Lecture 1: Introduction to complex numbers

Math 660—Jim Fowler

Monday, June 20, 2011

Syllabus

Syllabus — Math 660

Summer 2011 — Jim Fowler

This is a beginning graduate course in complex analysis; as such, we study *analysis* (e.g., the rigorous foundations of calculus) as it applies to *functions of a complex variable*. The resulting theory is strikingly beautiful.

Resources

We present 5 resources to help you to learn complex analysis.

Professor's office hours

If you have questions, want to work through problems, or just talk about mathematics, please attend office hours.

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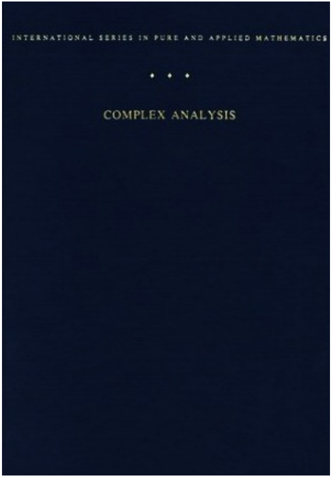
Office:	MW658 Mathematics Tower	Office Hours:	Monday 3:30–5:00P.M.
Phone:	(773) 809–5659		Wednesday 3:30–5:00P.M.
Email:	`fowler@math.osu.edu`		and by appointment
Website:	`http://www.math.osu.edu/~fowler/`		

Please email me with any concerns you have; the success of this course depends on open communication.



Today's Goal

Chapter 1 of *Complex Analysis*

We're going to move fast—some details won't be justified entirely for a while.

$$i = \sqrt{-1}$$

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$$i^2 =$$

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$$a + bi \in \mathbb{C}$$

$$a, b \in \mathbb{R}$$

The Complex Field

$$(a + bi) + (c + di) = (a + c) + (b + d)\, i$$

The Complex Field

$$(a+bi)+(c+di)=(a+c)+(b+d)\,i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)\,i$$

The Complex Field

$$
\begin{aligned}
(a + bi) + (c + di) &= (a + c) + (b + d)\, i \\
(a + bi) - (c + di) &= (a - c) + (b - d)\, i \\
(a + bi) \cdot (c + di) &= (ac - bd) + (ad + bc)\, i
\end{aligned}
$$

The Complex Field

$$
\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)\,i \\
(a+bi)-(c+di) &= (a-c)+(b-d)\,i \\
(a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)\,i \\
\frac{a+bi}{c+di} &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i
\end{aligned}
$$

Dividing Complex Numbers

$$\frac{a + bi}{c + di} =$$

Putting complex numbers in order

If $i > 0$, then $i^2 > 0$, meaning $-1 > 0$.

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Something is wrong...

The complex numbers aren't ordered.

Why write i? Why not $\sqrt{-1}$?

$$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$$

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But what if $a = -1$ and $b = -1 \ldots$?

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$$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$$

But what if $a = -1$ and $b = -1 \ldots$?

$$\sqrt{-1}\sqrt{-1} = -\sqrt{(-1)\cdot(-1)}$$

There are two square roots. This is confusing.
Choose one of them, and call it i.

Complex Conjugation

Definition

Define $\overline{a+bi} = a - bi$.

$\overline{\overline{z}} = z$. $\overline{a+b} = \overline{a} + \overline{b}$. $\overline{ab} = \overline{a} \cdot \overline{b}$.

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$\overline{\overline{z}} = z$. $\quad \overline{a + b} = \overline{a} + \overline{b}$. $\quad \overline{ab} = \overline{a} \cdot \overline{b}$.

Could you tell the difference between a world in which i was replaced by $-i$?

Absolute Value

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$|z|^2 = z \cdot \overline{z}$.

