Lecture 26: Removable singularities

Math 660—Jim Fowler

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Congratulations

I should have the midterms graded by tomorrow.

Your homework, however, might be a bit delayed (another faculty member is grading it—sorry).

Removable singularities

Think back to the theorems we proved for analytic functions with finitely many exceptional points.

Assuming $\lim_{z \to a}(z-a)f(z) = 0$, we could still make progress.

Suppose Ω is a region, and
$\Omega' = \Omega -$ point.
Suppose f is analytic in Ω'
There exists an analytic function in Ω extending f
if and only if
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Such singularities are called *removable*

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for all $z \neq a$ inside of the circle C.

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Application: Taylor's theorem

Apply the previous method to

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

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$$f(z) = f(a) + (z-a)f_1(a) + \cdots + (z-a)^{n-1}f_{n-1}(a) + (z-a)^n f_n$$

Taylor's theorem

Since $f^{(n)}(a) = n! f_n(a)$, we have proved

Theorem (Taylor's theorem)

If $f(z)$ is analytic in a region $\Omega \ni a$, we can write

$$f(z) = f(a) + (z-a) f_1(a) + \cdots + (z-a)^{n-1} f_{n-1}(a) + (z-a)$$

for a function $f_n(z)$ analytic in Ω.

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$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta) d\zeta}{\zeta - z}$$

we can substitute $f_n(\zeta)$, which involves $f(\zeta)$ and many terms of the form

$$\int_C \frac{d\zeta}{(\zeta - a)^m (\zeta - z)} = 0$$

so

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - a)^n (\zeta - z)}$$

Zeroes and poles

Now we proceed to study the zeroes and poles of an analytic function.

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and therefore $f(z) = 0$ inside of the circle C.

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Both E_1 and E_2 are open.
And $E_1 \neq \varnothing$.
So $E_1 = \Omega$ because Ω is connected.

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Theorem

There are no zeroes of infinite order.

Zeroes are isolated

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So $f(z)$ has isolated zeroes.

Theorem

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If $f(z) = g(z)$ for $z \in A \subset \Omega$,
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Analytic functions are determined by their values on sets with an accumulation point.

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Poles are isolated, because zeroes are.

Meromorphic functions

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Think of a meromorphic function as a holomorphic function from Ω to S^2, the Riemann sphere.