# Lecture 3: Limits

Math 153 Section 57

Friday October 3, 2008

We will be following chapter 11.3.

Here, we introduce limits, the **most important idea in calculus**, and that which distinguishes calculus from mere algebra.

## 1 Review of where we've been

Defined "bounded" for sets and sequences.

Defined "monotone" for sequences.

### 1.1 Some loose ends: recursively defined sequences

One person asked: is a sequence just a function? Yes! You've probably already studied functions $\mathbb{R} \to \mathbb{R}$, and a sequence is a function $\mathbb{N} \to \mathbb{R}$.

A sequence (or a function) is not necessarily given by a formula: A weirder example: $p_n$ is the $n$-th digit of $\pi$. $x_n$ is the number of digits in the decimal representation of $n$.

Another way to define sequences: define future terms by using past terms.

Example: $a_1 = 2$. $a_{n+1} = 2 \cdot a_n$. Increasing? This is just $a_n = 2^n$.

Example: $a_1 = 16$. $a_{n+1} = 16 - a_n$. Increasing? This alternates between 0 and 16.

Example: $a_1 = 1, a_2 = 1$. $a_{n+2} = a_{n+1} + a_n$. Increasing?

## 2 Limits

Formal definition: $\lim_{n\to\infty} a_n = L$ means for every $\epsilon > 0$, there exists $K$ such that $n \geq K$ implies $|a_n - L| < \epsilon$.

Intuitive definition: $\lim_{n\to\infty} a_n = L$ means that as close as you want $a_n$ to get to $L$, you can go far enough out in the sequence and stay that close.

If $a_n$ has a limit, then it is a **convergent** sequence, and we say it **converges**. If not, the sequence **diverges** (or **is divergent**).

## 2.1 Challenge-response

Think of the definition of limit as a challenge response game: the challenger gives you an $\epsilon$, and you must produce the $K$.

To prove that something converges to $L$, you must be find a $K$ for every $\epsilon$.

To prove that something doesn't converge at all? You have to show that no matter what $L$ you pick, there is some $\epsilon$ for which you can't find a suitable $K$. That sounds difficult.

## 2.2 Example

Useful fact: for any $b \in \mathbb{R}$, there is an $n \in \mathbb{N}$ with $n > b$.

Useful fact: for any $b \in \mathbb{R}$ with $b > 0$, there is an $n \in \mathbb{N}$ with $1/n < b$.

If $\lim_{n\to\infty}(2n+1)/n = 2$. Why? Set $L = 2$. We need to find $K$ so that for all $n > K$, we get $|2 + 1/n - 2| < \epsilon$.

If $a_n = 0.9999\cdots 9$, i.e., the $n$-term has $n$ nines, then $a_n = 1 - 10^{-n}$, and $\lim_{n\to\infty} a_n = 1$. To be precise? Set $L = 1$. Then need $K$ so that for all $n > K$, we get $|1 - 10^{-n} - 1| < \epsilon$. That is, we need $10^{-n} < \epsilon$, so we take $\log_{10} 10^{-n} < \log_{10} \epsilon$, so we take $n > -\log_{10} \epsilon$.

## 2.3 Theorems

Limits are unique: if $\lim_{n\to\infty} a_n = L$ and also $\lim_{n\to\infty} a_n = M$, then $L = M$.

Limits only depend on their tails: if $\lim_{n\to\infty} a_n = L$ and $m \in \mathbb{N}$, then $\lim_{n\to\infty} a_{n+m} = L$.

Convergent sequences are bounded.

Unbounded sequences are divergent (awesome—a situation where we can show that no matter what $L$ we pick, there is an $\epsilon$ for which we can't find a $K$).

# 3 Survey with closed eyes

One week is over: are we going too fast, too slowly, just right?