The reasonableness of the ridiculous.

If $\sum_{n=0}^{\infty} x^n$ converges to L, then consider the following.

$$x \cdot L = x \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n = L - 1.$$

So we can solve for L, to find that $L = \dfrac{1}{1-x}$. Of course, this argument **assumes that the series converges**.

What if we did this when the series did not converge? We might deduce that

$$\sum_{n=0}^{\infty} 10^n = \frac{1}{1-10} = -\frac{1}{9}.$$

Of course, this is incorrect, because the series $\sum_{n=0}^{\infty} 10^n$ diverges. But even if this is ridiculous, there is a way in which it makes sense.

The 10-adic numbers

We are comfortable with numbers that "go all the way to the right" (i.e., non-terminating decimals like $0.3333\cdots$) so why not numbers that go all the way to the **left**?

I mean, consider a "number" like $\cdots 999999$, meaning $\sum_{n=0}^{\infty} 9 \cdot 10^n$. Of course, this is meaningless, but if we **ignore convergence issues** and apply the formula for geometric series, we might be fooled into thinking $\cdots 999999 = -1$. After all, $\cdots 999999$ means $\sum_{n=0}^{\infty} 9 \cdot 10^n$.

This is less ridiculous than it seems, because

$$\begin{array}{lr} & \cdots 99999999 \\ + & 1 \\ \hline & \cdots 00000000 \end{array}$$

This also looks ridiculous, but just apply the usual algorithm for addition: add 9 and 1, get 10, write down the 0 and carry the 1—and repeat. The answer is all zeroes. A number which equal zero when we add 1 to it ought to be given a name: -1. For similar reasons, we might believe $\cdots 11111111 = -1/9$, because if we multiply $\cdots 11111111$ by 9, we get the number for -1.

We can show $-1 \times -1 = 1$, because

$$\begin{array}{lr} & \cdots 99999999 \\ \times & \cdots 99999999 \\ \hline & \cdots 99999991 \\ & \cdots 99999910 \\ & \cdots 99999100 \\ & \cdots 99991000 \\ & \vdots \\ \hline & \cdots 00000001 \end{array}$$

How about one third?

There are other examples in this crazy world, too. Because

$$\begin{array}{lr} & \cdots 66666667 \\ \times & 3 \\ \hline & \cdots 00000001 \end{array}$$

we decide that $\cdots 66666667$ deserves to be called $1/3$, since it is a multiplicative inverse for 3. But there is another reason why $\cdots 66666667$ deserves the name $1/3$. After all, if $\cdots 11111111 = -1/9$, then $\cdots 66666666$ is $-6/9 = -2/3$. And therefore,

$$\begin{array}{lrl} & \cdots 66666666 & \text{(think "}-2/3\text{")} \\ \times & 1 & \\ \hline & \cdots 66666667 & \text{(think "}1/3\text{")} \end{array}$$

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How about other negative numbers?

What happens if we multiply -1 by 17. We ought to get -17. And indeed, we do:

$$
\begin{array}{lr}
 & \cdots 99999999 \\
\times & 17 \\
\hline
 & \cdots 99999993 \\
+ & \cdots 99999990 \\
\hline
 & \cdots 99999983
\end{array}
$$

And of course, if we add 17 to $\cdots 99999983$, we get zero. This trick of handling negative numbers is called two's complement addition[1]

How about one seventh?

I wanted to write down $1/7$, so I started with a 3 (since $3 \times 7 = 21$, and this will give me the 1 on the right hand side). The next digit should be a 4, because $4 \times 7 = 28$, and since I had to carry that 2, I will get 30, which means I will write down a zero. Now I am carrying a 3; but if I put a 1 as the next digit, then $1 \times 7 + 3 = 10$, so I will write down a zero, and carry a 1. Each time the next digit is chosen so that I write down a zero, and carry something. Continuing in this way, I discover:

$$
\begin{array}{lr}
 & \cdots 285714285714285714285714 3 \\
\times & 7 \\
\hline
 & \cdots 0000000000000000000000001
\end{array}
$$

This is a repeating decimal: we might write it as $\overline{285714}3$, though here the digits repeat to the left. This means we could also write it as a series, formally:

$$3 + 10 \cdot \left(285714 \cdot \sum_{n=0}^{\infty} 1000000^n \right)$$

If we ignore convergence, and apply the formula for geometric series here, where it does not apply, we might be fooled into thinking

$$\cdots 2857142857142857143 = 3 + 10 \cdot \left(285714 \cdot \frac{1}{1 - 1000000} \right) = 1/7$$

The trouble is that these series do not converge. But if we changed our notion of convergence... That is a subject for a future course.

The point of all this?

Many mathematical advances started by taking a ridiculous idea (e.g., negative numbers, imaginary numbers) and making sense of the absurd. The numbers we have described here are called 10-adic numbers, and, I admit, they do not work very well (terrifyingly, there exist two non-zero 10-adic numbers which multiply to give zero); if we instead worked in base p for p a prime, we would get the p-adic numbers, and these numbers turn out to work much better.[2]

[1]For more, see `http://en.wikipedia.org/wiki/Twos_complement`

[2]For more, see `http://en.wikipedia.org/wiki/P-adic_number`

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