Calculus of Finite Differences

Mathematics is not just about solving problems: it is about analogies (and analogies between analogies). If two things are perfectly analogous, they are exactly the same, so analogies are most interesting when they are imperfect.

In this spirit, I present a "dictionary," the term mathematicians often use when speaking of an analogy between two fields of study. In this case, the analogy is between functions and sequences, between calculus and "the calculus of finite differences."

Dictionary

$$\begin{array}{rcl}
\text{function } f & \rightsquigarrow & \text{sequence } \mathbf{a} \\
f(x) & \rightsquigarrow & \mathbf{a}_n \\
\text{derivative } \left(\frac{d}{dx} f\right)(x) & \rightsquigarrow & \text{difference } (D\mathbf{a})_n = \mathbf{a}_{n+1} - \mathbf{a}_n \\
\text{integral } F(x) = \displaystyle\int_0^x f(x)\,dx & \rightsquigarrow & \text{sum } \mathbf{b}_n = (I\mathbf{a})_n = \displaystyle\sum_{k=0}^{n} \mathbf{a}_k \\
\text{fundamental theorem of Calculus} & \rightsquigarrow & \text{telescoping sum} \\
\text{differential equation} & \rightsquigarrow & \text{difference equation}
\end{array}$$

A true statement about functions and their derivatives can be "translated" into a possibly true statement about sequences and their differences.

Notation

Sequences have bold names, e.g., $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The terms of a sequence are indexed by a subscript, e.g., $\mathbf{a}_n$; the index may be any integer, e.g., $\mathbf{a}_{-23}$. Operators, which transform one sequence into another, are denoted by a capital letter. Some operators include:

Difference	**Sum**	**Shift**
$(D\mathbf{a})_n = \mathbf{a}_{n+1} - \mathbf{a}_n$	$(\Sigma\mathbf{a})_n = \sum_{i=0}^{n} \mathbf{a}_i$	$(S\mathbf{a})_n = \mathbf{a}_{n+1}.$

The sum rule

The "derivative" of a sum is the sum of the "derivatives," as follows:

$$\begin{aligned}
(D(\mathbf{a}+\mathbf{b}))_n &= (\mathbf{a}+\mathbf{b})_{n+1} - (\mathbf{a}+\mathbf{b})_n \\
&= \mathbf{a}_{n+1} + \mathbf{b}_{n+1} - \mathbf{a}_n - \mathbf{b}_n \\
&= (\mathbf{a}_{n+1} - \mathbf{a}_n) + (\mathbf{b}_{n+1} - \mathbf{b}_n) \\
&= (D\mathbf{a})_n + (D\mathbf{b})_n .
\end{aligned}$$

Formally, this resembles $(f+g)'(x) = f'(x) + g'(x)$.

The fundamental theorem of Calculus

Suppose $\mathbf{a} = D\mathbf{b}$, so $\mathbf{b}$ is an "antiderivative" for $\mathbf{a}$. Then,

$$\begin{aligned}
(\Sigma \mathbf{a})_n &= (\Sigma D\mathbf{b})_n \\
&= \sum_{i=0}^{n} (\mathbf{b}_{i+1} - \mathbf{b}_i) \\
&= (\mathbf{b}_{n+1} - \mathbf{b}_n) + (\mathbf{b}_n - \mathbf{b}_{n-1}) + (\mathbf{b}_{n-1} - \mathbf{b}_{n-2}) + \cdots + (\mathbf{b}_1 - \mathbf{b}_0) \\
&= \mathbf{b}_{n+1} - \mathbf{b}_0.
\end{aligned}$$

Formally, this resembles the fact that if $f(x) = F'(x)$, then $\int_0^x f(x)\,dx = F(x) - F(0)$.

