Lecture 27: Zeroes and poles

Math 660—Jim Fowler

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Theorem

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If $f(z) = g(z)$ for $z \in A \subset \Omega$,
and A has an accumulation point in Ω,
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Analytic functions are determined by their values on sets with an accumulation point.

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Poles are isolated, because zeroes are.

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Think of a meromorphic function as a holomorphic function from Ω to S^2, the Riemann sphere.

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So for all $\alpha > m-k$, the limit vanishes,
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Then $(z-a)^m f(z)$ has a removable singularity,
and $(z-a)^m f(z)$ has a pole of order *ell*.

So for all $\alpha < m+\ell$, the limit equals infinity,
and if $\alpha > m+\ell$, the limit vanishes.

Summarizing

So either $f(z) \equiv 0$,
or there is some h so that
if $\alpha > h$, then $\lim_{z\to a} |z-a|^\alpha |f(z)| = 0$
if $\alpha < h$, then $\lim_{z\to a} |z-a|^\alpha |f(z)| = \infty$
or maybe neither limit holds for any α.

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Apply Taylor's theorem to $(z-a)^h f(z)$, to find

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + \cdots + B_1 (z-a)^{h-1} + \varphi(z)(z-a)^h$$

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Divide both sides by $(z-a)^h$, to get

$$f(z) = \frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \cdots + \varphi(z)$$

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A pole comes with a well-defined singular part, and if you subtract off the singular part, you are left with an analytic function.

Essential singularity

If there is no such α satisfying those limits, then we say it is an essential singularity.

Theorem

An analytic function comes arbitrarily close to every complex value in a neighborhood of an essential singularity.