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- $|z + w| \leq |z| + |w|$.
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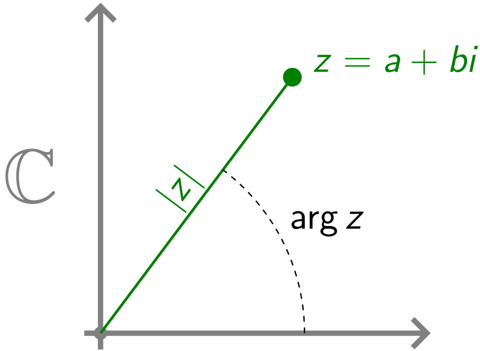
By Pythagoras' theorem, $|a + bi| = \sqrt{a^2 + b^2}$

Argument



Theorem

$\arg(z_1 z_2) \equiv \arg z_1 + \arg z_2 \bmod 2\pi.$

Absolute value and argument

$$z = |z| \cdot \left(\cos \arg z + i \sin \arg z\right).$$

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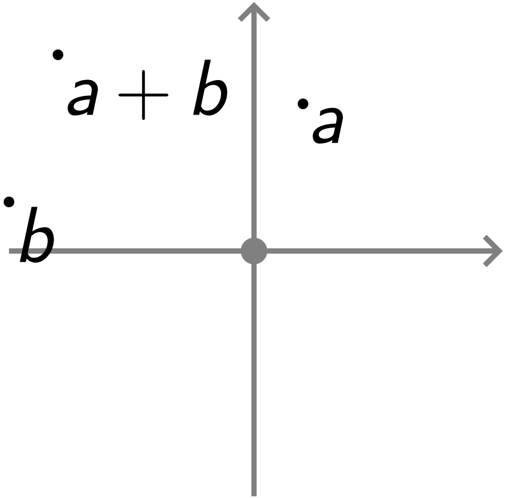
$$z = |z| \cdot (\cos \arg z + i \sin \arg z)\,.$$

$$\begin{aligned} zw = |z| \cdot |w| \cdot ((&\cos \arg z \cos \arg w - \sin \arg z \sin \arg w) \\ &+ (\cos \arg z \sin \arg w + \sin \arg z \cos \arg w)i). \end{aligned}$$

$$zw = |z| \cdot |w| \cdot (\cos(\arg z + \arg w) + i \sin(\arg z + \arg w))$$

Complex numbers as vectors



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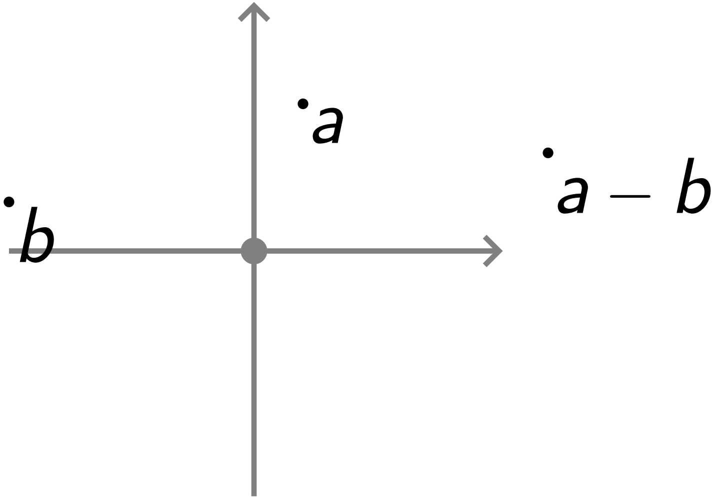

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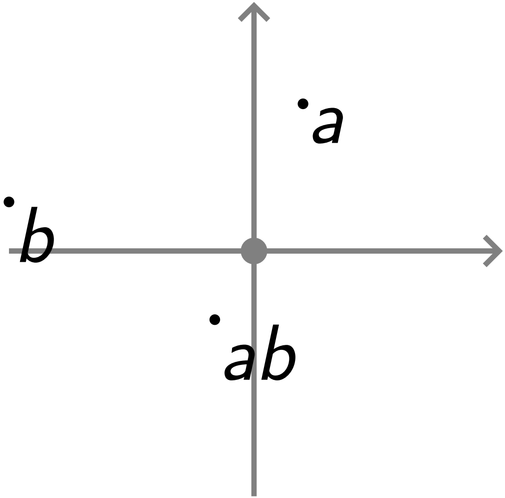

