Problem Set 3 — Math 765

Winter 2011 — Jim Fowler

This week, we study the smooth maps between manifolds. Please feel comfortable emailing me (`fowler@math.osu.edu`) if you are having trouble with homework problems. *The exercises below should be handed in on Monday, January 24, 2011.* These exercises are designed to get you to think about **transversality,** among the most powerful tools in the study of smooth manifolds.

Problem 3.1 (Lee 8–2)

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + xy + y^3$$

Which level sets of F are embedded submanifolds of $\mathbb{R}^2$?

Problem 3.2 (Lee 8–14 and Lee 8–15)

(8–14) If $S \subset M$ is an embedded submanifold and $\gamma : J \to M$ is a smooth curve whose image happens to lie in S, show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$.

(8–15) Give a counterexample if S is immersed but not embedded.

Problem 3.3 (Lee 8–16)

Suppose $f : M \to N$ is a smooth map and $S \subset N$ is an embedded submanifold. We say that f is transverse to S if, for every $p \in f^{-1}(S)$, the spaces $T_{f(p)}S$ and $f_\star T_p M$ together span $T_{f(p)}N$.

If f is transverse to S, show that $f^{-1}(S)$ is an embedded submanifold of M whose codimension[1] is equal to $\dim N - \dim S$.

[1]The codimension of a submanifold $N \subset M$ is $\dim M - \dim N$.

Problem 3.4 (Lee 8–17)

Let M be a smooth manifold. Two embedded submanifolds $S_1, S_2 \subset M$ are said to be *transverse* if, for each $p \in S_1 \cap S_2$, the tangent spaces T_pS_1 and T_pS_2 together span T_pM.

- If S_1 and S_2 are transverse, show that $S_1 \cap S_2$ is an embedded submanifold of M of dimension $\dim S_1 + \dim S_2 - \dim M$. [It will be easier to remember this if you think of it as saying $\operatorname{codim}(S_1 \cap S_2) = \operatorname{codim} S_1 + \operatorname{codim} S_2$.]

- Give a counterexample when S_1 and S_2 are not transverse.

Hint: You can invoke the previous problem, and an inclusion map, to make short work of this problem.

☕☕ Problem 3.5

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial so that its zero set $f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^2$.

Recall that $\mathbb{R}P^1$ parametrizes lines through the origin in $\mathbb{R}^2$; show that for all but finitely many points of $\mathbb{R}P^1$, the corresponding line through the origin intersects $f^{-1}(0)$ transversely.

Is this still true if f is not a polynomial, but merely smooth?