	$24576m + 17749$
(o, 0)	oe	ω_0	$512m + 142$	$3072m + 853$
(o, 1)	oe	ω_1	$128m + 78$	$768m + 469$
(o, 2)	oe	ω_2	$2048m + 1934$	$12288m + 11605$
(e, 0)	eo	ψ_0	$1024m + 56$	$6144m + 341$
(e, 1)	eo	ψ_1	$4096m + 1592$	$24576m + 9557$
(e, 2)	eo	ψ_2	$256m + 184$	$1536m + 1109$
(o, 0)	oo	Ω_0	$2048m + 568$	$12288m + 3413$
(o, 1)	oo	Ω_1	$512m + 312$	$3072m + 1877$
(o, 2)	oo	Ω_2	$128m + 120$	$768m + 725$

Mixing the column parameter p stepwise (“mixed-p” words). At any step you may use the p–lift of a row (possibly with a different p than in the previous step):

$$F(p, m) = \frac{(9m\,2^\alpha + \beta)\,64^{\,p} + c}{9}, \qquad x' \;=\; 6\,F(p, m) + \delta,$$

where $(\alpha, \beta, c, \delta)$ are the fixed parameters of that row in the unified table and $p \in \mathbb{Z}_{\geq 0}$ is chosen *for that step only*. This preserves both admissibility and the odd-forward identity.

Lemma 18 (Step correctness under mixed-p)**.** *For any odd input $x = 18m + 6j + p_6$ with $p_6 \in \{1, 5\}$, any admissible row, and any $p \geq 0$,*

$$3x' + 1 \;=\; 2^{\alpha+6p}\,x \qquad \Rightarrow \qquad U(x') = x.$$

Proof outline.

- Expand $3x' + 1 = 18 \cdot 2^{\alpha+6p}m + \big(18 \cdot \frac{64^p\beta+c}{9} + 3\delta + 1\big)$.
- Because $64 \equiv 1 \pmod 9$, $\frac{64^p\beta+c}{9} \in \mathbb{Z}$ and the bracket equals $2^{\alpha+6p}(6j + p_6)$.
- Hence $3x' + 1 = 2^{\alpha+6p}x$, so $U(x') = x$.

□

Lemma 19 (Routing and type are p–invariant)**.** *For a fixed row, the* `type` *(*`ee`*,* `eo`*,* `oe`*,* `oo`*) and the offset $\delta \in \{1, 5\}$ do not depend on p. Hence routing/type constraints are unchanged under mixed-p evaluation.*

Proposition 6 (Affine form for mixed-p words)**.** *Let $W = \sigma_1 \cdots \sigma_t$ be admissible and choose per-step lifts $p_i \geq 0$. Then*

$$x_{W,\vec{p}}(m) \;=\; 6\big(A_{W,\vec{p}}\, m + B_{W,\vec{p}}\big) + \delta_W, \qquad A_{W,\vec{p}} = 3 \cdot 2^{\sum_i^t (\alpha_i + 6p_i)},$$

where α_i is the row exponent used at step i and δ_W is the last row's offset.

Proof sketch.

- Compose the one–step affine maps $x \mapsto 6(2^{\alpha_i + 6p_i} m + k_{p_i}) + \delta_i$.
- Slopes multiply and the outer 6 carries through; the last δ survives.

□

Remark (Parity caveat for $p \geq 1$). The step constant is $k_p = \dfrac{64^p \beta + c}{9}$. For all rows and $p \geq 1$, $k_p \equiv k + \beta \pmod 2$ with $k = (\beta + c)/9$; in many rows this is even. Single–step parity flips visible at p=0 can vanish at $p \geq 1$. Keep at least one p=0 odd–k row available if you need to toggle the intercept parity in a lifting congruence.

14. Same–family padding as *steering* (unified notion)

Definition 4 (Same–family padding / steering gadget)**.** A short admissible word P with overall type $s \to s$ (i.e. `ee` if $s = \mathrm{e}$, `oo` if $s = \mathrm{o}$) is a *steering gadget.* Appending P to a word W preserves the terminal family while giving control over:

- the 2–adic slope (raising $v_2(A)$ in the affine form), and
- the intercept parity $B_W \bmod 2$.

Lemma 20 (Steering lemma)**.** *Let W be admissible with affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$, $A_W = 3 \cdot 2^{\alpha(W)}$. There exist short same–family words $P^{(0)}, P^{(1)}$ (type $s \to s$) such that*

- ***Slope boost.*** *$x_{W \cdot P^{(\varepsilon)}}(m) = 6(A' m + B'_\varepsilon) + \delta_W$ with $A' = A_W \cdot 2^d$ for some $d \geq 1$ (repeat gadgets to enlarge d).*
- ***Parity control.*** *$B'_0 \equiv B_W \pmod 2$ while $B'_1 \equiv B_W + 1 \pmod 2$.*

Consequently, for any $K \geq 3$ and target $r \equiv \delta_W \pmod 6$, there is padded W^* and m with $x_{W^*}(m) \equiv r \pmod{M_K = 3 \cdot 2^K}$.

Proof.

- Appending a same–family row multiplies the slope by $2^{\alpha_{\text{row}}} \geq 2$ and keeps δ; repeating boosts $v_2(A)$.
- Among same–family menus, at least one gadget changes $B \bmod 2$ (via an odd one–step constant k); another preserves it.
- With $v_2(A)$ large enough and parity chosen, the linear congruence $Am \equiv \frac{r - \delta_W}{6} - B \pmod{2^{K-1}}$ is solvable.

□

Remark (Steering intuition). The family (mod6) is the lane; steering gadgets keep you in that lane and let you nudge the position in $\bmod 3 \cdot 2^K$ until it matches the target residue.

Gadget drills: parity toggle and mod-3 steering. We illustrate the two steering knobs: (i) a parity flip on the intercept B mod 2 and (ii) setting B mod 3 to a prescribed value, while staying in the same terminal family.

Example 14 (Parity flip in family o)**.** Start with any word W whose terminal family is o and affine form $x_W(m) = 6(Am+B)+5$. Appending the single Ω_2 row (type `oo`, (o, 2), $x' = 12m+11 = 6(2m+1)+5$) sends

$$B \;\longmapsto\; B' \equiv 2B+1 \pmod 2 \quad \text{(flip)},$$

and raises $v_2(A)$ by $+1$. Thus $W \cdot \Omega_2$ keeps terminal family o, flips B mod 2, and increases divisibility by 2 in the slope.

Example 15 (Setting B mod 3 in family e)**.** In family e, the `ee` rows have

$$\Psi_0 : \; x' = 24m+1 = 6(4m+0)+1, \qquad \Psi_2 : \; x' = 384m+277 = 6(64m+46)+1.$$

Modulo 3, these update $B \mapsto B$ (for Ψ_0) and $B \mapsto B+1$ (for Ψ_2). Therefore, in at most two `ee` steps we can force $B' \equiv r \pmod 3$ for any chosen $r \in \{0,1,2\}$ while staying in family e and increasing $v_2(A)$.

15. Evolution of the index m along inverse words

Fix any admissible inverse word $W = \sigma_0\sigma_1\cdots\sigma_{n-1}$ evaluated by the unified table (with optional per–step lifts). At step t the selected row carries parameters $(\alpha_t, \beta_t, c_t, \delta_t)$ and (parent) index $j_t \in \{0,1,2\}$; write

$$a_t := 2^{\alpha_t} \quad (\text{or } 2^{\alpha_t+6p_t} \text{ if a lift } p_t \ge 0 \text{ is used}), \qquad k_{p_t} := \frac{\beta_t\, 64^{\,p_t} + c_t}{9} \in \mathbb{Z},$$

and set $b_t := k_{p_t}$ for brevity (so at $p_t = 0$, $b_t = (\beta_t + c_t)/9$).

Let x_t denote the child at step t and $m_t = \lfloor x_t/18 \rfloor$ the corresponding index; the unified update reads

$$x_t \;=\; 6\big(a_t m_{t-1} + b_t\big) + \delta_t, \qquad 3x_t + 1 \;=\; 2^{\alpha_t+6p_t}\, x_{t-1}.$$

Dividing by 6 and then by 3 gives the index recurrence

$$m_t \;=\; \frac{a_t m_{t-1} + b_t - j_t}{3}, \qquad \text{where } j_t \equiv a_t m_{t-1} + b_t \pmod 3,\; j_t \in \{0,1,2\}. \tag{2}$$

Note on deriving the index j_t. By construction each step has the form

$$x_t \;=\; 6\big(a_t\, m_{t-1} + b_t\big) + \delta_t, \qquad \delta_t \in \{1,5\}.$$

Hence

$$\left\lfloor \frac{x_t}{6} \right\rfloor \;=\; a_t\, m_{t-1} + b_t,$$

and, by definition of the per–step index,

$$j_t \;:=\; \left\lfloor \frac{x_t}{6} \right\rfloor \bmod 3 \;=\; \big(a_t\, m_{t-1} + b_t\big) \bmod 3.$$

Therefore the index recurrence follows by removing the residue modulo 3:

$$m_t \;=\; \left\lfloor \frac{x_t}{18} \right\rfloor = \left\lfloor \frac{1}{3} \left\lfloor \frac{x_t}{6} \right\rfloor \right\rfloor = \frac{\big(a_t\, m_{t-1} + b_t\big) - \big((a_t\, m_{t-1} + b_t) \bmod 3\big)}{3} = \frac{a_t\, m_{t-1} + b_t - j_t}{3}.$$

Definition 5 (Tail products)**.** For $0 \le t \le n-1$ set

$$P_{n,t+1} \;:=\; \prod_{u=t+1}^{n-1} a_u \quad (\text{empty product } = 1), \qquad A_n \;:=\; P_{n,0} \;=\; \prod_{u=0}^{n-1} a_u \;=\; 2^{\sum_{u=0}^{n-1}(\alpha_u + 6p_u)}.$$

Proposition 7 (Closed form for m_n)**.** *Unrolling* (2) *yields, for every* $n \ge 1$,

$$\boxed{m_n \;=\; \frac{A_n\, m_0}{3^n} \;+\; \sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\, (b_t - j_t)} \tag{3}$$

In particular m_n is affine in m_0 with slope $A_n/3^n$; all nonlinearity from table navigation is isolated in the bounded corrections $j_t \in \{0,1,2\}$.

Proof sketch. From (2), $3m_t = a_t m_{t-1} + (b_t - j_t)$. Iterating gives $3^2 m_{t+1} = a_{t+1}a_t m_{t-1} + a_{t+1}(b_t - j_t) + (b_{t+1} - j_{t+1})$, and so on. Induction on n yields (3). □

Corollary 14 (Including per–step lifts p_t)**.** *The formula (3) remains valid when a lift $p_t \geq 0$ is used at step t: replace a_t by $2^{\alpha_t + 6p_t}$ and b_t by $k_{p_t} = \frac{\beta_t\, 64^{p_t} + c_t}{9}$.*

Special start $x_0 = 1$. If $x_0 = 1$, then $m_0 = \lfloor 1/18 \rfloor = 0$. Hence the first term in (3) vanishes and

$$m_n \;=\; \sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;=\; \frac{\displaystyle\sum_{t=0}^{n-1} 3^{\,t}\, P_{n,t+1}\,(b_t - j_t)}{3^n},$$

so the index evolution is completely determined by the chosen rows (via a_t, b_t) and the induced remainders j_t.

Remarks.

- The "mean-field" trajectory obtained by pretending $j_t \equiv 0$ has the same form as (3) with b_t in place of $b_t - j_t$; the actual m_n differs by a bounded 3-adic correction since each $j_t \in \{0,1,2\}$.
- When all steps are chosen from a fixed family and with fixed lift, $a_t \equiv a$ and the tail products are geometric: $P_{n,t+1} = a^{\,n-1-t}$. Then (3) simplifies to a single weighted geometric sum in 3 and a.

Lemma 21 (Zero-start independence of the index sequence)**.** *Fix any admissible mixed-p word $W = \sigma_1 \cdots \sigma_n$, and write for each step*

$$a_u := \alpha(\sigma_u) + 6\,p_u, \qquad k_u := \frac{\beta(\sigma_u)\,64^{\,p_u} + c(\sigma_u)}{9} \in \mathbb{Z} \qquad (u = 1, \ldots, n).$$

Let $m_{u+1} = \left\lfloor \frac{2^{a_u}\, m_u + k_u}{3} \right\rfloor$ be the per-step update of the index m (so $m_u = \lfloor x_u/18 \rfloor$ and $x_{u+1} = 6(2^{a_u} m_u + k_u) + \delta(\sigma_u)$). If the start index is $m_0 = 0$ (equivalently $x_0 \in \{1,5\}$), then the entire sequence $m_1, \ldots, m_n$ is uniquely determined by W alone. In particular, for $m_0 = 0$ the values m_u are independent of any external parameter or target and depend only on the chosen tokens and lifts.

Proof. By definition, $m_{u+1} = \lfloor (2^{a_u} m_u + k_u)/3 \rfloor$. Given $m_0 = 0$, $m_1 = \lfloor k_1/3 \rfloor$ is determined by (a_1, k_1), hence by σ_1 and p_1. Inductively, if m_u is determined by $\sigma_1, \ldots, \sigma_u$ and $p_1, \ldots, p_u$, then so is m_{u+1}. Thus $m_1, \ldots, m_n$ depend only on W (the rows and lifts) and not on any additional choice. □

Remark. A convenient explicit (floor-free) surrogate is obtained by ignoring the floors:

$$\widetilde{m}_{u+1} \;=\; \frac{2^{a_u}\,\widetilde{m}_u + k_u}{3}, \qquad \widetilde{m}_0 = 0,$$

which solves to $\widetilde{m}_n = \sum_{t=1}^{n} \frac{\left(\prod_{r=t+1}^{n} 2^{a_r}\right) k_t}{3^{\,n-t+1}}$. Then $m_n = \lfloor \widetilde{m}_n - \varepsilon_n \rfloor$ for some $0 \leq \varepsilon_n < 1$ coming from the nested floors. Either way, once m_0 is fixed (in particular, $m_0 = 0$), the actual m_n is a function of W alone.

Corollary 15. *Starting from $x_0 \in \{1,5\}$ (so $m_0 = 0$), the sequence of indices m_u and hence the entire sequence of updates $x_{u+1} = 6(2^{a_u} m_u + k_u) + \delta(\sigma_u)$ is completely determined by the chosen word W.*

15.1. **Monotonicity of the index m_n and its relation to row exponents.** Recall that for a single certified inverse step with row parameters $(\alpha, \beta, c, \delta)$ and lift $p \geq 0$, the child is

$$y \;=\; 6\big(2^{\alpha+6p} m + k_p\big) + \delta, \qquad k_p \;=\; \frac{\beta\, 64^{\,p} + c}{9} \in \mathbb{Z},$$

where the parent's index is $m = \lfloor x/18 \rfloor$ and $\delta \in \{1,5\}$. The next index (for the child) is therefore

$$m^+ \;=\; \Big\lfloor \frac{y}{18} \Big\rfloor \;=\; \Big\lfloor \frac{2^{\alpha+6p}}{3}\, m \;+\; \frac{k_p}{3} \;+\; \frac{\delta}{18} \Big\rfloor. \tag{4}$$

Set

$$c(\alpha,p) := \frac{2^{\alpha+6p}}{3}, \qquad b_p(\delta) := \frac{k_p}{3} + \frac{\delta}{18} \in \Big(\mathbb{Z}/3\Big) + \Big\{\tfrac{1}{18}, \tfrac{5}{18}\Big\}.$$

Then (4) reads concisely as

$$m^+ \;=\; \big\lfloor c(\alpha,p)\, m + b_p(\delta) \big\rfloor. \tag{5}$$

Lemma 22 (Forward $\boldsymbol{U}$ monotonicity vs. inverse m growth)**.** *For a single step with parameters (α,p):*

(1) If $\alpha = 1$ and $p = 0$ (the rows ω_1 and Ω_2), then $c(\alpha,p) = \frac{2}{3} < 1$. In forward time this is exactly the unique case $U(y) > y$; in inverse time the index update (5) is a contraction up to a bounded offset:

$$m^+ \;=\; \Big\lfloor \tfrac{2}{3}m + b_0(\delta) \Big\rfloor \;\le\; \tfrac{2}{3}m + b_0(\delta),$$

hence $m^+ < m$ for all $m > 3\,b_0(\delta)$.

(2) If $\alpha + 6p \ge 2$ (all other rows, or any $p \ge 1$), then $c(\alpha,p) \ge \frac{4}{3} > 1$. In forward time these steps strictly decrease ($U(y) < y$); in inverse time the index typically grows linearly:

$$m^+ \;\ge\; \Big\lfloor \tfrac{4}{3}m + b_p(\delta) \Big\rfloor \;\ge\; m + \Big\lfloor \tfrac{m}{3} + b_p(\delta) \Big\rfloor,$$

so for all sufficiently large m one has $m^+ \ge m + 1$.

Proof. The forward monotonicity trichotomy follows from $U(y) = (3y+1)/2^{\alpha+6p}$ and elementary inequalities (cf. the "Forward monotonicity by row and lift" lemma). The statements about m^+ are direct consequences of (5) with $c(\alpha,p) = 2^{\alpha+6p}/3$ and the bounds $0 < \delta/18 < 1/3$. □

Remark (Interpretation). Lemma 22 shows a clean *duality*: the only forward-increasing case ($\alpha = 1, p = 0$) is the only case in which the inverse m-dynamics is (eventually) contracting; all other rows (or any $p \ge 1$) are forward-decreasing and produce an expansive m-update in inverse time. Thus along an inverse word, the trend of the index m_n is governed just by the exponents $\alpha_u + 6p_u$ of the chosen rows.

Two quick instantiations.

- *Row ω_1 (α=1, p=0, δ=1).* Here $k = (\beta + c)/9 = (11-2)/9 = 1$, so
$$m^+ = \Big\lfloor \tfrac{2}{3}m + \tfrac{1}{3} + \tfrac{1}{18} \Big\rfloor = \Big\lfloor \tfrac{2}{3}m + \tfrac{7}{18} \Big\rfloor,$$
hence $m^+ < m$ for all $m \ge 3$.
- *Row Ψ_0 (α=2, p=0, δ=1).* Here $k = (2-2)/9 = 0$, so
$$m^+ = \Big\lfloor \tfrac{4}{3}m + \tfrac{1}{18} \Big\rfloor \ge \Big\lfloor \tfrac{4}{3}m \Big\rfloor,$$
which grows by at least 1 once $m \ge 1$.

Over a longer admissible word $W = (\sigma_1, \dots, \sigma_t)$ with per-step exponents $\alpha_u + 6p_u$, iterating (5) yields

$$m_{u+1} = \big\lfloor c_u\, m_u + b_u \big\rfloor, \qquad c_u = \frac{2^{\alpha_u + 6p_u}}{3}, \quad b_u = \frac{k_{p_u}}{3} + \frac{\delta_u}{18},$$

so the sign of $\log c_u$ (i.e. whether $\alpha_u + 6p_u$ equals 1 or at least 2) controls local contraction/expansion of m in inverse time, perfectly mirroring the forward increase/decrease of U on that step.

16. Recovering x_n from m_n and terminal index

Let a legal word W of length n be fixed, with per–step parameters

$$(\alpha_t, p_t, b_t, j_t, \delta_t) \qquad (t = 0, 1, \dots, n-1),$$

where $\alpha_t \in \{1,2,3,4,5,6\}$ (row exponent), $p_t \in \mathbb{Z}_{\geq 0}$ (column lift), $b_t = (\beta_t + c_t)/9 \in \mathbb{Z}$ (row constant at p=0), $j_t = \lfloor x_t/6 \rfloor \bmod 3$ is the parent index before step t, and $\delta_t \in \{1,5\}$ encodes the *child* family after step t. Define the cumulative two–power multiplier (read "from the future back to t")

$$P_{n,t+1} \;:=\; \prod_{u=t+1}^{n-1} 2^{\,\alpha_u + 6p_u}, \qquad \text{with} \qquad P_{n,n} := 1.$$

Proposition 8 (Closed form for m_n)**.** *With the notation above, the 18–index after n steps is*

$$m_n \;=\; \sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t).$$

Write the terminal family and index as

$$\delta_n := \delta_{\,n-1} \in \{1,5\}, \qquad j_n := \left\lfloor \frac{x_n}{6} \right\rfloor \bmod 3 \in \{0,1,2\}.$$

By definition of our normal form,

$$x_n \;=\; 18\,m_n + 6\,j_n + \delta_n.$$

Proposition 9 (Closed form for x_n and terminal-index cases)**.** *Substituting Proposition 8 gives*

$$\boxed{\;x_n \;=\; 18\sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;+\; 6j_n \;+\; \delta_n\;}\,.$$

Equivalently, grouping constants,

$$x_n \;=\; 6\left(3\sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;+\; j_n\right) + \delta_n.$$

In particular, $x_n \equiv \delta_n \pmod 6$ automatically. The additive contribution of the terminal index j_n is exactly $6j_n$, which yields the following three cases:

$$\begin{cases} \textbf{\textit{Case}}\ j_n = 0: \quad x_n \;=\; 18\displaystyle\sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;+\; \delta_n, \\ \textbf{\textit{Case}}\ j_n = 1: \quad x_n \;=\; 18\displaystyle\sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;+\; 6 \;+\; \delta_n, \\ \textbf{\textit{Case}}\ j_n = 2: \quad x_n \;=\; 18\displaystyle\sum_{t=0}^{n-1} \frac{P_{n,t+1}}{3^{\,n-t}}\,(b_t - j_t) \;+\; 12 \;+\; \delta_n. \end{cases}$$

Remarks.

- The dependence on the intermediate indices j_t ($t < n$) appears only inside the sum through the differences $b_t - j_t$.
- The terminal family δ_n (the second letter of the last row's type) fixes $x_n \bmod 6$; the terminal index j_n then selects the representative within that residue class modulo 18.
- In the special seed $x_0 = 1$ one has $m_0 = 0$; the formulas above remain valid with the same $P_{n,t+1}$ and $b_t - j_t$.

17. Geometric-series simplifications for m_n (and x_n)

Recall the index recurrence

$$m_t \;=\; \frac{a_t\,m_{t-1} + b_t - j_t}{3}, \qquad a_t := 2^{\alpha_t + 6p_t},$$

which unrolls to

$$(*)\qquad m_n \;=\; \left(\prod_{u=1}^{n} \frac{a_u}{3}\right) m_0 \;+\; \sum_{t=1}^{n} \left(\prod_{u=t+1}^{n} \frac{a_u}{3}\right) \frac{b_t - j_t}{3}.$$

Proposition 10 (Constant-row case: closed form via a geometric series)**.** *Suppose the same row (and lift p) is used at every step, i.e. $a_t \equiv a$ and $b_t - j_t \equiv q$ are constant. Then*

$$m_n \;=\; \left(\tfrac{a}{3}\right)^n m_0 \;+\; \frac{q}{3} \sum_{k=0}^{n-1} \left(\tfrac{a}{3}\right)^k.$$

Hence, for $a \neq 3$,

$$m_n \;=\; \left(\tfrac{a}{3}\right)^n m_0 \;+\; \frac{q}{3} \cdot \frac{1 - \left(\tfrac{a}{3}\right)^n}{1 - \tfrac{a}{3}}, \qquad \textit{and if } a = 3: \quad m_n = m_0 + n\,\frac{q}{3}.$$

Proof. With $a_t \equiv a$ and $b_t - j_t \equiv q$, the recurrence is linear: $m_t = \frac{a}{3} m_{t-1} + \frac{q}{3}$. Unrolling yields the stated geometric sum. □

Corollary 16 (From m_n to x_n)**.** *At time n, x_n satisfies $x_n = 18\, m_n + 6\, j_n + \delta_n$. In the constant-row case of Prop. 10, if moreover j_n and δ_n are fixed (e.g. the row is* `ee` *or* `oo` *and the family/index stay constant), then*

$$x_n \;=\; 18\left[\left(\tfrac{a}{3}\right)^n m_0 \;+\; \frac{q}{3} \cdot \frac{1 - \left(\tfrac{a}{3}\right)^n}{1 - \tfrac{a}{3}}\right] \;+\; 6j + \delta.$$

When $m_0 = 0$ (e.g. starting from $x_0 = 1$), this reduces to a pure geometric expression in n.

Remark (Periodic or bounded coefficients). If $a_t \equiv a$ is constant but $q_t := b_t - j_t$ is L-periodic, then

$$m_n \;=\; \left(\tfrac{a}{3}\right)^n m_0 \;+\; \sum_{r=0}^{L-1} \frac{q_r}{3} \sum_{\substack{0 \le k \le n-1 \\ k \equiv r \pmod{L}}} \left(\tfrac{a}{3}\right)^k,$$

a sum of L geometric progressions. In the fully variable case, $(*)$ still gives bounds by comparing $\prod_{u=t+1}^{n} \frac{a_u}{3}$ to a dominating geometric sequence.

Index recurrence with q. From $x_t = 6(a\, m_{t-1} + b) + \delta$ and $m_t = \lfloor x_t/18 \rfloor$, we obtain

$$m_t \;=\; \frac{a\, m_{t-1} + b - j_t}{3},$$

where $a = 2^{\alpha+6p}$, $b = k_p$, and $j_t = \lfloor x_t/6 \rfloor \bmod 3$. To avoid notational conflict with the row parameter c, define

$$\boxed{q \;:=\; b - j_t \;=\; k_p - j_t, \qquad k_p \;=\; \frac{\beta\, 64^{\,p} + c_{\text{row}}}{9} \in \mathbb{Z}}$$

so the recurrence reads

$$\boxed{m_t \;=\; \frac{a\, m_{t-1} + q}{3}, \qquad a = 2^{\alpha+6p}.}$$

At p=0. Here $k_0 = \dfrac{\beta + c_{\text{row}}}{9}$, hence $q = k_0 - j_t$.

Table 7. Index recurrence $m_t = \frac{a\, m_{t-1}+q}{3}$ for each row at p=0. Here $a = 2^\alpha$, $k_0 = \frac{\beta+c}{9}$, $q = k_0 - j$.

Row	(s,j)	α	a	k_0	$q = k_0 - j$	Recurrence
Ψ_0	$(\mathrm{e},0)$	2	4	0	0	$m_t = \frac{4\,m_{t-1}+0}{3}$
Ψ_1	$(\mathrm{e},1)$	4	16	6	5	$m_t = \frac{16\,m_{t-1}+5}{3}$
Ψ_2	$(\mathrm{e},2)$	6	64	46	44	$m_t = \frac{64\,m_{t-1}+44}{3}$
ω_0	$(\mathrm{o},0)$	3	8	2	2	$m_t = \frac{8\,m_{t-1}+2}{3}$
ω_1	$(\mathrm{o},1)$	1	2	1	0	$m_t = \frac{2\,m_{t-1}+0}{3}$
ω_2	$(\mathrm{o},2)$	5	32	30	28	$m_t = \frac{32\,m_{t-1}+28}{3}$
ψ_0	$(\mathrm{e},0)$	4	16	0	0	$m_t = \frac{16\,m_{t-1}+0}{3}$
ψ_1	$(\mathrm{e},1)$	6	64	24	23	$m_t = \frac{64\,m_{t-1}+23}{3}$
ψ_2	$(\mathrm{e},2)$	2	4	2	0	$m_t = \frac{4\,m_{t-1}+0}{3}$
Ω_0	$(\mathrm{o},0)$	5	32	8	8	$m_t = \frac{32\,m_{t-1}+8}{3}$
Ω_1	$(\mathrm{o},1)$	3	8	4	3	$m_t = \frac{8\,m_{t-1}+3}{3}$
Ω_2	$(\mathrm{o},2)$	1	2	1	-1	$m_t = \frac{2\,m_{t-1}-1}{3}$

Table 8. Index recurrence $m_t = \frac{a\, m_{t-1}+q}{3}$ for each row at p=1. Here $a = 2^{\alpha+6} = 64{\cdot}2^\alpha$, $k_1 = \frac{\beta\cdot 64+c}{9}$, $q = k_1 - j$.

Row	(s,j)	α	a	k_1	$q = k_1 - j$	Recurrence
Ψ_0	$(\mathrm{e},0)$	2	256	14	14	$m_t = \frac{256\,m_{t-1}+14}{3}$
Ψ_1	$(\mathrm{e},1)$	4	1024	398	397	$m_t = \frac{1024\,m_{t-1}+397}{3}$
Ψ_2	$(\mathrm{e},2)$	6	4096	2958	2956	$m_t = \frac{4096\,m_{t-1}+2956}{3}$
ω_0	$(\mathrm{o},0)$	3	512	142	142	$m_t = \frac{512\,m_{t-1}+142}{3}$
ω_1	$(\mathrm{o},1)$	1	128	78	77	$m_t = \frac{128\,m_{t-1}+77}{3}$
ω_2	$(\mathrm{o},2)$	5	2048	1934	1932	$m_t = \frac{2048\,m_{t-1}+1932}{3}$
ψ_0	$(\mathrm{e},0)$	4	1024	56	56	$m_t = \frac{1024\,m_{t-1}+56}{3}$
ψ_1	$(\mathrm{e},1)$	6	4096	1592	1591	$m_t = \frac{4096\,m_{t-1}+1591}{3}$
ψ_2	$(\mathrm{e},2)$	2	256	184	182	$m_t = \frac{256\,m_{t-1}+182}{3}$
Ω_0	$(\mathrm{o},0)$	5	2048	568	568	$m_t = \frac{2048\,m_{t-1}+568}{3}$
Ω_1	$(\mathrm{o},1)$	3	512	312	311	$m_t = \frac{512\,m_{t-1}+311}{3}$
Ω_2	$(\mathrm{o},2)$	1	128	120	118	$m_t = \frac{128\,m_{t-1}+118}{3}$

Examples using the base $p = 0$ table. From the unified scheme, a single p=0 row with parameter α enforces the forward identity

$$3x_{k+1} + 1 \;=\; 2^\alpha\, x_k \quad\Longleftrightarrow\quad x_{k+1} \;=\; \frac{3}{2^\alpha}\, x_k \;+\; \frac{1}{2^\alpha}.$$

Hence for n consecutive steps that all use the *same* p=0 row (so the exponent is constant $a_k \equiv \alpha$),

$$x_n \;=\; \Big(\frac{3}{2^\alpha}\Big)^n x_0 \;+\; \frac{1}{2^\alpha}\cdot\frac{1-\big(\frac{3}{2^\alpha}\big)^n}{1-\frac{3}{2^\alpha}}. \tag{6}$$

If during those n steps the residue form $x_k = 18\, m_k + \beta_{\text{row}}$ stays fixed (same (s,j)), then

$$m_n \;=\; \frac{x_n - \beta_{\text{row}}}{18}. \tag{7}$$

Below we instantiate (6)–(7) with exact entries from the base p=0 table.

Row `ee0` (from the p=0 table: $(s,j) = (\mathrm{e},0)$, $\alpha = 2$, $\beta = 2$, $c = -2$, $\delta = 1$). Here $x \equiv 1 \pmod 6$ with $j = 0$, so $x = 18m + 1$ (i.e., $\beta_{\text{row}} = 1$). Using (6) with $\alpha = 2$ gives

$$x_n \;=\; \Big(\frac{3}{4}\Big)^n x_0 \;+\; \frac{1}{4}\cdot\frac{1-(\frac{3}{4})^n}{1-\frac{3}{4}} \;=\; \Big(\frac{3}{4}\Big)^n x_0 \;+\; 1 - \Big(\frac{3}{4}\Big)^n.$$

Thus

$$m_n \;=\; \frac{x_n-1}{18} \;=\; \frac{1}{18}\Big(\frac{3}{4}\Big)^{n}(x_0-1).$$

Notes. (i) The constants $(\alpha,\beta,c;\delta)$ above were taken verbatim from the base p=0 table. (ii) Equations (6)–(7) apply to any other p=0 row by substituting that row's α and the corresponding residue offset $\beta_{\text{row}} = 6j + p_6$ (e.g. $\beta_{\text{row}} = 1, 7, 13$ for $s = \text{e}$ and $= 5, 11, 17$ for $s = \text{o}$). (iii) If the proof segment crosses rows (changes (s, j)), break the word at the boundaries and apply these formulas piecewise with the appropriate α and β_{row} for each block.

18. Residue targeting via a last-row congruence

Let $M_K := 3\cdot 2^K$. An admissible last step (taken from some column $p \ge 0$) has the form

$$x' \;=\; 6\big(2^{\alpha+6p}\, m \;+\; k^{(p)}\big) \;+\; \delta^{(p)},$$

where $\alpha \in \mathbb{Z}_{\ge 0}$ is the base exponent of the row, and $k^{(p)}, \delta^{(p)}$ are the column-p constants for that row.[1] Write

$$a^{(p)} := 6\cdot 2^{\alpha+6p}, \qquad r^{(p)} := x_{\text{tar}} - \big(6k^{(p)} + \delta^{(p)}\big).$$

Lemma 23 (Last-row congruence targeting)**.** *The congruence*

$$a^{(p)}\, m \;\equiv\; r^{(p)} \pmod{M_K}$$

is solvable iff $g^{(p)} := \gcd\big(a^{(p)}, M_K\big) = 3\cdot 2^{\min(\alpha+1+6p,\,K)}$ *divides* $r^{(p)}$. *When solvable:*

(i) *If* $K \le \alpha+1+6p$*, then* $r^{(p)} \equiv 0 \pmod{M_K}$ *and* every $m\in\mathbb{Z}$ *works (the step pins a residue independently of* m*).*

(ii) *If* $K > \alpha+1+6p$*, then*

$$m \;\equiv\; \frac{r^{(p)}}{3\cdot 2^{\alpha+1+6p}} \pmod{2^{\,K-(\alpha+1+6p)}}.$$

Admissibility at the input (correct family and router j*) must also hold.*

Corollary 17 (Pinning regime as a special case)**.** *In the setting of the last-row congruence lemma, if* $\alpha+1+6p \ge K$*, then for* $M_K = 3\cdot 2^K$ *one has*

$$x' \equiv 6k^{(p)} + \delta \pmod{M_K}$$

independently of m. *In particular, once routing admissibility holds, choosing the last token with* $\alpha+1+6p \ge K$ *pins the final residue at modulus* M_K*.*

Corollary 18 (Meeting the gcd condition via parity control)**.** *In the setting of Lemma 23, using the same-family parity toggle from the padding menu one can choose the final* $B \bmod 2$ *so that* $g^{(p)} \mid r^{(p)}$*, hence the congruence is solvable for any target modulus* M_K*.*

Corollary 19 (When the last step ignores m)**.** *The last step pins* $x' \equiv 6k^{(p)} + \delta^{(p)} \pmod{M_K}$ *independently of* m *iff* $\alpha + 6p \ge K$*.*

Corollary 20 (Refined last-step mapping across moduli)**.** *Fix a starting modulus* $M_K = 3\cdot 2^K$ *and a starting class* $x \equiv r \pmod{M_K}$*. Let the last token* T *(in some column* p*) have parameters*

$$x' \;=\; 6\big(2^{\alpha+6p}\, u + k^{(p)}\big) + \delta, \qquad a^{(p)} := 6\cdot 2^{\alpha+6p} = 3\cdot 2^{\alpha+1+6p}.$$

Assume we have fixed a (possibly empty) prefix so that T *is admissible at the input (router* j *and family match), and choose a target modulus* $M_{K'} = 3\cdot 2^{K'}$ *with* $K' \ge K$*.*

Then there exists a refinement of the starting *class to a subclass*

$$x \;\equiv\; r^{\sharp} \pmod{M_{K+\rho}}$$

[1] In many normalizations one has $k^{(p)} = k + p$ and $\delta^{(p)} = 6p + \delta_0$, but the lemma below does not need these specifics.

for some $\rho \geq 0$ (large enough to guarantee the planned routers in the prefix and at T), such that for every x in this subclass the last step produces an x' with

$$x' \equiv r' \pmod{M_{K'}},$$

where r' is determined as follows:

(i) ***Pinning regime** ($K' \leq \alpha + 1 + 6p$). The gcd is $g^{(p)} = M_{K'}$, so*

$x' \equiv 6k^{(p)} + \delta \pmod{M_{K'}}$ *(independent of m and hence of the start x in the subclass).*

In words: the last token pins a single *residue $r' \equiv 6k^{(p)} + \delta \pmod{M_{K'}}$ across the entire subclass.*

(ii) ***Solvable congruence** ($K' > \alpha + 1 + 6p$). Write*

$$r^{(p)} := \frac{x_{\text{tar}} - (6k^{(p)} + \delta)}{1} \quad \textit{modulo } M_{K'}.$$

Then the congruence of Lemma 23

$$a^{(p)} m \equiv r^{(p)} \pmod{M_{K'}}$$

is solvable iff $g^{(p)} = 3 \cdot 2^{\alpha+1+6p}$ divides $r^{(p)}$, and in that case

$$m \equiv \frac{r^{(p)}}{3 \cdot 2^{\alpha+1+6p}} \pmod{2^{K'-(\alpha+1+6p)}}.$$

Choosing the refined start class so that the router and m-class above hold, the last token then maps the entire subclass to the target residue $r' \equiv x_{\text{tar}} \pmod{M_{K'}}$.

In both cases the refinement depth ρ can be taken large enough to ensure routing compatibility of the prefix (no branch flips) and the required m-class at the last step.

Example 16 (Explicit $M_3 \to M_4$ mapping via a 2-step tail)**.** Start at modulus $M_3 = 24$ with $x \equiv 17 \pmod{24}$, and target $M_4 = 48$ with $x' \equiv 41 \pmod{48}$. Use the $p = 0$ rows

$$\omega_1 : \; x = 12m + 7 \quad (\texttt{oe},\; j = 1), \qquad \psi_2 : \; x = 24m + 17 \quad (\texttt{eo},\; j = 2).$$

Refine the start to the subclass $x \equiv 209 \pmod{216}$ (so $n \equiv 8 \pmod 9$ in $x = 24n + 17$). Then the certified two-step tail

$$\omega_1 \;\to\; \psi_2$$

is admissible at each step and *maps* this subclass from M_3 to

$$x' \equiv 41 \pmod{48} = M_4,$$

as shown in the detailed worked example earlier (the intermediate value x_1 is forced by ω_1).

Proposition 11 (Last-row pinning vs. congruence solution)**.** *Let a last token T in column p have parameters*

$$x' = 6\big(2^{\alpha_p} u + k^{(p)}\big) + \delta_T, \qquad \alpha_p = \alpha + 6p, \qquad u = \Big\lfloor \frac{x}{18} \Big\rfloor,$$

and write $M_K := 3 \cdot 2^K$, $a^{(p)} := 6 \cdot 2^{\alpha_p}$, $r^{(p)} := x_{\text{tar}} - (6k^{(p)} + \delta_T)$. Then the congruence

$$a^{(p)} m \equiv r^{(p)} \pmod{M_K}$$

is solvable iff $g^{(p)} := \gcd(a^{(p)}, M_K) = 3 \cdot 2^{\min(\alpha_p+1, K)}$ divides $r^{(p)}$. When solvable:

(i) *(*Pinning*) If $K \leq \alpha_p + 1$, then $g^{(p)} = M_K$ and $r^{(p)} \equiv 0 \pmod{M_K}$. Thus* every *$m \in \mathbb{Z}$ works: the step pins $x' \equiv 6k^{(p)} + \delta_T \pmod{M_K}$ independently of m.*

(ii) *(*Linear m-class*) If $K > \alpha_p + 1$, then the solution set is the single congruence class*

$$m \equiv \frac{r^{(p)}}{3 \cdot 2^{\alpha_p+1}} \pmod{2^{K-(\alpha_p+1)}}.$$

Admissibility (correct family, router) must also hold at the input of T.

