Facts about limits.

Definition 1 (Formal)**.** We say $\lim_{n\to\infty} a_n = L$ if

for all $\epsilon > 0$,
there exists $K \in \mathbb{N}$,
so that if $n \geq K$,
then $|a_n - L| < \epsilon$.

Definition 2 (Intuitive)**.** We say $\lim_{n\to\infty} a_n = L$ if

however close we want to be,
there's a place we can go,
so that beyond that place,
we are that close.

Proposition 3 (Limits are unique)**.** *If* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} a_n = M$*, then* $L = M$*.*

Proposition 4. *If the sequence* a_n *converges, then* a_n *is bounded.*

Corollary 5. *If the sequence* a_n *is not bounded, then* a_n *diverges.*

Proposition 6. *If* $\lim_{n\to\infty} a_n = L$*, and* $c \in \mathbb{R}$*, then* $\lim_{n\to\infty} c \cdot a_n = c \cdot L$*.*

Proposition 7. *If* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$*, then* $\lim_{n\to\infty} a_n + b_n = L + M$*.*

Proposition 8. *If* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$*, then* $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$*.*

Proposition 9. *If* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$*, and* $b_n \neq 0$ *and* $M \neq 0$*, then* $\lim_{n\to\infty} a_n/b_n = L/M$*.*

Proposition 10. *If* $\lim_{n\to\infty} a_n = L$ *and* $m \in \mathbb{N}$*, then* $\lim_{n\to\infty} a_{n+m} = L$*.*

Proposition 11. *If* $\lim_{n\to\infty} a_n = L$*, and* b_n *is a sequence differing from* a_n *in finitely many terms, then* $\lim_{n\to\infty} b_n = L$*.*

Theorem 12. *If* a_n *is a nondecreasing bounded above sequence, then* a_n *converges.*

Theorem 13. *If* a_n *is a nonincreasing bounded below sequence, then* a_n *converges.*

Theorem 14 (Squeezing theorem)**.** *If* a_n*,* b_n*, and* c_n *are sequences of real numbers, and for all* $n \in \mathbb{N}$*, we have* $a_n \leq b_n \leq c_n$*, and* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} c_n = L$*, then* $\lim_{n\to\infty} b_n = L$*.*

Theorem 15 (Sequences and continuity)**.** *If a function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous at* L*, and* $\lim_{n\to\infty} a_n = L$*, then* $\lim_{n\to\infty} f(a_n) = f(L)$*.*

Example 16. For a real number $x > 0$,

$$\lim_{n\to\infty} x^{1/n} = 1.$$

Example 17. For a real number x with $-1 < x < 1$,

$$\lim_{n\to\infty} x^n = 0.$$

Example 18. For a real number $x > 0$,

$$\lim_{n\to\infty} \frac{1}{n^x} = 0.$$

Example 19. For $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Example 20. $\lim_{n\to\infty} \dfrac{\log n}{n} = 0.$

Example 21. $\lim_{n\to\infty} n^{1/n} = 0.$

Example 22. For $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$