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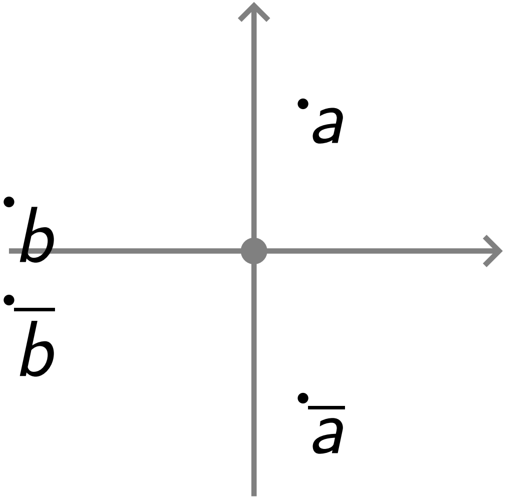

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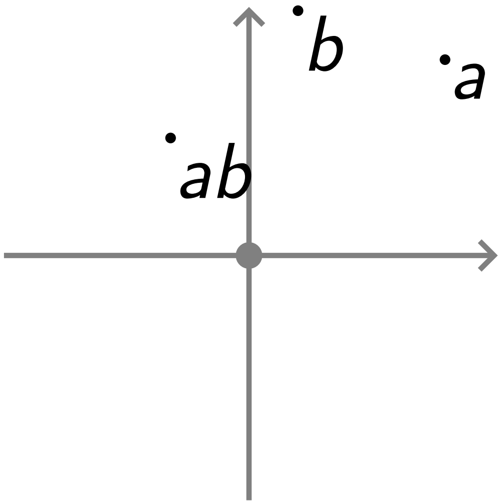

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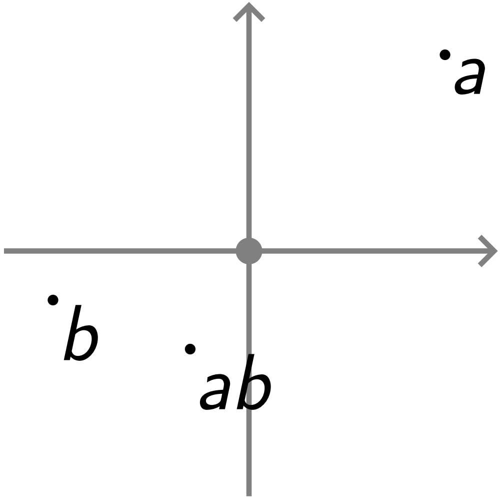

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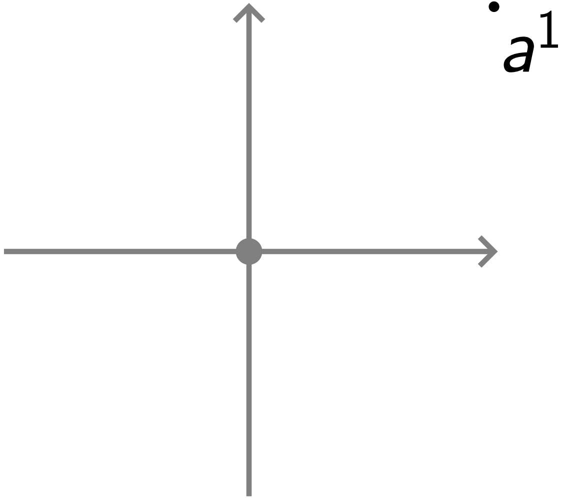

