Problem Set 7 — Math 765

Winter 2011 — Jim Fowler

Curves and flows. This week, we adopt a dynamic perspective: our vector fields give rise to flows on a manifold. Email me with questions at `fowler@math.osu.edu`. *The exercises below should be handed in on Monday, February 21, 2011.*

Problem 7.1 (Lee 17–4)

Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M. More precisely, for any two points $p, q \in M$, show that there is a diffeomorphism $F : M \to M$ such that $F(p) = q$.

Problem 7.2 (Lee 17–13 (Collar neighborhoods exist))

Let M be a smooth manifold with boundary. A subset $C \subset M$ containing ∂M is called a *collar* if C is the image of a smooth embedding $[0, 1] \times \partial M \to M$ that restricts to the obvious identification $\{0\} \times \partial M \to \partial M$. This problem shows that every smooth manifold with boundary has a collar.

(a) Show that here exists a vector field $N \in \mathcal{T}(M)$ whose restriction to ∂M is inward-pointing.

(b) For any positive real-valued function $\delta : \partial M \to \mathbb{R}$, define a subset $\mathcal{D}_\delta \subset [0, \infty) \times \partial M$ by

$$\mathcal{D}_\delta = \{(t, p) : p \in \partial M, 0 \leq t \leq \delta(p)\}.$$

Show that there are a smooth positive function $\delta : \partial M \to \mathbb{R}$ and a smooth map $\theta : \mathcal{D}_\delta \to M$ such that for each $p \in \partial M$, the map $t \mapsto \theta(t, p)$ is an integral curve of N starting at p. *Hint:* Use the ODE theorem in local coordinates around each point of ∂M, and define δ by means of a partition of unity.

(c) Show that θ is an embedding.

(d) Show that the image of θ is a collar.

Problem 7.3 (Lee Exercise 18.2)

Prove these statements for smooth vector fields V, W, X and a smooth function f.

(a) $\mathcal{L}_V W = -\mathcal{L}_W V$.

(b) $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.

(c) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.

(d) $\mathcal{L}_V (fW) = (Vf)W + f\mathcal{L}_V W$.

(e) If $F : M \to N$ is a diffeomorphism, then $F_\star(\mathcal{L}_V W) = \mathcal{L}_{F_\star V} F_\star W$.

Problem 7.4 (Cartan's formula)

For a smooth vector field X and a smooth form ω, prove that

$$\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega).$$

Problem 7.5 (Traveling along a vector field)

Does there exist a vector field on the plane $\mathbb{R}^2$ so that every two points in the plane are connected by an integral curve?

Problem 7.6 (Baker–Campbell–Hausdorff for the Heisenberg group)

Let H be the Lie group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } a, b, c \in \mathbb{R},$$

and $\mathfrak{h} = T_e H$ be its tangent space at the identity.

A vector $v \in \mathfrak{h}$ gives rise to the left-invariant vector field X_v with $X_v(e) = v$; the function $\theta_{X_v} : H \times \mathbb{R} \to H$ is the flow along this vector field, and the exponential map $\exp : \mathfrak{h} \to H$ is defined by $\exp(v) = \theta_v(e, 1)$.

Given $a, b \in \mathfrak{h}$, find a formula for a vector $c \in \mathfrak{h}$ in terms of a and b so that

$$\exp c = (\exp a) \cdot (\exp b).$$

Your formula will permit you to do calculations in the Lie group H by doing calculations in the Lie algebra $\mathfrak{h}$.