# Topology of Piecewise-Linear Manifolds

Jim Fowler

Lecture 1
Summer 2010

# Syllabus

## Syllabus — Piecewise-Linear Topology

Summer 2010 — Jim Fowler

Manifolds are spaces which are locally modeled on Euclidean space, but might be globally twisted in some way; two-dimensional examples include a sphere or a torus. In contrast to the usual introduction to manifolds based on calculus and charts (that is, smooth manifolds), this course will study manifolds as combinatorial objects (that is, piecewise-linear manifolds). Piecewise-linear manifolds are more general than smooth manifolds, and because the basic definitions involve combinatorics instead of calculus, we will find it easier to give rigorous proofs.

### Homework

Problem sets will be distributed during most lectures.

### Website

The course website is `http://www.math.osu.edu/~fowler/teaching/ross2010/`

### Lectures

We meet Mondays, Wednesdays, and Fridays, 1:30p.m.–2:30p.m. in CH240.

### Instructor

# Problem Set

## Problem Set 1 — Piecewise-Linear Topology

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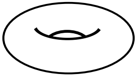
Problem Set 1 is intentionally vague: I want you to think a bit about how you would make these notions precise. **Problems marked with a • should be written up.**

**Problem 1.** Build a circle $S^1$ by gluing line segments together along their vertices. How many different ways are there of doing this?

**Problem 2.** Build a sphere $S^2$ by gluing triangles together along their boundaries. How are the various ways of doing this related to each other?

**Definition.** An object built by gluing together triangles is called a *simplicial complex*.

**Problem 3.** Build a torus $T^2$ by gluing triangles together.



a torus

**Definition.** A function $f : K \to L$ sending

- vertices to vertices,
- edges to either edges or vertices, and
- triangles to triangles, or
  to edges, or
  to vertices

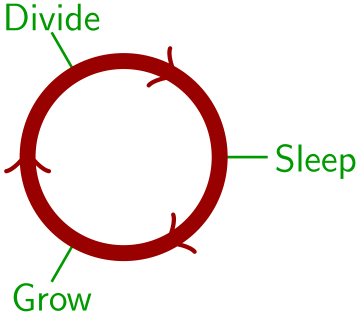
is called a *simplicial map*.

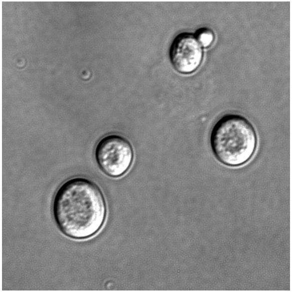
• **Problem 4.** Find a map $f : T^2 \to S^1$ so that, for every $x \in S^1$, the preimage $f^{-1}(x)$ is a circle.

• **Problem 5.** Let $k \in \mathbb{N}$. Find a simplicial map $f : T^2 \to T^2$ so that, for every $x \in T^2$, the preimage $f^{-1}(x)$ consists of $k$ points?

# Topology

# Yeast life cycle



Yeast life cycle is a **circle**.
$S^1$ = the circle.

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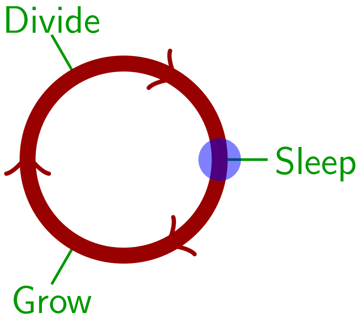


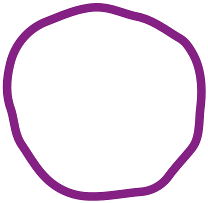

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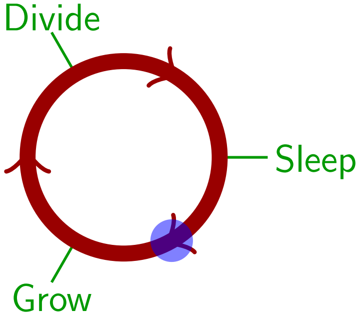


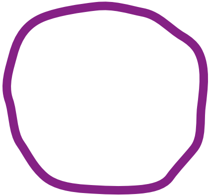

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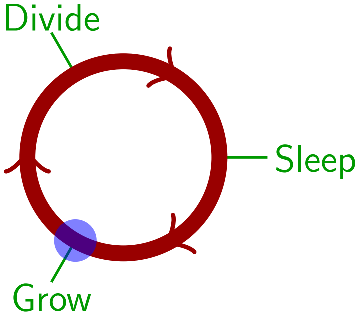


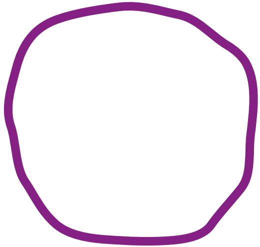

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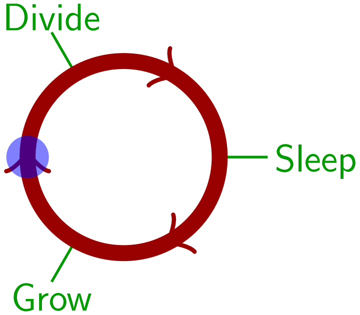


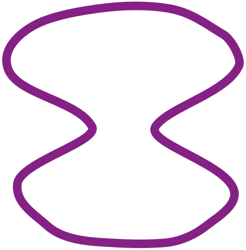

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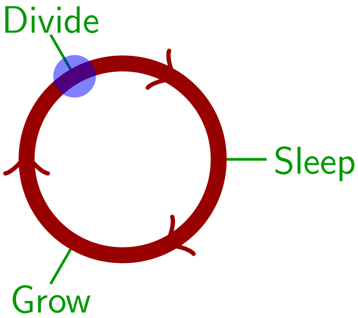


