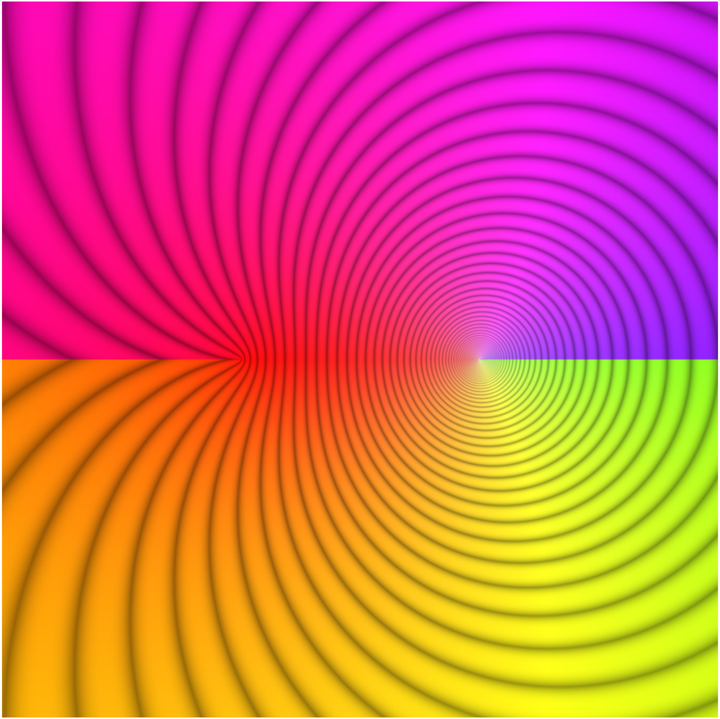
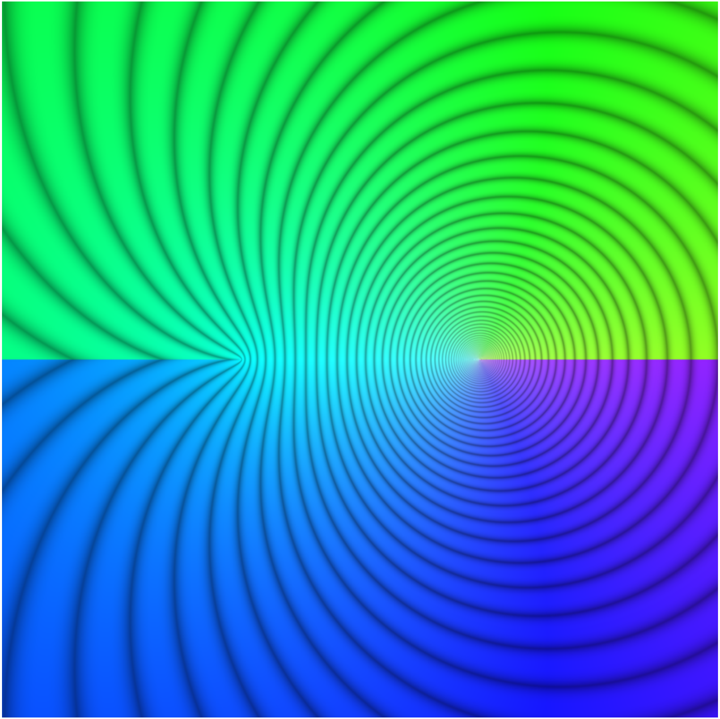
Lecture 23: Higher derivatives

