Lecture 20: Differentiating and integrating series

Math 153 Section 57

Wednesday November 12, 2008

Following chapter 12.9.

1 Differentiating term-by-term

Define $f : (-r, r) \to R$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-r, r)$, then

$$\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

the term-wise derivative, also converges on $(-r, r)$.

Moreover, f is differentiable on $(-r, r)$, and

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

Consequently: if you plug a power series into the machine that finds a Taylor series expansion, you get out the original power series. This is a key point, because if we can produce a power series by some other method, then we have found a Taylor series.

2 Radius of convergence, not same interval

$\sum_{n=1}^{\infty} x^n/n^2$ converges on $[-1, 1]$, but the dreivative only converges on $[-1, 1)$.

3 Examples

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} \sin x = \cos x$

4 Integrating term-by-term

Define $f : (-r, r) \to R$ by a series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

which converges for $x \in (-r, r)$

Define

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n\, x^n}{n+1}$$

which also converges for $x \in (-r, r)$, and

$$\int f(x)\, dx = F(x) + C$$

Remember the C.

5 Logs

Since $1/(1+x) = \sum_{n=0}^{\infty} (-1)^n x^n$, we have that

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

But $C = 0$.

6 Endpoints

Abel's theorem: if $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-r, r)$, and $f(x)$ equals the series there.

If f is left continuous at r and the series converges, then $f(r) = \sum_{n=0}^{\infty} a_n r^n$.

If f is right continuous at $-r$ and the series converges, then $f(r) = \sum_{n=0}^{\infty} a_n (-r)^n$.

So endpoints do get filled in correctly by series.

7 Not a correct argument whatsoever

$$\sin x = (1 - x/\pi)(1 + x/\pi)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

Multiply pairs of positive and negative roots, to get

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\cdots$$

Multiply it out?

$$1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{16\pi^2} + \cdots \right) x^2 + \cdots$$

But this should be the same as the Taylor series for $\frac{\sin x}{x}$. So, maybe

$$\sum_{n=1}^{\infty} \frac{1}{n^2\, \pi^2} = \frac{1}{3!}$$

So maybe $\sum_{n=1}^{\infty} 1/n^2 = 1/6$.

Well, no. We don't have any justification for making these arguments—why should that infinite product be equal to $(\sin x)/x$?