Practice Proof Problems

Below are a list of statements that you can either prove (if the claim is true) or disprove (by exhibiting a counterexample). These are just for fun—statements to ponder if you are looking for more things to think about.

Incidentally, there is a pattern (dare I say the beginning of a sequence!) to which statements are false, and which statements are true. And by my count, there are eighteen false statements in need of counterexamples, and seventeen true statements in need of a proof.

The Statements

Prove or Disprove 1. Suppose a_n is a sequence of real numbers, and $\lim_{n\to\infty} n^2 \cdot a_n = 0$. Then $\lim_{n\to\infty} n \cdot a_n = 0$.

Prove or Disprove 2. If a_n and b_n are sequences of real numbers, and $\lim_{n\to\infty} a_n = 0$, and b_n is bounded, then $\lim_{n\to\infty} (a_n \cdot b_n) = 0$.

Prove or Disprove 3. If a_n and b_n are bounded sequences of real numbers, then $c_n = a_n \cdot b_n$ is a bounded sequence.

Prove or Disprove 4. If a_n is an nondecreasing sequence of real numbers, then a_n has a subsequence which is increasing.

Prove or Disprove 5. If a_n is a decreasing sequence of real numbers, then a_n is bounded below.

Prove or Disprove 6. If a_n is an increasing sequence of real numbers, then $b_n = {a_n}^3$ is also increasing.

Prove or Disprove 7. There exists a sequence a_n of positive numbers, with $\lim_{n\to\infty} a_n < 0$.

Prove or Disprove 8. If a_n is a decreasing sequence of real numbers, then a_n is bounded above.

Prove or Disprove 9. There exists a sequence a_n of rational numbers, with $\lim_{n\to\infty} a_n = \sqrt{2}$.

Prove or Disprove 10. If a_n is a sequence of real numbers, and $\lim_{n\to\infty} \cos a_n = 1$, then $\lim_{n\to\infty} a_n = 0$.

Prove or Disprove 11. There exists a sequence a_n of irrational numbers, with $\lim_{n\to\infty} a_n = 3$.

Prove or Disprove 12. If a_n is a sequence of integers, and $\lim_{n\to\infty} a_n = 6$, then for all but finitely many values of $n \in \mathbb{N}$, we have $a_n = 6$.

Prove or Disprove 13. If a_n is an unbounded sequence, then

$$b_n = \frac{a_{n+1}}{\sqrt{n+1}}$$

is bounded.

Prove or Disprove 14. If a_n is an unbounded sequence and $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$b_n = \frac{a_{n+1}}{a_n}$$

is bounded.

Prove or Disprove 15. If a_n is a bounded sequence, then $\lim_{n\to\infty} a_n/n = 0$.

Prove or Disprove 16. If a_n is a sequence of real numbers, and $\lim_{n\to\infty} {a_n}^2 = 1$, then $\lim_{n\to\infty} a_n = 1$.

Prove or Disprove 17. If a_n is an unbounded sequence, then $b_n = \cos a_n$ is bounded, but does not converge.

Prove or Disprove 18. If a_n is a bounded sequence, and b_n is an unbounded sequence, then $\lim_{n\to\infty} a_n/b_n = 0$.

Prove or Disprove 19. If a_n is a sequence of positive real numbers, and $\lim_{n\to\infty} \log a_n = 0$, then $\lim_{n\to\infty} a_n = 1$.

Prove or Disprove 20. If a_n is an increasing sequence of positive real numbers, then $b_n = a_n/a_{n+1}$ is increasing.

Prove or Disprove 21. If a_n is a sequence of real numbers, and $\lim_{n\to\infty} {a_n}^2 = 0$, then $\lim_{n\to\infty} a_n = 0$.

Prove or Disprove 22. Let a_n be a sequence of non-zero numbers, and define $b_n = |a_n|/a_n$. If a_n converges, then b_n converges.

Prove or Disprove 23. If a_n and b_n are increasing sequences of positive real numbers, then

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even,} \\ b_n & \text{if } n \text{ is odd.} \end{cases}$$

is also increasing.

Prove or Disprove 24. If b_n is the n-th digit of π, and $a_n = (b_n/10)^n$, then $\lim_{n\to\infty} a_n = 0$.

Prove or Disprove 25. If a_n is a sequence of real numbers, and $b_n = a_n/n$ is bounded, then b_n converges.

Prove or Disprove 26. If a_n is an increasing sequence of positive real numbers, then $b_n = a_{2n}$ is increasing.

Prove or Disprove 27. If $\lim_{n\to\infty} a_n = L$, and there exists A and B so that for all $n \in \mathbb{N}$, we have $A \le a_n \le B$, then $A \le L \le B$.

Prove or Disprove 28. If a_n is an increasing sequence of positive real numbers, then $b_n = a_{n+1} - a_n$ is increasing.

Prove or Disprove 29. There exists a bounded sequence a_n so that $b_n = e^{a_n}$ converges to 0.

Prove or Disprove 30. Suppose $\lim_{n\to\infty} a_n = L$, and $\lim_{n\to\infty} b_n = L$. Define

$$c_n = \begin{cases} a_n & \text{if } n \text{ is even,} \\ b_n & \text{if } n \text{ is odd.} \end{cases}$$

Then $\lim_{n\to\infty} c_n = L$.

Prove or Disprove 31. Define $a_n = n^n$. Then $b_n = a_n/n!$ converges.

Prove or Disprove 32. There is a $K \in \mathbb{N}$ so that for every $\epsilon > 0$, if $n \ge K$, then

$$\left| \frac{1}{n} - 0 \right| < \epsilon.$$

Prove or Disprove 33. Let a_n be the biggest number that can be written down using n letters from the "alphabet" containing

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, \times.$$

Then $a_n > n$.

Prove or Disprove 34. There exists a sequence a_n of integers, with $\lim_{n\to\infty} a_n = \pi$.

Prove or Disprove 35. Let a_n be a sequence of real numbers, and define

$$b_n = \begin{cases} a_n & \text{if } a_n < 1/n, \\ 0 & \text{if } a_n \ge 1/n. \end{cases}$$

Then b_n converges.