Define an operator $\star$ (called the "Hodge star"),

$$\star : \bigwedge^k \mathbb{R}^n \to \bigwedge^{n-k} \mathbb{R}^n.$$

defined by

$$\star\left(e_1 \wedge \cdots \wedge e_k\right) = e_{k+1} \wedge \cdots \wedge e_n.$$

Our definition depends on choosing a basis.

**Exercise.** Compute $\dim \bigwedge^k \mathbb{R}^n$ and $\dim \bigwedge^{n-k} \mathbb{R}^n$.

**Exercise.** Is $\star : \bigwedge^k \mathbb{R}^n \to \bigwedge^{n-k} \mathbb{R}^n$ an isomorphism?

**Exercise.** Can you find a basis-free definition of $\star$?

**Exercise.** Compute $\star \star \omega$ in terms of $\omega$.

explain more here

For $n = 3$, the cross-product of vectors $v$ and $w$ is

$$v \times w := \star\left(v \wedge w\right).$$

The inner product can also be interpreted in this language, as

$$\star\langle v, w\rangle = v \wedge (\star w)$$

## A certain three tensor.

Here we define a map that takes three vectors in $\mathbb{R}^3$ and produces a number:

$$(a, b, c) \mapsto \langle a, b \times c\rangle.$$

**Exercise.** Check that this is trilinear (so it is, in fact, a 3-tensor).

We can deduce the true nature of this 3-tensor by using the Hodge star. Note that

$$\begin{aligned} \langle a, b \times c\rangle &= \star\left(a \wedge \star\left(b \times c\right)\right) \\ &= \star\left(a \wedge \star\star\left(b \wedge c\right)\right) \\ &= \star\left(a \wedge b \wedge c\right), \end{aligned}$$

and therefore $\langle a, b \times c\rangle$ is antisymmetric in the inputs $a, b, c$. In fact, it is

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

which explains a popular method for computing the cross product (by taking the determinant of a particular matrix).

## A certain four tensor.

An interesing four tensor on $\mathbb{R}^3$ is

$$(a, b, c, d) \mapsto \langle a \times (b \times c), d\rangle.$$

**Exercise.** Check that this is a four tensor.

In fact, this four tensor can be written in a different way.

$$\begin{aligned}
\star\langle a \times (b \times c), d\rangle &= (\star\,(a \wedge \star\,(b \wedge c))) \wedge (\star d) \\
&= (a \wedge \star\,(b \wedge c)) \wedge d \\
&= a \wedge (\star\,(b \wedge c)) \wedge d \\
&= -a \wedge d \wedge \star\,(b \wedge c) \\
&= -\,(\star\star\,(a \wedge d)) \wedge \star\,(b \wedge c) \\
&= -\langle a \times d, b \times c\rangle
\end{aligned}$$

$$(a \wedge \star c) \wedge (\star(\star b \wedge d)) + (a \wedge \star b) \wedge (\star(\star c \wedge d))$$

$$\star a \wedge (c \cdot \star(b \wedge \star d) - b \cdot \star(c \wedge \star d))$$

$$\star a \wedge (c \cdot \star(\star b \wedge d) - b \cdot \star(\star c \wedge d))$$

$$a \wedge (\star c \cdot \star(\star b \wedge d) - \star b \cdot \star(\star c \wedge d))$$

$$(a \wedge \star c) \cdot \star(\star b \wedge d) - (a \wedge \star b) \cdot \star(\star c \wedge d)$$

$$(a \wedge \star c) \cdot \star(\star b \wedge d) - (a \wedge \star b) \cdot \star(\star c \wedge d)$$

$$(a \wedge \star c)(b \wedge \star d) - (a \wedge \star b)(c \wedge \star d)$$

## reduction

$$(\star((\star b) \wedge (\star c))) \wedge d = c \wedge \star(b \wedge d) + b \wedge \star(c \wedge d)$$

$$(\star(b \wedge c)) \wedge d = (\star c) \wedge \star((\star b) \wedge d) + (\star b) \wedge \star((\star c) \wedge d)$$

## Birdtracks

Levi-Civita relation says

$$(a, b, c, d) \mapsto ((v \mapsto$$