Frequently Asked Questions about the Final Exam

When is the final exam?

The exam is on **Tuesday, December 9, 2008**. The exam starts at **4:00pm**, and ends at **6:00pm**, providing for **120 minutes** of examination.

Where is the exam?

The exam will be in the usual lecture location, namely **Kent 101**.

What are some questions that will be on the final exam?

These questions will absolutely appear on the exam.

- ☐ What is the definition of the limit of a sequence, $\lim_{n \to \infty} a_n = L$?
- ☐ What is the definition of the limit of a series, $\sum_{n=1}^{\infty} a_n = L$?
- ☐ Use an ϵ-K argument to prove that the sequence [redacted] converges.
- ☐ What letter grade do you believe you have earned this quarter?

There will, as usual, be extra credit questions at the end.

What do these boxes mean?

I have drawn boxes next to everything you should know, so you can easily check things off when you believe you know it.

How ought I to write down my answers?

You must not merely write down the answer; you must give the whole story, showing all the steps you took to arrive at your answer. You must **justify the arguments** you make in order to receive full credit.

What definitions must I know?

You may be asked to give definitions of the following terms:

- ☐ bounded below,
- ☐ bounded above,
- ☐ increasing,
- ☐ decreasing,
- ☐ non-increasing,
- ☐ non-decreasing,
- ☐ conditional convergence,
- ☐ absolute convergence,
- ☐ Taylor's theorem,
- ☐ Lagrange's theorem.

What convergence tests must I know?

You must be able to both **apply** and **state precise descriptions of** the following tests:

- ☐ the n^{th} term test,
- ☐ comparison test,
- ☐ limit comparison test,
- ☐ p-series test,
- ☐ geometric series test,
- ☐ harmonic series test,
- ☐ the ratio test,
- ☐ alternating series test,
- ☐ the root test,
- ☐ the integral test.

Use the **limit comparison test** to determine whether $\sum \frac{\text{polynomial}}{\text{polynomial}}$ converges.

What must I determine about *sequences* on the exam?

You must be able to determine whether a sequence is

- ☐ bounded, and
- ☐ monotone.

How will I evaluate *limits* on the exam?

In addition to giving an ϵ-K proof in certain cases, you must be able to evaluate limits by using

- ☐ algebraic manipulation,
- ☐ composition with continuous functions—i.e., $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$,
- ☐ squeezing,
- ☐ l'Hôpital's rule.

In particular, you must be able to evaluate limits having the following indeterminate forms:

☐ $\frac{0}{0}$, ☐ $\frac{\infty}{\infty}$, ☐ $0 \cdot \infty$, ☐ 0^0, ☐ 1^∞, ☐ ∞^0, ☐ $\infty - \infty$

What must I do with *series* on the exam?

Given a series you must be able to do the following:

- ☐ Determine whether the series converges absolutely,
- ☐ Determine whether the series converges conditionally,
- ☐ Evaluate the series in certain cases, like $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ provided $|x| < 1$.

Given a power series, you must be able to:

- ☐ Find the interval on which the power series converges,
- ☐ Differentiate and integrate the power series term-by-term.

With *Taylor series*, what must I be able to do?

Given a function $f : \mathbb{R} \to \mathbb{R}$, you must be able to:

- ☐ Write down the first few terms of the Taylor series for f around 0,
- ☐ Write down the first few terms of the Taylor series for f around $a \neq 0$,
- ☐ Find terms in a Taylor series using tricks like substitution,
- ☐ Find the Taylor series for f—i.e., if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find a formula for a_n,
- ☐ Use the Taylor series to find the values of derivatives of the function.

There may be problems about approximation.

- ☐ Given an alternating series, find an approximate value and estimate the error.
- ☐ Given a Taylor series, find an approximate value and estimate the error.

What sorts of *integration* must I perform?

There will be integrals and differential equations on the final exam. You must:

- ☐ Evaluate improper integrals,
- ☐ Compute $\int \sin(ax)\, \cos(bx)\, dx$ by using integration by parts,
- ☐ Compute $\int \sin^n(x)\, \cos^m(x)\, dx$ for natural numbers n and m,
- ☐ Use partial fractions to compute $\displaystyle\int \frac{\text{polynomial}}{\text{polynomial}}$,
- ☐ Solve inhomogeneous first-order linear differential equations by using an integrating factor,
- ☐ Solve homogeneous second-order linear differential equations by factoring the derivative.

Frequently Asked Questions on the Final Exam

Question 1. Give an ϵ-K argument to prove that $\lim_{n\to\infty} \frac{4}{n} = 0$.

Question 2. Give an ϵ-K argument to prove that $\lim_{n\to\infty} \frac{4}{n^2} = 0$.

Question 3. Give an ϵ-K argument to prove that $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$.

Question 4. Give an ϵ-K argument to prove that $\lim_{n\to\infty} \frac{4+2\,n^2}{n^2} = 2$.

Question 5. Give an ϵ-K argument to prove that $\lim_{n\to\infty} \frac{1+2\,n+3\,n^2}{n^2} = 3$.

Question 6. Is the sequence $a_n = n^2$ bounded? Is the sequence monotone?

