More organized facts about limits

Bare hands

Definition. We say $\lim_{n\to\infty} a_n = L$ if

> for all $\epsilon > 0$, there exists $K \in \mathbb{N}$,
> so that if $n \geq K$, then $|a_n - L| < \epsilon$.

We can give explicit ϵ-K arguments to prove

$$\lim_{n\to\infty} \frac{1}{n} = 0, \quad \text{and } \lim_{n\to\infty} x^n = 0 \text{ if } -1 < x < 1.$$

Algebraic operations

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty} (a_n + b_n) = L + M, \quad \lim_{n\to\infty} (a_n - b_n) = L - M, \quad \lim_{n\to\infty} (a_n \cdot b_n) = L \cdot M,$$

and if $M \neq 0$ and all $b_n \neq 0$, then $\lim_{n\to\infty} (a_n/b_n) = L/M$.

Squeezing theorem

If a_n, b_n, and c_n are sequences of real numbers, and for all $n \in \mathbb{N}$, we have $a_n \leq b_n \leq c_n$, and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Continuous functions

If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at L, and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Subsequences

If a_n is a sequence of real numbers, and b_n is a subsequence produced from a_n by throwing away finitely many terms, then a_n converges if and only if b_n converges.

If a_n is a convergent sequence of real numbers, and b_n is a subsequence, then b_n converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Existence without construction

Theorem. *If a_n is a nondecreasing bounded above sequence, then a_n converges.*

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