Problem Set 1 — Math 758

Spring 2012 — Jim Fowler

The only way to learn the game is to play the game. The following represents a *lower bound* on the number of exercises you should be doing; the textbook is full of great exercises, so I encourage you to do as many as possible.

Problem 1.1 (Hom is sometimes exact)

For which abelian groups G is $\mathrm{Hom}(G, -)$ an exact functor?

Problem 1.2 (Ext is functorial)

Let G be an abelian group; show that $\mathrm{Ext}(-, G)$ and $\mathrm{Ext}(G, -)$ are functors.

Problem 1.3 (Ext for cyclic groups)

Compute $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/n)$.

Problem 1.4 (Ext is extensions)

An extension of A by B is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

Two extensions $0 \to A \to E \to B \to 0$ and $0 \to A \to E' \to B \to 0$ are said to be equivalent if there is a map $f : E \to E'$ so that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commutes. Show that the set of equivalence classes of extensions of A by B is naturally isomorphic to $\mathrm{Ext}(B, A)$.

Problem 1.5 (Real projective plane)

Let X be $\mathbb{R}P^2$. Compute $H_\star(X; \mathbb{Z})$ and $H^\star(X; \mathbb{Z})$ via (simplicial or) cellular (co)homology.

Problem 1.6 (Jacob's ladder)

Consider the simplicial graph X with a vertex a_i and b_i for each $i \in \mathbb{Z}$, and edges

- between a_k and a_{k+1},
- between b_k and b_{k+1},
- between a_k and b_k, for each $k \in K$.

Compute $H^\star(X; \mathbb{Z})$.

Problem 1.7 (Isomorphic homology, isomorphic comohology?)

Let $f : X \to Y$ be a map of CW complexes and let G be an abelian group. Is it the case that if $f_\star : H_\star(X; G) \to H_\star(Y; G)$ is an isomorphism, then $f_\star : H^\star(Y; G) \to H^\star(X; G)$? If yes, prove it. If not, salvage the statement to make it true.

Problem 1.8 (Cohomology with coefficients and tensor product)

Suppose $H^\star(X; \mathbb{Z})$ is torsion-free and G is an abelian group. Is it then the case that $H^n(X; G) = H^n(X; \mathbb{Z}) \otimes G$? If yes, prove it; if not, salvage the statement to make it true.