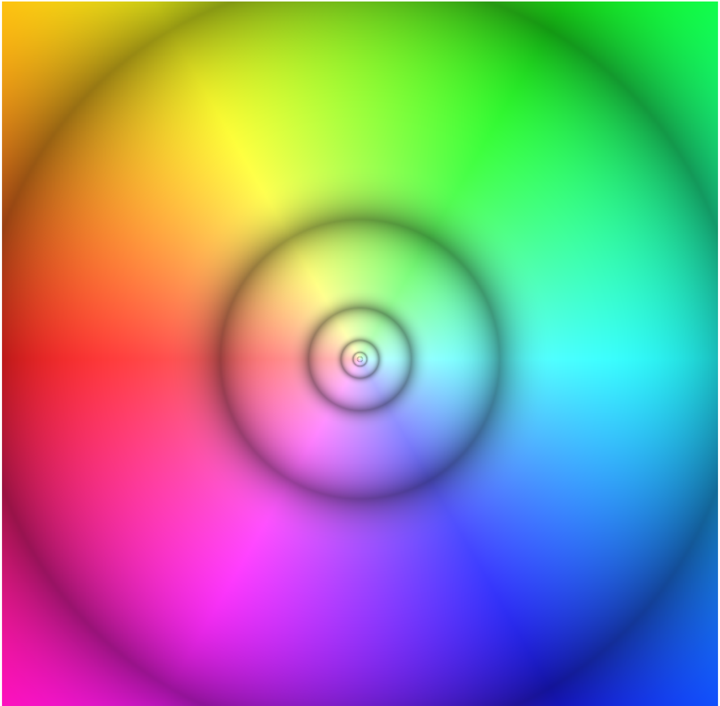
Lecture 4: Sequences and series

Math 660—Jim Fowler

Thursday, June 23, 2011



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$\omega^3 = \overline{\omega}^2$ and $\omega^4 = \overline{\omega}$, so

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Can we compute $\omega + \overline{\omega}$ and $\omega^2 + \overline{\omega}^2$?

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Can we compute $\omega + \overline{\omega}$ and $\omega^2 + \overline{\omega}^2$?

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$$(z - (\omega + \overline{\omega}))\left(z - (\omega^2 + \overline{\omega}^2)\right) = z^2 + z - 1,$$

$$\omega + \overline{\omega} = \frac{1}{2}\left(-1 + \sqrt{5}\right),$$

$$z^5 - 1 = (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$$

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Can we compute $\omega + \overline{\omega}$ and $\omega^2 + \overline{\omega}^2$?

$$(z-(\omega+\overline{\omega}))\left(z-(\omega^2+\overline{\omega}^2)\right) = z^2+z-1,$$

$$\omega+\overline{\omega} = \frac{1}{2}\left(-1+\sqrt{5}\right),$$

So a fifth root of unity is a root of the polynomial

$$z^2-(\omega+\overline{\omega})z+1 = z^2-\frac{-1+\sqrt{5}}{2}z+1,$$

So by the quadratic equation

$$\omega = \frac{-\frac{1-\sqrt{5}}{2} + \sqrt{\left(\frac{1-\sqrt{5}}{2}\right)^2 - 4}}{2}$$

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which we can simplify to

$$\omega = \frac{-1+\sqrt{5}}{4} + i\sqrt{1 - \frac{\left(1-\sqrt{5}\right)^2}{16}}$$