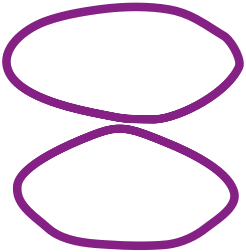

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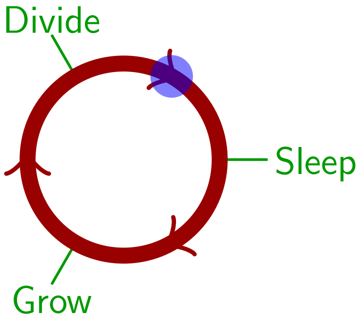


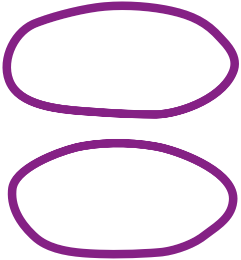

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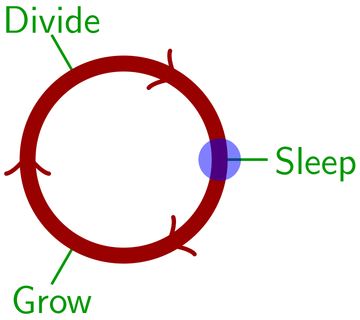


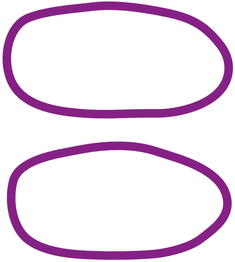

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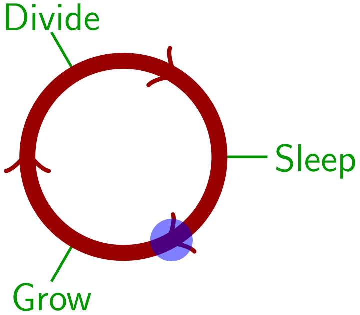


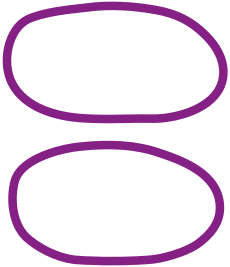

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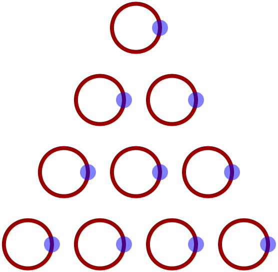
# Yeast synchronize with their neighbors

Cells signal each other
so they all stay at the same point in the life cycle.



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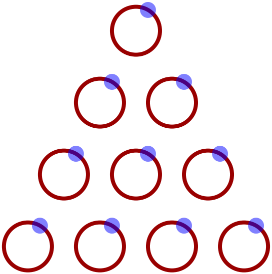

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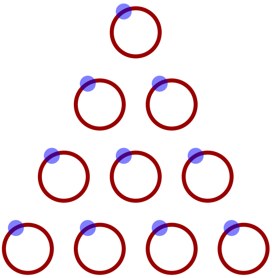

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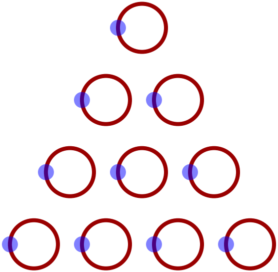

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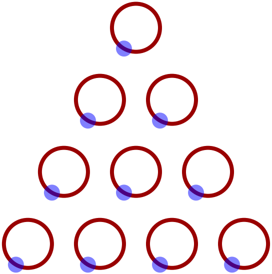

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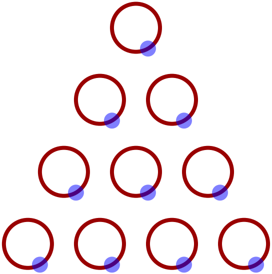

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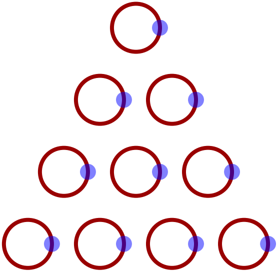
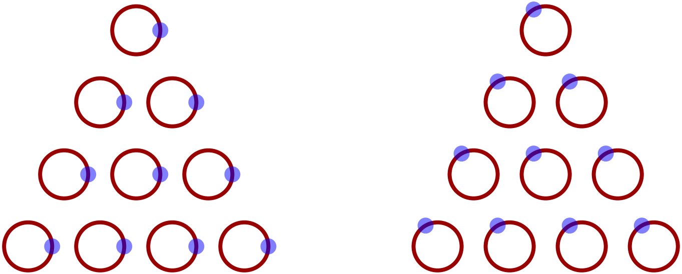


This raises a question. . .

# A Question

If two colonies are at different points in the cycle,



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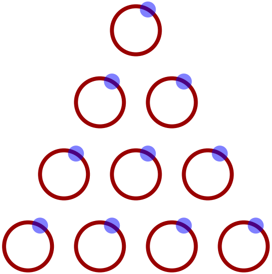

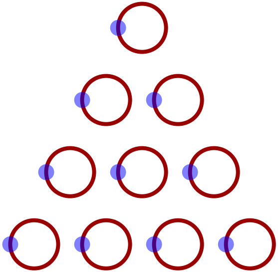

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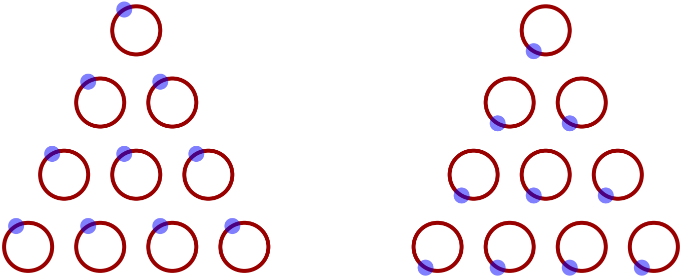