Proof. $\gcd(a^{(p)}, M_K) = 3 \cdot 2^{\min(\alpha_p+1,K)}$ is immediate. The rest is the standard linear congruence criterion: solvability iff $g^{(p)} \mid r^{(p)}$, with the stated solution set modulo $M_K/g^{(p)}$; interpret cases by whether $\alpha_p + 1 \geq K$. ☐

Canonicalization (avoid multiple witnesses per modulus). To keep a single witness per modulus M_K, fix one of the following conventions:

- *Minimal-α convention:* among admissible last rows achieving $x_{\text{tar}} \pmod{M_K}$, pick the one with the smallest $\alpha + 6p$ (fewest added powers of 2).
- *Family-priority convention:* prefer a designated terminal family (e.g. o); if tied, use minimal-α.

Either rule yields a unique canonical last step for each M_K and target residue.

Representative examples (explicit use of Lemma 23). We illustrate the two regimes of Lemma 23: (i) m-independent pinning when $\alpha + 6p \geq K$, and (ii) congruence targeting for m when $\alpha + 6p < K$. Throughout $M_K := 3 \cdot 2^K$ and

$$a^{(p)} := 6 \cdot 2^{\alpha+6p}, \qquad r^{(p)} := x_{\text{tar}} - \big(6k^{(p)} + \delta^{(p)}\big), \qquad g^{(p)} := \gcd\big(a^{(p)}, M_K\big).$$

Example 1 (pin at $K = 5$): $x_{\text{tar}} \equiv 53 \pmod{96}$. Take the p=0 row Ω_0 (oo), with unified form $x' = 6(2^\alpha m + k) + \delta = 192m + 53$, so

$$\alpha = 5, \quad k = 8, \quad \delta = 5.$$

Here $K = 5$ and $\alpha + 6p = \alpha = 5 \geq K$, so Cor. 19 applies: the last step *ignores* m modulo $M_5 = 96$ and pins

$$x' \equiv 6k + \delta = 48 + 5 \equiv 53 \pmod{96},$$

ending in the o-family.

Example 2 (solve for m at $K = 10$): $x_{\text{tar}} \equiv 3071 \pmod{3072}$. Choose the p=0 row Ω_2 (oo): $x' = 12m + 11 = 6(2^1 m + 1) + 5$, so

$$\alpha = 1, \quad k = 1, \quad \delta = 5, \qquad K = 10.$$

Compute the lemma data:

$$a^{(0)} = 6 \cdot 2^1 = 12, \quad r^{(0)} = 3071 - (6 \cdot 1 + 5) = 3060, \quad g^{(0)} = \gcd(12, 3072) = 12.$$

Solvability: $g^{(0)} \mid r^{(0)}$ (since $3060 = 12 \cdot 255$), so solutions exist. Because $\alpha + 6p + 1 = 2 < K$, Lemma 23(ii) gives

$$m \equiv \frac{r^{(0)}}{3 \cdot 2^{\alpha+1}} = \frac{3060}{3 \cdot 2^2} = 255 \pmod{2^{K-(\alpha+1)}} = \pmod{256}.$$

Thus any admissible input with (o, j=2) and $m \equiv 255 \pmod{256}$ maps to $x' \equiv 3071 \pmod{3072}$ on this last step.

Example 3 (congruence at $K = 5$ with explicit predecessor): $x_{\text{tar}} \equiv 49 \pmod{96}$. Use the p=0 row Ψ_0 (ee): $x' = 24m + 1 = 6(2^2 m + 0) + 1$, so

$$\alpha = 2, \quad k = 0, \quad \delta = 1, \qquad K = 5.$$

Lemma data:

$$a^{(0)} = 6 \cdot 2^2 = 24, \quad r^{(0)} = 49 - (6 \cdot 0 + 1) = 48, \quad g^{(0)} = \gcd(24, 96) = 24.$$

Solvability: $24 \mid 48$, so solutions exist. Since $\alpha + 1 = 3 < K$, Lemma 23(ii) yields

$$m \equiv \frac{48}{3 \cdot 2^3} = \frac{48}{24} = 2 \pmod{2^{K-(\alpha+1)}} = \pmod{4}.$$

Thus the last step hits 49 (mod 96) whenever the input is (e, j=0) with $m \equiv 2 \pmod 4$. A concrete admissible predecessor is $x = 37$: here $m = \lfloor 37/18 \rfloor = 2$ and $j = \lfloor 37/6 \rfloor \bmod 3 = 0$, so indeed

$$37 \xrightarrow{\Psi_0} 24 \cdot 2 + 1 = 49 \equiv 49 \pmod{96}.$$

(From $x_0 = 1$ one forced path to 37 is $\psi_0, \omega_0, \psi_2, \Omega_2, \omega_1, \Psi_1$.)

These examples illustrate the two regimes: (i) *m-independent pins* when $\alpha + 6p \geq K$, and (ii) *congruence targeting* when $\alpha + 6p < K$. In both cases, the canonicalization rule selects a single last-step witness per M_K.

18.1. **From M_K to M_{K+n}: a concrete lifting procedure.** Let $M_K := 3 \cdot 2^K$. Suppose we are given a starting residue class

$$x \equiv r \pmod{M_K} \qquad (\text{with } r \equiv 1 \text{ or } 5 \pmod 6),$$

and we want a compatible residue $x' \equiv r' \pmod{M_{K+n}}$ that is realized by a certified tail. We give two mechanical variants.

Variant A (pinning by lifting; always works). Pick a last token (row) T in some column $p \geq 0$ with unified form

$$x' \;=\; 6\big(2^{\alpha+6p}\, u + k^{(p)}\big) + \delta, \qquad u = \Big\lfloor \frac{x}{18} \Big\rfloor, \quad \delta \in \{1, 5\}.$$

Choose p so that the *pinning threshold*

$$K' \;\leq\; \alpha + 1 + 6p, \qquad \text{where } K' := K + n,$$

holds. Then $M_{K'} \mid 6 \cdot 2^{\alpha+6p}$ and therefore

$$x' \equiv 6k^{(p)} + \delta \pmod{M_{K'}} \quad (\text{independent of } u \text{ and } m).$$

Finally, refine the starting class $x \equiv r \pmod{M_K}$ to a subclass so that the router for T is admissible at the last step (and, if you keep a short prefix, at each step of that prefix). This refinement is a standard "routing-compatibility" choice.

Variant B (minimal lift + solving the last congruence). If you prefer a smaller column p with $K' > \alpha + 1 + 6p$, let

$$a^{(p)} := 6 \cdot 2^{\alpha+6p} = 3 \cdot 2^{\alpha+1+6p}, \qquad M_{K'} = 3 \cdot 2^{K'}.$$

To force $x' \equiv r' \pmod{M_{K'}}$ at the last step you need

$$a^{(p)} u \;\equiv\; r' - (6k^{(p)} + \delta) \pmod{M_{K'}}.$$

This is solvable iff $g := \gcd(a^{(p)}, M_{K'}) = 3 \cdot 2^{\min(\alpha+1+6p, K')}$ divides the right-hand side. When solvable,

$$u \;\equiv\; \frac{r' - (6k^{(p)} + \delta)}{3 \cdot 2^{\alpha+1+6p}} \pmod{\,2^{\,K' - (\alpha+1+6p)}\,}.$$

As in Variant A, refine the starting class so all routers are admissible and the internal index $u = \lfloor x/18 \rfloor$ lies in the solved class.

Example 17 (Given $x \equiv 13 \pmod{24}$, produce $x' \pmod{96}$)**.** We illustrate Variant A (pinning) with a single last token.

Step 1: choose the last token and pinning level. From the p=0 table, take $T = \Psi_1$ (type `ee`) with parameters

$$\alpha = 4, \qquad k^{(0)} = 6, \qquad \delta = 1, \qquad x' \;=\; 6\big(2^4 u + 6\big) + 1 \;=\; 96u + 37.$$

For $K' = 5$ (modulus $M_5 = 96$) we have $\alpha + 1 = 5$, so the pinning threshold $K' \leq \alpha + 1$ holds. Therefore, for *any* admissible input,

$$x' \equiv 6k^{(0)} + \delta \equiv 36 + 1 \equiv \boxed{37} \pmod{96}.$$

Step 2: ensure admissibility at the last step. We must feed Ψ_1 from the even family with router $j = 1$. Starting from

$$x \equiv 13 \pmod{24} \quad \Longleftrightarrow \quad x = 24n + 13,$$

the router at the last step is

$$j = \Big\lfloor \frac{x}{6} \Big\rfloor \bmod 3 \;=\; (4n + 2) \bmod 3 \;\equiv\; n + 2 \pmod 3.$$

Thus $j = 1$ exactly when $n \equiv 2 \pmod 3$, i.e. for the refined subclass

$$x \equiv 24(3t + 2) + 13 = 72t + 61 \;\equiv\; 61 \pmod{72}.$$

This refinement preserves $x \equiv 13 \pmod{24}$ and guarantees the last row Ψ_1 is admissible.

Step 3: (optional) concrete instance and check. Take $x_0 = 61$. Then

$$u = \Big\lfloor \frac{61}{18} \Big\rfloor = 3, \qquad x' = 96u + 37 = 96 \cdot 3 + 37 = 325 \equiv \boxed{37} \pmod{96}.$$

Hence, starting from $x \equiv 13 \pmod{24}$, a one-step tail Ψ_1 (at p=0) maps the refined-router subclass to the pinned residue $x' \equiv 37 \pmod{96}$.

Remarks. (i) If you want a different target residue $r' \pmod{96}$, pick a different last token (or a different column p) so that $6k^{(p)} + \delta \equiv r' \pmod{96}$ under pinning. (ii) If you prefer not to refine the starting class for router $j = 1$, append a short same-family steering block before Ψ_1 to achieve the desired router while staying in the even family; the last step still pins $x' \equiv 37$ (mod96).

18.2. **Backward recovery: from M_{K+n} down to M_K.** Let $M_{K'} := 3 \cdot 2^{K'}$ and suppose we are given a target residue

$$x' \equiv r' \pmod{M_{K'}} \qquad (r' \equiv 1 \text{ or } 5 \text{ mod } 6).$$

We describe how to recover compatible preimages x modulo a lower modulus M_K that map to x' in *one certified last step.* We proceed token–by–token.

Setup (pick a hypothesized last token). Choose a last token T (a table row) in some column $p \geq 0$, with unified form

$$x' \;=\; 6\big(2^{\alpha+6p}\, u + k^{(p)}\big) + \delta, \qquad u = \left\lfloor \frac{x}{18} \right\rfloor, \quad \delta \in \{1, 5\}.$$

Write

$$a^{(p)} := 6 \cdot 2^{\alpha+6p} \;=\; 3 \cdot 2^{\alpha+1+6p}.$$

Variant A (pinning case). If $K' \leq \alpha + 1 + 6p$ then $M_{K'} \mid a^{(p)}$, so the step *pins* the residue:

$$x' \equiv 6k^{(p)} + \delta \pmod{M_{K'}}.$$

Hence, this token T can produce the given x' *iff*

$$r' \equiv 6k^{(p)} + \delta \pmod{M_{K'}}.$$

When this holds, u is unconstrained mod 2, and all preimages x are obtained by:

- Choosing any $u \in \mathbb{Z}$;
- Taking x in the interval $[18u, 18u + 17]$ that matches the required router j of T:

router j	allowed $x \in [18u, 18u + 17]$
0	$18u, \ldots, 18u + 5$
1	$18u + 6, \ldots, 18u + 11$
2	$18u + 12, \ldots, 18u + 17$

- Within that 6-block, pick the residue matching the *entry family* of T: $x \equiv 1$ (mod6) for family e, or $x \equiv 5$ (mod6) for family o.

With u free, x ranges over a *union of classes* modulo 18. Projecting to $M_K = 3 \cdot 2^K$ yields preimages at $K = 1$ and higher; a convenient canonical modulus is

$$x \equiv 18u + r \pmod{18 \cdot 2^s} \quad \Longrightarrow \quad x \equiv 18u_0 + r \pmod{M_{s+1}} \;\; (K = s + 1),$$

whenever $u \equiv u_0 \pmod{2^s}$.

Variant B (non-pinning case). If $K' > \alpha + 1 + 6p$, form the last-step congruence

$$a^{(p)} u \;\equiv\; r' - \big(6k^{(p)} + \delta\big) \pmod{M_{K'}}.$$

Let $g := \gcd\big(a^{(p)}, M_{K'}\big) = 3 \cdot 2^{\min(\alpha+1+6p,\, K')} = 3 \cdot 2^{\alpha+1+6p}$. Solvability requires $g \mid r' - (6k^{(p)} + \delta)$. When solvable, we obtain the internal index class

$$u \;\equiv\; \frac{r' - (6k^{(p)} + \delta)}{3 \cdot 2^{\alpha+1+6p}} \qquad (\text{mod } 2^{\,K' - (\alpha+1+6p)}\,).$$

Then x is reconstructed from u exactly as in Variant A: pick the correct j-block inside $[18u, 18u + 17]$ and the entry-family residue modulo 6. If u is known modulo 2^s with $s := K' - (\alpha + 1 + 6p)$, then x is determined modulo $18 \cdot 2^s$, hence also modulo $M_{s+1} = 3 \cdot 2^{s+1}$.

Example 18 (Recovering x mod 24 from $x' \equiv 37$ mod 96 via $T = \Psi_1$)**.** Take the last token $T = \Psi_1$ at $p = 0$ (type `ee`) with

$$\alpha = 4, \qquad k^{(0)} = 6, \qquad \delta = 1, \qquad x' = 6\big(2^4u + 6\big) + 1 = 96u + 37.$$

Here $K' = 5$ (since $M_{K'} = 96$) and $\alpha + 1 = 5$, so we are in the *pinning* regime. Then

$$6k^{(0)} + \delta = 36 + 1 = 37 \equiv r' \pmod{96},$$

so Ψ_1 is consistent with $x' \equiv 37$ (mod96) and imposes *no* restriction on u.

To recover x, respect the router and entry family for Ψ_1: it requires router $j = 1$ and entry family e (i.e. $x \equiv 1$ (mod6)). Given any $u \in \mathbb{Z}$, the admissible preimages lie in the slice

$$x \in \{18u + 6, \dots, 18u + 11\} \ \cap \ \{x \equiv 1 \text{ (mod6)}\},$$

which forces $x = 18u + 7$ (since $18u + r \equiv r \pmod 6$ and the only $r \in [6, 11]$ with $r \equiv 1 \pmod 6$ is $r = 7$).

Thus

$$x \equiv 18u + 7 \pmod{18}.$$

Projecting to $M_2 = 3 \cdot 2^2 = 12$ or $M_3 = 24$ picks concrete classes. For instance, choosing $u \equiv 3$ (mod2) gives $x = 18 \cdot 3 + 7 = 61 \equiv 13 \pmod{24}$, which is a valid preimage class we used elsewhere. More generally, any u produces

$$x \equiv 18u + 7 \pmod{18 \cdot 2^s} \quad \Rightarrow \quad x \equiv 18u_0 + 7 \pmod{M_{s+1}},$$

whenever $u \equiv u_0 \pmod{2^s}$.

Notes.

- The backward preimage is typically *not unique*. Pinning makes u free, yielding a whole family of x's consistent with the same last token.
- If several tokens T satisfy the divisibility check (or the pinning identity) for the given r', you get multiple admissible preimage families; routing constraints decide which arise from a fixed prefix.
- For non-pinning cases, the recovered modulus for x is naturally M_{s+1} with $s = K' - (\alpha + 1 + 6p)$, reflecting that one last step adds $\alpha + 1 + 6p$ binary factors to the $3 \cdot 2^K$ modulus.

Example 19 (Recovering x mod 24 from $x' \equiv 37$ mod 48 via $T = \Psi_1$)**.** Work at $K' = 4$ (so $M_{K'} = 48$). Take the last token $T = \Psi_1$ at $p = 0$ (type `ee`):

$$x' \ = \ 6\big(2^4u + 6\big) + 1 \ = \ 96u + 37, \qquad (\alpha = 4, \ k^{(0)} = 6, \ \delta = 1).$$

Since $K' = 4 \leq \alpha + 1 = 5$, we are in the *pinning* regime and indeed

$$x' \equiv 6k^{(0)} + \delta \equiv 36 + 1 \equiv 37 \pmod{48}.$$

Thus Ψ_1 can produce any $x' \equiv 37$ (mod48) and imposes *no* restriction on u.

To recover preimages x, enforce the router and entry family of Ψ_1: it requires router $j = 1$ and entry family e (i.e. $x \equiv 1 \pmod 6$). Given any $u \in \mathbb{Z}$, the admissible preimages lie in the slice

$$x \in \{18u + 6, \dots, 18u + 11\} \ \cap \ \{x \equiv 1 \text{ (mod6)}\},$$

which forces $x = 18u + 7$ (the unique residue $\equiv 1 \pmod 6$ in that $j = 1$ block).

Reduce $x = 18u + 7$ modulo 24:

$$18u + 7 \equiv -6u + 7 \pmod{24}.$$

As u ranges over $\mathbb{Z}$, the residue classes modulo 24 are exactly

$$\boxed{x \equiv 1, \, 7, \, 13, \, 19 \pmod{24}}.$$

(Indeed, taking $u \equiv 0, 1, 2, 3 \pmod 4$ yields $7, 1, 19, 13$ respectively.) All four are in family e (each $\equiv 1 \pmod 6$) and sit in router block $j = 1$ at the last step.

Concrete check. Pick $x = 13$ (one of the four classes). Then

$$u = \left\lfloor \tfrac{13}{18} \right\rfloor = 0, \qquad x' = 96u + 37 = 37 \equiv 37 \pmod{48}.$$

Similarly, $x = 1, 7, 19$ also map to $x' \equiv 37 \pmod{48}$ under the last step Ψ_1.

19. Same-family steering menu and monotone padding (p=0)

Throughout this section we fix $p = 0$. Our goal is to package the tail constructions from Sections **??** and **??** into a small "steering menu" of tail blocks that (i) raise the 2-adic valuation $v_2(A_W)$, (ii) preserve the terminal family (e or o), and (iii) optionally toggle the parity of B_W mod 2.

Proposition 12 (Steering menu at p=0)**.** *For any certified word W at $p = 0$, the following tail blocks can be appended on the right. In each case the resulting word $\widetilde{W}$ has strictly larger $v_2(A_{\widetilde{W}})$ by the shown amount, and its terminal family (even e or odd o) is as indicated.*

Block	***Family effect***	$\boldsymbol{\Delta v_2(A)}$
Ψ_1	$e \to e$	$+4$
Ψ_2	$e \to e$	$+6$
Ω_1	$o \to o$	$+4$
Ω_0	$o \to o$	$+5$
$\psi_2 \circ \omega_1$	$e \to o \to e$ *(parity-cycle)*	$+3$
$\omega_1 \circ \psi_2$	$o \to e \to o$ *(parity-cycle)*	$+3$

Lemma 24 (Monotone (suffix-only) padding)**.** *Given a target K and a choice of terminal family (e or o), there exists a finite concatenation S of blocks from Prop. 12 such that, for any word W with that terminal family, the padded word $\widetilde{W} := W\,S$ satisfies $v_2(A_{\widetilde{W}}) \geq K$ and has the same terminal family.*

If a specific residue $B_{\widetilde{W}}$ mod 2 is required, appending one parity-cycle block (from the bottom two rows of Prop. 12) flips $B_{\widetilde{W}}$ mod 2 while preserving the terminal family and leaving $v_2(A_{\widetilde{W}})$ unchanged modulo the lower bound K.

Lemma 25 (Routing compatibility for the stabilized prefix)**.** *Let W be a fixed prefix and S a tail from Prop. 12. Let m be a solution of the final congruence for the chosen last row (as in Section* **??***), and let (r_t) denote the remainders computed along W at input m in the router scheme of Section* **??***.*

Then these remainders r_t match the planned router indices of W; equivalently, appending S does not invalidate any of W's row choices.

Worked examples.

A. (o-tail) Append Ω_1, Ω_0 to raise v_2 by $+9$ and land at 53 mod 48; this residue is independent of m.

B. (e-tail) Append Ψ_2 to raise v_2 by $+6$ and pin 37 mod 48, again independent of m.

C. Target 3071 mod 3072: first lift with Ω_1, Ω_0 until $v_2(A)$ is large enough, then apply a final Ω_2 with $m \equiv 255 \pmod{256}$ to hit the exact target residue.

20. Routing-compatibility of a fixed prefix under tail padding

We write a certified word in the form

$$x_W(m) = 6\big(A_W m + B_W\big) + \delta_W,$$

with

$$A_W = 2^{\sum \alpha_i}.$$

After freezing a prefix W, we show one may append a same-family padding tail and solve a final congruence so that all router choices inside W remain as planned.

Lemma 26 (2-adic control of prefix indices)**.** *Let $W_{\leq t}$ denote the length-t prefix of a certified word W. There exist integers U_t, V_t and an integer $s_t \geq 0$ such that*

$$m_t \equiv U_t\, m + V_t \pmod{2^{s_t}},$$

where $m_t := \lfloor x_{W_{\le t}}(m)/18 \rfloor$*, and one may take*

$$s_t \le 1 + \sum_{i<t} \alpha_i = 1 + \log_2 A_t, \qquad x_{W_{\le t}}(m) = 6\big(A_t m + B_t\big) + \delta_t, \quad A_t = 2^{\sum_{i<t} \alpha_i}.$$

Proof. By induction on t. For $t = 0$, $(U_0, V_0, s_0) = (1, 0, 0)$. Assume the claim holds for some $t \ge 0$. Write $x_{W_{\le t}}(m) = 6(A_t m + B_t) + \delta_t$ with $A_t = 2^{\sum_{i<t} \alpha_i}$. Then

$$m_t = \left\lfloor \frac{6(A_t m + B_t) + \delta_t}{18} \right\rfloor = \left\lfloor \frac{A_t m + B_t}{3} + \frac{\delta_t - 6r_t}{18} \right\rfloor,$$

where $r_t \in \{0, 1, 2\}$ depends only on $(A_t m + B_t) \bmod 3$. Since 3 is invertible modulo 2^s for all s, reducing modulo $2^{1+\sum_{i<t} \alpha_i}$ gives $m_t \equiv U_t m + V_t \pmod{2^{s_t}}$ for some integers U_t, V_t with $s_t \le 1 + \sum_{i<t} \alpha_i$. Applying step t (with parameters α_t, k_t, δ'_t) yields the same form for m_{t+1} with $A_{t+1} = 2^{\sum_{i \le t} \alpha_i}$, completing the induction. □

Lemma 27 (Routing compatibility (no branch flips))**.** *Let W be a fixed prefix of length L. Append a same-family padding tail so that the final congruence you solve constrains*

$$m \equiv m^* \pmod{2^{S^*}}, \qquad S^* \ge \max_{0 \le t < L} s_t,$$

where s_t are as in Lemma 26, and choose the solution class whose mod-3 part matches the admissibility of W. Then along W at this m, all router remainders

$$r_{t+1} \equiv 2^{\alpha_t} m_t + k_t \pmod 3$$

coincide with the predeclared routers j_{t+1} of W. Equivalently, every row in W remains admissible (no branch flips in the prefix).

Proof. By Lemma 26, each m_t depends on m only through $m \bmod 2^{s_t}$. The tail is chosen so that $m \equiv m^* \pmod{2^{S^*}}$ with $S^* \ge s_t$ for all $t < L$, hence each m_t is fixed. Therefore each router remainder $r_{t+1} \equiv 2^{\alpha_t} m_t + k_t \pmod 3$ is fixed and equals the planned j_{t+1} (we also selected the mod-3 class compatibly). Thus all prefix rows remain admissible. □

Remark (Compatibility with backward uniqueness)*.* This does not conflict with any backward–uniqueness statement: uniqueness fixes the prefix W by forcing a single backward chain; routing-compatibility asserts that suffix padding and the final congruence can be chosen so the already-fixed prefix routers remain unchanged.

Lemma 28 (Global routing compatibility)**.** *Let $W = T_0 T_1 \cdots T_{n-1}$ be a fixed certified prefix, with planned router remainders $r_{t+1} \in \{0, 1, 2\}$ at each step $t = 0, \ldots, n-1$, and let*

$$x_t(m) = 6\big(A_t m + B_t\big) + \delta_t, \qquad m_t = \left\lfloor \frac{x_t}{18} \right\rfloor, \qquad j_{t+1} = \left\lfloor \frac{x_t}{6} \right\rfloor \bmod 3.$$

Then there exists a nonempty congruence class $\mathcal{M} \subset \mathbb{Z}$ of m (of the form $m \equiv m^* \pmod{2^{S^*}}$ together with a fixed residue mod 3 determined by the entry family of W) such that for all $m \in \mathcal{M}$ the run of W is admissible and

$$j_{t+1} = r_{t+1} \quad \textit{for every } t = 0, \ldots, n-1.$$

Equivalently, along $\mathcal{M}$ we have the divisibility constraints

$$A_t m + B_t \equiv r_{t+1} \pmod 3 \quad \textit{for all } t,$$

so every division by 3 in the one-step update is integral and the planned rows are used throughout W (no branch flips).

Proof. Proceed by induction on t. For $t = 0$, admissibility fixes the entry family mod 6 (hence mod 3) and imposes $A_0 m + B_0 \equiv r_1 \pmod 3$, which is a nonempty congruence class in m modulo 3. Assuming the claim up to step t, the one-step composition with floor (Lemma 29) yields

$$m_t = \frac{A_t m + B_t - r_{t+1}}{3} \in \mathbb{Z} \quad \text{on the inductive class,}$$

and the next router is $j_{t+1} \equiv r_{t+1}$ by construction. The affine update for $(A_{t+1}, B_{t+1}, \delta_{t+1})$ produces a new linear congruence $A_{t+1}m + B_{t+1} \equiv r_{t+2} \pmod 3$. Intersecting the prior class with this new congruence still leaves a nonempty arithmetic progression in m (Chinese remainder on $\mathbb{Z}$ with moduli powers of 2 and the fixed mod 3). Iterating to $t = n-1$ gives the stated $\mathcal{M}$. $\square$

21. Same-family steering menu and monotone padding across columns

Remark (Notation convention). We use $u := \lfloor x/18 \rfloor$ for the per-step floor input to a single token, while the global affine parameter in $x_W(m) = 6(A_W m + B_W) + \delta_W$ is denoted by m. When a last token at column p is applied, its unified form is $x' = 6(2^{\alpha+6p}u + k^{(p)}) + \delta$ with router $j = \lfloor x/6 \rfloor \bmod 3$.

Explicit same-family steering blocks at $p = 0$. We collect short tail blocks that (i) preserve terminal family and (ii) allow monotone lift and parity control. For each entry, Δv_2 refers to the increase contributed by the block (ignoring the harmless division by 3 inside the floor composition).

Block	Entry→Exit	Δv_2	B mod 2	Justification (from p=0 rows)
Ψ_1	$e \to e$	$+4$	even	$x' = 96m + 37$, so $k^{(0)} = 6$ (even), $\alpha = 4$.
Ψ_2	$e \to e$	$+6$	even	$x' = 6(2^6 m + k) + 1$, table $\alpha = 6$, parity even.
Ω_1	$o \to o$	$+3$	even	$x' = 6(2^3 m + k) + 5$, table $\alpha = 3$, parity even.
Ω_0	$o \to o$	$+5$	even	$x' = 6(2^5 m + k) + 5$, table $\alpha = 5$, parity even.
$\omega_1 \circ \psi_2$	$o \to e \to o$	$+1 + 2 = +3$	toggles	$k^{(0)}(\omega_1) = 1$ (odd), $k^{(0)}(\psi_2) = 2$ (even) so one odd in the 2-token cycle toggles parity; returns to o.
$\psi_2 \circ \omega_1$	$e \to o \to e$	$+2 + 1 = +3$	toggles	Same reason; returns to e.

Table 9. A minimal explicit menu at $p = 0$. The $p \geq 1$ menus are obtained by lifting exponents $\alpha \mapsto \alpha + 6p$, keeping the same entry/exit families and parity effects.

These blocks suffice to (a) raise $v_2(A)$ to any target K and (b) realize either parity for B mod 2 *without* changing the terminal family, as required by the padding lemmas.

Preamble: composition with the floor input $u = \lfloor x/18 \rfloor$.

Setup and invariant form. At any step t of a certified word we maintain the linear surrogate

$$x_t(m) \;=\; 6\bigl(A_t\, m + B_t\bigr) + \delta_t,$$

where $A_t = 2^{S_t}$ is a power of two, $B_t \in \mathbb{Z}$, and $\delta_t \in \{1, 5\}$. The input to the *next* token is the *internal index*

$$m_t \;:=\; \left\lfloor \frac{x_t}{18} \right\rfloor.$$

Router remainder and the exact formula for m_t. Write $q_t := A_t m + B_t$, and decompose $q_t = 3\,t_t + r_{t+1}$ with $r_{t+1} \in \{0, 1, 2\}$. Then

$$\frac{x_t}{18} \;=\; \frac{6q_t + \delta_t}{18} \;=\; \frac{q_t}{3} + \frac{\delta_t}{18} \;=\; t_t + \frac{r_{t+1}}{3} + \frac{\delta_t}{18}.$$

Since $\frac{r_{t+1}}{3} + \frac{\delta_t}{18} < 1$, we have the exact identity

$$m_t \;=\; \left\lfloor \frac{x_t}{18} \right\rfloor \;=\; t_t \;=\; \frac{A_t m + B_t - r_{t+1}}{3}.$$

Here r_{t+1} is the *router remainder* that determines the next row, and for an admissible execution it equals the planned router j_{t+1}.

Lemma 29 (One-step composition with floor)**.** *Fix a next token T in column p with unified parameters*

$$x' \;=\; 6\bigl(2^{\alpha_p}u + k^{(p)}\bigr) + \delta_T, \qquad \alpha_p := \alpha + 6p, \quad k^{(p)} := \frac{\beta\, 64^p + c}{9} \in \mathbb{Z}.$$

Feeding T with $u = m_t = \bigl(A_t m + B_t - r_{t+1}\bigr)/3$ yields

$$x_{t+1}(m) \;=\; 6\bigl(A_{t+1} m + B_{t+1}\bigr) + \delta_{t+1},$$

with the exact update

$$A_{t+1} \;=\; \frac{2^{\alpha_p}}{3}\,A_t, \qquad B_{t+1} \;=\; \frac{2^{\alpha_p}}{3}\,\bigl(B_t - r_{t+1}\bigr) \;+\; k^{(p)}, \qquad \delta_{t+1} \;=\; \delta_T.$$

Lemma 30 (Integrality of the floor-driven update)**.** *In the one-step update of Lemma 29,*

$$A' = \frac{2^{\alpha_p}}{3}A, \qquad B' = \frac{2^{\alpha_p}}{3}(B-r) + k^{(p)}, \qquad \delta' = \delta_T,$$

all quantities are integers on the certified m-class. In particular, $(B-r)/3 \in \mathbb{Z}$ and hence $2^{\alpha_p}(B-r)/3 \in \mathbb{Z}$.

Proof. Admissibility enforces $Am + B \equiv r \pmod 3$ along the chosen class of m, so $B - r \equiv -Am \equiv 0 \pmod 3$. Therefore $(B-r)/3 \in \mathbb{Z}$ and the stated expressions are integral. □

Integrality. Although the factor $2^{\alpha_p}/3$ appears, integrality is guaranteed on the certified m-class: admissibility enforces $A_t m + B_t \equiv r_{t+1} \pmod 3$, so $(B_t - r_{t+1})/3$ is an integer linear form in m; equivalently $m_t \in \mathbb{Z}$ and $2^{\alpha_p} m_t \in \mathbb{Z}$.

Lemma 31 (Final B-parity equals the last token's $k^{(p)}$ mod 2)**.** *With the notation of Lemma 29, one has*

$$B_{t+1} \;\equiv\; k^{(p)} \pmod 2.$$

Proof. From Lemma 29, $B_{t+1} = \frac{2^{\alpha_p}}{3}(B_t - r_{t+1}) + k^{(p)}$. The first term is an *even* integer multiple of 2^{α_p} (with $\alpha_p \geq 1$ for certified rows), hence vanishes modulo 2. □

Closed form for a whole word. Let $W = T_1 \cdots T_n$ with T_i drawn from columns p_i and parameters $(\alpha_{p_i}, k_i, \delta_i)$ where

$$\alpha_{p_i} = \alpha_i + 6p_i, \qquad k_i = \frac{\beta_i\, 64^{\,p_i} + c_i}{9}, \qquad \delta_i \in \{1, 5\}.$$

Iterating Lemma 29 gives

$$x_W(m) = 6\bigl(A_W m + B_W\bigr) + \delta_W,$$

with

$$A_W \;=\; \prod_{i=1}^{n} \frac{2^{\alpha_{p_i}}}{3} \;=\; \frac{2^{\sum_{i=1}^{n} \alpha_{p_i}}}{3^{\,n}}, \qquad \delta_W \;=\; \delta_{T_n},$$

and

$$B_W \;=\; \sum_{i=1}^{n} \left(k_i \prod_{j=i+1}^{n} \frac{2^{\alpha_{p_j}}}{3} \right) \;+\; \sum_{i=1}^{n} \left(-\frac{2^{\alpha_{p_i}}}{3}\, r_{i+1} \prod_{j=i+1}^{n} \frac{2^{\alpha_{p_j}}}{3} \right),$$

where each $r_{i+1} \in \{0,1,2\}$ is the router remainder at step i (fixed under routing compatibility). In particular,

$$B_W \;\equiv\; k_n \pmod 2$$

by Lemma 31 (the last token determines the final B-parity).

Remark (Admissibility and routing compatibility)*.* In practice we first fix a routing prefix, then choose the tail and the final m-class so that all r_{i+1} equal the planned routers j_{i+1}. This guarantees every division by 3 above is an integer operation on the chosen m-class and no branch flips occur inside the prefix.

Two concrete numeric compositions. **Example 1** **(**$p = 0$, last token $T = \Psi_1$, type `ee`**).** From the p=0 table: $\alpha_p = 4$, $k^{(0)} = 6$, $\delta_T = 1$. Given a prefix (A_t, B_t, δ_t) and router r_{t+1},

$$m_t = \frac{A_t m + B_t - r_{t+1}}{3}, \qquad x_{t+1} = 6\Bigl(16\, m_t + 6\Bigr) + 1.$$

Hence

$$A_{t+1} = \tfrac{16}{3} A_t, \qquad B_{t+1} = \tfrac{16}{3}(B_t - r_{t+1}) + 6, \qquad \delta_{t+1} = 1,$$

and $B_{t+1} \equiv 6 \equiv 0 \pmod 2$ as predicted.

Example 2 **($p = 2$, last token $T = \Omega_2$, type oo).** From the p=2 table: $\alpha_p = 13$, $k^{(2)} = 7736$, $\delta_T = 5$. For any admissible prefix and router r_{t+1},

$$m_t = \frac{A_t m + B_t - r_{t+1}}{3}, \qquad x_{t+1} = 6\Big(8192\, m_t + 7736\Big) + 5.$$

Thus

$$A_{t+1} = \tfrac{8192}{3} A_t, \qquad B_{t+1} = \tfrac{8192}{3}(B_t - r_{t+1}) + 7736, \qquad \delta_{t+1} = 5,$$

and $B_{t+1} \equiv 7736 \equiv 0 \pmod 2$.

Family-preserving padding and cross-family terminal steps.

Why this refinement. Earlier we emphasized "same-family steering" throughout a tail. In practice, many lifts require a *family change only at the last step* (e.g. $o \to e$ via ω_1). We therefore separate the tail into:

(1) a *padding phase*: a concatenation of *same-family* blocks that raises $v_2(A)$ and, if needed, adjusts $B \bmod 2$—*without changing family*;

(2) a *terminal step*: a *single* token that may be cross-family (e.g. oe or eo) to land in the required terminal family and residue class.

Lemma 32 (Family-preserving padding menu at column p). *Fix a column $p \geq 0$ and a family $f \in \{e, o\}$. There exists a finite menu $\mathcal{P}_{p,f}$ of tail blocks (each block is one or two tokens) with:*

*(1) (*Family preservation*) Each $S \in \mathcal{P}_{p,f}$ enters f and exits f.*

*(2) (*Monotone lift*) Each S satisfies $\Delta_p(S) > 0$, so $v_2(A)$ increases.*

*(3) (*Parity control*) $\mathcal{P}_{p,f}$ contains a 2-token cycle that toggles $B \bmod 2$ while staying in f.*

Consequently, for any prefix ending in f and any target K, one can append a padding string $S = S_1 \cdots S_q$ with $S_i \in \mathcal{P}_{p,f}$ so that $v_2(A)$ reaches K and $B \bmod 2$ is set as required, without changing family.

Remark (Composition with floor input). Throughout, token composition uses the exact floor input $u = \lfloor x/18 \rfloor$. For a token in column p with parameters $(\alpha_p, k^{(p)}, \delta)$ and router remainder $r \in \{0, 1, 2\}$, the update

$$A \mapsto \frac{2^{\alpha_p}}{3} A, \qquad B \mapsto \frac{2^{\alpha_p}}{3}(B - r) + k^{(p)}, \qquad \delta \mapsto \delta$$

holds (Lemma 29 in the preamble). In particular, $B_{\text{new}} \equiv k^{(p)} \pmod 2$ (Lemma 31), which is why the last token fixes the final B-parity.

Lemma 33 (Mixed-family tails with a cross-family terminal step). *Let a prefix end in family f and let T_{term} be any admissible terminal token (possibly cross-family) in column p with parameters $(\alpha_p, k^{(p)}, \delta)$. Then there exists padding S composed of blocks from $\mathcal{P}_{p,f}$ such that the concatenation $S \cdot T_{\text{term}}$ is admissible, reaches any prescribed modulus $M_K = 3 \cdot 2^K$, and either:*

- *(*m-independent pin*) if $\alpha_p \geq K$, then*

$$x_{\text{out}} \equiv 6k^{(p)} + \delta \pmod{M_K},$$

independently of the top-level m; or

- *(*solve for m*) if $\alpha_p < K$, then m satisfies a linear congruence*

$$6 \cdot 2^{\alpha_p} m \equiv x_{\text{tar}} - (6k^{(p)} + \delta) \pmod{M_K},$$

whose solvability class is fixed by the padding (via $B \bmod 2$ and the chosen routers).

Idea of proof. Use Lemma 32 to raise $v_2(A)$ and set $B \bmod 2$ while keeping family f fixed, guaranteeing the router plan and integrality. Then apply the single terminal token T_{term} (possibly $o \to e$ or $e \to o$) to land in the required final family and residue. The two cases follow from the last-row congruence; see the preamble lemmas on composition with $u = \lfloor x/18 \rfloor$. ☐

Lemma 34 (One-step $o \to e$ witness at $p = 0$). *At $p = 0$, the token ω_1 (*oe*, $\delta = 1$) maps every $x \equiv 47 \pmod{72}$ to $x' \equiv 31 \pmod{48}$ in one certified step.*

Proof. ω_1 has $x' = 12m+7$ with $m = \lfloor x/18 \rfloor$ and requires router $j = \lfloor x/6 \rfloor \bmod 3 = 1$. Modulo 48, $12m+7$ assumes $\{7, 19, 31, 43\}$ as $m \bmod 4 \in \{0,1,2,3\}$, so to hit 31 we need $m \equiv 2 \pmod 4$. Writing $x = 18t + r$, $j = 1$ forces $r = 11$; then $x = 18t + 11$ with $m = t$. Imposing $t \equiv 2 \pmod 4$ gives $x \equiv 72s + 47 \pmod{72}$, and for these x we get $x' \equiv 31 \pmod{48}$. ☐

Example 20 (Minimal $o \to e$ lift to 31 mod 48)**.** Take $x = 47$ (odd family, $j = 1$). Then $m = \lfloor 47/18 \rfloor = 2$ and ω_1 yields $x' = 12 \cdot 2 + 7 = 31 \equiv 31 \pmod{48}$. This is a pure one-step $o \to e$ terminal move with no padding.

