Lecture 6: Abel's limit theorem

Math 660—Jim Fowler

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Homework questions

How did it go?

Abel's limit theorem

Suppose $\sum a_n$ converges and let $f(z) = \sum a_n z^n$.
Then $\lim_{z \to 1} f(z) = f(1)$ provided $z \to 1$ so that

$$\frac{|1-z|}{1-|z|}$$

remains bounded.

Wlog $\sum a_n = 0$ by changing a_0.

$$\begin{aligned} s_n(z) &= a_0 + a_1 z + \cdots + a_n z^n \\ &= s_0 + (s_1 - s_0)z + \cdots + (s_n - s_{n-1})z^n \\ &= s_0(1-z) + s_1(z - z^2) + \cdots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\ &= (1-z)(s_0 + \cdots + s_{n-1}z^{n-1}) + s_n z^n \end{aligned}$$

Since $s_n z^n \to 0$, we conclude

$$f(z) = (1-z) \sum s_n z^n$$

Then

$$|f(z)| \le |1-z| \left| \sum_{n=0}^{m-1} s_n z^n \right| + M\epsilon$$

Applications

$$\sum_{k=0}^{\infty}(-1)^n/(n+1) = \log 2$$

$$\sum_{k=0}^{\infty}(-1)^n/(2n+1) = \pi/4$$

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Abel summability

Let $\sum a_n$ be a series—**convergent or not!**
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for $|z| < 1$ than $\sum a_n$.
If $\sum a_n z^n$ converges for $|z| < 1$,
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How does Abel summability relate to usual summability?

Quiz

Tauber's theorem

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There are other such theorems,
known as Tauberian theorems
(and so-called Abelian theorems,
which go the other way).

Divergent series

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Abel's theorem: Abel summability is regular.

Use Abel summation to “sum”

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Amusingly, consider $(1 - 1 + 1 - 1 + \cdots)^2$.

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so $1 + 2 + 3 + 4 + \cdots = -1/12$.

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What about $1 + 2 + 4 + 8 + 16 + \cdots$?

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So $\sum = -1$? Does this make any sense?