Lecture 35: Some integrals

Math 660—Jim Fowler

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Generalizing argument principle

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If $g(z)$ is analytic in Ω, then $g(z)\dfrac{f'(z)}{f(z)}$ has residue $h\,g(a)$ at a zero a of order h and residue $-h\,g(a)$ at a pole a of order h. Therefore,

$$\frac{1}{2\pi i}\int_{\gamma} g(z)\frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j)g(a_j) - \sum_j n(\gamma, b_k)g(b_k)$$

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so $z_j(w)$ are roots of a polynomial with coefficients depending analytically on w.

Integration tricks!

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  need tricks for converting integrals over intervals to integrals over curves.
- The integrals we care about might be for real-valued functions;
  thankfully, many real integrals are actually of analytic functions.

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$$-i \int_{\text{unit circle}} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \frac{dz}{z}$$

Example 1

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} \text{ for } a > 1.$$

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$$\int_0^{2\pi} \cos^n \theta \, d\theta = -i \int_{\text{circle}} \frac{1}{2^n} \left(z + \frac{1}{z} \right)^n \frac{dz}{z}$$
$$= -i \int_{\text{circle}} \frac{1}{2^n} \left(\sum_m \binom{n}{m} z^{2m-n-1} \right) dz$$

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\int_0^{2\pi} \cos^n \theta \, d\theta &= -i \int_{\text{circle}} \frac{1}{2^n} \left(z + \frac{1}{z} \right)^n \frac{dz}{z} \\
&= -i \int_{\text{circle}} \frac{1}{2^n} \left(\sum_m \binom{n}{m} z^{2m-n-1} \right) dz \\
&= -i \int_{\text{circle}} \frac{1}{2^n} \binom{n}{n/2} z^{-1} dz \\
&= \pi \frac{1}{2^{n-1}} \binom{n}{n/2}
\end{aligned}$$

Rational functions

By integrating over large semicircles, compute

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} R(x)\,dx = \sum_{\operatorname{Im} z>0} \operatorname*{Res}_{z=a} R(z)$$

Rational functions times e^{iz}

By integrating over large semicircles, compute

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} R(x)\, e^{iz}\, dx = \sum_{\operatorname{Im} z>0} \operatorname*{Res}_{z=a} R(z)e^{iz}$$

provided $R(z)$ has a zero of order two at infinity.

Rational functions times e^{iz}

Integrate over rectangle with vertices $x_2, x_2 + iy, -x_1 + iy, -x_1$.

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} R(x)\, e^{iz}\, dx = \sum_{\operatorname{Im} z>0} \operatorname*{Res}_{z=a} R(z)e^{iz}$$

which holds provided $R(z)$ has a zero at infinity.

Powers of z

$$\int_0^{\infty} x^{\alpha} R(x)\, dx = 2 \int_0^{\infty} t^{2\alpha+1} R(t^2)\, dt$$

and then integrate over semicircle minus semicircle.

Example

$$\int_0^{\pi} \log \sin \theta \, d\theta$$