Lecture 3: Polynomials and rational functions

Math 660—Jim Fowler

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Review

Provided the limit exists, the derivative $f'(z)$ is

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Why the Cauchy-Riemann equations?

Compute limit in two different directions,

$$\lim_{\epsilon \to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon} = \frac{\partial f}{\partial x}(z)$$

versus

$$\lim_{\epsilon \to 0} \frac{f(z+i\epsilon) - f(z)}{i\epsilon} = \lim_{\epsilon \to 0} -i \frac{f(z+i\epsilon) - f(z)}{\epsilon}$$

and therefore

$$\frac{\partial f}{\partial x}(z) = -i \frac{\partial f}{\partial y}(z).$$

Cauchy-Riemann $\Rightarrow$ analytic

Suppose $f(z) = 0$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at zero. Then

$$f(z) = \left(\frac{\partial f}{\partial x}\right)(0)\,x + \left(\frac{\partial f}{\partial y}\right)(0)\,y + c(z)\,z,$$

where $c(z) \to 0$ as $z \to 0$. Now $z + \bar{z} = 2x$ and $z - \bar{z} = 2iy$, so $f(z)$ equals

$$\frac{\left(\frac{\partial f}{\partial x}\right)(0) - \left(\frac{\partial f}{\partial y}\right)(0)\,i}{2} z + \frac{\left(\frac{\partial f}{\partial x}\right)(0) + \left(\frac{\partial f}{\partial y}\right)(0)\,i}{2} \bar{z} + c(z)\,z$$

so

$$\frac{f(z)}{z} = \left(\frac{\partial f}{\partial z}\right)(0) + \left(\frac{\partial f}{\partial \bar{z}}\right)(0) \cdot \frac{\bar{z}}{z} + c(z)$$

- **Analytic** refers to power series.
- **Holomorphic** refers to differentiability.
- A theorem relates the two notions.

More on complex derivatives

$f : \mathbb{C} \to \mathbb{C}$ gives rise to $f : \mathbb{R}^2 \to \mathbb{R}^2$.
$i : \mathbb{R}^2 \to \mathbb{R}^2$ is the "rotate by $\pi/4$" map.

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}.$$

If f is holomorphic, then $Df \circ i = i \circ Df$.
If f is holomorphic, then Df is a combination of rotation and scaling.

$$\frac{\partial f}{\partial x} = -j\frac{\partial f}{\partial y}$$

$$j\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

Different, or the same?

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$
$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$
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All the same by the Cauchy-Riemann equations.

Cauchy-Riemann equations

Write a function $f : \mathbb{C} \to \mathbb{C}$ in terms of z and $\overline{z}$.

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

"*Analytic* means f doesn't depend on $\overline{z}$."

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- $f(z) = 17i$.
- $f(z) = \overline{z}$.
- $f(z) = \sin z + i \cos \operatorname{imag} z$.

A formal method

Goal: Given u, find v so that $f = u + iv$ is analytic.

$\overline{f(z)}$ is a function of $\overline{z}$.

$$u(x, y) = \frac{f(x + iy) + \overline{f}(x - iy)}{2}.$$

Substitute in $x = z/2$ and $y = -iz/2$. Guess

$$f(z) = 2u(z/2, iz/2).$$

Example: $u(x, y) = x^2 - y^2$; then $f(z) = z^2$.

Section 2.1.3 and 2.1.4

polynomials
and
rational functions

Polynomials

$$p(z) = c_n z^n + \cdots + c_1 z + c_0$$

Theorem (Fund. Theorem of Algebra)

Let $p(z)$ be a polynomial.
Then there exists $z \in \mathbb{C}$ so that $f(z) = 0$.

Theorem

Every polynomial $p(z)$ can be written as

$$p(z) = \lambda(z - a_1)(z - a_2) \cdots (z - a_n)$$

for complex numbers $\lambda, a_1, a_2, \ldots, a_n$.

Zeroes of a polynomial

A polynomial p has a zero of order n at w if

$$p(w) = 0, p'(w) = 0, \ldots, p^{(n-1)}(w) = 0$$

but $p^{(n)}(w) \neq 0$.

Equivalently, $(z - w)^n$ divides $p(z)$.

Theorem (Gauss–Lucas)

Let $p(z)$ be a polynomial;
if all zeroes of p lie in a half plane,
then all zeroes of p' lie in the same half plane.

Proof

$p(z) = \lambda(z - a_1)(z - a_2)\cdots(z - a_n).$

$$\frac{p'(z)}{p(z)} = \frac{1}{z - a_1} + \cdots + \frac{1}{z - a_n}$$

Let $H = \{z \in \mathbb{C} : \operatorname{imag}(z - a)/b < 0\}$.
If $a_k \in H$ but $z \notin H$, then

$$\operatorname{imag} \frac{z - a_k}{b} = \operatorname{imag} \frac{z - a}{b} - \operatorname{imag} \frac{a_k - a}{b} > 0$$

so $\operatorname{imag} b/(z - a_k) < 0$. So

$$\operatorname{imag} \frac{b\, p'(z)}{p(z)} = \sum \operatorname{imag} b/(z - a_k) < 0,$$

and $p'(z) \neq 0$.