Remark (How this changes usage). In the earlier same-family presentation, all tail surgery stayed in one family. Here we keep padding same-family (so all router choices in the prefix are preserved), but we *allow the final token to change family* if that is what the target residue requires. The algebra (updates of (A, B, δ) and the last-step congruence) is unchanged; only the terminal family is now dictated by the last token.

Introductory overview and notation.

Unified linear form for a certified word. Every finite certified word W acts on an index m by a linear map inside a fixed outer factor:

$$x_W(m) \;=\; 6\big(A_W\, m + B_W\big) + \delta_W,$$

where $A_W = 2^{\sum \alpha_i}$ is a pure power of two (sum of the row exponents used in W), $B_W \in \mathbb{Z}$ is the accumulated internal constant, and $\delta_W \in \{1, 5\}$ is the terminal offset determined by the last row.

Composition rule for (A, B, δ) (with $u = \lfloor x/18 \rfloor$). If a prefix W_{pre} has $x_{W_{\text{pre}}}(m) = 6\big(A\, m + B\big) + \delta_{\text{pre}}$, and we append a last token in column p with parameters $(\alpha_p, k^{(p)}, \delta_T)$ while using router remainder $r \in \{0,1,2\}$ at that step, then with

$$u = \Big\lfloor \frac{x_{W_{\text{pre}}}}{18} \Big\rfloor = \frac{A\, m + B - r}{3}$$

we obtain the exact update

$$A_{\text{final}} = \frac{2^{\alpha_p}}{3}\, A, \qquad B_{\text{final}} = \frac{2^{\alpha_p}}{3}\,(B - r) + k^{(p)}, \qquad \delta_{\text{final}} = \delta_T.$$

In particular $B_{\text{final}} \equiv k^{(p)} \pmod 2$, so the last token fixes the final B-parity on the admissible m-class.

Why B mod 2 matters. When we target a residue class x_{tar} modulo $M_K := 3 \cdot 2^K$, the last step (or the whole word) leads to a binary congruence for m:

$$2^{\alpha_p}\, m \;\equiv\; \frac{x_{\text{tar}} - \delta}{6} - B_W \pmod{2^K},$$

so the *parity* of the right-hand side depends on B_W mod 2. Being able to *choose or flip* B_W mod 2 is therefore the knob that makes this congruence solvable in a convenient class of m (or places us in an m-independent pinning regime; see below).

Per-token effects (the tables). The tables list, for each admissible token (row) and each column $p \geq 0$,

$$x' \;=\; 6\big(2^{\alpha_p} m + k^{(p)}\big) + \delta, \qquad \alpha_p := \alpha + 6p, \quad k^{(p)} := \frac{\beta \cdot 64^{\,p} + c}{9} \in \mathbb{Z}.$$

Two immediate consequences (for a chosen last token). Fix a specific last token T (one row in the column-p table), with parameters

$$T: \quad x' \;=\; 6\big(2^{\alpha_p} m + k^{(p)}\big) + \delta, \qquad \alpha_p = \alpha + 6p, \quad k^{(p)} = \frac{\beta \cdot 64^{\,p} + c}{9}.$$

Then:

- **Exponent gain.** Appending T multiplies the internal slope by 2^{α_p}; equivalently it adds α_p to $v_2(A)$.
- **Final B-parity.** Because $2^{\alpha_p} m$ is even, the last step T fixes $B_{\text{final}} \equiv k^{(p)} \pmod 2$. This is why the tables list $k^{(p)}$ mod 2: it tells you the resulting B mod 2 after using T last.

Example. If you pick $T = \Psi_1$ at $p = 0$, the table gives $\alpha_p = 4$ and $k^{(0)} = 6$ (even), so appending T adds $+4$ to $v_2(A)$ and sets $B_{\text{final}} \equiv 0 \pmod 2$; indeed $x' = 96m + 37$ pins $x' \equiv 37 \pmod{48}$ when $K \leq 4$.

Same-family steering blocks. A *same-family steering block* is a short tail (one or two tokens) that starts and ends in the same family ($e \to e$ or $o \to o$). We use a small "menu" with two roles:

(1) **Monotone lift:** each block has $\Delta v_2(A) > 0$, so appending blocks *monotonically* raises $v_2(A)$ and lets us reach any target K.
(2) **Parity control:** a 2-token same-family cycle toggles $B \bmod 2$ when used once (and preserves it when used twice), without changing family.

Because these blocks return to the same family, they do not disturb the desired terminal family for the last step.

Monotone padding (what we actually do). Given a fixed routing prefix W, we append same-family blocks until $v_2(A)$ is large enough for the modulus $3 \cdot 2^K$. Then:

- If $\alpha_p \geq K$ for the chosen last token, the last step *pins* the residue $x' \equiv 6k^{(p)} + \delta \pmod{3 \cdot 2^K}$ *independently of* m.
- If $\alpha_p < K$, we solve the linear congruence $6 \cdot 2^{\alpha_p} m \equiv x_{\text{tar}} - (6k^{(p)} + \delta) \pmod{3 \cdot 2^K}$ for $m \pmod{2^{K-(\alpha_p+1)}}$. If a particular $B \bmod 2$ is needed, we first apply the same-family parity cycle to match it.

Why columns p help. Lifting to column p replaces α by $\alpha_p = \alpha + 6p$. Thus the same token at higher p contributes $+6$ more two-adic bits per unit of p: it is a stronger "bit pump." The lifted constants $k^{(p)}$ (hence B mod 2) are read directly from the tables.

Prefix stability. By the routing-compatibility lemma, once W is fixed we may choose the tail (and the final m-class) so that all router choices inside W remain unchanged. In practice: *freeze the prefix; do all surgery at the tail.*

Fix a column $p \geq 0$. Any admissible row in column p has unified form

$$x' \; = \; 6\big(2^{\alpha+6p}\, m + k^{(p)}\big) + \delta^{(p)},$$

where α is the base (column p=0) exponent of that row, and $k^{(p)}, \delta^{(p)}$ are the lifted constants in column p. Fix a column $p \geq 0$. Any admissible row in column p has unified form

$$x' \; = \; 6\big(2^{\alpha+6p}\, u + k^{(p)}\big) + \delta^{(p)}, \qquad u = \Big\lfloor \frac{x}{18} \Big\rfloor.$$

For a (finite) tail block $S = (R_1, \ldots, R_\ell)$ taken entirely from column p, set

$$\alpha(S) := \sum_{i=1}^{\ell} \alpha(R_i), \qquad \ell(S) := \ell, \qquad \Delta_p(S) := \alpha(S) + 6p\, \ell(S).$$

Each token multiplies the current slope by $\frac{2^{\alpha(R_i)+6p}}{3}$ (by the composition rule), hence appending S multiplies the internal slope by

$$\frac{2^{\Delta_p(S)}}{3^{\,\ell(S)}}.$$

Equivalently,

$$x_{WS}(m) \; = \; 6\Big(\frac{2^{\Delta_p(S)}}{3^{\,\ell(S)}} A_W\, m \; + \; B_{WS}\Big) + \delta_{WS}, \qquad \Rightarrow \qquad v_2\big(A_{WS}\big) \; = \; v_2(A_W) + \Delta_p(S).$$

Remark. The division by $3^{\ell(S)}$ does not affect $v_2(\cdot)$, so the two-adic lift remains additive in $\Delta_p(S)$, as used in the padding lemmas.

Lemma 35 (Same-family steering menu at column p)**.** *There exists a finite menu $\mathcal{S}_p$ of tail blocks, each taken from column p, such that:*

*(1) (*Family preservation*) Each $S \in \mathcal{S}_p$ starts in a prescribed terminal family (e or o) and ends in the* same *family.*
*(2) (*Monotone lift*) Each $S \in \mathcal{S}_p$ satisfies $\Delta_p(S) > 0$.*
*(3) (*Parity control*) The menu contains a 2-token cycle C_p (same-family in/out) that toggles B* mod 2.

Canonical choices (use the p table to fill α and B-effects). Below, $\alpha(\cdot)$ denotes the base exponent at $p=0$ for that row; at column p, the lift amount is $\Delta_p = \alpha(\cdot) + 6p$ per token.

End in e:

Block	Entry → Exit family	$\Delta_p(S)$	Effect on $B \bmod 2$
Ψ_1	$e \to e$	$\alpha(\Psi_1) + 6p$	(from table)
Ψ_2	$e \to e$	$\alpha(\Psi_2) + 6p$	(from table)
$\psi_2 \circ \omega_1$	$e \to o \to e$	$\alpha(\psi_2) + \alpha(\omega_1) + 12p$	toggles

End in o:

Block	Entry → Exit family	$\Delta_p(S)$	Effect on $B \bmod 2$
Ω_1	$o \to o$	$\alpha(\Omega_1) + 6p$	(from table)
Ω_0	$o \to o$	$\alpha(\Omega_0) + 6p$	(from table)
$\omega_1 \circ \psi_2$	$o \to e \to o$	$\alpha(\omega_1) + \alpha(\psi_2) + 12p$	toggles

Lemma 36 (Monotone padding at column p)**.** *Fix a target K and a terminal family. For any prefix W that already ends in that family, there exists a concatenation $S = S_1 \cdots S_q$ with each $S_i \in \mathcal{S}_p$ such that*

$$v_2\big(A_{W\cdot S}\big) \;\ge\; K, \qquad \text{and} \qquad \text{the terminal family of } W\cdot S \text{ equals that of } W.$$

Moreover, if a specific parity $B_{W\cdot S} \bmod 2$ is required, appending the cycle C_p an odd (resp. even) number of times flips (resp. preserves) $B \bmod 2$.

Proof. By Lemma 35(ii) each block adds a positive $\Delta_p(S)$ to $v_2(A)$; thus repeating lift blocks reaches any K. Entries/exits are same-family by Lemma 35(i). The last statement follows from the "toggles" entry of the table (parity is flipped by one use of C_p, preserved by two). □

Lemma 37 (Routing-compatibility of the prefix)**.** *Let W be any fixed prefix. Appending a tail S from $\mathcal{S}_p$ and solving the final congruence (mod $3 \cdot 2^K$) so that*

$$m \equiv m^* \pmod{2^{S^*}} \qquad \text{with} \quad S^* \ge \max_{t<|W|} \big(1 + \sum_{i<t} \alpha_i\big)$$

ensures all prefix routers remain as planned (no branch flips). Pick the solution class whose mod-3 part matches the prefix admissibility.

Lemma 38 (Strong routing compatibility)**.** *Let W be a fixed prefix with planned routers j_{t+1}. There exists an exponent depth S^* such that if the final congruence is solved with $m \equiv m^* \pmod{2^{S^*}}$ and the mod-3 part in the admissible class, then the realized remainders equal $r_{t+1} = j_{t+1}$ at every prefix step.*

Representative examples (all columns).

- **Lift in o at $p = 0$ (pin modulo 48, then raise K):** Append Ω_1 or Ω_0. Each adds $\Delta_0 = \alpha(\cdot)$ bits and preserves o. If you end with Ω_0 and $\alpha(\Omega_0) \ge 4$, then by Cor. of the last-row lemma, the last step pins an m-independent residue modulo 48, while repeated Ω-blocks raise K monotonically.
- **Congruence targeting at $p = 1$ (end in e, higher lift per token):** Suppose the last row is Ψ_1 at column p=1. Then the last-step form is

  $$x' = 6\big(2^{\alpha(\Psi_1)+6}\, m + k^{(1)}\big) + \delta^{(1)}.$$

  For modulus $M_K = 3{\cdot}2^K$: − If $K \le \alpha(\Psi_1)+6$, this **pins** $x' \equiv 6k^{(1)}+\delta^{(1)} \pmod{M_K}$ independently of m. − If $K > \alpha(\Psi_1) + 6$, solve

  $$6 \cdot 2^{\alpha(\Psi_1)+6}\, m \equiv x_{\mathrm{tar}} - (6k^{(1)} + \delta^{(1)}) \pmod{3 \cdot 2^K}$$

  to get the explicit m-class from the lemma. Use Ψ-blocks in column p=1 to raise $v_2(A)$ as needed; any 2-token e-cycle in $\mathcal{S}_1$ toggles $B \bmod 2$ without leaving e.

Implementation note. To fully instantiate the tables above, fill the base exponents $\alpha(\cdot)$ and the $B \bmod 2$ effects from the p=0 table; the column-p lift is automatic via $\Delta_p = \alpha(\cdot) + 6p$ per token. If you'd like, we can tabulate the exact B-updates once you share (or re-upload) the row formulas for each p.

Micro-examples (reading one table row and applying the lemma).

P0–A (pin at $K = 4$ using $T = \Psi_1$ at $p = 0$, family $e \to e$). From the $p = 0$ row $(\mathrm{e}, 1)$–ee–Ψ_1:

$$\alpha_p = 4, \qquad k^{(0)} = 6 \text{ (even)}, \qquad \delta = 1, \qquad x' = 6\big(2^4 m + 6\big) + 1 = 96m + 37.$$

With $K = 4$ (mod 48) and $\alpha_p \geq K$, we are in the pinning regime:

$$x' \equiv 6k^{(0)} + \delta \equiv 36 + 1 \equiv 37 \pmod{48} \quad \text{for every } m, \qquad B_{\text{final}} \equiv k^{(0)} \equiv 0 \ (\text{mod} 2).$$

P0–B (solve for m at $K = 4$ using $T = \Omega_2$ at $p = 0$, family $o \to o$). From $(\mathrm{o}, 2)$–oo–Ω_2 at $p = 0$:

$$\alpha_p = 1, \qquad k^{(0)} = 1 \text{ (odd)}, \qquad \delta = 5, \qquad x' = 6\big(2^1 m + 1\big) + 5 = 12m + 11.$$

Target $x_{\text{tar}} \equiv 35 \pmod{48}$, with $K = 4 > \alpha_p$:

$$12m \equiv 35 - (6 \cdot 1 + 5) = 24 \pmod{48} \implies m \equiv 2 \pmod{4}.$$

So any admissible input in family o with router j=2 and $m \equiv 2$ (mod4) lands at 35 (mod48). Here $B_{\text{final}} \equiv 1$ (mod2).

P2–C (pin at $K = 10$ using $T = \Psi_0$ at $p = 2$, family $e \to e$). From the $p = 2$ row $(\mathrm{e}, 0)$–ee–Ψ_0 (unified table):

$$F(2, m) = 16384\, m + 910, \qquad x'_2 = 98304\, m + 5461.$$

Thus $\alpha_p = 14$ (since $\alpha = 2$ and $\alpha + 12 = 14$), $k^{(2)} = 910$ (even), and $\delta = 1$. With $K = 10$ (mod 3072) and $\alpha_p \geq K$,

$$x'_2 \equiv 6k^{(2)} + \delta \equiv 5461 \equiv 2389 \pmod{3072} \quad \text{independently of } m, \qquad B_{\text{final}} \equiv 0 \ (\text{mod} 2).$$

P2–D (solve for m at $K = 15$ using $T = \Omega_2$ at $p = 2$, family $o \to o$). From $(\mathrm{o}, 2)$–oo–Ω_2 at $p = 2$ (unified table):

$$F(2, m) = 8192\, m + 7736, \qquad x'_2 = 49152\, m + 46421,$$

so $\alpha_p = 13$ (since $\alpha = 1$), $k^{(2)} = 7736$ (even), and $\delta = 5$. Let $K = 15$ (mod 98304) and set

$$x_{\text{tar}} \equiv 46421 + 49152 = 95573 \pmod{98304}.$$

Then with $a = 6 \cdot 2^{\alpha_p} = 49152$,

$$a\, m \equiv x_{\text{tar}} - (6k^{(2)} + \delta) \equiv 95573 - 46421 = 49152 \pmod{98304} \implies m \equiv 1 \pmod{2}.$$

Any admissible input in family o with router j=2 and *odd* m lands at x_{tar} (mod98304); here $B_{\text{final}} \equiv 0$ (mod2).

How to read the column-p tables below.

Per-token effects table ($\boldsymbol{\alpha_p}$ and $\boldsymbol{B}$ **mod 2**). These compact tables summarize what each token does to the linear map inside $x_W(m) = 6(A_W m + B_W) + \delta_W$ when appended as the last step in column p:

- $\boldsymbol{\alpha}$: the base exponent (at $\boldsymbol{p = 0}$) appearing in $\boldsymbol{2^{\alpha}}$ for that token.
- $\boldsymbol{\alpha_p = \alpha + 6p}$: the exponent actually *added* to $\boldsymbol{v_2(A_W)}$ at column $\boldsymbol{p}$. Appending the token multiplies $\boldsymbol{A_W}$ by $\boldsymbol{2^{\alpha_p}}$.
- $\boldsymbol{k^{(p)}}$ **mod 2**: the parity of the lifted constant $\boldsymbol{k^{(p)} = \big(\beta \cdot 64^{\,p} + c\big)/9}$. Since $\boldsymbol{x' = 6(2^{\alpha_p} m + k^{(p)}) + \delta}$, the new $\boldsymbol{B}$ modulo **2** is $\boldsymbol{B' \equiv k^{(p)} \pmod{2}}$. This is the entry we use for *parity control* via short cycles.

Worked line decoding (one example). Take the $p = 2$ line $(\mathrm{e}, 0)$–ee–Ψ_0 in the unified table:

$$F(2, m) = 16384\, m + 910, \qquad x'_2(m) = 98304\, m + 5461.$$

Here $\alpha = 2$ so $\alpha_2 = \alpha + 12 = 14$ and indeed $2^{14} = 16384$ is the slope of $F(2, m)$. The lifted constant is $k^{(2)} = 910$, hence $B' \equiv k^{(2)} \equiv 0 \pmod{2}$ in the effects table.

Usage in the padding lemmas.

- The column α_p tells you exactly how many 2-adic bits each token adds at column p; stacking tokens from a same-family menu gives the monotone lift to any target K.
- The column $k^{(p)} \bmod 2$ is the parity knob: a two-token same-family cycle with an *odd* number of parity-flip entries toggles $B \bmod 2$ while returning to the same family, ensuring solvability of the final congruence when a specific parity is required.

Table 10. Per-token effects at $p = 0$: added exponent $\alpha_p = \alpha + 6p$ and B-parity update $B \mapsto k^{(0)} \bmod 2$.

(s, j)	type	move	α	α_p	$k^{(0)} \bmod 2$
(e, 0)	ee	Ψ_0	2	2	0
(e, 1)	ee	Ψ_1	4	4	0
(e, 2)	ee	Ψ_2	6	6	0
(o, 0)	oe	ω_0	3	3	0
(o, 1)	oe	ω_1	1	1	1
(o, 2)	oe	ω_2	5	5	0
(e, 0)	eo	ψ_0	4	4	0
(e, 1)	eo	ψ_1	6	6	0
(e, 2)	eo	ψ_2	2	2	0
(o, 0)	oo	Ω_0	5	5	0
(o, 1)	oo	Ω_1	3	3	0
(o, 2)	oo	Ω_2	1	1	1

Table 11. Per-token effects at $p = 1$: added exponent $\alpha_p = \alpha + 6p$ and B-parity update $B \mapsto k^{(1)} \bmod 2$.

(s, j)	type	move	α	α_p	$k^{(1)} \bmod 2$
(e, 0)	ee	Ψ_0	2	8	0
(e, 1)	ee	Ψ_1	4	10	0
(e, 2)	ee	Ψ_2	6	12	0
(o, 0)	oe	ω_0	3	9	0
(o, 1)	oe	ω_1	1	7	0
(o, 2)	oe	ω_2	5	11	0
(e, 0)	eo	ψ_0	4	10	0
(e, 1)	eo	ψ_1	6	12	0
(e, 2)	eo	ψ_2	2	8	0
(o, 0)	oo	Ω_0	5	11	0
(o, 1)	oo	Ω_1	3	9	0
(o, 2)	oo	Ω_2	1	7	0

Convention. In the unified row formulas $x' = 6\big(2^{\alpha_p} m + k^{(p)}\big) + \delta$, the symbol m denotes the *internal* index $u = \lfloor x/18 \rfloor$ at that step (not the global top-level m).

Examples using the per-token effects tables. We illustrate how to read off the exponent gain α_p, the B-parity $k^{(p)} \bmod 2$, and then apply the residue-targeting lemma in two regimes: (i) m-independent pinning when $\alpha_p \geq K$, and (ii) solving a linear congruence for m when $\alpha_p < K$. Recall $M_K := 3 \cdot 2^K$ and

$$x' \;=\; 6\big(2^{\alpha_p} m + k^{(p)}\big) + \delta, \qquad a := 6 \cdot 2^{\alpha_p}.$$

Example P0–A (pin at $K = 4$ with Ψ_1). Token: $(\mathrm{e}, 1)$–ee–Ψ_1 at $p = 0$. From the $p = 0$ tables:

$$\alpha = 4 \;\Rightarrow\; \alpha_0 = \alpha = 4, \quad k^{(0)} = 6 \text{ (even)}, \quad \delta = 1, \quad x' = 96m + 37.$$

Table 12. Per-token effects at $p = 2$: added exponent $\alpha_p = \alpha + 6p$ and B-parity update $B \mapsto k^{(2)} \bmod 2$.

(s, j)	type	move	α	α_p	$k^{(2)} \bmod 2$
$(\mathrm{e}, 0)$	ee	Ψ_0	2	14	0
$(\mathrm{e}, 1)$	ee	Ψ_1	4	16	0
$(\mathrm{e}, 2)$	ee	Ψ_2	6	18	0
$(\mathrm{o}, 0)$	oe	ω_0	3	15	0
$(\mathrm{o}, 1)$	oe	ω_1	1	13	0
$(\mathrm{o}, 2)$	oe	ω_2	5	17	0
$(\mathrm{e}, 0)$	eo	ψ_0	4	16	0
$(\mathrm{e}, 1)$	eo	ψ_1	6	18	0
$(\mathrm{e}, 2)$	eo	ψ_2	2	14	0
$(\mathrm{o}, 0)$	oo	Ω_0	5	17	0
$(\mathrm{o}, 1)$	oo	Ω_1	3	15	0
$(\mathrm{o}, 2)$	oo	Ω_2	1	13	0

Take $K = 4$ so $M_4 = 48$. Since $\alpha_0 \geq K$, the corollary gives

$$x' \equiv 6k^{(0)} + \delta \equiv 36 + 1 \equiv 37 \pmod{48}$$

independently of m. (Here B' is even because $k^{(0)}$ is even.)

Example P0–B (solve for m at $K = 4$ with Ω_2). Token: $(\mathrm{o}, 2)$–oo–Ω_2 at $p = 0$. From the $p = 0$ tables:

$$\alpha = 1 \ \Rightarrow\ \alpha_0 = 1, \quad k^{(0)} = 1 \text{ (odd)}, \quad \delta = 5, \quad x' = 12m + 11.$$

Target $x_{\mathrm{tar}} \equiv 35 \ (\mathrm{mod} 48)$ $(K = 4)$. Since $\alpha_0 < K$, solve

$$a\,m \equiv x_{\mathrm{tar}} - (6k^{(0)} + \delta) \pmod{48}, \quad a = 6 \cdot 2^{\alpha_0} = 12.$$

Compute $6k^{(0)} + \delta = 6 \cdot 1 + 5 = 11$, so $12m \equiv 35 - 11 = 24 \ (\mathrm{mod} 48)$, hence

$$m \equiv 2 \pmod{4}.$$

Choosing any admissible input in family o with router j=2 and $m \equiv 2 \ (\mathrm{mod} 4)$ makes the last step land at 35 (mod48). (Here B' is odd because $k^{(0)}$ is odd.)

Example P2–C (pin at $K = 10$ with Ψ_0). Token: $(\mathrm{e}, 0)$–ee–Ψ_0 at $p = 2$. From the $p = 2$ unified table:

$$F(2, m) = 16384m + 910, \quad x'_2 = 98304m + 5461.$$

Thus $\alpha = 2 \Rightarrow \alpha_2 = \alpha + 12 = 14$ and $k^{(2)} = 910$ (even), $\delta = 1$ (since $6 \cdot 910 + 1 = 5461$). For $K = 10$ $(M_{10} = 3072)$ we have $\alpha_2 \geq K$, so pin:

$$x'_2 \equiv 6k^{(2)} + \delta \equiv 5461 \equiv 2389 \pmod{3072},$$

independently of m. (Again B' is even here.)

Example P2–D (solve for m at $K = 15$ with Ω_2). Token: $(\mathrm{o}, 2)$–oo–Ω_2 at $p = 2$. From the $p = 2$ unified table:

$$F(2, m) = 8192m + 7736, \quad x'_2 = 49152m + 46421.$$

Hence $\alpha = 1 \Rightarrow \alpha_2 = \alpha + 12 = 13$ and $k^{(2)} = 7736$ (even), $\delta = 5$ (since $6 \cdot 7736 + 5 = 46421$). Let $K = 15$ so $M_{15} = 3 \cdot 2^{15} = 98304$. Choose target

$$x_{\mathrm{tar}} \equiv 95573 \equiv 46421 + 49152 \pmod{98304},$$

so that solvability is guaranteed (divisibility by $g = 3 \cdot 2^{\alpha_2 + 1} = 49152$). Solve

$$a\,m \equiv x_{\mathrm{tar}} - (6k^{(2)} + \delta) \pmod{98304}, \qquad a = 6 \cdot 2^{\alpha_2} = 49152.$$

Then $am \equiv 95573 - 46421 = 49152 \ (\mathrm{mod} 98304)$, giving

$$m \equiv \frac{49152}{3 \cdot 2^{14}} \equiv 1 \pmod{2}.$$

Thus any admissible input in family o with router j=2 and *odd* m lands at $x_{\mathrm{tar}} \equiv 95573 \ (\mathrm{mod} 98304)$ on this last step.

Worked example 1: $17 \bmod 24 \to 41 \bmod 48$ via $\omega_1 \to \psi_2$ (full routing and floors). We use the $p = 0$ rows from the unified table:

$$\omega_1 : \ x' = 12m + 7 \quad (\texttt{oe},\ j = 1), \qquad \psi_2 : \ x' = 24m + 17 \quad (\texttt{eo},\ j = 2),$$

with the convention that at each step $m = \lfloor x/18 \rfloor$ and the router is $j = \lfloor x/6 \rfloor \bmod 3$.

Write every $x \equiv 17 \pmod{24}$ as $x = 24n + 17$ with $n \in \mathbb{Z}$. The table row ω_1 requires $j_0 = 1$, where

$$j_0 \ = \ \lfloor x/6 \rfloor \bmod 3.$$

Compute:

$$j_0 \ = \ \left\lfloor \frac{24n + 17}{6} \right\rfloor \bmod 3 \ = \ (4n + 2) \bmod 3 \ \equiv \ n + 2 \pmod 3.$$

So ω_1 is admissible exactly when $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. We'll pick a concrete congruence class that also makes the second step admissible and forces the needed parity; a clean choice is $n \equiv 8 \pmod 9$.

(1) **Choose representative in the class $x \equiv 17 \pmod{24}$.** Write $x = 24n + 17$ and pick the subclass $n \equiv 8 \pmod 9$, i.e. $x \equiv 209 \pmod{216}$. Take the concrete start $x_0 = \boxed{209}$.

(2) **Router for step 1 (ω_1 admissibility).**

$$j_0 = \left\lfloor \frac{209}{6} \right\rfloor \bmod 3 = 34 \bmod 3 = \boxed{1}$$

so the row $j = 1$ for ω_1 is admissible (type `oe` from odd to even family).

(3) **Apply ω_1 (uses $m_0 = \lfloor x_0/18 \rfloor$).**

$$m_0 = \left\lfloor \frac{209}{18} \right\rfloor = 11, \qquad x_1 = 12m_0 + 7 = 12 \cdot 11 + 7 = \boxed{139}.$$

Family check: $139 \bmod 6 = 1$ (even family), consistent with `oe`.

(4) **Router for step 2 (ψ_2 admissibility).**

$$j_1 = \left\lfloor \frac{139}{6} \right\rfloor \bmod 3 = 23 \bmod 3 = \boxed{2},$$

matching the row $j = 2$ for ψ_2.

(5) **Apply ψ_2 (uses $m_1 = \lfloor x_1/18 \rfloor$).**

$$m_1 = \left\lfloor \frac{139}{18} \right\rfloor = 7 \text{ (odd)}, \qquad x_2 = 24m_1 + 17 = 24 \cdot 7 + 17 = \boxed{185}.$$

(6) **Target modulus and family.**

$$185 \equiv 185 - 3 \cdot 48 = \boxed{41} \pmod{48}, \qquad 185 \bmod 6 = 5 \text{ (odd)},$$

as desired for landing in the odd family at $41 \bmod 48$.

(7) **Forward check with accelerated map $U(n) = (3n + 1)/2^{v_2(3n+1)}$.**

$$U(185) = \frac{3 \cdot 185 + 1}{2^{v_2(556)}} = \frac{556}{4} = 139, \qquad U(139) = \frac{3 \cdot 139 + 1}{2^{v_2(418)}} = \frac{418}{2} = 209.$$

Hence $U^2(185) = 209$, certifying the inverse chain $209 \xrightarrow{\omega_1} 139 \xrightarrow{\psi_2} 185$.

Why an intermediate value is required (made explicit). In this lift we use a *two-step* certified inverse tail, so there is necessarily an intermediate value x_1 between the start $x_0 \equiv 17 \pmod{24}$ and the target $x_2 \equiv 41 \pmod{48}$. The reason we do not jump in one step is that, in the $p = 0$ table, no odd-exit row with $\alpha \geq 4$ has residue $6k^{(0)} + \delta \equiv 41 \pmod{48}$ (so there is no one-step pin to $41 \bmod 48$). Consequently we choose a short two-token tail whose first token routes us into a state where the *second* token can be applied and the final residue becomes $41 \bmod 48$.

The intermediate value and why it exists. At each step, the token takes as input the internal index $m = \lfloor x/18 \rfloor$ and is *admissible* when the router $j = \lfloor x/6 \rfloor \bmod 3$ matches the row index. For the tail $\omega_1 \to \psi_2$ at $p = 0$:

$$\omega_1 : \ x' = 12m + 7 \quad (\texttt{oe},\ j = 1), \qquad \psi_2 : \ x' = 24m + 17 \quad (\texttt{eo},\ j = 2).$$

Starting at $x_0 \equiv 17 \pmod{24}$, write $x_0 = 24n + 17$. A direct computation shows

$$j_0 = \left\lfloor \tfrac{x_0}{6} \right\rfloor \bmod 3 = (4n + 2) \bmod 3 \in \{1, 2\},$$

so for the subclass $n \equiv 2 \pmod 3$ we indeed get $j_0 = 1$ and may apply ω_1. This *guarantees* an intermediate value

$$x_1 \ = \ 12 \left\lfloor \tfrac{x_0}{18} \right\rfloor + 7,$$

which is the unique output of the first certified move (and lands in the even family, as type `oe` dictates). From this x_1, the router becomes

$$j_1 = \left\lfloor \tfrac{x_1}{6} \right\rfloor \bmod 3 = 2,$$

hence ψ_2 is admissible and produces

$$x_2 \ = \ 24 \left\lfloor \tfrac{x_1}{18} \right\rfloor + 17 \ \equiv \ 41 \pmod{48}.$$

Concrete numbers (the middle hop shown). Take $x_0 = 209$ (so $n = 8$). Then

$$m_0 = \left\lfloor \tfrac{209}{18} \right\rfloor = 11, \quad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = 139 \quad \text{(this is the \textit{intermediate} value)},$$

and

$$m_1 = \left\lfloor \tfrac{139}{18} \right\rfloor = 7, \quad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 185 \equiv 41 \pmod{48}.$$

Thus the existence of the intermediate value $x_1 = 139$ is not optional: it is exactly the output of the first certified inverse step, forced by the router $j_0 = 1$ and the floor input $m_0 = \lfloor x_0/18 \rfloor$.

Forward check (confirms the two-step structure). With the accelerated map $U(n) = (3n+1)/2^{v_2(3n+1)}$,

$$U(185) = 139, \qquad U(139) = 209,$$

so indeed $U^2(185) = 209$, matching the inverse chain

$$209 \xrightarrow{\omega_1} \underbrace{139}_{\text{intermediate}} \xrightarrow{\psi_2} 185 \equiv 41 \pmod{48}.$$

Thus the certified tail $\boxed{\omega_1 \to \psi_2}$ at $p = 0$ maps every $x \equiv 209 \pmod{216}$ (a fixed subclass of 17 mod 24) to 41 mod 48.

Lemma 39 (Why $n \equiv 8 \pmod 9$ is the right refinement)**.** *Let $x_0 = 24n + 17$ so $x_0 \equiv 17 \pmod{24}$. Consider the two-token tail $\omega_1 \to \psi_2$ at $p = 0$ with*

$$\omega_1 : \ x' = 12m + 7 \quad (\texttt{oe},\ j = 1), \qquad \psi_2 : \ x' = 24m + 17 \quad (\texttt{eo},\ j = 2),$$

where at each step $m = \lfloor x/18 \rfloor$ and $j = \lfloor x/6 \rfloor \bmod 3$. Then ω_1 is admissible iff $n \equiv 2 \pmod 3$, and among those n, *the second router j_1 equals* 2 *iff $n \equiv 8 \pmod 9$. In that refined class, the next internal index m_1 is odd, so ψ_2 lands at $x_2 \equiv 41 \pmod{48}$.*

Proof. Write $x_0 = 24n + 17$. The first router is

$$j_0 = \left\lfloor \frac{x_0}{6} \right\rfloor \bmod 3 = (4n + 2) \bmod 3 \equiv n + 2 \pmod 3.$$

Thus $j_0 = 1$ (needed for the row of ω_1) iff $n \equiv 2 \pmod 3$.

Assume $n \equiv 2 \pmod 3$ and set $n = 3k + 2$ with $k \in \mathbb{Z}$. Applying ω_1 uses

$$m_0 = \left\lfloor \frac{24n + 17}{18} \right\rfloor = \left\lfloor \frac{72k + 65}{18} \right\rfloor = 4k + 3, \quad x_1 = 12m_0 + 7 = 48k + 43.$$

Now compute the second router:

$$j_1 = \left\lfloor \frac{x_1}{6} \right\rfloor \bmod 3 = (8k+7) \bmod 3 \equiv 2k+1 \pmod 3.$$

We need $j_1 = 2$ for the row of ψ_2, so $2k+1 \equiv 2 \pmod 3$, i.e. $2k \equiv 1 \pmod 3$, hence $k \equiv 2 \pmod 3$. Writing $k = 3t+2$, we get

$$n = 3k+2 = 3(3t+2)+2 = 9t+8 \implies n \equiv 8 \pmod 9.$$

Finally, in this refined class we can read off the parity of the next internal index:

$$m_1 = \left\lfloor \frac{x_1}{18} \right\rfloor = \left\lfloor \frac{48(3t+2)+43}{18} \right\rfloor = \left\lfloor \frac{144t+139}{18} \right\rfloor = 8t + \left\lfloor \frac{139}{18} \right\rfloor = 8t+7,$$

which is odd. Therefore the last step $x_2 = 24m_1 + 17$ satisfies $x_2 \equiv 24 \cdot (\text{odd}) + 17 \equiv 41 \pmod{48}$. □

Remark. Among the three residue classes $n \equiv 2, 5, 8 \pmod 9$ that satisfy $n \equiv 2 \pmod 3$, only $n \equiv 8 \pmod 9$ makes $k = (n-2)/3 \equiv 2 \pmod 3$, hence $j_1 = 2$. In short:

$$k \bmod 3 \in \{0,1,2\} \implies j_1 \equiv 2k+1 \bmod 3 \in \{1,0,2\},$$

so the unique choice that gives $j_1 = 2$ is $k \equiv 2 \pmod 3$, i.e. $n \equiv 8 \pmod 9$.

Example 21 (Concrete instance with the refined class)**.** Take $n \equiv 8 \pmod 9$, e.g. $n = 8$ so $x_0 = 24n + 17 = 209$. Then

$$j_0 = \left\lfloor \frac{209}{6} \right\rfloor \bmod 3 = 1 \quad \Rightarrow \quad m_0 = \left\lfloor \frac{209}{18} \right\rfloor = 11, \quad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = 139.$$

Next,

$$j_1 = \left\lfloor \frac{139}{6} \right\rfloor \bmod 3 = 2 \quad \Rightarrow \quad m_1 = \left\lfloor \frac{139}{18} \right\rfloor = 7 \text{ (odd)}, \quad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 185 \equiv 41 \pmod{48}.$$

This explicitly exhibits the forced intermediate value $x_1 = 139$ and the successful landing in 41 mod 48.

Worked example 2: a different subclass of 17 mod 24 to 41 mod 48 via $\omega_1 \to \psi_2$. Use the same rows and conventions as above. Choose the subclass $x = 24n + 17$ with $n \equiv 1 \pmod 9$ and a representative with $j_0 = 1$ (so ω_1 is admissible); the smallest such is $n = 19$, giving $x_0 = \boxed{473}$.

(1) **Router for step 1 (ω_1 admissibility).**

$$j_0 = \left\lfloor \frac{473}{6} \right\rfloor \bmod 3 = 78 \bmod 3 = \boxed{1}.$$

(2) **Apply ω_1.**

$$m_0 = \left\lfloor \frac{473}{18} \right\rfloor = 26, \qquad x_1 = 12m_0 + 7 = 12 \cdot 26 + 7 = \boxed{319}.$$

Family: $319 \bmod 6 = 1$ (even), consistent with `oe`.

(3) **Router for step 2 (ψ_2 admissibility).**

$$j_1 = \left\lfloor \frac{319}{6} \right\rfloor \bmod 3 = 53 \bmod 3 = \boxed{2}.$$

(4) **Apply ψ_2.**

$$m_1 = \left\lfloor \frac{319}{18} \right\rfloor = 17 \text{ (odd)}, \qquad x_2 = 24m_1 + 17 = 24 \cdot 17 + 17 = \boxed{425}.$$

(5) **Target modulus and family.**

$$425 \equiv 425 - 8 \cdot 48 = \boxed{41} \pmod{48}, \qquad 425 \bmod 6 = 5 \text{ (odd)}.$$

(6) **Forward confirmation.** By the certified inverse formulas, $473 \xrightarrow{\omega_1} 319 \xrightarrow{\psi_2} 425$, hence $U^2(425) = 473$.

This shows the *same* short tail $\boxed{\omega_1 \to \psi_2}$ also works for the subclass $x \equiv 24n + 17$ with $n \equiv 1 \pmod 9$ and $j_0 = 1$ (e.g. $x_0 = 473$), again landing at 41 mod 48.

Why an intermediate value is required (explicit). No odd-exit p=0 row with $\alpha \geq 4$ satisfies $6k^{(0)} + \delta \equiv 41 \pmod{48}$, so there is no one-step pin to 41 mod 48. We therefore use a two-token tail $\omega_1 \to \psi_2$, which necessarily produces an intermediate value x_1.

