Topology of Piecewise-Linear Manifolds

Jim Fowler

Lecture 5
Summer 2010

Why do you keep hitting your hand with a hammer?

Where are we?

Granby
116
Boul. Laurier
141
1 km
20
OUEST
Montréal
211
SUD
St-Valérien-
de-Milton
143
SORTIE
45
km/h

Where are we?

Objects

Where are we?

Objects

PL manifolds?

Where are we?

Objects

PL manifolds?

simplicial complexes

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simplicial complexes

Maps

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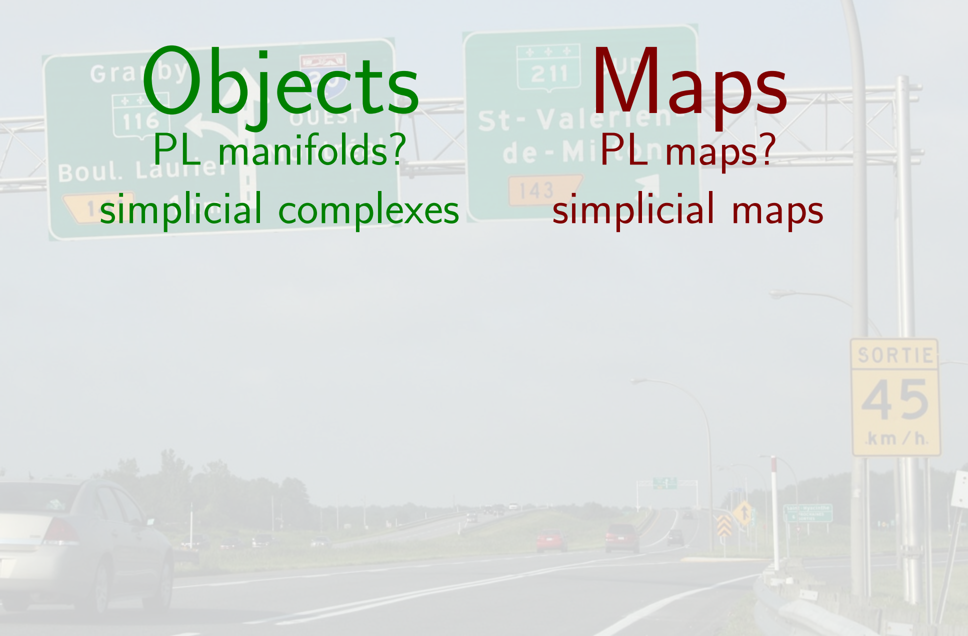

Objects

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Where are we?

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Invariants

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Maps

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simplicial maps

Invariants

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Where are we?

Objects

PL manifolds?

simplicial complexes

Maps

PL maps?

