Lecture 37: More on Harmonic functions

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Harmonic functions in polar coordinates

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So if $\frac{\partial u}{\partial \theta}=0$, then $u=a\log r+b$.

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$$dv = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy.$$

But in practice, there may be no conjugate harmonic function,
so we take as a definition

$$*du = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy.$$

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And so $\int_\gamma *du = 0$ for cycles homologous to zero.

Simply connected regions

In a simply connected regions, $\int_\gamma *du = 0$ for all cycles, so u has a single-valued conjugate function v.

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Proof: If conjugate harmonic functions exist (e.g., in a simply connected region like a rectangle), then

$$\begin{aligned} u_1 * du_2 - u_2 * du_1 &= u_1 dv_2 - u_2 dv_1 \\ &= u_1 dv_2 + v_1 du_2 - d(u_2 v_1) \\ &= (\mathrm{im}(u_1 + iv_1)(du_2 + idv_2)) - d(u_2 v_1 \end{aligned}$$

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so the integral vanishes.

$\square$

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Apply this to $u_1 = \log r$ and $u_2 = u$.

Mean-value property

Suppose u is harmonic in a punctured disk; then the arithmetic mean of u over concentric circles is a linear function of $\log r$,

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = a\log r + \beta$$

and if u is harmonic in the whole disk, then $a = 0$.

Maximum principle

A nonconstant harmonic function has neither a maximum nor a minimum in the interior of a disk; the maximum and minimum on a closed bounded set occur on the boundary.

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Can we get a formula for the function on the interior, given the values on the boundary?

Recall

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Consider $S_a(z) = \frac{R(R\,z+a)}{R+\bar{a}z}$.

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$$u(a) = \frac{1}{2\pi} \int_{|z|=1} u(S_a(e^{i\theta}))\, d\theta.$$