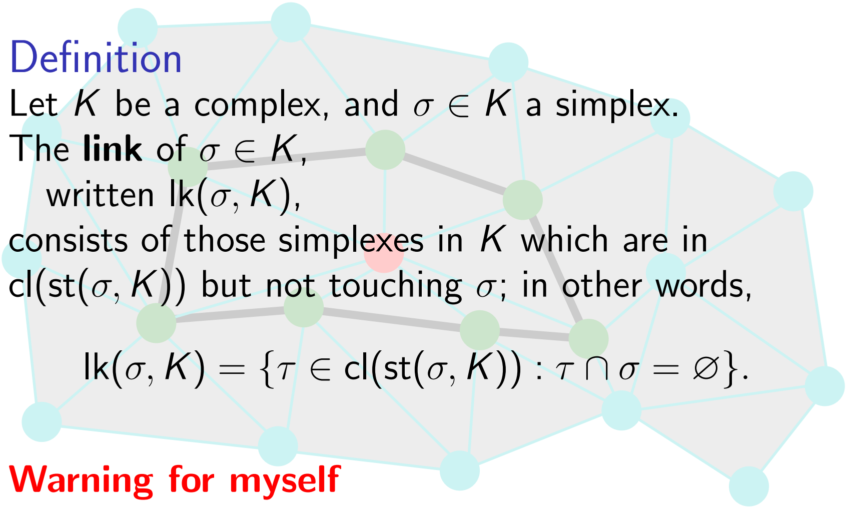
Topology of Piecewise-Linear Manifolds

Jim Fowler

Lecture 4
Summer 2010

Link



Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **link** of $\sigma \in K$,
written $\mathrm{lk}(\sigma, K)$,
consists of those simplexes in K which are in $\mathrm{cl}(\mathrm{st}(\sigma, K))$ but not touching σ; in other words,

$$\mathrm{lk}(\sigma, K) = \{\tau \in \mathrm{cl}(\mathrm{st}(\sigma, K)) : \tau \cap \sigma = \varnothing\}.$$

Warning for myself

The other definition also appears in the literature.

Stellar Subdivision

Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **stellar subdivision** of K at σ is a new complex K_σ with:

- the vertices of K with a new vertex v.
- the simplexes of K not in $\mathrm{st}(\sigma, K)$, along with the simplexes in $v * (\partial\sigma) * \mathrm{lk}(\sigma, K)$.

We might say:

$$K_\sigma := (K - \mathrm{st}(\sigma, K)) \cup (v * (\partial\sigma) * \mathrm{lk}(\sigma, K))$$

Some names

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Can you compute $\chi(CX)$?

$$\chi(CX) = \chi(\text{point}) + \chi(X) - \chi(\text{point})\chi(X) = 1.$$

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ST^2 is not a manifold.

Homeomorphisms

Problem 5. Is PL homeomorphism an equivalence relation? That is, is it:

reflexive, meaning, is A homeomorphic to A,

symmetric, meaning, if $A \cong B$, is it true that $B \cong A$, and

transitive, meaning, if $A \cong B$ and $B \cong C$, is it true that $A \cong C$?

- **Problem 6.** Is join well-defined with respect to homeomorphism? That is, if we have PL homeomorphic complexes $K \cong K'$ and $L \cong L'$, is it then the case that $K * L \cong K' * L'$?

from the last

homework

Problem 7. Let X_n consist of n points. For which $n \in \mathbb{N}$ is it the case that for every two injective maps $f, g : X_n \to S^1$ there is a homeomorphism $h : S^1 \to S^1$ so that $h \circ f = g$?

Problem 8. Let X_n be the disjoint union of n circles (e.g., $X_3 = S^1 \cup S^1 \cup S^1$). For this problem, call two maps $f, g : X_n \to S^2$ "equivalent" if there exists a homeomorphism $h : S^2 \to S^2$ so that $h \circ f = g$. Count the equivalence classes of maps from X_5 to S^2?

Problem 9. To the right, three curves are pictured on $T^2 \# T^2$, which is our notation for a two-holed surface. For which pairs of curves α and β does there exist a homeomorphism

$$f : T^2 \# T^2 \to T^2 \# T^2$$

so that $f(\alpha) = \beta$?

