Fake Final

Math 345

December 2010

Das Unendliche hat wie keine andere Frage von jeher so tief das Gemüt der Menschen bewegt. *The infinite, has, like no other question, moved the human mind so deeply.*

—David Hilbert

Name: ______________________________

Lecture time (circle one): 12:30–1:18 P.M.

2:30–3:18 P.M.

1. Write your name above.
2. Calculators are forbidden.
3. Look inside the fake exam before taking the real exam.
4. Justify your answers.
5. Show your work.
6. Write your answers down to practice.
7. Answer all questions.
8. To prevent fire, do not divide by zero.

Problem 1	/350
Problem 2	/350
Problem 3	/350
Problem 4	/350
Problem 5	/350
Problem 6	/350
Problem 7	/350
Problem 8	/350
Problem 9	/350
Problem 10	/350
Problem 11	/350
Problem 12	/350
Problem 13	/350
Problem 14	/350
Problem 15	/350
Problem 16	/350
Problem 17	/350
Problem 18	/350
Problem 19	/350
Problem 20	/350
Problem 21	/350
Problem 22	/350
Problem 23	/350
Problem 24	/350
Problem 25	/350
Problem 26	/350
Problem 27	/350
Problem 28	/350
Problem 29	/350
Problem 30	/350
Problem 31	/350
Problem 32	/350
Problem 33	/350
Problem 34	/350

Problem 1 /350

Let A be the set of English words, and B be the set of words (nonsense or not) made up of (possibly zero!) letters in the Latin alphabet. Consider the function $f : A \to B$ which sends an English word to the word without its last letter.

Is f injective? Is f surjective?

Solution

Problem 2 /350

Let A be the set of all strings of at least 2 letters in the Latin alphabet. Let $r : A \to A$ be the function which reverses its input (so $r(\text{hello}) = \text{olleh}$) and $s : A \to A$ be the function which takes the first letter and makes it the last letter (so $s(\text{hello}) = \text{elloh}$)).

By applying r and s repeatedly, is it possible to transform game into mage? Is it possible to transform pets to pest? Is it possible to transform maple into ample?

Solution

Problem 3 /350

Let $A = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $B = \{x \in \mathbb{Z} : x \text{ is odd}\}$. Describe a bijection $f : A \to B$.

Be sure to justify your answer completely.

Solution

Problem 4 /350

Use complete induction to prove that every natural number can be written as a product of prime numbers.

Solution

Problem 5 /350

Consider the statement: if $f : A \to A$ is injective, then f is surjective. What is the converse of this statement? The contrapositive of this statement? If A is a finite set, which of these statements are true?

Solution

Problem 6 /350

Let $f : \mathbb{R} \to \mathbb{R}$, and suppose $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} \exists z \in \mathbb{R} f(x+z) = y$. Does it follow that f is surjective?

Solution

Problem 7 /350

Suppose $f : \mathbb{N} \to \mathbb{N}$ is given by

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$$

Is f bijective? If so, find an inverse function.

Solution

Problem 8 /350

Suppose A, B, C are sets, and $f : A \to B$ and $g : B \to C$ are functions. If f is surjective, and g is injective, what can you say about the composition $g \circ f$? Need it be injective? Need it be surjective?

Solution

Problem 9 /350

Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function with the properties that

- $f(xy) = f(x) \cdot f(y)$ and
- $f(p) = 2$ if p is prime.

Among two-digit numbers x, how large can $f(x)$ be?

Solution

Problem 10 /350

Find three sets of real numbers, A, B, and C, so that $A \cap B \neq \varnothing$, and $A \cap C \neq \varnothing$, and $B \cap C \neq \varnothing$, but $A \cap B \cap C = \varnothing$.

Solution

Problem 11 /350

Let $P_2(A)$ be the set of two element subsets of A. If A is an n-element set, show that $P_2(A)$ has $\binom{n}{2}$ elements.

Solution

Problem 12 /350

Let A be the set of natural numbers which only use the digits 1 and 2. Define B by

$$\{n \in \mathbb{N} : 9n \in A\}.$$

Does B have a least element?

Solution

Problem 13 /350

Let A be a set. Suppose $P(n)$ is the proposition $(n \in A) \rightarrow ((n+1) \in A)$. If $P(n)$ is true for all n, does it follow that $A = \mathbb{N}$?

Solution

Problem 14 /350

Let P, Q, R, S be propositions.

Rewrite $(\neg(P \vee (Q \wedge R))) \wedge S$ in a simpler way.

