Problem Set 2 — Math 765

Winter 2011 — Jim Fowler

You have long understood a single vector space; now you will study vector spaces parametrized by a manifold—*a bundle.* Please feel comfortable emailing me (`fowler@math.osu.edu`) if you are having trouble with homework problems. *The exercises below should be handed in on Monday, January 17, 2011—but that's a Holiday, so let's make the deadline Wednesday, January 19, 2011, instead.*

Problem 2.1 (Lee 4–3)

Let M be a nonempty manifold of dimension $n \geq 1$. Show that $\operatorname{Vect}(M)$ is infinite dimensional [as an $\mathbb{R}$-vector space].

Problem 2.2 (Lee 4–9)

Let $F : M \to N$ be a local diffeomorphism. Show that for every $Y \in \operatorname{Vect}(N)$, there is a unique smooth vector field on M that is F-related to Y.

Problem 2.3

Show that $TS^1 \cong S^1 \times \mathbb{R}$ and $TS^3 \cong S^3 \times \mathbb{R}^3$.

Problem 2.4

Describe a nontrivial vector bundle over S^1.

Problem 2.5

Show that $TS^2 \ncong S^2 \times \mathbb{R}^2$.

Problem 2.6 (Parallel parking is possible)

Consider $M = S^1 \times \mathbb{R}^2$; a point $(\theta, x) \in M$ corresponds to a position of the automobile $x \in \mathbb{R}^2$, and the angle θ of its wheels.

Describe two vector fields on M: "drive in direction of wheels" and "turn steering wheel." What, in words, is the Lie bracket of these two vector fields?

Problem 2.7 (Lee 5–5)

Let $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M. Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M such that both E and $\tilde{E}$ admit smooth local trivializations over U_α.

Let $\{\tau_{\alpha\beta}\}$ and $\{\tilde{\tau}_{\alpha\beta}\}$ denote the transition funcctions determined by the given local trivializations of E and $\tilde{E}$, respectively.

Show that E and $\tilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)^{-1}\tau_{\alpha\beta}(p)\sigma_\beta(p), \quad p \in U_\alpha \cap U_\beta.$$

Problem 2.8 (Lee 6–9)

Suppose $F : M \to N$ is a smooth map, ω is a smooth covector field, and γ a piecewise smooth curve segment in M. Show that

$$\int_\gamma F^\star\omega = \int_{F\circ\gamma} \omega.$$

Problem 2.9 (Lee 6–13)

(a) If M is a compact manifold, show that every exact covector field on M vanishes at least at two points.

(b) Is there a smooth covector field on S^2 that vanishes at exactly one point?