Lecture 32: Cauchy's theorem

Math 660—Jim Fowler

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and the integral vanishes.

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Theorem

If $f(z)$ is analytic in Ω, then

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for every cycle γ which is homologus to zero in Ω.

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Suppose $f(z)$ is analytic in Ω, a simply connected region. Then

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This is a theorem that probably belongs more properly to a topology course.

Homology basis

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for cycles γ_i which "go around" A_i.
The collection of γ_i are a **homology basis** for Ω.

Periods

If $[\gamma] = [c_1\gamma_1 + \cdots c_n\gamma_n]$, then

$$\int_\gamma f\,dz = \sum_j c_j \int_{\gamma_i} f\,dz$$

so every integral is a sum of $\int_{\gamma_i} f\,dz$ (the "periods") with integral coefficients.

Compute some periods

Residues