Composition is key

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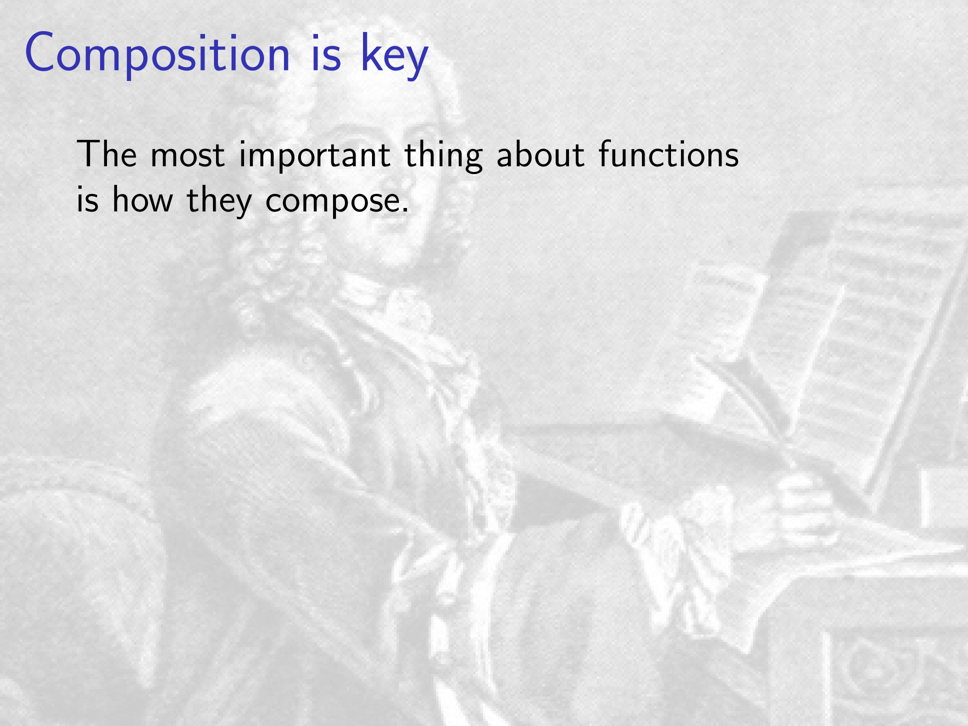
The most important thing about functions is how they compose.



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Evaluating a function $f : A \to B$ at a point $x \in A$ is really composition of a "point function"

$$\star \to A$$

with f, to get a new point function $\star \to B$.

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To know how the function composes is to know the function.

Composing Maps

Problem

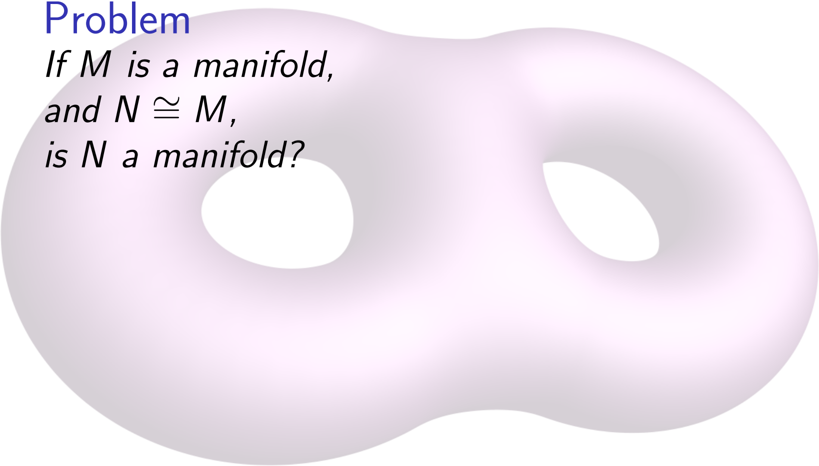
Let K, L, M be complexes.
If $f : K \longrightarrow L$ and $g : L \longrightarrow M$ are PL maps,
how should we define the PL map $g \circ f : K \longrightarrow M$?

We need a more general notion of "PL map"
in order to do this!