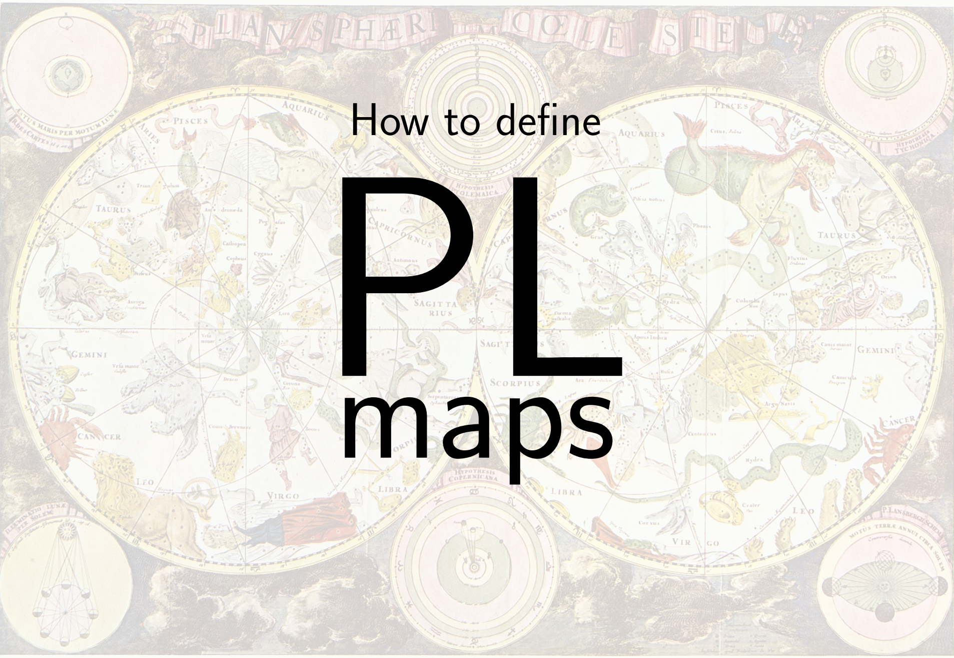
simplicial maps

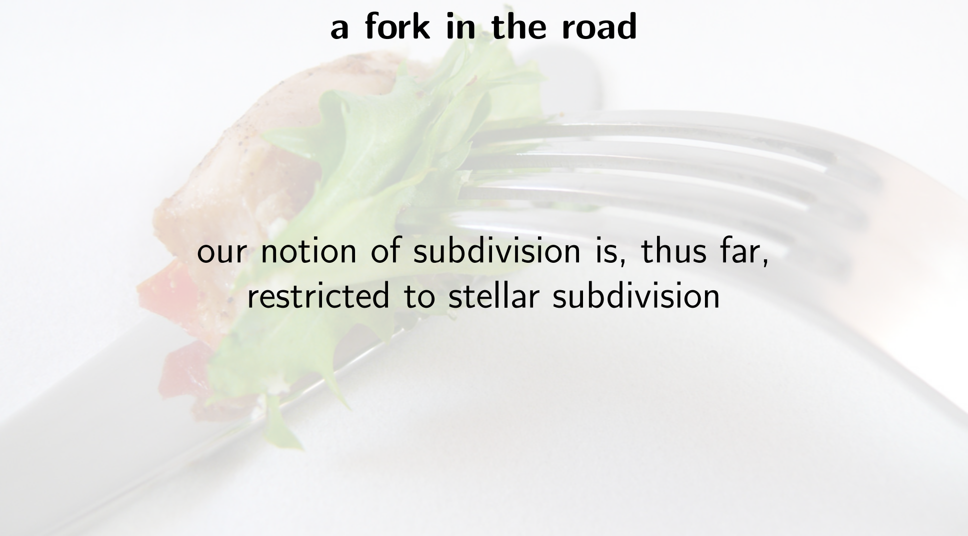
Invariants

χ b_0

How to define

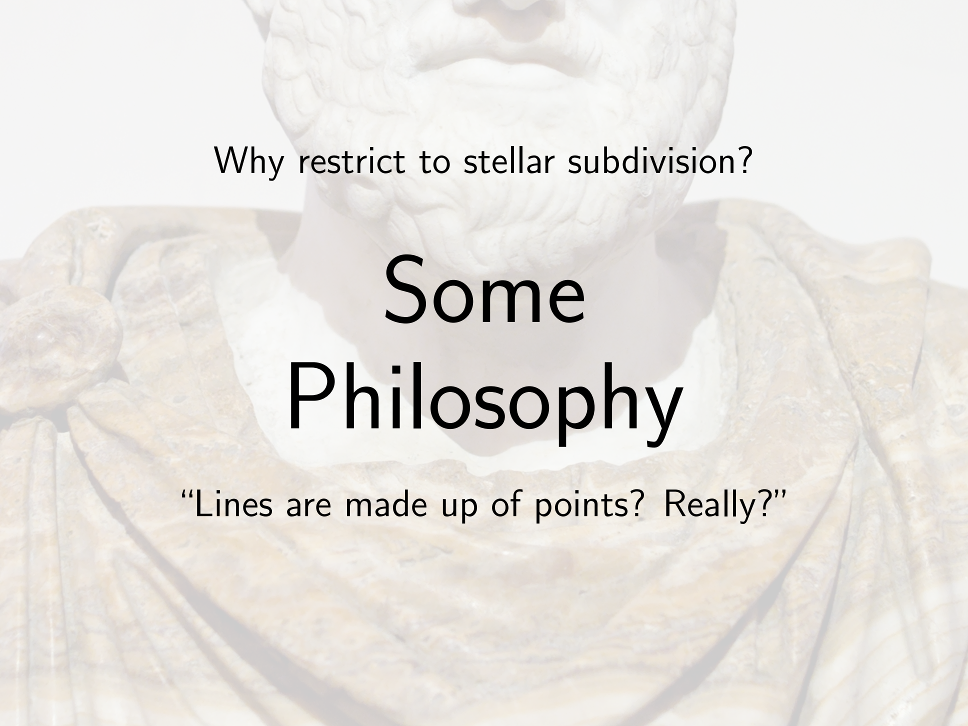
PL maps





a fork in the road

our notion of subdivision is, thus far,
restricted to stellar subdivision



Why restrict to stellar subdivision?

Some Philosophy

“Lines are made up of points? Really?”

State of the art

Theorem

Two n-dimensional simplicial complexes
are PL homeomorphic
if and only if
they are stellar equivalent.

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A PL map $f : K \to L$ is a simplicial map from a subdivision of K to a stellar subdivision of L.

