Topology of Piecewise-Linear Manifolds

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Lecture 8
Summer 2010

Knotting S^2 in S^4

Problem

Suppose $S^1 \subset S^3$ is a knot;
show that the suspension
gives a knotted S^2 in S^4.

This is called K **derived away** from L.

On the other hand, the derived of K **near** L is obtained by deriving K away from $L \cup C(L,K)$, i.e., subdividing those simplexes meeting $|L|$ but not in L.

If K' is K derived near L, then $N(L,K')$ is a **derived neighborhood** of L in K.

Definition (Regular neighborhood)**.** Suppose $X \subset Y$ are polyhedra.

- $|K|$ a neighborhood of X in Y.
- $|L| = X$.
- $L \Subset K$.
- K' derived of K near L.

Then $|N(L,K')|$ is called a **regular neighborhood** of X in Y.

Theorem. If N_1 and N_2 are regular neighborhoods of X in Y, then there is a homeomorphism $h : Y \to Y$, which throws N_1 onto N_2, and which is the identity on X, and the identity outside a compact subset of Y.

Theorem. A regular neighborhood N of a polyhedron X in a manifold M is a manifold with boundary.

Theorem (Simplicial neighborhood theorem)**.** Suppose X is a compact polyhedron, M is a manifold, and $X \subset \operatorname{int} M$. Then a polyhedral neighborhood N of X in $\operatorname{int} M$ is a regular neighborhood if and only if

- N is a compact manifold with boundary
- there are triangulations (K, L, J) of $(N, X, \partial N)$ with $L \Subset K$, $K = N(L,K)$ and $J = \partial N(L,K)$.

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Proof.

Links are preserved. $\square$

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Problem

Find a knotted $S^2 \subset S^4$ which is locally flat.

Philosophy

Why is unknotting easier than knotting?

(Why is it easier to show that a pair is unknotted than to show that it is knotted?)

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Existential versus universal quantifiers

E. C. Zeeman

Goal: *"Unknotting combinatorial balls"*

a paper in the **Annals**

Main result

Theorem

Any embedding $S^p \subset S^q$ is unknotted, if $q - p \geq 3$.

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Corollary

Any embedding $S^1 \subset S^4$ is unknotted.

Advertisement

There are now other ways to prove Zeeman's result.

The technical lemmas do some "heavy lifting" and I am curious if there are other applications of the machinery in Zeeman's paper.