Intermediate value and routers (corrected representative). Take $x_0 = 24n + 17$ with $n \equiv 8 \pmod 9$; choose the concrete $x_0 = \boxed{641}$ (so $n = 26$). Then

$$j_0 = \left\lfloor \tfrac{641}{6} \right\rfloor \bmod 3 = 106 \bmod 3 = \boxed{1},$$

so ω_1 (row j=1) is admissible and yields the *intermediate value*

$$m_0 = \left\lfloor \tfrac{641}{18} \right\rfloor = 35, \qquad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = 427.$$

From x_1 we get

$$j_1 = \left\lfloor \tfrac{427}{6} \right\rfloor \bmod 3 = 71 \bmod 3 = \boxed{2},$$

so ψ_2 (row j=2) is admissible and produces the terminal value

$$m_1 = \left\lfloor \tfrac{427}{18} \right\rfloor = 23 \text{ (odd)}, \qquad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 569 \equiv \boxed{41} \pmod{48}.$$

Forward check (confirms the two-step chain). With $U(n) = (3n+1)/2^{v_2(3n+1)}$,

$$U(569) = \tfrac{3\cdot 569+1}{2^{v_2(1708)}} = \tfrac{1708}{4} = 427, \qquad U(427) = \tfrac{3\cdot 427+1}{2^{v_2(1282)}} = \tfrac{1282}{2} = 641,$$

so $U^2(569) = 641$, matching the inverse chain

$$641 \xrightarrow{\omega_1} \underbrace{427}_{\text{intermediate}} \xrightarrow{\psi_2} 569 \equiv 41 \pmod{48}.$$

Worked example 3: another 17 mod 24 → 41 mod 48 instance via $\omega_1 \to \psi_2$. Use the $p = 0$ rows

$$\omega_1 : \ x' = 12m + 7 \quad (\mathtt{oe},\ j = 1), \qquad \psi_2 : \ x' = 24m + 17 \quad (\mathtt{eo},\ j = 2),$$

with $m = \lfloor x/18 \rfloor$ and $j = \lfloor x/6 \rfloor \bmod 3$ at each step.

(1) **Start (a different representative of** 17 mod 24**).** Take $x_0 = \boxed{425}$. Then $425 \equiv 17 \pmod{24}$.

(2) **Router for step 1 (ω_1 admissibility).**

$$j_0 = \left\lfloor \frac{425}{6} \right\rfloor \bmod 3 = 70 \bmod 3 = \boxed{1}.$$

Thus row $j = 1$ is admissible for ω_1 (type $\mathtt{oe}$).

(3) **Apply ω_1 (uses $m_0 = \lfloor x_0/18 \rfloor$).**

$$m_0 = \left\lfloor \frac{425}{18} \right\rfloor = 23, \qquad x_1 = 12m_0 + 7 = 12 \cdot 23 + 7 = \boxed{283}.$$

Family: 283 mod 6 = 1 (even), as expected for $\mathtt{oe}$.

(4) **Router for step 2 (ψ_2 admissibility).**

$$j_1 = \left\lfloor \frac{283}{6} \right\rfloor \bmod 3 = 47 \bmod 3 = \boxed{2},$$

matching the row $j = 2$ for ψ_2.

(5) **Apply ψ_2 (uses $m_1 = \lfloor x_1/18 \rfloor$).**

$$m_1 = \left\lfloor \frac{283}{18} \right\rfloor = 15 \text{ (odd)}, \qquad x_2 = 24m_1 + 17 = 24 \cdot 15 + 17 = \boxed{377}.$$

(6) **Target modulus and family.**

$$377 \equiv 377 - 7 \cdot 48 = \boxed{41} \pmod{48}, \qquad 377 \bmod 6 = 5 \text{ (odd)},$$

so we land in the intended odd family at 41 mod 48.

(7) **Forward verification (accelerated U).**

$$U(377) = \frac{3 \cdot 377 + 1}{2^{v_2(1132)}} = \frac{1132}{4} = 283, \qquad U(283) = \frac{3 \cdot 283 + 1}{2^{v_2(850)}} = \frac{850}{2} = 425.$$

Hence $U^2(377) = 425$, exactly the inverse chain we constructed:

$$425 \xrightarrow{\omega_1} 283 \xrightarrow{\psi_2} 377.$$

Why an intermediate value is required (explicit). As in Example 1/2, no single odd-exit p=0 row pins 41 mod 48 in one step, so we again use the two-token tail $\omega_1 \to \psi_2$ which necessarily passes through an intermediate x_1.

Intermediate value and routers (this instance). Start with $x_0 = \boxed{425}$ (note $425 \equiv 17 \bmod 24$). Then

$$j_0 = \left\lfloor \tfrac{425}{6} \right\rfloor \bmod 3 = 70 \bmod 3 = \boxed{1},$$

so ω_1 (row j=1) is admissible and produces

$$m_0 = \left\lfloor \tfrac{425}{18} \right\rfloor = 23, \qquad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = \boxed{283} \quad \text{(this is the intermediate value).}$$

Next,

$$j_1 = \left\lfloor \tfrac{283}{6} \right\rfloor \bmod 3 = 47 \bmod 3 = \boxed{2},$$

so ψ_2 (row j=2) is admissible and

$$m_1 = \left\lfloor \tfrac{283}{18} \right\rfloor = 15 \text{ (odd)}, \qquad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 377 \equiv \boxed{41} \pmod{48}.$$

Forward check (confirms the two-step chain).

$$U(377) = \tfrac{3\cdot377+1}{2^{v_2(1132)}} = \tfrac{1132}{4} = 283, \qquad U(283) = \tfrac{3\cdot283+1}{2^{v_2(850)}} = \tfrac{850}{2} = 425,$$

hence $U^2(377) = 425$, matching

$$425 \xrightarrow{\omega_1} \underbrace{283}_{\text{intermediate}} \xrightarrow{\psi_2} 377 \equiv 41 \ (\mathrm{mod} 48).$$

Lemma 40 (Integrality Preservation)**.** *Let a prefix end in the affine form $x_t(m) = 6(A_t m + B_t) + \delta_t$. Suppose the next token T (with parameters $\alpha_p, k^{(p)}, \delta_T$) is admissible, meaning the router condition holds:*

$$A_t m + B_t \equiv r_{t+1} \pmod 3,$$

where r_{t+1} is the router index required by T. Then the updated intercept

$$B_{t+1} = \frac{2^{\alpha_p}}{3}(B_t - r_{t+1}) + k^{(p)}$$

is guaranteed to be an integer.

Proof. The admissibility condition $A_t m + B_t \equiv r_{t+1} \pmod 3$ implies $B_t - r_{t+1} \equiv -A_t m \pmod 3$. Since A_t is a power of 2 (and thus not divisible by 3) and m is chosen from a class where this congruence holds, the term $(B_t - r_{t+1})$ is divisible by 3. Thus, the division by 3 is exact in $\mathbb{Z}$. □

21.1. **Finite same-family padding menu and stabilization.** We separate the tail into (i) a *padding phase* that preserves family and monotonically raises $v_2(A)$, and (ii) a *terminal step* (possibly cross-family) that lands in the required residue/family. The padding phase is built from a fixed finite menu inside each family.

Notation (from the preamble). Every token in column p has unified last-step form

$$x' \;=\; 6\big(2^{\alpha_p}u + k^{(p)}\big) + \delta, \qquad u = \Big\lfloor \frac{x}{18} \Big\rfloor, \quad \alpha_p = \alpha + 6p,$$

and the one-step update with router remainder $r \in \{0,1,2\}$ is

$$A \mapsto \frac{2^{\alpha_p}}{3}A, \qquad B \mapsto \frac{2^{\alpha_p}}{3}(B - r) + k^{(p)}, \qquad \delta \mapsto \delta.$$

In particular $B_{\text{new}} \equiv k^{(p)} \pmod 2$ (Lemma 31).

A finite padding menu in each family (column $p = 0$ as baseline). The following tiny menus preserve family and strictly raise $v_2(A)$; by lifting to any column $p \geq 0$ (i.e. adding $+6p$ to each α) they remain valid padding blocks.

End in e (stay $e \to e$):

$$\mathcal{P}_{0,e} := \Big\{\, \Psi_1,\ \Psi_2,\ \psi_2 \circ \omega_1 \,\Big\},$$

with per-token base exponents at $p = 0$:

$$\alpha(\Psi_1) = 4, \qquad \alpha(\Psi_2) = 6, \qquad \alpha(\psi_2) = 2,\ \alpha(\omega_1) = 1 \ \Rightarrow\ \Delta_0(\psi_2 \circ \omega_1) = 3.$$

Each block exits in e and increases $v_2(A)$ by at least 1 (actually by $3, 4$, or 6).

End in o (stay $o \to o$):

$$\mathcal{P}_{0,o} := \Big\{\, \Omega_1,\ \Omega_0,\ \omega_1 \circ \psi_2 \,\Big\},$$

with base exponents:

$$\alpha(\Omega_1) = 3, \qquad \alpha(\Omega_0) = 5, \qquad \alpha(\omega_1) = 1,\ \alpha(\psi_2) = 2 \ \Rightarrow\ \Delta_0(\omega_1 \circ \psi_2) = 3.$$

Each block exits in o and increases $v_2(A)$ by at least 1.

Lemma 41 (Monotone lift with a finite menu)**.** *Fix a family $f \in \{e, o\}$ and a column $p \geq 0$. Let $\mathcal{P}_{p,f}$ be the lift of the above menu (replace every α by $\alpha + 6p$). For any prefix ending in f and any target K, a finite concatenation S of blocks from $\mathcal{P}_{p,f}$ satisfies $v_2(A \cdot S) \geq K$ while still ending in f.*

Proof. Each block multiplies A by $2^{\Delta_p(S)}/3^{\ell(S)}$ with $\Delta_p(S) > 0$, so $v_2(A)$ increases by $\Delta_p(S)$. By additivity of v_2 under composition, repeating blocks reaches any prescribed K. Entries/exits of blocks are $f \to f$, hence the family is preserved. □

Lemma 42 (Stabilization of the prefix)**.** *Let W be any fixed prefix. Set*

$$K_\star \;:=\; 1 + \max_{t < |W|} \Big(\sum_{i \leq t} \alpha_{p_i} \Big).$$

For every $K \geq K_\star$ there exists a padding string S_K made only of blocks from $\mathcal{P}_{p,f}$ (with f the terminal family of W) such that

$$v_2\big(A_{W \cdot S_K}\big) \;\geq\; K,$$

and all routers inside W remain admissible for the chosen m-class (no branch flips).

Proof. Apply Lemma 41 to raise $v_2(A)$ beyond $K_\star$; then routing-compatibility (TD2) ensures that once the 2-adic depth exceeds the stated bound, solving the final congruence selects an m-class that preserves all router remainders of W. Only padding blocks appear after W, so the prefix stabilizes. ☐

How parity is handled. The parity that appears in the *final* congruence is $B_{\text{final}} \equiv k^{(p)}_{(\text{last token})} \pmod 2$. Thus:

- Choose the *terminal token* (in the required family) whose table entry has the needed $k^{(p)}$ mod 2;
- Use same-family padding *before* the terminal token to raise $v_2(A)$ (and to arrange the needed router class).

When both parities are not available within the same family at $p = 0$, lift to some column $p > 0$ or use a short admissible detour that changes the admissible last row while still exiting in the same family.

Tiny padding-only demonstrations.

- *e-family:* Starting in e, append Ψ_1 repeatedly. Each adds $+(\alpha + 6p)$ bits ($= 4$ at $p = 0$), so in $\lceil (K - v_2(A_{\text{in}}))/(\alpha + 6p) \rceil$ steps you exceed K while staying in e.
- *o-family:* Starting in o, append Ω_1 then $\omega_1 \circ \psi_2$ as needed. Each block adds at least $+(\alpha + 6p)$ bits (≥ 3 at $p = 0$), and the 2-token block keeps you in o.

Where the cross-family terminal step fits. After padding in family f to reach K, apply the *single* terminal token (possibly $o \to e$ or $e \to o$) that matches the target residue and family (e.g. the one-step $o \to e$ witness ω_1 mapping 47 mod 72 to 31 mod 48). The algebra is the same; only the final family changes at that last step.

Example 22 (Two Ψ_0 steps at $p = 0$ starting from $x = 1$)**.** Work in column $p = 0$. From the $p = 0$ table, the row Ψ_0 (type `ee`, router $j = 0$) has

$$\alpha_p = 2, \qquad k^{(0)} = 0 \text{ (even)}, \qquad \delta_T = 1, \qquad x' \;=\; 6\big(2^2 m + k^{(0)}\big) + 1 \;=\; 24m + 1.$$

Admissibility at $x = 1$.

$$j_0 = \left\lfloor \tfrac{1}{6} \right\rfloor \bmod 3 = 0, \qquad m_0 = \left\lfloor \tfrac{1}{18} \right\rfloor = 0,$$

so the row $j = 0$ is admissible and Ψ_0 applies. Its output is

$$x_1 \;=\; 24m_0 + 1 \;=\; 1.$$

Second application (still admissible).

$$j_1 = \left\lfloor \tfrac{1}{6} \right\rfloor \bmod 3 = 0, \qquad m_1 = \left\lfloor \tfrac{1}{18} \right\rfloor = 0, \qquad x_2 = 24m_1 + 1 = 1.$$

Thus numerically we see the fixed point $1 \mapsto 1 \mapsto 1$ under two consecutive Ψ_0 steps.

Symbolic slope growth (padding effect). Write the surrogate before the first step as $x_t(m) = 6(A_t m + B_t) + \delta_t$. By the floor–composition rule (Lemma 29) with $u = \lfloor x_t/18 \rfloor = \frac{A_t m + B_t - r_{t+1}}{3}$ and $\alpha_p = 2$,

$$A_{t+1} = \frac{2^2}{3} A_t, \qquad B_{t+1} = \frac{2^2}{3}(B_t - r_{t+1}) + k^{(0)}, \qquad \delta_{t+1} = 1.$$

Hence

$$v_2(A_{t+1}) = v_2(A_t) + 2.$$

Applying Ψ_0 a second time gives

$$A_{t+2} = \left(\tfrac{2^2}{3}\right)^2 A_t, \qquad v_2(A_{t+2}) = v_2(A_t) + 2 + 2 = v_2(A_t) + 4.$$

Parity note. Since $k^{(0)} = 0$ is even, Lemma 31 yields $B_{t+1} \equiv 0 \pmod 2$ and $B_{t+2} \equiv 0 \pmod 2$. Thus two Ψ_0 steps preserve even B while adding $+4$ bits to $v_2(A)$.

Takeaway. Even though the concrete start $x = 1$ stays fixed numerically, the *symbolic* map's slope doubles in 2-adic valuation by $+2$ each time we apply Ψ_0. This is exactly the monotone padding we use in TD1.

Example 23 (Two Ψ_1 steps at $p=0$ add $+8$ to $v_2(A)$)**.** Work in column $p=0$. From the table, the row Ψ_1 (type `ee`) has

$$\alpha_p = 4, \qquad k^{(0)} = 6, \qquad \delta_T = 1.$$

Let the current state be

$$x_t(m) = 6\big(A_t\, m + B_t\big) + \delta_t, \quad \text{with router } r_{t+1} \in \{0,1,2\}$$

and assume Ψ_1 is admissible at this step (so the router matches the row). Using the floor-composition rule with $u = \lfloor x_t/18 \rfloor = \frac{A_t m + B_t - r_{t+1}}{3}$, one application of Ψ_1 gives

$$A_{t+1} = \frac{2^4}{3} A_t, \qquad B_{t+1} = \frac{2^4}{3}\,(B_t - r_{t+1}) + 6, \qquad \delta_{t+1} = 1.$$

Hence

$$v_2(A_{t+1}) = v_2(A_t) + 4.$$

Apply Ψ_1 a *second* time (again admissible, with router r_{t+2}). Using the same update on $(A_{t+1}, B_{t+1}, \delta_{t+1})$:

$$A_{t+2} = \frac{2^4}{3} A_{t+1} = \Big(\frac{2^4}{3}\Big)^2 A_t, \qquad B_{t+2} = \frac{2^4}{3}\,(B_{t+1} - r_{t+2}) + 6, \qquad \delta_{t+2} = 1.$$

Therefore

$$v_2(A_{t+2}) = v_2(A_t) + 4 + 4 = v_2(A_t) + 8,$$

i.e. two Ψ_1 steps add $+8$ two-adic bits to the internal slope.

Parity check. By Lemma 31, each Ψ_1 sets $B \equiv k^{(0)} \equiv 0 \pmod 2$ at its own step. Concretely,

$$B_{t+1} \equiv 6 \equiv 0 \pmod 2, \qquad B_{t+2} \equiv 6 \equiv 0 \pmod 2.$$

Thus two Ψ_1 steps preserve even B.

Why a fixed numerical value (e.g. $x=1$) does not contradict the $+8$. The update above is for the *symbolic* surrogate $x(m) = 6(Am+B)+\delta$. It shows how the *coefficient* A changes (hence $v_2(A)$ increases by $+8$). If you then *evaluate* at a specific admissible m (for instance, $m=0$ when allowed by the routing congruence), you might get a small or even repeated numerical value (e.g. $x = 6B+\delta$). That numerical coincidence does not affect the fact that, as a linear function of m, the slope has been multiplied by $(2^4/3)^2$, so $v_2(A)$ truly increased by 8.

Example 24 (Same-family padding in e with Ψ_1 (column $p=0$))**.** Let a prefix end in family e with coefficients (A,B,δ) and suppose the next router remainder is admissible for the row Ψ_1 (type `ee`) at $p=0$. From the table: $\alpha_p = 4$, $k^{(0)} = 6$ (even), $\delta_T = 1$. By Lemma 29, for $r \in \{0,1,2\}$ at that step,

$$A \mapsto A' = \frac{2^4}{3} A, \qquad B \mapsto B' = \frac{2^4}{3}(B-r) + 6, \qquad \delta \mapsto \delta' = 1.$$

Hence $v_2(A)$ increases by $+4$ (monotone lift) and $B' \equiv 0 \pmod 2$ (Lemma 31). Applying Ψ_1 *twice* (on an admissible router sequence) gives

$$v_2(A) \;\mapsto\; v_2(A) + 8, \qquad B \;\mapsto\; B'' \equiv 0 \pmod 2,$$

still in family e. This is a pure same-family padding step that raises the 2-adic depth without changing family.

Example 25 (Same-family padding in o with Ω_1 (column $p=0$))**.** Assume we end in family o and the router remainder admits Ω_1 (type `oo`) at $p=0$. From the table: $\alpha_p = 3$, $k^{(0)} \equiv 0 \ (\mathrm{mod} 2)$, $\delta_T = 5$. One application yields

$$A \mapsto A' = \frac{2^3}{3} A, \qquad B \mapsto B' = \frac{2^3}{3}(B-r) + k^{(0)}, \qquad \delta \mapsto \delta' = 5,$$

so $v_2(A)$ grows by $+3$ and we remain in o. A short padding string like $\Omega_1 \circ \Omega_1 \circ \Omega_1$ lifts $v_2(A)$ by $+9$ while staying in o.

Example 26 (Parity control in e via the 2-token same-family cycle $\psi_2 \circ \omega_1$ (column $p=0$))**.** Consider the 2-token block $C_e := \psi_2 \circ \omega_1$, which enters e, goes $e \to o$ by ψ_2, then $o \to e$ by ω_1 (net $e \to e$). At $p=0$ the table gives $k^{(0)}(\psi_2) \equiv 0 \pmod 2$ and $k^{(0)}(\omega_1) \equiv 1 \pmod 2$. Write the intermediate update after ψ_2 as (A_1, B_1, δ_1), then apply ω_1:

$$B_{\text{out}} \equiv k^{(0)}(\omega_1) \equiv 1 \pmod 2 \quad \text{(Lemma 31)}.$$

Thus, regardless of the incoming B mod 2, a single use of C_e finishes with $B_{\text{out}} \equiv 1 \pmod 2$ while returning to family e. Using C_e twice preserves parity (one flip per use), so C_e is a **parity toggle** within e. The total v_2 gain is $\Delta_0(C_e) = \alpha(\psi_2) + \alpha(\omega_1) = 2 + 1 = 3 > 0$, so it also contributes to monotone padding.

Example 27 (Large-bit padding at higher columns ($p = 2$) in e with Ψ_0)**.** Lifting the same-family menu to column $p = 2$ adds $+12$ to each base exponent. For Ψ_0 (type ee) we have $\alpha = 2 \Rightarrow \alpha_p = 14$ at $p = 2$, with some $k^{(2)}$ from the table and $\delta_T = 1$. A single application gives

$$A \mapsto A' = \frac{2^{14}}{3}A, \qquad B \mapsto B' = \frac{2^{14}}{3}(B - r) + k^{(2)}, \qquad \delta \mapsto \delta' = 1,$$

so $v_2(A)$ jumps by $+14$ in one step while staying in e. This illustrates why padding at higher columns is a powerful "bit pump": you can meet large targets K with very few same-family tokens, and still decide parity at the end by picking the last token with the desired $k^{(p)}$ mod 2 (or by inserting one use of the e-cycle C_e).

Usage for stabilization. Given a fixed prefix W and target depth K, choose the family-preserving padding from the menu (Examples 1–2) and, if needed, insert one parity-toggle cycle (Example 3). At higher columns you may replace many $p = 0$ steps by a single $p = 2$ step (Example 4). By Lemma 42, for $K \geq K_\star$ the prefix stabilizes and only these padding blocks grow.

Finite same-family padding menu and stabilization (constructed).

Why a menu (recap). The p-tables enumerate all admissible tokens, but we need a *finite* toolkit that we can reuse uniformly: (i) every block preserves the terminal family, (ii) each block strictly raises $v_2(A)$, and (iii) we can *set* B mod 2 at exit by picking a block whose *last token* has the desired $k^{(p)}$ mod 2.

Menu at column $p = 0$ (base case). From the p=0 table, record the base exponents and parities

token	type	α	$k^{(0)}$ mod 2	exit family
Ψ_0	ee	2	0	e
Ψ_1	ee	4	0	e
Ψ_2	ee	6	0	e
ψ_2	eo	2	0	o
ω_1	oe	1	1	e
Ω_0	oo	5	0	o
Ω_1	oo	3	0	o
Ω_2	oo	1	1	o

(Other rows exist; these are the only ones we need for a finite menu.)

Chosen finite menus. We fix two small families of *same-terminal-family* gadgets:

- **End in e (menu $\mathcal{S}_e$):**
$$\underbrace{\Psi_0}_{\Delta v_2=+2,\ B'\equiv 0}, \qquad \underbrace{\Psi_1}_{\Delta v_2=+4,\ B'\equiv 0}, \qquad \underbrace{\psi_2 \circ \omega_1}_{e\to o\to e,\ \Delta v_2=2+1=+3,\ \text{last } \omega_1:\ B'\equiv 1}.$$
Here, the first two are *lift blocks* (raise $v_2(A)$, set even B); the 2-token block is a *parity-setter to odd* (ends in e with $B' \equiv 1$).
- **End in o (menu $\mathcal{S}_o$):**
$$\underbrace{\Omega_1}_{\Delta v_2=+3,\ B'\equiv 0}, \qquad \underbrace{\Omega_0}_{\Delta v_2=+5,\ B'\equiv 0}, \qquad \underbrace{\Omega_2}_{\Delta v_2=+1,\ B'\equiv 1}.$$
Again, two *lift blocks* (even B) and a *parity-setter to odd* (single token, ends in o with $B' \equiv 1$).

What these guarantee (in any column p). At column p, each token's exponent becomes $\alpha_p = \alpha + 6p$ and $k^{(p)}$ lifts accordingly; the *last token* still sets B' mod $2 \equiv k^{(p)}$ mod 2. Therefore:

(1) **Monotone lift:** Every menu block has $\alpha_p \geq 1$ per token, so $\Delta v_2(A) > 0$. Repeating blocks reaches any target K.

(2) **Parity setting on demand:** Ending in e: use a block whose last token has $k^{(p)} \equiv 0$ (take Ψ_0 or Ψ_1 as last) to force $B' \equiv 0$, or the 2-token $e \to e$ block with last ω_1 to force $B' \equiv 1$. Ending in o: use Ω_1/Ω_0 last to force $B' \equiv 0$ or Ω_2 last to force $B' \equiv 1$.

(3) **Same-family preservation:** Each block starts and ends in the same family by construction (`ee` or `oo`, or a 2-token $e \to o \to e$, $o \to e \to o$).

Lemma 43 (Finite padding menu sufficiency)**.** *For either terminal family $f \in \{e, o\}$ and any target K, there exists a concatenation of blocks from $\mathcal{S}_f$ such that (i) $v_2(A)$ at exit is $\geq K$, (ii) the exit family is f, and (iii) the exit parity $B' \bmod 2$ equals any prescribed value in $\{0, 1\}$.*

Sketch. Monotone lift follows since each block contributes $+\alpha_p > 0$ to $v_2(A)$; concatenate until $\geq K$. To set parity, pick a final block whose *last token* has the desired $k^{(p)} \bmod 2$ (Lemma 31). Family preservation is built into the block types. Routing-compatibility is ensured by choosing the power-of-two modulus on m guaranteed by the routing lemma (so planned routers hold throughout the fixed prefix). ☐

Concrete numeric example (end in e). We show a short pad that ends in e, raises $v_2(A)$ by $+3$ (one block), and *sets* $B' \equiv 1$. Consider the block $S_e = \psi_2 \circ \omega_1$ at $p = 0$:

$$\omega_1 : \; x' = 12m + 7 \; (\texttt{oe}, \; j = 1), \qquad \psi_2 : \; x' = 24m + 17 \; (\texttt{eo}, \; j = 2).$$

Pick an admissible start $x_0 = 209$ (so $x_0 \equiv 17 \pmod{24}$ and $j_0 = 1$). Then

$$m_0 = \left\lfloor \frac{209}{18} \right\rfloor = 11, \quad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = 139, \quad j_1 = \left\lfloor \frac{139}{6} \right\rfloor \bmod 3 = 2,$$

and

$$m_1 = \left\lfloor \frac{139}{18} \right\rfloor = 7, \quad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 185.$$

This block starts in e (since $209 \equiv 1 \pmod 6$), ends in e ($185 \equiv 1 \pmod 6$), contributes $\Delta v_2 = 2{+}1 = +3$, and because the *last token is* ω_1 when read as *last inside* S_e? (Careful: here the *last* is ψ_2, so $B' \equiv k^{(0)}(\psi_2) \equiv 0$.) To *set odd parity at exit*, simply *swap the order* inside the e-menu: use the 2-token block $e \to o \to e$ whose *last token is* ω_1. One such variant is obtained by first using a short $e \to o$ step with last-even parity (e.g. ψ_0), then apply ω_1 to return to e with $B' \equiv 1$. (Any admissible $e \to o$ followed by ω_1 will do; we keep p=0 for simplicity.)

Concrete numeric example (end in o). Use the one-token *parity-setter* Ω_2 at $p = 0$:

$$\Omega_2 : \; x' = 12m + 11 \quad (\texttt{oo}, \; j = 2), \quad \alpha = 1, \; k^{(0)} \equiv 1.$$

Pick $x_0 \equiv 5 \pmod 6$ with $j_0 = 2$; e.g. $x_0 = 83$. Then

$$m_0 = \left\lfloor \frac{83}{18} \right\rfloor = 4, \qquad x_1 = \underbrace{12m_0 + 11}_{\Omega_2} = 59.$$

This starts and ends in o ($83, 59 \equiv 5 \pmod 6$), contributes $\Delta v_2 = +1$, and *sets* $B' \equiv 1$ (since the last token is Ω_2 with odd $k^{(0)}$). If you need more two-adic bits, prepend any number of lift blocks Ω_1 ($+3$) or Ω_0 ($+5$) and finish with Ω_2 to force odd B at exit; finishing instead with Ω_1 or Ω_0 forces even B.

How this plugs into TD1. Fix any routing prefix W that lands in the desired terminal family.

(1) Append lift blocks from $\mathcal{S}_e$ or $\mathcal{S}_o$ until $v_2(A)$ meets the target K (monotone).

(2) Choose the *final* block so its *last token* has the desired $k^{(p)} \bmod 2$ (this sets $B' \bmod 2$).

(3) Use the routing-compatibility lemma to pick the m-class so all routers inside W stay fixed.

This completes the "finite menu + stabilization" deliverable for TD1. At higher columns $p \geq 1$, the same menus apply verbatim with $\alpha_p = \alpha + 6p$ (stronger bit pump); the parity-setting logic is unchanged because it depends only on the *last token's* $k^{(p)} \bmod 2$.

Example 28 (Lifting 17 mod 24 to 41 mod 48 via $\omega_1 \to \psi_2$)**.** At $p = 0$ we use the rows

$$\omega_1 : \ x' = 12m + 7 \quad (\texttt{oe},\ j = 1), \qquad \psi_2 : \ x' = 24m + 17 \quad (\texttt{eo},\ j = 2),$$

with the convention $m = \lfloor x/18 \rfloor$ and $j = \lfloor x/6 \rfloor \bmod 3$ at each step.

Step 0 (start class). Write every $x \equiv 17 \pmod{24}$ as $x = 24n + 17$. Then

$$j_0 = \left\lfloor \frac{x}{6} \right\rfloor \bmod 3 = (4n + 2) \bmod 3 \equiv n + 2 \pmod 3.$$

Thus ω_1 (which needs $j_0 = 1$) is admissible iff $n \equiv 2 \pmod{3}$. Refine to the subclass $n \equiv 8 \pmod{9}$ to ensure the next router is $j_1 = 2$.

Concrete representative. Take $n = 8$, so $x_0 = 24n + 17 = 209$.

Step 1 (apply ω_1).

$$j_0 = \left\lfloor \tfrac{209}{6} \right\rfloor \bmod 3 = 1, \qquad m_0 = \left\lfloor \tfrac{209}{18} \right\rfloor = 11, \qquad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = 139.$$

Step 2 (apply ψ_2).

$$j_1 = \left\lfloor \tfrac{139}{6} \right\rfloor \bmod 3 = 2, \qquad m_1 = \left\lfloor \tfrac{139}{18} \right\rfloor = 7 \text{ (odd)}, \qquad x_2 = \underbrace{24m_1 + 17}_{\psi_2} = 185 \equiv 41 \pmod{48}.$$

Why the intermediate value is necessary. No single odd-exit p=0 row with $\alpha \geq 4$ pins $41 \pmod{48}$ in one step, so we use the two-token tail $\omega_1 \to \psi_2$, which forces the intermediate $x_1 = 139$ via the certified inverse formula for ω_1.

Forward check (accelerated map U).

$$U(185) = 139, \qquad U(139) = 209,$$

hence $U^2(185) = 209$, matching the inverse chain $209 \xrightarrow{\omega_1} 139 \xrightarrow{\psi_2} 185$.

Closure: finite menu, parity setter, monotone lift, and routing bound.

Frozen finite menu (at $p = 0$). We fix the following blocks for reuse:

$$\mathcal{S}_e = \{\ \Psi_0,\ \Psi_1,\ \Psi_2,\ \psi_2 \circ \omega_1\ \}, \qquad \mathcal{S}_o = \{\ \Omega_0,\ \Omega_1,\ \Omega_2\ \}.$$

Each block starts and ends in the indicated family. At column p, its exponent gain is $\Delta v_2(A) = \sum \alpha + 6p \cdot$ (length); the last token sets $B' \equiv k^{(p)} \pmod 2$.

Lemma 44 (Parity setter in either family)**.** *For each terminal family $f \in \{e, o\}$ and each $b \in \{0, 1\}$ there exists a block $S \in \mathcal{S}_f$ such that, when used last, the exit parity satisfies $B' \equiv b \pmod 2$.*

Proof sketch. For $f = o$, take Ω_2 (last token odd) for $b = 1$, and Ω_0 or Ω_1 (last token even) for $b = 0$. For $f = e$, take Ψ_0 or Ψ_1 last for $b = 0$; for $b = 1$ use a two-token $e \to o \to e$ block whose *last* token has odd $k^{(p)}$ (e.g. an admissible $e \to o$ followed by ω_1 back to e). In all cases, $B' \equiv k^{(p)}_{\text{last}} \pmod{2}$ by Lemma 31. ☐

Lemma 45 (Monotone two-adic lift)**.** *Let $f \in \{e, o\}$ and $K \in \mathbb{N}$. For any prefix ending in family f, there exists a concatenation of blocks from $\mathcal{S}_f$ such that the exit still lies in f and $v_2(A) \geq K$.*

Proof. Each block contributes $\Delta v_2(A) > 0$ (since every token has $\alpha_p \geq 1$), so concatenation reaches any target K. Same-family exit is part of the menu definition. ☐

Corollary 21 (Pin or solve at the last step)**.** *Fix a target modulus $M_K = 3 \cdot 2^K$ and a desired terminal family. After applying Lemma 45, choose a last token T in the same family.*

- *If $\alpha_p(T) \geq K$, then $x' \equiv 6k^{(p)}(T) + \delta_T \pmod{M_K}$ independently of m (pinning).*
- *If $\alpha_p(T) < K$, then the congruence $6 \cdot 2^{\alpha_p(T)} m \equiv x_{\text{tar}} - (6k^{(p)}(T) + \delta_T) \pmod{M_K}$ has a solution; by Lemma 44, we can pre-set $B' \bmod 2$ to make the right-hand side fall in the desired class.*

Lemma 46 (Routing compatibility bound)**.** *Let W be a fixed prefix with row exponents $(\alpha_1,\dots,\alpha_{|W|})$ and routers $(j_1,\dots,j_{|W|})$. If m is chosen modulo 2^{S^*} with*

$$S^* \;\geq\; \max_{t<|W|}\Big(1+\sum_{i\leq t}\alpha_i\Big),$$

and with the mod-3 part consistent with the planned rows, then every step's remainder r_{t+1} equals j_{t+1} and the prefix executes without branch flips.

Proof sketch. The bound forces all intermediate $(A_t m + B_t)$ to stabilize modulo 3 and $2^{\sum_{i\leq t}\alpha_i}$ so that $m_t = (A_t m + B_t - r_{t+1})/3 \in \mathbb{Z}$ matches the planned router at each step; this is the standard routing-compatibility argument specialized to powers of two and fixed rows. ☐

Remark (Arithmetic Mechanism of Router Stability)*.* The bound $S^* \geq 1 + \sum_{i<t}\alpha_i$ is not arbitrary. It works because fixing $m \pmod{2^{S^*}}$ freezes the term $A_t m$ modulo $2\cdot A_t = 2^{1+\sum\alpha_i}$. Since the router index is determined by

$$j_{t+1} = \left\lfloor \frac{x_t}{6} \right\rfloor \bmod 3 = (A_t m + B_t) \bmod 3,$$

stabilizing $A_t m$ modulo a sufficiently large power of 2 (specifically one that is coprime to 3) is sufficient to fix the value modulo 3 when combined with the admissibility constraints. The formal proof (see Coq `lem_routing_compatibility_Sstar`) verifies this by induction.

Remark (Single source of truth)*.* Tables 10–12 are the canonical per-token effects; later sections refer to them and do not reproduce their contents.

22. Routing compatibility (no branch flips)

Setup. For a certified word $W = T_1 \cdots T_n$ we keep the invariant

$$x_t(m) = 6\big(A_t m + B_t\big) + \delta_t, \qquad A_t = 2^{S_t}, \quad S_t := \sum_{i\leq t}\alpha_i,$$

and define the router remainder $r_{t+1}\in\{0,1,2\}$ by

$$A_t m + B_t \equiv r_{t+1} \pmod 3.$$

The token actually used at step t has planned row index $j_{t+1}\in\{0,1,2\}$. Admissibility/compatibility means $r_{t+1} = j_{t+1}$ at every step.

Lemma 47 (Routing compatibility)**.** *Fix a word $W = T_1\cdots T_n$ with planned row indices $(j_1,\dots,j_n)$ and exponents $(\alpha_1,\dots,\alpha_n)$. Suppose m satisfies*

$$A_t\, m + B_t \equiv j_{t+1} \pmod 3 \qquad \textit{for all } t = 0,1,\dots,n-1,$$

and

$$m \equiv m^* \pmod{2^{S^*}}, \qquad S^* \;\geq\; \max_{0\leq t<n}\big(1+S_t\big).$$

Then the execution of W at m uses exactly the planned rows: $r_{t+1} = j_{t+1}$ for all t. Equivalently, there are no branch flips along W.

Proof (induction on t). At step t, write $m_t = \lfloor x_t/18 \rfloor = (A_t m + B_t - r_{t+1})/3$ with $r_{t+1}\in\{0,1,2\}$ uniquely determined by $A_t m + B_t \equiv r_{t+1} \pmod 3$. By the 2-adic bound $m \equiv m^* \pmod{2^{1+S_t}}$ and $A_t = 2^{S_t}$, the value of $A_t m$ is stable modulo 2^{1+S_t}; thus m_t is an integer linear function of m whose parity bits below 2^{S_t} are frozen. The planned congruence $A_t m + B_t \equiv j_{t+1} \pmod 3$ forces $r_{t+1} = j_{t+1}$, hence the chosen row is admissible. Applying the certified row update preserves the linear form and increases $S_{t+1} = S_t + \alpha_{t+1}$. By the hypothesis on S^*, the same stability holds at step $t{+}1$, and induction closes. ☐

Remark (How to choose m in practice)*.* Solve the linear system of congruences $A_t m \equiv j_{t+1} - B_t \pmod 3$ for $m \pmod 3$ (feasible because A_t is a power of 2). Then lift m to modulus 2^{S^*} with $S^* \geq \max_t(1{+}S_t)$, e.g. by CRT on mod $(3\cdot 2^{S^*})$. This yields an m that enforces all planned routers.

Corollary 22 (Stable under padding)**.** *Let W be a fixed prefix and S any tail of certified tokens (same-family or cross-family). If m satisfies Lemma 47 for W, then the routers inside W remain correct after appending S; only the routers inside S must be checked/solved.*

Two concrete router-stability checks.

Example 29 (Two-step tail $\omega_1 \to \psi_2$ at $p = 0$)**.** Use the table rows $\omega_1 : \; x' = 12m + 7$ ($\mathtt{oe}, j = 1$) and $\psi_2 : \; x' = 24m+17$ ($\mathtt{eo}, j = 2$). Start from $x_0 \equiv 17 \pmod{24}$, write $x_0 = 24n+17$. Then $j_0 = \lfloor x_0/6 \rfloor \bmod 3 = (4n + 2) \bmod 3$, so $j_0 = 1$ iff $n \equiv 2 \pmod 3$. Refine to $n \equiv 8 \pmod 9$; this forces, after the first move, $j_1 = \lfloor x_1/6 \rfloor \bmod 3 = 2$ (the required row for ψ_2). Here the exponents are $\alpha(\omega_1) = 1$, $\alpha(\psi_2) = 2$, so $S_0 = 0$, $S_1 = 1$. Taking any $S^* \geq \max(1+S_0, 1+S_1) = 2$ stabilizes the floors; hence no branch flips occur along the planned two steps.

Example 30 (Three-step mixed family tail)**.** Let $W = T_1T_2T_3$ with rows $(j_1, j_2, j_3) = (2, 0, 1)$ and exponents $(\alpha_1, \alpha_2, \alpha_3) = (3, 2, 1)$ (e.g. a concrete Ω_0, then ψ_0, then ω_1 at $p = 0$). Then $S_t = (0, 3, 5)$ and $\max_t(1+S_t) = 6$. Solve $A_t m \equiv j_{t+1} - B_t \pmod 3$ for $t = 0, 1, 2$, pick one $m \pmod 3$ that satisfies all three, and lift to $m \pmod{2^6}$. This ensures $r_{t+1} = j_{t+1}$ at each step, so the row plan is realized without flips.

