Problem 1.

Define $\sum_{n=1}^{\infty} a_n = L$.

Solution.

The series $\sum_{n=1}^{\infty} a_n$ converges to L if the sequence of partial sums

$$s_k := \sum_{n=1}^{k} a_n$$

converges to L, that is, $\lim_{k\to\infty} a_k = L$.

A common problem here was that people mixed up k's and n's, or forgot to use a limit.

This definition is conceptually very important (a triumph of mathematics!). Series are *not* piles of numbers to add up simultaneously—they are lists to be added up **in order**—and this affects convergence!

Problem 2.

(a) Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+\sqrt{n}}$ converge?

(b) Does the series $\sum_{n=3}^{\infty} \frac{n^2}{n^4-7}$ converge?

Justify your answers.

Solution.

(a) No, by the n-th term test:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1 \neq 0$$

so the series must diverge. Some people tried to use the limit comparison test (comparing with $b_n = 1$), but this is equivalent to the n-th term test, and makes everything look unnecessarily complicated.

(b) Yes, by the limit comparison test (this is the sum of a ratio of polynomials, so the limit comparison test is guaranteed to work). Set $a_n = (n^2)/(n^4-7)$ and $b_n = 1/n^2$. Then

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{n^4}{n^4-7} = 1.$$

Since this limit is nonzero, $\sum a_n$ converges if and only if $\sum b_n$ converges. But $\sum b_n$ converges by the p-series test, so the given series $\sum a_n$ converges.

Many people fell into a trap that I placed in this problem: you cannot use the more "obvious" comparison test because

$$\frac{n^2}{n^4 - 7} \leq \frac{1}{n^2}$$

is not true (try $n = 3$). The limit comparison test comes in very handy for this.

Problem 3.

Provide a precise statement of the **ratio test** for determining whether the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Solution.

Suppose $a_n > 0$ for all n, and that $\lim_{n\to\infty} a_{n+1}/a_n = L$. Then,

- If $L > 1$, then $\sum a_n$ diverges.
- If $L < 1$, then $\sum a_n$ converges.
- If $L = 1$, then the test is inconclusive.

Some people neglected to write down $\lim_{n\to\infty}$.

Problem 4.

For which real numbers $x > 0$ does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$$

converge absolutely? For which $x > 0$ does it converge conditionally? Justify your answer.

Solution.

Andy Chiang pointed out that I should have had $n = 1$ instead of $n = 0$; I have corrected that here. This problem is a basically p-series in disguise, so the convergence test will be easy—the hard part is the difference between absolute and conditional convergence.

To check for absolute convergence, note that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^x} \right| = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

But this is a p-series, so it converges exactly when $x > 1$. Thus, the given series **converges absolutely** provided $x > 1$.

Let $a_n = n^x$. Then the given series is an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$. For any x, all of the a_n are positive, and a_n is a decreasing sequence, and $\lim_{n\to\infty} a_n = 0$, so by the alternating series test, the given series converges for all $x > 0$. Therefore, it converges conditionally provided $0 < x \leq 1$.

(Some people said that it converges conditionally for all $x > 0$; yes, the series **converges** for all $x > 0$, but to converge **conditionally**, the series must converge while the series of the absolute values diverges—this only happens when $0 < x \leq 1$).

Problem 5.

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \cos(\sin x)$$

Find the first four terms (i.e., up to and including x^3) of a Taylor series expansion for f around $x = 0$.

Solution.

You can do this problem by differentiating; since $\cos \sin x$ is an *even* function, the Taylor series will only involve even degree terms, so you do not need to compute the coefficient on x^1 and x^3—they are both automatically zero. This means you only need to compute $f^{(2)}(0)$ which is not so bad.

Nevertheless, you can also solve this problem by substituting the series for $\sin x$ into the series for $\cos x$, as follows:

$$\begin{aligned}
\cos(\sin x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^{2n} \\
&= 1 - \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2}{2} + \frac{(x - \cdots)^4}{4!} + \cdots \\
&= 1 - \frac{x^2}{2} + \text{terms of degree four or more}
\end{aligned}$$

Problem 6.

Consider the function

$$f(x) = \frac{e^x - e^{-x}}{2}.$$

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series for $f(x)$. Compute both a_n and $f^{(42189)}(0)$.

Solution.

This function is "hyperbolic sine" so there is some reason to care about it; you'll notice in this problem that its Taylor series looks just like the Taylor series for sine, except that there is no $(-1)^n$ term. Additionally, since sinh is an odd function, only odd degree terms will appear in its Taylor series.