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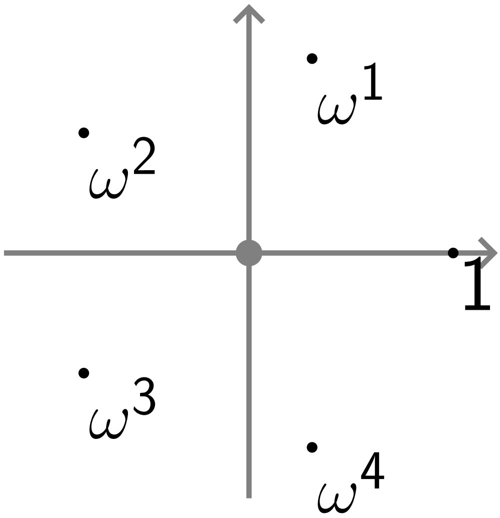
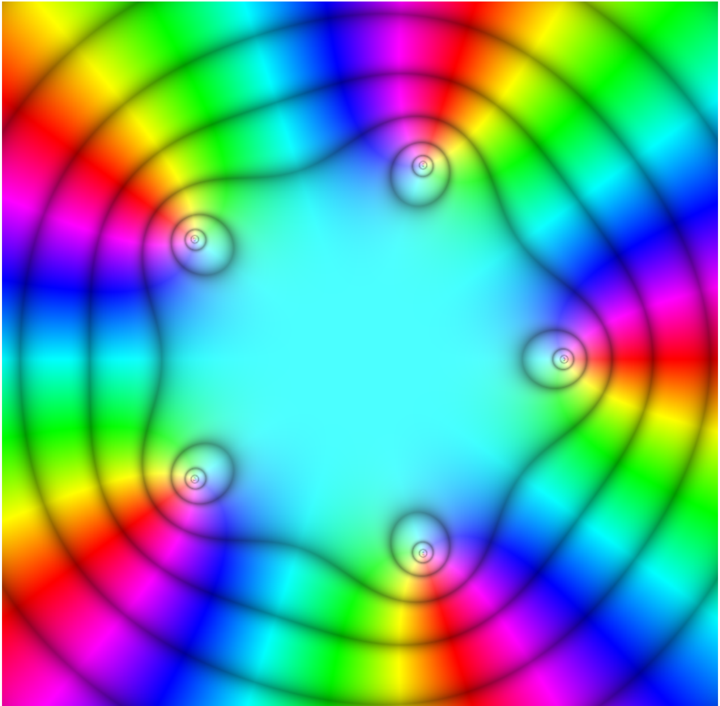
and further into

$$\omega = \frac{-1+\sqrt{5}}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}$$


ω^1
ω^2
ω^3
ω^4
1



So we conclude

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$$

$$\sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

So we conclude

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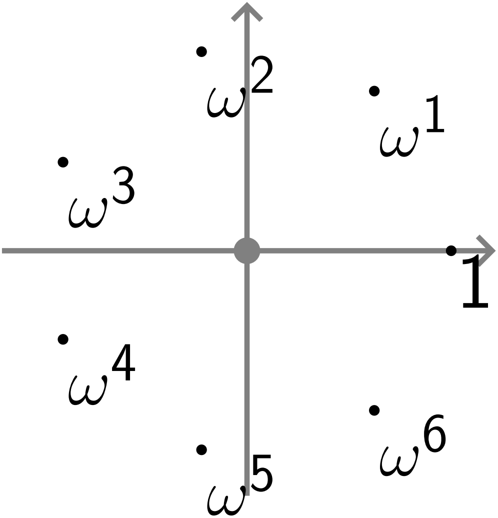
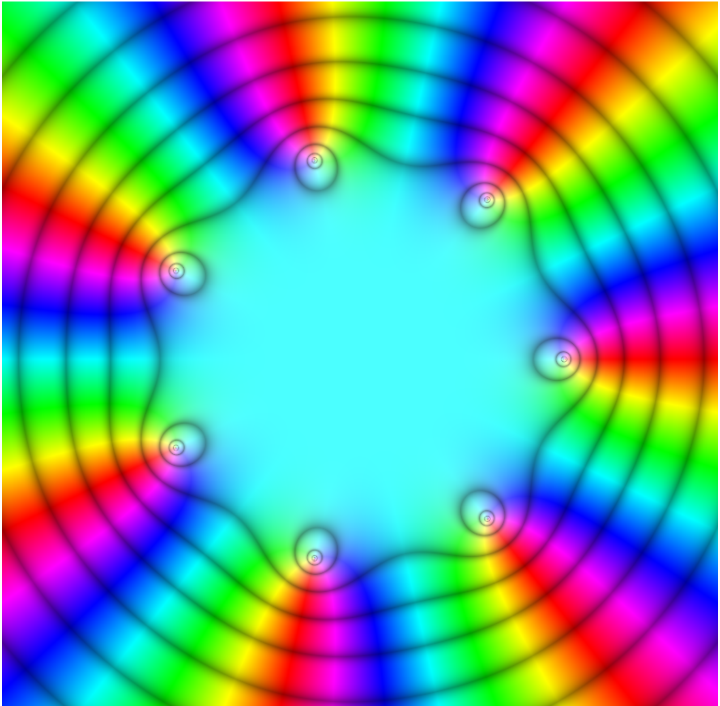
$$\sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