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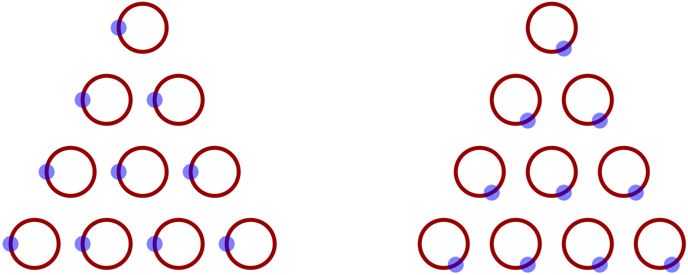

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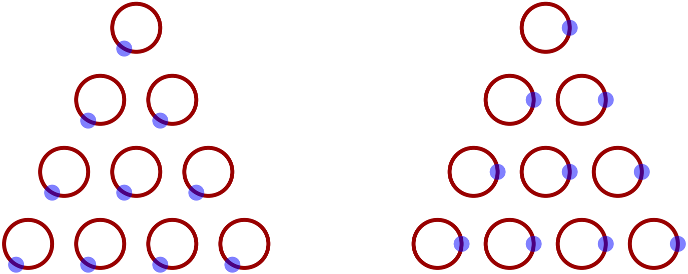

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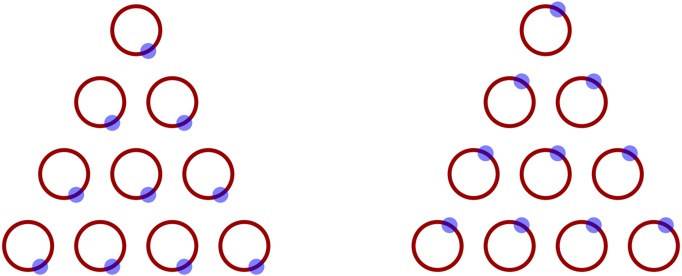

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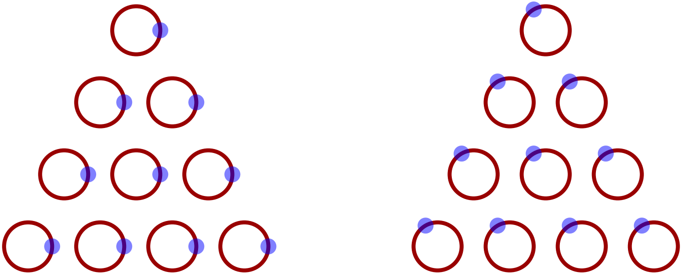
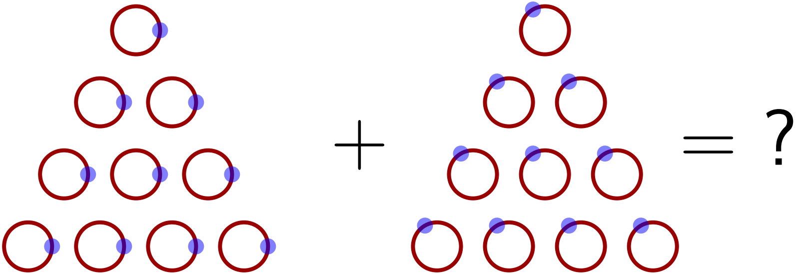


# A Question

If two colonies are at different points in the cycle, what happens when they are combined?



# Modelling the answer

$f(a,b) =$ result of combining colonies at $a$ and $b$.

$f : S^1 \times S^1 \to S^1$

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**But such a function does not exist!**

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**But such a function does not exist!** *Why not?*

# Obstructing existence

## Question

*Why shouldn't such an $f$ exist?*

## Ingredients for exploring this

| circles | symmetry |
| --- | --- |
| $S^1$ | $f(a, b) = f(b, a)$ |

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**manifolds**

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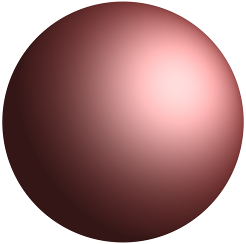
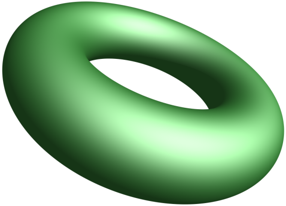
| **manifolds** | **maps** |

# Manifold: definition

A **manifold** is a topological space which near each point looks like $\mathbb{R}^n$.





These locally resemble $\mathbb{R}^2$.
They are called 2-manifolds.

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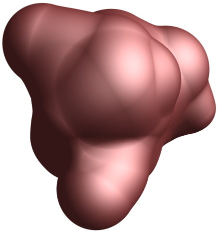

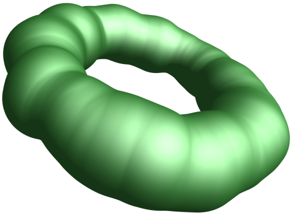

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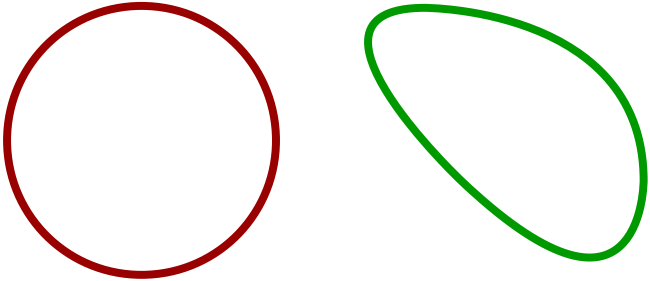


These locally resemble $\mathbb{R}^1$.
They are called 1-manifolds.

# Maps between manifolds

It is never enough to define objects—
we must also define relationships between objects.

