Take-Home Quiz 6

Math 133 Section 22

Due Wednesday, May 31

Problem 1. (5 points). In this problem, we will develop an explicit formula for the sequence

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+2} = a_{n+1} + a_n,$$

namely, the **Fibonacci sequence**. We work in a few steps:

Step 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Explain why $f(x) = \dfrac{1}{1-x-x^2}$.

Step 2. Define $\phi_1 = \dfrac{1+\sqrt{5}}{2}$, $\phi_2 = \dfrac{1-\sqrt{5}}{2}$. The number ϕ_1 is the **golden ratio**. Prove

$$\frac{1}{1-x-x^2} = \frac{1}{x\sqrt{5}} \cdot \left(\frac{x\phi_1}{1-x\phi_1} - \frac{x\phi_2}{1-x\phi_2} \right).$$

Step 3. Use the fact that $\dfrac{x\phi_1}{1-x\phi_i} = \sum_{n=1}^{\infty} {\phi_i}^n x^n$ to show

$$\frac{1}{1-x-x^2} = \frac{1}{x\sqrt{5}} \cdot \left(\sum_{n=1}^{\infty} {\phi_1}^n x^n - \sum_{n=1}^{\infty} {\phi_2}^n x^n \right).$$

Rearrange to prove

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x-x^2} = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left({\phi_1}^{n+1} - {\phi_2}^{n+1} \right) x^n.$$

Equate corresponding coefficients to conclude

$$a_n = \frac{1}{\sqrt{5}} \left({\phi_1}^{n+1} - {\phi_2}^{n+1} \right).$$

Step 4. Use the formula and a calculator to compute a_{15}.

Step 5. For three extra credit points, compute $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ using the formula.

Problem 2. **(3 points).** Find a Taylor series expansion for $f(x) = x^2 + 3x + 1$ around the point $a = 2$.

Problem 3. **(3 points).** Find the first three terms of the Taylor series expansion for $f(x) = \sqrt[3]{x}$ around the point $a = 1$. Use this to approximate $\sqrt[3]{1.1}$.

Problem 4. **(2 points).** Use the first three nonzero terms of the Maclaurin series for $\sin x$ to approximate $\sin 1$.

Problem 5. **(2 points).** Use the Maclaurin series expansion for $f(x) = e^{x^2}$ to compute $f^{(100)}(0)$.