Group actions on tensor products

Math 205

May 12, 2009

The footnotes in this document are only for flavor.

Left and right group actions

One thing we have not really paid much attention to yet is that a group acts from either the left, or from the right.

Definition. A left action of a group G on a set X is a map $G \times X \to X$, which we write as $g \cdot x$, with the property that

$$g \cdot (h \cdot x) = (gh) \cdot x.$$

On the other hand (and I mean that literally),

Definition. A right action of a group G on a set X is a map $X \times G \to X$, which we write as $x \cdot g$, with the property that

$$(x \cdot g) \cdot h = x \cdot (gh).$$

These look similar, but they are not at all the same—in a left action, the meaning of "gh" is "do h, then do g." In a right action, "gh" means "do g, then do h." It is perhaps significant to reflect on my use of the word "means"—it is precisely the group action that gives a meaning to a group element.

Philosophically, a **group is a symmetric thing without the thing**—only the symmetry* remains. A group action is what glues this abstract symmetry back to something in the real world. The handedness, this choice of a left or a right action, gives an interpretation of the group operation: does it mean "first do this, then do that" or does it mean "first do that, then do this"? These are not the same.

Yet, they are not so different. You can always turn a left action into a right action (and vice versa) with a trick: if G acts on the left on a set X, then G acts on the right after taking inverses. Specifically, define the right action $x \cdot g$ to be $g^{-1} \cdot x$.

Exercise. Check that this is a right action.

Very concretely, what this says is this: if you want to read a sequence of commands backwards, you should interpret each command as an "undo."

*that is, the beauty

Group action example

All this theory is useless without something concrete, so here is an example: a group* acting on a square piece of paper with a front and a back side.

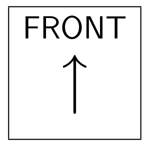
The group has eight elements, which are

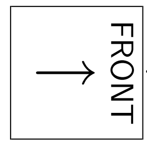
$$\mathbf{e} = \mathbf{X}^2, \quad \mathbf{R}, \quad \mathbf{R}^2, \quad \mathbf{R}^3, \quad \mathbf{X}, \quad \mathbf{XR} = \mathbf{R}^3\mathbf{X}, \quad \mathbf{XR}^2 = \mathbf{R}^2\mathbf{X}, \quad \mathbf{XR}^3 = \mathbf{RX}$$

You can do calculations in this group, like

$$\mathbf{RXR} = \mathbf{RR}^3\mathbf{X} = \mathbf{R}^4\mathbf{X} = \mathbf{X}.$$

This group *acts* on the set of configurations of a piece of square paper.

$\mathbf{R}$ acts by rotating clockwise by $90°$, turning 

into 

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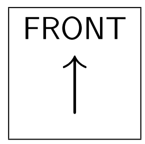
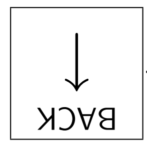
$\mathbf{X}$ acts by flipping across the x-axis, turning 

into 

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Once you decide whether to read the list of instructions from left-to-right (or from right-to-left), you will have decided that this is a right (or a left) action, respectively.

For English readers, you might scoff and say "Why would I want to read the instructions from right to left?" Let me remind you that you already know any example of such a situation! The notation $f \circ g$ means "first apply g, then apply f" which is arguably backwards from what you might have wanted§.

Actions on tensors

The goal is to prove the following.

Theorem. *Let* $S \in \mathcal{T}^k V$ *and* $T \in \mathcal{T}^\ell V$. *If* $\operatorname{Alt} S = 0$, *then* $\operatorname{Alt}(S \otimes T) = 0$.

*A popular application of this group is to the ancient problem of *square mattress flipping*. One would like a single group element (e.g., $\mathbf{RX}$) which, as it is applied repeatedly to the mattress, produces each of the 8 possible configurations of the square mattress. This would be helpful because you could decide "every month I will apply $\mathbf{RX}$ to my square mattress" and over the course of 8 months you would have rotated your mattress through all configurations.

Tragically, it turns out that there is no such an element—or in more intimidating language "the group of symmetries of the square is not cyclic." It is therefore a rather complicated matter to flip one's square mattress through all 8 configurations. This remains problematic even in the smaller group of symmetries of rectangular mattresses.

§This is why I would be happier putting the arguments to functions on the left hand side—i.e., we should all be writing $(x)f$ instead of $f(x)$.

