Topology of Piecewise-Linear Manifolds

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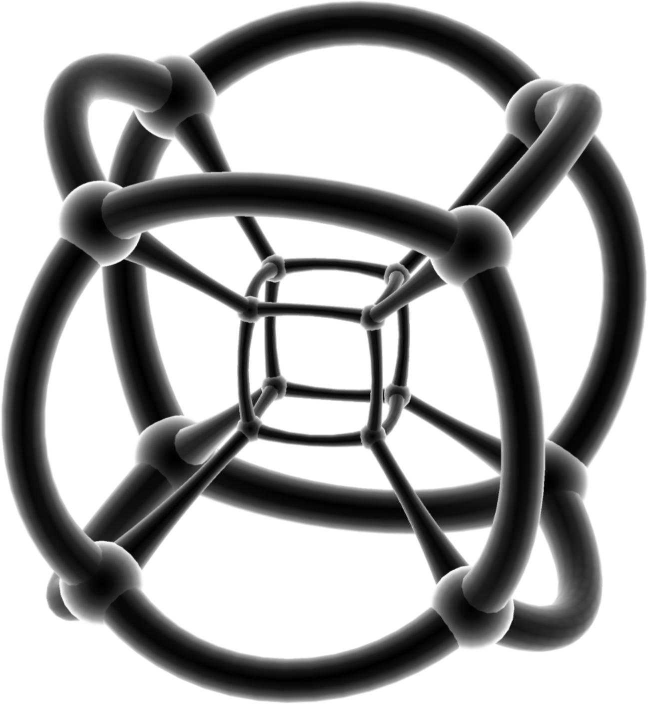
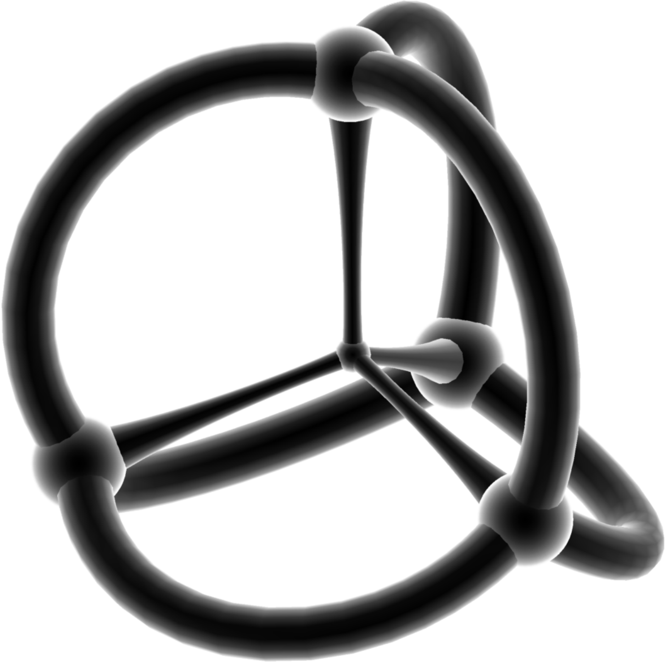
Lecture 2
Summer 2010

Goal

Definitions?





A **simplicial complex** K is
a collection of finite sets (called the "simplexes"),
with the property that

$$\text{if } \sigma \in K, \text{ and } \tau \subset \sigma, \quad \text{then } \tau \in K.$$

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This definition is pure combinatorics,
but we will think of this as a geometric object.

Examples

$$= \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$$

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Shouldn't $V \cong I$?

Non-examples

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Examples

The n-simplex Δ^n is a complex: label the $n+1$ vertices of Δ^n using the set $V = \{0, 1, 2, \ldots, n\}$, then the simplexes of Δ^n are all 2^{n+1} subsets of V.

The n-sphere S^n consists of all simplexes in Δ^n, except for the top dimensional simplex $\{0, 1, \ldots, n\}$.

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Problem

Calculate $\chi(S^n)$.

Simplicial Maps

A **simplicial map** $f : K \longrightarrow L$ is a function $f : \text{vert}(K) \longrightarrow \text{vert}(L)$ so that, whenever $\sigma \in K$, then $f(\sigma) \in L$.

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Problem

Find a simplicial map $f : T^2 \to S^2$ which doesn't crush any edges (i.e., edges are sent to edges, not to vertices).

Think "4-color the vertices of T^2."

Conversely. . .

Problem

Find a simplicial map $f : S^2 \to T^2$ *which doesn't crush any edges.*

Conversely...

Problem

Find a simplicial map $f : S^2 \to T^2$ which doesn't crush any edges.

Think "Skewer and fold."

Problem

Do there exist simplicial maps $f : T^2 \longrightarrow S^2$ and $g : S^2 \longrightarrow T^2$ which are inverses of each other?

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No. **And why not?**

- Euler characteristic.
- Separation via curves.

Problem

Suppose $f : S^1 \to S^2$ is an injective simplicial map (i.e., distinct vertices are sent to distinct vertices). Does the image of f necessarily separate S^2 into two pieces?

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Suppose $f : S^1 \longrightarrow S^2$ *is an injective simplicial map (i.e., distinct vertices are sent to distinct vertices). Does the image of* f *necessarily separate* S^2 *into two pieces?*

Yes.

Proof

A problem with our definitions: we want to talk about S^2, but there are so many simplicial complexes which deserve to be called S^2.

Our notion of simplicial complex is too rigid to be the right notion topologically.

Example

Let K be a circle with three arcs
and L be a circle with four arcs
(i.e., the boundary of a square).
Then K and L are not simplicially isomorphic.

Fixing the theory

Need to define a few things first...

- star
- closure
- link
- subdivision

Star

Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **star** of σ in K,
written $\mathrm{st}(\sigma, K)$,
is defined by

$$\mathrm{st}(\sigma, K) = \{\tau \in K : \sigma < \tau\},$$

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Is the star of a simplex a complex?

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Relate $\text{cl}(\text{cl}(S))$ *and* $\text{cl}(S)$.

Link

Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **link** of $\sigma \in K$,
 written $\mathrm{lk}(\sigma, K)$,
consists of those simplexes in K which are in $\mathrm{cl}(\mathrm{st}(\sigma, K))$ but not touching σ; in other words,

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Is the link of a simplex a complex?

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Let K be a complex, and $\sigma \in K$ a simplex.
The **stellar subdivision** of K at σ is a new complex K_σ with:

- the vertices of K along with a brand new vertex v.
- the simplexes of K not in $\mathrm{st}(\sigma, K)$, along with the simplexes in $v * (\partial\sigma) * \mathrm{lk}(\sigma, K)$.

We might say:

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Problem

Stellar subdivision of a complex is a complex?

subdivision = repeated stellar subdivision

Definition

Let K, L be complexes.
If K can be produced through a (possibly empty) sequence of stellar subdivisions of L,
we say that K is a **subdivision** of L,
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As we will see, the *real* definition of subdivision is more general than this.

Piecewise linear maps

Definition

Let K, K', L, L' be complexes, with $K' \lhd K$ and $L' \lhd L$. If $f : K' \to L'$ is a simplicial map, we call $f : K \to L$ a **piecewise linear map** (or a **PL map** for short). We call $f : K' \to L'$ an **underlying simplicial map**.

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We write $f : K \longrightarrow L$ for a PL map, but such a map does not send simplexes in K to simplexes in L

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The *real* definition of PL map is more general than this.

Going back, rethinking everything...

Are S^2 and T^2 the same?

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We will check that χ is well-defined.