Taylor Series You Should Know

You should have the following series memorized

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{(when } -1 < x < 1\text{)} \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\end{aligned}
$$

Note the similarity between $\sin x$, $\cos x$ and e^x.

How to Find a Taylor Series

Given a smooth function f, we can always write down a Taylor series; there is no guarantee that the series converges to anything, let alone to the function. Given a smooth function $f : \mathbb{R} \to \mathbb{R}$, its Taylor series (around 0) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!}$$

A common mistake is to use $f^{(n)}(x)$ instead of $f^{(n)}(0)$.

Given a smooth function $f : \mathbb{R} \to \mathbb{R}$, its Taylor series expanded around a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

The most important example of this sort of Taylor series is

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

How to *More Easily* Find a Taylor Series

There are tricks: you can add, subtract, multiply, and divide (!) power series. You can substitute one series into another.

Why to Find a Taylor Series

Taylor series are good for:

Estimating values by cutting the series off after a few terms and bounding the remainder;

Seeing a snapshot by looking at the first few terms, you can get a sense of what the function is doing near zero;

Proving facts about functions like $\cos(-x) = \cos x$, which follows from the fact that the Taylor series for cosine only includes even degree terms;

Doing calculus because it is easy to differentiate and integrate term-by-term;

Solving differential equations for which it is often easier to find a Taylor series for a solution.