# Intersections, Linking **Math 758**

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First, something that came up during office hours.

**Problem 1.** Let $M^n$ be a closed manifold with a fixed triangulation; let $M'$ be its barycentric subdivision, and let $w_1$ be the $\mathbb{Z}/2$-chain given by adding up all the edges in $M'$. Is $w_1$ a cocycle? When is $w_1$ a coboundary?

Next, we'll talk about submanifolds, intersections, and linking. At this point, the "special topics" start, so if there are particular things you're eager to learn about, we should talk about those things.

**Remark 2.** For an oriented manifold $M^m$ with submanifolds $A^a$ and $B^b$ intersecting transversely, then

$$[A \cap B] = ([A]^\star \smile [B]^\star)^\star$$

where $\star$ denotes Poincaré duality, and the inclusion maps are not mentioned.

One proof of this goes via the **Thom isomorphism theorem** which is the purview of a future course—we haven't defined what "transversely" means, even.

**Problem 3.** When $a + b = m$, we can regard $[A \cap B]$ as a number. Is this number the same as the cardinality of the set $A \cap B$? If we count with orientation? Can we isotope $A$ and $B$ to make $|A \cap B| = [A \cap B]$?

**Problem 4.** Compute $\pi_1$ of the complement of the "unlink" and the "Hopf link." Why might this fact might put a mountain climber in danger?

**Problem 5.** Use Alexander duality to compute $H_1(S^3 - A)$ for a curve $A$. Can you describe a generator?

**Definition 6.** For two curves $A$ and $B$ in $S^3$, their **linking number** $\operatorname{link}(A, B)$ is defined to be

$$[A] = \operatorname{link}(A, B)[m] \in H_1(S^3 - B)$$

where $m$ is the **meridian** of the curve $B$.

**Problem 7.** Let $N(B)$ be a thickened, closed neighborhood of the curve $B$, and let $\Sigma$ be an oriented connected surface with $\partial\Sigma = B$. Show that $\operatorname{link}(A, B) = [A \cap \Sigma]$ by using a certain geometric fact about the meridian.

**Problem 8.** How does $\operatorname{link}(A, B)$ relate to $\operatorname{link}(B, A)$?

**Problem 9.** Compute linking number of the Hopf link.

**Problem 10.** Compute linking number of the Whitehead link.

**Problem 11.** Suppose $B$ is a curve, and that there is a surface $\Sigma$ in $S^3 - B$ with boundary $A_1 \sqcup A_2$ with $A_1$ and $A_2$ connected curves; how does $\operatorname{link}(A_1, B)$ relate to $\operatorname{link}(A_2, B)$?