Facts about series

We are comfortable with numbers that "go all the way to the right" (i.e., non-terminating decimals like $0.3333\cdots$) so why not numbers that go all the way to the **left**?

I mean, consider a "number" like $\cdots 999999$, meaning $\sum_{n=0}^{\infty} 9 \cdot 10^n$. Of course, this is meaningless, but if we **ignore convergence issues** and apply the formula for geometric series, we might be fooled into thinking $\cdots 999999 = -1$. This is possibly less ridiculous than it seems, because

$$\begin{array}{rr} & \cdots 99999999 \\ + & 1 \\ \hline & \cdots 00000000 \end{array}$$

which we see by adding 9+1, getting 10, writing down a 0 and carrying the 1—and repeating. A number which vanishes when we add 1 to it ought to be called "-1." For similar reasons, we might believe $\cdots 11111111 = -1/9$, because if we multiply $\cdots 11111111$ by 9, we get the number for -1.

We can show $-1 \times -1 = 1$, because

$$\begin{array}{rr} & \cdots 99999999 \\ \times & \cdots 99999999 \\ \hline & \cdots 99999991 \\ & \cdots 99999910 \\ & \cdots 99999100 \\ & \cdots 99991000 \\ & \vdots \\ \hline & \cdots 00000001 \end{array}$$

How about one third?

There are fancier examples in this crazy world, too. Because

$$\begin{array}{rr} & \cdots 66666667 \\ \times & 3 \\ \hline & \cdots 00000001 \end{array}$$

so $\cdots 66666667$ deserves to be called $1/3$, since it is a multiplicative inverse for 3. But there is another reason why $\cdots 66666667 = 1/3$. After all, if $\cdots 11111111 = -1/9$, then $\cdots 66666666$ is $-6/9 = -2/3$. And therefore,

$$\begin{array}{rrl} & \cdots 66666666 & (\text{“}-2/3\text{”}) \\ \times & 1 & \\ \hline & \cdots 66666667 & (\text{“}1/3\text{”}) \end{array}$$

How about one seventh?

I wanted to write down $1/7$, so I started with a 3 (since $3 \times 7 = 21$, and this will give me the 1 on the right hand side). The next digit should be a 4, because $4 \times 7 = 28$, and since I had to carry that 2, I will get 30, which means I will write down a zero. Now I am carrying a 3; but if I put a 1 as the next digit, then $1 \times 7 + 3 = 10$, so I will write down a zero, and carry a 1. Each time the next digit is designed so that I write down a zero, and carry something. I discover:

$$\begin{array}{rr} & \cdots 285714285714285714285714 3 \\ \times & 7 \\ \hline & \cdots 0000000000000000000000001 \end{array}$$

This is a repeating decimal: we might write it as $\overline{285714}3$, though here the digits repeat to the left. This means we could also write it as a series, formally:

$$3 + 10 \cdot \left(285714 \cdot \sum_{n=0}^{\infty} 1000000^n \right)$$

If we ignore convergence, and apply the formula for geometric series here, where it does not apply, we might be fooled into thinking

$$3 + 10 \cdot \left(285714 \cdot \frac{1}{1 - 1000000} \right) = 1/7$$

And even though this series does not converge, it does have the appearance of being equal to $1/7$.

How about $\sqrt{3}$?

$$\begin{array}{rr} & \cdots 285714285714285714285714 3 \\ \times & \cdots 7 \\ \hline & \cdots 0000000000000000000000001 \end{array}$$