Problem Set 8 — Math 758

Spring 2012 — Jim Fowler

This week's theme is H-spaces, spaces X with a multiplication map $X \times X \to X$ and a two-sided "homotopy" identity. *The exercises below should be handed in on Monday.*

Problem 8.1 (Hatcher exercise 2 page 291)

Show that a retract of an H-space is an H-space if it contains the identity element.

Problem 8.2 (Hatcher exercise 5 page 291)

Show that if (X, e) is an H-space, then $\pi_1(X, e)$ is abelian.

Problem 8.3 (Hatcher exercise 7 page 291)

What are the primitive elements of the Hopf alegebra $\mathbb{Z}_p[x]$ for p prime?

Problem 8.4 (Hatcher exercise 10 page 291–292)

(This is a sort of **fundamental theorem of algebra** for quaternions.) Let X be a path-connected H-space with $H^\star(X; R)$ free and finitely generated in each dimension. For maps $f, g : X \to X$, the product $fg : X \to X$ is defined by $(fg)(x) = f(x)\,g(x)$ using the H-space product.

- Show that $(fg)^\star(\alpha) = f^\star(\alpha) + g^\star(\alpha)$ for primitive $\alpha \in H^\star(X; R)$.
- Deduce that the k-th power map $x \mapsto x^k$ induces the map $\alpha \mapsto k\alpha$ on primitive elements α. In particular, show that the quaternionic k-th power map $S^3 \to S^3$ has degree k.
- Show that every polynomial $a_n x^n b_n + \cdots + a_1 x b_1 + a_0$ of nonzero degree with coefficients in $\mathbb{H}$ has a root in $\mathbb{H}$.

Problem 8.5 (Hatcher exercise 3 page 302)

Compute the Pontrjagin ring structure in $H_\star(\mathrm{SO}(5); \mathbb{Z})$.