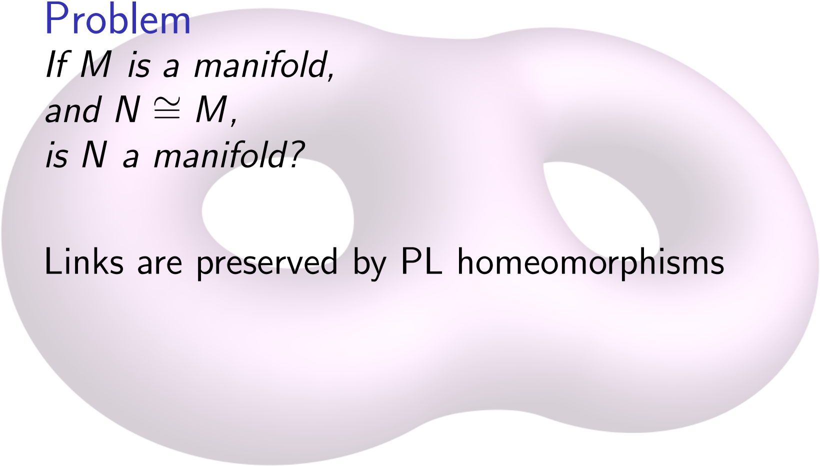
Manifoldness preserved?

Problem

If M is a manifold,
and $N \cong M$,
is N a manifold?



Manifoldness preserved!



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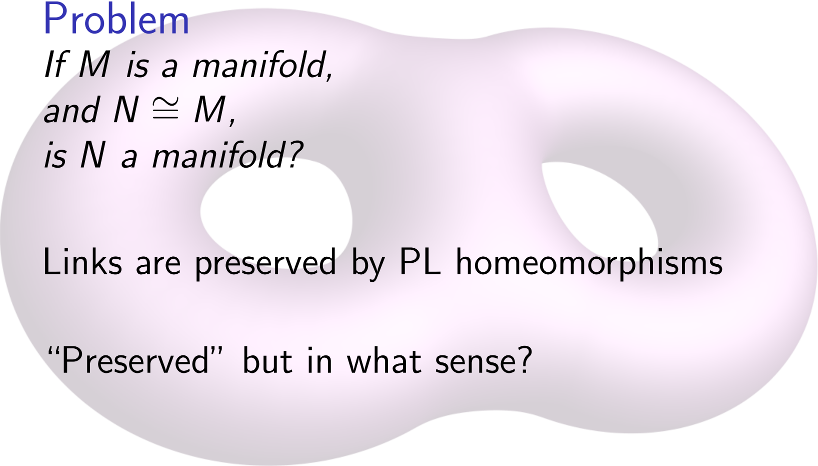
Links are preserved by PL homeomorphisms

Manifoldness preserved!



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“Preserved” but in what sense?

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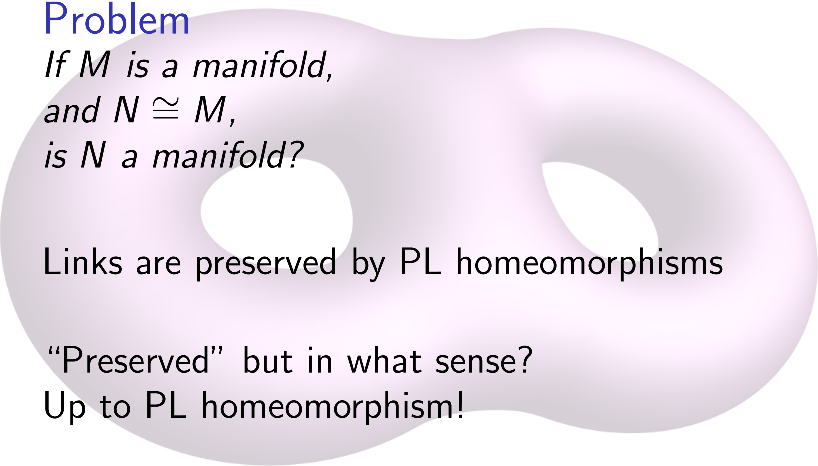

Problem

If M is a manifold,
and $N \cong M$,
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Links are preserved by PL homeomorphisms

"Preserved" but in what sense?
Up to PL homeomorphism!

Where are we?

Granby
116
Boul. Laurier
141
1 km
20
OUEST
Montréal
211
SUD
St-Valérien-de-Milton
143
SORTIE
45
km/h

Where are we?

Objects

Where are we?

Objects

PL manifolds

Where are we?

Objects

PL manifolds

simplicial complexes

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simplicial complexes

Maps

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Invariants

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simplicial complexes

Maps

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simplicial maps

Invariants

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Where are we?

