Midterm 1 Solution Set

Math 345

October 2010

> We often hear that mathematics consists mainly of "proving theorems." Is a writer's job mainly that of "writing sentences?"
>
> —Gian-Carlo Rota

Name: ____________________________________

Lecture time (circle one): 12:30–1:18P.M.

2:30–3:18P.M.

1. Write your name above.
2. Calculators are forbidden (and useless, anyhow).
3. Do not look inside the exam until instructed to do so.
4. You have **48 minutes** for this exam.
5. Justify your answers for full credit.
6. Show your work for generous partial credit.
7. Write your answers on the included pages, or request additional paper.
8. Answer all questions asked.
9. To prevent fire, do not divide by zero.

Problem 1	/360
Problem 2	/360
Problem 3	/360
Problem 4	/360
Problem 5	/360
Total	/1800

Problem 1 /360

Consider the proposition:

$$(Q \Rightarrow R) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

Solution

Claim. *The given proposition is a tautology.*

Proof. Assume $Q \Rightarrow R$. (A1)
We want to prove that $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$.

> To prove this latter statement, we will assume $P \Rightarrow Q$. (A2)
> we want to prove $P \Rightarrow R$.
>
> > To prove this final statement, we will assume P. (A3)
> > We want to prove R.
> >
> > Assumptions (A3) and (A2) are that P is true and that $P \Rightarrow Q$
> > so by modus ponens, Q is true.
> >
> > But assumption (A1) is that $Q \Rightarrow R$,
> > so by modus ponens, R is true.
> >
> > We have proved R is true, which is what we wanted to prove. $\square$

Commentary

Many people submitted solutions for this problem that were much more complicated than the proof I offer above; the problem text asks you not to split into cases. The structure of the proof is modeled on the structure of the proposition: the proof begins by considering the outermost implication arrow, which involves proving another implication, which itself involves proving an implication.

Problem 2 /360

Let x be an integer.

Write down the **contrapositive** of the conditional sentence

> If x is even, then x^2 is even.

Is this a true statement? Is the contrapositive a true statement?
If yes, prove it. If not, find a counterexample.

Solution

Let P be the proposition "x is even" and Q be the proposition "x^2 is even." The contrapositive of $P \Rightarrow Q$ is $(\neg Q) \Rightarrow (\neg P)$, so the contrapositive of the given statement is "If x^2 is not even, then x is not even."

The original statement is true.

Claim. *If x is even, then x^2 is even.*

Proof. Assume x is even.
Then there exists an integer k so that $x = 2k$,
and so $x^2 = (2k)^2 = 4k^2$.

But $4k^2 = 2(2k^2)$,
so x^2 is also twice an integer, and therefore even. □

The contrapositive is also true
(because an implication holds if and only if the contrapositive of the implication holds).

Commentary

Many people submitted separate proofs for the original statement and the contrapositive: this is unnecessary. It is easier to prove the original statement than the contrapositive in my opinion.

Problem 3 /360

Consider the four propositions

$$P \vee (Q \wedge R), \tag{1}$$
$$(P \wedge Q) \Rightarrow R, \tag{2}$$
$$(\neg P) \vee (\neg Q) \vee R, \text{ and} \tag{3}$$
$$(P \vee Q) \wedge (P \vee R). \tag{4}$$

Exactly which of these propositions are logically equivalent to which other propositions?

Provide justification for all the claims you make; in particular, if you claim that two propositions are logically equivalent, you must prove this, and if you claim that they are *not* equivalent, you must explain why not.

I prefer arguments that don't involve cases.

Solution

I claim that (1) and (4) are equivalent, and (2) and (3) are equivalent, but neither (1) nor (4) are equivalent to (2) or (3).

Claim. *(1) $\equiv$ (4).*

Proof. This is precisely the distributive law. □

Claim. *(2) $\equiv$ (3).*

Proof.

$$\begin{aligned}
(2) &\equiv (P \wedge Q) \Rightarrow R \\
&\equiv (\neg (P \wedge Q)) \vee R && \text{(definition of } \Rightarrow) \\
&\equiv ((\neg P) \vee (\neg Q)) \vee R && \text{(de Morgan's law)} \\
&\equiv (3).
\end{aligned}$$

□

Claim. *(1) $\not\equiv$ (3).*

Proof. If P is true, Q is true, and R is false, then (3) is false, but (1) is true. So (1) $\not\equiv$ (3). Providing a specific example is the quickest way to verify this. □

Commentary

A complete solution to this problem requires proving three statements (namely that (1) $\equiv$ (4), that (2) $\equiv$ (3), and that (1) $\not\equiv$ (3)). You received 120 points for each claim you proved. Many people failed to discuss why (1) $\not\equiv$ (3) and lost points because of this; the problem text asks "exactly which" propositions are equivalent, so you must discuss both which are equivalent and which are inequivalent.

Problem 4 /360

Let x and y be real numbers, and consider the following proposition:

> If x is rational and y is irrational, then $x + 2y$ is irrational.

If the proposition is true, prove it; if not, give a counterexample.

Solution

This is a true proposition.

Claim. *If x is rational and y is irrational, then $x + 2y$ is irrational.*

Proof. Assume that x is rational, y is irrational. For a contradiction, we assume that $x+2y$ is rational.

Since the difference of rational numbers is rational, $(x + 2y) - x$ is rational, so $2y$ is rational. But $1/2$ is rational, and since the product of rational numbers is rational, $(1/2) \cdot 2y = y$ is rational. But this is a contradiction—y is irrational. □

For completeness, I include proofs of two results that I used.

Claim. *The difference of rational numbers is rational.*

Proof. Assume $x, y \in \mathbb{Q}$. Then there exist integers $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$ so that

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

and, combining denominators,

$$x - y = \frac{ad - bc}{bd}$$

is a rational number, since $ad - bc \in \mathbb{Z}$, $bd \in \mathbb{Z}$, and $bd \neq 0$. □

Claim. *The product of rational numbers is rational.*

Proof. Assume $x, y \in \mathbb{Q}$. Then there exist integers $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$ so that

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

and, multiplying,

$$xy = \frac{ac}{bd}$$

is a rational number, since $ac, bd \in \mathbb{Z}$ and $bd \neq 0$. □

Commentary

A number of people stated that $2y$ is irrational by claiming that a rational number times an irrational number is irrational; this is not a true statement (consider when the rational number is zero).

Problem 5 /360

Is the statement

$$\forall x \in \mathbb{R} \left(\exists y \in \mathbb{R} \left(y^2 < x\right)\right)$$

true or false? For full credit, justify your answer.

Solution

The proposition is false.
I claim that the negation of $\forall x \in \mathbb{R}\ (\exists y \in \mathbb{R}\ (y^2 < x))$ is true, namely

Claim. $\exists x \in \mathbb{R}\ (\forall y \in \mathbb{R}\ (y^2 \geq x))$

Proof. Set $x = -1$.
Let y be a real number.
Then $y^2 \geq 0$, so $y^2 \geq 0 > -1 = x$, which is what I wanted to prove. $\square$

Commentary

Many students gave a specific choice of x and y for which $y^2 \not< x$; this is not enough; you need to explain why there is a value of x for which no value of y will satisfy $y^2 < x$.