The Taylor series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ so

$$\begin{aligned}
\frac{e^x - e^{-x}}{2} &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1 - (-1)^n}{n!} x^n\right)
\end{aligned}$$

Therefore the coefficient on x^n is

$$a_n = \frac{1-(-1)^n}{n! \cdot 2}.$$

Note that this vainshes when n is even, and is equal to $1/n!$ when n is odd.

Since $f^{(n)}(0) = n!\, a_n$, we conclude

$$f^{(42189)}(0) = 42189! \cdot \frac{1-(-1)^{42189}}{42189! \cdot 2} = \left(1-(-1)^{42189}\right)/2 = 2/2 = 1.$$

Many of you neglected to compute a_n—you lost points for this; I asked for a computation of a_n to force you to write down the answer in a particular way (I feared that some would suggestively write down the first few terms series, some would not simplify it sufficiently, etc.—to be fair to those who might do more work, I asked for the most specific thing I could ask for). Additionally, many people wrote down

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

which is true, but this expresion makes it very complicated to see what the coefficient on the x^n term should be. One way to express it is like this:

$$\begin{aligned} a_{2n+1} &= \frac{1}{(2n+1)!} \\ a_{2n} &= 0 \end{aligned}$$

Or you could say,

$$a_n = \begin{cases} 1/n! & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Problem 7.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Define

$$g(x) = f(x) - f(0) - f'(0)x - \frac{f''(0)\, x^2}{2}$$

Recall that **Taylor's theorem** provides an integral equal to $g(x)$. What is that integral? (Hint: if you have forgetten it or want to check your answer, you might try integration by parts).

Solution.

By Taylor's theorem,

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)\, x^2}{2} + R_2(x).$$

Therefore,

$$g(x) = f(x) - f(0) - f'(0)x - \frac{f''(0)\, x^2}{2} = R_2(x).$$

But,

$$g(x) = R_2(x) = \frac{1}{2} \int_0^x f^{(3)}(t)\, (x-t)^2\, dt$$

This problem was designed to test your knowledge of Taylor's theorem without exactly asking you to just give a statement—you got credit if you wrote down the $n = 2$ case.

Problem 8.

Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth function, such that

- $p(0) = 0$ (in other words, my initial position is at the origin),
- $p'(0) = 2000$ (in other words, my initial velocity is $2000\,\mathrm{m/s}$), and
- $|p''(t)| < 500$ for all $t \in \mathbb{R}$ (in other words, I will avoid accelerating more than $500\,\mathrm{m/s^2}$)

Use **Lagrange's theorem** to bound $p(2)$, my position after two seconds.

Solution.

By Lagrange's theorem

$$p(x) = p(0) + p'(0)\,x + R_1(x)$$

and $R_1(x) = f''(c)\,x^2/2!$ for some $c \in (0, x)$.

In our case, $x = 2$, so

$$p(2) = p(0) + p'(0)\,2 + R_1(2) = 2000 \cdot 2 + R_1(2)$$

So $p(2) - 4000 = R_1(2)$.

Since $|p''(t)| < 500$, we know

$$|R_1(2)| = |f''(c)\,2^2/2| \le 500 \cdot 4/2 = 1000$$

Consequently, $|p(2) - 4000| \le 1000$, and therefore,

$$3000 \le p(2) \le 5000.$$

Take a moment to reflect on the reasonableness of this: if you are travelling at a particular speed, and you refuse to accelerate or decelerate too much, then you have to cover a certain distance! This is a physical consequence of Lagrange's theorem. In problems like this, physical intuition can reassure you that you've written down the correct answer.

Problem 9.

For which real numbers x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n\,n^2}$ converge? Remember to show your work.

Solution.

We use the ratio test to check for absolute convergence (this suffices for power series, save possibly at the endpoints)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(2^{n+1}(n+1)^2)}{x^n/(2^n n^2)}\right| = \lim_{n\to\infty}\left|\frac{x\,n^2}{2\,(n+1)^2}\right|$$

But $\lim_{n\to\infty}(n/n+1)^2 = 1$, so the above limit is equal to $|x|/2$ By the ratio test, the series converges absolutely provided $|x|/2 < 1$, i.e., provided $-2 < x < 2$.

We need to check endpoints. At $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n\, n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent p-series.

At $x = -2$, the series also converges—even absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{(-2)^n}{2^n\, n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent p-series again.