Objects

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Invariants

χ b_0

Next steps

Next steps

- New constructions

Next steps

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  Cartesian product

Next steps

- New constructions
  Cartesian product and
  connected sum

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- New constructions
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- New invariant
  orientability, w_1
- New kind of equivalence
  simple homotopy equivalence

$$\#$$

connected sum

Connected sum

Definition

Suppose K and L are two n-manifolds.

Let K' be K with one n-simplex removed.
$\quad\;\; L' \qquad L$

Define $K\#L = K' \cup_{\partial\Delta^n} L'$.

Connected sum

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What is $T^2 \# T^2$?

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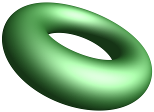
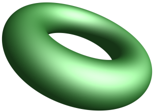
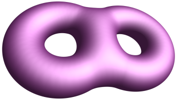
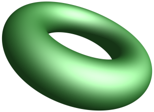
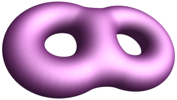
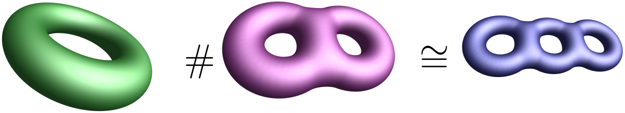
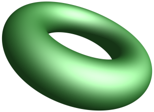
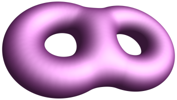
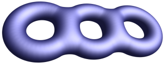
Problem

What is $S^2 \# T^2$?

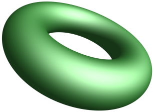
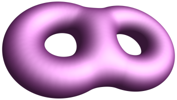
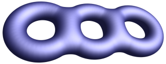
 $\#$  $\cong$ 

 $\#$  $\cong$



 $\#$  $\cong$ 

In general, $\Sigma_g =$ genus g surface,

 $\#$  $\cong$ 

In general, $\Sigma_g =$ genus g surface,

and $T^2 \# \Sigma_g = \Sigma_{g+1}$.

Covering map

Definition

$f : M \longrightarrow N$ is an n-fold **covering map**
if there is a subdivision N' of N,
so that for every vertex $v \in N'$
$f^{-1}(\mathrm{cl}\,\mathrm{st}(v, N)) \cong n$ copies of $\mathrm{cl}\,\mathrm{st}(v, N)$,
and f is a homeo onto each copy of $\mathrm{cl}\,\mathrm{st}(v, N)$.

Gift-wrapping surfaces

Problem

When is there an n-fold covering map from a genus g surface to a genus g' surface?

Gift-wrapping surfaces

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When is there an n-fold covering map
from a genus g surface
to a genus g′ surface?

Theorem

If $f : M \to N$ is an n-fold covering map
then $\chi(M) = n \cdot \chi(N)$.

Gift-wrapping surfaces

Problem

When is there an n-fold covering map from a genus g surface to a genus g' surface?

Theorem

If $f : M \to N$ is an n-fold covering map then $\chi(M) = n \cdot \chi(N)$.

Problem

What is χ(genus g surface)?

$$\times$$

Cartesian product

Products, geometrically

Definition

Suppose $K \subset \mathbb{R}^n$, and $L \subset R^m$.
The product of K and L is

$$K \times L = \{(k, \ell) \in \mathbb{R}^{n+m} : k \in K, \ell \in L\}.$$

Products, geometrically

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How to do this for a simplicial complex?

Product of simplexes

How do we take a product of Δ^1 and Δ^1?

Product of simplexes

How do we take a product of Δ^1 and Δ^2?

Product of simplexes

How do we take a product of Δ^2 and Δ^2?

Product of simplexes

How do we take a product of Δ^2 and Δ^2?
Hard to see, so let's take a step back.

Properties of products

The product of Δ^n and Δ^m should have projection maps

$$p_1 : \Delta^n \times \Delta^m \to \Delta^n$$
$$p_2 : \Delta^n \times \Delta^m \to \Delta^m$$

so that a point in $\Delta^n \times \Delta^m$ is determined by where it lands under the two projection maps.

Vertices of $\Delta^n \times \Delta^m$ are pairs (i, j) with $0 \leq i \leq n+1$ and $0 \leq j \leq m+1$.

