Lecture 44: Partial fractions and Infinite Products

Math 660—Jim Fowler

Friday, August 20, 2010

Final exam

11:30A.M.–1:18P.M. on Tuesday, August 24, 2010.

Review session

During lecture on Monday.

Partial fractions

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

Infinite products

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By convention, the infinite product $\prod_{n=1}^{\infty} p_n$ converges if only finitely many factors are zero, and if the product converges (in the above sense) after removing those zeroes.

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So we may write $p_n = 1 + a_n$ with $\lim a_n = 0$.

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Why?

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Convergence of $\prod(1+a_n)$ implies convergence of $\sum \log(1+a_n)$, although $\sum \log(1+a_n)$ may not converge to the principal branch log of $\prod(1+a_n)$, it converges to some logarithm of $\prod(1+a_n)$.

Infinite products analyzed as series

In fact, $\sum \log(1 + a_n)$ converges absolutely iff $\sum |a_n|$ converges.

Expressing functions as products

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$\square$

Function with finitely many zeroes

If $f(z)$ has finitely many zeroes $a_1, \ldots, a_N$ and a zero of order m at the origin, then we can write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right).$$

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In general, need to include extra “convergence producing” factors.

Weierstrass showed that

$$f(z) = z^m \; e^{\phi(z)} \; \prod_{n=1}^{\infty} \left(1 - \frac{z}{u_n}\right) e^{\frac{z}{u_n} + \frac{1}{2}\left(\frac{z}{u_n}\right)^2 + \cdots + \frac{1}{\lambda_n}\left(\frac{z}{u_n}\right)^{\lambda_n}}$$

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Corollary

Every function meromorphic in the whole plane is a quotient of entire functions.

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

with $x = \pi/2$, gives

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2 \cdot 4}\right)\left(1 - \frac{1}{2^2 \cdot 9}\right)\cdots = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right),$$

and taking the reciprocal,

$$\begin{aligned}
\frac{\pi}{2} &= \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2 - 1}\right) \\
&= \prod_{n=1}^{\infty}\frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots.
\end{aligned}$$