Can we compute

$$\cos\left(\frac{2\pi}{7}\right) \text{ and } \sin\left(\frac{2\pi}{7}\right)?$$



ω^2
ω^1
ω^3
1
ω^4
ω^6
ω^5



Rational Functions

These are quotients of polynomials.

$$r(z) = \frac{p(z)}{q(z)}, \text{ for polynomials } p \text{ and } q.$$

The zeroes of p are the zeroes of r;
The zeroes of q are the poles of r.

Define $r(\infty) = \tilde{r}(0)$, where $\tilde{r}(z) = r(1/z)$.

We can compute the order of a pole or zero at infinity.

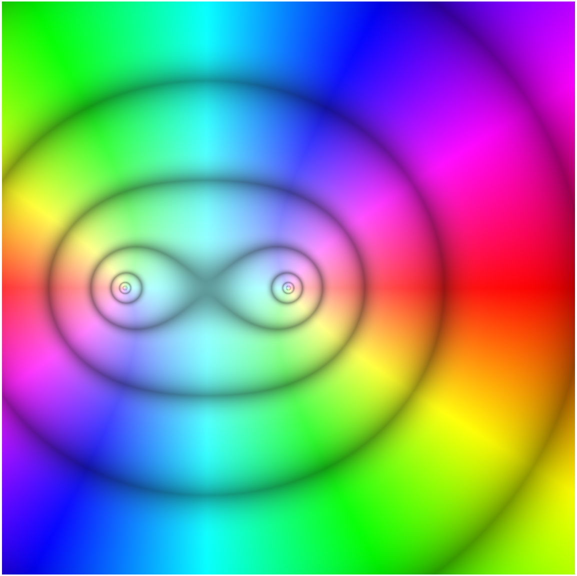
If $\deg p < \deg q$, then r has a zero of order $\deg q - \deg p$ at infinity.
If $\deg p > \deg q$, then r has a pole of order $\deg p - \deg q$ at infinity.

The number of poles of a rational function
(including a pole at infinity)
is the same as
the number of zeroes of a rational function
(including a zero at infinity).

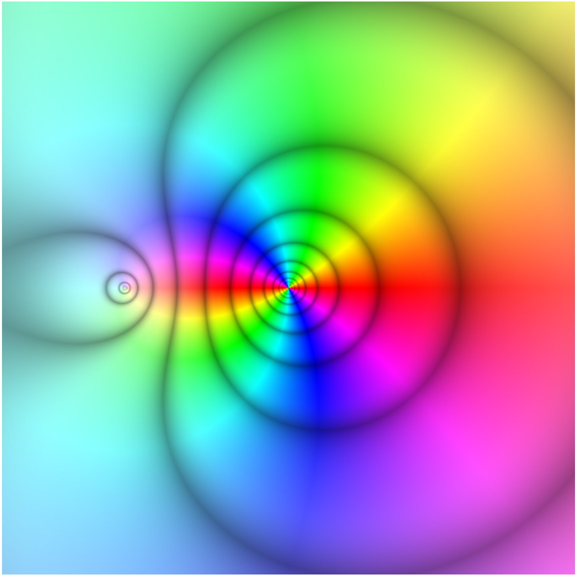
Both are equal to the larger of $\deg p$ and $\deg q$.

Call this common quantity the order of the rational function.