Warning: The subdivision of K needn't be stellar.

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Two n-dimensional simplicial complexes are PL homeomorphic if and only if they are stellar equivalent.

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Warning: The subdivision of K needn't be stellar.
Question: What does "subdivision" mean?

The combinatorial part

- simplicial complex
- join
- link
- star
- Euler characteristic
- PL manifold: a complex K the link of each vertex PL homeomorphic (meaning stellar equivalent) to a sphere.

The non-combinatorial part

The notion of PL map is not quite general enough:
a simplicial map from a stellar subdivision of K
to a stellar subdivision of L is a PL map,
but we will need more general maps.

To do this, we will place our simplexes in $\mathbb{R}^n$.

$$\mathbb{R}^n$$

The join of $A, B \subset \mathbb{R}^n$ is

$$\{\lambda a + (1 - \lambda) b : a \in A, b \in B, \lambda \in [0, 1]\}$$

Polyhedron

$P \subset \mathbb{R}^n$ is a **polyhedron**
if for each $p \in P$
there is a neighborhood $N \ni p$
so that $N = p * L$,
with L closed and bounded.

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N is called a **closed star** around p.
L is a **link** of p.

Piecewise linear map

Let P, Q be polyhedra.
$f : P \longrightarrow Q$ is a PL map
if each point $p \in P$
has a closed star $N = p * L$
so that $f(\lambda p + (1-\lambda)x) = \lambda f(p) + (1-\lambda) f(x)$
for $x \in L$ and $\lambda \in [0,1]$.

In short, it locally maps conical rays to conical rays.

Next steps

Next steps

- New constructions

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  connected sum

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- New constructions
  connected sumand
  Cartesian product, using polyhedra

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  orientability, w_1
- New kind of equivalence
  simple homotopy equivalence

$$\#$$

connected sum

Connected sum

Definition

Suppose K and L are two n-manifolds.

Let K' be K with one n-simplex removed.
$\quad\; L' \quad\;\; L$

Define $K\#L = K' \cup_{\partial\Delta^n} L'$.

Connected sum

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L' L

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Problem

What is $T^2 \# T^2$?

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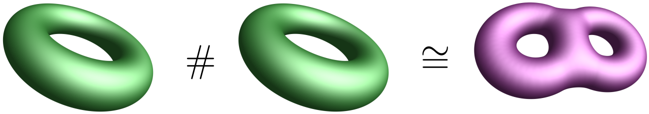
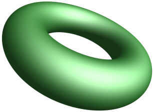
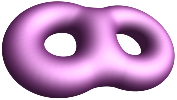
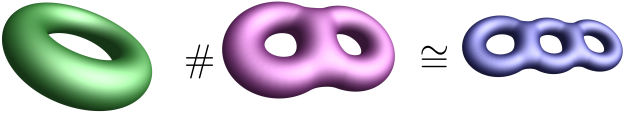
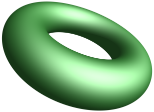
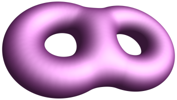
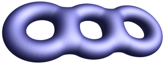
Problem

What is $S^2 \# T^2$?

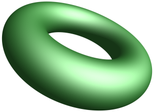
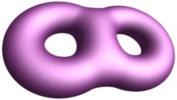
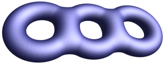


 $\#$  $\cong$



 $\#$  $\cong$ 

In general, $\Sigma_g =$ genus g surface,

 $\#$  $\cong$ 

In general, $\Sigma_g =$ genus g surface,

and $T^2 \# \Sigma_g = \Sigma_{g+1}$.

Covering map

Definition

$f : M \to N$ is an n-fold **covering map**
if for every k-simplex $\tau \in N$,
$f^{-1}(\tau)$ consists of n copies of a k-simplex.

Gift-wrapping surfaces

Problem

When is there an n-fold covering map from a genus g surface to a genus g' surface?

Gift-wrapping surfaces

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When is there an n-fold covering map
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Theorem

If $f : M \to N$ is an n-fold covering map
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Problem

What is $\chi(\text{genus } g \text{ surface})$?

$$\times$$

Cartesian product

Products, geometrically

Definition

Suppose $K \subset \mathbb{R}^n$, and $L \subset R^m$.
The product of K and L is

$$K \times L = \{(k, \ell) \in \mathbb{R}^{n+m} : k \in K, \ell \in L\}.$$

Products, geometrically

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Could we do this for a simplicial complex?

Product of simplexes

How do we take a product of Δ^1 and Δ^1?

Product of simplexes

How do we take a product of Δ^1 and Δ^2?

Product of simplexes

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Product of simplexes

How do we take a product of Δ^2 and Δ^2?
Hard to see, so let's take a step back.

