Homework 3

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1. State whether the following sequences converge (and, if so, state the limit).

(a) $a_n = \sqrt{n}$. Since a_n is not bounded above, it diverges.

(b) $b_n = 2^{-n} + \frac{1}{n}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 2^{-n} + \lim_{n\to\infty} \frac{1}{n} = 0 + 0 = 0.$$

(c) $c_n = \dfrac{n+1}{n^2}$.

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \left(\frac{n}{n^2} + \frac{1}{n^2} \right) = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0 + 0 = 0.$$

(d) $d_n = \dfrac{n + (-1)^n}{n}$.

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} \left(\frac{n}{n} + \frac{(-1)^n}{n} \right) = \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{n} \right) = 1.$$

(e) $e_n = (-1)^n \cdot n^3$. Since e_n is not bounded (because, for instance, $e_{2n} > 2n$), e_n diverges.

(f) $f_n = \cos \dfrac{1}{n}$. Because cos is continuous,

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \cos \frac{1}{n} = \cos \left(\lim_{n\to\infty} \frac{1}{n} \right) = \cos 0 = 1.$$

(g) $g_n = |10 - n| - n$. If $n > 10$, then $10 - n < 0$, so $|10 - n| = n - 10$. But then $g_n = (n-10) - n = -10$. So for large values of n, $g_n = -10$, and therefore, $\lim_{n\to\infty} g_n = -10$.

(h) $h_n = \dfrac{3^n}{4^n + 1}$. We can use squeezing here. For all $n \in \mathbb{N}$,

$$0 \leq \frac{3^n}{4^n+1} \leq \frac{3^n}{4^n} = \left(\frac{3}{4} \right)^n$$

But the left-hand and right-hand sequences both converge to zero, so $\lim_{n\to\infty} h_n = 0$.

(i) $i_n = \log(n+1) - \log n$. Note that $\lim_{n\to\infty} \frac{n+1}{n} = 1$. But log is continuous, so $\lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = \log 1 = 0$. Since

$$i_n = \log(n+1) - \log n = \log\left(\frac{n+1}{n}\right),$$

we may conclude $\lim_{n\to\infty} i_n = 0$.

(j) $j_n = \sqrt{n^2+n} - n$.[‡] The limit is $1/2$, which I will prove by squeezing. First, for all $n \in \mathbb{N}$,

$$j_n = \sqrt{n^2+n} - n < \sqrt{n^2+n+1/4} - n = \sqrt{\left(n+\frac{1}{2}\right)^2} - n = n + \frac{1}{2} - n = \frac{1}{2}.$$

On the other hand,

$$\begin{aligned} \lim_{n\to\infty} j_n &= \lim_{n\to\infty} \sqrt{n^2+n} - n \\ &= \lim_{n\to\infty} \sqrt{n^2+n} - n \end{aligned}$$

2. Give an ϵ-K proof that the sequence $a_n = \frac{3}{n}$ converges,
3. Prove that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} a_n + b_n = L + M$.
4. Let a_n be a sequence of real numbers, and set $b_n = |a_n|$. If b_n converges, does a_n converge? If so, prove it; if not, provide a counterexample.
5. Let a_n and b_n be sequences of real numbers; suppose $\lim_{n\to\infty} a_n = 0$. Is it the case that

$$\lim_{n\to\infty} a_n \cdot b_n = 0?$$

If so, prove it; if not, provide a counterexample.

[‡]This is rather tricky; if you don't answer it, you will not be penalized.