Lecture 11: Analytic functions in regions

Math 660—Jim Fowler

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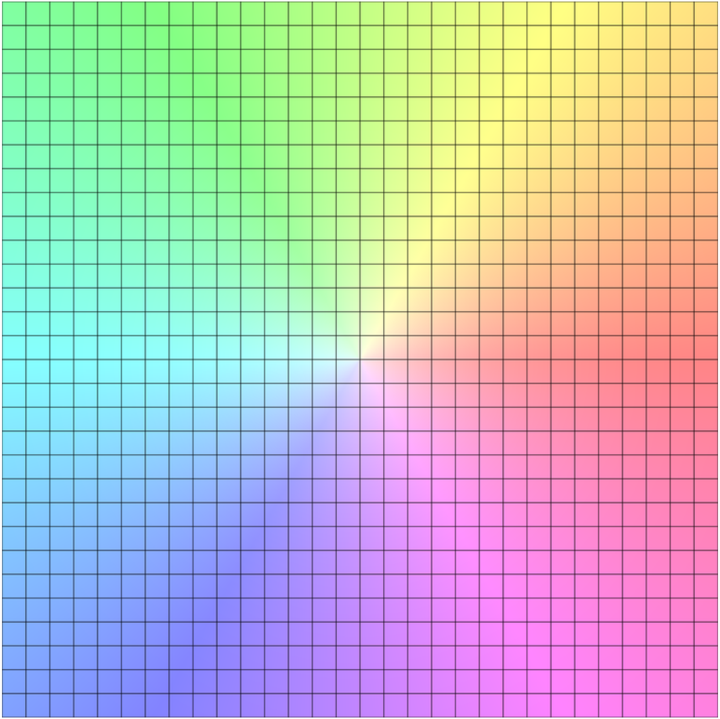
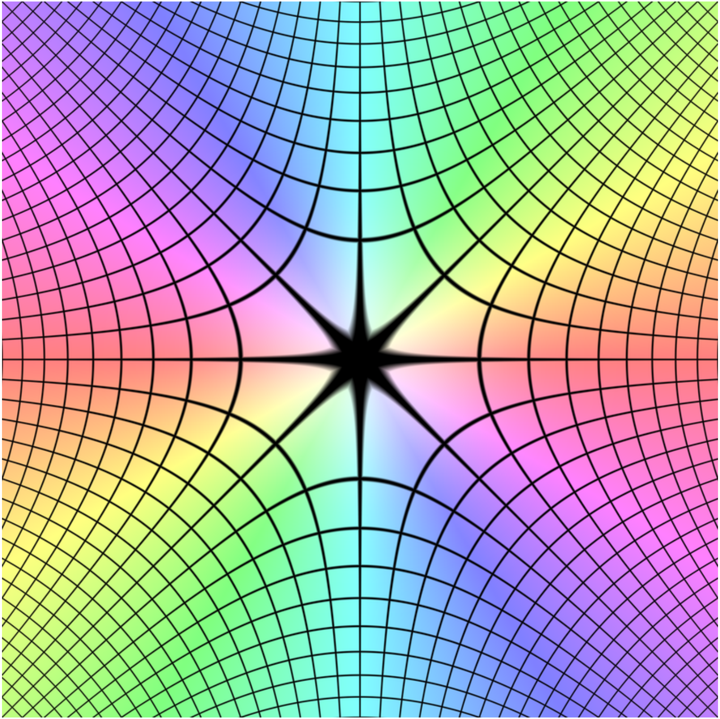
Multi-valued functions

How to talk about $\sqrt{z}$? Or $\log z$?