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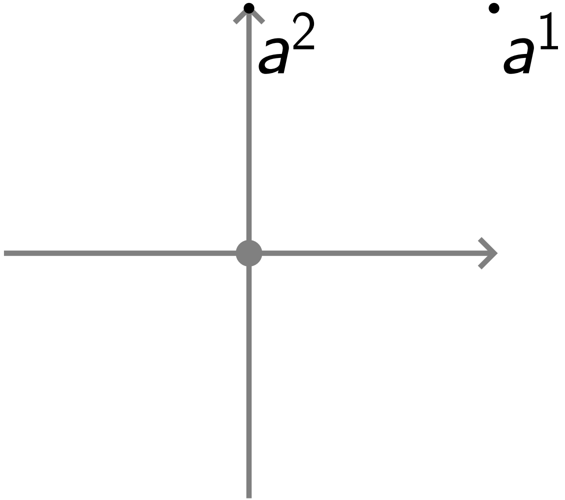

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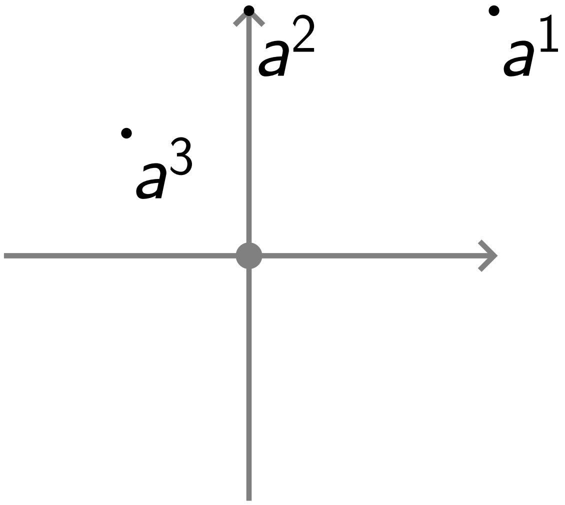

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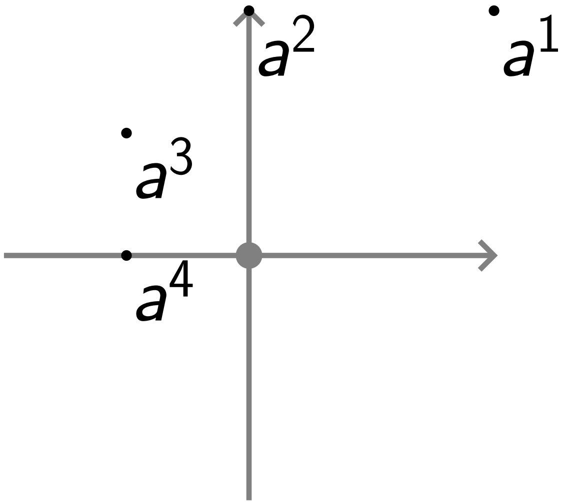

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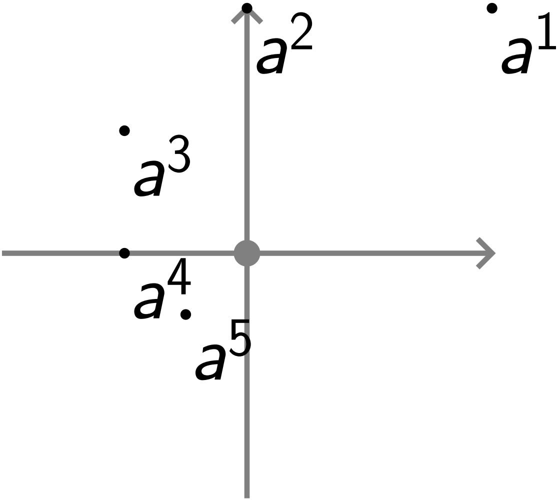

