Problem Set 5 — Piecewise-Linear Topology

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Ceci n'est pas une Problem Set. Instead of questions, this sheet records the definitions we will be working with; here, we leave the realm of abstract simplicial complexes, and embrace the geometry—our simplexes are now situated in space.

Definition. The **join** of $A, B \subset \mathbb{R}^n$ is

$$A * B = \{\lambda a + \mu b : a \in A, \quad b \in B, \quad \lambda \in [0,1], \quad \mu \in [0,1], \quad \lambda + \mu = 1\}$$

Imagine this as connecting A and B by including all line segments between points in A and points in B.

A and B are **independent** if each point in $A * B$ can be written uniquely as $\lambda a + \mu b$, for $a \in A$, $b \in B$, $\lambda, \mu \in [0,1]$, $\lambda + \mu = 1$.

Polyhedra

Definition. $P \subset \mathbb{R}^n$ is a **polyhedron** if, for each $p \in P$, there is a neighborhood $N \ni p$, so that $N = p * L$, with L closed and bounded (in other words, compact).

In this case, N is called a **closed star** around p, and L is a **link** of p.

A **subpolyhedron** of P is a subset $Q \subset P$ which is also a polyhedron.

Definition. Let P, Q be polyhedra. Then $f : P \to Q$ is a **piecewise-linear map** (a PL map, for short) if each point $p \in P$ has a closed star $N = p * L$ so that $f(\lambda p + \mu x) = \lambda f(p) + \mu f(x)$ for $x \in L$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$. A **PL homeomorphism** is a PL map with a PL inverse.

Definition. A PL map $f : P \to Q$ is a **piecewise linear embedding** if $f(P)$ is a subpolyhedron of Q, and $f : P \to f(P)$ is a PL homeomorphism.

Manifolds

Definition. A polyhedron P is an n-dimensional **PL manifold** if each $p \in P$ has an open neighborhood $N \ni p$ which is PL homeomorphic to an open set in $\mathbb{R}^n$. In this case, we call N with the PL homeomorphism a **coordinate neighborhood**.

A polyhedron P is an n-dimensional PL manifold **with boundary** if each point $p \in P$ has an open neighborhood $N \ni p$ which is PL homeomorphic to an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$. The **boundary** of P (written ∂P) consists of points $p \in P$ which are identified with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.

A manifold P is **closed** if $\partial P = \varnothing$ and P is compact.

Definition. The n-ball B^n (sometimes called the n-disk D^n) is any manifold PL homeomorphic to $[0,1]^n$. The n-sphere is any manifold PL homeomorphic to $\partial[0,1]^{n+1}$.

Cell complexes

Definition. A subset $C \in \mathbb{R}^n$ is **convex** if for any $p, q \in C$, the segment $\{p\} * \{q\}$ is contained in C.

Definition. A compact convex subset $C \in \mathbb{R}^n$ is a k-dimensional **cell** if it spans a k-dimensional subspace.

For $x \in C$, define $\langle x, C \rangle$ to be the union of $\{x\}$ and all lines L such that $L \cap C$ contains x in its interior. The subset $C_x = C \cap \langle x, C \rangle$ is the **face** of C containing x. Write $D < C$ if D is a face of C.

Definition. A **cell complex** is a finite collection K of cells such that

- If $C \in K$, and $D < C$, then $D \in K$.
- If $C, D \in K$, then $C \cap D$ is a face of C and D.

The **underlying polyhedron**, $|K|$ is the union of all cells in K.

A **cellular map** $f : K \to L$ is a PL map $|f| : |K| \to |L|$ which is linear on cells of K, and sends cells to cells.

Definition. A cell complex L is a **subdivision** of K if $|L| = |K|$ and each cell of L is contained in a cell K. We write $L \triangleleft K$ if L is a subdivision of K.

Simplicial complexes

Our original notion of simplicial complex should now be called an **abstract simplicial complex**, to emphasize the fact that we originally did not place our simplexes in $\mathbb{R}^n$. In an abstract simplicial complex, only the relationship between the vertices and the faces is recorded.

Definition. A cell complex is a **simplicial complex** if each $C \in K$ is a **simplex** (i.e., an n-cell which is the join of $n+1$ independent points). A **triangulation** of a polyhedron P is a simplicial complex K with a PL homeomorphism $f : |K| \to P$.

Definition. Suppose $L \subset K$ are simplicial complexes. Define $f : K \to [0,1]$ on vertices by

$$f(v) = \begin{cases} 0 & \text{if } v \in L \\ 1 & \text{if } v \notin L \end{cases}$$

and extending linearly to simplexes. If $L = f^{-1}(0)$ we say L is a **full subcomplex** of K, and write $L \Subset K$.