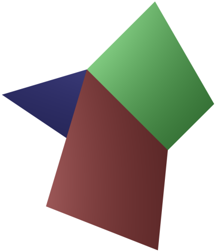
A map $f$ from a manifold $M$ to a manifold $N$

$$f : M \longrightarrow N$$

is a continuous function.

# Manifold: non-examples

Not every space is a manifold.



$Y \times I$

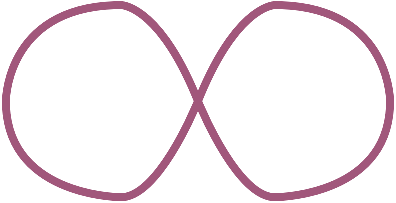
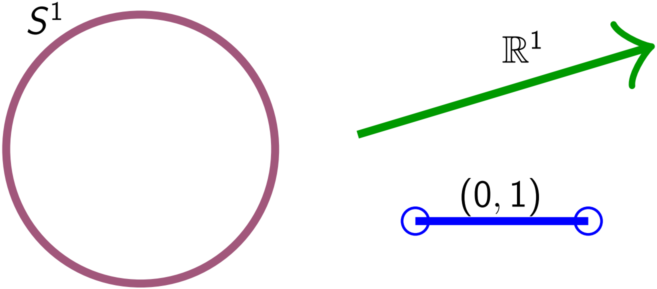


Figure Eight

These objects have **singularities**.

# 1-manifolds



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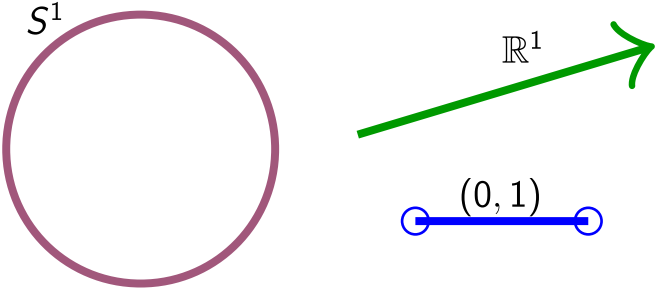
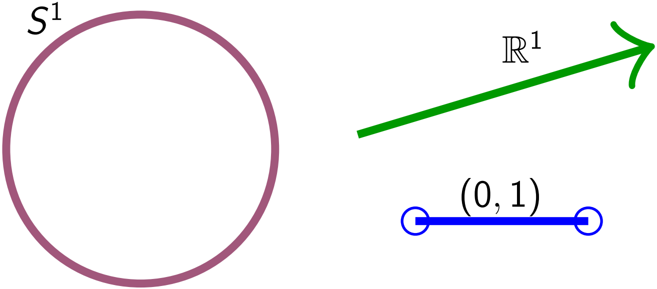



Warning

The closed interval $[0, 1]$ is not a manifold, while the open interval $(0, 1)$ is a manifold.

# 1-manifolds

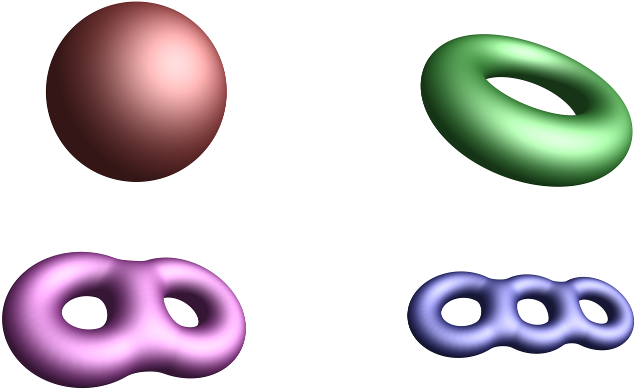



**Warning**

The closed interval $[0,1]$ is not a manifold, while the open interval $(0,1)$ is a manifold.

Notice also that $(0,1) \cong \mathbb{R}$.

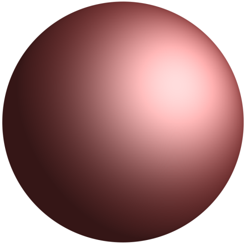
# 2-manifolds



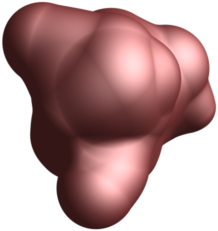
And there are many more 2-manifolds.

# 2-manifolds

Different looking manifolds may be the same.
This is the $4 = \mathbf{IV} = \text{four} = 3.\overline{9}$ problem.



$\cong$



Both are $S^2$, the two-dimensional sphere.

# 2-manifolds

# What does "the same" mean?

### Definition

A **homeomorphism** $f : M \to N$ is a continuous function with a continuous inverse.

Two manifolds $M$ and $N$ are **homeomorphic** if there exists a homeomorphism $f : M \to N$.
We write $M \cong N$.

### Remark

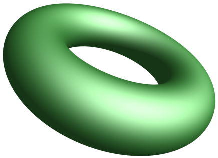
*Homeomorphic spaces are topologically indistinguishable.*

# Distinguishing spaces

Is there a homeomorphism between

X and Y ?

# 2-manifolds: the torus



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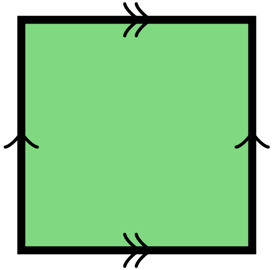


Glue the sides marked $\rightarrow$ together

Glue the sides marked $\twoheadrightarrow$ together

Imagine living inside the torus.

