Lecture 33: Residue theorem

Math 660—Jim Fowler

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Theorem (Cauchy)

If $f(z)$ is analytic in a region Ω, then

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z - a}$$

if γ is homologous to zero in Ω.

The **residue** of $f(z)$ at an isolated singularity a is the complex number $R = \operatorname*{Res}_{z=a} f(z)$ which makes

$$f(z) - \frac{R}{z-a} = F'(z)$$

for some single-valued analytic function F defined in a small annulus around a.

Suppose Ω is simply connected, and
Suppose $\Omega' = \Omega - \{a_1, \ldots, a_n\}$.

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Then for any $\gamma \in \Omega'$,

$$[\gamma] = [c_1\gamma_1 + \cdots + c_n\gamma_n] \in H_1(\Omega')$$

for curves γ_i around a_i, with $c_i = n(\gamma, a_i)$.

$$\frac{1}{2\pi i} \int_{\gamma} f \, dz$$

$$\frac{1}{2\pi i}\int_{\gamma} f\,dz = \sum_{j} \frac{c_j}{2\pi i}\int_{\gamma_j} f\,dz$$

$$
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\frac{1}{2\pi i}\int_{\gamma} f\,dz &= \sum_j \frac{c_j}{2\pi i}\int_{\gamma_j} f\,dz \\
&= \sum_j \frac{c_j}{2\pi i}\int_{\gamma_j} \frac{\operatorname*{Res}_{z=a_i} f(z)}{z-a}\,dz
\end{aligned}
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This is Cauchy's **residue theorem** for when there are finitely many singularities.

If there are infinitely many singularities, still only finitely many have nonzero winding number with γ, If there are infinitely many singularities, still only finitely many have nonzero winding number with γ, and therefore

Theorem (Residue theorem)

Let $f(z)$ be analytic in Ω, except for isolated singularities a_j. Then,

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_j n(\gamma, a_j)\cdot \operatorname*{Res}_{z=a_i} f(z)$$

for any cycle which is homologous to zero in Ω and does not pass through any of the a_j.

If only we had a procedure for computing $\operatorname*{Res}_{z=a} f(z)$...

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Wait, we do!

Consider $f(z) = b_k(z-a)^{-k} + \cdots b_1(z-a)^{-1} + \varphi(z)$, for φ analytic.

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Then $\operatorname*{Res}_{z=a} f(z) = b_1$.

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For *simple* poles, it is even easier: then

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) \cdot f(z).$$

Example 1

$$f(z) = \frac{e^z}{(z-a)(z-b)} \text{with } a \neq b.$$

Compute $\int_\gamma f\, dz$ for a curve around a and b.

Example 2

$$f(z) = \frac{1}{z^2 + z - 1}.$$

Compute $\int_\gamma f \, dz$ for $\gamma = \partial B_3(0)$.

Example 3

$$f(z) = \frac{1}{\sin z}$$

Compute $\int_\gamma f\, dz$ for γ a circle of radius one around the origin.

Example 3

$$f(z) = \frac{1}{z^4 + 4}$$

Compute $\int_\gamma f\, dz$ for γ a circle of radius two and center 1.

Example 4

Evaluate

$$\int_{-\infty}^{\infty} e^{-z^2},\ dz$$