Lecture 29: The maximum principle

Math 660—Jim Fowler

Friday, July 30, 2010

Corollary

A nonconstant analytic function is an open map (meaning it maps open sets to open sets).

The maximum principle

Theorem

If $f(z)$ is analytic and non-constant in a region Ω, then $|f(z)|$ does not attain a maximum in Ω.

The maximum principle

Theorem

If $f(z)$ is analytic and non-constant in a region Ω, then $|f(z)|$ does not attain a maximum in Ω.

Proof.

A nonconstant analytic function is an open map. $\square$

The maximum principle

Theorem

If $f(z)$ is analytic and non-constant in a region Ω, then $|f(z)|$ does not attain a maximum in Ω.

Proof.

A nonconstant analytic function is an open map. $\square$

Note that the maximum principle holds generally for open maps.

The maximum principle, version two

Theorem

If $f(z)$ is continuous on the closed, bounded set $\overline{\Omega}$, and analytic in the region Ω, then the maximum of $|f(z)|$ is attained somewhere on $\partial\Omega$.

Analytic proof

Analytic proof

Apply Cauchy's theorem, get

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

Analytic proof

Apply Cauchy's theorem, get

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

so

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta$$

Analytic proof

Apply Cauchy's theorem, get

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

so

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|\, d\theta$$

and the theorem follows.

Lemma of Schwarz

Theorem

Suppose $f(z)$ is analytic in the unit disk, and $|f(z)| \leq 1$, and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Lemma of Schwarz

Theorem

Suppose $f(z)$ *is analytic in the unit disk, and* $|f(z)| \leq 1$*, and* $f(0) = 0$*. Then* $|f(z)| \leq |z|$ *and* $|f'(0)| \leq 1$*.*
In fact, if $|f(z)| = |z|$ *for some* z *or if* $|f'(0)| = 1$*, then* $f(z) = \lambda z$ *for* $|\lambda| = 1$*.*

Lemma of Schwarz

Theorem

Suppose $f(z)$ is analytic in the unit disk, and $|f(z)| \leq 1$, and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
In fact, if $|f(z)| = |z|$ for some z or if $|f'(0)| = 1$, then $f(z) = \lambda z$ for $|\lambda| = 1$.

Apply maximum principle to $F(z) = f(z)/z$.

Lemma of Schwarz

Theorem

Suppose $f(z)$ is analytic in the unit disk, and $|f(z)| \leq 1$, and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
In fact, if $|f(z)| = |z|$ for some z or if $|f'(0)| = 1$, then $f(z) = \lambda z$ for $|\lambda| = 1$.

Apply maximum principle to $F(z) = f(z)/z$.
On the circle $|z| = r$, then $|F(z)| \leq 1/r$, so $|F(z)| \leq 1$ for all $|z| \leq 1$.

Lemma of Schwarz

Theorem

Suppose $f(z)$ is analytic in the unit disk, and $|f(z)| \leq 1$, and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
In fact, if $|f(z)| = |z|$ for some z or if $|f'(0)| = 1$, then $f(z) = \lambda z$ for $|\lambda| = 1$.

Apply maximum principle to $F(z) = f(z)/z$.
On the circle $|z| = r$, then $|F(z)| \leq 1/r$, so $|F(z)| \leq 1$ for all $|z| \leq 1$.
If $F(z)$ attains its maximum, $F(z)$ is a constant, so $f(z) = F(0)z$.