Lecture 28: Differential equations

Math 153 Section 57

Monday December 1, 2008

Following chapter 9.

Differential equations are important: we often know something about the derivatives of a function, and we want to recover the function.

Easy to verify a solution; hard to find.

The technique from the homework

You already solved a diff equ on your homework; this was using the technique of "separation of variables" which we will talk about more on Wednesday.

Melting icecube

The change in volume of a melting object is proportional to its surface area. That is,

$$\frac{dV}{dt} = k\,S(t)$$

For a cube of side length r, $V = r^3$ and $S = 6r^2$, so

$$3\,r(t)^2\frac{dr}{dt} = k \cdot 6r(t)^2$$

Divide both sides by $r(t)$ and get

$$3\,\frac{dr}{dt} = 6k$$

so $\frac{dr}{dt} = 2k$.

Homogeneous first-order linear equations

Consider:

$$a(x)\,f'(x) + b(x)\,f(x) = 0$$

Might as well divide everything by $q(x)$, so we are left with

$$f'(x) + h(x)\,f(x) = 0.$$

Note that if f_1 and f_2 are both solutions, then $f_1 + f_2$ is a solution.

Note that if f_1 is a solution, so is $\alpha \cdot f_1$.

Can we solve this equation? Yes. Rewrite it as

$$\frac{f'(x)}{f(x)} = -h(x)$$

We integrate both sides

$$\log f(x) = -\int h(x)\,dx + C$$

But then

$$f(x) = e^{-\int h(x)\,dx + C} = ke^{-H(x)}.$$

This is a solution, and indeed, every solution has this form.

Example: the change in the value of my company is proportional to its current value.

$$f'(x) = k\,f(x)$$

which we can solve. Get e^{kx+C}.

Nonlinear equations

Solve something like

$$f''(x) + f(x)^2 = 0.$$

Note that if f_1 and f_2 are solutions, there is no reason to believe $f_1 + f_2$ would be a solution.

You could solve these by finding the Taylor coefficients!

Homogeneous second-order linear diff equ

Suppose we want to solve

$$f''(x) + b(x)\,f'(x) + c(x)\,f(x) = 0$$

In other words

$$\left(\frac{d^2}{dx^2} + b(x)\,\frac{d}{dx} + c(x)\right) f(x) = 0$$

Suppose we could "factor" the operator into

$$\left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_2\right) f(x) = 0$$

But then solutions include

$$f(x) = e^{\int r_1(x)\,dx} \text{ and } f(x) = e^{\int r_2(x)\,dx}$$

Very concretely:

$$f''(x) - 5f(x) + 6 = 0$$

This can be factored as

$$\left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} - 3\right) f(x) = 0$$

So solutions include e^{2x} and e^{3x}. Check?

More surprisingly, the spring equation:

$$f''(x) = -f(x)$$

can be written as

$$\left(\frac{d^2}{dx^2} + 1\right) f(x) = 0$$

and then written as

$$\left(\frac{d}{dx} - i\right)\left(\frac{d}{dx} + i\right) f(x) = 0$$

where $i^2 = -1$. But then the solutions we might guess would include

$$f(x) = C \cdot e^{\pm ix}$$

This includes sine and cosine! Why?

(Problems appear if there are repeated roots)

And the third degree case? And fourth?

What about the nonhomogeneous case?

What if we have

$$f'(x) + h(x)\, f(x) = g(x)?$$

Can we find a solution?

Note that if f_1 and f_2 are solutions, it is no longer the case that $f_1 + f_2$ is a solution.

Trick: integrating factor. Set $H(x) = \int h(x)\, dx$, and multiply by $e^{H(x)}$.

Then we want to solve

$$e^{H(x)} f'(x) + h(x)\, e^{H(x)}\, f(x) = e^{H(x)}\, g(x).$$

But this is the same as

$$\frac{d}{dx}\left(e^{H(x)} f(x)\right) = e^{H(x)}\, g(x).$$

And so

$$e^{H(x)} f(x) = \int e^{H(x)}\, g(x)\, dx + C$$

General trick: sometimes multiplying by something (the “integrating factor”) lets us solve a previously unsolvable differential equation.

What about nonhomogeneous second order equations? The method of “undetermined coefficients” (i.e., make an educated guess).