Lecture 31: Homology

Math 660—Jim Fowler

Tuesday, August 3, 2010

Theorem

A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points $a \notin \Omega$.

Homology

A cycle in a region Ω is homologous to zero inside Ω if $n(\gamma, a)$ for all $a \notin \Omega$.

Homology

A cycle in a region Ω is homologous to zero inside Ω if $n(\gamma, a)$ for all $a \notin \Omega$.

We write $[\gamma_1] = [0] \in H_1(\Omega)$.

Homology

A cycle in a region Ω is homologous to zero inside Ω if $n(\gamma, a)$ for all $a \notin \Omega$.

We write $[\gamma_1] = [0] \in H_1(\Omega)$.

We write $[\gamma_1] = [\gamma_2] \in H_1(\Omega)$ if $[\gamma_1 - \gamma_2] = [0]$.

Cauchy's theorem

Theorem

If $f(z)$ is analytic in Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ which is homologus to zero in Ω.

Cauchy's theorem

Theorem

If $f(z)$ is analytic in Ω, then

$$\int_{\gamma} f(z)\,dz = 0$$

for every cycle γ which is homologus to zero in Ω.

In other words, if the property holds for $1/(z-a)$ with $a \notin \Omega$, then it holds for all analytic f.

Theorem

If $f(z)$ is analytic in Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ which is homologus to zero in Ω.

Theorem

Suppose $f(z)$ is analytic in Ω, a simply connected region.

Theorem

If $f(z)$ is analytic in Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ which is homologus to zero in Ω.

Theorem

Suppose $f(z)$ is analytic in Ω, a simply connected region. Then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ in Ω.

By Cauchy's theorem, there exists a single-valued analytic function $F(z)$ so that $F'(z) = f(z)$.

By Cauchy's theorem, there exists a single-valued analytic function $F(z)$ so that $F'(z) = f(z)$. Then,

Corollary

If $f(z)$ is analyic and nonzero in a simply connected region Ω, then it is possible to define single valued analytic branches of $\log f(z)$ and $\sqrt[n]{f(z)}$.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j \in J}$ be squares inside Ω.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j \in J}$ be squares inside Ω.
$\Gamma_\delta = \sum \partial Q_j$.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j \in J}$ be squares inside Ω.
$\Gamma_\delta = \sum \partial Q_j$.
$\Omega_\delta = \operatorname{int} \bigcup_{j \in J} Q_j$.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j \in J}$ be squares inside Ω.
$\Gamma_\delta = \sum \partial Q_j$.
$\Omega_\delta = \operatorname{int} \bigcup_{j \in J} Q_j$.

Suppose γ with $[\gamma] = [0] \in H_1(\Omega)$.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j\in J}$ be squares inside Ω.
$\Gamma_\delta = \sum \partial Q_j$.
$\Omega_\delta = \operatorname{int} \bigcup_{j\in J} Q_j$.

Suppose γ with $[\gamma] = [0] \in H_1(\Omega)$.
Choose δ small so that $\gamma \in \Omega_\delta$.

Proof of Cauchy's theorem

Suppose Ω is a bounded region.
Cover $\mathbb{C}$ by squares of side length δ.
Let $\{Q_j\}_{j \in J}$ be squares inside Ω.
$\Gamma_\delta = \sum \partial Q_j$.
$\Omega_\delta = \operatorname{int} \bigcup_{j \in J} Q_j$.

Suppose γ with $[\gamma] = [0] \in H_1(\Omega)$.
Choose δ small so that $\gamma \in \Omega_\delta$.
Check that $n(\gamma, a) = 0$ if $a \in \Gamma_\delta$.

Suppose f is analytic in Ω.

Suppose f is analytic in Ω.
If $z \in \operatorname{int} Q_j$, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

Suppose f is analytic in Ω.
If $z \in \operatorname{int} Q_j$, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

and by continuity, this holds for all $z \in \Omega_\delta$.

Suppose f is analytic in Ω.
If $z \in \operatorname{int} Q_j$, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

and by continuity, this holds for all $z \in \Omega_\delta$. Therefore,

$$\int_\gamma f(z)\, dz = \int_\gamma \left(\frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{\zeta - z} \right) dz$$

Suppose f is analytic in Ω.
If $z \in \operatorname{int} Q_j$, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

and by continuity, this holds for all $z \in \Omega_\delta$. Therefore,

$$\int_\gamma f(z)\, dz = \int_\gamma \left(\frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{\zeta - z} \right) dz$$

and reversing the order of integration,

$$\int_\gamma f(z)\, dz = \int_{\Gamma_\delta} \left(\frac{1}{2\pi i} \int_\gamma \frac{dz}{\zeta - z} \right) f(\zeta) d\zeta$$

Suppose f is analytic in Ω.
If $z \in \operatorname{int} Q_j$, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

and by continuity, this holds for all $z \in \Omega_\delta$. Therefore,

$$\int_\gamma f(z)\, dz = \int_\gamma \left(\frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{\zeta - z} \right) dz$$

and reversing the order of integration,

$$\int_\gamma f(z)\, dz = \int_{\Gamma_\delta} \left(\frac{1}{2\pi i} \int_\gamma \frac{dz}{\zeta - z} \right) f(\zeta) d\zeta$$

and the integral vanishes.

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$for R large enough that $B_R(0) \supset \gamma$.

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$for R large enough that $B_R(0) \supset \gamma$.

Then $n(\gamma, a) = 0$ for all $a \notin B_R(0)$ or $a \notin \Omega$,

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$for R large enough that $B_R(0) \supset \gamma$.

Then $n(\gamma, a) = 0$ for all $a \notin B_R(0)$ or $a \notin \Omega$,so $[\gamma] = [0] \in H_1(\Omega')$.

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$for R large enough that $B_R(0) \supset \gamma$.

Then $n(\gamma, a) = 0$ for all $a \notin B_R(0)$ or $a \notin \Omega$,so $[\gamma] = [0] \in H_1(\Omega')$.

Thus, $\int_\gamma f(z)\, dz = 0$, so the theorem holds for arbitrary Ω.

If Ω is unbounded?

If Ω is unbounded, consider $\Omega' = \Omega \cap B_R(0)$for R large enough that $B_R(0) \supset \gamma$.

Then $n(\gamma, a) = 0$ for all $a \notin B_R(0)$ or $a \notin \Omega$,so $[\gamma] = [0] \in H_1(\Omega')$.

Thus, $\int_\gamma f(z)\, dz = 0$, so the theorem holds for arbitrary Ω. $\square$

Theorem

If $f(z)$ is analytic in Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ which is homologus to zero in Ω.

Theorem

Suppose $f(z)$ is analytic in Ω, a simply connected region. Then

$$\int_\gamma f(z)\,dz = 0$$

for every cycle γ in Ω.

Locally exact differentials