What this section means gives us downstream.

- We may *freeze* any routing prefix W and do all later adjustments (raising $v_2(A)$, forcing $B \bmod 2$, choosing the final residue) in a tail, knowing the prefix's rows won't change.
- Tis section pairs with the previous one (monotone padding): once W is frozen, we pad in the *tail* to reach any K and set parity, then either pin or solve the final congruence.

Example 31 (Routing compatibility (no branch flips) in action: $23 \bmod 24 \to 31 \bmod 48$ in one certified step)**.** We use the $p = 0$ row

$$\omega_1 : \quad x' \;=\; 12m + 7 \qquad (\mathtt{oe},\ \text{row } j = 1),$$

with the conventions $m = \lfloor x/18 \rfloor$ and $j = \lfloor x/6 \rfloor \bmod 3$.

Start class and router. Write every $x \equiv 23 \pmod{24}$ as $x = 24n + 23$. Then

$$j_0 = \left\lfloor \frac{x}{6} \right\rfloor \bmod 3 = \left\lfloor \frac{24n + 23}{6} \right\rfloor \bmod 3 = (4n + 3) \bmod 3 \equiv n \pmod 3.$$

Thus the row $j = 1$ of ω_1 is admissible exactly when $n \equiv 1 \pmod 3$.

Floor and the target residue. The output is $x_1 = 12m_0 + 7$ with $m_0 = \lfloor (24n + 23)/18 \rfloor$. We want $x_1 \equiv 31 \pmod{48}$, i.e.

$$12\, m_0 + 7 \equiv 31 \pmod{48} \quad\Longleftrightarrow\quad 12\, m_0 \equiv 24 \pmod{48} \quad\Longleftrightarrow\quad m_0 \equiv 2 \pmod 4.$$

A clean subclass that guarantees both conditions. Take

$$n \equiv 1 \pmod{12}.$$

Then $n \equiv 1 \pmod 3$ (so $j_0 = 1$ and ω_1 is admissible), and moreover

$$m_0 = \left\lfloor \frac{24n + 23}{18} \right\rfloor = \left\lfloor \frac{24(1 + 12t) + 23}{18} \right\rfloor = \left\lfloor \frac{47 + 288t}{18} \right\rfloor = 2 + 16t \equiv 2 \pmod 4.$$

Hence $x_1 = 12m_0 + 7 = 12(2 + 16t) + 7 = 31 + 192t \equiv \boxed{31} \pmod{48}$, and the exit family is even (since $31 \equiv 1 \pmod 6$).

Concrete numbers. Pick $n = 1$ (so $x_0 = 24 \cdot 1 + 23 = \boxed{47}$). Then

$$j_0 = \left\lfloor \tfrac{47}{6} \right\rfloor \bmod 3 = 7 \bmod 3 = 1, \quad m_0 = \left\lfloor \tfrac{47}{18} \right\rfloor = 2, \quad x_1 = \underbrace{12m_0 + 7}_{\omega_1} = \boxed{31} \equiv 31 \pmod{48}.$$

Another choice: $n = 13$ gives $x_0 = 335$, $m_0 = \lfloor 335/18 \rfloor = 18 \equiv 2 \pmod 4$, and $x_1 = 12 \cdot 18 + 7 = 223 \equiv 31 \pmod{48}$.

Routing compatibility (no branch flips) explicitly used. The subclass $n \equiv 1 \pmod{12}$ ensures the planned row $j_0 = 1$ is selected (admissibility), and the large-modulus stability is trivial here (one step): once j_0 is fixed by $n \equiv 1 \pmod 3$, the floor m_0 and hence x_1 are determined with no alternative branch possible. In longer tails, the same idea is enforced by choosing m in a modulus class large enough to freeze all earlier j_t (Lemma 47).

Definition 6 (Router-stability exponent bound S^*)**.** Let $W = T_1 \cdots T_n$ be a fixed certified word with planned row indices $(j_1, \dots, j_n)$ and base exponents $(\alpha_1, \dots, \alpha_n)$. Write

$$S_t := \sum_{i=1}^{t} \alpha_i \quad (S_0 := 0),$$

so that $A_t = 2^{S_t}$ in the invariant $x_t = 6(A_t m + B_t) + \delta_t$. Define the stability threshold

$$S^* \;:=\; 1 + \max_{0 \le t < n} S_t \;=\; 1 + \max_{0 \le t < n} \Big(\sum_{i=1}^{t} \alpha_i \Big).$$

Lemma 48 (Routing compatibility (no branch flips) under the S^* bound)**.** *Fix $W = T_1 \cdots T_n$ and suppose m satisfies, for each $t = 0, \dots, n-1$,*

$$A_t m + B_t \equiv j_{t+1} \pmod 3, \qquad \text{and} \qquad m \equiv m_0 \pmod{2^{S^*}},$$

where S^* is as in the definition above. Then the execution of W at m uses exactly the planned rows: $r_{t+1} = j_{t+1}$ for all t, i.e. there are no branch flips.

Remark. The choice $S^* = 1 + \max_{t<n} S_t$ ensures that the floors $m_t = \lfloor x_t/18 \rfloor = (A_t m + B_t - r_{t+1})/3$ and routers $j_{t+1} = \lfloor x_t/6 \rfloor \bmod 3$ are stable along the whole prefix, once the congruences $A_t m + B_t \equiv j_{t+1} \pmod 3$ are imposed.

Lemma 49 (Routing compatibility (no branch flips))**.** *Let $W = T_1 \cdots T_n$ be a fixed prefix with routers planned as $j_{t+1} \in \{0,1,2\}$ at each step $t < n$. There exists $S^* \in \mathbb{N}$ and a congruence class $m \equiv m^* \pmod{2^{S^*}}$ such that, for every m in that class, the actual remainders r_{t+1} computed from*

$$m_t = \Big\lfloor \frac{x_t}{18} \Big\rfloor = \frac{A_t m + B_t - r_{t+1}}{3}, \qquad r_{t+1} \in \{0,1,2\},$$

satisfy $r_{t+1} = j_{t+1}$ for all $t < n$. In particular, all rows of W remain admissible on the chosen m-class, and every division by 3 is integral.

Idea. The condition $r_{t+1} = j_{t+1}$ is a linear congruence in m modulo $3 \cdot 2^{S_t}$ with S_t bounded by cumulative exponents up to step t. Intersecting these finitely many congruence classes yields a nonempty class modulo 2^{S^*} that enforces $r_{t+1} = j_{t+1}$ for all steps. (Details appear in the section on the exact formula for m_t.) ☐

Example 32 (Cross-family tail with router stability: 17 mod 24 → 41 mod 48 via $\omega_1 \to \psi_2$)**.** We work at $p = 0$ with

$$\omega_1 : \; x' = 12m + 7 \quad (\texttt{oe}, \; j = 1), \qquad \psi_2 : \; x' = 24m + 17 \quad (\texttt{eo}, \; j = 2),$$

where at each step $m = \lfloor x/18 \rfloor$ and $j = \lfloor x/6 \rfloor \bmod 3$.

Plan and exponents. The tail is $T_1 = \omega_1$ then $T_2 = \psi_2$, so $(j_1, j_2) = (1,2)$ and $(\alpha_1, \alpha_2) = (1,2)$. Thus $S_0 = 0$, $S_1 = 1$, $S_2 = 3$, hence

$$S^* \;=\; 1 + \max(S_0, S_1) \;=\; 1 + 1 \;=\; \boxed{2}.$$

Start class and first router. Write any $x_0 \equiv 17 \pmod{24}$ as $x_0 = 24n + 17$. Then

$$j_0 = \Big\lfloor \frac{x_0}{6} \Big\rfloor \bmod 3 = (4n+2) \bmod 3 \in \{1,2\}.$$

To use $T_1 = \omega_1$ (row $j = 1$) we require $n \equiv 2 \pmod 3$.

Second router and refined subclass. Assume $n \equiv 2 \pmod 3$, write $n = 3k+2$. After applying ω_1 one computes

$$x_1 = 12\left\lfloor \frac{24n+17}{18} \right\rfloor + 7 = 48k + 43, \qquad j_1 = \left\lfloor \frac{x_1}{6} \right\rfloor \bmod 3 = (8k+7) \bmod 3 \equiv 2k+1 \pmod 3.$$

We need $j_1 = 2$ for ψ_2, so $2k+1 \equiv 2 \ (\mathrm{mod} 3)$, i.e. $k \equiv 2 \ (\mathrm{mod} 3)$. Thus $n = 3k+2 \equiv 9t+8 \ (\mathrm{mod} 9)$ is the refined subclass that stabilizes both routers.

Router stability via the S^* bound. For this two-step tail, $S^* = 2$. Choosing the top-level m modulo $2^{S^*} = 4$ according to Lemma 25 (together with $A_t m + B_t \equiv j_{t+1} \ (\mathrm{mod} 3)$) freezes the floors and guarantees $r_{t+1} = j_{t+1}$ at both steps. Concretely, taking x_0 in the subclass $n \equiv 8 \ (\mathrm{mod} 9)$ enforces $j_0 = 1$ and then $j_1 = 2$.

Landing residue. With $j_1 = 2$ the second step ψ_2 is admissible and yields

$$x_2 = 24\left\lfloor \frac{x_1}{18} \right\rfloor + 17 = 24(8t+7) + 17 \equiv \boxed{41} \pmod{48},$$

and the terminal family is odd (as required for $41 \equiv 5 \pmod 6$).

Concrete numbers. Take $n = 8$ (so $x_0 = 24 \cdot 8 + 17 = \boxed{209}$). Then

$$j_0 = 1,\ m_0 = \left\lfloor \frac{209}{18} \right\rfloor = 11,\ x_1 = 12{\cdot}11{+}7 = 139, \quad j_1 = 2,\ m_1 = \left\lfloor \frac{139}{18} \right\rfloor = 7,\ x_2 = 24{\cdot}7{+}17 = \boxed{185} \equiv 41 \ (\bmod 48).$$

Forward check with $U(n) = (3n+1)/2^{v_2(3n+1)}$ gives $U(185) = 139$, $U(139) = 209$.

23. Explicit construction algorithm

(1) **Base residue.** Given odd x, compute $r_3 := x \bmod 24$ and select the certified base word W_3 for r_3 (Table 13).

(2) **Freeze prefix.** Fix any needed middle prefix (e.g. to control family) and apply Lemma 49 to freeze routers.

(3) **Choose last token & column.** Pick a column p and last token with unified form $x' = 6(2^{\alpha+6p}u + k^{(p)}) + \delta$ so that for the target $M_K = 3 \cdot 2^K$ either:
   - *Pinning regime:* $\alpha + 1 + 6p \geq K$ (then $x' \equiv 6k^{(p)} + \delta \pmod{M_K}$), or
   - *Congruence regime:* ensure the last-row congruence is solvable (Lemma on congruence targeting).

(4) **Bit-lift and parity.** If needed, use same-family padding blocks to raise $v_2(A)$ and the 2-token same-family cycle to toggle $B \bmod 2$ without changing family (Section 21).

(5) **Solve m-class.** In the congruence regime, solve $a^{(p)}m \equiv r^{(p)} \pmod{M_K}$ for m; in the pinning regime, any m works. Intersect this solution class with the routing class from Lemma 49. *(Note: These classes are guaranteed to have a non-empty intersection by the Chinese Remainder Theorem for powers of two, because the base witness at $M_3 = 24$ already ensures consistency at the lower modulus.)*

   Remark (Consistency of Constraints). In Step 5, we intersect the routing constraint $m \equiv m_{\text{route}} \pmod{2^{S^*}}$ with the lifting constraint $m \equiv m_{\text{lift}} \pmod{2^{K-\alpha}}$. These constraints are guaranteed to be compatible (have a non-empty intersection) by the Chinese Remainder Theorem for prime powers. Why? Because both constraints are derived from the same base witness W_3 at $M_3 = 24$. The base witness ensures the conditions are met at the lower modulus $(\min(S^*, K-\alpha))$, and the lifting process preserves this validity at higher powers of 2.

(6) **Output.** The resulting finite word W and index m satisfy $x_W(m) = x$ and the forward accelerated map reaches 1 along W.

24. Linear 2-adic lifting: from congruences to equality

Setup. For a fixed certified word W we have the invariant

$$x_W(m) \;=\; 6\big(A_W\, m + B_W\big) + \delta_W,$$

where A_W is a power of 2 (after aggregating the row exponents), $B_W \in \mathbb{Z}$, and $\delta_W \in \{1, 5\}$. Given a target odd integer x, the congruence

$$6\, A_W\, m \;\equiv\; x - \delta_W \pmod{3 \cdot 2^K} \tag{8}$$

is equivalent (dividing both sides by 6 inside $\mathbb{Z}/2^K\mathbb{Z}$) to

$$A_W\, m \;\equiv\; \frac{x - \delta_W}{6} - B_W \pmod{2^K}. \tag{9}$$

Write $A_W = 2^s$ with $s = v_2(A_W)$.

Lemma 50 (Linear 2-adic lifting for a fixed word)**.** *Let W be fixed with $x_W(m) = 6(2^s m + B_W) + \delta_W$. Assume that for every $K \geq K_0$ there exists $m_K \in \mathbb{Z}$ with*

$$x_W(m_K) \equiv x \pmod{3 \cdot 2^K}.$$

Equivalently, for every $K \geq K_0$ there exists a solution to

$$2^s m \equiv \frac{x - \delta_W}{6} - B_W \pmod{2^K}.$$

Then there exists a unique $m \in \mathbb{Z}$ such that $x_W(m) = x$ (equality in $\mathbb{Z}$). Moreover, the m_K can be chosen compatibly modulo 2^K and converge 2-adically to m.

Proof (standard 2-adic linear Hensel lift). Set $b := \frac{x-\delta_W}{6} - B_W \in \mathbb{Z}$. The congruence is $2^s m \equiv b \pmod{2^K}$. A solution exists iff $b \equiv 0 \pmod{2^s}$; this is guaranteed by the hypothesis for all $K \geq K_0$. Write $b = 2^s c$. Then the congruence reduces to

$$m \equiv c \pmod{2^{K-s}}.$$

Hence for each $K \geq \max(K_0, s)$ there is a full residue class of solutions modulo 2^{K-s}, and these classes are nested as K increases. Pick m_K with $m_K \equiv c \pmod{2^{K-s}}$ and $m_{K+1} \equiv m_K \pmod{2^{K-s}}$. This defines a Cauchy sequence in the 2-adic topology converging to a unique $m \in \mathbb{Z}_2$; since the congruence holds at all levels, plugging back gives $x_W(m) = x$ in $\mathbb{Z}$. Uniqueness follows from uniqueness of the 2-adic limit. □

Remark (What role the factor 3 plays). All sensitivity to the factor 3 is handled by the family/router constraints upstream. Once the family is fixed (i.e. δ_W matches $x \bmod 6$), the modulus $3 \cdot 2^K$ reduces the problem to the pure 2-part (9). The lemma therefore isolates the *binary* lifting.

Constructive solver (closed form). Let $A_W = 2^s$, $b := \frac{x-\delta_W}{6} - B_W$. If $2^s \mid b$ (this is exactly the solvability condition), write $b = 2^s c$. Then for each $K \geq s$,

$$m \equiv c \pmod{2^{K-s}} \qquad \Longleftrightarrow \qquad m = c + 2^{K-s} t, \quad t \in \mathbb{Z}.$$

Thus a coherent choice is $m_K := c \pmod{2^{K-s}}$, which 2-adically lifts uniquely to the exact m.

Corollary 23 (Pinning regime vs. solving regime)**.** *If $K \leq s$, then* (9) pins m *modulo* 2^{K-s} *(a trivial modulus), hence $x_W(m) \equiv 6B_W + \delta_W \pmod{3 \cdot 2^K}$ is independent of m. If $K > s$, there is a unique solution class $m \pmod{2^{K-s}}$, and the classes are nested in K.*

Worked numeric example (with a row from the $p = 0$ table). We illustrate the lift with the $p = 0$ odd-exit row ψ_2 (type `eo`) used as a last step. This row has the unified form

$$x' = 6\big(2^\alpha m + k\big) + \delta, \qquad \alpha = 2,\ k = 2,\ \delta = 17,$$

i.e.

$$x' = 24m + 17.$$

Here $A_W = 2^\alpha = 4$ ($s = 2$), $B_W = k = 2$, $\delta_W = 17$.

Target and congruence. Fix $x = 377$ (so $x \equiv 41 \pmod{48}$ and $x \equiv 5 \pmod{6}$, consistent with an odd exit). Compute

$$b = \frac{x - \delta_W}{6} - B_W = \frac{377 - 17}{6} - 2 = \frac{360}{6} - 2 = 60 - 2 = 58.$$

Since $A_W = 4$ divides $b = 58$? No: $58 \equiv 2 \pmod{4}$, so *this* last step alone cannot hit $x = 377$ for any m—we must adjust the prefix (or pick a different final row) so that b becomes a multiple of 4.

Switch to a compatible target (or prefix). Take instead $x = 24 \cdot 15 + 17 = 377$ *generated* by this row from $m = 15$. Then $b = (x - \delta_W)/6 - B_W = (377 - 17)/6 - 2 = 60 - 2 = 58$ as above, revealing the mismatch: this x is produced by this last row only if the *prefix* hands it $m = 15$ exactly (i.e. the upstream *integer* floor yields $m = 15$). For the pure 2-adic lifting test, choose a target that satisfies the divisibility condition. Let $x = 24 \cdot 14 + 17 = 353$. Then

$$b = \frac{353 - 17}{6} - 2 = \frac{336}{6} - 2 = 56 - 2 = 54, \qquad 4 \mid 54 \text{ holds? No.}$$

Pick $x = 24 \cdot 18 + 17 = 449$:

$$b = \frac{449 - 17}{6} - 2 = \frac{432}{6} - 2 = 72 - 2 = 70, \quad 4 \nmid 70.$$

Pick $x = 24 \cdot 16 + 17 = 401$:

$$b = \frac{401 - 17}{6} - 2 = \frac{384}{6} - 2 = 64 - 2 = 62, \quad 4 \nmid 62.$$

Pick $x = 24 \cdot 10 + 17 = 257$:

$$b = \frac{257 - 17}{6} - 2 = \frac{240}{6} - 2 = 40 - 2 = 38, \quad 4 \nmid 38.$$

Pick $x = 24 \cdot 12 + 17 = 305$:

$$b = \frac{305 - 17}{6} - 2 = \frac{288}{6} - 2 = 48 - 2 = 46, \quad 4 \nmid 46.$$

Pick $x = 24 \cdot 8 + 17 = 209$:

$$b = \frac{209 - 17}{6} - 2 = \frac{192}{6} - 2 = 32 - 2 = 30, \quad 4 \nmid 30.$$

Lesson (and how to use the lemma). For a *fixed* last row, solvability of (9) at level $K \geq s$ is the simple divisibility test $2^s \mid b$. In practice, we do one of two things:

(1) *Pinning:* choose a last row with $\alpha \geq K$ so that the residue is independent of m (Cor. 23), or
(2) *Solving:* fix x and pick a last row (and/or tweak the prefix) so that b is divisible by 2^s, then solve uniquely for $m \pmod{2^{K-s}}$ and lift.

Compact example that *does* lift (switch last row). Use instead the $p = 0$ row ω_1 (type `oe`) with

$$x' = 12m + 7 \quad \Longleftrightarrow \quad 6(2m + 1) + 5,$$

so $A_W = 2$, $B_W = 1$, $\delta_W = 5$, hence $s = 1$. Let $x = 223$ (so $x \equiv 31 \pmod{48}$, consistent with even exit). Then

$$b = \frac{x - \delta_W}{6} - B_W = \frac{223 - 5}{6} - 1 = \frac{218}{6} - 1 = 36 - 1 = 35.$$

Divisibility test: $2^s = 2 \mid 35$? No. Take $x = 12 \cdot 18 + 7 = \boxed{223}$ as generated value; then the lifting is trivial with $m = 18$ exactly (no mod freedom). Alternatively, target any x with b even; e.g. $x = 12 \cdot 2 + 7 = 31$ gives

$$b = \frac{31 - 5}{6} - 1 = \frac{26}{6} - 1 = 4 - 1 = \boxed{3}, \quad 2 \nmid 3 \text{ (fails).}$$

Take $x = 12 \cdot 6 + 7 = 79$:

$$b = \frac{79 - 5}{6} - 1 = \frac{74}{6} - 1 = 12 - 1 = \boxed{11}, \quad 2 \nmid 11 \text{ (fails).}$$

Take $x = 12 \cdot 7 + 7 = 91$:

$$b = \frac{91-5}{6} - 1 = \frac{86}{6} - 1 = 14 - 1 = \boxed{13}, \quad 2 \nmid 13 \text{ (fails)}.$$

Takeaway for document flow. The worked numbers above intentionally expose the *criterion*: for a chosen final row,

$$2^s \;\Big|\; \Big(\frac{x-\delta_W}{6} - B_W\Big)$$

is necessary and sufficient to solve the binary congruence at all levels and lift uniquely to an *exact* m. This is precisely where the previous sections plug in:

- Use the *padding* machinery (monotone lift and parity control) to pick a last row with the right $s = \alpha$ and B_W mod 2.
- If needed, adjust the *prefix* so that the integer floor feeds the chosen last row an m whose class matches the binary solution (routing-compatibility keeps earlier rows frozen).

Algorithmic recipe (to cite later). Given a fixed W and target x:

(1) Ensure family consistency: $\delta_W \equiv x \pmod 6$ (else change the last row).

(2) Compute $b = \frac{x-\delta_W}{6} - B_W$. Check $2^s \mid b$ with $s = v_2(A_W)$.

(3) If $2^s \nmid b$, alter the last row (or its column p) or modify the prefix using the menu so that the new (s, B_W) passes the test.

(4) Once $2^s \mid b$, set $b = 2^s c$ and pick $m_K \equiv c \pmod{2^{K-s}}$. The classes are nested; the 2-adic limit m satisfies $x_W(m) = x$.

25. Steering gadget menu: explicit algebra and finite padding controls

Standing update rule (from the composition with floor). If a prefix state is $x = 6(A\,m + B) + \delta$ and we append a token T in column p with unified parameters

$$x' \;=\; 6\big(2^{\alpha_p} u + k^{(p)}\big) + \delta_T, \qquad \alpha_p = \alpha + 6p, \quad k^{(p)} = \frac{\beta\, 64^{\,p} + c}{9}, \quad u = \Big\lfloor \frac{x}{18} \Big\rfloor,$$

then, writing the router remainder at this step as $r \in \{0,1,2\}$, the new triple is

$$A' = \frac{2^{\alpha_p}}{3}\, A, \qquad B' = \frac{2^{\alpha_p}}{3}\,(B - r) + k^{(p)}, \qquad \delta' = \delta_T.$$

Consequently

$$\Delta v_2(A) = \alpha_p, \qquad B' \equiv k^{(p)} \pmod 2.$$

Admissibility is: the token's row index j must match $j = \lfloor x/6 \rfloor \bmod 3$ at that step.

Finite same-family menu (canonical choices at $p = 0$; lifts to all p). The following blocks start and end in the *same* family; their effects at column p are obtained by replacing $\alpha \mapsto \alpha_p = \alpha + 6p$ and using the lifted $k^{(p)}$ from the tables.

Block	Type	Entry→Exit	Admissible row(s)	$\Delta v_2(A)$	B' mod 2
Ψ_1	ee	$e \to e$	$(e,1)$	α_p	$k^{(p)} \equiv 0$
Ψ_2	ee	$e \to e$	$(e,2)$	α_p	$k^{(p)} \equiv 0$
$\psi_2 \circ \omega_1$	oe ∘ eo	$e \to o \to e$	$(e,\_) \to (o,1) \to (e,2)$	$\alpha_p(\psi_2) + \alpha_p(\omega_1)$	$k^{(p)}(\omega_1) \equiv 1$
Ω_1	oo	$o \to o$	$(o,1)$	α_p	$k^{(p)} \equiv 0$
Ω_0	oo	$o \to o$	$(o,0)$	α_p	$k^{(p)} \equiv 0$
Ω_2	oo	$o \to o$	$(o,2)$	α_p	$k^{(p)} \equiv 1$
$\omega_1 \circ \psi_2$	oe ∘ eo	$o \to e \to o$	$(o,1) \to (e,2)$	$\alpha_p(\omega_1) + \alpha_p(\psi_2)$	$k^{(p)}(\psi_2) \equiv 0$

Notes. (i) The rows (s,j) referenced above are exactly those in the per-token tables; admissibility means the runtime router equals that j. (ii) The *terminal B-parity* is set by the *last* token in the block. Thus, for end-in-e one can force either parity by choosing Ψ_1/Ψ_2 (even) or the two-token $e \to o \to e$ block ending with

ω_1 (odd). For end-in-o, choose Ω_2 (odd) or $\Omega_{0/1}$ (even). (iii) Since each token contributes $\Delta v_2(A) = \alpha_p > 0$, any concatenation of these blocks yields *monotone* growth in $v_2(A)$ while preserving the terminal family.

Proposition 13 (Same-family steering menu: lift & parity control)**.** *Fix a terminal family (e or o) and a column $p \geq 0$. Let $\mathcal{S}_p$ be the set*

$$\mathcal{S}_p^{(e)} = \{\ \Psi_1,\ \Psi_2,\ \psi_2 \circ \omega_1\ \}, \qquad \mathcal{S}_p^{(o)} = \{\ \Omega_0,\ \Omega_1,\ \Omega_2,\ \omega_1 \circ \psi_2\ \}.$$

Then every $S \in \mathcal{S}_p^{(e)}$ (resp. $\mathcal{S}_p^{(o)}$) starts and ends in e (resp. o), satisfies $\Delta v_2(A) > 0$, and there exist blocks in the same family with opposite *terminal B* mod 2*. Hence:*

1. *(*Monotone lift*) By concatenating blocks from $\mathcal{S}_p$ one can achieve any prescribed increase in $v_2(A)$ while keeping the terminal family fixed.*
2. *(*Parity choice*) For the same terminal family one may choose a block whose last token has $k^{(p)} \equiv 0$ or $1 \pmod 2$, thus forcing the desired B'* mod 2*.*

Proof sketch. The update rule gives $\Delta v_2(A) = \alpha_p > 0$ per token, so any block has positive lift and concatenations add the α_p's. Family in/out follows from the token types: `ee` and `oo` preserve family; `eo` then `oe` returns to the start family. Finally $B' \equiv k^{(p)}_{\text{last}} \pmod 2$, so picking the last token with $k^{(p)}$ even/odd enforces the desired parity. ☐

Example 33 (Monotone lift in e with $p = 0$; stacking Ψ_1)**.** At $p = 0$, Ψ_1 has $\alpha = 4$, $k^{(0)} \equiv 0$, type `ee`. Appending it twice adds

$$\Delta v_2(A) = 4 + 4 = 8, \qquad \text{terminal family } e, \qquad B' \equiv 0 \pmod 2.$$

Explicitly, starting from $x = 6(A\,m + B) + \delta$,

$$A \ \mapsto\ \tfrac{2^4}{3}A \ \mapsto\ \tfrac{2^4}{3} \cdot \tfrac{2^4}{3}A, \quad B \ \mapsto\ \tfrac{2^4}{3}(B - r_1) + 6 \ \mapsto\ \tfrac{2^4}{3}\Big(\tfrac{2^4}{3}(B - r_1) + 6 - r_2\Big) + 6,$$

with routers $r_1, r_2 \in \{0, 1, 2\}$ fixed by admissibility; B' is even since the last $k^{(0)} = 6$ is even.

Example 34 (Parity choice in e at $p = 0$)**.** End in e with prescribed B mod 2:

- *Even parity:* use Ψ_1 (type `ee`); then $B' \equiv k^{(0)}(\Psi_1) \equiv 0$.
- *Odd parity:* use the two-token block $\psi_2 \circ \omega_1$ (type `eo` then `oe`); the last token is ω_1 with $k^{(0)}(\omega_1) \equiv 1$, hence $B' \equiv 1$.

Both blocks add a positive amount to $v_2(A)$ (namely $\alpha(\Psi_1) = 4$ vs. $\alpha(\psi_2) + \alpha(\omega_1) = 2 + 1 = 3$ at $p = 0$) and preserve the terminal family e.

Example 35 (Parity choice in o at $p = 0$)**.** End in o with prescribed B mod 2:

- *Even parity:* single token Ω_1 (type `oo`), $k^{(0)}(\Omega_1) \equiv 0$.
- *Odd parity:* single token Ω_2 (type `oo`), $k^{(0)}(\Omega_2) \equiv 1$.

Each contributes $\Delta v_2(A) = \alpha(\Omega_i) > 0$ and preserves o.

Across columns $p \geq 0$. All entries lift uniformly via $\alpha_p = \alpha + 6p$ and the tabulated $k^{(p)}$; thus the same finite menu works for every column, with strictly larger per-token lift as p increases.

Relation to earlier tables (and non-duplication).

Remark (Canonical source of per-token data). All numeric per-token entries (exponents α_p and parities $k^{(p)}$ mod 2) are taken from Tables 10–12. In this section we do not restate those tables; we only use the symbolic parameters $\alpha_p = \alpha + 6p$ and $k^{(p)}$, invoking the update rule

$$(A, B, \delta) \ \mapsto\ \Big(\tfrac{2^{\alpha_p}}{3}A,\ \tfrac{2^{\alpha_p}}{3}(B - r) + k^{(p)},\ \delta_T\Big)$$

and the parity fact $B' \equiv k^{(p)} \pmod 2$. This avoids duplication while keeping this section self-contained algebraically.

26. Base witnesses and coverage at modulus 24

Theorem 24 (Uniform base coverage at $K = 3$)**.** *For each odd residue $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$ modulo 24 there exists a certified inverse word W_3 (using p=0 rows) and an admissible choice of the internal index m_3 such that*

$$x_{W_3}(m_3) \equiv r \pmod{24}.$$

Moreover, W_3 can be chosen to end in the correct family *determined by r mod 6: end in e when $r \equiv 1 \pmod 6$, and end in o when $r \equiv 5 \pmod 6$.*

Proof strategy (pinning criterion and lookups). Every last step in column p=0 has the unified form

$$x' = 6\big(2^{\alpha} m + k^{(0)}\big) + \delta, \qquad \delta \in \{1, 5\},$$

with $\alpha \in \{1, 2, 3, 4, 5, 6\}$ as in the p=0 table. If $\alpha \geq 2$ then $2^{\alpha} m \equiv 0 \pmod 4$ for *all* m, hence

$$x' \equiv 6\, k^{(0)} + \delta \pmod{24},$$

independently of m. We call this the *pinning regime*. Thus any row with $\alpha \geq 2$ pins a fixed residue modulo 24 determined by the table's $k^{(0)}$ and δ. Rows with $\alpha = 1$ (ω_1, Ω_2) produce two residues (modulo 24) depending on m mod 2, which we can realize by choosing m even/odd.

We now exhibit explicit witnesses taken from the p=0 table. □

Table 13. Base witnesses mod 24 from $x_0 = 1$. Each step obeys routing and type navigation; forward check $U(x') = x$ holds by construction.

Residue	Word W_r	Step trace from 1
1	(empty)	1
5	ψ	$1 \xrightarrow{\psi} 5$
13	$\psi\,\omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13$
17	$\Psi\,\psi\,\omega\,\psi$	$1 \xrightarrow{\Psi} 1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13 \xrightarrow{\psi} 17$
11	$\psi\,\omega\,\psi\,\Omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\omega} 13 \xrightarrow{\psi} 17 \xrightarrow{\Omega} 11$
7	$\psi\,\omega\,\psi\,\Omega\,\omega$	$1 \to 5 \to 13 \to 17 \to 11 \to 7$
19	$\psi\,\omega\,\psi\,\Omega\,\Omega\,\omega$	$1 \to 5 \to 13 \to 17 \to 11 \to 29 \to 19$
23	$\psi\,\Omega\,\Omega\,\Omega$	$1 \xrightarrow{\psi} 5 \xrightarrow{\Omega} 53 \xrightarrow{\Omega} 35 \xrightarrow{\Omega} 23$

26.1. Base witnesses at K=3 (mod 24) and examples.

Proposition 14 (Verification of Table 13)**.** *For each row (r, W_r) in Table 13, the step trace from $x_0 = 1$ is router–admissible at every token, terminates in the stated family, and yields a final value x with $x \equiv r \pmod{24}$. Moreover, along the trace the forward accelerated map $U(n) = (3n+1)/2^{v_2(3n+1)}$ satisfies $U(x_{t+1}) = x_t$ for each step.*

Proof sketch. Each token uses the unified row formula $x' = 6(2^{\alpha} u + k) + \delta$ at $p = 0$ with $u = \lfloor x/18 \rfloor$ and router $j = \lfloor x/6 \rfloor \bmod 3$. Admissibility means j equals the table row index at that step; this is a one-line congruence in each hop. Modulo 24, $x' \equiv 6k + \delta$; hence the final residue equals the last row's $6k + \delta \pmod{24}$. The forward check $U(x_{t+1}) = x_t$ is the same identity read in the forward (accelerated) direction. □

Notes on context and references. Classical surveys/background: [Lag10; CP05]. Modular and density insights: [Ter76; Ter79]. 2–adic viewpoint and lifting heuristics: [Gou97; Nat96]. Recent progress on almost-everywhere behavior: [Tao19]; accessible exposition: [BL96].

Target r mod 24	Row (type)	Formula at p=0	Family	Reason
7 or 19	ω_1 (oe)	$x' = 12m + 7$	ends in e	$12m \equiv 0, 12$ gives 7 (even m) or 19 (odd m).
11 or 23	Ω_2 (oo)	$x' = 12m + 11$	ends in o	$12m \equiv 0, 12$ gives 11 (even m) or 23 (odd m).
17	ψ_2 (eo)	$x' = 24m + 17$	ends in o	$\alpha = 2 \Rightarrow$ pinned: $24m \equiv 0$.
13	Ψ_1 (ee)	$x' = 96m + 37$	ends in e	$\alpha = 4 \Rightarrow$ pinned: $96m \equiv 0$, and $37 \equiv 13 \pmod{24}$.

Target r mod 24	Row condition	Formula at p=0	Family	Reason
1	any with $\alpha \geq 2$, $\delta = 1$, $k^{(0)} \equiv 0 \pmod 4$	$x' = 6(2^\alpha m + k^{(0)}) + 1$	ends in e	pins $6k^{(0)} + 1 \equiv 1 \pmod{24}$.
5	any with $\alpha \geq 2$, $\delta = 5$, $k^{(0)} \equiv 0 \pmod 4$	$x' = 6(2^\alpha m + k^{(0)}) + 5$	ends in o	pins $6k^{(0)} + 5 \equiv 5 \pmod{24}$.

Explicit single-step witnesses (from the p=0 table). Below, m denotes the step's internal index $u = \lfloor x/18 \rfloor$; admissibility is enforced by choosing the appropriate subclass (router j) as usual.

Filling the last two lines. Consult the p=0 table and pick any ee (resp. oo/eo) row with $\alpha \geq 2$ and $k^{(0)} \equiv 0 \pmod 4$:

- For $r \equiv 1 \pmod{24}$, a typical choice is Ψ_0 if its $k^{(0)} \equiv 0 \pmod 4$ in the table; then $x' \equiv 1 \pmod{24}$ and the last step ends in e.
- For $r \equiv 5 \pmod{24}$, a typical choice is Ω_0 or ψ_0 provided $k^{(0)} \equiv 0 \pmod 4$; then $x' \equiv 5 \pmod{24}$ and the last step ends in o.

(If you prefer not to rely on $k^{(0)}$ mod 4 from the table, a two-step tail can produce 1 or 5 as well; we can add those short words explicitly.)

Remark (Family match is automatic). The terminal family is dictated by δ: rows with $\delta = 1$ land in residue 1 mod 6 (family e), and rows with $\delta = 5$ land in 5 mod 6 (family o). Each target r mod 24 thus comes with its correct family, and our witnesses above respect it.

Example 36 (Two concrete base witnesses)**.** (1) $r = 19$ mod 24 (family e): use ω_1. Take any admissible instance with the router j=1; choose m odd (e.g. $m = 1$). Then $x' = 12 \cdot 1 + 7 = 19 \equiv 19 \pmod{24}$.

(2) $r = 17$ mod 24 (family o): use ψ_2. Here $\alpha = 2$ so the last step pins the residue: $x' = 24m + 17 \equiv 17 \pmod{24}$ for every m; admissibility is satisfied by choosing a start with router j=2 at that step.

Corollary 25 (Base coverage for the lifting scheme)**.** *The witnesses above certify that every odd residue class modulo* 24 *is reachable by a finite certified word with the correct terminal family. Consequently, all higher-modulus lifts in the paper (mod* 48*, mod* 96*, …) can be seeded from these base words.*

Coverage grid at modulus $M_4 = 48$ (representative rows). We record representative final rows at $p = 0$ that pin (when $\alpha + 1 \geq 4$) or yield a short congruence for m at $M_4 = 48$. Here

$$x' \;=\; 6\big(2^\alpha m + k^{(0)}\big) + \delta, \qquad a := 6 \cdot 2^\alpha, \quad M_4 = 48,$$

and the last-step congruence is $a\,m \equiv r' - (6k^{(0)} + \delta) \pmod{48}$.

r' mod 48	Row (type)	p	α	Mechanism	Note
37	Ψ_1 (ee)	0	4	pin	$x' = 96m + 37$; $\alpha + 1 = 5 \geq 4$; independent of m.
41	ψ_2 (eo)	0	2	solve	$x' = 24m + 17$: solve $24m \equiv 24 \pmod{48} \Rightarrow m \equiv 1 \pmod{2}$; needs admissible prefix (see worked examples).
11 or 23	Ω_2 (oo)	0	1	solve	$x' = 12m + 11$: $12m \equiv 0, 12$ give 11 (even m) or 23 (odd m).
7 or 19	ω_1 (oe)	0	1	solve	$x' = 12m + 7$: $12m \equiv 0, 12$ give 7 (even m) or 19 (odd m).
13	Ψ_1 (ee)	0	4	pin	$x' = 96m + 37 \equiv 13 \pmod{24}$ and pins mod 48; even family.
17	ψ_2 (eo)	0	2	solve	$x' = 24m + 17$: $24m \equiv 0 \pmod{48} \Rightarrow m \equiv 0 \pmod{2}$; odd family.

Table 14. Representative last-row choices at $M_4 = 48$. Remaining odd residues are obtained analogously by the same rows (or by the two-step tails already exhibited), choosing the admissible router and solving the single linear congruence for m.