# 2-manifolds: the torus

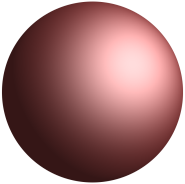
$$\begin{aligned}\{\text{points on the torus}\} &= \{\text{points on a pair of circles}\}\\ &= \{(x, y) : x, y \in S^1\}\end{aligned}$$

$$T^2 = S^1 \times S^1$$

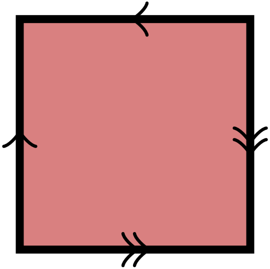
# 2-manifolds: other examples?

## Question

*What happens if I glue the sides of a square together in a different way?*

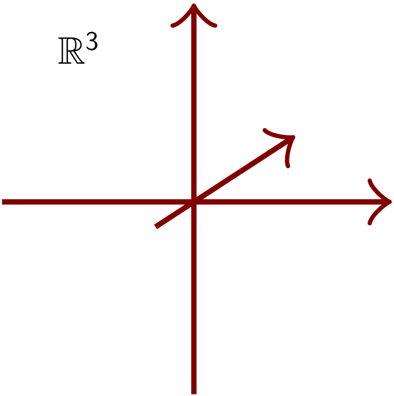


$\cong$



There are other possibilities—and other polygons!

# 3-manifolds

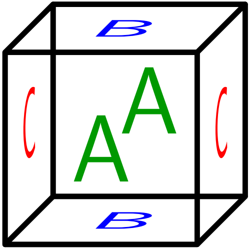



Three-dimensional
Euclidean space
is a 3-manifold,

but there are other
3-manifolds.

# 3-manifolds

It is tricky to draw pictures of 3-manifolds; instead, imagine living inside the 3-manifold.




Cube with A glued to A,
B glued to B, and
C glued to C.

This is $T^3$, the three-torus.

# 4-manifolds? *n*-manifolds?

?

How can we visualize a 4-manifold?<br>
a 17-manifold?

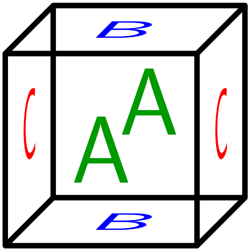
# 4-manifolds? *n*-manifolds?

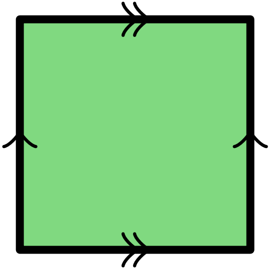
?

How can we visualize a 4-manifold?
a 17-manifold? 8 weeks to go!...

# How to build manifolds? Glue!

We can build manifolds by gluing together sides



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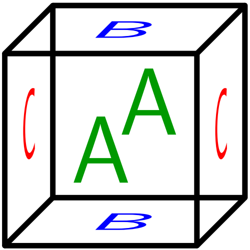
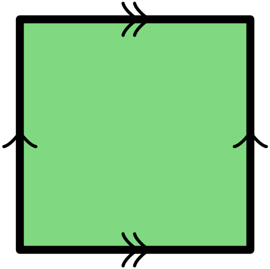


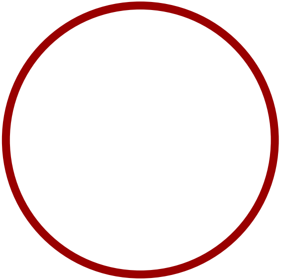



Warning

Not every use of glue results in a manifold.

# How to build manifolds? Glue!

How to build $S^1$,
the circle?



Start with an interval $I = [0, 1]$

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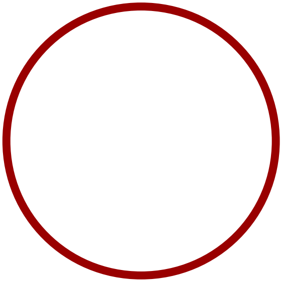

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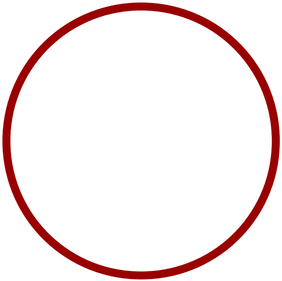

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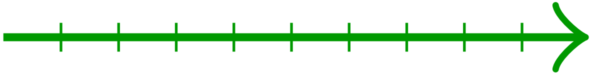
But there is another perspective on $S^1$...

# Build $S^1$ from $\mathbb{R}$

Consider the 1-manifold $\mathbb{R}$



Identify $x \in \mathbb{R}$ with $x + n$ for any $n \in \mathbb{Z}$.
Name the result $\mathbb{R}/\mathbb{Z}$.

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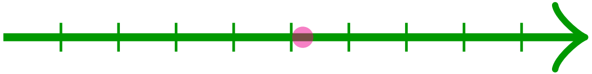

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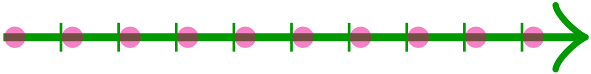

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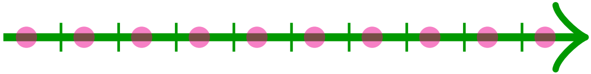

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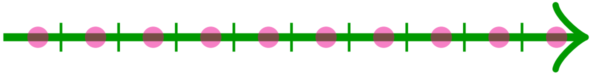

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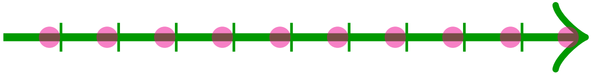

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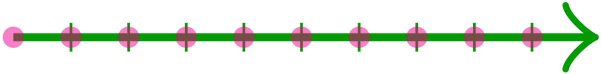

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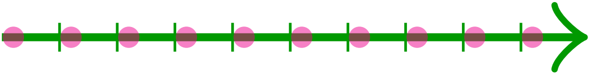

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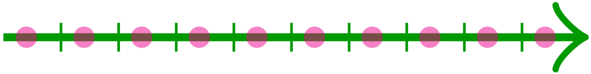

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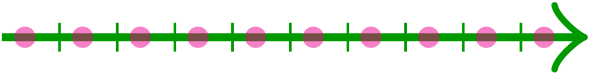

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$\mathbb{R}/\mathbb{Z} \cong S^1$.

