
Problem 1. (20 points).

Define $\lim_{n\to\infty} a_n = L$.

Solution.

For every $\epsilon > 0$, there exists $K \in \mathbb{N}$, so that if $n \geq K$, then $|a_n - L| < \epsilon$.

Problem 2. (25 points).

A mathematician often gives a definition, and then provides an example of the definition in action.

Part (i). Define what it means for a sequence a_n to be **bounded**.

Part (ii). Give an example of sequences x_n and y_n so that neither x_n nor y_n is bounded, but the sequence $z_n = x_n + y_n$ is bounded.

Solution.

Part (i). A sequence a_n is bounded if there exists real numbers U and L so that for all natural numbers n,

$$L \leq a_n \leq U.$$

Part (ii). As one of infinitely many posible examples, set $x_n = n$ and $y_n = -n$. Then neither x_n nor y_n is bounded, but $z_n = 0$ is bounded.

Problem 3. (25 points).

Provide an ϵ-K proof of the fact that $\lim_{n\to\infty} \frac{1+n^5}{n^5} = 1$.

Solution.

Let $\epsilon > 0$. Pick K, a natural number, larger than $1/\sqrt[5]{\epsilon}$. So if $n \geq K$, then $n > 1/\sqrt[5]{\epsilon}$. Rearranging, $n \geq K$ implies

$$\frac{1}{n^5} < \epsilon.$$

Further algebraic manipulation proves that $n \geq K$ implies

$$0 < \frac{1}{n^5} + \frac{n^5}{n^5} - 1 = \frac{1+n^5}{n^5} - 1 < \epsilon$$

And therefore, if $n \geq K$, then

$$\left|\frac{1+n^5}{n^5} - 1\right| < \epsilon.$$

Problem 4. (20 points).

Evaluate $\lim\limits_{x\to\infty} \cos\left((1/x)^x\right)$.

Solution.

First note that $\lim_{x\to\infty}(1/x)^x = 0$, by, say, squeezing: for large x,

$$0 \leq (1/x)^x \leq 1/x$$

but $\lim_{x\to\infty} 1/x = 0$, so $\lim_{x\to\infty}(1/x)^x = 0$.

Since cos is continuous,

$$\lim_{x\to\infty} \cos\left((1/x)^x\right) = \cos\left(\lim_{x\to\infty}(1/x)^x\right) = \cos 0 = \boxed{1}.$$

Problem 5. (20 points).

Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin\cos^2\cos^3\sin^4\cos^5\sin^6\cos^7\sin^8 x}{x}$.

Solution.

Note that

$$\frac{-1}{x} \leq \frac{\sin(\text{anything})}{x} \leq \frac{1}{x}$$

But $\lim_{x\to\infty} 1/x = 0$ and $\lim_{x\to\infty} -1/x = 0$, so by squeezing,

$$\lim_{x\to\infty} \frac{\sin(\text{anything})}{x} = 0.$$

Problem 6. (20 points).

Give an example of a sequence a_n so that

$$\lim_{n\to\infty} a_n = 1$$

but $\lim\limits_{n\to\infty} (a_n)^n = 17$. Justify your choice.

Solution.

An example of such a sequence is

$$a_n = \left(1 + \frac{\log 17}{n}\right)^n.$$

Since a_n is the indeterminate form 1^∞, we replace a_n with $e^{\log a_n}$ to deduce

$$\lim_{n\to\infty} a_n = e^{\lim_{n\to\infty}\left(n\cdot\log\left(1+\frac{\log 17}{n}\right)\right)}.$$

But the latter involves the indeterminate $\infty \cdot 0$, which we transform as

$$\lim_{n\to\infty}\left(n\cdot\log\left(1+\frac{\log 17}{n}\right)\right)=\lim_{n\to\infty}\frac{\log\left(1+\frac{\log 17}{n}\right)}{1/n}$$

to which we may appliy l'Hôpital to get

$$\lim_{n\to\infty}\frac{\log\left(1+\frac{\log 17}{n}\right)}{1/n}=\lim_{n\to\infty}\frac{\frac{1}{1+\frac{\log 17}{n}}\cdot\frac{-\log 17}{n^2}}{-1/n^2}=\lim_{n\to\infty}\left(\frac{1}{1+\frac{\log 17}{n}}\cdot\log 17\right)=\log 17.$$

And therefore,

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}e^{\log 17}=17.$$

Problem 7. (20 points).