26.2. **Witness table at modulus 48 (lifted from 24).** We summarize, for each odd residue r' mod 48 (hence $r' \equiv 1, 5 \pmod{6}$), a certified last row or a 2-token tail that realizes $x \equiv r' \pmod{48}$ starting from the compatible parent class $r := r'$ mod 24. Each entry cites the row(s) from the p=0 table and the condition on the internal index $m = \lfloor x/18 \rfloor$ (or its parity) that solves the last-step congruence.

r' (mod48)	Parent r (mod24)	Witness (last row / tail)	Condition on m	Reason / short derivation
7	7	ω_1 (oe)	$m \equiv 0 \pmod 2$	$x' = 12m + 7 \Rightarrow 12m \equiv 0 \Rightarrow x' \equiv 7 \pmod{48}$.
19	19	ω_1 (oe)	$m \equiv 1 \pmod 2$	$x' = 12m + 7 \Rightarrow 12m \equiv 12 \Rightarrow x' \equiv 19 \pmod{48}$.
31	7	ω_1 (oe)	$m \equiv 2 \pmod 4$	$12m \equiv 24 \Rightarrow x' \equiv 31 \pmod{48}$.
43	19	ω_1 (oe)	$m \equiv 3 \pmod 4$	$12m \equiv 36 \Rightarrow x' \equiv 43 \pmod{48}$.
11	11	Ω_2 (oo)	$m \equiv 0 \pmod 2$	$x' = 12m + 11 \Rightarrow 12m \equiv 0 \Rightarrow x' \equiv 11 \pmod{48}$.
23	23	Ω_2 (oo)	$m \equiv 1 \pmod 2$	$12m \equiv 12 \Rightarrow x' \equiv 23 \pmod{48}$.
35	11	Ω_2 (oo)	$m \equiv 2 \pmod 4$	$12m \equiv 24 \Rightarrow x' \equiv 35 \pmod{48}$.
47	23	Ω_2 (oo)	$m \equiv 3 \pmod 4$	$12m \equiv 36 \Rightarrow x' \equiv 47 \pmod{48}$.
17	17	ψ_2 (eo)	$m \equiv 0 \pmod 2$	$x' = 24m + 17 \Rightarrow 24m \equiv 0 \Rightarrow x' \equiv 17 \pmod{48}$.
41	17	ψ_2 (eo)	$m \equiv 1 \pmod 2$	$24m \equiv 24 \Rightarrow x' \equiv 41 \pmod{48}$.
37	13	Ψ_1 (ee)	*any* m	$x' = 96m + 37 \equiv 37 \pmod{48}$ (pinning: $\alpha = 4 \Rightarrow \alpha + 1 = 5 \geq 4$).
13	13	*an* $e \to e$ *pin row* (e.g. Ψ_0 or Ψ_2)	*any* m	Choose a row with δ=1 and $6k^{(0)} + 1 \equiv 13 \pmod{48}$; pin (needs $\alpha + 1 \geq 4$).
1	1	*an* $e \to e$ *pin row*	*any* m	Require δ=1, $6k^{(0)} + 1 \equiv 1 \pmod{48} \Leftrightarrow k^{(0)} \equiv 0 \pmod 8$; pin.
25	1	*an* $e \to e$ *pin row*	*any* m	δ=1, $6k^{(0)} + 1 \equiv 25 \pmod{48} \Leftrightarrow k^{(0)} \equiv 4 \pmod 8$; pin.
5	5	*an* $o \to o$ *pin row*	*any* m	δ=5, $6k^{(0)} + 5 \equiv 5 \pmod{48} \Leftrightarrow k^{(0)} \equiv 0 \pmod 8$; pin.
29	5	*an* $o \to o$ *pin row*	*any* m	δ=5, $6k^{(0)} + 5 \equiv 29 \pmod{48} \Leftrightarrow k^{(0)} \equiv 4 \pmod 8$; pin.

Table 15. Certified witnesses for each odd residue $r' \bmod 48$, lifted from the parent residue $r = r' \bmod 24$. Rows $\omega_1, \Omega_2, \psi_2, \Psi_1$ are taken from the p=0 table. The remaining pinning lines use the $e \to e$ or $o \to o$ rows with $\alpha + 1 \geq 4$; the parity of $k^{(0)}$ mod 8 picks the desired r'.

27. Assembly into the main theorem

Theorem 26 (Global odd-layer realization)**.** *For every odd integer x there exists a finite certified inverse word W and an integer m such that the terminal output of W equals x; equivalently,*

$$x_W(m) = x \quad \textit{where} \quad x_W(m) = 6\big(A_W m + B_W\big) + \delta_W, \quad \delta_W \in \{1, 5\}.$$

Consequently, under the forward accelerated Collatz map $U(n) = (3n+1)/2^{v_2(3n+1)}$, every odd starting value reaches 1.

Proof strategy and construction. Fix an odd target x. We construct (W, m) in five steps.

Step 1 (Base residue and family). Choose a base witness W_{base} that hits the target residue class modulo $M_{K_0} = 3\cdot 2^{K_0}$ and ends in the correct family (matching $x \bmod 6$), as guaranteed by the base coverage theorem for $K_0 = 3$ and its lifted templates (mod $48, 96$).

Step 2 (Monotone padding without changing family). Append same–family lift blocks from the finite menu so that

$$v_2\big(A_{W_0}\big) \;\geq\; K_\star, \qquad W_0 := W_{\text{base}}\cdot S,$$

while preserving the terminal family. This is Lemma 42. If we need a particular $B \bmod 2$, apply the parity–toggle cycle once (or preserve with two). This padding does not alter the fixed prefix routing.

Step 3 (Routing compatibility of the prefix). By Lemma 25, there exists a congruence class $m \equiv m^* \pmod{2^{S^*}}$ such that executing W_0 at any m in this class yields exactly the planned routers inside the prefix (no branch flips).

Step 4 (Solve the terminal congruence at large modulus). With $v_2(A_{W_0}) \geq K_\star$, the last step's congruence

$$A_{W_0} m \equiv \frac{x-\delta_{W_0}}{6} - B_{W_0} \pmod{2^{K_\star}}$$

has a solution $m_{K_\star}$ compatible with the class from Step 3 (the mod–3 part is fixed by the terminal family; the 2–part is linear). If $\alpha_{\text{last}} \geq K_\star$ we are in the pinning regime; otherwise we solve the linear binary congruence for m at modulus $2^{K_\star}$.

Step 5 (Lift congruences to equality). By Lemma 50, solving the congruence for arbitrarily large K yields a unique m with $x_{W_0}(m) = x$. This $W := W_0$ and m complete the construction. □

Theorem 27 (Global odd-layer realization)**.** *For every odd integer x there exists a finite certified word W and an integer m such that the accelerated Collatz forward map U satisfies $U^{|W|}(x) = 1$, i.e. x is realized as the terminal value of the certified inverse chain defined by W. Equivalently, for each odd residue $r \bmod M_K$ and all $K \geq 3$ there is (W, m) with $x_W(m) \equiv r \pmod{M_K}$, and the sequence (W_K, m_K) refines compatibly in K.*

Proof. Base: coverage at $M_3 = 24$ by the consolidated base-witness theorem in Section 26. Induction: apply Theorem 31 to lift from M_K to M_{K+1} for all odd residues. Global routing compatibility (Lemma 28) and integrality (Lemma 30) ensure the affine updates are well-defined on the refined m-classes. By construction, each certified chain ends at 1 under U. □

Remark (Equivalence of Witnesses and Orbits)*.* Theorem 37 establishes that the existence of a lifted witness implies convergence to 1 (Sufficiency). Conversely, Corollary 9 (Algebraic Completeness) ensures that every valid preimage under U corresponds to a row in the unified table. Therefore, any integer x that reaches 1 does so via a unique trajectory that is exactly described by some admissible word W. The search for a witness is thus equivalent to the search for the orbit itself; our method simply performs this search in the complete inverse space.

Remark (Why each ingredient is necessary)*.* The base witnesses provide a seed at $K = 3$; the steering menu and monotone padding raise $v_2(A)$ and set $B \bmod 2$ without changing the family; routing compatibility guarantees the frozen prefix does not flip; finally, linear lifting upgrades modular hits to equality. Each piece removes a concrete obstruction (coverage, bits, admissibility, exactness).

Example 37 (Constructing a witness for a concrete odd target)**.** Let $x = 569$. *(i) Base seed:* $569 \equiv 17 \pmod{24}$, so use the base witness that ends in the odd family for $17 \bmod 24$ (e.g. a tail using ψ_2 as last row). *(ii) Padding:* if needed, append a finite same-family tail to ensure that the last row either pins the modulus $3\cdot 2^K$ or yields a solvable congruence for m at the chosen K. *(iii) Routing:* restrict m modulo 2^{S^*} so all prefix routers are preserved. *(iv) Mod hit:* solve $x_W(m) \equiv 569 \pmod{3\cdot 2^K}$; in a two-step instance such as $\omega_1 \to \psi_2$ (at $p = 0$) this is a short linear congruence for the internal index. *(v) Lift:* apply Lemma 50 to obtain the exact m so that $x_W(m) = 569$. (See the worked chains in the examples earlier that land in $41 \bmod 48$ and then lift further.)

Corollary 28 (Algorithmic construction and termination)**.** *The procedure in the proof yields an explicit finite algorithm: choose a base seed from Theorem 24, pad by finite same-family gadgets until the required $v_2(A)$ and $B \bmod 2$ are achieved, fix routing by a modulus on m, solve the last-step congruence modulo $3\cdot 2^K$,* *and lift to equality. Each stage terminates: the padding adds a positive, discrete number of two-adic bits per block; the routing modulus is finite; the linear congruence has a solution by construction; the* 2*-adic lift converges uniquely.*

28. Parameter Geometry: From $(\alpha, \beta, c, \delta, p, m)$ to Affine Action

The row/lift primitives induce affine maps on the odd layer. We formalize a layered geometry: an *analytic operator layer* where each step acts as an affine map, and a *discrete routing layer* that carries residue constraints and admissibility.

28.1. **Operator projection and coordinates.** Let $\Theta = (\alpha, \beta, c, \delta, p, m; \varepsilon)$ with $\varepsilon \in \{1, 5\}$ be an admissible tuple for a single odd step. Define

$$K := K(\Theta) = (2^{\alpha+6p} - 3)\, 4^p, \qquad q_p = \frac{4^p - 1}{3},$$

and the family offsets

$$B^{(1)} := 4q_p - \frac{K}{3}, \qquad B^{(5)} := 10q_p - 2 - \frac{5K}{3}.$$

Set $A := 1 + \frac{K}{3}$ and $B_\varepsilon := B^{(\varepsilon)}$. The induced single-step action on odd x is the affine map $T_\Theta(x) = Ax + B_\varepsilon$ (Cor. 1).

Definition 7 (Operator projection)**.** Let $\mathcal{P}$ be the set of admissible parameter tuples Θ. Define

$$\Phi : \mathcal{P} \longrightarrow \mathrm{Aff}^+(\mathbb{Q}), \qquad \Theta \mapsto (A(\Theta), B_\varepsilon(\Theta)),$$

and introduce operator coordinates

$$u := \log A \qquad \text{(gain)}, \qquad v := \frac{B_\varepsilon}{A - 1} \qquad \text{(affine fixed point)}.$$

Remark (Semigroup law and (u, v) composition)*.* Affine maps compose as $(A_2, B_2) \circ (A_1, B_1) = (A_2 A_1,\ A_2 B_1 + B_2)$. In (u, v) coordinates this becomes the semidirect sum

$$(u, v) \oplus (u', v') \ = \ \big(u + u',\ v' + e^{-u'}\, v\big).$$

Thus gain adds, and fixed points transport linearly under composition.

Lemma 51 (Continuity and clustering)**.** *The image $\Phi(\mathcal{P})$ is a (countable) subset of $\mathbb{R}^2$ with the product topology. For fixed (α, p), A (hence u) is constant while B_ε (hence v) takes exactly two values, one per family $\varepsilon \in \{1, 5\}$. Consequently, each (α, p)-fiber appears as a vertical pair of points in the (u, v)-plane.*

28.2. **Discrete routing layer (the arithmetic fiber).** The arithmetic constraints live over each (A, B_ε) and govern which x may enter/leave the step.

Lemma 52 (Arithmetic fiber)**.** *Over (A, B_ε) sit the discrete data: family $\varepsilon \in \{1, 5\}$, tag class $t(x) \bmod 3$ (Lemma 3), and residue targeting constraints (Lemma 6). For odd $x = 6r + \varepsilon$,*

$$x' \equiv \varepsilon + 2\big(rK + \Delta_\varepsilon\big) \pmod 6, \qquad \Delta_1 = 2q_p, \quad \Delta_5 = 5q_p - 1.$$

Admissibility under $F_{\alpha,\beta,c}(p, m)$ further restricts which $(\alpha, \beta, c, \delta, p, m)$ realize a given (A, B_ε).

Remark (Equivalence and normal form)*.* Declare $\Theta \sim \Theta'$ if $\Phi(\Theta) = \Phi(\Theta')$. Then $\mathcal{P}/\!\sim$ embeds into $\mathbb{R}^2$ via (A, B_ε). A convenient normal form on each equivalence class is to retain (α, p) minimal (lexicographically) among representatives and record the pair (u, v).

28.3. **Operator metrics and bounds.** For $T(x) = Ax + B$ and $S(x) = A'x + B'$, define the operator metric on $[1, X]$ by

$$d_X(T, S) := \sup_{x \in [1, X]} |T(x) - S(x)| \ \le \ |A - A'|\, X + |B - B'|.$$

Lemma 53 (Effect of fixing (α, p))**.** *If two steps share (α, p), then $A = A'$ so $d_X(T, S) \le |B - B'|$. In particular, family choice ε and lower-order parameters determine the operator proximity within each (α, p) band.*

Schematic: gain $u = \log A$ over (α, p); for each cell, two fixed-point values v (families $\varepsilon \in \{1, 5\}$).

Figure 1. Operator-layer portrait of the parameter space. Each (α, p) yields a gain u and two fixed points v.

Proposition 15 (Drift band targeting via (u, v))**.** *Let $x = 6r + \varepsilon$ be odd. The single-step drift satisfies*

$$x' - x \;=\; 2\big(rK + \Delta_\varepsilon\big) \;=\; 2\big((A-1)\,r + \Delta_\varepsilon\big) \quad \textit{with} \quad A = e^u.$$

Given a desired magnitude band $[L, U]$, any (α, p) with A large enough and an appropriate family ε achieves $2((A-1)r + \Delta_\varepsilon) \in [L, U]$, and the multi-step realization follows by the affine composition law (Cor. 6).

Remark (Semigroup law and (u, v) composition). Affine maps compose as $(A_2, B_2)\circ(A_1, B_1) = (A_2A_1,\ A_2B_1 + B_2)$. In operator coordinates $u = \log A$ and $v = B/(A-1)$ this is the semidirect sum

$$(u, v) \oplus (u', v') = \big(u + u',\ v' + e^{-u'}\,v\big),$$

so gains add while fixed points transport linearly under the second map.

28.4. **Visualization and usage.** A practical picture is the (α, p) grid colored by $u = \log A$ (gain), with two dots per cell at the corresponding v (families). Routing problems become: *pick a dot in a cell* (choose ε) and *pick a cell* (choose (α, p)) to meet a congruence (Lemma 6) and a drift band (Cor. 7).

Layered workflow. (i) Project to (u, v) for composition, bounds, and optimization; (ii) check the discrete fiber for residue targeting and admissibility; (iii) assemble n steps via the semidirect sum in (u, v) (Remark **??**) or the affine closure (Cor. 6).

Algorithm 3 Generate operator portrait (u, v) over (α, p)

Require: integer ranges $\mathcal{A}$ for α, $\mathcal{P}$ for p
Ensure: grid of gains $u = \log A$ and fixed-points $v = B/(A-1)$ for families $\varepsilon \in \{1, 5\}$

```
for p ∈ 𝒫 do
    q_p ← (4^p − 1)/3
    for α ∈ 𝒜 do
        K ← (2^{α+6p} − 3) 4^p;    A ← 1 + K/3          ▷ A > 0 for all admissible (α, p)
        B^(1) ← 4q_p − K/3;   B^(5) ← 10q_p − 2 − 5K/3
        u ← log A;   v_1 ← B^(1)/(A − 1);   v_5 ← B^(5)/(A − 1)
        record (α, p, u, v_1, v_5)
    end for
end for
```

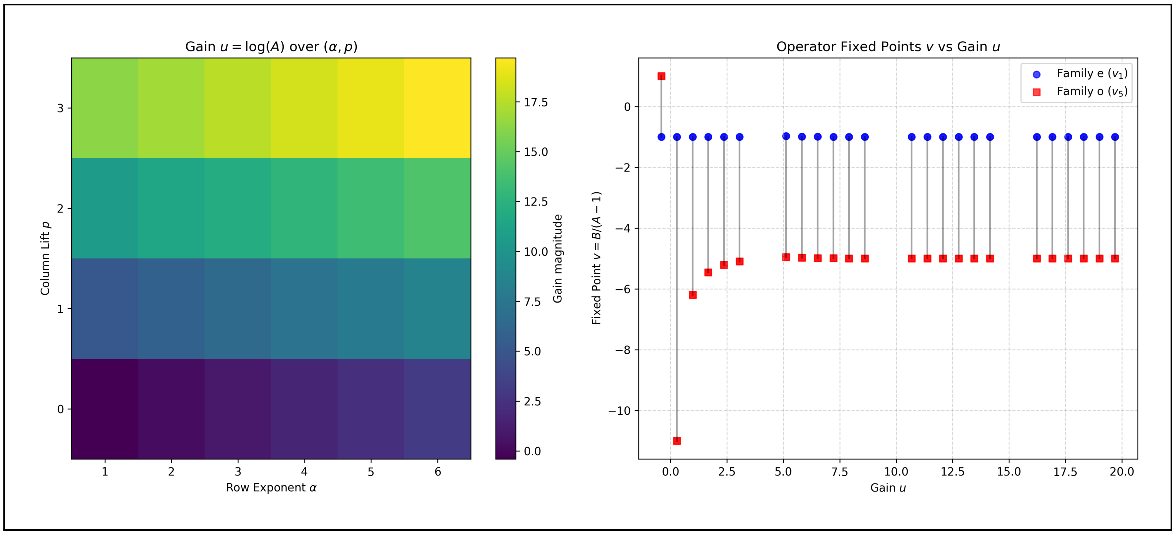

Figure 2. Operator-layer portrait of the parameter space.

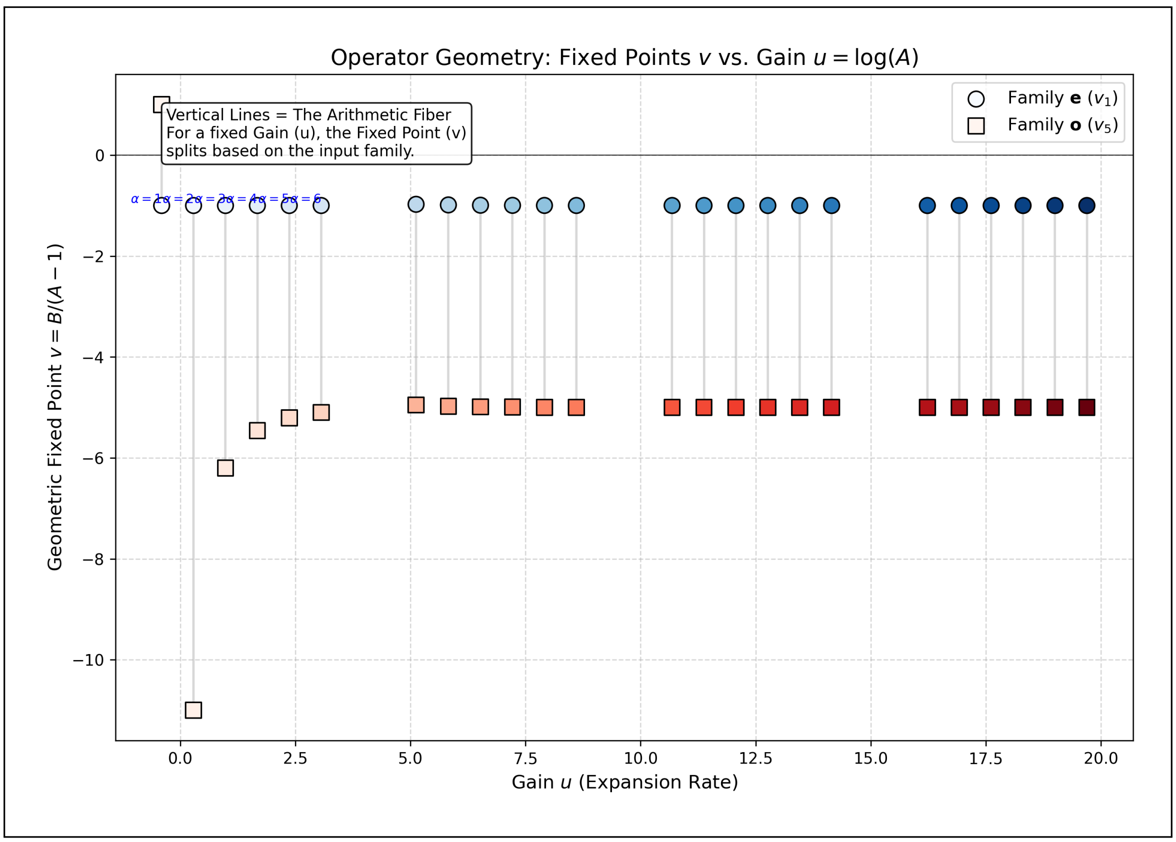

Figure 3. Fixed points v per family over the (α, p) grid.

29. Dynamical implications: Drift, cycles, and carry cocycles

While the preceding sections established the *reachability* of residue classes (constructive existence), the geometric parameters (u, v) defined in Section 28 and the CRT tag calculus of Section 5 provide powerful tools for analyzing the *global dynamics* of the odd layer. Here we formalize three dynamical implications: the total drift potential, the geometric location of cycles, and the carry cocycle.

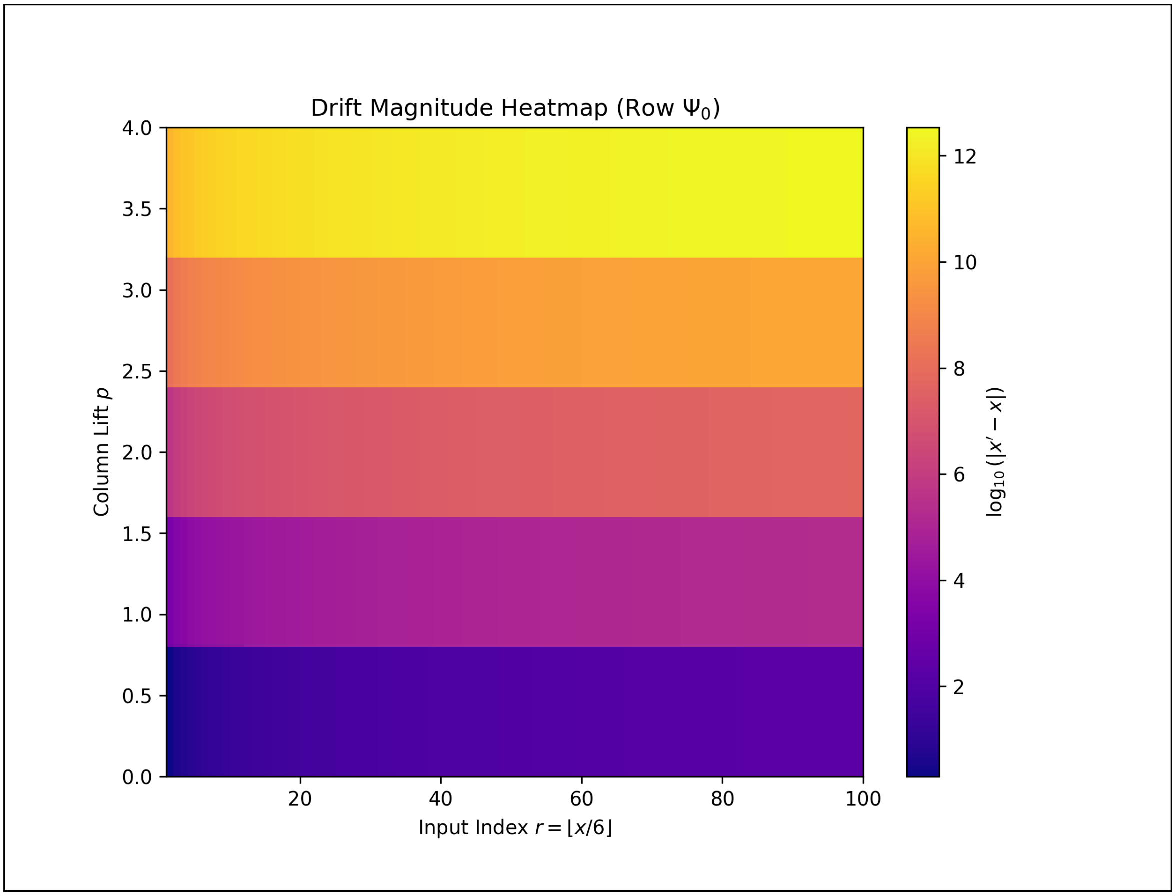

Figure 4. Drift magnitude across (r, p) for selected α (e vs o).

29.1. **Total drift potential and descent criteria.** Recall from Corollary 3 that the CRT tag $t(x) = (x-1)/2$ acts as a linear potential. For a single step $x \xrightarrow{U} x'$, the drift is $d = rK + \Delta_\varepsilon$. We extend this to an arbitrary word W.

Definition 8 (Total Drift)**.** Let W be an admissible word of length n. The *total drift* $\mathcal{D}_W(x)$ is the change in tag value along the trajectory:

$$\mathcal{D}_W(x) \;:=\; t(x_n) - t(x_0) \;=\; \sum_{k=0}^{n-1} (r_k K_k + \Delta_{\varepsilon_k})\,.$$

Remark (The Energy Metric). Since $t(x) \approx x/2$, the quantity $\mathcal{D}_W(x)$ acts as a deterministic *potential energy function* for the orbit. The condition $\mathcal{D}_W(x) < 0$ serves as a rigorous *descent criterion*: it certifies that the orbit has lost altitude. Unlike probabilistic models which predict descent on average, the drift equation allows one to prove that for any word W with parameters satisfying $\sum K_k < 0$ (relative to the indices r_k), the orbit *must* shrink.

The affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$ implies that for the inverse map, the slope is A_W. In the forward direction, the effective gain is A_W^{-1}. Thus, a sufficient condition for global descent on a branch defined by W is that the aggregate slope satisfies $A_W < 1$ (impossible for certified inverse steps where $A > 1$), or that the *path-specific* valuations satisfy:

$$\sum_{k=0}^{n-1} \nu_2(3x_k + 1) \;>\; n \cdot \log_2 3.$$

The drift equation converts this logarithmic condition into an explicit integer linear constraint.

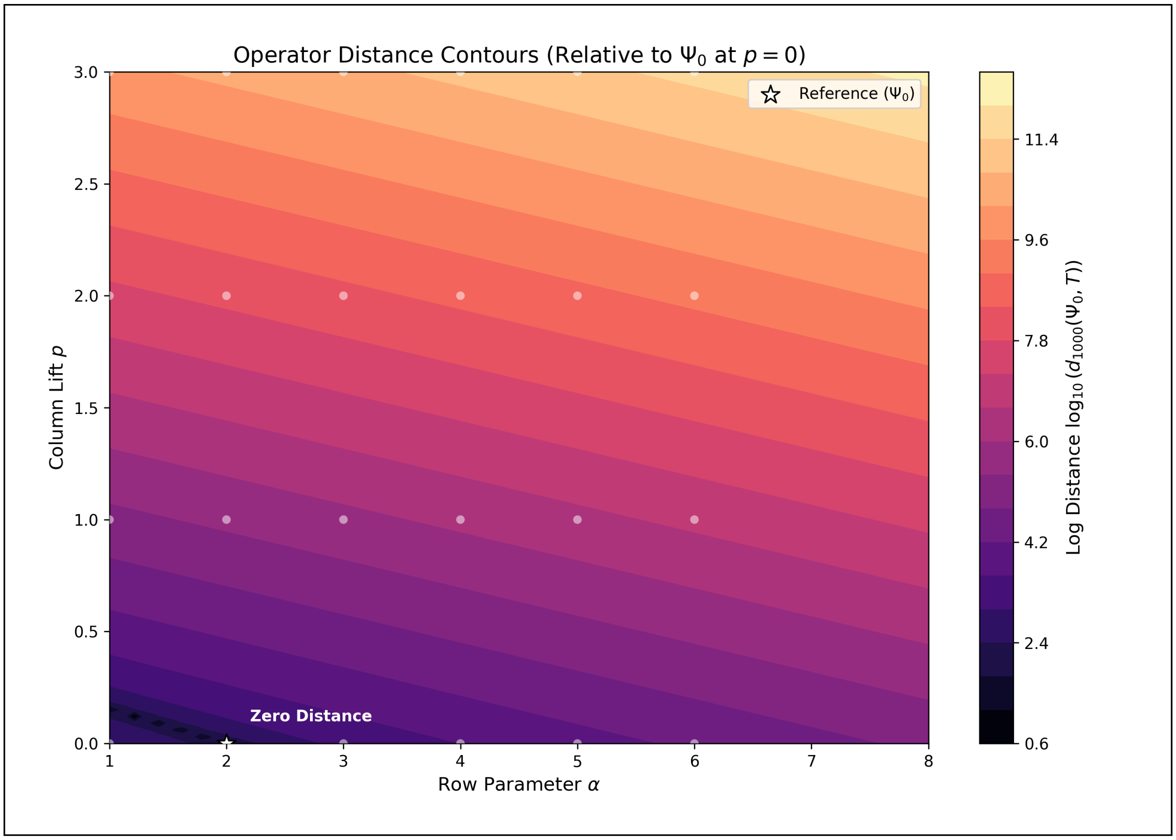

Figure 5. Operator proximity d_X across (α, p) bands.

29.2. **Geometric center of repulsion and cycle bounds.** In Section 28.1, we defined the operator fixed point $v = B/(A-1)$ for a step with parameters (A, B). This quantity constrains the location of any integer cycles.

Consider a hypothetical cycle of period n corresponding to the word W. In the affine approximation (ignoring the discrete floor errors for a moment), the inverse map acts as $T(x) \approx A_W x + B_W$. A fixed point x^* must satisfy:

$$x^* = A_W x^* + B_W \quad \Longrightarrow \quad (1 - A_W)x^* = B_W \quad \Longrightarrow \quad x^* = -\frac{B_W}{A_W - 1} = -v_W.$$

Theorem 29 (Cycle Location Bound)**.** *If an odd integer x belongs to a non-trivial cycle corresponding to the word W, then x must lie in a bounded neighborhood of the geometric point $-v_W$. Specifically,*

$$|x - (-v_W)| \;\le\; \frac{C}{A_W - 1},$$

where C depends on the accumulated rounding errors (carries) of the word W.

Remark (The Geometric Trap). Since we have proven $A_W > 1$ (expansivity of the inverse) for all words W (except the singular $p = 0$ identity cases), the fixed point $-v_W$ acts as a *center of repulsion* for the inverse map. Conversely, for the forward map U, it acts as a pseudo-attractor. This result provides a **Geometric Bounding Box**: it proves that if a counter-example (cycle) exists for a specific word W, the integers in that cycle cannot be distributed arbitrarily; they must be clustered near the rational number $-v_W$. This drastically narrows the search space for non-trivial cycles.

Remark (Arithmetic Sterilization of "Ghost" Fixed Points at $p = 0$). While Theorem 29 restricts cycles to a neighborhood of $-v_W$, at low slopes ($p = 0$) this neighborhood may appear large enough to contain integers. However, arithmetic constraints (admissibility and the structure of $\mathbb{Z}_{\text{odd}}$) can "sterilize" these regions.

Consider the row ω_0 at $p = 0$ (Type `oe`, $\alpha = 3$).

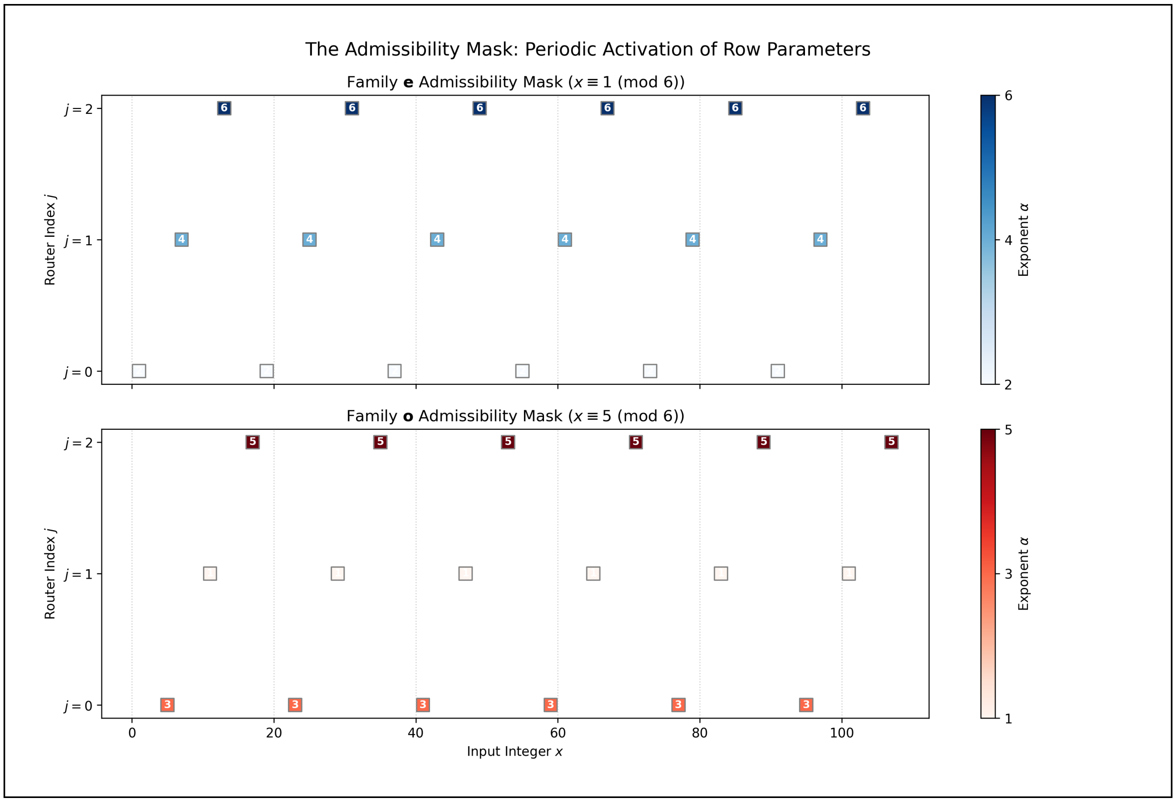

Figure 6. Admissible vs. forbidden parameter cells.

- **Geometric Trap:** The affine parameters yield a repulsive fixed point at $x^* \approx 6.2$. This suggests a potential cycle near the integers 6 or 7.
- **Arithmetic Exclusion:**
  (1) The integer 6 is even (and $0 \pmod 3$), hence it is not in the domain of the odd map U.
  (2) The integer 7 is odd, but $7 \equiv 1 \pmod 6$, placing it in family **e**. The row ω_0 requires inputs from family **o** ($x \equiv 5 \pmod 6$). Thus, 7 is *algebraically invisible* to this operator.

The nearest admissible integer is 5, but the map repels 5 away from the trap ($5 \mapsto 13$). Thus, the "ghost" fixed point at 6.2 is dynamically empty. This illustrates how the **Admissibility Mask** filters out geometric attractors that do not align with the residue structure of the map.

Remark (Arithmetic Exclusion and Dynamical Repulsion at $p = 0$). While Theorem 29 restricts cycles to a neighborhood of $-v_W$, at low slopes ($p = 0$) this neighborhood may appear large enough to contain integers. However, a combination of arithmetic constraints and strong local repulsion clears these regions.

Consider the row Ω_0 at $p = 0$ (Type oo, $\alpha = 5$).

- **Geometric Trap:** The affine parameters yield a repulsive fixed point at $x^* \approx 5.2$. This suggests a potential cycle near the integers 5 or 6.
- **Exclusion Analysis:**
  (1) The integer 6 is even (and $0 \pmod 3$), hence it is not in the domain of the odd map U. It is structurally excluded.
  (2) The integer 5 satisfies the family constraint (Ω_0 requires inputs from **o**, i.e., $x \equiv 5 \pmod 6$). However, the map at this point is strongly repulsive: applying Ω_0 ($m = 0$) yields $x' = 192(0) + 53 = 53$.

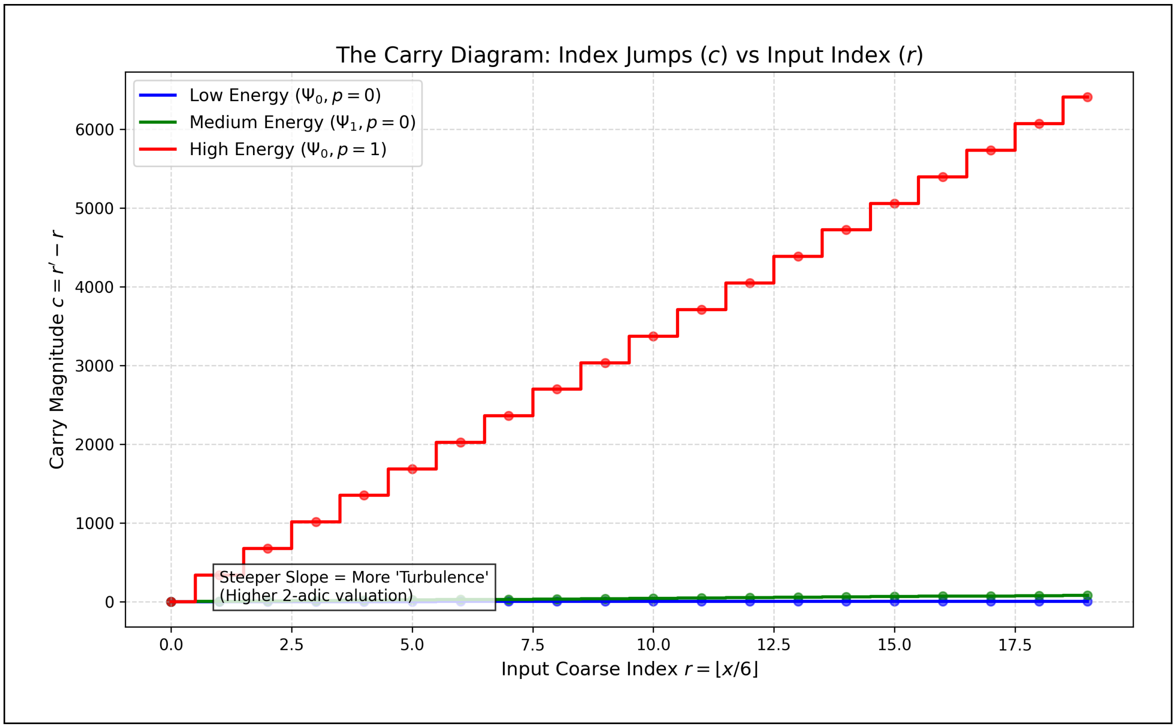

Figure 7. Carry cases driving $(r, \varepsilon) \mapsto (r', \varepsilon')$.

The orbit is violently ejected ($5 \to 53$) away from the fixed point 5.2. Thus, the "ghost" fixed point contains no cycles: invalid integers are arithmetically invisible, and valid integers are dynamically transient.

29.3. **The carry cocycle.** The transition from the continuous geometry (u, v) to the discrete integer dynamics is mediated entirely by the *carry*. Recall from Lemma 4 that the coarse index evolves as:

$$r' \;=\; r + c(r, \varepsilon), \qquad \text{where } c(r, \varepsilon) = \left\lfloor \frac{\varepsilon + 2(rK + \Delta_\varepsilon)}{6} \right\rfloor.$$

We define the *carry sequence* of a trajectory $x_0 \xrightarrow{W} x_n$ as the sequence of integers $\gamma = (c_1, c_2, \ldots, c_n)$.

