Final Exam — Math 758

Spring 2012 — Jim Fowler

Thank you for a great quarter: I hope you've enjoyed cohomology and that you'll be able to use (co)homological arguments to do great things. *Here are five problems to write up for next Thursday; each is worth eighty points.*

Problem 1. Let Σ_2 be the closed surface of genus 2; find a map $f : \Sigma_2 \to \Sigma_2$ which is not homotopic to the identity, but for which $f^\star : H^\star(\Sigma_2) \to H^\star(\Sigma_2)$ is the identity. This is an element of the **Torelli group**.

Problem 2. Sometimes manifolds have finite covers which are homeomorphic to the original manifolds, e.g., S^1 or the n-torus. Is it possible that a closed manifold M^n has an infinite-sheeted cover which is homotopy equivalent to itself?

Problem 3. Calculate $H^n(\mathbb{R}P^7 \times S^3; R)$ and compare it to $H^n(\mathrm{SO}(5); R)$ for $R = \mathbb{Z}$ and $R = \mathbb{Z}/2$. Is $\mathbb{R}P^7 \times S^3$ homotopy equivalent to $\mathrm{SO}(5)$?

Problem 4. Does Alexander duality hold for "homology spheres"? Suppose M^n is a closed n-manifold with $H_\star(M^n) \cong H_\star(S^n)$, and K a compact, locally contractible, nonempty, proper subspace of M^n. Is it true that $\tilde{H}_i(M^n - K) \cong \tilde{H}^{n-i-1}(K)$?

Problem 5. The **Segre map** $\sigma : \mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{(n+1)(m+1)-1}$ is defined by

$$\sigma([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) = [x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_m].$$

Describe the map of rings $\sigma^\star : H^\star(\mathbb{C}P^{(n+1)(m+1)-1}) \to H^\star(\mathbb{C}P^n \times \mathbb{C}P^m)$.