To prove this, we will define an additional concept (namely, an action of the symmetric group S_k on $\mathcal{T}^k V$) which should make it clearer what exactly is happening.

Recall that an element $S \in \mathcal{T}^k V$ is a k-linear functional

$$S : \overbrace{V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R}$$

A permutation $\sigma \in S_k$ will *act* on $S \in \mathcal{T}^k V$ to produce a new k-linear functional, $\sigma^\star S \in \mathcal{T}^k V$. But how? How do we define this?

"Definition". Let $S \in \mathcal{T}^k V$ and $\sigma \in S_k$. Then define

$$(\sigma^\star S)(v_1, \ldots, v_k) = S\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}\right).$$

I put this in quotation marks not because it isn't a definition (it will be the definition we will use, as it agrees with what we have been using thus far), but because it is more of a convention—there ar other choices that are (arguably) better.

A good thing about the action of S_k on $\mathcal{T}^k V$ is a concise definition of our friend Alt.

Definition. $\operatorname{Alt} S = \dfrac{1}{k!} \displaystyle\sum_{\sigma \in S_k} (-1)^\sigma \, \sigma^\star S.$

Here I have used a convention that $(-1)^\sigma$ means the sign of the permutation σ. I think this makes the formulas nicer looking (and, as good notation ought, suggests that this is a homomorphism).

A bad thing about our "definition" is shown in the next pair of exercises.

Exercise. Show that $(\sigma\tau)^\star S \neq \sigma^\star \tau^\star S$.

Exercise. Show that $(\sigma\tau)^\star S = \tau^\star \sigma^\star S$.

In fact, the whole reason I put the little stars as a superscript instead of as a subscript is because of this fact: the superscript star reminds me that something contravariant ("backwards") is taking place. In other words, what we have built is actually a *right action* of the symmetric group S_k on tensors $\mathcal{T}^k V$. If we want a left action, we could take inverses—but in order to keep consistent with the conventions we have already chosen, we will not do this.

Exercise. The action of S_k on $\mathcal{T}^k V$ is by linear maps, that is,

$$\sigma^\star(S + T) = \sigma^\star S + \sigma^\star T$$

for $\sigma \in S_k$ and $S, T \in \mathcal{T}^k V$.

I want to define another operation: a sort of "tensor product" on the symmetric group.

Definition. The operation $\otimes : S_k \times S_\ell \to S_{k+\ell}$ is defined by...

I will let you complete the definition appropriately; using the $\otimes$ symbol is merely suggestive—of course this isn't a tensor product at all.

We are now ready to "prove" the theorem.

$$
\begin{aligned}
(k+\ell)! \cdot \operatorname{Alt}(S \otimes T) &= \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \, \sigma^\star (S \otimes T) \\
&= \sum_\tau \sum_\sigma (-1)^{\sigma\tau} \, (\sigma\tau)^\star (S \otimes T) \\
&= \sum_\tau \sum_\sigma (-1)^\sigma \, (-1)^\tau \, \tau^\star \sigma^\star (S \otimes T) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \left(\sum_\sigma (-1)^\sigma \, \sigma^\star (S \otimes T) \right) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \left(\sum_{\sigma \in S_k} (-1)^{\sigma \otimes \mathrm{id}} \, (\sigma \otimes \mathrm{id})^\star (S \otimes T) \right) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \left(\sum_{\sigma \in S_k} (-1)^\sigma \, (\sigma^\star S) \otimes \mathrm{id}^\star \, T \right) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \left(\sum_{\sigma \in S_k} (-1)^\sigma \, (\sigma^\star S) \otimes T \right) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \left(\left(\sum_{\sigma \in S_k} (-1)^\sigma \, \sigma^\star S \right) \otimes T \right) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \, (k! \cdot \operatorname{Alt} S \otimes T) \\
&= \sum_\tau (-1)^\tau \, \tau^\star \, (0 \otimes T) = 0.
\end{aligned}
$$

I have supressed exactly what we are summing over (i.e., where do τ and σ take values?). I hope you will take a moment to reflect on this notation—its benefits, its drawbacks. There is a great discussion to be had about the purpose of proof and excessive machinery and such.

There is also a theme illustrated here: we have worked with the tensors directly, without considering their inputs. This technique—reasoning about things which take parameters without referring to the parameters—can be quite powerful.