Lecture 7: More indeterminate forms

Math 153 Section 57

Monday October 13, 2008

Following chapter 11.6.

1 Trig formulas

Introduce $e^{i\theta} = \cos\theta + i\sin\theta$. Derive all the rules for sine and cosine.

With the correct perspective, you need to remember nothing, and derive everything. Practically, in your future lives, you probably wouldn't remember tables of sines and cosines and derivatives and blah, but if you learn the ideas, then you won't have.

2 Trig functions and l'Hôpital's rule

We want to calculate $\lim_{x\to 0} \frac{\sin x}{x}$. Let's use l'Hôpital's rule.

Now we need to know the derivative of sine. So by definition,

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin(h)}{h}$$

I guess now we use l'Hôpital's rule again? Uhm...

3 What does converge to ∞ mean?

We say $f(x) \to \infty$ as $x \to a$ if for all N, there is δ, so that if $0 < |x-a| < \delta$, then $f(x) > N$.

As big as we want f to be, there's a δ so that if we're δ close to a, then $f(x)$ is that big.

4 Tips about L'Hôpital's rule

Only apply it to $0/0$ or ∞/∞. If you can't do the limit, and it isn't in that form, transform it into that form.

Do not apply it to $0/1$ or $1/0$ or $0/\infty$ or $\infty/0$.

Sometimes you need to do it more than once: e.g., $\sin^2 x/x^2$ (though here you could use the product formula).

5 L'Hôpital's rule backwards

$f(x) = x + \sin x$ and $g(x) = x$. Then $f(x)/g(x) \to 1$ as $x \to \infty$, but $f'(x)/g'(x)$ does not converge.

6 Review L'Hôpital's rule

If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, and $g'(x) \neq 0$ for all x near a, then if $f'(x)/g'(x) \to L$, then $f(x)/g(x) \to L$.

Intuition: limits of fractions measure how fast one is growing compared to the other.

Examples: $\log x/x^n \to 0$ as $x \to \infty$. Think: logrithms grow very slowly.

Example: $e^x/x^n \to \infty$. Apply L'Hôpital many times.

7 $\infty - \infty$

Example: $x - \log x = x(1 - \log x/x)$.

8 0^0

Show $\lim_{x\to 0^+} x^x = 1$. Take logs to do $\lim_{x\to 0^+} x \log x = \lim \log x/(1/x) = \lim(1/x)/(-1/x^2) = \lim -x = 0$.

9 Proof

Apply mean-value theorem to

$$H(x) = (g(b) - g(a))(f(x) - f(a)) - (g(x) - g(a))(f(b) - f(a))$$

on the interval (a, b). Get a point where $H'(r) = 0$, which means

$$f'(r)/g'(r) = (f(b) - f(a))/(g(b) - g(a))$$

Suppose $f(a) = 0$ and $g(a) = 0$. Then on the interval $(a, a + \epsilon)$ we have some number $r \in (a, a + \epsilon)$ so that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(r)}{g'(r)}$$

10 The 0^0 controversy

What does 0^0 mean? Well, what does $0/0$ mean...