Pairs

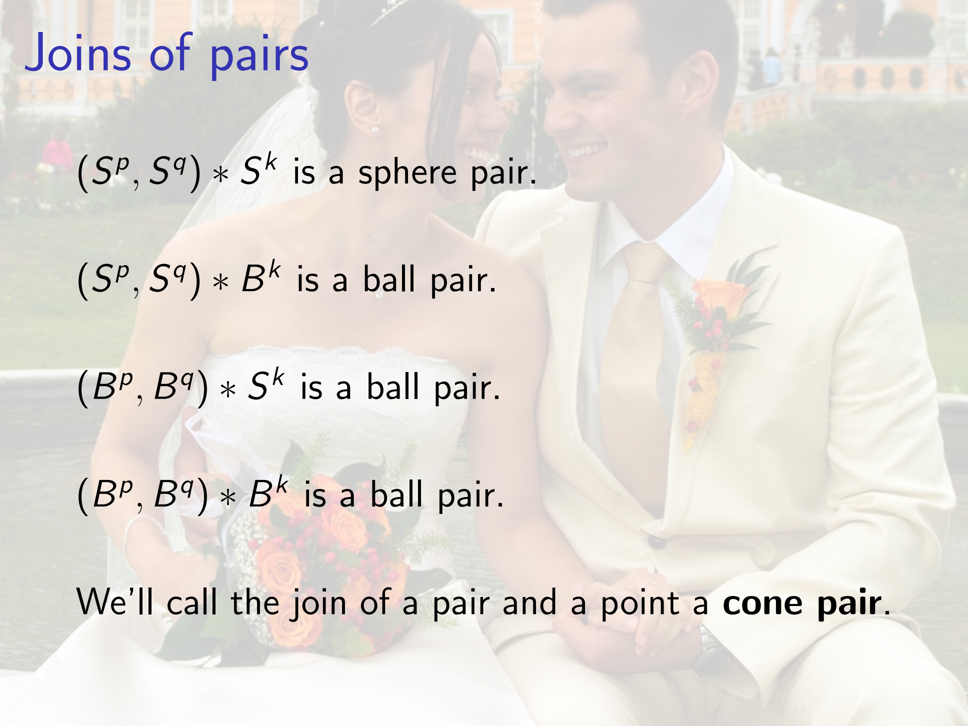
Definition

"(p, q)-sphere pair" means $S^q \subset S^p$.
We sometimes write (S^p, S^q).

"(p, q)-ball pair" means $B^q \subset B^p$,
with B^q properly embedded in B^p,
meaning $\partial B^q \subset \partial B^p$ and $\operatorname{int} B^q \subset \operatorname{int} B^p$.
We sometimes write (B^p, B^q).

"Pair" means either a ball-pair or sphere-pair.

$\partial(B^p, B^q) = (S^{p-1}, S^{q-1})$.



Joins of pairs

$(S^p, S^q) * S^k$ is a sphere pair.

$(S^p, S^q) * B^k$ is a ball pair.

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We'll call the join of a pair and a point a **cone pair**.

Subpairs

$X = (X^p, X^q)$ and $Y = (Y^r, Y^s)$ are pairs
we say Y is a subpair of X
(written $Y \subset X$ or $X \supset Y$)
if $Y^r \subset X^p$ and $T^s = X^q \cap Y^r$.

If $P = (S^p, S^q) \supset Q = (B^p, B^q)$, then

$$P - \operatorname{int} Q = (S^p - \operatorname{int} B^p, S^q - \operatorname{int} B^q)$$

is a ball pair (via regular neighborhood machinery).

Faces of pairs

If $Q' = (B^{p-1}, B^{q-1})$ is
contained in the boundary of $Q = (B^p, B^q)$,
we call Q' a **face** of Q.

Theorem

If ball pairs intersect in their common boundary, their union is a sphere pair.

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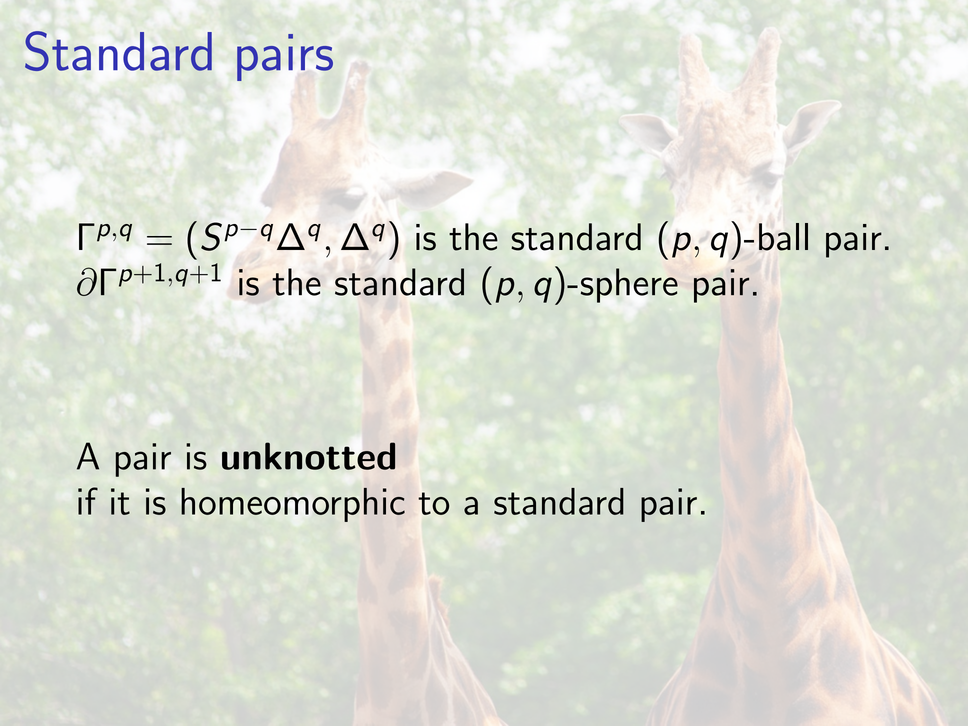
If ball pairs intersect in a face, their union is a ball pair.



