Cup Products Commute — Math 758

Summer 2013 — Jim Fowler

I'll update the calendar soon to account for the changes in the schedule.

Remark 1. Our goal is to show that for $\alpha \in H^i(X)$ and $\beta \in H^j(X)$, show that the cup product satisfies
$$\alpha \smile \beta = (-1)^{ij} \beta \smile \alpha.$$

Definition 2. Let $\epsilon_n = (-1)^{n \cdot (n+1)/2}$.

Subproblem 3. Verify $\epsilon_{i+j} = (-1)^{ij} \epsilon_i \epsilon_j$.

Definition 4. Let $\tau : C_n(X) \to C_n(X)$ be the chain map which, on a simplex σ, is defined as
$$\tau(\sigma) = \epsilon_n (\sigma \circ \text{reverse}),$$
where reverse is the affine map $[v_0, \ldots, v_n] \to [v_n, \ldots, v_0]$.

Subproblem 5. Verify that τ is a chain map.

Subproblem 6. Verify $\tau^\star \alpha \smile \tau^\star \beta = \pm \tau^\star (\beta \smile \alpha)$ and determine the sign.

Subproblem 7. Consider $X \times I$ as a simplicial complex; in particular, for $\Delta^n \times I$,
$$\begin{aligned} [v_0, \ldots, v_n] &= \Delta^n \times \{0\} \\ [w_0, \ldots, w_n] &= \Delta^n \times \{1\} \\ \Delta^n \times I &= \bigcup [v_0, \ldots, v_i, w_n, \ldots, w_i]. \end{aligned}$$

Draw a picture to illustrate the resulting simplicial structure on the prism $\Delta^2 \times I$.

Subproblem 8. Recall, from Math 757, the **prism operator** $P : C_n(X) \to C_{n+1}(X)$ defined via
$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i} (\sigma \circ \text{proj})|_{[v_0, \ldots, v_i, w_n, \ldots, w_i]},$$
where proj is the projection $\Delta^n \times I \to \Delta^n$. Show that $\partial \circ P + P \circ \partial = \tau - \text{Id}$.

Subproblem 9. Conclude that $\smile$ is (graded) commutative.

Problem 10. Let Σ_g be the oriented surface of genus g; describe a four-fold covering map $f : \Sigma_5 \to \Sigma_2$, and explain how the map $f_\star$ maps the ring $H^\star(\Sigma_2)$ to the ring $H^\star(\Sigma_5)$.

Problem 11. For which i and j are there maps $f : \Sigma_i \to \Sigma_j$ and $g : \Sigma_j \to \Sigma_i$ so that $f \circ g$ is the identity?