Midterm 2 Solution Set — Math 345

November 2010

The essence of mathematics is
not to make simple things complicated,
but to make complicated things simple.
—Stan Gudder

Name: ______________________________

Lecture time (circle one): 12:30–1:18 P.M.

2:30–3:18 P.M.

1. Write your name above.
2. Calculators are forbidden (and useless, anyhow).
3. Do not look inside the exam until instructed to do so.
4. You have **48 minutes** for this exam.
5. Justify your answers for full credit.
6. Show your work for generous partial credit.
7. Write your answers on the included pages, or request additional paper.
8. Answer all questions asked.
9. To prevent fire, do not divide by zero.

Problem 1	/360
Problem 2	/360
Problem 3	/360
Problem 4	/360
Problem 5	/360
Total	/1800

Problem 1 /360

State the binomial theorem, and then apply it to prove that, for nonnegative integers n,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Be sure to explain your argument carefully.

Solution

First, I state the binomial theorem:
For $x, y \in \mathbb{R}$ and a nonnegative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Claim.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Let $x = y = 1$; then the binomial theorem gives

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k.$$

But $1^k = 1$ and $1^{n-k} = 1$, so

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k},$$

which is what we wanted to prove. $\square$

Commentary

Some people gave an inductive proof of this fact; this is not needed. All you need to do is apply the binommial theorem.

Problem 2 /360

Prove, by complete induction, that every integer $x \geq 2$ is either prime, or is the product of primes.

Solution

Let $P(x)$ be the statement that x is either prime, or a product of primes. In other words, $P(x)$ is the statement that $x = p_1 \cdots p_n$ for primes p_i.

Claim. *For all integers $x \geq 2$, the statement $P(x)$ is true.*

Proof. We proceed by induction.

Base case. Let $x = 2$. Then two is prime, so $P(2)$ is true.

Inductive step. We assume $P(k)$ holds for $2 \leq k \leq x$. We want to show $P(x+1)$.

If $x+1$ is prime, then $P(x+1)$ is true.

If $x+1$ is not prime, then $x+1 = ab$ for $a, b \in \mathbb{N}$ with $a \neq 1$ and $b \neq 1$. Then $2 \leq a < x+1$ and $2 \leq b < x+1$, so $P(a)$ and $P(b)$ are true. This means

$$\begin{aligned} a &= p_1 \dots p_n \text{ and} \\ b &= q_1 \cdots q_m \end{aligned}$$

for primes p_i and q_j, so

$$x + 1 = ab = (p_1 \dots p_n)(q_1 \cdots q_m)$$

so $P(x+1)$ is true.

Thus, by strong induction, the statement $P(x)$ holds for all integers $x \geq 2$. $\square$

Commentary

Some people did not use complete induction—you really need to use complete induction here.

Problem 3 /360

Prove that there are infinitely many prime numbers.

Solution

Claim. *There are infinitely primes numbers.*

Proof. Suppose not—then there are finitely many prime numbers, say $p_1, \ldots, p_n$.

Consider $x = p_1 \cdots p_n + 1$. By the previous problem, x can be written as a product of primes, so in particular, there is a prime number p_i which divides x. But p_i also divides the product of all the primes, $p_1 \cdots p_n$. Therefore, by the theorem that an integer dividing a and b divides $a - b$, we conclude that p_i divides $x - p_1 \cdots p_n$. But then p_i divides 1, but $p_i > 0$, so $p_i \leq 1$. This implies $p_i = 1$, which is a contradiction. $\square$

Commentary

Many people gave very nicely written solutions for this problem.

Problem 4 /360

Let $A = \{k \in \mathbb{Z} \mid k \geq 3\}$. Prove by induction for each $n \in A$ that

$$\binom{n}{3} = \frac{n\,(n-1)\,(n-2)}{6}.$$

You may use the recurrence relation for binomial coefficients, and the fact that

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

You will get more points if you do the algebra in the induction efficiently than if you do it correctly but inefficiently—don't just expand everything in sight. Look for a way to apply the distributive law to make your work easier to understand.

Solution

Let $P(n)$ be the statement that

$$\binom{n}{3} = \frac{n\,(n-1)\,(n-2)}{6}.$$

I claim, for an integer n with $n \geq 3$, the statement $P(n)$ holds.

Proof. We proceed by induction.

Base case. Consider $n = 3$. The statement $P(3)$ is

$$\binom{3}{3} = \frac{3\,(3-1)\,(3-2)}{6} = 1,$$

which is true by inspecting Pascal's triangle.

Inductive step. Assume $P(n)$; we will show $P(n+1)$. By the recurrence relation for binomial coefficients,

$$\begin{aligned}
\binom{n+1}{3} &= \binom{n}{3} + \binom{n}{2} \\
&= \binom{n}{3} + \frac{n(n-1)}{2} && \text{(as we are told we may assume)} \\
&= \frac{n\,(n-1)\,(n-2)}{6} + \frac{n(n-1)}{2} && \text{(by the inductive hypothesis)} \\
&= (n)(n-1)\left(\frac{n-2}{6} + \frac{1}{2}\right) = (n)(n-1)\frac{n+1}{6} && \text{(by the distributive law)}
\end{aligned}$$

which is the statement $P(n+1)$. Therefore, by induction, $P(n)$ holds for all $n \geq 3$. $\square$

Commentary

Many people used the fact that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

to prove this; however, the problem requires that you do this by induction.

Problem 5 /360

Give a proof by induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Then, explain why $1^3 + 2^3 + 3^3 + \cdots + (2^{10})^3 \equiv 1 \pmod 3$.

Solution

Let $P(n)$ be the statement that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

I claim $P(n)$ holds for all $n \in \mathbb{N}$.

Proof. We proceed by induction. **Base case.** Let $n = 1$. Then $P(1)$ asserts

$$\sum_{k=1}^{1} k^3 = \frac{1^2(1+1)^2}{4} = 1,$$

which is true. **Inductive step.** Suppose $P(n)$; we prove $P(n+1)$. Since $P(n)$, we know

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Adding $(n+1)^3$ to both sides yields

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= (n+1)^2 \left(\frac{n^2}{4} + (n+1) \right) \\
&= (n+1)^2 \left(\frac{n^2}{4} + \frac{4n+4}{4} \right) \\
&= (n+1)^2 \left(\frac{n^2+4n+4}{4} \right) \\
&= (n+1)^2 \left(\frac{(n+2)^2}{4} \right),
\end{aligned}$$

which is the statement $P(n+1)$, so, by induction, we may conclude $\forall n \in \mathbb{N}\, P(n)$. $\square$

Claim. $1^3 + 2^3 + 3^3 + \cdots + (2^{10})^3 \equiv 1 \pmod 3$

Proof. By the preceeding claim, $1^3 + 2^3 + 3^3 + \cdots + (2^{10})^3 = \dfrac{(2^{10})^2(2^{10}+1)^2}{4}$, and

$$\begin{aligned}
\frac{(2^{10})^2(2^{10}+1)^2}{4} &= \frac{2^{20}(2^{20} + 2 \cdot 2^{10} + 1)}{4} \\
&= 2^{18}(2^{20} + 2 \cdot 2^{10} + 1) \\
&= 4^9(4^{10} + 2 \cdot 4^5 + 1) \\
&\equiv 1^9(1^{10} + 2 \cdot 1^5 + 1) \pmod 3 \\
&\equiv 1(1 + 2 + 1) \pmod 3 \\
&\equiv 4 \equiv 1 \pmod 3
\end{aligned}$$

$\square$