A cool trick for limits of sequences

Slogan. *Taking successive differences of the terms in a sequence is vaguely like differentiating a function.*

Taking this slogan seriously, the following might be called a "l'Hôpital's rule for sequences." Here, the differences of successive terms take the place of the derivative:

Theorem (Stolz-Cesàro)**.** *Let a_n and b_n be sequences of real numbers, both unbounded. Assume b_n is positive and increasing. If*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$$

then $\lim_{n\to\infty} a_n/b_n = L$.

A cleaner proof to an old problem

You might remember

$$L = \lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right).$$

This problem, done in the straightforward way, required four applications of l'Hôpital to increasingly ugly denominators. There is a trick—it doesn't require any less differentiation, but the differentiation is much easier. Since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

we can multiply the above by this, without changing the limit L. So

$$L = \lim_{x\to 0} \left(\frac{\sin^2 x}{x^2} \right) \cdot \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right)$$

Gathering these terms together gives

$$L = \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{\sin^2 x}{x^4} \right),$$

which we put over a common denominator

$$L = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^4}.$$

But this is pretty easy to apply l'Hôpital to:

$$L = \lim_{x\to 0} \frac{2x - 2\sin x \cos x}{4x^3} = \lim_{x\to 0} \frac{2x - \sin(2x)}{4x^3}.$$

And again,

$$L = \lim_{x\to 0} \frac{2 - 2\cos(2x)}{12x^2}.$$

And again,

$$L = \lim_{x\to 0} \frac{4\sin(2x)}{24x}.$$

And once more,

$$L = \lim_{x\to 0} \frac{8\cos(2x)}{24} = \frac{8}{24} = \frac{1}{3}.$$

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