Lecture 24: Integrals with tricks

Math 153 Section 57

Friday November 21, 2008

Following chapter 8.3.
There are many other tricks not in the book; we will cover some of these.
Also, I will try to do examples that you might actually care about.

Products of sines and cosines

From your homework you know how to do

$$\int \sin(2x)\,\cos(3x)\,dx$$

by using parts.

You can also do

$$\int \sin^2 x\,dx$$

by using a trig substitution.

You can also do

$$\int \sin^n x\,\cos^m x\,dx$$

by making a substitution $\cos^2 x + \sin^2 x = 1$.

Repeated integration

$$\int uv = uv_1 - u'v_2 + u''v_3 - \cdots + (-1)^n\, u^{(n)}\, v_{n+1}$$

Integrals of products of sines and cosines

Do

$$\int \sin(ax)\,\cos(bx)\,dx$$

by using an angle sum formula? No! Use parts twice.

Powers of trig functions? Replace $\sin^2 x$ with $1-\cos^2 x$ as follows

$$\int \cos^3 x\,\sin^5 x\,dx = \int \cos x(1-\sin^2 x)\,\sin^5 x\,dx = \int (1-u^2)\,u^5\,du =$$

Trig identities

$1-\sin^2 x=\cos^2 x$.

$1+\tan^2 x=\sec^2 x$.

$\sec^2 x-a=\tan^2 x$.

So if you see $\sqrt{a^2-x^2}$, set $x=a\sin u$.

For example,

$$\int_0^r \sqrt{r^2-x^2}\,dx$$

which we know is $\pi r^2/4$. But let's check it.

Set $x=r\sin u$ and $dx=r\cos u\,du$. Then the integral above is

$$\int_0^{\pi/2} \sqrt{r^2-r^2\sin^2 u}\,r\ \cos u\,du$$

But this becomes

$$\int_0^{\pi/2} r^2\sqrt{1-\sin^2 u}\ \cos u\,du$$

which becomes

$$\int_0^{\pi/2} r^2\cos^2 u\,du$$

which we can do using $\cos^2 u=(1+\cos(2u))/2$, so get

$$\int_0^{\pi/2} r^2\frac{1+\cos(2u)}{2}\,du=\frac{r^2}{2}\int_0^{\pi/2}(1+\cos(2u))\,du$$

which is $\pi r^2/4$, as we suspected.

Volume of a sphere

Even easier to compute the volume of a sphere this way! A hemisphere has volume

$$\int_0^r \pi(\sqrt{r^2-x^2})^2\,dx=\frac{2}{3}\pi r^3$$

So the whole sphere has voluem $4\pi r^3/3$.

Volume of a hypersphere

A hemi-hypersphere has volume

$$\int_0^r \frac{4}{3}\pi(\sqrt{r^2-x^2})^3\,dx$$

If we make a trig substitution $x = r\sin u$ and $dx = r\cos u$ we get

$$V = \frac{4r^4\pi}{3}\int_0^{\pi/2} \cos^4 u\, du$$

This is an integral we can do!

Replace $\cos^2 u$ with $(1+\cos(2u))/2$, and get

$$V = \frac{4r^4\pi}{3}\int_0^{\pi/2} \left(\frac{1+\cos(2u)}{2}\right)^2 du$$

or equivalently

$$V = \frac{4r^4\pi}{12}\int_0^{\pi/2} \left(1 + 2\cos(2u) + \cos^2(2u)\right) du$$

Now do it again!

$$V = \frac{4r^4\pi}{12}\int_0^{\pi/2} \left(1 + 2\cos(2u) + \frac{1+\cos(4u)}{2}\right) du$$

I'll clean it up a bit

$$V = \frac{4r^4\pi}{24}\int_0^{\pi/2} \left(3 + 4\cos(2u) + \cos(4u)\right) du$$

But the integrals of $\cos(2u)$ and $\cos(4u)$ vanish (do not do more work than you need to do!). We end up with,

$$V = \frac{4r^4\pi}{24}(3\pi/2) = \frac{r^4\pi^2}{4}$$

This is a volume for a hemisphere, so the whole sphere has volume

$$V = \frac{2\,r^4\pi^2}{4} = \frac{r^4\pi^2}{2}$$

Yeah, this seems a bit *ad hoc* but in a multivariable course, you'll see a more conceptual route to these calculations.

Is this right?

B^2 sits inside a box built out of two B^1's; so $\pi < 2\cdot 2$. In the same way, B^4 sits inside a box built out of two B^2's, so $\pi^2/2 < \pi^2$. Not exactly a big surprise.

Using power series

$$\int_0^\infty \log(1+e^{-x})\, dx$$

This is

$$\int_0^\infty \sum_{n=1}^\infty \frac{(e^{-x})^n(-1)^{n+1}}{n}\, dx$$

which we can integrate term by term (why?!)

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{(e^{-nx})(-1)^{n+1}}{n}\, dx$$

which gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

which is $\pi^2/12$. Why is that?

Let $A = \sum_{\text{even}} 1/n^2$ and $B = \sum_{\text{odd}} 1/n^2$. We want to compute $B-A$. But $B+A = 4A$, so $B = 3A$. And $B + A = \pi^2/6 = 4A$, so $A = \pi^2/24$ and $B = \pi^2/8$. Then $B - A = \pi^2/12$.