Lecture 24: Removable singularities

Math 660—Jim Fowler

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Removable singularities

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can be extended to $\Omega \to \mathbb{C}$
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iff $\lim_{z \to a} f(z)\,(z - a) = 0$.
Why? Cauchy's formula is valid.

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and get $f(z) = f(a) + (z - a)F(z)$.

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Rinse, repeat.

Consequently. . .

If $f : \Omega \to \mathbb{C}$ is analytic,
for any $a \in \Omega$, we can write

$$f(z) = f(a) + \sum_{n=1}^{k} \frac{f^{(n)}(a)}{n!}(z-a)^n + F(z)(z-a)^{k+1}$$

for some analytic $F : \Omega \to \mathbb{C}$.

Say more about F

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for the analytic $F : \Omega \to \mathbb{C}$.

$$F(z) = \frac{1}{2\pi i \eta(z,\gamma)} \int_\gamma \frac{f(w)}{(w-a)^n(w-z)}\, dw.$$

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$f(z)$ vanishes on Ω.

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If $f^{(n)}(a) = 0$ for $n \in \{0, \ldots, k-1\}$,
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It's like a polynomial!

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So if $f(z) = g(z)$ for $z \in S$,
(what sort of set is S?)
then $f \equiv g$ on Ω.

Classifying singularities

Δ a disk, with center a.
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Consider $1/f(z)$,
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now called the order of the pole.

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Meromorphic means holomorphic except for poles.

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Quotients of analytic functions

Poles of meromorphic functions are isolated

Think of the relationship between $\mathbb{Z}$ and $\mathbb{Q}$.

$$Z = \{\alpha \in \mathbb{R} : \lim_{z \to a} |(z-a)^\alpha f(z)| = 0\}$$
$$P = \{\alpha \in \mathbb{R} : \lim_{z \to a} |(z-a)^\alpha f(z)| = \infty\}$$

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If $\alpha \in Z$ and $\beta > \alpha$, then $\beta \in Z$.
If $\alpha \in P$ and $\beta < \alpha$, then $\beta \in P$.

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- $Z = \mathbb{R}$ and $P = \varnothing$,
- $Z = (n, \infty)$ and $P = (-\infty, n)$,
- $Z = \varnothing$ and $P = \varnothing$.

Partial fractions redux

If $f(z)$ has a pole of order k, consider

$$(z-a)^k f(z) = a_0 + a_1(z-a) + \cdots + a_{k-1}(z-a)^{k-1} + F(z)(z-$$

and divide by $(z-a)^k$ to get the **singular part**.

$Z = \varnothing$ and $P = \varnothing$

These are **essential singularities.**

Theorem (Weierstrass)

$f : \Omega - \{a\} \to \mathbb{C}$ analytic
with essential singularity at a.
For every $w \in \mathbb{C}$ and $\epsilon > 0$,
there is $z \in B_\epsilon(a) \cap \Omega$
so that $f(z) = w$.

Singularities at infinity

To determine the behavior of $f(z)$ at infinity, consider instead $f(1/z)$.

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What sort of singularity does $\sin z$ have near infinity?