Lecture 32: Cauchy's theorem

Math 660—Jim Fowler

Wednesday, August 4, 2010

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and reversing the order of integration,

$$\int_{C} f(z) dz = \int_{C} \left(\frac{1}{2\pi i} \int_{C} \frac{dz}{\zeta - z} \right) f(\zeta) d\zeta$$

and the integral vanishes.

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Theorem

If f(z) is analytic in Ω , then

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This is a theorem that probably belongs more properly to a topology course.

Homology basis

Suppose $\mathbb{C} - \Omega$ has components A_0, \ldots, A_n with $\infty \in A_0$.

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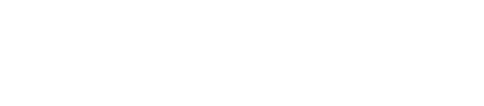
for cycles γ_i which "go around" A_i . The collection of γ_i are a **homology basis** for Ω .

Periods

If
$$[\gamma] = [c_1\gamma_1 + \cdots c_n\gamma_n]$$
, then

$$\int_{\gamma} f \, dz = \sum_{j} c_{j} \int_{\gamma_{i}} f \, dz$$

so every integral is a sum of $\int_{\gamma_i} f \, dz$ (the "periods") with integral coefficients.



Compute some periods