Question 7. Is the sequence $a_n = n\,\cos n$ bounded? Is the sequence monotone?

Question 8. Is the sequence $a_n = \cos(\pi n)$ bounded? Is the sequence monotone?

Question 9. Is the sequence $a_n = n\cos^2 n + n\sin^2 n$ bounded? Is the sequence monotone?

Question 10. Is the sequence $a_n = \sin(\pi n)$ bounded? Is the sequence monotone?

Question 11. Is the sequence $a_n = \sin(\pi n)$ bounded? Is the sequence monotone?

Question 12. Is the sequence $a_n = \sin\left(\frac{1}{n}\right)$ bounded? Is the sequence monotone?

Question 13. Suppose a_n and b_n are bounded sequences. Is the sequence $c_n = a_n + b_n$ necessarily bounded?

Question 14. Suppose a_n and b_n are bounded sequences. Is the sequence $c_n = a_n \cdot b_n$ necessarily bounded?

Question 15. Suppose a_n and b_n are monotone sequences. Is the sequence $c_n = a_n + b_n$ necessarily monotone?

Question 16. Evaluate $\lim_{n\to\infty} \frac{n+\sqrt{n}}{1+n^2}$.

Question 17. Evaluate $\lim_{n\to\infty} \frac{n^2+\sin n + n\,\cos n}{(1+n)^3 - n^3}$.

Question 18. Evaluate $\lim_{n\to\infty} \cos\left(\sin\left(1/n^n\right)\right)$.

Question 19. Evaluate $\lim_{n\to\infty} \cos\left(\left(1+\frac{1}{n}\right)^n\right)$.

Question 20. Evaluate $\lim_{n\to\infty} (\pi/4)^{1/n}$.

Question 21. Evaluate $\lim_{n\to\infty} (1/2)^n$.

Question 22. Evaluate $\lim_{n\to\infty} (\sin(1/n))^n$.

Question 23. Evaluate $\lim_{n\to\infty} (\sin(1/n)+1)^n$.

Question 24. Evaluate $\lim_{n\to\infty} \left(\sin^2(1/n)+1\right)^n$.

Question 25. Evaluate $\lim_{n\to\infty} \left(\sin^2(1/n)+2\sin(1/n)+1\right)^n$.

Question 26. Evaluate $\lim_{n\to\infty} \left(1+\frac{\log 123456789}{n}\right)^n$.

Question 27. Search for the value of $\lim_{n\to\infty} \left(1+\frac{50\log 10}{n}\right)^{2n}$ on the Internet.

Question 28. Evaluate $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$.

Question 29. Evaluate $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt[3]{n}}$.

Question 30. Evaluate $\lim_{n\to\infty} \frac{\left(10^{\left(10^{\left(10^{10}\right)}\right)}\right)^n}{n!}$.

Question 31. Evaluate $\lim_{n\to\infty} \frac{\left(100^{\left(100^{\left(100^{100}\right)}\right)}\right)^n}{\sqrt{n!}}$.

Question 32. Evaluate $\lim_{n\to\infty} \frac{\log n}{n}$.

Question 33. Evaluate $\lim_{n\to\infty} (2n)^{3/n}$.

Question 34. For each of the seven indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 0^0,\ 1^\infty,\ \infty^0,\ \infty-\infty$$

find a limit exhibiting the form.

Question 35. Evaluate $\lim_{n\to\infty} \left(\frac{\cos n+\sin n+\sin^2 n}{4}\right)^n$.

Question 36. By Taylor's theorem,

$$\sin\frac{1}{2} = (1/2) - \frac{(1/2)^3}{3!} + R_3(1/2) = \frac{23}{48} + R_3(1/2)$$

I have built a right triangle, having a hypotenuse 48 cm, and height 23 cm. Use Lagrange's theorem to bound $R_3(1/2)$, and thereby justify that this is a good way to build a triangle having an angle of 1/2 radians.

Question 37. Use Taylor's theorem to show that $\log 2 \approx 7/12$ and estimate the error.

Question 38. Recall that (as you should have memorized for the exam)

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, x^n}{n}.$$

Musically, an important fact is that $2^{19} \approx 3^{12}$. Let's see if we can show this using Taylor series. Taking logarithms, we find

$$19 \cdot \log 2 \approx 12 \cdot \log 3$$

But $\log 3 = \log 1.5 + \log 2$, so

$$19 \cdot \log 2 \approx 12 \cdot (\log 1.5 + \log 2)$$

which means that

$$7 \cdot \log 2 \approx 12 \cdot \log 1.5$$

or by the Taylor series above

$$7\,(1 - 1/2 + 1/3 + R_3(1)) \approx 12\left((1/2) - (1/2)^2/2 + (1/2)^2/3 + R_3(1/2)\right).$$

Use a theorem on alternating series to bound $R_3(1)$ and $R_3(1/2)$ to show that this is possible.

Question 39. Does the series $\sum_{n=1}^{\infty} n^4/4^n$ converge?

Question 40. Does the series $\sum_{n=1}^{\infty} n!$ converge?

Question 41. Does the series $\sum_{n=1}^{\infty} (n+1)/(n)$ converge?

