Problem Set 9 — Math 765

Winter 2011 — Jim Fowler

Immersion theory. This week, we'll study $\mathrm{Imm}(M, N)$, the space of immersions from M to N. The most important result is

Theorem (Smale–Hirsch)**.** Let M be a closed smooth manifold, and N a smooth manifold. Then map $\mathrm{Imm}(M, N) \to \mathrm{Mono}(TM, TN)$ which sends f to Df is a homotopy equivalence. Here, $\mathrm{Mono}(TM, TN)$ consists of pairs of continuous maps $f : M \to N$ and a smooth bundle map $TM \to f^\star TN$ which is fiberwise injective. Note that an element of $\mathrm{Mono}(TM, TN)$ need not arise from an immersion and its derivative.

Email me with questions at `fowler@math.osu.edu`. *The exercises below should be handed in on Monday, March 7, 2011.*

Problem 9.1 (Bicycle chain redux)

Compute $\pi_0(\mathrm{Mono}(TS^1, T\mathbb{R}^2))$, and use this result to say something about (regular homotopy[1] classes of) immersions of the circle in the plane.

Problem 9.2 (Sphere eversion is possible)

Let $i : S^2 \to \mathbb{R}^3$ be the standard inclusion, and let $r : S^2 \to S^2$ be the map given by $r(v) = -v$. Show that the immersions i and $i \circ r$ are homotopic through immersions.

For a bonus point, for which $S^n \hookrightarrow \mathbb{R}^{n+1}$ is sphere eversion possible?

[1] A *regular homotopy* is a homotopy through immersions which extends continuously to a homotopy of the tangent bundles.