Lecture 39: Schwarz' theorem

Math 660—Jim Fowler

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Poisson integral

Given a continuous function $U : [0, 2\pi] \to \mathbb{R}$, consider

$$P_*(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{re} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(\theta)\, d\theta.$$

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- $P_{kU} = kP_U$
- $P_c = c$
- $P_U \geq 0$ if $U \geq 0$.

Schwarz' theorem

The function $P_U(z)$ is harmonic in the interior of the disk $B_1(0)$,
and continuous on the closed disk (provided U is continuous).

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Consider $f(z) - \overline{f(\overline{z})}$.

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Suppose $v(x)$ is continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+, and zero on σ.

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In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\overline{z})}$.