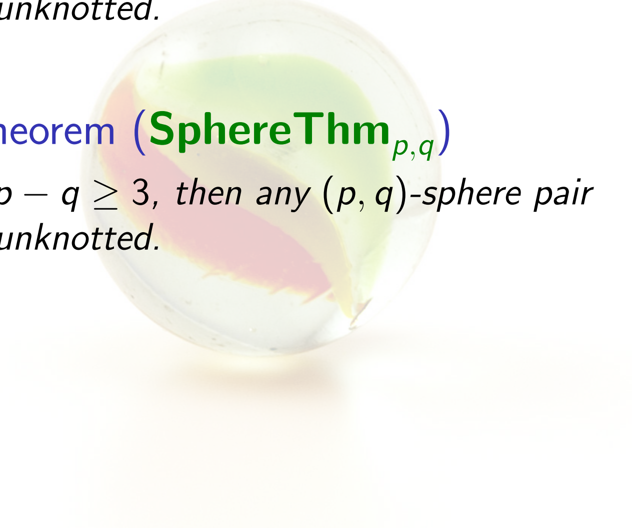
Standard pairs

$\Gamma^{p,q} = (S^{p-q}\Delta^q, \Delta^q)$ is the standard (p,q)-ball pair.
$\partial\Gamma^{p+1,q+1}$ is the standard (p,q)-sphere pair.

A pair is **unknotted**
if it is homeomorphic to a standard pair.

Theorem ($\mathbf{BallThm}_{p,q}$)

If $p - q \geq 3$, then any (p,q)-ball pair is unknotted.



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Theorem ($\mathbf{SphereThm}_{p,q}$)

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Proof.

By induction. $\square$

Overview of the induction

Prove $\mathbf{BallThm}_{p,0}$ by hand.

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The last step

$$\mathbf{BallThm}_{p-1,q-1} \text{ and } \mathbf{SphereThm}_{p-1,q-1}$$
$$\Downarrow$$
$$\mathbf{BallThm}_{p,q}$$

This will require more machinery,
building on simplicial collapse
and regular neighborhoods.

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Lemma

If $p-q \geq 3$ *and* (B^p, B^q) *is any ball pair, then* $B^p \searrow B^q$.

But first. . .

Before we can proceed,
we will prove a couple of helpful lemmas.

Lemma

Q_1 and Q_2 are unknotted (p, q)-ball pairs.
Any homeomorphism $h : \partial Q_1 \xrightarrow{\cong} \partial Q_2$
extends to a homeomorphism $h' : Q_1 \xrightarrow{\cong} Q_2$.

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Proof (via *Alexander trick*).

$y =$ point in interior of Δ^q;
then since Q_i is unknotted, we have maps
$f_i : Q_i \xrightarrow{\cong} \{y\} * \partial\Gamma^{p,q}$

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$$h' : Q_1 \xrightarrow{f_1} \{y\} * \partial\Gamma^{p,q} \xrightarrow{\text{cone}\, g} \{y\} * \partial\Gamma^{p,q} \xrightarrow{{f_2}^{-1}} Q_2 \quad \square$$

In short, **radial extension**.