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A simplex of $\Delta^n \times \Delta^m$ is [redacted]

A homework problem!

Some counting problems

$\Delta^1 \times \Delta^n$ has $(n+1)!$ triangulations.

$\Delta^2 \times \Delta^n$ has [redacted] triangulations.

Some 4-manifolds

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$$S^3 \times S^1$$

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$$S^3 \times S^1$$
$$S^2 \times S^2$$

Some 4-manifolds

$$S^3 \times S^1$$
$$S^2 \times S^2$$
$$\Sigma_g \times \Sigma_h$$

w_1

orientability

The torus versus the Klein bottle

Since $\chi(T^2) = \chi(K) = 0$,
we can't distinguish T^2 and K using χ.

The torus versus the Klein bottle

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we can't distinguish T^2 and K using χ.

We need a new invariant.

Orientation

An orientation of a simplex is
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An orientation on a manifold M^n
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so that neighboring n-simplexes
induce opposite orientations on the shared
$(n-1)$-simplex.

If M admits an orientation, $w_1(M) = 0$.
Otherwise, $w_1(M) = 1$.

Orientability for surfaces

Theorem

A surface Σ is not orientable
iff Σ contains a Möbius strip.

Problem

What is $w_1(T^2)$?

Problem

What is $w_1(K)$?

Problem

Is w_1 a PL homeomorphism invariant?

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

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No

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

No...

We need more invariants,

Distinguishing other manifolds?

Can we distinguish T^3 and S^3 with Euler characteristic?

No...

We need more invariants,
or a looser notion of “the same.”

Simplicial Collapse

Principal Simplexes

Definition

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$\sigma \in K$ a simplex.
Call σ a **principal simplex**
if the only simplex containing σ
is σ itself
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Problem

Does every complex have a principal simplex?

Free faces

Definition

Let K be a complex,
and $\sigma \in K$ a simplex,
and $\tau < \sigma$ a face.

Call τ a **free face** of σ
if the only simplexes containing τ
are τ and σ.

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Problem

Does every complex have a simplex with a free face?

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Problem

Does every complex have a simplex with a free face?
Does any complex have a simplex with a free face?

Elementary simplicial collapse

Definition

Let L and $K = L \cup \mathrm{cl}\{\sigma, \tau\}$ be complexes

If σ is a principal simplex of K, and
τ is a free face of σ, then
L is an **elementary simplicial collapse** of K.

Simplicial collapse

Definition

Let $K_1, K_2, \ldots, K_n$ be complexes, with K_{i+1} an elementary simplicial collapse of K_i.

Call K_n a **simplicial collapse** of K_1, and write $K_1 \searrow K_n$.

Call K_1 a **simplicial expansion** of K_n, and write $K_n \nearrow K_1$.

Simple homotopy equivalence

Definition

K is **simple homotopy equivalent** to L
(sometimes abbreviated s.h.e.)
if you can reach transform K into L
via a sequence of

- PL homeomorphisms,
- simplicial collapses,
- simplicial expansions.

In this case, we write $K \searrow\!\!\nearrow L$.

Next time

UNKNOTTING SPHERES IN FIVE DIMENSIONS

BY E. C. ZEEMAN

Communicated by S. Eilenberg, December 26, 1959

Given a semi-linear embedding of S^2 in euclidean 5-space, we show that it is unknotted.

Join it up to a vertex V in general position. If the cone VS^2 is non-singular we are finished. Otherwise, for dimensional reasons, there are at most a finite number of singularities, where just two points of S^2 are collinear with V. Let's have V away on one side, so that at each singularity we can call one point "near" and the other point "far." Now separate the near and far points by an equator S^1, so that all the near points lie in the northern hemisphere A, and all the far points lie in the southern hemisphere B.

Let $\hat{S}^2$ be the sphere $VS^1 \cup B$. Then $\hat{S}^2$ is equivalent to S^2, because they differ by the boundary of the ball VA, whose interior does not meet them. But $\hat{S}^2$ is unknotted because it bounds, and does not meet the interior of, the ball VB. Hence S^2 is unknotted.

REMARK 1. The argument generalizes to unknotting S^n in k-space, $k \geq (3/2)(n+1)$.

Next time

- Regular neighborhoods
- Knotting S^1 in S^3
- Unknotting S^1 in S^4
- How to unknot with simplicial collapse

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