Fake Midterm 1

Math 345

October 2010

Music is the pleasure the human soul experiences from counting without being aware that it is counting.

—Leibniz

Name: ______________________________

Lecture time (circle one): 12:30–1:18 P.M.

2:30–3:18 P.M.

1. Do not write your name above.
2. Calculators are forbidden (and useless, anyhow).
3. Look inside the exam before you are instructed to do so.
4. Give yourself have **48 minutes** for five problems on this fake exam.
5. Justify your answers.
6. Show your work.
7. Write your answers down.
8. Answer all questions.
9. To prevent fire, do not divide by zero.

Problem 1	/360
Problem 2	/360
Problem 3	/360
Problem 4	/360
Problem 5	/360
Problem 6	/360
Problem 7	/360
Problem 8	/360
Problem 9	/360
Problem 10	/360
Problem 11	/360
Problem 12	/360
Problem 13	/360
Problem 14	/360
Problem 15	/360
Problem 16	/360
Problem 17	/360
Problem 18	/360
Problem 19	/360
Problem 20	/360
Problem 21	/360
Problem 22	/360
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Total	/10800

Problem 1 /360

Write down a truth table for the proposition

$$(P \wedge (Q \Rightarrow R)) \Rightarrow (P \wedge Q \wedge R)$$

Solution

Problem 2 /360

Consider the proposition:

$$(P \vee Q) \Rightarrow (((P \Rightarrow R) \wedge (Q \Rightarrow R)) \Rightarrow R)$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Problem 3 /360

Consider the proposition:

$$\neg(P \Rightarrow (\neg P))$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, prove it.

Solution

Problem 4 /360

Consider the proposition:

$$(P \lor Q) \Rightarrow (P \Rightarrow Q)$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Problem 5 /360

Consider the proposition:

$$(P \Rightarrow (Q \Rightarrow P)) \Rightarrow Q$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Problem 6 /360

Consider the proposition:

$$(Q \Rightarrow R) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Problem 7 /360

Consider the proposition:

$$((P \Rightarrow Q) \vee (P \Rightarrow R)) \Rightarrow (P \Rightarrow (Q \vee R))$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Problem 8 /360

Write down the contrapositive of the conditional sentence:

If $x > 2$, then $x^2 > 4$.

Solution

Problem 9 /360

Write down the contrapositive of the conditional sentence:

If $x > 2$ and $x < 17$, then $x^2 > 4$ or $x^2 < 289$.

Solution

Problem 10 /360

Write down the contrapositive of the conditional sentence:

$$(P \vee Q) \Rightarrow (R \wedge S).$$

Solution

Problem 11 /360

Consider the three propositions

$$P \wedge (Q \vee \neg(R \wedge S)) \tag{1}$$
$$(P \wedge Q) \vee (P \wedge ((\neg R) \vee (\neg S))) \tag{2}$$
$$P \wedge (Q \vee (\neg R) \vee (\neg S)) \tag{3}$$

Which of these propositions are logically equivalent to which other propositions? Provide justifications for any claims you make; in particular, if you claim that two propositions are logically equivalent, you must prove this, and if you claim that they are *not* equivalent, you must explain why not.

Solution

Problem 12 /360

Consider the four propositions

$$P \wedge \neg(Q \Rightarrow R) \tag{1}$$

$$P \wedge ((\neg Q) \vee R)) \tag{2}$$

$$P \wedge Q \wedge \neg R \tag{3}$$

$$(P \wedge (\neg Q)) \vee (P \wedge R) \tag{4}$$

Which of these propositions are logically equivalent to which other propositions? Provide justifications for any claims you make; in particular, if you claim that two propositions are logically equivalent, you must prove this, and if you claim that they are *not* equivalent, you must explain why not.

Solution

Problem 13 /360

Write down a proposition logically equivalent to

$$\neg\left((P \vee Q) \Rightarrow (Q \wedge R)\right)$$

without using the symbol "$\Rightarrow$."

Solution

Problem 14 /360

Is the proposition

$$\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, (x + y = 0)$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 15 /360

Is the proposition

$$\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, (xy = 1)$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 16 /360

Is the proposition

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (xy = 1)$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 17 /360

Is the proposition

$$\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, (x > y)$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 18 /360

Is the proposition

$$\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\, \exists z \in \mathbb{R}\ ((x < z) \wedge (z < y))$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 19 /360

Is the proposition

$$\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\ ((x < y) \Rightarrow \exists z \in \mathbb{R}\ ((x < z) \wedge (z < y)))$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 20 /360

Is the proposition

$$\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, \forall z \in \mathbb{R}\ \left(x + y < z^2\right)$$

true or false? If it is false, give a counterexample. If it is true, give a proof. You may use the fact that the square of a real number is nonnegative.

Solution

Problem 21 /360

Let $P(x, y)$ be a proposition with free variables x, y. Is the statement

$$(\exists x \, \forall y \, P(x, y)) \Rightarrow (\forall y \, \exists x \, P(x, y))$$

true or false? If it is false, give a counterexample. If it is true, give a proof.

Solution

Problem 22 /360

Consider the following proposition:

> If x and y are irrational, then $x + y$ is irrational.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 23 /360

Consider the following proposition:

> If x and y are rational, then $x + 2y$ is rational.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 24 /360

Consider the following proposition:

> Let $a, b, c \in \mathbb{Z}$. If a divides b and a divides c, then a divides $17a + 13b$.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 25 /360

Consider the following proposition:

> Let $a \in \mathbb{Z}$. The integer $a^3 + 3a^2 + 2a$ is divisible by three.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 26 /360

Consider the following proposition:

> Let $a, b \in \mathbb{Z}$. If a and b are both even or both odd, then $a + b$ is odd.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 27 /360

Consider the following proposition:

> Let $a, b \in \mathbb{Z}$. If a and b are both odd, then ab is odd.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 28 /360

Consider the following proposition:

> Let $a \in \mathbb{Z}$. If a is even, then a^6 is even.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 29 /360

Consider the following proposition:

> Let $a, b \in \mathbb{Z}$. If $a + b$ is odd, then a is even or b is even.

If the proposition is true, prove it; if not, give a counterexample.

Solution

Problem 30 /360

Let $P(x)$ and $Q(x)$ be propositions, and consider

$$((\forall x\, P(x)) \wedge (\forall x\, Q(x))) \quad \Rightarrow \quad (\forall x\, (P(x) \wedge Q(x))) \tag{1}$$
$$((\exists x\, P(x)) \wedge (\exists x\, Q(x))) \quad \Rightarrow \quad (\exists x\, (P(x) \wedge Q(x))) \tag{2}$$

Is (1) true or false? Is (2) true or false? Explain your answer.

Solution