Lecture 36: Harmonic functions

Math 660—Jim Fowler

Tuesday, August 10, 2010

#### **Example**

$$\int_0^\pi \log \sin \theta \, d\theta = -\pi \log 2$$

## Single-valued branch

Define a single-valued branch of  $\sqrt{1-x^2}$  for any region  $\Omega$ , so that  $\pm 1$  are in the same component of  $\mathbb{C}-\Omega$ .



#### **Harmonic functions**

A function  $u: \mathbb{R}^2 \to \mathbb{R}$  is **harmonic** if u is continuous, with continuous seconds partial derivatives, and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial v^2} = 0.$$



**Example of harmonic functions** 

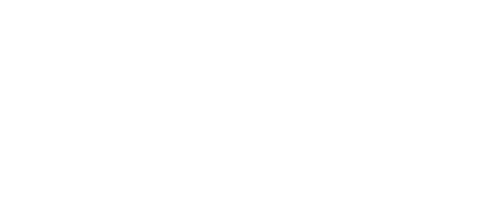
## **Example of harmonic functions**

$$u(x,y) = ax + by$$
.

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u(x,y)=ax+by.

The real part of an analytic function.



Harmonic functions in polar coordinates

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So if  $\frac{\partial u}{\partial \theta} = 0$ , then  $u = a \log r + b$ .

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But in practice, there may be no conjugate harmonic function, so we take as a definition

$$*du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy.$$

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And so  $\int_{\gamma} *du = 0$  for cycles homologous to zero.

# Simply connected regions

In a simply connected regions,  $\int_{\gamma} *du = 0$  for all cycles, so u has a single-valued conjugate function v.

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$$\int_{\gamma} u_1 * du_2 - u_2 * du_1 = 0$$

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**Proof:** If conjugate harmonic functions exist (e.g., in a simply connected region like a rectangle), then

$$u_1 * du_2 - u_2 * du_1 = u_1 dv_2 - u_2 dv_1$$

$$= u_1 dv_2 + v_1 du_2 - d(u_2 v_1)$$

$$= (im(u_1 + iv_1)(du_2 + idv_2)) - d(u_2 v_1),$$

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so the integral vanishes.