Lecture 14: Root and ratio tests

Math 153 Section 57

Wednesday October 29, 2008

Following chapter 12.4.

Ratio Test

This is a test I do like, and it is easy to apply!

If $a_n > 0$, and $\lim a_{n+1}/a_n = L$, then if $L < 1$, $\sum a_n$ converges, if $L > 1$, then $\sum a_n$ diverges, and if $L = 1$, the test is inconclusive.

Proof: If $L < 1$, choose ϵ so small that $L + \epsilon < 1$. Then there exists K so that for $n \geq K$, we have $|a_{n+1}/a_n - L| < \epsilon$. In other words, $a_{n+1}/a_n < L + \epsilon < 1$.

So $a_{n+1} < (L + \epsilon) \cdot a_n$. Therefore, $a_{n+k} < (L + \epsilon)^n a_k$. But $\sum (L + \epsilon)^n a_k$ converges, and therefore, by comparison, so does $\sum_{n=k}^{\infty} a_n$.

0.1 Best use of the ratio test

We know that $\sum 1/n!$ converges, because for n large

$$\frac{1}{n!} < \frac{1}{n \cdot (n-1)} < \frac{1}{(n-1)^2}$$

But we can also check this using the ratio test.

Set $a_n = 1/n!$. Then $a_{n+1}/a_n = 1/(n+1)$ which converges to zero, and we are done.

What do we know about $\sum 1/n!$? It equals e. We'll see that soon.

Anti-example: $\sum n^n/n!$. But we already knew that since $\lim n^n/n! = \infty$.

Example: $\sum n^{n/2}/n!$. Apply ratio test:

$$\frac{\sqrt{(n+1)^{n+1}}}{(n+1)!} \cdot \frac{n!}{\sqrt{n^n}} = \frac{\sqrt{n+1} \cdot \sqrt{(\frac{n+1}{n})^n}}{n+1} = \frac{\sqrt{(\frac{n+1}{n})^n}}{\sqrt{n+1}}$$

which converges.

0.2 Bad uses of the ratio test

example: $\sum 1/(3n + 17)$.

0.3 Sometuimes we have a parameter

Which test to apply?

$\lim a_n = 0$ is always a good thing to check!

Integral test is good for things you can integrate.

Root test is really only good for things with a powers of n.

Limit comparison takes care of all rational functions (i.e., polynomial over polynomial).

Ratio test is good for factorials.

Absolute and conditional convergence

Sometimes series have both positive and negative terms.

Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof: If $\sum |a_n|$, then $\sum 2|a_n|$ conv, then $\sum a_n + |a_n|$ conv, so $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$ conv.

The proof doesn't work backwards!

Definition: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ conv.

Absolute convergence implies convergence, but it is not the same.

Example: alternating series.