Problem Set 3 — Piecewise-Linear Topology

Summer 2010 — Jim Fowler

This is the end of the beginning. Problem Set 3 introduces *manifolds*, and with that, the last of the survey topics. Please submit answers to problems marked with a •.

Definition. Let K be a complex, and $\sigma \in K$ a simplex. The **stellar subdivision** of K at σ is a new complex K_σ with:

- the vertices of K with a new vertex v.
- the simplexes of K not in $\mathrm{st}(\sigma, K)$,
  along with the simplexes in $v * (\partial\sigma) * \mathrm{lk}(\sigma, K)$.

In symbols, we say

$$K_\sigma := (K - \mathrm{st}(\sigma, K)) \cup (v * (\partial\sigma) * \mathrm{lk}(\sigma, K))$$

If K can be produced through a (possibly empty) sequence of stellar subdivisions of L, we say that K is a **subdivision** of L, and write $K \triangleleft L$.

If K and L are complexes, with $K' \triangleleft K$ and $L' \triangleleft L$, and K' and L' are simplicially isomorphic, then we say that K and L are **PL homeomorphic**, and write $K \cong L$.

Problem 1. For $n \geq 3$, let P_n be the boundary of an n-gon. Prove that $P_n \cong P_m$ for $n, m \geq 3$. Thus, any P_n is topologically an S^1.

Definition. A complex K is **path-connected** if for any two vertices $v, w \in K$, there exists a sequence of edges

$$[v, v_0], [v_0, v_1], [v_1, v_2], [v_2, v_3], \ldots, [v_n, w]$$

connecting v and w.

Problem 2. Prove that being path-connected is a topological property, meaning that a space which is PL homeomorphic to a path-connected space is itself path-connected.

Homeomorphisms

Problem 3. Is PL homeomorphism an equivalence relation? That is, is it:

reflexive, meaning, is A homeomorphic to A,

symmetric, meaning, if $A \cong B$, is it true that $B \cong A$, and

transitive, meaning, if $A \cong B$ and $B \cong C$, is it true that $A \cong C$?

• **Problem 4.** Is join well-defined with respect to homeomorphism? That is, if we have PL homeomorphic complexes $K \cong K'$ and $L \cong L'$, is it then the case that $K * L \cong K' * L'$?

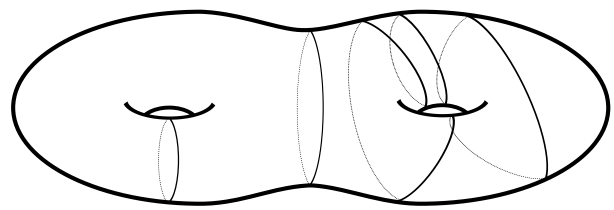
Problem 5. Let X_n consist of n points. For which $n \in \mathbb{N}$ is it the case that for every two injective maps $f, g : X_n \to S^1$, there is a homeomorphism $h : S^1 \to S^1$ so that $h \circ f = g$? For that matter, in what sense can we think of PL homeomorphic spaces as having a "homeomorphism" between them?

Problem 6. Let X_n be the disjoint union of n circles (e.g., $X_3 = S^1 \cup S^1 \cup S^1$). For this problem, call two maps $f, g : X_n \to S^2$ "equivalent" if there exists a homeomorphism $h : S^2 \to S^2$ so that $h \circ f = g$. Count the equivalence classes of maps from X_5 to S^2?

Problem 7. To the right, three curves are pictured on $T^2 \# T^2$, which is our notation for a two-holed surface. For which pairs of curves α and β does there exist a homeomorphism

$$f : T^2 \# T^2 \to T^2 \# T^2$$

so that $f(\alpha) = \beta$?



Manifolds

Definition. A complex M is an n-dimensional **PL manifold** (for short, an n-manifold) if for every vertex v of M, we have that $\mathrm{lk}(v, M)$ is PL homeomorphic to S^{n-1}.

Problem 8. List all of the 0-manifolds.

• **Problem 9.** Show that any 1-manifold is a disjoint union of circles.

Problem 10. Show that there are infinitely many path-connected 2-manifolds.

• **Problem 11.** If M is a manifold, and $N \cong M$, is N a manifold? In other words, is "being a manifold" preserved by PL homeomorphisms?

Problem 12. If σ is a k-simplex in an n-manifold M, show that $\mathrm{lk}(\sigma, M) \cong S^{n-k-1}$.

Problem 13. Suppose M^m is an m-manifold, and N^n is an n-manifold. Is their join, $M * N$, a manifold?

• **Problem 14.** Identify the manifold $S^n * S^m$.

Definition. If a complex N is a subcomplex of a complex M, and both N and M are also manifolds, then we say that N is a **submanifold** of M.

Problem 15. Suppose $f : M \to N$ is a simplicial map between manifolds, M and N. If K is a submanifold of M, is $f(K)$ a submanifold of N?

Problem 16. Suppose $f : M \to N$ is a simplicial map between manifolds, M and N. Is it always the case, for a vertex $v \in N$, that $f^{-1}(v)$ a submanifold of M? Is that never the case?

Problem 17. Suppose M is a 3-manifold. Calculate $\chi(M)$.

Problem 18. If you glue together Δ^n and Δ^n using a homeomorphism between their boundaries, do you necessarily get S^n? (You probably can't answer this question yet, but I think it is worth considering).

Manifolds with boundary

Definition. A complex M is an n-dimensional **PL manifold with boundary** (for short, and confusingly, also called an n-manifold) if, for every vertex v of M, we have that $\mathrm{lk}(v, M)$ is PL homeomorphic to either S^{n-1} or to Δ^{n-1}.

The **boundary** of an n-manifold M^n is the closure of the set of $(n-1)$-simplexes $\sigma \in M$, for which there is a unique n-simplex $\tau \in M$ with $\sigma < \tau$. We write ∂M for the boundary of M. Finally, we say that M^n is a **manifold without boundary** if $\partial M = \varnothing$, i.e., if the link of every vertex is a sphere.

Problem 19. Show that ∂ is well-defined (up to PL homeomorphism—in other words, why is $\partial M \cong \partial N$ for PL homeomorphic manifolds M and N?).

Problem 20. Prove that ∂M is a submanifold of M, and that $\partial\partial M = \varnothing$.

Problem 21. Suppose M and N are n-manifolds with PL homeomorphic boundaries. Define what one might mean by gluing M and N along their common boundary—is the result a manifold?

Problem 22. Does there exist a complex K and a number n so that, for every vertex v of K, $\mathrm{lk}(\{v\}, M) \cong \Delta^{n-1}$?