Question 42. Does the series $\sum_{n=1}^{\infty} (n!+1)/(n!)$ converge?

Question 43. Does the series $\sum_{n=1}^{\infty} (1/3)^n$ converge?

Question 44. Does the series $\sum_{n=1}^{\infty} \left((1/3)^n + (1/4)^n\right)$ converge?

Question 45. Does the series $\sum_{n=1}^{\infty} \left((1/3)^n + (1/3)^{n-1}(n/4) + (1/4)^n\right)$ converge? For a very different method, try comparison with $((1/3) + (1/4))^n$.

Question 46. Does the series $\sum_{n=1}^{\infty} (n\,(1/3)^n)$ converge? Did you do it without using the ratio test? Can you estimate its limit?

Question 47. Does the series $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+n+1}$ converge?

Question 48. Does the series $\sum_{n=1}^{\infty} \frac{4\,n^{10}+n+1}{(n^2+1)^6}$ converge?

Question 49. Does the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+10}\right)^n$ converge?

Question 50. Does the series $\sum_{n=1}^{\infty} \frac{n+1}{n!}$ converge?

Question 51. Does the series $\sum_{n=1}^{\infty} \frac{3\,n^6+n^4}{(n!)^2}$ converge?

Question 52. Does the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!}$ converge?

Question 53. Does the series $\sum_{n=1}^{\infty} \frac{(2n)!+13}{(3n)!+12}$ converge?

Question 54. Does the series $\sum_{n=1}^{\infty} \frac{(2n)!-13}{(3n)!+12}$ converge?

Question 55. Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Question 56. Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Question 57. If a_n converges absolutely, does a_n converge?

Question 58. Does the series $\sum_{n=1}^{\infty}(1/n! - 1/n)$ converge absolutely? Converge conditionally?

Question 59. Does the series $\sum_{n=1}^{\infty} 2^{-n}$ converge absolutely? Converge conditionally?

Question 60. Does the series $\sum_{n=1}^{\infty}(-2)^{-n}$ converge absolutely? Converge conditionally?

Question 61. Does the series $\sum_{n=1}^{\infty}(-1)^n/n$ converge absolutely? Converge conditionally?

Question 62. Does the series $\sum_{n=1}^{\infty}(-1)^n \frac{\log n}{n}$ converge absolutely? Converge conditionally?

Question 63. Does the series $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n}$ converge absolutely? Converge conditionally?

Question 64. Does the series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converge absolutely? Converge conditionally?

Question 65. Does the series $\sum_{n=1}^{\infty} (\sin n)/n^2$ converge absolutely? Converge conditionally?

Question 66. What are the first five terms of the Taylor series for $f(x) = \sin x \cos x$ around zero?

Question 67. What are the first four terms of the Taylor series for $f(x) = e^{\sin x}$ around zero?

Question 68. What are the first five terms of the Taylor series for $f(x) = \cos^2 x$ around π?

Question 69. Find the Taylor series for $f(x) = e^{3x}$ around $x = 0$.

Question 70. Find the Taylor series for $f(x) = e^x - e^{-x}$ around $x = 0$.

Question 71. Find the Taylor series for $f(x) = e^x - e^{-x}$ around $x = 0$.

Question 72. Find the Taylor series for $f(x) = (1 + x)^2$ around $x = 0$.

Question 73. Find the Taylor series for $f(x) = (1 + x)^{100}$ around $x = 0$.

Question 74. Find the Taylor series for $f(x) = -x \cos x + \sin x$ around $x = 0$.

Question 75. Find the Taylor series for $f(x) = -x \cos x + \sin x$ around $x = 0$.

Question 76. Find the Taylor series for $f(x) = \cos^2 x \sin^3 x$ around $x = 0$.

Question 77. What are the first five terms of the Taylor series for $f(x) = \sin x \cos x$ around π?

Question 78. Evaluate the integral

$$\int_3^{\infty} \frac{dx}{(1-x)^2}$$

Question 79. Evaluate the integral

$$\int \frac{1}{x^2 - 9x + 20}\, dx$$

Question 80. Evaluate the integral

$$\int \frac{x^3 - 1}{1 + x + x^2}\, dx$$

Question 81. Evaluate the integral

$$\int \frac{x^3 - 2}{1 + x + x^2}\, dx$$

Question 82. Evaluate the integral

$$\int \sin(3x)\, \cos(3x)\, dx.$$

Question 83. Evaluate the integral

$$\int \sin^3 x\, \cos^{10} x\, dx.$$

Question 84. Evaluate the integral

$$\int \sin^5 x\, \cos^5 x\, dx.$$

Question 85. Find a two nonconstant polynomials $p(x)$ and $q(x)$ so that

$$\int_0^1 p(x)\,(1 + x^2)dx = 0 \text{ and } \int_0^1 q(x)\,(1 + x^2)dx = 0$$

and $p(x)$ is not a multiple of $q(x)$.

Question 86. Find the general solution to the differential equation

$$f'(x) + \frac{1}{x} f(x) = \sin x$$

Question 87. Find the general solution to the differential equation

$$f'(x) - \tan x\, f(x) = \sin x$$