Lecture 36: Harmonic functions

Math 660—Jim Fowler

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Example

$$\int_0^{\pi} \log \sin \theta \, d\theta = -\pi \log 2$$

Single-valued branch

Define a single-valued branch of $\sqrt{1-x^2}$ for any region Ω, so that ± 1 are in the same component of $\mathbb{C} - \Omega$.

Harmonic functions

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A function $u : \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** if u is continuous, with continuous seconds partial derivatives, and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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The real part of an analytic function.

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So if $\frac{\partial u}{\partial \theta} = 0$, then $u = a \log r + b$.

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and we can find $f\, dz$.

$$dv = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy.$$

But in practice, there may be no conjugate harmonic function,
so we take as a definition

$$*du = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy.$$

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And so $\int_\gamma *du = 0$ for cycles homologous to zero.

Simply connected regions

In a simply connected regions, $\int_\gamma *du = 0$ for all cycles, so u has a single-valued conjugate function v.

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$$\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$$

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Proof: If conjugate harmonic functions exist (e.g., in a simply connected region like a rectangle), then

$$\begin{aligned} u_1 * du_2 - u_2 * du_1 &= u_1 dv_2 - u_2 dv_1 \\ &= u_1 dv_2 + v_1 du_2 - d(u_2 v_1) \\ &= (\mathrm{im}(u_1 + iv_1)(du_2 + idv_2)) - d(u_2 v_1), \end{aligned}$$

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so the integral vanishes.

$\square$