Lecture 2: Analytic functions

Math 660—Jim Fowler

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Stereographic projection

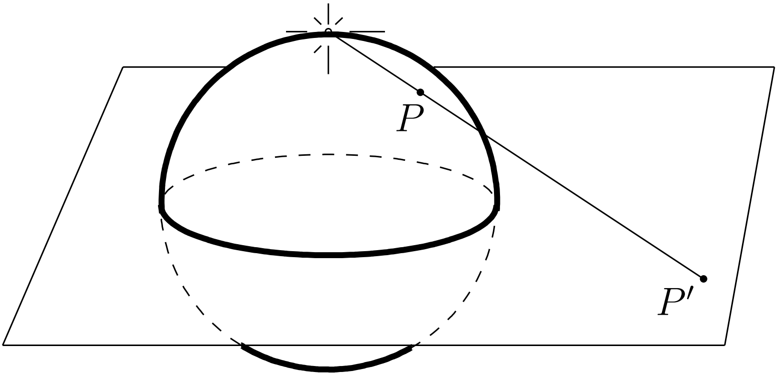
Extend $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$.

Consider $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_i {x_i}^2 = 1\}$,

Define $f : S^2 \to \mathbb{C} \cup \{\infty\}$ by

$$z = x + iy = f(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

The points $(0, 0, 1)$, $(x, y, 0)$, and (x_1, x_2, x_3) are collinear.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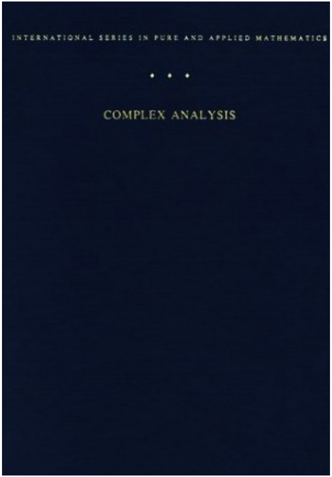

Today's Goal

§2.1.1 and 2.1.2 of *Complex Analysis*

Derivatives for $\mathbb{C}$-valued functions

Review of Calculus

- Limit
- Derivative
- Integral

Limits

Definition

$\lim_{x \to a} f(x) = L$ if
for all $\epsilon > 0$,
there exists $\delta > 0$, such that
$|f(x) - L| < \epsilon$ whenever
$|x - a| < \delta$.

"Absolute value" makes sense in $\mathbb{C}$.

Limits

Theorem

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

Theorem

$$\lim_{x \to a} \overline{f(x)} = \overline{\lim_{x \to a} f(x)}$$

Derivative

Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Warning: $h \in \mathbb{C}$. This has deep consequences.

Theorem

Suppose $f : \mathbb{C} \to \mathbb{R}$ *is complex differentiable.*
Then $f' \equiv 0$.

Terminology

Definition

A complex differentiable function f is called *analytic* or *holomorphic*.

If $f : \mathbb{C} \to \mathbb{C}$ and f is everywhere complex differentiable, we call it *entire*.

Examples of entire functions include polynomials.

Analytic Functions

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Compare linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ to linear maps $\mathbb{C} \to \mathbb{C}$.

The sum	of analytic functions is analytic.
The difference	of analytic functions is analytic.
The product	of analytic functions is analytic.
The quotient	of analytic functions is analytic,
	where the denominator is nonzero.

Cauchy-Riemann equations

 

Cauchy-Riemann equations

$f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Jacobian

$$
\begin{aligned}
|f'(z)|^2 &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\
&= \text{Jacobian of } u \text{ and } v \text{ with respect to } x \text{ and } y \\
&= \text{infinitesimal change in area}
\end{aligned}
$$

Complex Analysis is Amazing

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The derivative of a analytic function is itself differentiable.

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This is incredible. We will prove this later.

Harmonic functions

A function $f : U \to \mathbb{R}$ is *harmonic* if

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \text{ on } U \subset \mathbb{R}^n.$$

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The real and imaginary parts of an analytic function are harmonic.

Cauchy-Riemann equations

If u, v are a harmonic functions ($\Delta u = 0$) and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then v is called a *conjugate harmonic function* for u.

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So $v(x, y) = 2xy + C$ for some constant C.

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Note that $z^2 = u(x, y) + i \cdot 2xy$.

Cauchy-Riemann equations

Theorem

If $u(x, y)$ and $v(x, y)$ have continuous first partials, and u and v satisfy the Cauchy-Riemann equations, then

$$f(a + bi) = u(a, b) + i\, v(a, b)$$

is analytic.

Cauchy-Riemann equations

Alternatively, the C-R equations can be written as

$$\frac{\partial f}{\partial \overline{z}} = 0$$

where $z = x + iy$ and $\overline{z} = x - iy$.

In some sense, analytic functions are truly functions of z, and not of $\overline{z}$.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$