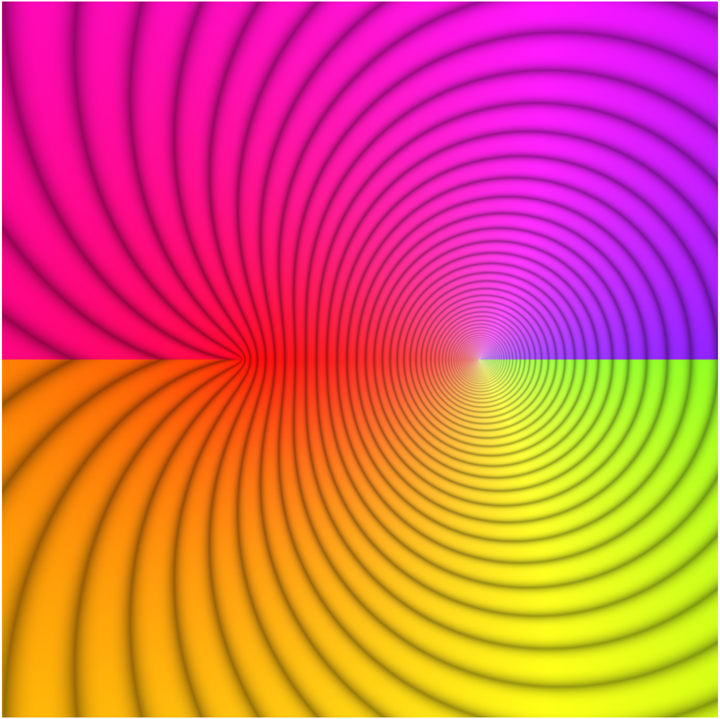
Math 660—Jim Fowler

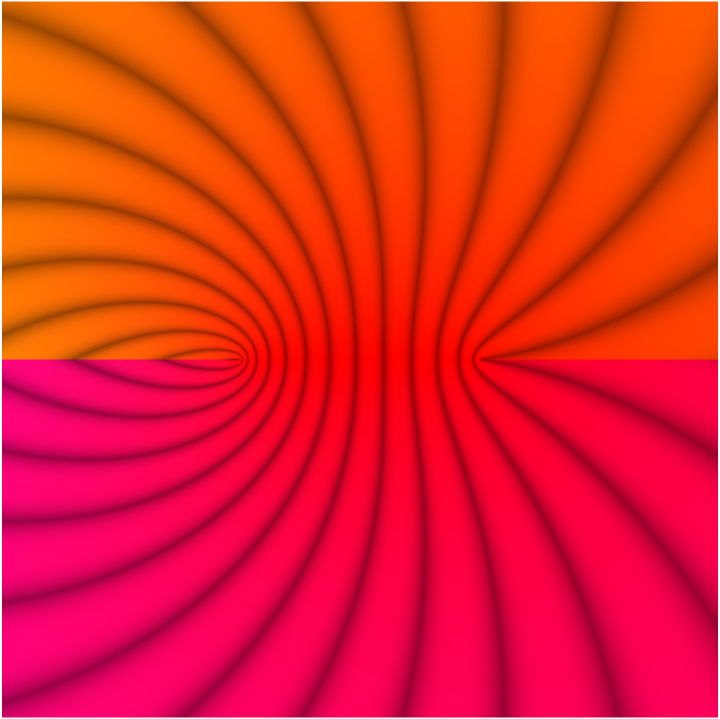
Thursday, July 21, 2011

Midterm tomorrow!









Partial fractions

Partial fractions

Ahlfors' technique:

subtract off the poles

$$f(z) = \frac{1}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

$$f(z) = \frac{1}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

Consequently,

analytic functions are determined by their values on a circle.

$$f(z) = \frac{1}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

Consequently,
analytic functions are determined
by their values on a circle.
By a fractional linear transformation,
analytic functions are determined
by their value on the real line.

$$f^{(n)}(z) = \frac{n!}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Theorem (Liouville)

If $f : \mathbb{C} \to \mathbb{C}$ analytic and bounded, then $f \equiv$ constant.

Theorem (Morera's theorem)

If $f : \Omega \to \mathbb{C}$ *is continuous,*
and $\int_\gamma f(z)\, dz = 0$ *for all closed* γ*,*
then $f(z)$ *is analytic.*

Theorem (Morera's theorem)

If $f : \Omega \to \mathbb{C}$ *is continuous,*
and $\int_\gamma f(z)\, dz = 0$ *for all closed* γ*,*
then $f(z)$ *is analytic.*

Proof.

So there exists $F : \Omega \to \mathbb{C}$ with $F' = f$.

Theorem (Morera's theorem)

If $f : \Omega \to \mathbb{C}$ is continuous,
and $\int_\gamma f(z)\,dz = 0$ for all closed γ,
then $f(z)$ is analytic.

Proof.

So there exists $F : \Omega \to \mathbb{C}$ with $F' = f$.
But then F is twice differentiable, so f is analytic.

Theorem (Morera's theorem)

If $f : \Omega \to \mathbb{C}$ is continuous,
and $\int_\gamma f(z)\,dz = 0$ for all closed γ,
then $f(z)$ is analytic.

Proof.

So there exists $F : \Omega \to \mathbb{C}$ with $F' = f$.
But then F is twice differentiable, so f is analytic.

$\square$

Cauchy's estimate

$$f^{(n)}(z) = \frac{n!}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Cauchy's estimate

$$f^{(n)}(z) = \frac{n!}{2\pi i \eta(\gamma, z)} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, dw.$$

Considering a circle of radius R around z,
if $|f(z)|$ is bounded by C on the circle,

$$|f^{(n)}(z)| \leq C \cdot n! \cdot R^{-n}.$$

Consequences

Consequences

If $|f(Re^{i\theta})| \sim R^n$,
what can you say about the derivatives of f?

Removable singularities

$f : \Omega - \{a\} \to \mathbb{C}$ analytic
can be extended to $\Omega \to \mathbb{C}$
iff $\lim_{z \to a} f(z)(z-a) = 0$.

Removable singularities

$f : \Omega - \{a\} \to \mathbb{C}$ analytic
 can be extended to $\Omega \to \mathbb{C}$
 iff $\lim_{z \to a} f(z)\,(z - a) = 0$.
Why? Cauchy's formula is valid.

Apply this trick to

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

and get $f(z) = f(a) + (z - a)F(z)$.

Apply this trick to

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

and get $f(z) = f(a) + (z - a)F(z)$.
Rinse, repeat.

Consequently...

If $f : \Omega \to \mathbb{C}$ is analytic,
for any $a \in \Omega$, we can write

$$f(z) = f(a) + \sum_{n=1}^{k} \frac{f^{(n)}(a)}{n!}(z-a)^n + F(z)(z-a)^{k+1}$$

for some analytic $F : \Omega \to \mathbb{C}$.