Also, rewrite $(\neg(P \vee (Q \wedge R))) \wedge P$ in a simpler way.

Solution

Problem 15 /350

Show that every k^{th} Fibonacci number is a multiple of F_k.

Solution

Problem 16 /350

For which $n \in \mathbb{Z}$ is there a number $x \in \mathbb{Z}$ so that $x^2 \equiv n \pmod 1$1?

Solution

Problem 17 /350

For which prime numbers p is it possible to find an integer x so that $x^2 \equiv -1 \pmod{p}$? You might not be able to prove your statement, but you will probably be able to come up with enough evidence to support your conjecture.

Solution

Problem 18 /350

Solution

Problem 19 /350

Let A be the set

$$\{n \in \mathbb{N} : n \text{ is prime and } n \equiv 3 \pmod{4}\}.$$

Prove that A is infinite.

Solution

Problem 20 /350

Let A be the set of prime numbers. Can you find a million consecutive numbers in the set $\mathbb{N} \setminus A$?

Solution

Problem 21 /350

Explainw why there is a bijection between $\mathbb{N}$ and $\mathbb{Q}$.

Solution

Problem 22 /350

Prove that, if A is an infinite set, then there exists a proper subset $B \subset A$ with B equinumerous to A.

Solution

Problem 23 /350

Let A, B, C be sets. Prove that $A \cap (B \cup C)$ is equal to $(A \cap B) \cup (A \cap C)$.

Solution

Problem 24 /350

For $q \in \mathbb{Q}$, let A_q be the interval $(q - \epsilon, q + \epsilon)$. Prove that $\bigcup_{q \in \mathbb{Q}} A_q = \mathbb{R}$.

Solution

Problem 25 /350

Show that 101 divides $(15+17)^{101} - 15 - 17$.

Solution

Problem 26 /350

Describe a collection of eight sets, any seven of which intersect, but for which all eight do not intersect.

Describe a collection of k sets, any $(k-1)$ of which intersect, but for which all k do not intersect.

Solution

Problem 27 /350

Define functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = n + 1$ and

$$g(n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n = 1, \\ n & \text{otherwise.} \end{cases}$$

Describe a bijective function $h : \mathbb{Z} \to \mathbb{Z}$ which cannot be written as a composition of f and g and f^{-1}.

Solution

Problem 28 /350

In the statement

$$\forall x \in \mathbb{R}\ \big((\exists y \in \mathbb{R}\,(x^2 > y)) \vee (\exists z \in \mathbb{R}\,(z^2 < w))\big)\,,$$

which variables are bound? Which variables are free?

Solution

Problem 29 /350

Prove the binomial theorem.

Solution

Problem 30 /350

Show that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

Solution

Problem 31 /350

Describe a proposition $P(x, y)$ so that

$$\forall x \, \exists y \, P(x, y)$$

is true, but

$$\exists x \, \forall y \, P(x, y)$$

is false.

Solution

Problem 32 /350

Let ω be the set of nonnegative integers, and define a function $A : \omega \times \omega \to \omega$,

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Compute $A(4,4)$.

Solution

Problem 33 /350

Define a sequence a_n be the rule that $a_0 = 1$ and $a_1 = 2$ and $a_{n+2} = 2a_{n+1} + a_n$. For which values of n is a_n a multiple of 7?

Solution

Problem 34 /350

Show that the set of functions $f : \mathbb{N} \to \mathbb{N}$ is not countable.

Solution

Problem 35 /350

Let A be the set of functions $\mathbb{N} \to \mathbb{N}$, and let $g : \mathbb{N} \to A$ be a function. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = g(n)\,(n) + 1$. Does there exists a number $F \in \mathbb{N}$ so that $g(F) = f$?

Solution

Problem 36 /350

Define $A = \{x \in x \in \mathbb{R} : 0 \leq x \leq 1\}$, and suppose B_x is the interval $(x, 10)$. Determine

$$\bigcup_{x \in A} B_x \text{ and } \bigcap_{x \in A} B_x \text{ and}$$

Solution

Problem 37 /350

Let A, B, C be three subsets of $\mathbb{R}^2$. Rewrite

$$\mathbb{R}^2 \setminus ((\mathbb{R}^2 \setminus (A \cup B)) \cap C)$$

in a shorter form.

Solution

Problem 38 /350

Let A be the power set of $\mathbb{R}$; show that there is no bijection between A and $\mathbb{R}$.

Solution