Lecture 16: Taylor series

Math 153 Section 57

Monday November 3, 2008

Following chapter 12.6.

0.1 Administrative stuff

Many thanks to those who filled out an online anonymous survey; very helpful! I especially noticed a consensus on doing trickier problems during lecture.

0.2 Recall

Series are not piles of numbers that we add up—they are **lists** of numbers that we add up, in order.

0.3 Overview

We have paid a high price (limits, series) and now we reap the rewards (Taylor series).

0.4 Taylor series

Goal: approximate e^x (really, any function) by a polynomial.

$p_0(x) = 1$ is a good approximation (gets the value correct at 0).

$p_1(x) = 1 + x$ is better approximation (gets the value and first derivative correct at 0).

$p_2(x) = 1 + x + x^2/2$ is yet better approximation (gets the value and first and second derivative correct at 0).

$p_3(x) = 1 + x + x^2/2 + x^3/6$ is even yet a better approximation (gets the value and first and second and third derivative correct at 0).

General pattern: $p_k(x) = \sum_{n=0}^{k} \frac{x^n}{n!}$.

Very generally:

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)\, x^n}{n!}$$

0.5 Is this useful?

Approximations are useless without error estimates. People often say $\pi \approx 3.14$, but without knowing how good or bad an approximation this is, it is useless. I could just as well say $\pi \approx 5$.

0.6 Theorem

If $f(x) : (-a, a) \to R$, with $n+1$ cts derivatives, then $f(x) = p_n(x) + R_n(x)$, where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)\,(x-t)^n\,dt$$

This is the exact error term—and if we know the error exactly, it isn't much of an error—if we could calculate $R_n(x)$ we could calculate the function.

0.7 Bounding the remainder

Lagrange's theorem:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where c is some number between 0 and x.

0.8 Remainder vanishes for large n

For instance, for $f(x) = e^x$, the remainder looks like $e^c \cdot x^{n+1}/(n+1)!$, which goes to zero.

Consequence: $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all x.

0.9 Other functions

We can find series for $\sin x$ and $\cos x$—and these series converge to the corresponding function for all x.

0.10 Horrible truth

For any smooth function, we can write down a Taylor series—but we can't be sure that it will converge unless we can show that the remainder term gets small.

Example: $f(x) = 1/(1-x)$.