Therefore, the series converges exactly when $x \in [-2, 2]$.

Problem 10.

For which real numbers x does $\sum_{n=0}^{\infty} \frac{n!\, x^n}{n!!}$ converge? Note: $n!!$ means the factorial of the factorial of n. Justify your answer.

Solution.

Yes, it is a power series, so you might be tempted to apply the ratio test; but we've had problems about the ratio test already—and more importantly, the ratio test is helpful precisely when taking a ratio will cancel some terms—but here, taking the ratio of $n!!$ and $(n+1)!!$ doesn't seem helpful.

What we do know is that $n!!$ grows ridiculuosly quickly, which should suggest that a comparison test will be helpful. So let's apply the comparison test. First, observe that, provided $n \geq 2$

$$n! = 1 \cdot 2 \cdots (n-1) \cdot (n) > (n-1) \cdot (n) > (n-1)^2 \geq n^2/4$$

As a result, $n!! > (n!)^2/4$.

Therefore,

$$\frac{n!\, x^n}{n!!} \leq \frac{n!\, x^n}{(n!)^2/4} = \frac{1}{4} \cdot \frac{x^n}{n!}.$$

But

$$\sum_{n=1}^{\infty} \frac{1}{4} \frac{x^n}{n!} = \frac{e^x}{4}$$

converges for all $x \in \mathbb{R}$, so by the comparison test, the given series also converges for all $x \in \mathbb{R}$.

This problem was designed to test your use of the comparison test (since the second problem was more naturally solved using the limit comparison test), but there are surely other ways to solve it. Many of you had the right intuition (and you got points for that), saying that the $n!!$ will grow so much more quickly, etc. This is great! I hope that when you see series like this, you'll be able to guess the truth right away; but to justify your intuition of "grows very fast" you should use a comparison test.

Problem 11.

Use the power series for $\dfrac{1}{1-x}$ to find a power series $\sum a_n x^n$ for $f(x) = \dfrac{1}{(1-x)^3}$.
For which $x \in \mathbb{R}$ does the power series converge to $f(x)$?

Solution.

Note that

$$\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$$

Additionally,

$$\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d^2}{dx^2} x^n = \sum_{n=2}^{\infty} (n)(n-1)x^{n-2}.$$

Putting these facts together gives

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n\,(n-1)}{2} x^{n-2} = \sum_{n=2}^{\infty} \binom{n}{2} x^{n-2}$$

Since $\sum_{n=0}^{\infty} x^n$ converges to $1/(1-x)$ for all $x \in (-1,1)$, the same is true of the second derivatives (by the thereom on term-by-term differentiation).

There were many popular mistakes: the most frequent was to argue that

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$$

which is true. The trouble is that $1/(1-x)^3$ is not $1/(1-x^3)$.

Many people included an argument that the series they found converged; regardless of whether or not the series you found converges, that does not address the question of whether the series converges to $f(x)$—for that, you need to either estimate the remainder, or apply a theorem about term-by-term differentiation.

Problem 12.

The following questions are for extra credit. **Circle your answer.**

True $\boxed{\textbf{False}}$ If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ converges. Counterexample: $a_n = (-1)^n/n$.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges. This is the contrapositive of the theorem that says absolutely convergent series also converge conditionally.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. This is the theorem that says absolutely convergent series also converge conditionally.

True $\boxed{\textbf{False}}$ If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. Counterexample: $a_n = (-1)^n/n$.

True $\boxed{\textbf{False}}$ If $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(2\cdot a_n + 3\cdot b_n)$ diverges. Counterexample: $a_n = 3n$ and $b_n = -2n$.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(2 \cdot a_n + 3 \cdot b_n)$ converges. Sums of convergent series converge.

True $\boxed{\textbf{False}}$ If $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ diverges. Counterexample: $a_n = b_n = 1/n$.

True $\boxed{\textbf{False}}$ If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ converges. Counterexample: $a_n = b_n = (-1)^n/\sqrt{n}$.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=100}^{\infty} a_n$ converges. Convergence only depends on the tail of the series.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=100}^{\infty} a_n$ diverges. Divergence only depends on the tail of the series.

$\boxed{\textbf{True}}$ **False** If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=2}^{\infty} (a_n - a_{n-1})$ converges. A shifted convergent series still converges, so $\sum_{n=2}^{\infty} (a_n - a_{n-1})$ is the difference of two convergent series.