Lemma (assume **BallThm**$_{p-1,q-1}$)

If Q_1, Q_2 are unknotted (p,q)-ball pairs, and $Q_3 = Q_1 \cap Q_2$ is a face, then $Q_1 \cup Q_2$ is unknotted.

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Proof.

Choose $Q_3 \xrightarrow{\cong} \Gamma^{p-1,q-1}$, extend over $\partial Q_1 - \operatorname{int} Q_3$ to $h : \partial Q_1 \xrightarrow{\cong} \Gamma^{p-1,q-1} \cup C\partial\Gamma^{p-1,q-1}$,

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Similarly produce $Q_2 \to C\Gamma^{p-1,q-1}$

Glue together $Q_1 \cup Q_2 \cong S\Gamma^{p-1,q-1}$. $\square$

Regular neighborhoods

M, an n-manifold, $X \subset M$ a polyhedron
a **regular neighborhood** of X in M
is a subpolyhedron $N \subset M$ such that

- N is a closed neighborhood of X
- N is an n-manifold
- $N \searrow X$.

Theorem

If N_1 and N_2 are regular neighborhoods of $X \subset M$, there's a homeomorphism $N_1 \longrightarrow N_2$ keeping X fixed.

Lemma

Assuming $\mathbf{BallThm}_{p-1,q-1}$ *and* $\mathbf{SphereThm}_{p-1,q-1}$, (B^p, B^q) *with* $p - q \geq 3$ *is unknotted provided* $B^p \searrow B^q$.

Lemma

Assuming **BallThm**$_{p-1,q-1}$ *and* **SphereThm**$_{p-1,q-1}$*,*
(B^p, B^q) *with* $p - q \geq 3$ *is unknotted*
provided $B^p \searrow B^q$*.*

Warning: The lemma is false if $p - q = 2$.
If $(B^4, B^2) = \mathrm{cone}(S^3, S^1)$,
then $B^4 \searrow B^2$ because cones collapse to a subcone

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$Q_0 = \{x\} * L$ where $L = (\mathrm{lk}(x, B^p), \mathrm{lk}(x, B^q))$,

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Proceed by induction.

$Q_0 = \{x\} * L$ where $L = (\mathrm{lk}(x, B^p), \mathrm{lk}(x, B^q))$,
and L is unknotted by either **BallThm**$_{p-1,q-1}$ or **SphereThm**$_{p-1,q-1}$.

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Proof Technique.

Sunny collapse. $\square$

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STOP

Stop and back up a bit.

Shadows

$I^p = I^{p-1} \times I$,
thinking of I^{p-1} as horizontal
and I as vertical.

X a polyhedron, $X \subset I^p$.
a point $y \in I^p$ lies in the **shadow** of X, if y lies below a point of X.

Definition

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Lemma

$X^\star$ is a subpolyhedron of X.

Lemma

If X is a polyhedron,
$X \subset I^p$,
$\dim X = q \leq p$, *and*
$\dim(X \cap \partial I^p) < p - 1$,
then there exists
$h : I^p \xrightarrow{\cong} I^p$,
$h(X) = X$,
$h(X) \cap \left(I^{p-1} \times \{1\}\right) = \varnothing$,
$h(X) \cap \left(I^{p-1} \times \{0\}\right) = \varnothing$,
$h(X) \cap$ *vertical line* $=$
finite set, and
$\dim h(X)^\star \leq 2q - p + 1$.

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Proof.

Since $\partial I^p \not\subset X$,
pick vertical top-dim face
$I^{p-2} \times I$ of I^p.
Throw $X \cap \partial I^p$ into the
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Shift vertices into general position.

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Shift vertices into general position. $\square$

Sunny collapse

$Y \subset X \subset I^p$.
An elementary collapse $X \searrow Y$ is **sunny**
if no point of $X - Y$ is in the shadow of X.

A **sunny collapse** is a sequence of sunny elementary collapses.