Define $\displaystyle\sum_{n=1}^{\infty}a_n=L$.

Solution.

The series $\displaystyle\sum_{n=1}^{\infty}a_n$ converges to L if the sequence of partial sums

$$s_k:=\sum_{n=1}^{k}a_n$$

converges to L, that is, $\lim_{k\to\infty}s_k=L$.

Problem 8. (25 points).

Part (i). What does it mean for the series $\displaystyle\sum_{n=1}^{\infty}a_n$ to **converge absolutely**?

Part (ii). What does it mean for the series to **converge conditionally**?

Part (iii). Does the series $\displaystyle\sum_{n=2009!}^{\infty}\left(\frac{(-1)^n}{n}\right)$ converge absolutely? Converge conditionally?

Solution.

Part (i). The series $\displaystyle\sum_{n=1}^{\infty}a_n$ is said to converge absolutely provided $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges.

Part (ii). The series $\displaystyle\sum_{n=1}^{\infty}a_n$ is said to converge conditionally when $\displaystyle\sum_{n=1}^{\infty}a_n$ converges but $\displaystyle\sum_{n=1}^{\infty}|a_n|$ does not converge.

Part (iii). The first 2009! terms of a series do not affect convergence, so we need only know whether or not

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \right)$$

converges absolutely or conditionally. But

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, and so, diverges. Nevertheless, the original series satisfies the alternating series test (since the terms are decreasing in magnitude, and $\lim (-1)^n/n = 0$) and so the series converges **conditionally**.

Does the series $\displaystyle\sum_{n=2009!}^{\infty} \left(\frac{(-1)^n}{n} \right)$ converge absolutely? Converge conditionally?

Problem 9. (20 points).

Recall that

$$0.\overline{9} = 0.99999 \cdots = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots = \sum_{n=1}^{\infty} \frac{9}{10^n}$$

Use the formula for a **geometric series** to evaluate this series.

Solution.

Recall that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consequently,

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

Plugging in $x = 1/10$ gives

$$\sum_{n=1}^{\infty} 10^{-n} = \frac{1/10}{1 - 1/10} = \frac{1/10}{9/10} = \frac{1}{9}.$$

Multiplying by nine gives

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = 9 \cdot \frac{1/10}{9/10} = 9 \cdot \frac{1}{9} = \boxed{1}.$$

Problem 10. (25 points).

Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n^3+n+1} + \frac{1}{n!} \right)$ converge?

Solution.

The series converges, because it is the sum of two convergent series.

First, $\sum \frac{n+1}{n^3+n+1}$ converges by the limit comparison test with $\sum \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^3+n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+n+1} = 1$$

and since $\sum \frac{1}{n^2}$ converges (as its a p-series with $p > 1$), then by the limit comparison test, $\sum \frac{n+1}{n^3+n+1}$ also converges.

Additionally, $\sum \frac{1}{n!}$ converges by, say, the ratio test, since

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Problem 11. (20 points).

Define a function $f : \mathbb{N} \to \mathbb{R}$ by

$$f(n) = \begin{cases} 0 & \text{if } \sin^2 n \le 1/2, \\ 1 & \text{if } \sin^2 n > 1/2. \end{cases}$$

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^{(14+f(n))}}$ converge?

Solution.

It does converge. We check convergence by comparison. Since $0 \le f(n) \le 1$, we have

$$0 \le \frac{f(n)}{n^{(14+f(n))}} \le \frac{1}{n^{(14+f(n))}} \le \frac{1}{n^{14}}$$

But $\sum \frac{1}{n^{14}}$ converges, as it is a p-series with $p > 1$. Therefore, by comparison, the original series converges.

Problem 12. (20 points).

Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$ converge?

Solution.

This series diverges by the n-th term test. I claim that

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \neq 0.$$

But this is a 0^0 indeterminate form, so we note that

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = e^{\lim_{n\to\infty} \frac{1}{n}\cdot\log\left(\frac{1}{n}\right)},$$

but the exponent is a $0 \cdot \infty$ indeterminate form, so we transform it

$$\lim_{n\to\infty} \frac{1}{n} \cdot \log\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\log\left(\frac{1}{n}\right)}{n}$$

which is an ∞/∞ indeterminate form, and we can apply l'Hôpital to get

$$\lim_{n\to\infty} \frac{\log\left(\frac{1}{n}\right)}{n} = \lim_{n\to\infty} \frac{1}{1/n} \cdot \frac{-1}{n^2} = 0.$$

And therefore,

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = e^0 = 1 \neq 0$$

so the series diverges by the n-th term test.

Problem 13. (25 points).

Consider the power series $\displaystyle\sum_{n=0}^{\infty} n^3\, x^n$. For which $x \in \mathbb{R}$ does the series converge?

Solution.

This series converges provided $x \in (-1, 1)$, and diverges otherwise.

By the ratio test, the series converges absolutely provided

$$\lim_{n\to\infty} \left| \frac{(n+1)^3\, x^{n+1}}{n^3\, x^n} \right| = |x| < 1.$$

Now we need only check the endpoints. If $x = 1$, then the series becomes

$$\sum_{n=0}^{\infty} n^3$$

which diverges (by the n-th term test, say). If $x = -1$, then the series becomes

$$\sum_{n=0}^{\infty} n^3\, (-1)^n$$

which diverges (again by the n-th term test).

Problem 14. (20 points).

Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = (1 + \sin x)^2.$$

Write down the the first four terms (i.e., x^0 through x^3) of the Taylor series for f expanded around $x = 0$.

Solution.

We could differentiate f a few times to find the Taylor series, but it is a bit faster just to plug in the first few terms of $\sin x$. That is,

$$f(x) = (1+\sin x)^2 = \left(1 + x - x^3/6 + \mathcal{O}(x^5)\right)^2$$

which we expand out to get

$$f(x) = 1 + 2x + x^2 - \frac{x^3}{3} + \mathcal{O}(x^4).$$

Problem 15. (25 points).

For positive real numbers x, define

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\, n!\, 2^{2n}} = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\, dt$$

This is the **Bessel function**.

Part (i): You may assume that $|J^{(3)}(x)| \leq 1$. Use this fact and Lagrange's theorem to find an approximation of $J(1/2)$ and estimate the error in your approximation.

Part (ii): What is $J^4(0)$?

Solution.

Part (i): By Taylor's theorem,

$$J(1/2) = 1 - \frac{(1/2)^2}{4} + R_2(1/2) = \frac{15}{16} + R_2(1/2)$$

But by Lagrange's theorem, $R_2(1/2) = J^{(3)}(c)(1/2)^3/6$ for some $c \in [0, 1/2]$. Since $|J^{(3)}(x)| \leq 1$, we can conclude

$$|R_2(1/2)| < \frac{1}{2^3 \cdot 6} = \frac{1}{48}$$

So $J(1/2) \approx 15/16$ with error no worse than $1/48$. In other words,

$$J(1/2) \in (15/16 - 1/48, 15/16 + 1/48) = (22/24, 23/24).$$

Indeed $J(1/12) \approx 22.5233\ldots/24$.

Part (ii): The coefficient on x^4 in the Taylor series for $J(x)$ is

$$\frac{(-1)^2}{2!\,2!\,2^4} = \frac{J^{(4)}(0)}{4!}$$

Therefore,

$$J^{(4)}(0) = \frac{24}{2 \cdot 2 \cdot 16} = \frac{3}{8}.$$

Problem 16. (20 points).

Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = xe^x + e^x = \frac{d}{dx}\left(x\, e^x\right).$$

Let $\sum_{n=0}^{\infty} a_n\, x^n$ be the Taylor series for f expanded around $x = 0$. Find a formula for a_n.