Vertices of $\Delta^n \times \Delta^m$ are pairs
(i, j) with $0 \leq i \leq n+1$ and $0 \leq j \leq m+1$.

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A simplex of $\Delta^n \times \Delta^m$ is [redacted]

Some counting problems

$\Delta^1 \times \Delta^n$ has $(n+1)!$ triangulations.

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So maybe it would be nice if we could handle these cases without resorting to simplexes.

Product of polyhedra is a polyhedron

P, Q polyhedra.
Is $P \times Q$ a polyhedron?

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Need neighborhood of $(p, q) \in P \times Q$.

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Need neighborhood of $(p, q) \in P \times Q$.

$p \in P$ has a neighborhood $p * L_p$
$q \in Q$ has a neighborhood $q * L_q$

Product of polyhedra is a polyhedron

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Is $P \times Q$ a polyhedron?

Need neighborhood of $(p, q) \in P \times Q$.

$p \in P$ has a neighborhood $p * L_p$
$q \in Q$ has a neighborhood $q * L_q$

Use $(p, q) * (L_p * L_q)$.

Some 4-manifolds

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$$S^3 \times S^1$$

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$$S^2 \times S^2$$

Some 4-manifolds

$$S^3 \times S^1$$
$$S^2 \times S^2$$
$$\Sigma_g \times \Sigma_h$$

w_1

orientability

The torus versus the Klein bottle

Since $\chi(T^2) = \chi(K) = 0$,
we can't distinguish T^2 and K using χ.

The torus versus the Klein bottle

Since $\chi(T^2) = \chi(K) = 0$,
we can't distinguish T^2 and K using χ.

We need a new invariant.

Orientation

An orientation of a simplex is
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If M admits an orientation, $w_1(M) = 0$.
Otherwise, $w_1(M) = 1$.

Orientability for surfaces

Theorem

A surface Σ is not orientable
iff Σ contains a Möbius strip.

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What is $w_1(K)$?

Problem

Is w_1 a PL homeomorphism invariant?

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

Distinguishing other manifolds?

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No

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

No...

We need more invariants,

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

No. . .

We need more invariants,
or a looser notion of “the same.”

Simplicial Collapse

Principal Simplexes

Definition

Let K be a complex, and
$\sigma \in K$ a simplex.
Call σ a **principal simplex**
if the only simplex containing σ
is σ itself
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Problem

Does every complex have a principal simplex?

Free faces

Definition

Let K be a complex,
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Elementary simplicial collapse

Definition

Let L and $K = L \cup \mathrm{cl}\{\sigma, \tau\}$ be complexes

If σ is a principal simplex of K, and
τ is a free face of σ, then
L is an **elementary simplicial collapse** of K.

Simplicial collapse

Definition

Let $K_1, K_2, \ldots, K_n$ be complexes, with K_{i+1} an elementary simplicial collapse of K_i.

Call K_n a **simplicial collapse** of K_1, and write $K_1 \searrow K_n$.

Call K_1 a **simplicial expansion** of K_n, and write $K_n \nearrow K_1$.

Simple homotopy equivalence

Definition

K is **simple homotopy equivalent** to L
(sometimes abbreviated s.h.e.)
if you can reach transform K into L
via a sequence of

- PL homeomorphisms,
- simplicial collapses,
- simplicial expansions.

In this case, we write $K \searrow\!\nearrow L$.

Next time

UNKNOTTING SPHERES IN FIVE DIMENSIONS

BY E. C. ZEEMAN

Communicated by S. Eilenberg, December 26, 1959

Given a semi-linear embedding of S^2 in euclidean 5-space, we show that it is unknotted.

Join it up to a vertex V in general position. If the cone VS^2 is non-singular we are finished. Otherwise, for dimensional reasons, there are at most a finite number of singularities, where just two points of S^2 are collinear with V. Let's have V away on one side, so that at each singularity we can call one point "near" and the other point "far." Now separate the near and far points by an equator S^1, so that all the near points lie in the northern hemisphere A, and all the far points lie in the southern hemisphere B.

Let $\hat{S}^2$ be the sphere $VS^1 \cup B$. Then $\hat{S}^2$ is equivalent to S^2, because they differ by the boundary of the ball VA, whose interior does not meet them. But $\hat{S}^2$ is unknotted because it bounds, and does not meet the interior of, the ball VB. Hence S^2 is unknotted.

REMARK 1. The argument generalizes to unknotting S^n in k-space, $k \geq (3/2)(n+1)$.

Next time

- Regular neighborhoods
- Knotting S^1 in S^3
- Unknotting S^1 in S^4
- How to unknot with simplicial collapse

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