Lecture 2: Sequences

Math 153 Section 57

Wednesday October 1, 2008

We will be following chapter 11.2.

Here we introduce sequences (some "nouns"), define some properties (bounded above, bounded below, increasing, decreasing, non-decreasing, non-increasing; the "adjectives"), and prove that some sequences satisfy some of the properties.

Remember: $\mathbb{N}$ is for the natural numbers, and $\mathbb{R}$ is for the real numbers.

1 Sequences

Intuitive definition: a list of numbers that goes on forever.

Precise definition: a function $a : \mathbb{N} \to \mathbb{R}$, but instead of writing $a(n)$, we normally write it with subscripts, like $a_1, a_2, \ldots$.

Some terminology: the numbers in the sequence are "terms," e.g., the first term, second term, third term.

1.1 Building sequences

Define a sequence by giving a formula for the n-th term.

Examples of sequences: $a_n = n$. $b_n = 2n$. $c_n = 2^n$. $d_n = (-1)^n$.

Modify sequences: $f_n = 17 \cdot a_n$.

Combine sequences: $g_n = b_n + c_n$, or $h_n = c_n \cdot d_n$. Then $g_n = n + 2n$, and $h_n = (-2)^n$.

1.2 Why do we care?

Mathematically: A fun object to play with, sort of like numbers (we can add, subtract, multiply, divide).

Practically: sequences come up all the time: a_n might be how much money I have on the n day of trading stocks (decreasing), the number of rabbits in a field after n generations (unbounded, increasing), etc.

1.3 Boundedness

A sequence a_n is **bounded above** if there is a number b such that $b \geq a_n$ for all $n \in \mathbb{N}$.

A sequence a_n is **bounded below** if there is a number b such that $b \leq a_n$ for all $n \in \mathbb{N}$.

A sequence is **bounded** if it is bounded above, and bounded below.

Examples: $a_n = 17$. $a_n = 2^n$. $a_n = -n$. $a_n = (-1)^n$. $a_n = (-1)^n \cdot n$.

1.4 Monotonicity

Increasing? Decreasing? Nonincreasing? Nondecreasing?

Increasing means for all $n \in \mathbb{N}$ that $a_n < a_{n+1}$.

Decreasing means for all $n \in \mathbb{N}$ that $a_n > a_{n+1}$.

Non-increasing means for all $n \in \mathbb{N}$ that $a_n \geq a_{n+1}$.

Non-decreasing means for all $n \in \mathbb{N}$ that $a_n \leq a_{n+1}$.

Examples of increasing sequences: $a_n = 2n$, $a_n = n^2$.

Proofs.

Example of a decreasing sequence: $a_n = 1/n$.

Example of a decreasing sequence: $a_n = 1/n^2$.

Proofs.

Examples of non-increasing sequences: $a_n = 17$.

Examples of non-decreasing sequences:

$$a_n = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases}$$

Another example: $a_n = (-1)^n + n$.

1.5 Some remarks

To prove something is increasing, you must check infinitely many things.

To prove something is not increasing, you need a single counterexample.

"Not increasing is not the same thing as non-increasing."

1.6 Some basic theorems.

The sum of two bounded sequences is bounded.

The sum of two unbounded sequences is unbounded?