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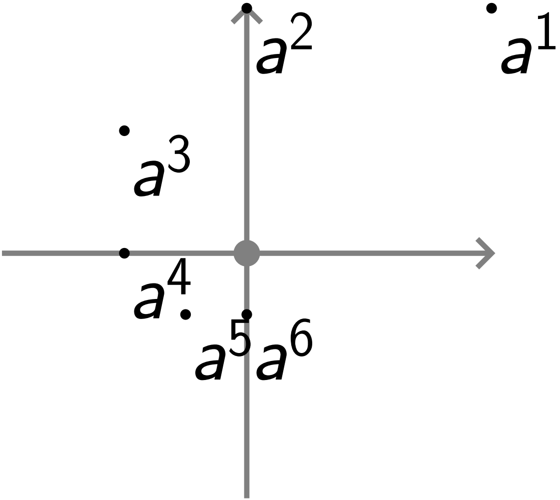

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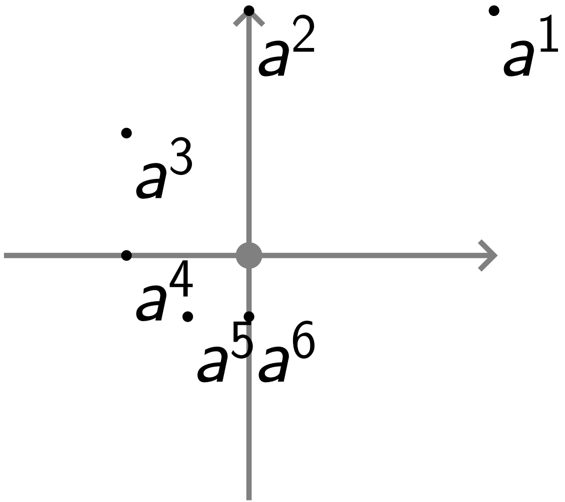

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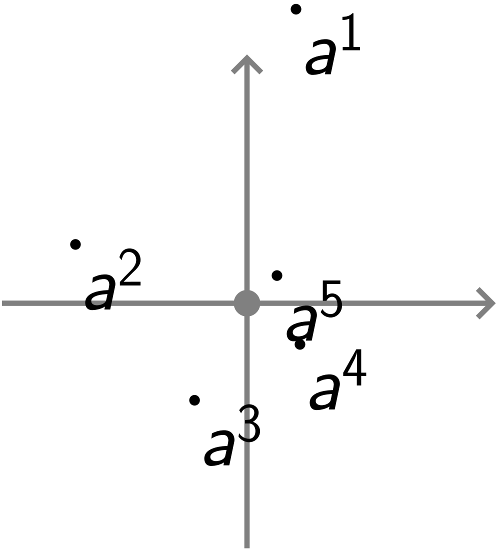

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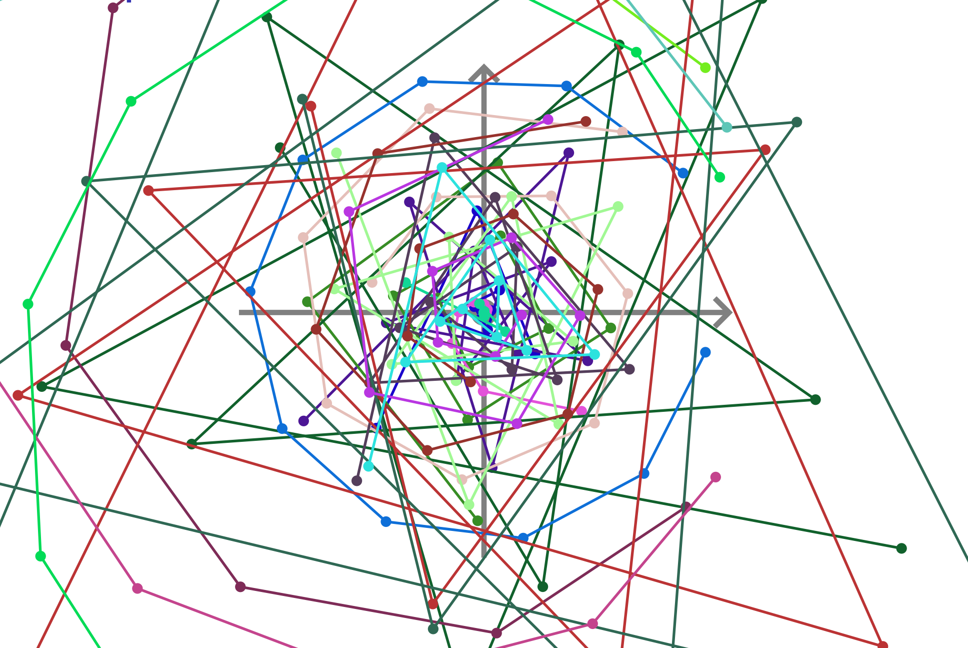

