Take-Home Final Exam — Math 765

Winter 2011 — Jim Fowler

The final exam. The course is ended, but there is still one task that remains—this, the take-home final exam. Please feel free to email me with questions at `fowler@math.osu.edu`, and feel free to discuss the problems with your friends—but write up your own solutions. *The final exam below should be handed in 100 hours from now—on Tuesday, March 15, 2011, at 5:30* P.M.

Problem 1

Consider $G_2(\mathbb{R}^4)$, the Grassmannian of planes in $\mathbb{R}^4$ (that is, a point in $G_2(\mathbb{R}^4)$ corresponds to a subspace $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$). Show that the map

$$\bigsqcup_{\substack{V \subset \mathbb{R}^4 \\ \dim V = 2}} V \to G_2(\mathbb{R}^4)$$

sending a vector in V to the point $[V] \in G_2(\mathbb{R}^4)$ is a smooth vector bundle.

Problem 2

Let M be a closed smooth n-manifold without boundary; let g be a Riemannian metric on M; the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$ is given by

$$\Delta(u) = -\operatorname{div}(\operatorname{grad} u).$$

- Show that, if $\Delta u \equiv 0$, then u is constant.
- Show that, if $\Delta u = \lambda u$ and u is not constant, then $\lambda > 0$.

Problem 3

Suppose that $f : S^n \to S^n$ is a diffeomorphism; show that $X = D^{n+1} \cup_f D^{n+1}$ is a smooth manifold.

Problem 4

Suppose $f : M^n \to \mathbb{R}$ is a proper smooth map for which there are no critical points. Is it the case that $M = N \times \mathbb{R}$ for some smooth manifold N?

Problem 5

Consider the tangent distribution on $\mathbb{R}^3$ given by $\ker(dz - y\,dx)$.

- Is this distribution involutive?
- For any two points $p, q \in \mathbb{R}^3$, can you find a curve $\gamma : I \to \mathbb{R}^3$ so that $\gamma'(t)$ lies in this distribution?