Problem Set 4 — Math 765

Winter 2011 — Jim Fowler

Submanifolds. This week, we study submanifolds, embedded or immersed. I have heard many complaints about the difficulty of the problem sets, so I hope you find this problem set more concrete and tractable—let me know at `fowler@math.osu.edu`. *The exercises below should be handed in on Monday, January 31, 2011.*

Problem 4.1

Let $\mathrm{Mat}_{n,n}(\mathbb{R})$ denote $n \times n$ matrices with real entries. Show that $\mathrm{SO}(n)$ is a smooth submanifold of $\mathrm{Mat}_{n,n}(\mathbb{R})$ in the following way. Show that the map $f : \mathrm{Mat}_{n,n}(\mathbb{R}) \to \mathrm{SymMat}_{n,n}(\mathbb{R})$ given by $f(A) = A^\intercal A$ has the identity matrix as a regular value. The matrix $A^\intercal$ is the transpose of A, and $\mathrm{SymMat}_{n,n}(\mathbb{R})$ are symmetric $n \times n$ matrices.

Problem 4.2 (Gimbal lock is possible)

For a nonzero vector $v \in \mathbb{R}^3$, let $R(v, \theta)$ denote counterclockwise rotation around the axis v. Define a map $f : S^1 \times S^1 \times S^1 \to \mathrm{SO}(3)$ by sending $(\theta_1, \theta_2, \theta_3) \in (S^1)^3$ to the rotation

$$R(\mathbf{x}, \theta_1) \circ R(\mathbf{y}, \theta_2) \circ R(\mathbf{x}, \theta_3) \in \mathrm{SO}(3).$$

Here, $\mathbf{x}$ and $\mathbf{y}$ denote the unit vectors $(1, 0, 0)$ and $(0, 1, 0)$, respectively. *Show that f is not a submersion.* When you are controlling a system, you may have only infinitesimal control over the inputs—and if the function is not a submersion, you then do not have complete (albeit, infinitesimal!) control over the outputs. This problem was experienced by the astronauts of Apollo 11.

For bonus points, avoid gimbal lock by adding a fourth gimbal (i.e., describe a map $(S^1)^4 \to \mathrm{SO}(3)$ which sends the four angles to a product of rotations, and show that your map is a submersion).

Problem 4.3 (Lee 10–5)

Suppose M is a smooth, compact manifold that admits a nowhere vanishing vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points. *[Hint: use the tubular neighborhood theorem.]*

Problem 4.4 (Lee 10–3)

If $M \subset \mathbb{R}^m$ is an embedded submanifold and $\epsilon > 0$, let M_ϵ be the set of points in $\mathbb{R}^m$ whose distance from M is less than ϵ. If M is compact, show that for sufficiently small ϵ, the topological boundary ∂M_ϵ is a compact embedded submanifold of $\mathbb{R}^m$, and M_ϵ is a smooth manifold with boundary.

Problem 4.5 (Square roots)

Let $\mathrm{Mat}_{2,2}(\mathbb{R})$ be the four-dimensional vector space of 2×2 matrices; let $f : \mathrm{Mat}_{2,2}(\mathbb{R}) \to \mathrm{Mat}_{2,2}(\mathbb{R})$ be the function $f(A) = A \cdot A$.

(a) Compute $Df^{-1}(\mathrm{Id})$.

(b) Use part (a) to approximate

$$\sqrt{\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.1 \end{bmatrix}}$$

(c) When is Df invertible? This will tell us when perturbations of matrices having square roots also have square roots.

Problem 4.6 (Characteristic polynomials)

Let $\mathrm{Mat}_{n,n}(\mathbb{R})$ be the n^2-dimensional vector space of $n\times n$ matrices; let P_n be the $(n+1)$-dimensional vector space of degree n polynomials in a variable λ. The characteristic polynomial of a matrix A is $\det(A - \lambda I)$; let $f : \mathrm{Mat}_{n,n}(\mathbb{R}) \to P_n$ be the function sending a matrix to its characteristic polynomial. Is it the case for some polynomials $p \in P_n$ that $f^{-1}(p)$ is a submanifold of $\mathrm{Mat}_{n,n}(\mathbb{R})$?

Problem 4.7 (Polynomials with specified roots)

Let P_n be the vector space of degree n polynomials in a variable λ. Define a function $f : \mathbb{R}^n \to P_n$ given by

$$f(x_1, \ldots, x_n) = (\lambda - x_1)\cdots(\lambda - x_n)$$

and define a function $g : P_n - \{0\} \to \mathbb{R}P^n$ which sends a nonzero polynomial $a_n\lambda^n + \cdots + a_0$ to the point $[a_n : \cdots : a_0] \in \mathbb{R}P^n$.

Consider $g \circ f : \mathbb{R}^n \to \mathbb{R}P^n$. Show that, for $(x_1, \ldots, x_n)$ with x_i pairwise distinct, there is an open neighborhood $U \ni (x_i)$ on which $g \circ f$ is a diffeomorphism.