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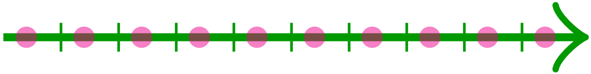

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### Claim

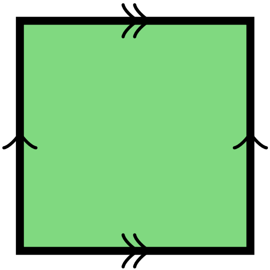
$\mathbb{R}/\mathbb{Z} \cong S^1$.

### Proof.

Define $f : \mathbb{R}/\mathbb{Z} \to S^1$ by sending a real number $r \in \mathbb{R}$ to a point $2\pi r$ radians around the circle. $\square$

# A beautiful example

$\mathbb{Z}_2$ acts on $T^2$ by $0 \cdot (x, y) = (x, y)$
$1 \cdot (x, y) = (y, x)$



$T^2 = S^1 \times S^1$

The **quotient** is a Möbius strip.

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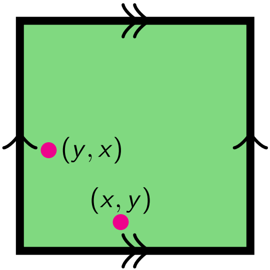


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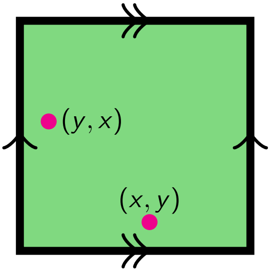


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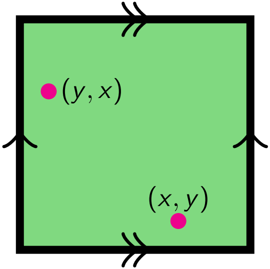


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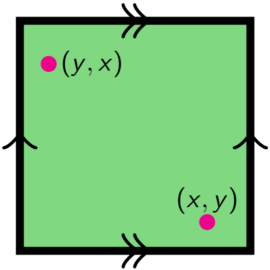


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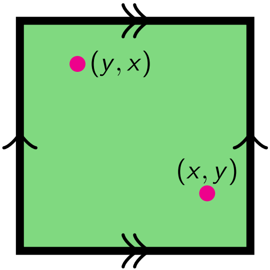


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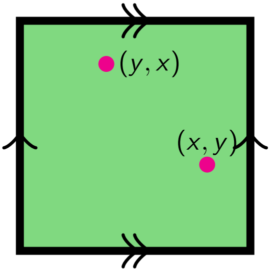


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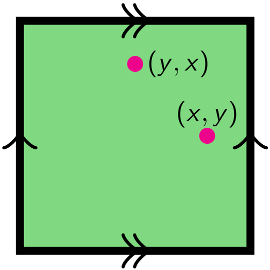


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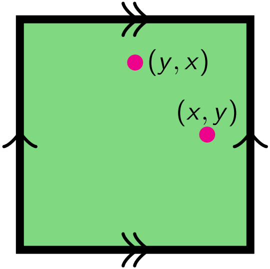


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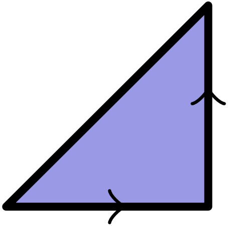
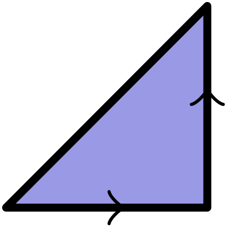


$T^2/\mathbb{Z}_2$

The **quotient** is a Möbius strip.

# $T^2/\mathbb{Z}_2 =$ Möbius strip



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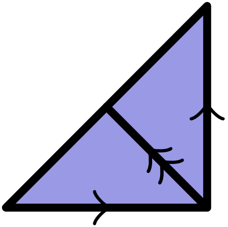

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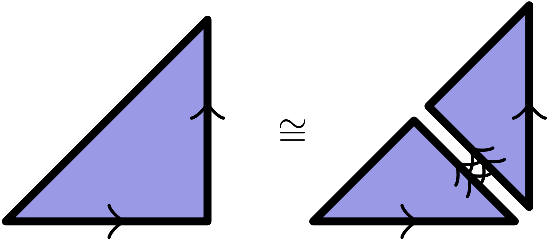

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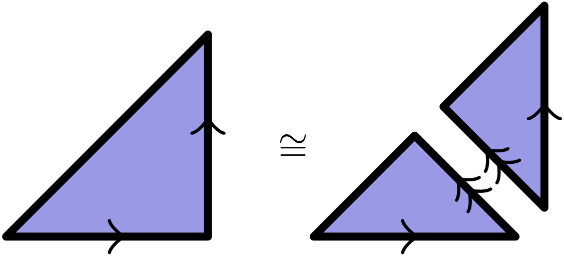

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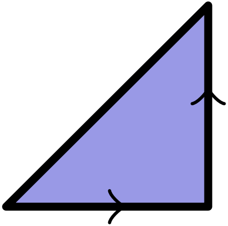

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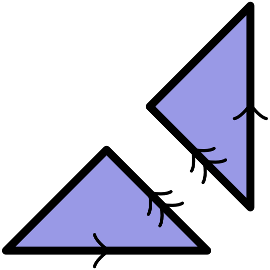