Solution.

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we know that

$$x\, e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

But differentiating gives

$$\frac{d}{dx}\left(x\, e^x\right) = \sum_{n=0}^{\infty} \frac{(n+1)\, x^n}{n!}$$

We conclude that

$$a_n = \frac{n+1}{n!}.$$

Problem 17. (20 points).

Use the method of **partial fractions** to evaluate

$$\int \frac{1}{x\,(x-1)}\, dx.$$

Remember to show your work.

Solution.

We must find real numbers A and B so that

$$\frac{A}{x} + \frac{B}{x-1} = \frac{1}{x\,(x-1)}.$$

Multiply both sides by $x\,(x-1)$ to get

$$A(x-1) + B(x) = 1$$

and so $A = -1$, and $A + B = 0$, so $B = 1$. Thus,

$$\int \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx = \int \frac{1}{x\,(x-1)}\, dx$$

And so,

$$\int \frac{1}{x\,(x-1)}\, dx = -\log x + \log(x-1) + C.$$

Problem 18. (25 points).

Find a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$x \cdot f'(x) + f(x) = \sin^2 x \, \cos^3 x.$$

Remember to show your work.

Solution.

Divide by x, to get the equivalent equation

$$f'(x) + \frac{f(x)}{x} = \frac{\sin^2 x \, \cos^3 x}{x}$$

Since $\int \frac{1}{x} \, dx = \log x + C$, so an integrating factor is $e^{\log x} = x$. Multiplying by the integrating factor gives

$$\frac{d}{dx} \left(x \, f(x) \right) = x f'(x) + f(x) = \sin^2 x \, \cos^3 x$$

Therefore,

$$x \, f(x) = \int \sin^2 x \, \cos^3 x \, dx.$$

Exchange $\cos^2 x$ for $1 - \sin^2 x$ to get

$$x \, f(x) = \int \sin^2 x \, (1 - \sin^2 x) \, \cos x \, dx = \int (\sin^2 x - \sin^4 x) \, \cos x \, dx$$

but now we can do a u-substitution (with $u = \sin x$) to get

$$x \, f(x) = \frac{\sin^3 x}{3} - \frac{\sin^5}{5} + C$$

and therefore, the general solution is

$$f(x) = \frac{\sin^3 x}{3x} - \frac{\sin^5 x}{5x} + \frac{C}{x}.$$

Problem 19. (0 points).

Please fill in the square next to the correct answer.

- ■ True
- □ False

If a_n and b_n are bounded, then $c_n = a_n + b_n$ is bounded.

- □ True
- ■ False

If a_n and b_n are monotone, then $c_n = a_n + b_n$ is monotone.

- □ True
- ■ False

If $b_n = {a_n}^2$ converges, then a_n converges.

- □ True
- ■ False

If a_n is decreasing, then $b_n = a_{n+1} - a_n$ is decreasing.

- ■ True
- □ False

If $\lim\limits_{n\to\infty} \log a_n = 0$, then $\lim\limits_{n\to\infty} a_n = 1$.

- ■ True
- □ False

There exists a sequence a_n of irrational numbers, with $\lim\limits_{n\to\infty} a_n = 0$.

- □ True
- ■ False

There exists a sequence a_n of even integers, with $\lim\limits_{n\to\infty} a_n = 1$.

- ■ True
- □ False

If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} |a_n|$ diverges.

- □ True
- ■ False

If $\sum\limits_{n=1}^{\infty} a_n$ converges and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} (a_n \cdot b_n)$ converges.

- ■ True
- □ False

If $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=2}^{\infty} (a_n - a_{n-1})$ converges.

Problem 20. (5 points).

Congratulations—you have finished Math 153, the third and final quarter of a year-long course in Calculus.

Take a moment to reflect on this past quarter: how did you do on the sixteen homeworks, the two midterms, and now this final exam? **What letter grade (including $\pm$) do you believe you have earned this quarter?**

Solution.

Leading you through this course has been tremendously enjoyable for me. I wish you many blessings as you continue your studies.