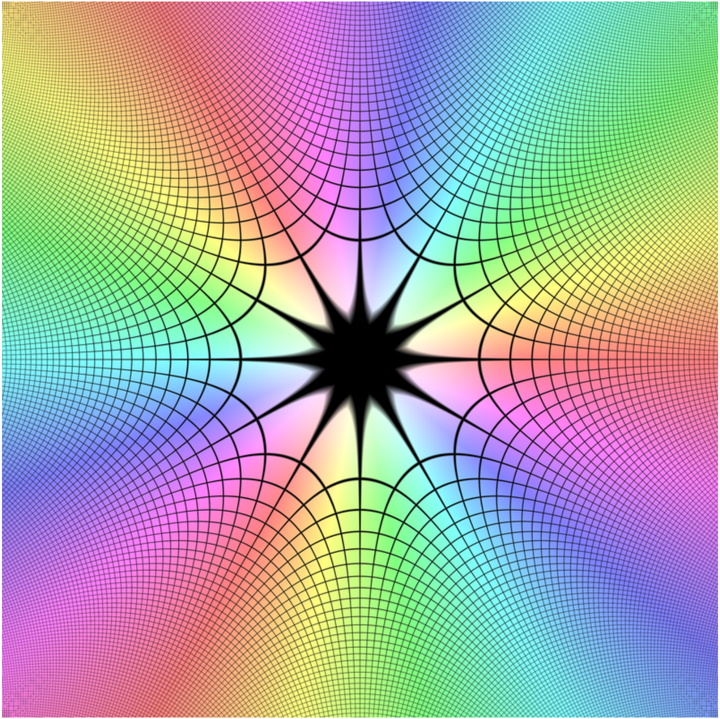
Conformal maps



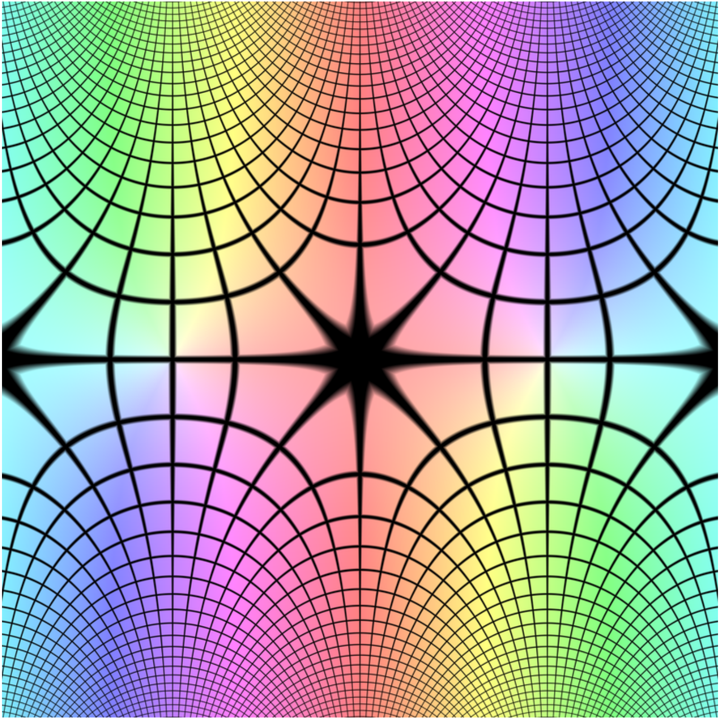
z



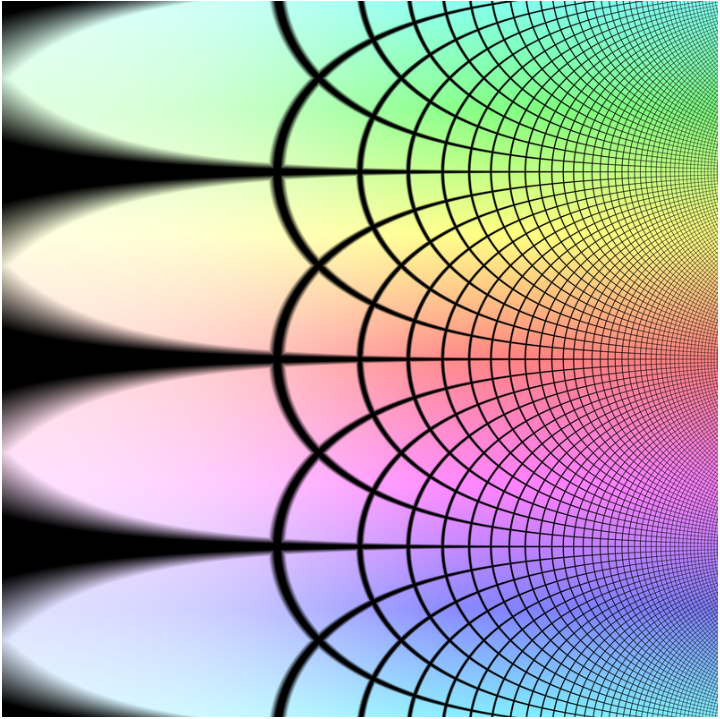
z^2



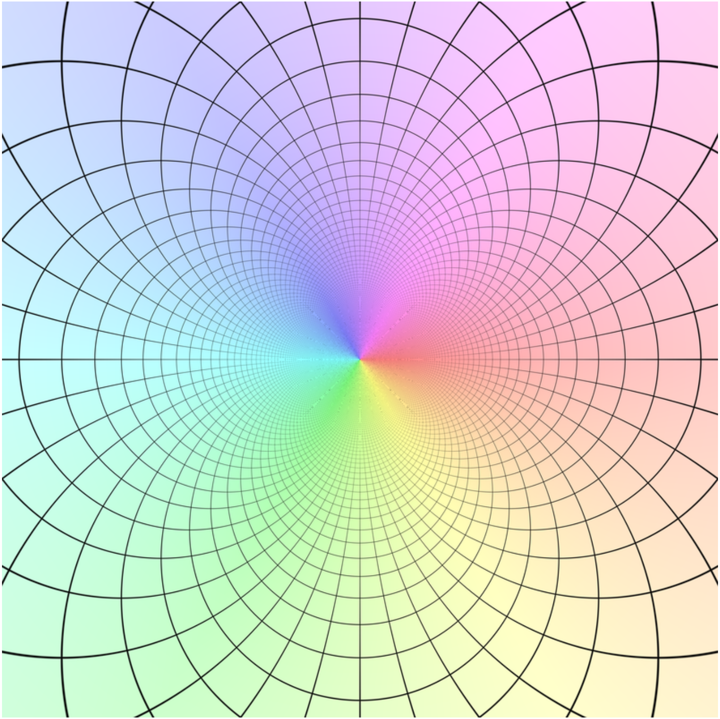
z^3



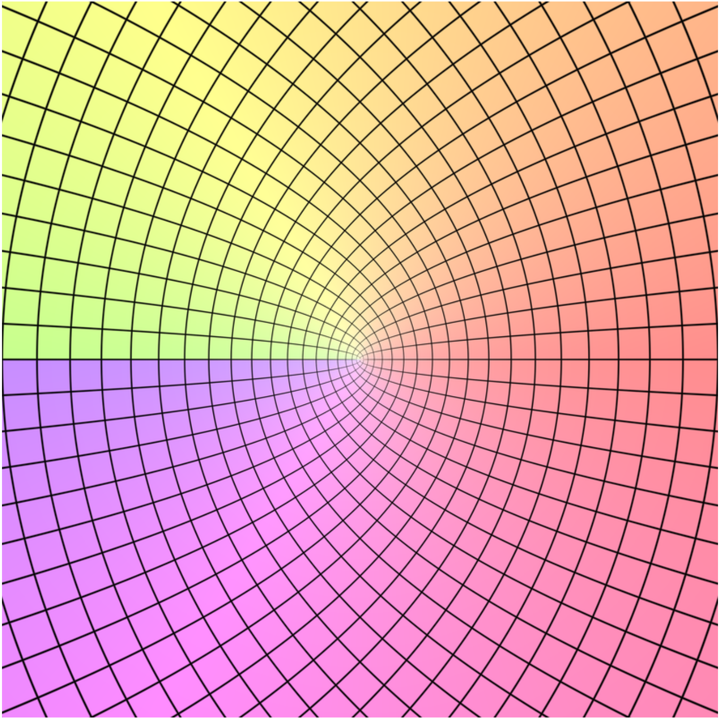
cos z



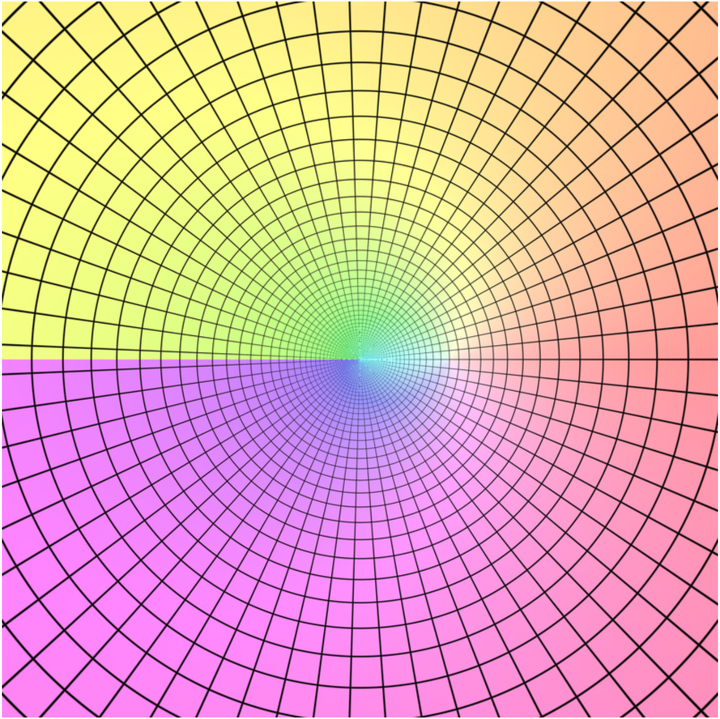
e^z



$1/z$



$\sqrt{z}$



$\log z$

z

z^2

z^3

cos z

sin z

e^z

$1/z$

$\log z$

Proof that analytic functions are conformal

A converse?

Suppose $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.
Set $w(t) = f(z(t))$, so

$$
\begin{aligned}
w'(t) &= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) z'(t) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \overline{z'(t)}.
\end{aligned}
$$

If angles are preserved,

$$
\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \frac{\overline{z'(t)}}{z'(t)}.
$$

has a constant argument, so $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$.

A converse?

What about if the change in scale is isotropic?

Riemann surfaces

We won't go into these in any detail,
but a bit of thinking about this will help.

Riemann surface for $\sqrt{z}$?

Riemann surface for $\sqrt{z^4 - 1}$?