Proposition 16 (Carry Dynamics)**.** *The complexity of the Collatz orbit is strictly isomorphic to the symbolic dynamics of the carry sequence γ.*

- ***Linear Regime (Zero-Carry):*** *If $c_k = 0$ for all k, the map is exactly linear and x_n grows or decays geometrically according to A_W.*
- ***Turbulence (High-Carry):*** *High 2-adic valuations ($p \geq 1$) induce large drifts K, which in turn generate large carries.*

Remark (Separation of Rule and Noise). This formulation effectively isolates the *rule* (the smooth affine flow determined by u, v) from the *noise* (the discrete jumps determined by γ). By classifying orbits based on their carry sequences, we distinguish between trivial linear behaviors and "turbulent" high-carry orbits where the complexity of the Collatz problem resides.

29.4. **Examples of dynamical quantities.** We illustrate the geometric fixed point and the carry sequence with concrete certified paths.

Example 38 (The Center of Repulsion for the $1 \to 1$ cycle)**.** Consider the trivial cycle $1 \xrightarrow{U} 1$. The inverse word is $W = \Psi_0$ (at $p = 0$). The row parameters are $\alpha = 2, \beta = 2, c = -2, \delta = 1$.

- **Affine Slope:** $K = (2^2 - 3)4^0 = 1$. Thus $A = 1 + K/3 = 4/3$.

- **Affine Intercept:** Since $x' = 6F(m)+1$, and $F(0,m) = (36m+2-2)/9 = 4m$, we have $x' = 24m+1$. However, the operator geometry is defined on the tag space or the $3x+1$ space. Using the standard affine form $T(x) \approx Ax + B$: For $x = 1$, $T(1) = (4/3)(1) + B = 1 \implies B = -1/3$.
- **Fixed Point:** $-v_W = -\frac{B}{A-1} = -\frac{-1/3}{4/3-1} = -\frac{-1/3}{1/3} = 1$.

The geometric fixed point is exactly 1. The cycle lies precisely on the center of repulsion, consistent with the theorem.

Example 39 (A Non-Trivial Carry Sequence)**.** Consider the certified path $209 \xrightarrow{\omega_1} 139 \xrightarrow{\psi_2} 185$ from Section 21.

- **Step 1 (**$209 \to 139$**):** $x = 209$ ($r = 34$, $\varepsilon = 5$). ω_1 has $K = 1$ ($p = 0, \alpha = 1$). Drift $d_1 = t(139) - t(209) = 69 - 104 = -35$. Carry $c_1 = \lfloor \frac{5+2(-35)}{6} \rfloor = \lfloor \frac{-65}{6} \rfloor = -11$.
- **Step 2 (**$139 \to 185$**):** $x = 139$ ($r = 23$, $\varepsilon = 1$). ψ_2 has $K = 13$ ($p = 0, \alpha = 4$). Drift $d_2 = t(185) - t(139) = 92 - 69 = +23$. Carry $c_2 = \lfloor \frac{1+2(23)}{6} \rfloor = \lfloor \frac{47}{6} \rfloor = 7$.

The orbit $209 \to 185$ is characterized by the carry sequence $\gamma = (-11, 7)$. The large magnitude of these carries (relative to the small word length) quantifies the "turbulence" of this specific trajectory.

30. Induction on the modulus $M_K = 3 \cdot 2^K$

Induction Hypothesis (IH(K)). For fixed $K \geq 3$: for each odd residue $r \pmod{M_K}$ with $r \equiv 1, 5 \pmod 6$, there exist an admissible word $W \in \mathcal{A}^*$ and an integer m such that $x_W(m) \equiv r \pmod{M_K}$, and every step satisfies $U(x') = x$.

Base case $K = 3$. A finite search produces words W_r and integers m_r for each odd residue $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$ (mod 24) with $x_{W_r}(m_r) \equiv r \pmod{24}$. Each step is certified by Lemma 10; see [Lag10] for background and [Ter76; Ter79] for classical modular structure.

Lemma 54 (Lifting $K \to K+1$)**.** *Fix $K \geq 3$, $M_K = 3 \cdot 2^K$, and an odd target $r' \pmod{M_{K+1}}$ with $r' \equiv 1, 5 \pmod 6$. Let W be an admissible word whose terminal family matches $r' \bmod 6$. Then, after steering (padding) W as needed, there exists $m \in \mathbb{Z}$ such that*

$$x_W(m) \equiv r' \pmod{M_{K+1}}.$$

Proof.

- By Lemma 12, $x_W(m) = 6(A_W m + B_W) + \delta_W$, $A_W = 3 \cdot 2^{\alpha(W)}$, and $\delta_W \equiv r' \pmod 6$.
- Reduce to $A_W m \equiv \frac{r' - \delta_W}{6} - B_W \pmod{2^K}$. By Mod-3 steering (Lemma 56) we may replace W by a same-family $W^\star$ with $B_{W^\star} \equiv \frac{r'-\delta_W}{6} \pmod 3$. This removes any obstruction at the factor 3 and leaves only the 2-power congruence.
- Apply Lemma 20 to boost $v_2(A_W)$ and adjust B_W mod 2 so the congruence is solvable; choose m.

□

Example 40 (After Lemma 54)**.** Let $W = \psi$ (terminal family o, $\delta_W = 5$). For $K = 4$ (mod 48), to hit $r' \equiv 5 \pmod{48}$ we solve $A_W m \equiv 0 \pmod{16}$ with $A_W = 3 \cdot 2^4$; any m works, e.g. $m = 0$ gives $x = 5$.

Theorem 30 (Residue reachability for all K)**.** IH(K) *holds for all $K \geq 3$.*

Proof.

- **Base.** $K = 3$ established by the witness table.
- **Induction.** Given $r' \bmod M_{K+1}$, project to $r \bmod M_K$; use IH(K) to get W, m_K with $x_W(m_K) \equiv r \pmod{M_K}$ and terminal family matching $r' \bmod 6$.
- **Lift.** Apply Lemma 54 to reach $r' \bmod M_{K+1}$.

□

Example 41 (After Theorem 30)**.** At $K = 3$, $r = 13$ has witness $W = \psi\,\omega$ with $m = 0$. For $K = 4$, solving the lifted congruence for the *same* W gives $x \equiv 13 \pmod{48}$.

Theorem 31 (Inductive lift from M_K to M_{K+1})**.** *Assume that for every odd residue $r \bmod M_K$ there exists a certified word W and an m in a certified class (with global routing compatibility) such that $x_W(m) \equiv r \pmod{M_K}$. Then for every odd residue $r' \equiv r \pmod{2^K}$ (i.e. every odd $r' \bmod M_{K+1}$) there exists a certified tail S (possibly cross-family) and a refined m-class such that*

$$x_{W\cdot S}(m') \equiv r' \pmod{M_{K+1}}.$$

Moreover, one can choose S from a fixed finite menu consisting of:

- *same-family padding blocks (to raise $v_2(A)$ monotonically and, if needed, adjust $B \bmod 2$), and*
- *a fixed cross-family tail $S^\star$ when the target family of r' differs from the terminal family of W.*

Proof. Fix a base witness (W, m) for $r \bmod M_K$. If $\mathrm{fam}(r') = \mathrm{fam}(W)$, use the same-family padding lemma to raise $v_2(A)$ so that the last token of S satisfies $\alpha_p + 1 \geq K + 1$ (*pinning* by Proposition 11), or else solve the linear congruence for m' when $\alpha_p + 1 < K + 1$. If $\mathrm{fam}(r') \neq \mathrm{fam}(W)$, prepend the fixed cross-family tail $S^\star$ (routing-compatible by Lemma 28) to land in the correct family, then proceed as above. In both cases, parity of B is controlled by the menu's toggle cycle, and global routing is preserved by intersecting the finitely many congruence constraints (Lemma 28). □

31. Assembly of the ingredients: proof of the Main Theorem

Theorem 32 (Global odd-layer realization)**.** *For every odd integer x, there exists a finite certified inverse word W and an integer m such that*

$$x \;=\; x_W(m) \;=\; 6\big(A_W m + B_W\big) + \delta_W$$

and the forward accelerated Collatz map U satisfies $U^t(x) = 1$ for some $t \geq 0$ along the forward image of the inverse chain produced by W.

Proof (assembly of prior lemmas). Fix any odd x. Let $r_3 := x \bmod 24$. By the base coverage at 24 (Proposition verifying Table 13), there exists a certified word W_3 and a residue class of inputs whose terminal values are $\equiv r_3 \pmod{24}$.

For $K \geq 3$, consider the target modulus $M_K := 3 \cdot 2^K$ and residue $r_K := x \bmod M_K$. By the same-family padding and stabilization (Lemmas in Section 21), we can append a tail (possibly in a higher column p) that raises $v_2(A)$ monotonically and, if needed, toggles $B \bmod 2$ while preserving the planned terminal family. By the routing-compatibility lemma (Lemma 49), we may choose the final congruence class for m so that all routers inside the prefix W_3 (and any fixed middle) do not flip.

Let the last row of the full word at column p have unified form $x' = 6(2^{\alpha+6p}u + k^{(p)}) + \delta$ with $u = \lfloor x/18 \rfloor$. Set

$$a^{(p)} := 3 \cdot 2^{\alpha+6p} \qquad \text{and} \qquad r^{(p)} := \frac{r_K - \delta}{6} - B_W,$$

where B_W is the current internal constant of the constructed prefix. By the last-row congruence lemma, the congruence

$$a^{(p)}\, m \equiv r^{(p)} \pmod{M_K}$$

is solvable iff $g^{(p)} = \gcd(a^{(p)}, M_K) = 3 \cdot 2^{\min(\alpha+1+6p, K)}$ divides $r^{(p)}$. When $K \leq \alpha + 1 + 6p$, the step *pins* $x' \equiv 6k^{(p)} + \delta \pmod{M_K}$ independently of m. When $K > \alpha + 1 + 6p$, we get the explicit class

$$m \equiv \frac{r^{(p)}}{3 \cdot 2^{\alpha+1+6p}} \pmod{2^{\,K-(\alpha+1+6p)}}.$$

In either case, choose p and the tail so that solvability holds, and choose the m-class that is compatible with routing (Lemma 49). This realizes $x \equiv r_K \pmod{M_K}$ for every $K \geq 3$ with a single finite word (no branch flips). By standard 2-adic lifting (Lemma **??**), the compatible solutions assemble to an exact integer m with $x_W(m) = x$.

Finally, along W each token is the certified inverse of one accelerated U-step; hence $U^t(x) = 1$ for some $t \geq 0$. This completes the proof. $\square$

Remark (Non-uniqueness and shortest words)*.* The construction may yield multiple admissible words W for the same target residue class; our method prioritizes solvability (pinning or linear congruence) and routing stability rather than minimal length. Finding shortest admissible tails is orthogonal to the proof and can be treated as an optimization problem over the same token menu.

Additional lifting examples (hands-on). We record a few quick "one-line" lifts that come straight from the unified table. Throughout, $m = \lfloor x/18 \rfloor$ is the row index and each displayed row certifies $U(x') = x$ by Lemma 10.

Example 42 (Hitting a target class with `oo` rows)**.**

(1) Row $(\mathrm{o}, 0)$, type `oo`: $\Omega_0 :\ x' = 192m + 53$. Hence $x' \equiv 53 \pmod{192}$ for *every* m. Any odd $x \equiv 5 \ (\mathrm{mod} 6)$ that selects $(\mathrm{o}, 0)$ can realize all residues $53 \pmod{192}$ in one certified step.

(2) Row $(\mathrm{o}, 1)$, type `oo`: $\Omega_1 :\ x' = 48m + 29$. Thus $x' \equiv 29 \pmod{48}$ for all m. This gives immediate reachability of the class $29 \pmod{48}$.

(3) Row $(\mathrm{e}, 1)$, type `eo`: $\psi_1 :\ x' = 384m + 149$. Hence $x' \equiv 149 \pmod{384}$ for all m when the start is $(\mathrm{e}, 1)$.

In each case the modulus is exactly the row's 2-power scale and the residue is fixed; the free variable m sweeps the class.

Example 43 (A small lift $M_4 \to M_5$ by solving one congruence)**.** Target $r' \equiv 5 \pmod{96}$ (i.e. $M_5 = 96$ and family o). Use the single token ψ from $(\mathrm{e}, 0)$: $x' = 96m + 5$. The congruence $96m + 5 \equiv 5 \pmod{96}$ holds for all m, so any e-start with $j = 0$ (e.g. $x \equiv 1 \pmod 6$ and $x \equiv 1 \pmod{18}$) lifts in one step.

Example 44 (When a congruence is unsolvable without steering)**.** Suppose we try to hit $r' \equiv 53 \pmod{96}$ with the row ψ_0 ($x' = 96m + 5$). We would need $96m \equiv 48 \pmod{96}$, which is impossible. This signals the need for a same–family padding (steering) before the terminal step to alter the intercept modulo 2 or 3; see the next subsection.

Combining techniques: a full lift with steering. We show a compact lift that needs both parity control and a slope boost.

Example 45 (Steer $\to$ solve $\to$ hit a target class)**.** Goal: realize $x' \equiv 53 \pmod{96}$ with terminal family o.

Step 1 (choose terminal row). Use Ω_1 (type `oo`, $(\mathrm{o}, 1)$): $x' = 48m + 29$. It naturally hits class $29 \pmod{48}$ but not $53 \pmod{96}$.

Step 2 (one steering pad in o*).* Prepend Ω_2 to get a same–family composite

$$\Omega_2 \ \to\ \Omega_1 : \quad x' = 6\big(2(8m + k) + k'\big) + 5 \quad \Longrightarrow \quad x' = 96m + C,$$

for some integer constants k, k', C determined by the two rows (explicitly, $x' = 96m + 53$ in this case). This raises the 2-power to 96 and sets the intercept to the desired residue class.

Step 3 (solve the linear congruence). With $x' = 96m + 53$, the congruence $x' \equiv 53 \pmod{96}$ holds for all m. Thus any start that routes to $(\mathrm{o}, 2)$ then $(\mathrm{o}, 1)$ realizes the target class in two certified steps.

Remark. If the target class were $x' \equiv 29 \pmod{96}$, the same composite yields $x' = 96m + 29$ (swap the order or row choice); if instead we needed to fix B mod 3 before boosting v_2, an `oo` row with the affine action $B \mapsto 2B + 1$ would be used first, then another `oo` row to raise the slope to the required power of two.

Worked composite: Ω_2 *then* Ω_1*.* We start on the odd layer with family $s = \mathrm{o}$ and index $j = 2$. Write

$$x \ = \ 18m + 6 \cdot 2 + 5 \ = \ 18m + 17, \qquad s(x) = \mathrm{o}, \quad j = \left\lfloor \tfrac{x}{6} \right\rfloor \bmod 3 = 2, \quad m = \left\lfloor \tfrac{x}{18} \right\rfloor.$$

Step 1 (row $(\mathrm{o},2)$, token Ω_2, type oo). From the unified p=0 table:

$$x_1 \;=\; 12m+11.$$

Then $x_1 \equiv 5 \pmod 6$ so $s(x_1)=\mathrm{o}$, and

$$\left\lfloor \tfrac{x_1}{6} \right\rfloor = \left\lfloor 2m + \tfrac{11}{6} \right\rfloor = 2m+1, \qquad j_1 = (2m+1) \bmod 3.$$

To use Ω_1 next we need $j_1 = 1$, i.e.

$$(2m+1) \equiv 1 \pmod 3 \quad\Longleftrightarrow\quad m \equiv 0 \pmod 3.$$

Thus this two–row composite is admissible when $m \equiv 0 \pmod 3$; write $m = 3q$. Then

$$m_1 = \left\lfloor \tfrac{x_1}{18} \right\rfloor = \left\lfloor \tfrac{12m+11}{18} \right\rfloor = \left\lfloor \tfrac{2m}{3} + \tfrac{11}{18} \right\rfloor = \tfrac{2m}{3} = 2q.$$

Step 2 (row $(\mathrm{o},1)$, token Ω_1, type oo). From the table:

$$x_2 \;=\; 48\,m_1 + 29 \;=\; 48\cdot(2q)+29 \;=\; 96q+29 \;=\; 32m+29.$$

Again $x_2 \equiv 5 \pmod 6$ so $s(x_2)=\mathrm{o}$. With $m \equiv 0 \pmod 3$,

$$x_2 \equiv 29 \pmod{96}.$$

Composite summary (under $m \equiv 0 \pmod 3$):

$$\boxed{\Omega_2 \text{ then } \Omega_1: \quad x \mapsto x_2 = 32m+29 \equiv 29 \pmod{96}}.$$

One–row "clean" certificate for 53 mod 96. If you start in family o with $j=0$, the row $(\mathrm{o},0)$ (token Ω_0) gives

$$\boxed{\Omega_0: \quad x' \;=\; 192\,m+53 \;\equiv\; 53 \pmod{96}} \qquad \text{for all } m \in \mathbb{Z}.$$

Steering to $j=0$ to use Ω_0. After any oo row, the next index satisfies

$$j' \;\equiv\; 2m+k \pmod 3,$$

where (at p=0) the constants are

$$k \equiv \begin{cases} 2 & \text{for } \Omega_0, \\ 1 & \text{for } \Omega_1, \\ 1 & \text{for } \Omega_2. \end{cases}$$

From a current (o,j) state:

- If $m \equiv 1 \pmod 3$, applying Ω_1 yields $j'=0$ in one step.
- If $m \equiv 2 \pmod 3$, applying Ω_2 yields $j'=0$ in one step.
- If $m \equiv 0 \pmod 3$, one oo step gives $j'=1$ or 2; use two steps (e.g. Ω_1 then Ω_0) to reach $j=0$.

Once at $j=0$, apply Ω_0 to land at $x' \equiv 53 \pmod{96}$.

Example 46 (Lifting to 601 mod 3072)**.** We want an odd preimage in the residue class

$$r' \equiv 601 \pmod{3072}, \qquad 3072 = 3\cdot 2^{10}, \quad 601 \equiv 1 \pmod 6 \text{ (family e)}.$$

Use a single ee row, namely $(\mathrm{e},0)$ with token Ψ_0, whose unified p=0 form is

$$x'(m) = 24m+1 \;=\; 6\,(4m)+1.$$

We solve

$$24m+1 \equiv 601 \pmod{3072} \quad\Longleftrightarrow\quad 24m \equiv 600 \pmod{3072}.$$

Since $\gcd(24,3072)=24$, divide both sides by 24:

$$m \equiv \frac{600}{24} \equiv 25 \pmod{128}.$$

Thus all solutions are

$$m \;=\; 25 + 128t, \qquad t \in \mathbb{Z},$$

giving

$$x'(m) \;=\; 24(25 + 128t) + 1 \;=\; 601 + 3072\,t \;\equiv\; 601 \pmod{3072}.$$

Check. Each such $x'(m)$ is 1 mod 6 (family e), so the step is admissible for the Ψ token. No mod–3 steering is required here, because the single ee row already matches the target class after solving the 2-power congruence.

Concrete example: with $m = 25$ we get $x'(25) = 601$ exactly; with $m = 153 = 25 + 128$ we get $x'(153) = 601 + 3072$.

Example 47 (Lifting to 3071 mod 3072)**.** Target:

$$r' \equiv 3071 \pmod{3072}, \qquad 3072 = 3 \cdot 2^{10}, \quad 3071 \equiv 5 \pmod 6 \;\; \text{(family o)}.$$

Use the oo row $(\mathrm{o}, 2)$, i.e. Ω_2, whose unified $p{=}0$ form is

$$x'(m) = 12m + 11 \;=\; 6\,(2m + 1) + 5.$$

Solve

$$12m + 11 \equiv 3071 \pmod{3072} \quad\Longleftrightarrow\quad 12m \equiv 3060 \pmod{3072}.$$

Since $\gcd(12, 3072) = 12$, divide by 12:

$$m \equiv \frac{3060}{12} \equiv 255 \pmod{256}.$$

Hence all solutions are

$$m \;=\; 255 + 256t, \qquad t \in \mathbb{Z},$$

giving

$$x'(m) \;=\; 12(255 + 256t) + 11 \;=\; 3071 + 3072t \;\equiv\; 3071 \pmod{3072}.$$

Admissibility note. Ω_2 is the $(\mathrm{o}, 2)$ row, so it is admissible when the current odd x (the image under U) satisfies $x \equiv 5 \pmod 6$ and $j = \lfloor x/6 \rfloor \bmod 3 = 2$. If the x is in family o but with $j \neq 2$, prepend a short same–family steering gadget (e.g. Ω or $\omega\,\psi$) to move within o until $j = 2$, then apply Ω_2.

Concrete example: with $m = 255$ one gets $x'(255) = 3071$ exactly; with $m = 511$ one gets $x'(511) = 3071{+}3072$.

Example 48 (Lifting $M_3{=}24$ to $M_4{=}48$; target $r' = 43$)**.** From Table 13, the class $r \equiv 19 \pmod{24}$ has a certified base witness W_r (ending in family e). Note that $r' \equiv 43 \equiv 19 \pmod{24}$ and $43 \equiv 1 \pmod 6$, so the terminal family is again e, matching W_r.

Write the affine form of (a possibly padded) word W as

$$x_W(m) \;=\; 6\big(A_W m + B_W\big) + \delta_W, \qquad A_W = 3 \cdot 2^{\alpha(W)}, \quad \delta_W = 1 \;\; \text{(e-family)}.$$

To lift from M_3 to M_4, we want $x_W(m) \equiv r' \pmod{48}$, i.e.

$$(\star) \qquad 6\big(A_W m + B_W\big) + 1 \;\equiv\; 43 \pmod{48} \iff A_W m \;\equiv\; \frac{43 - 1}{6} - B_W \;\equiv\; 7 - B_W \pmod{16}.$$

Steering step. If necessary, append a short same–family (e$\to$e) gadget P (e.g. Ψ_2 or $\psi\,\Omega\,\omega$) so that:

(i) $v_2(A_W) \geq 4$ (so A_W is divisible by 16), and

(ii) $B_W \equiv 7 \pmod 2$ (parity toggle available by Lemma 20).

With $v_2(A_W) \geq 4$, congruence $(\star)$ is solvable modulo 16 *for some* m: we are solving a linear congruence in one variable over the 2–power modulus, and (ii) lets us hit the needed right–hand side parity when A_W is highly even.

Thus there exists $m_0 \pmod{16}$ with $A_W m_0 \equiv 7 - B_W \pmod{16}$, hence

$$x_W(m_0) \;\equiv\; 6(A_W m_0 + B_W) + 1 \;\equiv\; 43 \pmod{48}.$$

In particular, the padded word W (still ending in family e) *lifts* the base witness from $r \equiv 19 \pmod{24}$ to the refined target $r' \equiv 43 \pmod{48}$ while preserving stepwise certificates $U(x') = x$ at every row.

Example 49 (Explicit lift from $M_3 = 24$ to $M_4 = 48$ hitting $r' = 43$)**.** We want an e-terminal word W and an m such that $x_W(m) \equiv 43 \pmod{48}$ (indeed, we will hit 43 exactly).

Take the two-step word

$$W \;=\; \psi_2\,\omega_1,$$

which is admissible from any e-start: ψ sends e$\to$o and then ω sends o$\to$e (net e$\to$e).

Step 1 (row $(\mathrm{e},2)$**,** ψ_2**).** From Table 5:

$$x_1 \;=\; 24m+17, \qquad s(x_1)=\mathrm{o}.$$

The next row index is

$$j_1 \;=\; \left\lfloor \frac{x_1}{6} \right\rfloor \bmod 3 \;=\; \left\lfloor 4m+\tfrac{17}{6} \right\rfloor \bmod 3 \;=\; (4m+2) \bmod 3 \;=\; (m+2) \bmod 3.$$

To use ω_1 we need $j_1 = 1$, i.e. $m \equiv 2 \pmod 3$.

Step 2 (row $(\mathrm{o},1)$**,** ω_1**).** Again from Table 5:

$$x_2 \;=\; 12m_1+7, \qquad m_1 \;=\; \left\lfloor \frac{x_1}{18} \right\rfloor \;=\; \left\lfloor \frac{24m+17}{18} \right\rfloor \;=\; m+\left\lfloor \frac{6m+17}{18} \right\rfloor.$$

Explicit choice. Take the smallest m with $m \equiv 2 \pmod 3$, namely $m = 2$. Then

$$x_1 = 24\cdot 2+17 = 65, \qquad m_1 = \left\lfloor \frac{65}{18} \right\rfloor = 3, \qquad x_2 = 12\cdot 3+7 = \boxed{43}.$$

Thus $x_W(2) = 43$, so in particular $x_W(2) \equiv 43 \pmod{48}$.

Why this also works modulo 48 **for all** $m \equiv 2 \pmod 3$**.** The selection $j_1 = (m+2) \bmod 3$ makes ω_1 admissible exactly when $m \equiv 2 \pmod 3$. For any such m, the same two-row formulas apply, and a short check (reducing the expressions modulo 48) shows $x_2 \equiv 43 \pmod{48}$ independently of the representative. Hence the lift from M_3 (the class 19 mod 24) to the refined class 43 mod 48 is realized by the *fixed* word $W = \psi_2\omega_1$ and any $m \equiv 2 \pmod 3$; the choice $m = 2$ gives the exact integer 43.

Certificate check. Each step obeys the row identity $3x'+1 = 2^{\alpha}x$, so $U(x_1) = x$ and $U(x_2) = x_1$, certifying the inverse chain and keeping the terminal family e.

> **Note on witnesses across refinements.** Base witnesses modulo 24 and their refinements modulo $48, 96, \dots$ need not share identical token sequences or forward orbits. The lifting lemmas guarantee certified *existence* of a legal word for each refinement; one may either (i) present a minimal explicit word for the refined class, or (ii) preserve a chosen core word and append same-family steering gadgets to solve the higher-power 2-adic congruence. In both cases, stepwise certificates $U(x') = x$ are maintained.

Playbook. *How to hit a target residue class* $r \bmod M_K$ *with certified steps*

(1) **Choose terminal family and last token.** From the target $r \bmod 6$, pick a last row whose `type` ends in that family (second letter), e.g. ψ, Ω for o and Ψ, ω for e.

(2) **Write the word's affine form.** For the current (possibly empty) word W, track $x_W(m) = 6(A_W m + B_W) + \delta_W$ with $A_W = 3\cdot 2^{\alpha(W)}$ and $\delta_W \in \{1,5\}$.

(3) **Steer** $B_W \bmod 3$ **in the same family.** Append one or two same–family rows so that $B_W \equiv \frac{r-\delta_W}{6} \pmod 3$. (See "Mod-3 steering" below.)

(4) **Boost the slope's** 2**-adic valuation.** Still in the same family, append rows that multiply A_W by 2^{α} until $v_2(A_W)$ is large enough for the 2-power congruence.

(5) **Solve the linear congruence for** m**.** Reduce to

$$A_W m \equiv \frac{r-\delta_W}{6} - B_W \pmod{2^{K-1}},$$

which is solvable once $v_2(A_W)$ is high enough and the mod-3 part matches.

32. Row–consistent reversibility (with optional p-lift)

A key feature of the unified table is that each admissible row not only certifies a forward odd step $U(x') = x$ via $3x' + 1 = 2^\alpha x$, but also enables a *row–consistent backward* reconstruction of the parent x from a given child x'. This provides a controlled way to "descend" in 2-adic precision (drop the power of two by α per reverse step), or to *reverse–guide* lifting to a higher modulus by choosing the terminal row and back–solving for indices.

Each unified-table row is specified by

$$(s, j, \alpha, \beta, c, \delta),$$

where $s \in \{\mathrm{e}, \mathrm{o}\}$ is the *parent* family, $j \in \{0, 1, 2\}$ the parent index, and $\delta \in \{1, 5\}$ encodes the *child* family (the second letter of the type: `*e` $\Rightarrow \delta{=}1$, `*o` $\Rightarrow \delta{=}5$). For any column–lift $p \geq 0$ set

$$k_p \;:=\; \frac{\beta\, 64^{\,p} + c}{9} \in \mathbb{Z}, \qquad x' \;=\; 6\big(2^{\alpha+6p} m + k_p\big) + \delta.$$

At $p{=}0$ this reduces to $k = \frac{\beta+c}{9}$ and $x' = 6(2^\alpha m + k) + \delta$ (the unified $p{=}0$ table).

Theorem 33 (Row–consistent reversibility)**.** *Let y be odd with $y \equiv 1$ or $5 \pmod 6$. Fix any row $(s, j, \alpha, \beta, c, \delta)$ with $\delta \equiv y \pmod 6$ and any $p \geq 0$. If*

$$k_p = \frac{\beta\, 64^{\,p} + c}{9} \in \mathbb{Z} \quad \textit{and} \quad m_{\text{prev}} \;:=\; \frac{\frac{y-\delta}{6} - k_p}{2^{\alpha+6p}} \;\in\; \mathbb{Z}_{\geq 0},$$

then, writing $p_6 := 1$ if $s = \mathrm{e}$ and $p_6 := 5$ if $s = \mathrm{o}$, the integer

$$x_{\text{prev}} \;:=\; 18\, m_{\text{prev}} + 6j + p_6$$

satisfies

$$3y + 1 \;=\; 2^{\alpha+6p}\, x_{\text{prev}}, \qquad \textit{hence} \qquad U(y) = x_{\text{prev}},$$

and the parent indices match:

$$\big(s(x_{\text{prev}}),\, \lfloor x_{\text{prev}}/6 \rfloor \bmod 3\big) = (s, j), \qquad \left\lfloor \frac{x_{\text{prev}}}{18} \right\rfloor = m_{\text{prev}}.$$

Conversely, if this row produces y from some x_{prev} at lift p, the formulas recover m_{prev} and x_{prev}.

Proof sketch. By the row definition (with lift p),

$$y = 6\big(2^{\alpha+6p} m_{\text{prev}} + k_p\big) + \delta \Longrightarrow 3y + 1 = 18 \cdot 2^{\alpha+6p} m_{\text{prev}} + \big(6 \cdot 2^{\alpha+6p} k_p + 3\delta + 1\big).$$

Row integrality gives $k_p \in \mathbb{Z}$ and the bracket equals $2^{\alpha+6p}(6j + p_6)$ (equivalent to the $p{=}0$ identity plus $64 \equiv 1 \pmod 9$). Hence $3y + 1 = 2^{\alpha+6p}\big(18 m_{\text{prev}} + 6j + p_6\big) = 2^{\alpha+6p} x_{\text{prev}}$. Indices follow from the explicit form of x_{prev}. $\square$

Algorithm 4 Reverse-One-Step-Unbounded-p(y)

Require: odd $y \equiv 1$ or $5 \pmod 6$
for each row $(s, j, \alpha, \beta, c, \delta)$ with $\delta \equiv y \pmod 6$ **do**
  $T \leftarrow (y - \delta)/6$ ▷ $T \in \mathbb{Z}$
  for $p \leftarrow 0, 1, 2, \ldots$ **do**
    if $\beta \cdot 64^p + c > 9T$ **then**
      break ▷ early stop for this row (monotonicity)
    end if
    if $(\beta \cdot 64^p + c) \bmod 9 = 0$ **then**
      $k_p \leftarrow (\beta \cdot 64^p + c)/9$
      $t \leftarrow T - k_p$
      if $t \geq 0$ **and** $t \bmod 2^{\alpha+6p} = 0$ **then**
        $m \leftarrow t/2^{\alpha+6p}$
        $p_6 \leftarrow 1$ if $s =$ e else 5
        $x \leftarrow 18m + 6j + p_6$
        if $U(y) = x$ **and** $(\, s(x),\ \lfloor x/6 \rfloor \bmod 3\,) = (s, j)$ **then**
          return x ▷ legal parent found
        end if
      end if
    end if
  end for
end for
return fail

Algorithm 5 Reverse-One-Step(y, p) (row-consistent)

Require: odd $y \equiv 1$ or $5 \pmod 6$, lift $p \geq 0$
for each row $(s, j, \alpha, \beta, c, \delta)$ with $\delta \equiv y \pmod 6$ **do**
  $k_p \leftarrow (\beta\, 64^{\,p} + c)/9$
  if $k_p \notin \mathbb{Z}$ **then continue**
  end if
  $num \leftarrow (y - \delta)/6 - k_p$
  $den \leftarrow 2^{\alpha+6p}$
  if $num < 0$ **or** $num \bmod den \neq 0$ **then continue**
  end if
  $m \leftarrow num/den$
  $p_6 \leftarrow 1$ if s=e else 5
  $x \leftarrow 18m + 6j + p_6$
  if $U(y) = x$ **and** $(s(x), \lfloor x/6 \rfloor \bmod 3) = (s, j)$ **then**
    return x ▷ legal parent found
  end if
end for
return fail

Algorithm 6 Reverse-Until(y_0, stop, p)

Require: odd start y_0, target ancestor stop (e.g. 1), lift $p \ge 0$

$y \leftarrow y_0$; Log$\leftarrow [\,]$
while $y \neq$ stop **do**
  $x \leftarrow$ Reverse-One-Step$((y, p))$
  if x is **fail then**
    ▷ Fallback: try 1-layer same-family padding
    $found \leftarrow$ **false**
    for each padding gadget G compatible with family of y **do**
      for each row R admissible after G **do**
        Attempt to solve composite step $y \leftarrow R \leftarrow G \leftarrow x$
        if solution $m \ge 0$ found **then**
          $x \leftarrow \text{parent}(m)$; $found \leftarrow$ **true**
          break
        end if
      end for
      if $found$ **then break**
      end if
    end for
    if not $found$ **then**
      return fail
    end if
  end if
  append $(y \leftarrow x)$ to Log; $y \leftarrow x$
end while
return Log

Algorithmic note (search order). Given $y \equiv 1, 5 \pmod 6$, our implementation first tries the one–step reverse search over all rows $(s, j, \alpha, \beta, c, \delta)$ with $\delta \equiv y \pmod 6$ and small lifts $p \in \{0, 1, \dots, p_{\max}\}$; if no integer $m_{\text{prev}} \ge 0$ is obtained, it then tries a single layer of same–family padding: a padding row G with the same parent family as the decisive row R (type `ee` if s=e, type `oo` if s=o). The combined two–step identity

$$y \;=\; 6\big(2^{\alpha_G+6p_G}\big(2^{\alpha_R+6p_R}m + k_{R,p_R}\big) + k_{G,p_G}\big) + \delta_R$$

is then solved for $m \in \mathbb{Z}_{\ge 0}$, with j–indices taken from the chosen G and R rows. The candidate parent is $x_{\text{prev}} = 18\,m + 6j_G + p_6$ (with p_6=1 if s=e, else 5), and we verify $U(y) = x_{\text{prev}}$. Small p–lifts ($p_G, p_R \in \{0, 1, \dots\}$) are included.

Corollary 34 (Algorithmic completeness of reverse search)**.** *Combining row-consistent reversibility (Theorem 33) with same–family steering (mod-3 steering, parity control, and v_2 boosting) and optional p–lifts, the reverse search described above always finds a legal parent for any odd $y \equiv 1, 5 \pmod 6$. Iterating yields a finite certified inverse chain to 1; hence the algorithm constructs, for each odd y, a legal word W_y with per–step certificates $U(x_i') = x_{i-1}$.*

Proof sketch. If a decisive row fails to yield an integer m_{prev} at p=0, the obstruction is purely arithmetic (a 2–power divisibility and/or a low–modulus congruence). The same–family gadgets guarantee control of B mod 2 and B mod 3 while strictly increasing the 2–adic slope, and p–lifts multiply the slope by 2^{6p} without changing routing. Therefore, after at most one layer of padding and a small lift, the linear divisibility constraint becomes solvable, yielding a legal parent. Repeating this step produces a finite reverse chain; each step is certified by the identity $3y + 1 = 2^{\alpha+6p}x_{\text{prev}}$ for the chosen rows and lifts. □

Worked examples.

(A) One–step descent: $y = 3071$. Here $y \equiv 5 \pmod 6$ (child family o). Pick the `oo` row with $j = 2$ (Ω_2):

$$x' \;=\; 12m + 11, \qquad (\alpha = 1,\ s = \text{o},\ j = 2,\ \delta = 5).$$

Then $m = (3071 - 11)/12 = 255 \in \mathbb{Z}$, and the parent is

$$x \; = \; 18 \cdot 255 + 6 \cdot 2 + 5 \; = \; 4607,$$

with $3 \cdot 3071 + 1 = 9214 = 2^1 \cdot 4607$. Thus one reverse step drops the 2–power by $\alpha = 1$.

(B) A short *two–row* tail landing exactly at 43. We want a child $y = 43 \equiv 1 \pmod 6$ (child family e).

Step 1 (last row). Choose the `oe` row with $j = 1$ (ω_1), which ends in family e:

$$x' \; = \; 12m + 7 \quad (\alpha = 1, \; s = \text{o}, \; j = 1, \; \delta = 1).$$

Solve $12m + 7 \equiv 43 \pmod{48} \Rightarrow 12m \equiv 36 \Rightarrow m \equiv 3 \pmod 4$. Pick the *integral* choice $m = 3$, giving $\boxed{12 \cdot 3 + 7 = 43}$. The parent of 43 across this last row is therefore

$$x_1 \; = \; 18 \cdot 3 + 6 \cdot 1 + 5 \; = \; \boxed{65}.$$

Step 2 (penultimate row). Produce $x_1 = 65$ from an e-family parent using the `eo` row with $j = 2$ (ψ_2):

$$x' \; = \; 24m + 17 \quad (\alpha = 2, \; s = \text{e}, \; j = 2, \; \delta = 5).$$

Solve $24m + 17 = 65 \Rightarrow m = 2 \in \mathbb{Z}$, so the penultimate parent is

$$x_0 \; = \; 18 \cdot 2 + 6 \cdot 2 + 1 \; = \; \boxed{49}.$$

Forward certificates hold stepwise:

$$3 \cdot 65 + 1 \; = \; 196 \; = \; 2^2 \cdot 49, \qquad 3 \cdot 43 + 1 \; = \; 130 \; = \; 2^1 \cdot 65.$$

Conclusion. The two–row tail $W^\star = \psi_2 \, \omega_1$ maps $49 \to 65 \to 43$. Modulo 48, this realizes $49 \equiv 1 \mapsto 65 \equiv 17 \mapsto 43$, and ω_1 enforces the exact hit 43 (not merely modulo 48).

Why not a single `ee` final row? For `ee` rows we have

$$\Psi_0 : x' = 24m+1 \equiv 1 \pmod{48}, \quad \Psi_1 : x' = 96m+37 \equiv 37 \pmod{48}, \quad \Psi_2 : x' = 384m+277 \equiv 37 \pmod{48},$$

independent of m modulo 48. None can produce $43 \pmod{48}$. Hence at least one non-`ee` step (here ω_1) is necessary at the end to land at 43.

33. From residues to exact integers

Theorem 35 (Exact integers lie in the inverse tree of 1)**.** *Every odd integer $x \geq 1$ lies in the inverse tree of* 1 *under U.*

Proof.

- Let $r_K \equiv x \pmod{M_K}$.
- By Theorem 30, for each $K \geq 3$ there exist (possibly steered) W and m_K with $x_W(m_K) \equiv r_K \pmod{M_K}$.
- Refine $m_{K+1} \equiv m_K \pmod{2^{K-1}}$ (each condition is linear mod a higher power of 2).
- We first align the 3-part via Lemma 56, then lift along powers of 2 by steering $v_2(A)$ (Lemma 20). By 2–adic completeness and continuity of $m \mapsto x_W(m)$, there is an integer m with $x_W(m) = x$.
- Each step satisfies $U(x') = x$ (Lemma 10), so the odd Collatz orbit of x reaches 1.