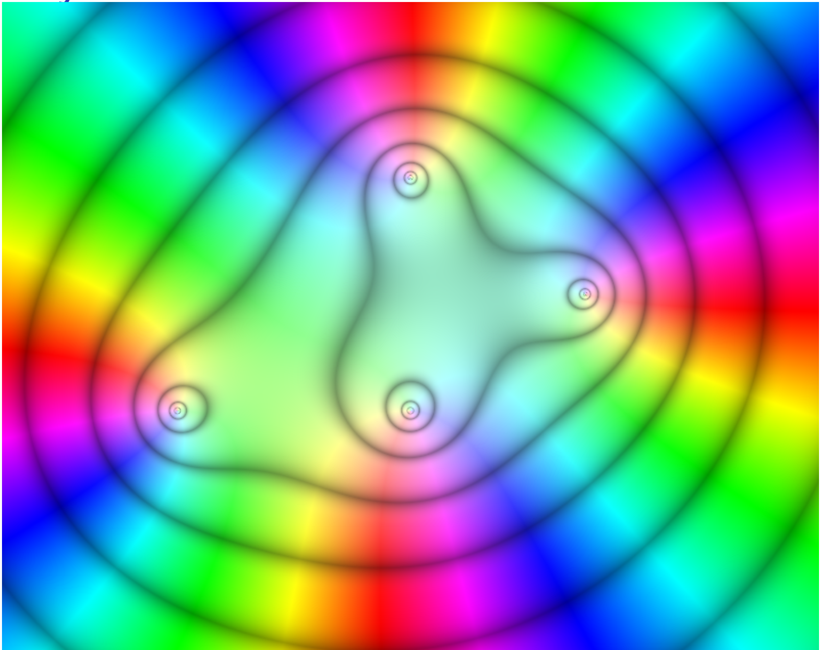
As a consequence of the preceding theorem:

Theorem

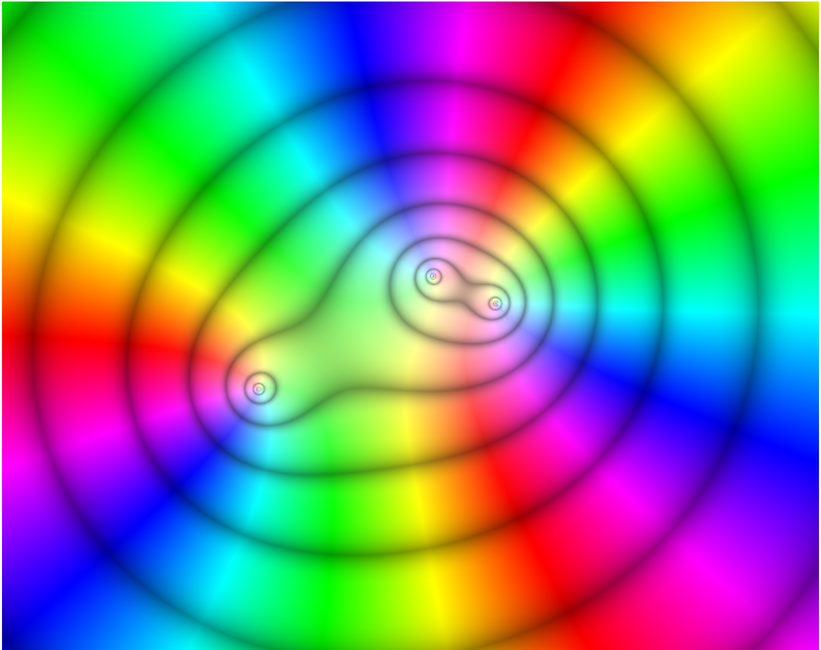
Let $p(z)$ be a polynomial;
the convex hull of the zeroes of p
contains the zeroes of p'.

Vaguely, the zeroes of p' are between the zeroes of p,
like in Rolle's theorem.

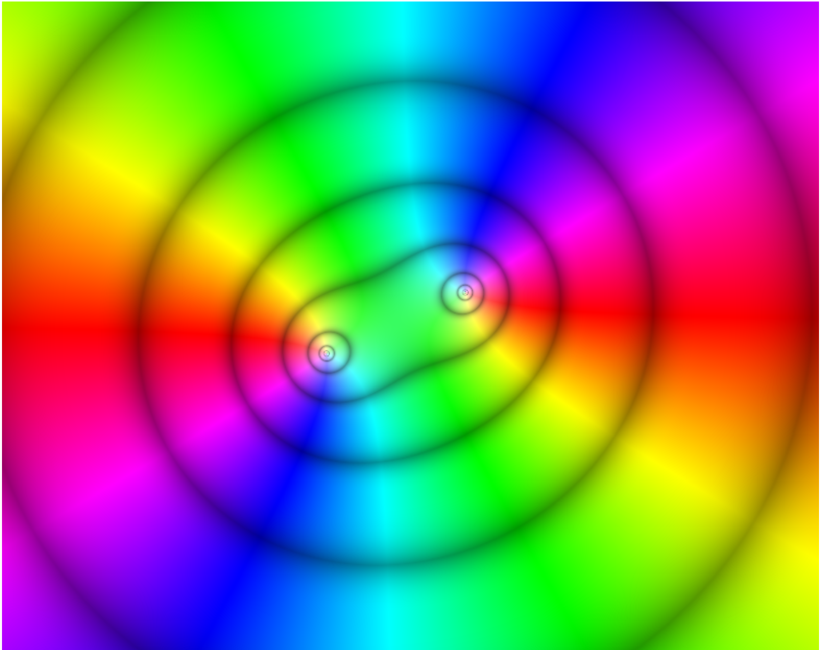
a polynomial with four roots



the derivative



the second derivative



the third derivative