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An elementary collapse $X \searrow Y$ is **sunny**
if no point of $X - Y$ is in the shadow of X.

A **sunny collapse** is a sequence of sunny elementary collapses. Write $X \overset{☼}{\searrow} Y$.

If $X \overset{☼}{\searrow}$ pt, call X sunny collapsible.

All that remains

If we could prove

Lemma

If $p - q \geq 3$ *and* (B^p, B^q) *is any ball pair, then* $B^p \searrow B^q$.

we would finish the argument.

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Lemma

Suppose (I^p, X) is a (p, q)-ball pair,
$p - q \geq 3$
$X \cap \left(I^{p-1} \times \{1\}\right) = \varnothing$,
$X \cap \left(I^{p-1} \times \{0\}\right) = \varnothing$,
$X \cap$ *vertical line* $=$ *finite set, and*
$\dim X^{\star} \leq 2q - p + 1$.
Then $X \overset{\text{☼}}{\searrow}$ *point.*

Sunny collapse in codimension two can be obstructed

Proof postponed.

Putting together all the pieces.

If $X \subset I^p$, define $X^\sharp$ to be all points in I^p lying in shadow of X.

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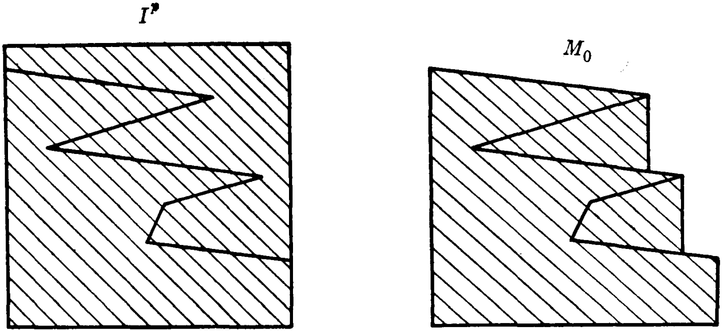
First show $I^p \searrow M$.


I'
M_0

If $X \subset I^p$ and $X \overset{\sun}{\searrow}$ point, then

$$X = X_0 \overset{\sun}{\searrow} X_1 \overset{\sun}{\searrow} \cdots \overset{\sun}{\searrow} K_n = \text{point}.$$

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Define $M_i = \left(I^{p-1} \times \{0\}\right) \cup X \cup X_i^{\sharp}$.

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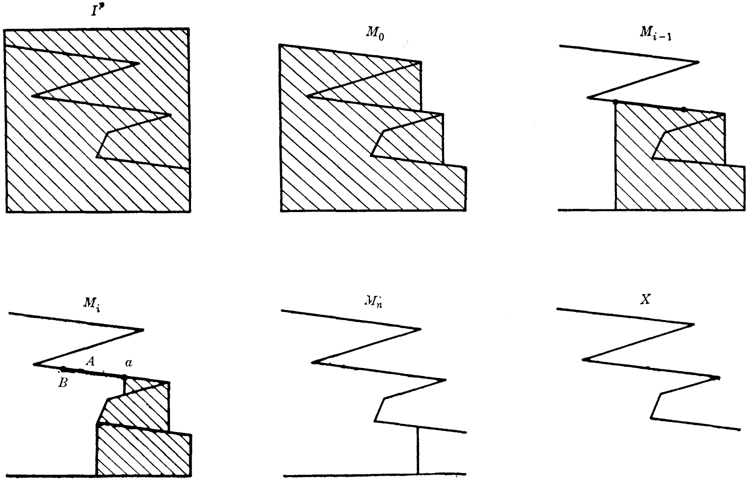
Define $M_i = \big(I^{p-1} \times \{0\}\big) \cup X \cup X_i{}^{\sharp}$.

Goal: $I^p \searrow M_0 \searrow \cdots \searrow M_n \searrow X$.


I'
M0
Mi-1
Mi
A
a
B
Mn
X