Lecture 34: Argument principle

Math 660—Jim Fowler

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Example 1

$$f(z) = \frac{1}{z^4 + 4}$$

Compute $\int_\gamma f\, dz$ for γ a circle of radius two and center 1.

Counting zeroes

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$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

Now $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$, so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n(\gamma, z_1) + \cdots + n(\gamma, z_n)$$

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In fact, this is true even if there are infintely many zeroes in D^2.

Counting zeroes, again

Theorem ( )

If $f(z)$ is meromorphic in Ω, with zeroes a_j and poles b_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

for every γ homologous to zero in Ω.

Counting zeroes, again

Theorem (Argument principle)

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Corollary (Rouché's theorem)

Suppose $\gamma \in \Omega$ *with* $[\gamma] = [0] \in H_1(\Omega)$, *and* $n(\gamma, a) = 0$ *or* 1 *for* $a \notin \omega$.
Suppose $f(z)$ *and* $g(z)$ *are analytic in* Ω, *and satisfy* $|f(z) - g(z)| < |f(z)|$ *on* γ.
Then $f(z)$ *and* $g(z)$ *have the same number of zeroes inside* γ.

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If we can prove $R^n|f_n(z)| < |p(z)|$, then we need only determine solutions of $p(z) = 0$.

Example

How many roots does $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have inside the disk $|z| < 1$?

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Therefore,

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j)g(a_j) - \sum_j n(\gamma, b_k)g(b_k)$$

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so $z_j(w)$ are roots of a polynomial with coefficients depending analytically on w.