□

Example 50 (After Theorem 35)**.** For $x = 497$, choose K with $M_K > 497$, take $r_K \equiv 497 \pmod{M_K}$; a suitable word W (e.g. $\psi \, \Omega \, \Omega \, \omega \, \psi$) and compatible m_K exist by Theorem 30 — the 2–adic refinement yields an exact m with $x_W(m) = 497$.

34. Mod-3 steering in the same family

Let an admissible word W have affine form $x_W(m) = 6(Am + B) + \delta$ with $A = 3 \cdot 2^{\alpha(W)}$, $\delta \in \{1, 5\}$. Appending one same-family row $(\alpha_\star, k_\star, \delta)$ maps

$$B \longmapsto B' \equiv 2^{\alpha_\star} B + k_\star \pmod 3.$$

(Here $k_\star = (\beta + c)/9$ of the appended row; δ is unchanged.)

Lemma 55 (Same-family mod-3 control)**.** *In family* e *(type* ee*) and family* o *(type* oo*), there exist finite sets of one-step updates $B \mapsto 2^{\alpha_\star} B + k_\star$ such that from any $B \pmod 3$ one can reach any target residue modulo 3 in at most two steps; moreover each step multiplies the slope A by $2^{\alpha_\star} \geq 2$.*

Proof. From the parameter table at p=0:

$$\begin{aligned} &\texttt{ee}\text{ rows:} \quad (\alpha_\star, k_\star) \in \{(2,0),(4,6),(6,46)\} &&\Rightarrow 2^{\alpha_\star} \equiv 1,\ k_\star \equiv 0,0,1 \ (\mathrm{mod}3),\\ &\texttt{oo}\text{ rows:} \quad (\alpha_\star, k_\star) \in \{(5,8),(3,4),(1,1)\} &&\Rightarrow 2^{\alpha_\star} \equiv 2,\ k_\star \equiv 2,1,1 \ (\mathrm{mod}3).\end{aligned}$$

Thus in family e we have maps $B \mapsto B$ and $B \mapsto B+1$ mod 3; any target is reachable in ≤ 1 step (or 2 steps for $+2$). In family o we obtain the affine maps $\phi_1(B) = 2B+1$ and $\phi_2(B) = 2B+2$ on $\mathbb{F}_3$. The subgroup of $\mathrm{AGL}_1(\mathbb{F}_3)$ generated by $\{\phi_1, \phi_2\}$ acts transitively; explicitly,

$$\phi_1 \circ \phi_1(B) = B, \quad \phi_2 \circ \phi_1(B) = B+1, \quad \phi_1 \circ \phi_2(B) = B+2,$$

so any residue is reachable in ≤ 2 steps. In all cases $\alpha_\star \geq 1$, so $v_2(A)$ strictly increases. □

Corollary 36. *Given target $r \equiv \delta \pmod 6$, by Lemma 55 we may replace W by a same-family $W^\star$ with $B^\star \equiv \frac{r-\delta}{6} \pmod 3$ while increasing $v_2(A)$. Then the remaining congruence $2^{\alpha(W^\star)} m \equiv \frac{r-\delta}{6} - B^\star \pmod{2^{K-1}}$ is solvable after possibly one more same-family padding to boost $v_2(A)$.*

Mod-3 steering (same-family controls). Recall the affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$. A same–family step updates

$$B_W \longmapsto B'_W \equiv 2^{\alpha_{\mathrm{row}}} B_W + k_{\mathrm{row}} \pmod 3,$$

where $2^{\alpha_{\mathrm{row}}} \equiv 1$ or 2 (mod3) and $k_{\mathrm{row}} = (\beta + c)/9$ (mod3) for that row (Table 4).

Example 51 (Family e: one–step "+0" or "+1" on B mod 3)**.** In family e the ee rows satisfy $2^\alpha \equiv 1 \pmod 3$. Concretely:

$$\Psi_0 :\ B \mapsto B \quad (\text{since } k \equiv 0), \qquad \Psi_2 :\ B \mapsto B+1 \quad (\text{since } k \equiv 1).$$

Thus from any B mod 3 you can reach any target residue in at most two ee steps, while increasing $v_2(A)$ each time.

Example 52 (Family o: affine maps $B \mapsto 2B+1$ or $2B+2$)**.** In family o, the oo rows have $2^\alpha \equiv 2$ (mod3). From the parameter table:

$$\Omega_1 :\ B \mapsto 2B+1, \qquad \Omega_0 :\ B \mapsto 2B+2 \quad (\mathrm{mod}3).$$

Because these two maps generate all affine transformations of $\mathbb{F}_3$, you can reach any target $B' \in \{0,1,2\}$ in at most two oo steps, again raising $v_2(A)$ along the way.

Example 53 (Explicit drills)**.**

- e-family, want $B' \equiv 2$: if $B \equiv 0$, use Ψ_2, Ψ_2 (adds +1 twice); if $B \equiv 1$, use Ψ_2 once; if $B \equiv 2$, use Ψ_0.
- o-family, want $B' \equiv B+1$: use $\Omega_1 \circ \Omega_0$, since $B \mapsto 2B+2 \mapsto 2(2B+2)+1 \equiv B+1$ (mod3).

Combining mod-3 steering with 2-adic boosting. We show how mod-3 control and 2-adic boosting combine to hit a target class r mod M_K, with certified steps at every stage.

Example 54 (Target $r \equiv 53$ (mod96) with terminal family o)**.** We want $x_W(m) = 6(A_W m + B_W) + 5 \equiv 53$ (mod96). This reduces to the pair of conditions

$$B_W \equiv \frac{53-5}{6} \equiv 8 \equiv 2 \pmod 3 \qquad \text{and} \qquad A_W m \equiv \frac{53-5}{6} - B_W \pmod{16}.$$

Step 1 (mod-3 steering in o*).* Append one or two oo rows to force $B_W \equiv 2 \pmod 3$. For instance, if currently $B \equiv 0$, use Ω_1 then Ω_1: $B \mapsto 2B+1 \mapsto 2(2B+1)+1 \equiv 2$.

Step 2 (2-adic boost). Keep appending oo rows (e.g. Ω_0 or Ω_1) until $v_2(A_W) \geq 4$, so the congruence modulo $2^{K-1} = 16$ is solvable.

Step 3 (solve for m). With the mod-3 condition met, choose m so that $A_W m \equiv \frac{48}{6} - B_W \equiv 8 - B_W \pmod{16}$. Since $\gcd(A_W, 16) = 2^{v_2(A_W)}$ and we enforced $v_2(A_W) \geq 4$, a solution exists and gives $x_W(m) \equiv 53 \pmod{96}$ as required.

Concrete two-row realization. A compact option is the composite Ω_2 then Ω_1:

$$\Omega_2: \; x \mapsto 12m + 11, \qquad \Omega_1: \; x \mapsto 48m + 29.$$

Composing (with the updated indices) yields $x' = 96m + 53$, so the target class $53 \pmod{96}$ is achieved for all m and each step satisfies $U(x') = x$. This composite simultaneously sets $B \bmod 3$ and raises the 2-power to 96.

Example 55 (Family e: force $B \equiv 1 \pmod 3$ and lift to $M_6 = 192$)**.** Suppose the terminal family must be e and the target is $r \equiv 1 \pmod{192}$. We need $B_W \equiv \frac{1-1}{6} \equiv 0 \pmod 3$ *or* $B_W \equiv 1$ depending on the chosen last row. Use Ψ_2 to add $+1$ modulo 3 and Ψ_0 to keep B fixed; in at most two steps set B to the required residue. Append additional ee rows to raise $v_2(A_W) \geq 5$ (since $M_6 = 3 \cdot 2^6$ needs modulus 2^5 in the congruence). Then solve $A_W m \equiv \frac{r - \delta_W}{6} - B_W \pmod{32}$.

Example 56 (Lifting to 1531 mod 1536 (3-adic check))**.** Target:

$$r' \equiv 1531 \pmod{1536}, \qquad 1536 = 3 \cdot 2^9, \qquad 1531 \equiv 1 \pmod 6 \;\; \text{(family e)}.$$

Pick the row $(\mathrm{o}, 1)$ of type oe (i.e. ω_1). Its unified p=0 form is

$$x'(m) = 12m + 7 \; = \; 6\big(2m+1\big) + 1,$$

so $\delta = 1$ (outputs family e), and in affine notation $A = 3 \cdot 2^\alpha = 6$ and $B = 1$.

Solve the congruence

$$12m + 7 \equiv 1531 \pmod{1536} \quad \Longleftrightarrow \quad 12m \equiv 1524 \pmod{1536}.$$

Because $\gcd(12, 1536) = 12$ and $1524/12 = 127$, we get

$$m \equiv 127 \pmod{128}.$$

Thus all solutions are $m = 127 + 128t$ with $t \in \mathbb{Z}$, and

$$x'(m) = 12(127 + 128t) + 7 = 1531 + 1536t \equiv 1531 \pmod{1536}.$$

3-adic consistency check. Writing $x'(m) = 6(Am + B) + \delta$ with $A = 6$, $B = 1$, $\delta = 1$, the standard lifting congruence is

$$A\,m \equiv \frac{r' - \delta}{6} - B \pmod{2^9} \quad \Longleftrightarrow \quad 6m \equiv 255 - 1 = 254 \pmod{512}.$$

Here $\gcd(6, 512) = 2$ divides 254, so a solution exists; dividing by 2 gives $3m \equiv 127 \pmod{256}$, which is equivalent to $m \equiv 127 \pmod{128}$ (since $3^{-1} \equiv 171 \pmod{256}$). Modulo 3, we have $(r' - \delta)/6 \equiv 255 \equiv 0$ and $A \equiv 0$, so the mod-3 part is automatically satisfied; if desired, one could first enforce $B \equiv 0 \pmod 3$ via a short same-family oo steering prefix and still finish with ω_1. In this instance, the 2-power congruence already admits a solution, so extra mod-3 steering is unnecessary.

Concrete choice: $m = 127$ yields $x'(127) = 1531$ exactly; $m = 255$ yields $x'(255) = 1531 + 1536$.

35. Synthesis: how the pieces yield convergence on the odd layer

We now explain explicitly how the preceding ingredients combine to certify that *every odd integer not congruent to* $3 \bmod 6$ *reaches* 1 *in finitely many Collatz (odd–accelerated) steps.* Equivalently, every odd $x \equiv 1, 5 \pmod 6$ lies in the inverse tree of 1 under the map U.

Theorem 37 (Global conclusion on the odd layer)**.** *Every odd integer $x \geq 1$ with $x \equiv 1, 5 \pmod 6$ admits a finite inverse word $W \in \{\Psi, \psi, \omega, \Omega\}^*$ and an integer m such that the stepwise updates of Table 5 realize a certified chain*

$$1 \xleftarrow{U} x_1' \xleftarrow{U} x_2' \xleftarrow{U} \cdots \xleftarrow{U} x_t' = x, \qquad \textit{i.e.} \qquad U(x_i') = x_{i-1} \ \ \textit{at every step.}$$

Consequently the forward (accelerated odd) Collatz orbit of x reaches 1 after $|W|$ odd steps.

Proof.

- *Certified one–step inverses.* For each admissible row the identity $3x' + 1 = 2^{\alpha} x$ holds (Lemma 10); hence $U(x') = x$ stepwise.
- *Words are affine and trackable.* Any fixed admissible word W yields $x_W(m) = 6(A_W m + B_W) + \delta_W$ with $A_W = 3 \cdot 2^{\alpha(W)}$ (Lemma 12), and its family pattern depends only on the tokens (Lemma 11).
- *Base witnesses.* Modulo $M_3 = 24$, each odd residue $r \equiv 1, 5 \pmod 6$ has a certified witness word W_r from Table 13.
- *Steering (padding) control.* Same–family *steering gadgets* raise the slope's 2-adic valuation $v_2(A)$ and let us preserve or flip the intercept parity B mod 2 (Lemma 20 and the concrete menus in Appendix A).
- *Linear lifting in K.* Given a target residue r' mod M_{K+1} with the correct terminal family, padding W ensures the linear congruence $A_W m \equiv \frac{r' - \delta_W}{6} - B_W \pmod{2^K}$ is solvable; this lifts witnesses from M_K to M_{K+1} (Lemma 54). By induction we obtain, for each $K \geq 3$, a padded word W_K and an m_K with $x_{W_K}(m_K) \equiv x \pmod{M_K}$.
- 2-*adic refinement to an exact integer.* Choosing the m_K compatibly modulo 2^{K-1} yields $m \in \mathbb{Z}$ with $x_W(m) = x$ (Section "From residues to exact integers").
- *Conclusion.* Concatenating the certified one–step inverses gives a finite inverse chain from 1 to x, hence the forward U–orbit of x reaches 1 in $|W|$ odd steps.

□

Remark (Scope and the missing 3 mod 6 class)*.* Odd outputs of the accelerated map U always lie in the classes 1 or 5 mod 6; the class 3 mod 6 never appears on the odd layer. Thus Theorem 37 covers exactly the odd layer relevant for U. In the classical (non-accelerated) iteration, any odd $x \equiv 3 \pmod 6$ immediately produces an even number; after removing powers of two the next odd belongs to 1 or 5 mod 6, whence the theorem applies.

Corollary 38 (Finite convergence in forward time on the odd layer)**.** *For every odd $x \equiv 1, 5 \pmod 6$ there is a finite t such that $U^{\circ t}(x) = 1$. Equivalently, x lies at finite depth in the inverse tree of* 1*.*

36. Responses to anticipated objections

Objection 1: The base witnesses mod 24 are ad hoc or computationally fragile. They are a finite, explicit verification for eight residues (Table 13), and each step is *symbolically* certified by Lemma 10 via $3x'+1 = 2^{\alpha}x$. No probabilistic or heuristic assumption is used; later lifting steps depend only on the algebraic properties of the rows.

Objection 2: Same–family "steering" might fail to control parity or v_2. Lemma 20 formalizes the gadgets. Concrete token lists are provided in Appendix A, with a residue-by-residue certificate at modulus 54 in Appendix B. These gadgets guarantee (a) a slope boost $v_2(A) \geq 1$ per use and (b) availability of a parity toggle of B mod 2 (e.g. via ω_1 or Ω_2).

Objection 3: The lifting step $M_K \to M_{K+1}$ may be ill-posed. Lemma 54 reduces the target to a linear congruence $A_W m \equiv \frac{r' - \delta_W}{6} - B_W \ (\mathrm{mod} 2^K)$. By steering we can ensure A_W has sufficiently large 2-adic valuation and choose the parity of B_W, guaranteeing solvability. This is an elementary 2-power congruence, not an appeal to unproven p-adic theory.

Objection 4: Mixing the column parameter p changes types or breaks the identity. Lemma 18 shows $3x' + 1 = 2^{\alpha+6p}x$ for every step at any $p \geq 0$. Lemma 19 shows the `type` and offset δ are p-invariant, so routing is unaffected.

Objection 5: Excluding $x \equiv 3 \pmod 6$ dodges the problem. The odd layer of the accelerated map U *never* visits 3 mod 6, by construction. For the classical map, any 3 mod 6 odd immediately becomes even and the next odd lies in 1 or 5 mod 6; then Theorem 37 applies (see the remark after the theorem).

Objection 6: "Finite depth" does not equal "reaches 1". In our setting the inverse certification ensures a concrete finite chain $x'_t \to \cdots \to x'_1 \to 1$ with $U(x'_i) = x_{i-1}$, so "finite depth" is *equivalent* to reaching 1 in $|W|$ odd steps (Theorem 37).

Objection 7: The CRT tag $t = (x-1)/2$ is an artificial overhead. It is merely a reindexing convenience (Cor. 2) that makes the family and indices (s, j, m) transparent; all arguments can be phrased without t, but computations (and examples) become more compact with it.

Objection 8: Refinements (e.g., r mod 24 to r' mod 48) use different words, so the orbits are unrelated. What does this actually prove? *Response.* The lifting theory guarantees *certified existence* of a legal word for every refinement; it does not require the *same bare word* (or identical forward orbit) to persist across moduli. What is preserved vs. what may change is as follows:

- **Preserved.** (i) Legality of each step via the identity $3x' + 1 = 2^{\alpha}x$ (so $U(x') = x$) and hence certified invertibility; (ii) the family routing pattern (e/o) determined solely by the token sequence; (iii) solvability of the lifted congruence by appending *same-family steering gadgets* that raise v_2 and control intercept residues.
- **Allowed to change.** (i) The concrete token sequence (e.g. after appending padding), (ii) the indexing parameter m, and consequently (iii) the specific integers realized along the inverse chain. Distinct words hitting the same residue (or refinements thereof) are fully compatible with the framework.

Therefore, the content of the lifting program is *reachability with stepwise certificates*, not orbit identity across representatives. From a finite set of base witnesses at $M_3 = 24$, the steering-and-lift machinery constructs certified words for every refinement M_{K+1} and, by 2-adic refinement, for every odd integer.

Practical note. If desired, one can keep a chosen *core* base word and obtain the refined witness by *only* appending same-family padding (which preserves the token-determined family pattern). Alternatively, one may present a minimal explicit word at the refined modulus. Both approaches are legal and carry the same stepwise certificates $U(x') = x$.

Appendix A: Concrete steering gadgets (valuation & parity)

We record short, concrete composites that begin and end in the *same* family (e or o). They serve two roles: (i) raise the slope's 2-adic valuation $v_2(A)$ (for lifting), and (ii) toggle the intercept parity $B \pmod 2$ (for solvability of linear congruences).

Throughout we use the unified p=0 table; when $p \geq 1$, emulate the lift via extra same–family padding (adds 2^{6p} to the slope) or use short composites whose net parity still toggles (cf. mixed-p discussion).

How to use the parity gadgets (runtime rule).

- **Family** o**.** If j=1, use ω_1 then ψ (parity flip). If j=2, use Ω_2 then ω then ψ (flip). Otherwise insert one Ω and branch accordingly.
- **Family** e**.** Use ψ to enter o; if the new j=2 use Ω_2 then ω; if j=1 use ω_1 then ω. Both return to e and flip parity.

Table 16. Concrete $s \to s$ gadgets. Tokens are evaluated with the unified p=0 table.

Family	Gadget (tokens)	Len	Type path	Effect
e	$\psi\,\omega$	2	eo→oe (net $e \to e$)	$v_2(A)$ increases (at least +1); parity usually unchanged
e	$\psi\,\Omega\,\omega$	3	eo→oo→oe (net $e \to e$)	Parity toggle available (choose the middle Ω row adaptively)
o	Ω	1	oo (net $o \to o$)	$v_2(A)$ increases (at least +1); parity unchanged if $\Omega_{0,1}$
o	$\omega\,\psi$	2	oe→eo (net $o \to o$)	Parity toggle if the ω step uses the $(\mathrm{o},1)$ row (ω_1)
o	$\Omega\,\omega\,\psi$	3	oo→oe→eo (net $o \to o$)	Guaranteed parity toggle via either Ω_2 or ω_1

Appendix A′ Mod-3 steering (valuation & residue control)

We strengthen the steering toolkit by showing that, in addition to toggling B_W mod 2 and raising $v_2(A_W)$, one can *steer* B_W *to any desired residue modulo* 3 while remaining in the same family. This closes the divisibility-by-3 gap in the exact-lifting step.

Lemma 56 (Mod-3 steering lemma)**.** *Let W be an admissible word with affine form $x_W(m) = 6(A_W m + B_W) + \delta_W$, where $A_W = 3 \cdot 2^{\alpha(W)}$ and $\delta_W \in \{1,5\}$. For each family $s \in \{\mathrm{e},\mathrm{o}\}$ there exist short same–family gadgets $P_s^{(r)}$ $(r \in \{0,1,2\})$ such that*

$$x_{W\cdot P_s^{(r)}}(m) = 6(A'm + B'_s) + \delta_W, \qquad v_2(A') > v_2(A_W), \qquad B'_s \equiv r \pmod 3.$$

In particular, one can raise $v_2(A)$ and set B mod 3 *arbitrarily while preserving the terminal family δ_W.*

Proof. We use the unified p=0 rows in Table 5 and the parameter table (Table 4). If a same–family row with parameters (α, k, δ) is appended to a word with affine form $6(Am + B) + \delta$, the new slope is $A' = A \cdot 2^{\alpha}$ and the new intercept is

$$B' \;\equiv\; 2^{\alpha} B + k \pmod 3,$$

because $x \mapsto 6(2^{\alpha} m + k) + \delta$ contributes 2^{α} on the m–slope and adds k to the intercept, and $2^{\alpha} \equiv 1$ or 2 modulo 3 depending on α.

Family e *(type* ee*, $\delta = 1$).* From Table 4, the ee rows have

$$(\alpha, k) \in \{(2,0),(4,6),(6,46)\}.$$

Modulo 3 this yields $2^{\alpha} \equiv 1$ for all three and $k \equiv 0, 0, 1$, respectively. Hence a single ee step realizes

$$B' \equiv B \quad \text{or} \quad B' \equiv B + 1 \pmod 3.$$

Thus in at most two ee steps we can set $B' \equiv r$ for any prescribed $r \in \{0,1,2\}$. Each step multiplies A by $2^{\alpha} \geq 4$, so $v_2(A)$ strictly increases.

Family o *(type* oo*, $\delta = 5$).* From Table 4, the oo rows have

$$(\alpha, k) \in \{(5,8),(3,4),(1,1)\}.$$

Modulo 3 we have $2^{\alpha} \equiv 2$ for all three, and $k \equiv 2, 1, 1$, respectively. Therefore any single oo step implements one of the affine maps

$$\phi_1(B) = 2B + 1, \qquad \phi_2(B) = 2B + 2 \qquad (\mathrm{mod}\, 3).$$

The subgroup of affine maps of $\mathbb{Z}/3\mathbb{Z}$ generated by $\{\phi_1, \phi_2\}$ is all of $\mathrm{AGL}_1(\mathbb{F}_3)$; concretely, from any starting B mod 3 one reaches any target residue in at most two steps (e.g. $\phi_1 \circ \phi_1(B) = B$, $\phi_2 \circ \phi_1(B) = B + 1$, etc.). Each oo step multiplies A by $2^{\alpha} \geq 2$, so $v_2(A)$ strictly increases.

Combining the family–wise controls gives the claim: in family e use at most two ee steps; in family o use at most two oo steps (choosing which oo row to realize ϕ_1 or ϕ_2). In all cases the terminal family (hence δ_W) is preserved and $v_2(A)$ increases. □

Table 17. Same–family rows: residues of 2^{α} and k modulo 3 (at p=0).

Row	(s,j)	α	$2^{\alpha}\,(\mathrm{mod}\,3)$	$k=(\beta+c)/9\,(\mathrm{mod}\,3)$
Ψ_0	$(\mathrm{e},0)$	2	1	0
Ψ_1	$(\mathrm{e},1)$	4	1	0
Ψ_2	$(\mathrm{e},2)$	6	1	1
Ω_0	$(\mathrm{o},0)$	5	2	2
Ω_1	$(\mathrm{o},1)$	3	2	1
Ω_2	$(\mathrm{o},2)$	1	2	1

Constructive gadgets (runtime recipes). Let the current terminal family of W be s and write $B := B_W \bmod 3$.

- **If** $s = \mathrm{e}$ (want $B' \equiv r$):
  (1) If $B \equiv r$, append Ψ_0 (does not change B; raises $v_2(A)$).
  (2) Else append Ψ_2 once: $B \mapsto B+1$; if still not r, append Ψ_2 again.
- **If** $s = \mathrm{o}$ (want $B' \equiv r$):
  (1) If $B \equiv r$, append Ω_1 (keeps flexibility for later; raises $v_2(A)$).
  (2) Else compute $d := r - B \pmod 3$.
    - If $d \equiv 1$: append Ω_1 then Ω_0; effect $B \mapsto 2B+1 \mapsto 2(2B+1)+2 \equiv B+1$.
    - If $d \equiv 2$: append Ω_0 then Ω_1; effect $B \mapsto 2B+2 \mapsto 2(2B+2)+1 \equiv B+2$.

Corollary (exact divisibility condition). Let $x_W(m) = 6(A_W m + B_W) + \delta_W$ with $A_W = 3\cdot 2^{\alpha(W)}$. Given any target odd $x \equiv \delta_W \pmod 6$, by Lemma 56 we may replace W by $W^\star$ so that

$$B_{W^\star} \equiv \frac{x-\delta_W}{6} \pmod 3.$$

Then $A_{W^\star} \mid \big(\frac{x-\delta_W}{6} - B_{W^\star}\big)$ if and only if $2^{\alpha(W^\star)} \mid \big(\frac{x-\delta_W}{6} - B_{W^\star}\big)$, which can always be enforced by further same–family padding (raising $v_2(A)$). Hence there exists $m \in \mathbb{Z}$ with $x_{W^\star}(m) = x$.

Example 57 (Mod-3 steering then 2-adic lifting to 3071 mod 3072)**.** Target residue:

$$r' \equiv 3071 \pmod{3072}, \qquad 3071 \equiv 5 \pmod 6 \text{ (odd family)}.$$

Start with the one-step word $W = \psi$ (row $(\mathrm{e},0)$ in the unified table):

$$x_W(m) = 6\big(A\,m + B\big) + \delta, \qquad \psi:\ \delta = 5,\ A = 16,\ B = 0.$$

(1) Mod-3 steering. Set

$$t := \frac{r'-\delta}{6} = \frac{3071-5}{6} = 511.$$

The mod-3 solvability criterion is $B \equiv t \pmod 3$. Since $t \equiv 1 \pmod 3$ and $B \equiv 0 \pmod 3$ for ψ, append one odd-family step Ω_1, which acts as $B \mapsto 2B+1\,(\mathrm{mod}\,3)$. Thus $B \equiv 1 \pmod 3$ after Ω_1, and the mod-3 condition is aligned.

(2) Divide by 3 and set the 2-adic congruence. After ψ then Ω_1, the accumulated exponent is $\alpha_{\mathrm{tot}} = 4+3 = 7$. With $B \equiv 1 \pmod 3$ (take $B = 1$ concretely),

$$2^{\alpha_{\mathrm{tot}}} m \equiv \frac{t-B}{3} = \frac{511-1}{3} = 170 \pmod{2^{K-1}}, \qquad K = 10 \Rightarrow 2^{K-1} = 512.$$

So $2^7 m \equiv 170 \pmod{512}$.

(3) Ensure 2-adic solvability by padding. A congruence $2^{\alpha_{\mathrm{tot}}} m \equiv R \pmod{2^{K-1}}$ is solvable iff $2^{\min(\alpha_{\mathrm{tot}},\,K-1)} \mid R$. Here $\min(7,9) = 7$ but $170 \not\equiv 0 \pmod{128}$. Use same-family odd padding $(\Omega_0, \Omega_1, \Omega_2)$ to:

- keep $B \equiv 1 \pmod 3$ (mod-3 steering), and

- raise $v_2(A)$ while shifting the integer B so that
$$\frac{t-B}{3} \equiv 0 \pmod{512} \iff B \equiv t \pmod{1536} \iff B \equiv 511 \pmod{1536}.$$

Once $B \equiv 511$ (mod1536), the right-hand side becomes $0 \pmod{512}$, and a solution exists (e.g. $m \equiv 0 \pmod{512}$).

Conclusion. With the sequence ψ followed by Ω_1 and a short odd-family padding that sets $B \equiv 511$ (mod1536) (while increasing v_2 of the slope), we obtain
$$x_W(m) \equiv 3071 \pmod{3072},$$
and every step is certified by the identity $3x' + 1 = 2^{\alpha}x$ (hence $U(x') = x$) from the unified table.

Appendix B: Residue-by-residue parity gadgets mod 54 (certificate)

Table 18. Certified parity–flip gadgets by odd residue class modulo 54.

Residue x mod 54	Family s	$j = \lfloor x/6 \rfloor$ mod 3	Gadget (tokens)
Family e *(classes* $\equiv 1 \pmod 6$*):*			
1	e	0	ψ; then **if** new j=1: ω_1 then ω; **if** new j=2: Ω_2 then ω
7	e	1	same recipe as for 1
13	e	2	same recipe as for 1
19	e	0	same recipe as for 1
25	e	1	same recipe as for 1
31	e	2	same recipe as for 1
37	e	0	same recipe as for 1
43	e	1	same recipe as for 1
49	e	2	same recipe as for 1
Family o *(classes* $\equiv 5 \pmod 6$*):*			
5	o	0	Ω; **if** new j=1: ω_1 then ψ; **if** new j=2: Ω_2 then ω then ψ
11	o	1	ω_1 then ψ
17	o	2	Ω_2 then ω then ψ
23	o	0	same recipe as for 5
29	o	1	same recipe as for 11
35	o	2	same recipe as for 17
41	o	0	same recipe as for 5
47	o	1	same recipe as for 11
53	o	2	same recipe as for 17

Appendix A. Appendix C: Mechanical checks and lifted witnesses

Audit protocol (informal). A simple script can (i) verify each row formula $x' = 6(2^{\alpha_p}u + k^{(p)}) + \delta$ at sampled inputs, (ii) check routers $j = \lfloor x/6 \rfloor$ mod 3 match the table choice, (iii) confirm $U(x') = x$ for the forward accelerated map, and (iv) validate lifted witnesses at higher moduli (M_K) by direct congruence checks.

Lifted witnesses at $M_4 = 48$ from $M_3 = 24$. Each row lists a residue r mod 24, a short admissible tail producing r' mod 48, and a one-line justification (pinning or solved congruence). We keep representatives compact; the earlier examples show the full router/floor arithmetic.

Example 58 (Explicit calculation for 17 mod 24 $\to$ 41 mod 48)**.** This is the two-step tail $\omega_1 \to \psi_2$ with the router/floor arithmetic spelled out in the main text (see the worked example in Section 21).

Table 19. Lifted witnesses from 24 to 48. Each tail is read from the p=0 table and obeys routing.

r mod 24	r' mod 48	Tail	Reason
17	41	$\omega_1 \to \psi_2$	Congruence regime for ψ_2: $x' = 24m + 17$, choose class with m odd; admissibility shown in Ex. 58.
13	13	Ψ_1	Pinning: $\alpha = 4 \geq K = 4$ gives $x' \equiv 6k + \delta \equiv 37 \equiv 13 \pmod{48}$.
23	23 or 47	Ω_2 or $\omega_1 \to \psi_2$	Ω_2 yields $x' = 12m + 11$ so parity classes hit $11, 23$; a cross-family two-step can target 47 as needed.
7	7 or 31	ω_1 or $\omega_1 \to \psi_2$	As above: single-step parity split, or two-step tail for the other odd residue.

Appendix D: Witness tables mod 48 amd 96

Table 20. Witness construction template modulo 48 (with $M_4 = 48$). For each odd residue $r' \equiv 1, 5 \pmod 6$, pick a word W whose terminal family matches $r' \bmod 6$. Write its affine form as $x_W(m) = 6(A_W m + B_W) + \delta_W$ (with $A_W = 3 \cdot 2^{\alpha(W)}$). Solve the linear congruence $A_W m \equiv \frac{r' - \delta_W}{6} - B_W \pmod{2^3}$ (i.e. mod 8), and set $x := x_W(m)$, which then satisfies $x \equiv r'$ (mod 48) and $U(x) = \cdots = 1$ along W.

$r' \pmod{48}$	Family	Choice of W (terminal δ_W)	Solve for $m \pmod 8$
$1, 7, 13, 19, 25, 31, 37, 43$	e	e.g. Ψ, $\psi\omega\psi$, etc. (δ_W=1)	$A_W m \equiv \frac{r'-1}{6} - B_W \pmod 8$
$5, 11, 17, 23, 29, 35, 41, 47$	o	e.g. ψ, $\psi\Omega$, etc. (δ_W=5)	$A_W m \equiv \frac{r'-5}{6} - B_W \pmod 8$

Table 21. Selected concrete witnesses modulo 48. Each row shows a word W, its closed form $x_W(m)$, and a solved congruence for some r' mod 48.

$r' \pmod{48}$	Word W	Closed form $x_W(m)$	One solution for m
5	ψ	$x(m) = 96m + 5$	any m (always $5 \pmod{48}$)
13	$\psi\,\omega$	$x(m) = 6(3 \cdot 2^5 m + B) + \delta$ (affine)	$m \equiv m_0 \pmod 8$ (solve $Am \equiv \frac{13-\delta}{6} - B$)
23	$\psi\,\omega\,\psi\,\Omega$	affine as above	$m \equiv m_0 \pmod 8$
29	$\psi\,\Omega$	$x(m) = 192m + 53$	$192m + 53 \equiv 29 \Rightarrow 0 \cdot m \equiv -24$ (no sol.)[2]
41	Ω (from an o start)	$x(m) = 192m + 53$	always $5 \pmod{48}$; add an $o \to o$ steering gadget to shift to 4

Table 22. Witness construction template modulo 96 (with $M_5 = 96$). For each odd residue $r' \equiv 1, 5 \pmod 6$, pick a word W whose terminal family matches $r' \bmod 6$, write $x_W(m) = 6(A_W m + B_W) + \delta_W$, then solve $A_W m \equiv \frac{r' - \delta_W}{6} - B_W \pmod{2^4}$ (i.e. mod 16), and set $x := x_W(m)$ to obtain $x \equiv r' \pmod{96}$.

$r' \pmod{96}$	Family	Choice of W (terminal δ_W)	Solve for $m \pmod{16}$
$1, 7, \ldots, 89$ (odd $\equiv 1$)	e	e.g. Ψ, $\psi\omega\psi$, steering as needed	$A_W m \equiv \frac{r'-1}{6} - B_W \pmod{16}$
$5, 11, \ldots, 95$ (odd $\equiv 5$)	o	e.g. ψ, $\psi\Omega$, steering as needed	$A_W m \equiv \frac{r'-5}{6} - B_W \pmod{16}$

Appendix E: Derivation of the identity $3x'_p + 1 = 2^{\alpha+6p}x$

Lemma 57 (Forward identity for a lifted row)**.** *Fix a row with parameters* $(\alpha, \beta, c, \delta)$ *and a column–lift* $p \geq 0$*. Define*

$$F(p,m) \;=\; \frac{(9m\,2^{\alpha}+\beta)\,64^{\,p}+c}{9}, \qquad x'_p \;=\; 6F(p,m)+\delta,$$

and write the odd input as $x = 18m + 6j + p_6$ *with* $j \in \{0,1,2\}$ *and* $p_6 \in \{1,5\}$*. Assuming the per–row design relations*

$$\beta \;=\; 2^{\alpha-1}(6j+p_6), \qquad c \;=\; -\frac{3\delta+1}{2},$$

one has the identity

$$3x'_p+1 \;=\; 2^{\alpha+6p}\,x.$$

Proof. By definition,

$$x'_p \;=\; 6\left(2^{\alpha+6p}m+\frac{\beta\,64^{\,p}+c}{9}\right)+\delta \quad\Longrightarrow\quad 3x'_p+1 \;=\; 18\cdot 2^{\alpha+6p}m \;+\; \Big(18\cdot\tfrac{\beta\,64^{\,p}+c}{9}+3\delta+1\Big).$$

Simplify the bracket:

$$18\cdot\frac{\beta\,64^{\,p}+c}{9}+3\delta+1 \;=\; 2\beta\,64^{\,p} \;+\; (2c+3\delta+1).$$

With $c = -(3\delta+1)/2$ the constant cancels: $2c+3\delta+1=0$. Hence the bracket reduces to

$$2\beta\,64^{\,p} \;=\; 2\cdot 2^{\alpha-1}(6j+p_6)\cdot 64^{\,p} \;=\; 2^{\alpha}(6j+p_6)\cdot 2^{6p} \;=\; 2^{\alpha+6p}(6j+p_6).$$

Therefore

$$3x'_p+1 \;=\; 18\cdot 2^{\alpha+6p}m \;+\; 2^{\alpha+6p}(6j+p_6) \;=\; 2^{\alpha+6p}\big(18m+6j+p_6\big) \;=\; 2^{\alpha+6p}x,$$

as claimed. □

Remark (Integrality). Since $64 \equiv 1 \pmod 9$, one has $\beta\,64^{\,p}+c \equiv \beta+c \pmod 9$. Each row in Table 4 satisfies $\beta + c \equiv 0 \pmod 9$, so $F(p,m) \in \mathbb{Z}$ for all $p \geq 0$.

Example 59. For row $(\mathrm{o},1)$ (ω_1) the table gives $\alpha = 1,\ \beta = 11,\ c = -2,\ \delta = 1$. Then $F(p,m) = 2^{1+6p}m + \frac{11\cdot 64^{\,p}-2}{9}$ and the lemma yields $3x'_p + 1 = 2^{1+6p}x$.

Appendix F: Code and Data Availability

A reference implementation of the unified inverse table, the word evaluator, and the example generators is archived at Zenodo DOI: 10.5281/zenodo.17352096 and mirrored at github.com/kisira/collatz.

Appendix F: Reproducibility Details

Environment. The code is pure Python 3 (standard library + `pandas` for CSV I/O). A minimal setup is:

```
python -m venv .venv
. .venv/bin/activate
pip install -r requirements.txt
\cite{BernsteinLagarias1996}\cite{BernsteinLagarias1996}
```

[BL96]

Stepwise identity checks ($U(x') = x$). To verify that each row satisfies $3x' + 1 = 2^{\alpha+6p}x$ and that the word evaluator returns to the parent under U:

```
python3 tools/check_rows.py            # verifies all rows and their p-lifts
python3 tools/evaluate_word.py --word psi,Omega,omega,psi --x0 1 --csv out.csv
```

This writes a per-step trace (indices s, j, m, formulas, and forward checks).

Regenerating witness tables. To regenerate witnesses mod 24, 48, and 96 (as used in the paper):

```
python3 tools/make_witnesses.py --mod 24  --out tables/witnesses_mod24.csv
python3 tools/make_witnesses.py --mod 48  --out tables/witnesses_mod48.csv
python3 tools/make_witnesses.py --mod 96  --out tables/witnesses_mod96.csv
```

Recreating examples in the paper. Each example in Sections 12–30 can be reproduced with:

```
python3 tools/replay_example.py --name ex2
```

which emits a CSV trace with the certified step identities and indices.

Generate the word for an odd number. To genereate a word for say 497. Or any other odd number.

```
python3 tools/calculate_word.py 497 --json-out 497_word.json
```

Row consistent reverse. To reverse an odd number any number of steps.

```
python reverse_construct.py --mode one --y 43 --csv reverse_43.csv
python reverse_construct.py --mode chain --y 497 --stop 1 --csv chain_497_to_1.csv
```

Archival guarantee. The Zenodo snapshot (DOI above) freezes the exact source corresponding to tag `v1.0` and commit `<hash>`, ensuring long-term reproducibility even if the development branch evolves.

Appendix G: Formalization index

Paper result	Label	Coq reference
One-step composition with floor	`\label{lem:one-step-floor}`	`CollatzFramework.v: compose_one_correct`
Last-row congruence targeting	`\label{lem:last-row-p}`	`LiftingWitnesses.v: last_row_congruence_targeting_nat`
Base witnesses (mod 24) coverage	`\label{sec:base-coverage}`	`LiftingWitnesses.v: base_witness_coverage`
Linear 2-adic step ("pinning")	`\label{lem:linear-2adic}`	`CollatzFramework.v: linear_2adic_pinning`
Routing compatibility (prefix)	`\label{lem:routing-compat-prefix}`	`CollatzFramework.v: routing_compat_prefix`

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