Binomial theorem

$$z = r\left(\cos\theta + i\sin\theta\right).$$

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$$\begin{aligned} z &= r(\cos\theta + i\sin\theta). \\ z^n &= r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \end{aligned}$$

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z &= r\big(\cos\theta + i\sin\theta\big). \\
z^n &= r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \\
\sqrt[n]{z} &= \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)
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\sqrt[n]{z} &= \sqrt[n]{r}\left(\cos\frac{\theta + k\cdot 2\pi}{n} + i\sin\frac{\theta + k\cdot 2\pi}{n}\right),\\
&\text{for } k \in \{0,\ldots,n-1\}.
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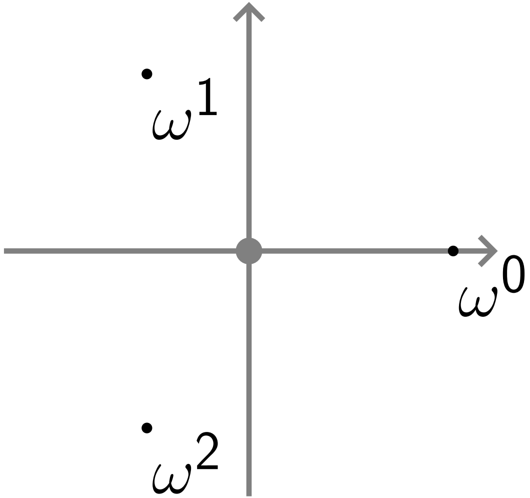
\end{aligned}
$$

So n^{th} roots exist; they form vertices of a regular polygon.

Roots of unity



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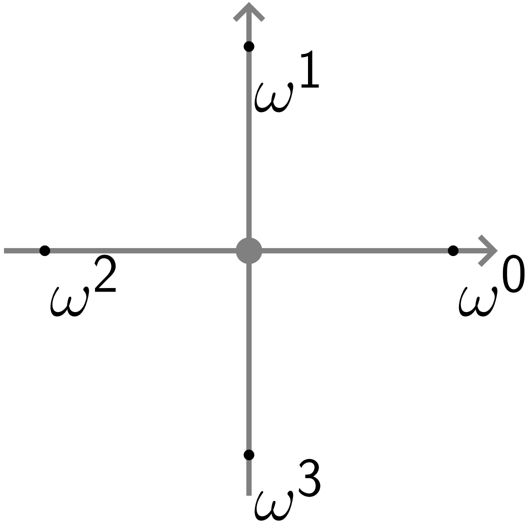

