# Poincaré Duality **Math 758**

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**Problem 1.** The **Euler characteristic** of a space $X$ with finite dimensional homology is
$$\chi(X) = \sum_n (-1)^n \dim H_n(X;\mathbb{Q}).$$
If $X$ is a finite simplicial complex, show that this is the same as
$$\chi(X) = \sum_n (-1)^n \dim C_n(X;\mathbb{Q}),$$
where $C_n(M;\mathbb{Q})$ is the $n$-dimensional simplicial chain groups with rational coefficients.

**Problem 2.** Let $M^3$ be a closed oriented 3-manifold with a given PL triangulation (so the **link** of each vertex is a sphere); show, by using the formula for $\chi$ in terms of chains, that $\chi(M) = 0$.

**Problem 3.** If $M$ is an odd-dimensional closed oriented manifold, show that $\chi(M) = 0$.

**Problem 4.** Let $M^n$ be an oriented closed $n$-manifold. Show that the pairing
$$\smile : H^k(M;\mathbb{Q}) \times H^{n-k}(M;\mathbb{Q}) \to H^n(M;\mathbb{Q}) = \mathbb{Q}$$
is nonsingular form by considering the adjoint map
$$H^k(M;\mathbb{Q}) \to \operatorname{Hom}(H^{n-k}(M;\mathbb{Q}), H^n(M;\mathbb{Q})).$$

**Problem 5.** Suppose $M^6$ is an oriented closed 6-manifold. Show that the pairing
$$\smile : H^3(M;\mathbb{Q}) \to H^3(M;\mathbb{Q}) \to H^6(M;\mathbb{Q})$$
is a nonsingular skew-symmetric bilinear form over $\mathbb{Q}$. What does this imply about $\dim H^3(M;\mathbb{Q})$?

**Problem 6.** Use Poincaré duality to compute the cup product structure on
$$H^\star(\mathbb{R}P^m;\mathbb{Z}/2).$$

**Problem 7.** Let $X$ and $Y$ be simplicial complexes with disjoint sets of vertices; in what follows, we regard a simplicial complex as simply a set of subsets of a vertex set, closed under taking of subsets.

The **join** of $X$ and $Y$ is denoted by $X \star Y$, and is defined as follows
$$X \star Y = \{\sigma \cup \tau : \sigma \in X \text{ and } \tau \in Y\},$$
If $X$ is homeomorphic to the $n$-sphere $S^n$ and $Y$ is homeomorphic to the $m$-sphere $S^m$, describe the homeomorphism type of $X \star Y$.

**Problem 8.** Consider a $\mathbb{Z}/5$ action on $S^1$; by taking the join of $S^1$ with itself, produce a $\mathbb{Z}/5$ action on $S^{2n+1}$. The quotient of this sphere by the $\mathbb{Z}/5$ action is a **lens space** $L^{2n+1}$.

Pick a generator $\alpha \in H^1(L^{2n+1};\mathbb{Z}/5)$ and a generator $\beta \in H^2(L^{2n+1};\mathbb{Z}/5)$. Show that $\alpha$ and $\beta$ generate $H^\star(L^{2n+1};\mathbb{Z}/5)$ as a ring.

**Problem 9.** Exhibit two noncompact surfaces which are not homeomorphic.