Lecture 8: Improper integrals

Math 153 Section 57

Wednesday October 15, 2008

Following chapter 11.7.

1 Reminder about integrals

The importance of the fundamental theorem.

2 Logarithms

Define $\log a = \int_1^a dx/x$. What is $\lim_{a\to\infty} \log a$? This is an improper integral.

Incidentally, why is $\log a + \log b = \log(ab)$? On the second integral in

$$\int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x}$$

do a substitution: $u = ax$. Then $dx = u/a$ and $x = u/a$, so $dx/x = du/u$. But

$$\int_1^a \frac{dx}{x} + \int_a^{ab} \frac{du}{u} = \int_1^{ab} \frac{dx}{x}$$

Hand out slide rules to demonstrate the power of this fact.

3 Two bad things

Either unbounded width, or unbounded height. Both are "improper."

4 Unbounded intervals

When we write $\int_a^\infty$ we mean $\lim_{b\to\infty} \int_a^b$.

If the limit exists, the improper integral converges.

If not, it "diverges."

Examples: $\int_1^\infty dx/x$. $\int_1^\infty dx/x^2$. $\int_0^\infty \sin x\, dx$. Different reasons for why these integrals diverge.

Remark: $\int_1^\infty dx/x^p$ converges if $p > 1$, and diverges if $0 < p \le 1$.

5 Unbounded on both sides

When we write $\int_{-\infty}^{\infty}$ we mean $\lim_{b\to\infty}\int_0^{\infty} + \lim_{b\to-\infty}\int_b^0$.

This is not the same as $\lim_{b\to\infty}\int_{-b}^{b}$. Example: $\sin x$.

6 Unbounded integrand

If f is continuous on $[a, b)$ but not defined at b, we can still integrate, by defining

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^b f(x)\,dx$$

Because for each number $c < b$, the integral over $[a, c] \subset [a, b)$ makes sense.

Example: $\int_0^2 dx/x$.