Root Systems and Coxeter Groups

Undergraduate Lie Theory Seminar

> The belief that all simple (having no continuous moduli) objects in the nature are controlled by the Coxeter groups is a kind of religion.
> —V.I. Arnold, *First steps of local contact algebra.*

1 Root systems

Unfortunately, I again define root system without much motivation. Nonetheless, I hope you will be surprised to discover how restrictive the definition of root system is—there are four infinite families, and five "exceptional" root systems that just don't belong.

Definition 1.1 (Hyperplanes and reflections)**.** Let V be a vector space with inner product, and $v \in V$ a vector. Then the **hyperplane** normal to v is

$$H_v := \{w \in V : \langle w, v \rangle = 0\}.$$

The **reflection** through the hyperplane H_v is the linear map $s_v : V \to V$ defined by

$$s_v(w) = w - 2\frac{\langle v, w \rangle}{\langle v, v \rangle}v.$$

Exercise 1.2. Check that $s_v{}^2$ is the identity.

Definition 1.3 (Root system.)**.** Let V be a vector space. A **root system** is subset $\Phi \subset V$ satisfying:

- The set Φ is finite.

- The span of Φ is V.

- If $v \in \Omega$, then $s_v(\Phi) = \Phi$.

- If $v \in \Phi$, then $\Phi \cap \mathbb{R}v = \{v, -v\}$.

The elements of a root system are called **roots**.

Definition 1.4 (Isomorphism)**.** Two root systems $\Phi \subset V$ and $\Phi' \subset V'$ are **isomoprhic** if there is an isometry $f : V \to V'$ with $f(\Phi) = \Phi'$.

Definition 1.5 (Direct sums)**.** Let Φ and Φ' be root systems in V and V', respectively. Then $\Phi \oplus \Phi'$ is a root system in the vector space $V \oplus V'$. A root system Φ is **reducible** if it can be written as such an (orthogonal) direct sum, and **irreducible** otherwise.

Exercise 1.6. For a vector space V with orthogonal basis $\{e_i\}$, the set $\{\pm e_i\}$ is a root system. This is a reducible root system.

Definition 1.7 (Crystallographic root system.)**.** A root system Φ is **crystallographic** if, for all $v, w \in \Phi$, we have

$$2\frac{\langle v, w \rangle}{\langle v, v \rangle} \in \mathbb{Z}.$$

Exercise 1.8. Play around with this crystallographic condition; what are the possible angles between $v, w \in \Phi$ when Φ is crystallographic?

Remark 1.9. A Coxeter group does not in general have a crystallographic root system; but the groups that we will be working with in the classification of Lie groups are all crystallographic, so we might as well simplify our work by considering only the special sort of Coxeter group arising here.

Exercise 1.10. Find four non-isomorphic crystallographic root systems in $\mathbb{R}^2$.

2 Weyl Groups

Definition 2.1 (Weyl group.)**.** Let $\Phi \subset V$ be a root system. Then the **Weyl group** of Φ is the group generated by $s_v : V \to V$ for all $v \in \Phi$.

Exercise 2.2. Is the Weyl group well-defined (i.e., do isomorphic root systems give isomorphic Weyl groups?).

Exercise 2.3. Is the Weyl group finite?

Exercise 2.4. What are the Weyl groups of the four crystallographic root systems in $\mathbb{R}^2$?

Exercise 2.5. Find a root system having the symmetric group on four letters, S_4, as its Weyl group.

3 Positive systems

Definition 3.1 (Positive system)**.** Let $\Phi \subset V$ be a root system, and choose $w \in V$. Define

$$\Phi^+(w) := \{v \in \Phi : \langle v, w \rangle > 0\}.$$

We say that w is **regular** if $\Phi = \pm\Phi^+(w)$, and **singular** otherwise. If w is regular, we call $\Phi^+(w)$ a **positive system** for Φ.

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This is only one way to define a positive system (it makes clearer the relationship between positive systems and Weyl chambers, defined later). Alternatively, we can say that a positive system is any subset $\Phi^+ \subset \Phi$ so that Φ is the disjoint union of Φ^+ and $-\Phi^+$, and so that Φ^+ is closed under sums.

Exercise 3.2. Give examples of positive systems for some root systems in $\mathbb{R}^2$.

Definition 3.3 (Decomposing roots)**.** Let Φ^+ be a positive system for a root system Φ. Then $v \in \Phi^+$ is **decomposable** if it is the sum of two other roots in Φ^+, and **indecomposable** otherwise.

4   Simple systems

Definition 4.1 (Positive cone)**.** Given vectors $v_1, \ldots, v_n \in V$, the **positive cone** of $\{v_1, \ldots, v_n\}$ are the nonnegative linear combinations of the v_i.

