Lecture 4: More about limits

Math 153 Section 57

Monday October 6, 2008

Continuing with chapter 11.3.

1 Limits

Repeat the definition: $\lim_{n\to\infty} a_n = L$ means for every $\epsilon > 0$, there exists K such that $n \geq K$ implies $|a_n - L| < \epsilon$.

This **will** be on the first test.

Whether or not a sequence converges will be a big deal to us. Why? The answer to many of life's problems are be constructed as the limit of a sequence.

Give some intuition for what the definition means.

2 Examples

The sequence $a_n = (-1)^n$ diverges.

3 Useful theorems

Limits are unique: if $\lim_{n\to\infty} a_n = L$ and also $\lim_{n\to\infty} a_n = M$, then $L = M$.

Limits only depend on their tails: if $\lim_{n\to\infty} a_n = L$ and $m \in \mathbb{N}$, then $\lim_{n\to\infty} a_{n+m} = L$.

3.1 Sometimes we get convergence for free.

Theorem: Nondecreasing bounded above sequences converge.

Theorem: Nonincreasing bounded below sequences converge.

Proofs are tricky: require the "least upper bound" axiom. We will assume that they hold.

Example?

Let $a_1 = 2$. Let $a_{n+1} = a_n/2 + 1/a_n$. The sequence is decreasing and bounded below, and $\lim a_n = \sqrt{2}$.

Easy examples: $a_1 \in (0, 1)$ and $a_{n+1} = a_n(1 - a_n)$. Very easy dynamics—$\lim a_n = 0$.

Weird examples: $a_1 \in (0, 1)$ and $a_{n+1} = 3.5a_n(1 - a_n)$. Very complicated dynamics!

3.2 Continuity of addition, etc.

If $\lim a_n = L$ and $\lim b_n = M$, then $\lim(a_n + b_n) = L + M$.

If $\lim a_n = L$ $\lim(c \cdot a_n) = c \cdot L$.

If $\lim a_n = L$ and $\lim b_n = M$, then $\lim(a_n \cdot b_n) = L \cdot M$.

Example: rational functions.

Some poetry: chiastic rules: the limit of the sum is the sum of the limits.

3.3 Squeezing theorem

If there exists a K so that $n \geq K$ implies $a_n \leq b_n \leq c_n$, and if $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Proof.

$\lim_{n\to\infty} 1/n^2 = 0$.