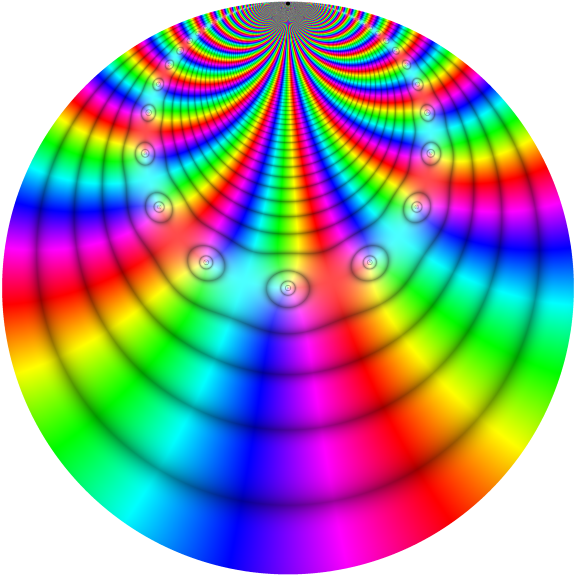
Lecture 28: Local mapping

Math 660—Jim Fowler

Thursday, July 29, 2010

$f(z) = \sin(z - i)$ viewed in a disk



Algebraic order

in the second case, the cut-off α is an integer.

Algebraic order

in the second case, the cut-off α is an integer.
Apply Taylor's theorem to $(z-a)^h f(z)$, to find

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + \cdots + B_1(z-a)^{h-1} + \varphi(z)(z-$$

with $\varphi(z)$ analytic at $z = a$.
Divide both sides by $(z-a)^h$, to get

$$f(z) = \frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \cdots + \varphi(z)$$

The part before $\varphi(z)$ is the **singular part** of f.

Algebraic order

in the second case, the cut-off α is an integer.
Apply Taylor's theorem to $(z-a)^h f(z)$, to find

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + \cdots + B_1(z-a)^{h-1} + \varphi(z)(z-$$

with $\varphi(z)$ analytic at $z = a$.
Divide both sides by $(z-a)^h$, to get

$$f(z) = \frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \cdots + \varphi(z)$$

The part before $\varphi(z)$ is the **singular part** of f.
A pole comes with a well-defined singular part, and if you subtract off the singular part, you are left with an analytic function.

Essential singularity

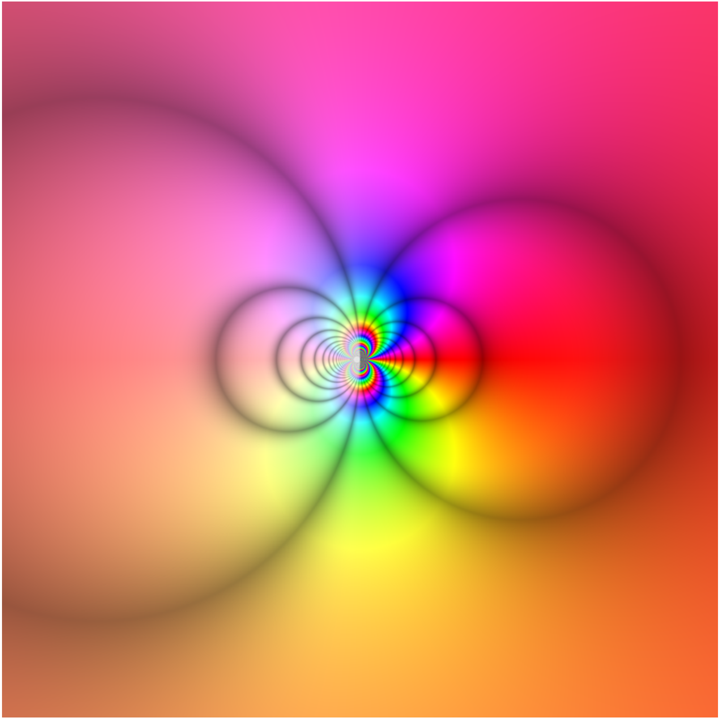
If there is no such α satisfying the limits

$$\lim_{z \to a} |z-a|^{\alpha}|f(z)| = 0$$
$$\lim_{z \to a} |z-a|^{\alpha}|f(z)| = \infty$$

then we say it is an **essential singularity**.

Theorem

An analytic function comes arbitrarily close to every complex value in a neighborhood of an essential singularity.



counting the zeroes and poles of an analytic function

Counting zeroes

Let $D^2 = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D^2 \to \mathbb{C}$ analytic. Assume $f(z) \not\equiv 0$.

Counting zeroes

Let $D^2 = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D^2 \to \mathbb{C}$ analytic. Assume $f(z) \not\equiv 0$.

Assume $f(z)$ has only finitely many zeroes, $z_1, \ldots, z_n$, with multiplicity.

Counting zeroes

Let $D^2 = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D^2 \to \mathbb{C}$ analytic. Assume $f(z) \not\equiv 0$.

Assume $f(z)$ has only finitely many zeroes, $z_1, \ldots, z_n$, with multiplicity.

$f(z) = (z - z_1) \cdots (z - z_n) g(z)$ for $g(z)$ analytic and nowhere zero.

Counting zeroes

Let $D^2 = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D^2 \to \mathbb{C}$ analytic. Assume $f(z) \not\equiv 0$.

Assume $f(z)$ has only finitely many zeroes, $z_1, \ldots, z_n$, with multiplicity.

$f(z) = (z - z_1) \cdots (z - z_n) g(z)$ for $g(z)$ analytic and nowhere zero.

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

Now $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$, so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n(\gamma, z_1) + \cdots + n(\gamma, z_n)$$

Now $\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$, so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = n(\gamma, z_1) + \cdots + n(\gamma, z_n)$$

In fact, this is true even if there are infintely many zeroes in D^2

Now $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$, so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n(\gamma, z_1) + \cdots + n(\gamma, z_n)$$

In fact, this is true even if there are infintely many zeroes in D^2—why?

Theorem

$f : D^2 \to \mathbb{C}$ analytic with zeroes z_i.
For closed curves γ not passing through a zero,

$$\sum_i n(\gamma, z_i) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)\, dz}{f(z)}$$

Theorem

$f : D^2 \to \mathbb{C}$ analytic with zeroes z_i.
For closed curves γ not passing through a zero,

$$\sum_i n(\gamma, z_i) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)\, dz}{f(z)}$$

In particular, if we let $\Gamma = f(\gamma)$, then

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \frac{f'(z) dz}{f(z)}$$

so $n(\Gamma, 0) = \sum_i n(\gamma, z_i)$.

Local correspondence

Instead of counting zeroes, count how many times $f(z)$ takes on the value a.

Local correspondence

Instead of counting zeroes, count how many times $f(z)$ takes on the value a.

If γ is a circle and $n(\Gamma, a) = n(\Gamma, b)$, then $f(z)$ takes on the value a as many times as the value b inside γ.

Local correspondence

Instead of counting zeroes, count how many times $f(z)$ takes on the value a.

If γ is a circle and $n(\Gamma, a) = n(\Gamma, b)$, then $f(z)$ takes on the value a as many times as the value b inside γ.

Theorem

Suppose $f(z)$ is analytic at z_0, $f(z_0) = w_0$,a nd $f(z) - w_0$ has a zero of order n at z_0.
For all sufficiently small $\epsilon > 0$, there exists $\delta > 0$, so that for all a within δ of w_0, the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \epsilon$.

Corollary

A nonconstant analytic function is an open map (meaning it maps open sets to open sets).