Definition 4.2 (Simple systems)**.** Let Φ^+ be a positive system for a root system Φ. A **simple system** Π for Φ^+ is a minimal subset of Φ^+ generating the same positive cone as Φ^+. The elements of a simple system are called **simple roots**.

This definition of simple system depends on a choice of positive system, but we can also define simple systems intrinsically—a simple system Π is a subset of Φ spanning V, so that every root is a non-positive or a non-negative linear combination of roots in Π.

Exercise 4.3. Let Φ^+ be a positive system for a root system Φ. Show that the indecomposable roots form a simple system.

Exercise 4.4. Consider a simple system Π for a root system $\Phi \subset V$. Then the span of Π is V, so maps between simple systems determine maps between vector spaces (and it will therefore be enough to classify simple systems).

Definition 4.5 (Weyl chambers)**.** Let $\Phi \subset V$ be a root system. The connected components of $V - \bigcup_{v \in \Phi} H_v$ are the **Weyl chambers** of Φ.

Exercise 4.6. There is a bijection between Weyl chambers of Φ and positive systems, and between positive systems and simple systems. In fact, the Weyl group acts transitively on positive systems and simple systems.

There's a lot more to say about the action of the Weyl group on the positive systems, but that is for another day.

5   Reducibility

Exercise 5.1. Give examples of reducible and irreducible root systems.

Exercise 5.2. A simple system Π for a root system Φ is reducible if and only if Φ is reducible.

6 Cartan matrix

Definition 6.1 (Cartan matrix)**.** Given a simple system $\Pi = \{a_i\}$ for Φ, we define the **Cartan matrix** $A = (A_{ij})$ by

$$A_{ij} = 2\frac{\langle a_i, a_j \rangle}{\langle a_i, a_i \rangle}.$$

Exercise 6.2. What are the four possible values of A_{ij} in the special case of a crystallographic root system?

Exercise 6.3. What are the Cartan matrices for the root systems in $\mathbb{R}^2$?

Exercise 6.4. The Cartan matrix determines Φ up to isomorphism.

Exercise 6.5. What does the Cartan matrix for a reducible root system look like?

7 Coxeter graph

Definition 7.1 (Coxeter graph)**.** Given a Cartan matrix A for a simple system $\Pi = \{a_i\}$, its associated **Coxeter graph** has a vertex for each simple root, and a $A_{ij}A_{ji}$ edges between roots a_i and a_j.

Exercise 7.2. Draw the Coxeter graphs for your favorite root systems. Does the Coxeter graph determine the root system?

Definition 7.3 (Dynkin Diagram)**.** Given a Cartan matrix A for a simple system $\Pi = \{a_i\}$, its associated **Dynkin diagram** is its Coxeter graph, decorated with an arrow on double and triple edges, drawn from a_i to a_j whenever a_i is longer than a_j.

Exercise 7.4. What does the Coxeter graph for a reducible root system look like? Does the Dynkin diagram determine the root system? *Hint:* does the Dynkin diagram determine the Cartan matrix?

8 Summarizing the situation

From root systems, we produced positive systems, which gave rise to simple systems, which allowed us to define the Cartan matrix, and finally, the Coxeter graph—which doesn't quite determine the root system—and the Dynkin diagram—which does determine the root system.

Consider a finite undirected graph $\Gamma = (V, E)$. Let $V = \{v_1, \ldots, v_n\}$, and define c_{ij} to be the number of edges connecting v_i and v_j. On the real vector space spanned by the vertices of V, define a bilinear form (which we call the **adjacency form**) by $b(v_i, v_j) = -\sqrt{c_{ij}}$.

Definition 8.1 (Positive graphs)**.** A graph $\Gamma = (V, E)$ is **positive** if the adjacency form b is positive definite.

Exercise 8.2. A Coxeter graph is a positive graph.

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9 Sketch: Classifying positive graphs

Definition 9.1 (Subgraph). Given a graph $\Gamma = (V, E)$, a graph $\Gamma' = (V', E')$ is a **subgraph** of Γ, written $\Gamma' \subset \Gamma$, if $V' \subset V$ and $E' \subset E$.

Exercise 9.2. A subgraph of a positive graph is positive.

Exercise 9.3. Check that the Coxeter graphs A_i, B_i, D_i, E_6, E_7, E_8, F_4, and G_2 are all positive graphs.

Exercise 9.4. Find a family P of "poisonous" graphs so that

- No $\Gamma \in P$ is positive.

- If Γ is any non-positive connected graph, then there exists $\Gamma' \subset \Gamma$ with $\Gamma' \in P$.

Having finished this exercise, you will have shown that every positive graph is one of the Coxeter graphs we know about.

Exercise 9.5. Decorate your Coxeter graphs to find the possible Dynkin diagrams.

Exercise 9.6. Find root systems having each of the possible Dynkin diagrams.