Dissection Problems

Math 205

May 27, 2009

Polygons

Problem 1. Let P and Q be triangles having the same area; show that P can be cut into smaller triangles, which can be rearranged to form Q.

Problem 2 (Bolyai-Gerwien Theorem)**.** Let P and Q be polygons having the same area; show that P can be cut into smaller polygons, which can be rearranged to form Q.

Remark. It is (surprisingly!) possible to partition $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ into about 10^{50} subsets, translate (without rotations) these subsets, and form a square having the same area. This was proved by Miklós Laczkovich in 1990.

Banach-Tarski Paradox

Definition 3. The **free group on two generators**, denoted F_2, consists of all words written using

$$a, \quad b, \quad a^{-1}, \quad b^{-1}$$

where a never appears next to a^{-1}, and b never appears next to b^{-1}. We can **multiply** two words by concatenating them, and cancelling a's with a^{-1} and b with b^{-1}; the **empty word** is (ironically?) written e; multiplying a word w by e leaves w unchanged (so e is the **identity**).

Problem 4. For every word $w \in F_2$, does there exist a word $w' \in F_2$ so that $ww' = e$?

Problem 5. For three words $w_1, w_2, w_3 \in F_2$, is it the case that $(w_1 w_2) w_3 = w_1 (w_2 w_3)$?

Remark. If you answered **YES!** to the previous two quesions, you have shown that F_2 is a **group**—a set with an binary associative operation having an identity and inverses.

Definition 6. The set $S(\ell)$ consists of words in F_2 starting with the letter ℓ.

Problem 7. Show that $F_2 = \{e\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$.

Definition 8. Let S be a set of words in F_2, and ℓ a letter. Define

$$\ell \cdot S = \{\ell \cdot s : s \in S\}.$$

Problem 9. "Paradoxically" prove that $F_2 = a \cdot S(a^{-1}) \cup S(a)$.
Similarly, prove that $F_2 = b \cdot S(b^{-1}) \cup S(b)$.

Remark. Here we see the beginnings of a paradoxical decomposition: we have divided F_2 into five sets, and using four of them, built two copies of F_2. If only there were a way to convert our paradoxical decomposition of F_2 into a paradoxical decomposition of the sphere...

Problem 10. Let

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}.$$

Both A and B are linear maps $\mathbb{R}^3 \to \mathbb{R}^3$. Are these linear maps invertible? Are they **rotations**?

Problem 11. Any word in F_2 can be rewritten as a product of the matrices A, B, A^{-1}, and B^{-1}; that is, there is a function $\phi : F_2 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$. For which $w \in F_2$ does it happen that $\phi(w)$ is the identity matrix?

Definition 12. Write S^2 for the two-sphere, i.e., the set of points $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$.

Problem 13. The linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ can be restricted to give a function $A : S^2 \to S^2$. The analogous statement holds for B, indeed, for any $\phi(w)$. This means that $\phi(w)$ can be regarded as function taking points on the sphere to other points on the sphere.

Definition 14. Let $D \subset S^2$ consist of those points $p \in S^2$ for which there is some $w \in F_2$, with $w \neq e$ but $\phi(w)(p) = p$. These are the points **fixed by some element** of F_2.

Problem 15. D is countable. *Hint:* if p is fixed by a rotation, then p is on the axis of the rotation; how many points on the sphere lie on a given axis of rotation?

Definition 16. Points $p, q \in S^2 - D$ are in the same **orbit** if there is some $w \in F_2$ so that $\phi(w)(p) = q$. Let R be a set containing a representative from each orbit.

Problem 17. Every point in $S^2 - D$ can be written *uniquely* as $\phi(w)(r)$ for some $w \in F_2$ and some $r \in R$.

Problem 18. Define

$$\begin{aligned} R_a &= \phi(S(a))(R), \\ R_b &= \phi(S(b))(R), \\ R_{a^{-1}} &= \phi(S(a^{-1}))(R), \\ R_{b^{-1}} &= \phi(S(b^{-1}))(R). \end{aligned}$$

Show that $S^2 - D = R_a \cup R_b \cup R_{a^{-1}} \cup R_{b^{-1}}$.
Also show that $S^2 - D = \phi(a)(R_{a^{-1}}) \cup R_a$.
But also show that $S^2 - D = \phi(b)(R_{b^{-1}}) \cup R_b$.

Remark. We have decomposed $S^2 - D$ into four pieces, thrown away two of the pieces, rotated one the remaining pieces by $\phi(a)$, and ended with $S^2 - D$. This is *paradoxical*. Next we will show that $S^2 - D$ can be cut up and rearranged to get S^2.

Problem 19. Pick some point $x \in S^2$ with $x \notin D$.

Let J consist of the **bad angles**—angles θ so that rotation by $n\theta$ (for some $n \in \mathbb{N}$) around axis x takes some point $d \in D$ to some other point in D. Prove J is countable.

Problem 20. Let θ be a good angle (i.e., not in J). Why does a good angle exist? Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation around the axis $x \in S^2$ through angle θ.

Problem 21. Define $D_n = G(G(G(\cdots G(D)$, i.e., n applications of G to D. Prove that all the D_n are disjoint; let $E = \bigcup_{i=0}^{\infty} D_n$.

Problem 22. Show that $G(E) = E - D$.

Problem 23. Show that S^2 can be cut into two pieces, $S^2 - E$ and E, which can be rearranged to form $S^2 - D$.

Remark. Altogether, we have shown that S^2 can be cut into two pieces, which can be rearranged to form $S^2 - D$. We earlier showed that $S^2 - D$ can be cut into four pieces, which can be rearrnaged to make two copies of $S^2 - D$. But each of these copies of $S^2 - D$ can be cut up, make two copies of S^2.

Remark. This decomposition did not even require translation! We take S^2, throw away some pieces, and rotate the remaining pieces (perhaps through other pieces!) into new positions to reform S^2.

Problem 24. Let $B^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$; this is the solid ball. Use the paradoxical decomposition of S^2 to build a paradoxical decomposition of $B^3 - \{(0, 0, 0)\}$.
Hint: Extend radially.

Problem 25. Use a trick (like the one we used to convert $S^2 - D$ into S^2) and some translations to build a paradoxical decomposition of B^3 into two copies of B^3.