The product rule

We can take successive differences of the product of two sequences as follows:

$$\begin{aligned}
D(\mathbf{a} \cdot \mathbf{b})_n &= (\mathbf{a} \cdot \mathbf{b})_{n+1} - (\mathbf{a} \cdot \mathbf{b})_n \\
&= \mathbf{a}_{n+1} \cdot \mathbf{b}_{n+1} - \mathbf{a}_n \cdot \mathbf{b}_n \\
&= \mathbf{a}_{n+1} \cdot \mathbf{b}_{n+1} - \mathbf{a}_n \cdot \mathbf{b}_{n+1} + \mathbf{a}_n \cdot \mathbf{b}_{n+1} - \mathbf{a}_n \cdot \mathbf{b}_n \\
&= (\mathbf{a}_{n+1} - \mathbf{a}_n) \cdot \mathbf{b}_{n+1} + \mathbf{a}_n \cdot (\mathbf{b}_{n+1} - \mathbf{b}_n) \\
&= (D\mathbf{a})_n \cdot \mathbf{b}_{n+1} + \mathbf{a}_n \cdot (D\mathbf{b})_n .
\end{aligned}$$

Formally, this resembles $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Exercises

Exercise 1. Let $\mathbf{a}_n = n$. What is $D\mathbf{a}$?

Exercise 2. Let $\mathbf{a}_n = n \cdot n$. What is $D\mathbf{a}$?

Exercise 3. Find a sequence $\mathbf{a}$ solving the difference equation $D\mathbf{a} = 2$.

Exercise 4. Find a sequence $\mathbf{a}$ solving the difference equation $D^2\mathbf{a} = 2$. Can you find another sequence solving the equation (besides $\mathbf{a} + C$)? Can you find a third solution?

Exercise 5. Find a nonzero sequence $\mathbf{a}$ solving the difference equation $D\mathbf{a} = \mathbf{a}$.

Exercise 6. Find a nonzero sequence $\mathbf{a}$ solving the difference equation $D\mathbf{a} = c \cdot \mathbf{a}$.

Exercise 7. Find a nonzero sequence $\mathbf{a}$ solving the difference equation $D\mathbf{a} = c \cdot \mathbf{a}$. Is this always possible?

Exercise 8. Find a nonzero sequence $\mathbf{a}$ solving the difference equation $D^2\mathbf{a} = -\mathbf{a}$. This is quite interesting; a sequence which begins

$$2, 2, 0, -4, -8, -8, 0, 16, 32, 32, 0, -64, -128, -128, 0, 256, 512, \ldots$$

is a solution—but can you find the general pattern?

Shift equations

Another equation you might enjoy solving is a "shift equation" (these are usually called **recurrence relations**): can you find a sequence $\mathbf{a}_n$ which solves the equation

$$S\mathbf{a} = 4 \cdot \mathbf{a}?$$

The Fibonacci numbers $\mathbf{f}_n$ are a solution to a shift equation:

$$S^2\mathbf{f} = S\mathbf{f} + \mathbf{f}.$$

Solving a shift equation

As a toy problem to consider, find a sequence $\mathbf{a}_n$ which solves

$$S^2\mathbf{a} = 5 \cdot S\mathbf{a} - 6 \cdot \mathbf{a}$$

by the following method. Rearrange this equation to have the form

$$S^2\mathbf{a} - 5 \cdot S\mathbf{a} + 6 \cdot \mathbf{a} = 0.$$

"Factor" the operators to get

$$(S - 3)\,(S - 2)\,\mathbf{a} = 0.$$

But $(S - 2)$ kills the sequence $\mathbf{a}_n = 2^n$, and $(S - 3)$ kills the sequence $\mathbf{a}_n = 3^n$. Is it the case that

$$\mathbf{a}_n = \alpha \cdot 2^n + \beta \cdot 3^n$$

is a solution to $S^2\mathbf{a} = 5 \cdot S\mathbf{a} - 6 \cdot \mathbf{a}$, for any constants α and β? What happens if you use this same trick on the Fibonacci sequence?