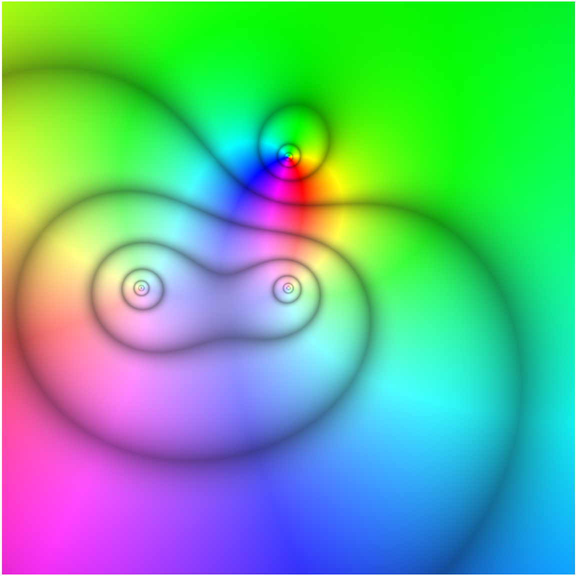
$$f(z) = z(z+1)$$



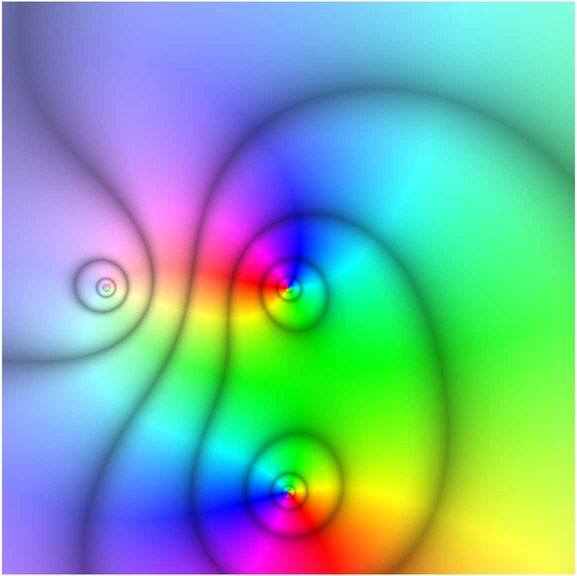
$$\tilde{f}(z) = \frac{1}{z}\left(\frac{1}{z} + 1\right)$$



$$f(z) = z(z + 0.9)/(z - 0.8i)$$



$$\tilde{f}(z) = \frac{1}{z}\left(\frac{1}{z} + 0.9\right) / \left(\frac{1}{z} - 0.8i\right)$$



Section 2.2.1–2.2.3 of the textbook

sequences
and
series

Polynomials and rational functions are good.
To expand the functions available for consideration, take limits.
Heading towards *Power series* on Friday.

