Problem Set 4 — Piecewise-Linear Topology

Summer 2010 — Jim Fowler

A new equivalence. Problem Set 4 introduces *simple homotopy equivalence*, an equivalence relation between simplicial complexes which is coarser than simplicial isomorphism or PL homeomorphism; Please write down answers to problems labeled with •.

Definition. Let K be a complex, and $\sigma \in K$ a simplex. Call σ a **principal simplex** if the only simplex containing σ is σ itself (i.e., it isn't contained in a larger simplex).

For a simplex $\sigma \in K$, call $\tau < \sigma$ a **free face** of σ if the only simplexes containing τ are τ and σ.

If L and $K = L \sqcup \{\sigma, \tau\}$ are simplicial complexes, and σ is a principal simplex of K, and τ is a free face of σ, then L is an **elementary simplicial collapse** of K. If $K_1, K_2, \ldots, K_n$ are complexes, with K_{i+1} an elementary simplicial collapse of K_i, then K_n is a **simplicial collapse** of K_1, denoted $K_1 \searrow K_n$; conversely, K_1 a **simplicial expansion** of K_n, denoted $K_n \nearrow K_1$.

Finally, K is **simple homotopy equivalent** to L if you can reach transform K into L via a sequence of

- PL homeomorphisms,
- simplicial collapses,
- simplicial expansions.

In this case, we write $K \searrow\!\nearrow L$.

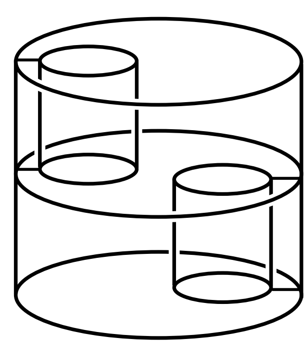
• **Problem 1.** Find two manifolds with boundary which are simple homotopy equivalent, but not PL homeomorphic.

Definition. Let K be a complex. If $K \searrow \Delta^0$, i.e., it collapses to a point, then we call the complex **collapsible**.

• **Problem 2.** Prove that the complex Δ^n is collapsible.

Problem 3. Construct infinitely many collapsible complexes, no two of which are PL homeomorphic.

Problem 4. Show that *Bing's House with Two Rooms,* as shown on the right, is simple homotopy equivalent to a point, but is not collapsible.



Problem 5. Show that the *dunce cap* is not collapsible, but is simple homotopy equivalent to a point; the dunce cap is constructed by starting with triangle ABC, and identifying edge AB with edge AC, and also with edge CB.

- **Problem 6.** Is $\chi(K) = \chi(L)$ if K and L are simple homotopy equivalent?

Definition. The **cone** of a complex K (written CK) is $K * \Delta^0$, i.e., the join of K with a point.

- **Problem 7.** Show that the cone of any complex is collapsible.

Definition. The **suspension** of a complex K (written SK) is $K * S^0$, where $S^0 = \partial\Delta^1$, that is, two disjoint points. In this notation, $SS^n \cong S^{n+1}$.

- **Problem 8.** Find a complex K so that $SK \searrow\!\nearrow K$.

Problem 9. Is K (non-)collapsible precisely when SK is (non-)collapsible?

Problem 10. Is the join of collapsible complexes necessarily collapsible?

Problem 11. Is join well-defined on simple homotopy types? That is, if $K \searrow\!\nearrow K'$ and $L \searrow\!\nearrow L'$, is it then the case that $K * L \searrow\!\nearrow K' * L'$?

3-manifolds

Problem 12. Prove that there exist two distinct 3-manifolds.

Problem 13. Find a copy of $T^2 \# T^2 \# T^2$ inside T^3, which divides T^3 into two components, homeomorphic to each other.

Problem 14. Let $f : T^2 \to T^2$ be a homeomorphism, and use f to glue together two copies of $S^1 \times \Delta^2$. Describe two distinct 3-manifolds that can result from this procedure (though it might be hard to prove that they are distinct, given what we know thus far!).

Connected sum

Problem 15. Given n-manifolds M and N, compute $\chi(M\#N)$ in terms of $\chi(M)$ and $\chi(N)$.

- **Problem 16.** Show that T^2 and $T^2 \# T^2$ are not PL homeomorphic.

Problem 17. Let K be the Klein bottle; determine whether $K\#K$ and $T^2\#K$ are PL homeomorphic.

Problem 18 (Hard)**.** Suppose that $M\#N = M$. Does it follow that N is a sphere?

Orientability

• **Problem 19.** Construct an orientation on S^n.

Problem 20. Does there exist a connected 2-manifold M^2 with

$$\chi(M) > 0 \text{ and } w_1(M) = 1.$$

Problem 21. Give an example of a non-orientable 3-manifold M^3.

Problem 22. Let K be the Klein bottle; is there an orientation on the 4-manifold $K \times K$?

Problem 23. Consider how $w_1(M \times N)$ might relate to $w_1(M)$ and $w_1(N)$.

Products

Problem 24. Show that if M is a manifold, then $M \times S^1$ is a manifold. What does $M \times S^1$ even mean?

Problem 25. Compute $\chi(K \times L)$ in terms of $\chi(K)$ and $\chi(L)$.

Problem 26. Show that there exist infinitely many distinct 4-manifolds.