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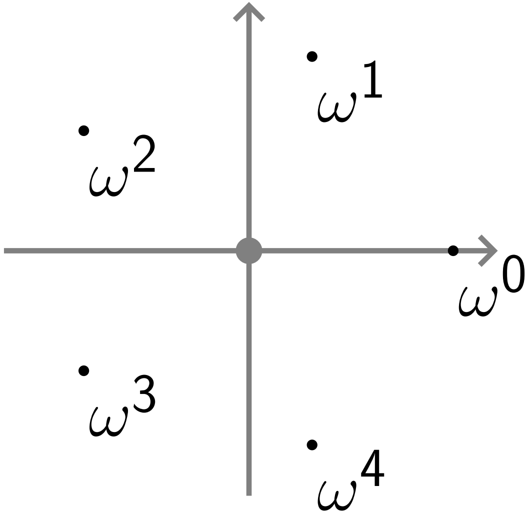

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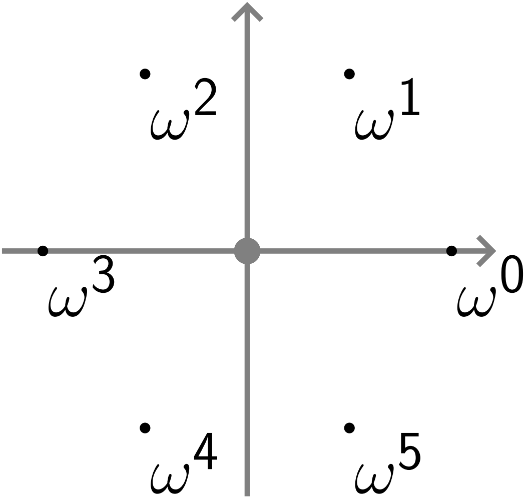

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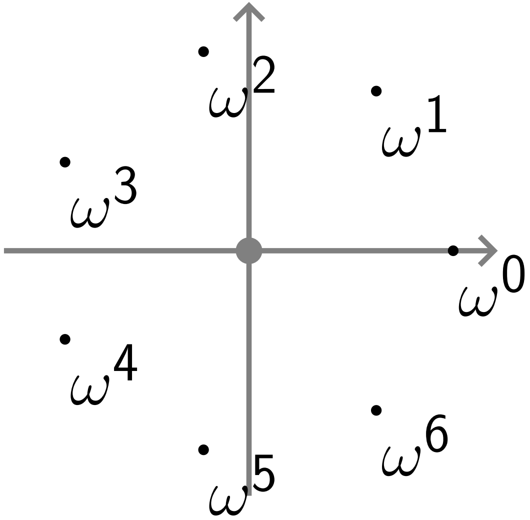

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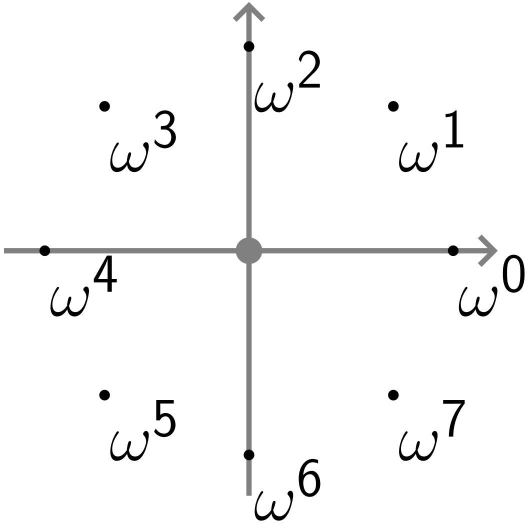

Powers of a complex number

We can now compute $z^{p/q}$.
By continuity, we could define z^x for $x \in \mathbb{R}$.

What about, say, z^i? This will have to wait.

Stereographic projection

Extend $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$.

Consider $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_i {x_i}^2 = 1\}$,

Define $f : S^2 \to \mathbb{C} \cup \{\infty\}$ by

$$z = x + iy = f(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

The points $(0, 0, 1)$, $(x, y, 0)$, and (x_1, x_2, x_3) are collinear.

Powers on the Riemann sphere