Lecture 18: Generalizing Taylor series

Math 153 Section 57

Friday November 7, 2008

Following chapter 12.7.

1 how to expand Taylor series around other values.

Two perspectives: the formal answer, and the trick by replacing x by $x-a$.

2 examples

example: log around $x=1$
example: $1/x$ around $x=1$

3 euler's formula

exponentials and trig functions: $e^{i\theta} = \cos\theta + i\sin\theta$

4 tricks

tricks for Taylor series: do $2\sin x\cos x$ and discover that this is $\sin(2x)$:

$$2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

which equals

$$2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - 2\left(\frac{x^3}{2} - \frac{x^5}{12} + \cdots\right) + 2\left(\frac{x^5}{24} - \cdots\right)$$

which simplifies to

$$2x - 2\left(\frac{1}{6} + \frac{1}{2}\right)x^3 + 2\left(\frac{1}{120} + \frac{1}{12} + \frac{1}{24}\right)x^5 - \cdots$$

or equivalently

$$2x - \frac{4}{3}x^3 + \left(\frac{4}{15}\right)x^5 - \cdots$$

But this is the same as

$$\frac{1}{2}\left((2x) - \frac{(2x)^3}{6} + \frac{(2x)^5}{5!} - \cdots\right)$$

which is the series for $\sin(2x)$

Very formally:

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}x^{2n+2m+1}}{(2n+1)!\,(2m)!}$$

5 trick again

$\sin x/x$ is very easy to do

What about $x/\sin x$? We could differentiate, but that would be painful. Instead, assume it has a Taylor series, and do long division to find it.

$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \frac{127x^8}{604800} + \frac{73x^{10}}{3421440} + \cdots$$

same trick works on $1/(1-x)$.

6 proving facts about functions

$e^a \cdot e^b = e^{a+b}$