Little o notation

Math 205

1 Overview

There is no *new* mathematics in this handout, but there is a slightly different perspective. We think of the linear approximation (not the value of the derivative) as the fundamental object.

2 Introduction

Something that can make differentiation more beautiful is *little o notation*.

Definition 2.0.1. We say "f is little-oh of h as x approaches x_0" and write

$$f(x) = o(h(x)) \text{ as } x \to x_0$$

to mean that

$$\lim_{x \to x_0} \frac{f(x)}{h(x)} = 0.$$

Intuitively, this means $f(x)$ is *much* smaller than $h(x)$ for x near x_0.

Warning. If $f(x) = o(h(x))$, it is not the case that $h(x) = o(f(x))$.

Example 2.0.1. For instance,

- $x^2 = o(x)$ as $x \to 0$.
- $x \neq o(x^2)$ as $x \to 0$.
- $x - \sin x = o(x)$ as $x \to 0$.
- $x - \sin x = o(x^2)$ as $x \to 0$.

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- $x - \sin x = o(x^2)$ as $x \to 0$.
- $x - \sin x \neq o(x^3)$ as $x \to 0$.

Definition 2.0.2. For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) \sim_{x_0} g(x)$ provided

$$f(x) - g(x) = o(x - x_0) \text{ as } x \to x_0.$$

Unwrapping the definitions, this just means that

$$\lim_{x \to x_0} \frac{f(x) - g(x)}{x - x_0} = 0$$

Intuitively, $f(x) \sim_{x_0} g(x)$ means f and g look "linearly" the same around x_0. More formally (or when trying to intimidate others), we might say that the "linear germ of f and g around x_0 is the same."

Exercise 2.0.1. Prove that $\sim_{x_0}$ is an equivalence relation.

Definition 2.0.3. A **linear approximation** for $f(x)$ at x_0 is a linear function

$$L(x) = ax + b$$

so that $f(x) \sim_{x_0} L(x)$.

Exercise 2.0.2. If $ax + b \sim_{x_0} cx + d$, then $a = c$ and $b = d$.

Exercise 2.0.3. Let $f : (a, b) \to \mathbb{R}$. If f has a linear approximation at c, then it is unique.

Exercise 2.0.4. Let $f : (a, b) \to \mathbb{R}$. Then f is differentiable at c if and only if f has a linear approximation at c.

Exercise 2.0.5. Let f_i be functions, and L_i be linear functions, so that $f_1 \sim_{x_0} L_1$ and $f_2 \sim_{x_0} L_2$.

Then,

- $f_1 + f_2 \sim_{x_0} L_1 + L_2$.
- $f_1 \cdot f_2 \sim_{x_0} L_1 \cdot L_2$.

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Remark 1. The latter "explains" the product rule:

$$\begin{aligned} f_1(x) &\sim_{x_0} {f_1}'(x_0)\,(x-x_0)+f_1(x_0) \\ f_2(x) &\sim_{x_0} {f_2}'(x_0)\,(x-x_0)+f_2(x_0), \end{aligned}$$

and therefore,

$$\begin{aligned} f_1(x)\,f_2(x) &\sim_{x_0} \left({f_1}'(x_0)\ (x-x_0)+f_1(x_0)\right)\left({f_2}'(x_0)\ (x-x_0)+f_2(x_0)\right) \\ f_1(x)\,f_2(x) &\sim_{x_0} \left({f_1}'(x_0)f_2(x_0)+f_1(x_0){f_2}'(x_0)\right)\ (x-x_0)+f_1(x_0)f_2(x_0). \end{aligned}$$

You might have noticed that the ${f_1}'(x_0){f_2}'(x_0)(x-x_0)^2$ term is missing—why are we justified in ignoring it?

3 Higher derivatives

Definition 3.0.4. For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) \sim_{x_0}^{n} g(x)$ provided

$$f(x) - g(x) = o\left((x-x_0)^n\right) \text{ as } x \to x_0.$$

Intuitively, this means that f and g are the "same" through degree n. Note that, as before, this is an equivalence relation.

Definition 3.0.5. A **quadratic approximation** for $f(x)$ at x_0 is a quadratic function

$$L(x) = ax^2 + bx + c$$

so that $f(x) \sim_{x_0}^{2} L(x)$.

Remark 2. The quadratic approximation packages together both the first and second derivative of f at x_0.

Exercise 3.0.6. Let L_1 and L_2 be quadratic functions such that $L_1 \sim_{x_0}^{2} L_2$. Prove $L_1 = L_2$.

Exercise 3.0.7. Let $f : (a,b) \to \mathbb{R}$. If f has a quadratic approximation at c, then it is unique.

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