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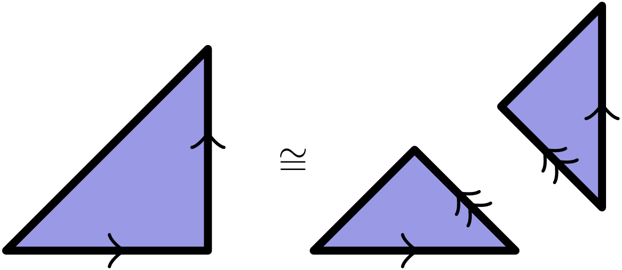

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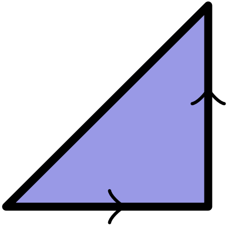

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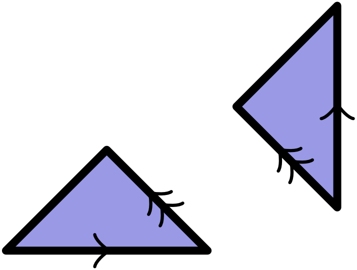

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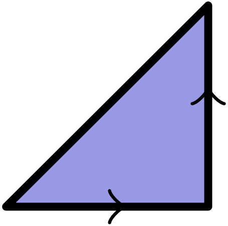

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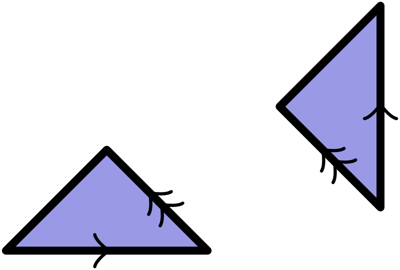

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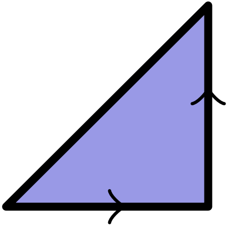

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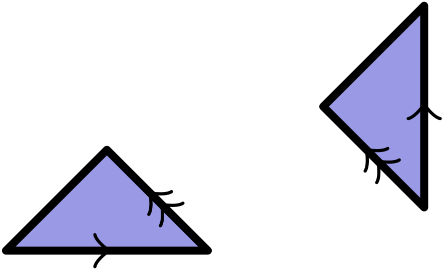

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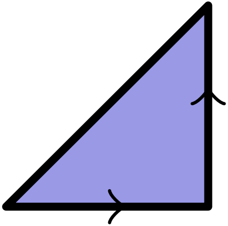

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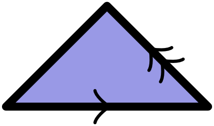

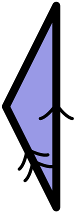

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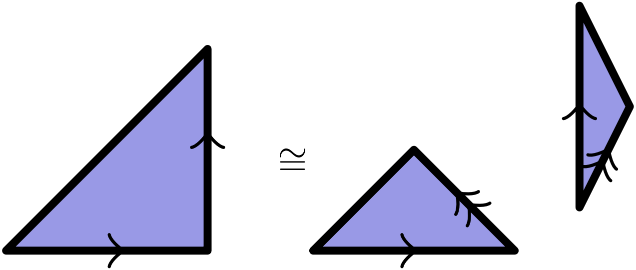

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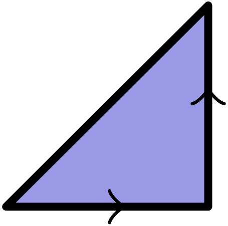

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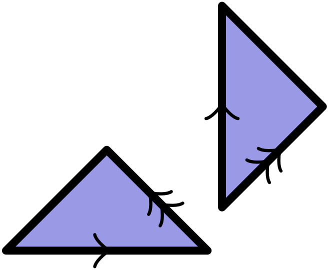

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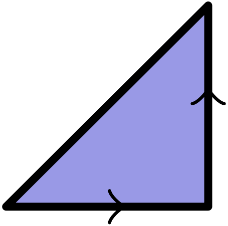

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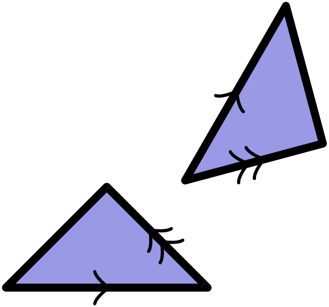

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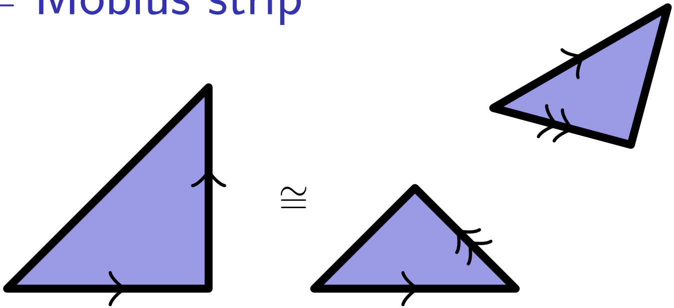

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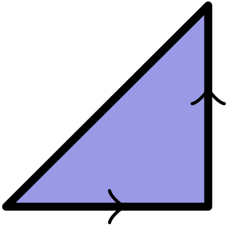

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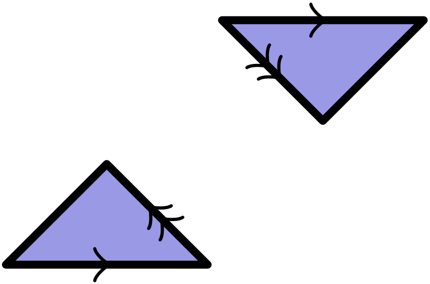