Sequences

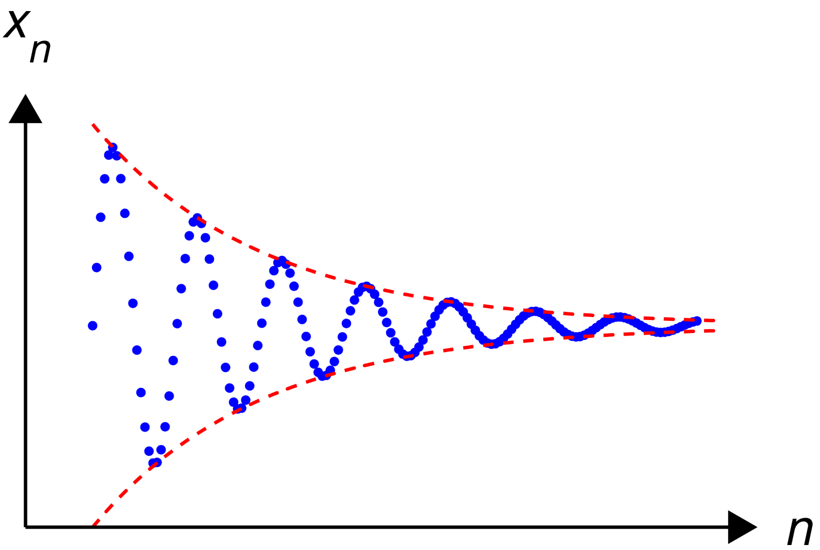
$$a_n$$

Limits

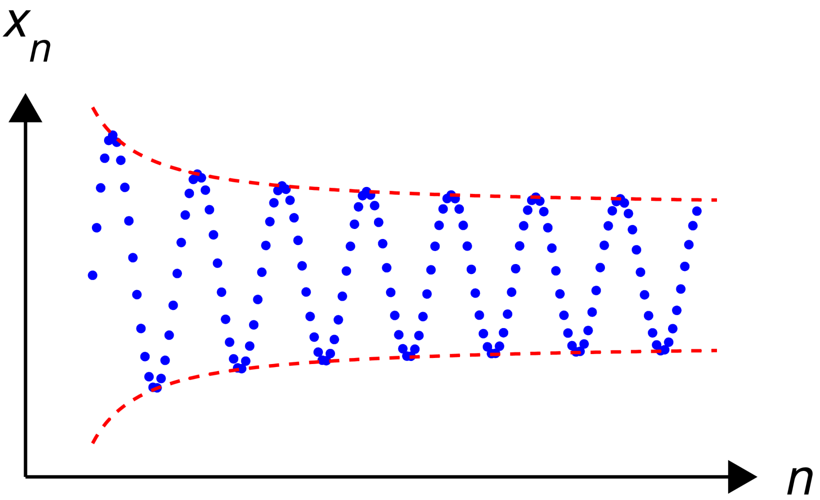
$\lim_{n\to\infty} a_n = L$ means that,
for every $\epsilon > 0$,
there exists N,
so that whenever $n > N$,
then $|a_n - L| < \epsilon$.

Usually, we must prove convergence
without being able to compute the limit.

Convergent sequence



Not a convergent sequence



A sequence $(a_n)_{n\in\mathbb{N}}$ is **Cauchy** if,
for every $\epsilon > 0$,
there exists N,
so that whenever $n, m > N$,
then $|a_n - a_m| < \epsilon$.

A sequence $(a_n)_{n\in\mathbb{N}}$ is **Cauchy** if,
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A sequence of real numbers is convergent
if and only if
it is Cauchy.

Series

$$\sum$$

To evaluate the series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

we form a sequence $s_k = a_1 + \cdots + a_k$, and define

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} s_k.$$

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We call $\sum a_n$ *absolutely convergent* if $\sum |a_n|$ converges.

Formulate a condition on the a_n so that the sequence s_k is Cauchy.

If a series $\sum a_n$ convergences, then $\lim_{n \to \infty} a_n = 0$.

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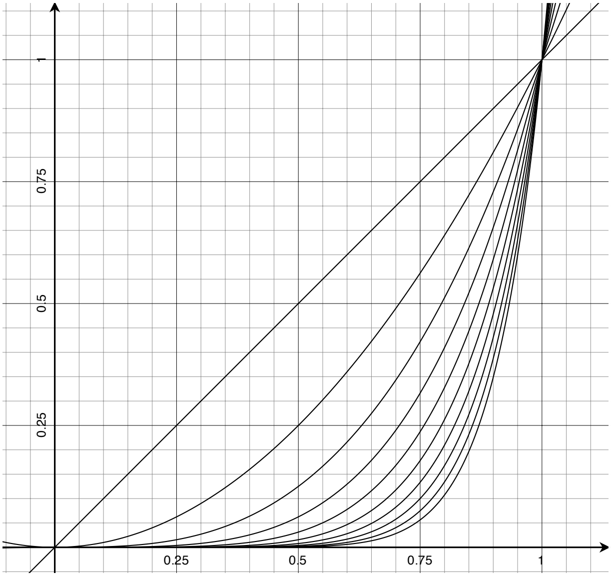
The converse is false.

Uniform convergence



A sequence of functions $f_n : U \to V$,
converges *pointwise* to $f : U \to V$ if
for every $x \in U$,
for every $\epsilon > 0$,
there is an N,
so that if $n > N$, then
$|f_n(x) - f(x)| < \epsilon$.

A sequence of functions $f_n : U \to V$,
converges *uniformly* to $f : U \to V$ if
for every $\epsilon > 0$,
there is an N,
so that if $n > N$, then
for every $x \in U$,
$|f_n(x) - f(x)| < \epsilon$.

Why care about uniform convergence?

The limit of a uniformly convergent sequence of continuous functions is itself continuous.

Cauchy sequences of functions?

A sequence of functions $f_n : U \to V$ is *uniformly Cauchy* if
for every $\epsilon > 0$,
there exists N, so that
whenever $n, m > N$, then
for all $x \in U$,
$|f_n(x) - f_m(x)| < \epsilon$.
Uniform convergence iff uniformly Cauchy.

Weierstrass M-test

The series $\sum a_n$ *majorizes* a series $\sum f_n(x)$
if there exists M so that
for n sufficiently large,
$|f_n(x)| \leq M a_n$.

If $\sum a_n$ converges, then $\sum f_n(x)$ converges uniformly.

Speed of course?

The Goldilocks question:
Too fast? Too slow? Just right?