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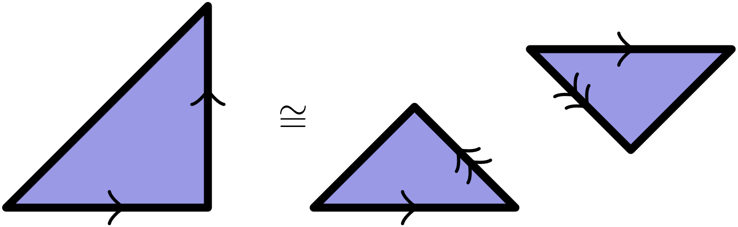

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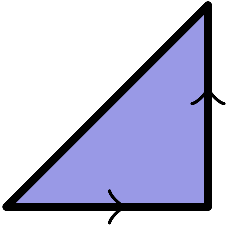

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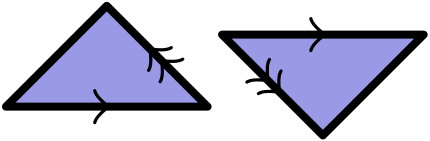

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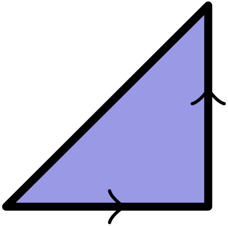

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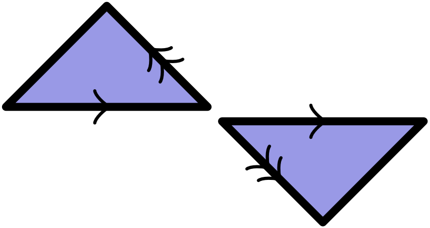

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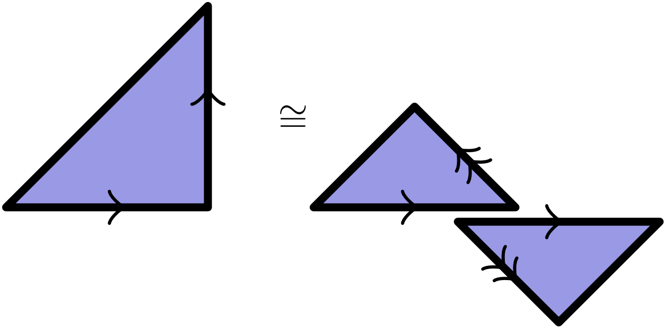

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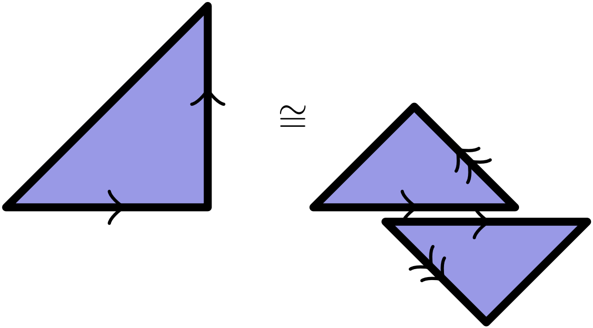

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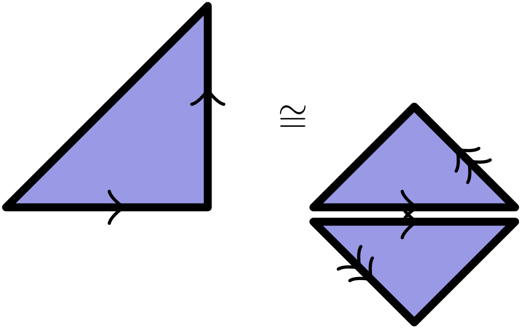

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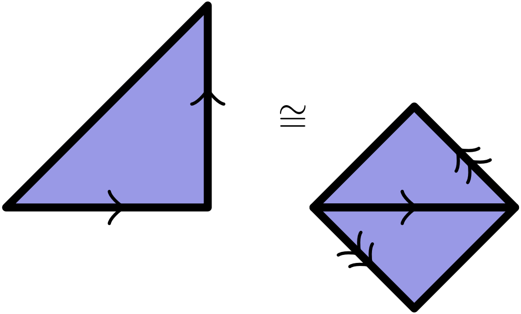

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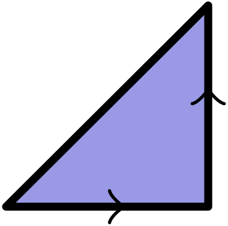

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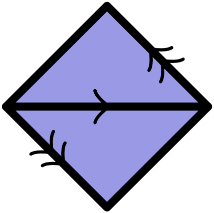
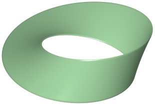


This is  or rather 

$$\mathbb{Z}_2 \text{ acts on } T^2 \text{ by } \begin{aligned} 0 \cdot (x,y) &= (x,y) \\ 1 \cdot (x,y) &= (y,x) \end{aligned}$$

Since $T^2 =$ points on two circles

$T^2/\mathbb{Z}_2 =$ two indistinguishable points on a circle


How does this relate to yeast?

# Modelling the answer

$f(a, b) =$ result of combining colonies at $a$ and $b$.

$f : S^1 \times S^1 \rightarrow S^1$

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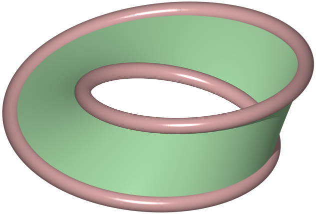
- $f$ is continuous.

**But such a function does not exist!** *Why not?*

# Reinterpretation

$f : S^1 \times S^1 \to S^1$ means $f : T^2 \to S^1$
$f(a, b) = f(b, a)$ means $f : T^2/\mathbb{Z}_2 \to S^1$
$f(x, x) = x$ means boundary goes to $S^1$



$\to S^1$

This is impossible.
*Is this a proof?*

# Simplicial Complexes

# Thank You