Topology of Piecewise-Linear Manifolds

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Lecture 6
Summer 2010

Where are we now?

Where are we now?

Objects

Where are we now?

Objects

polyhedra

Where are we now?

Objects

polyhedra

simplicial complexes

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simplicial complexes

Maps

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PL maps

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χ $\quad b_0 \quad$ w_1

Piecewise linear map

Let P, Q be polyhedra.
$f : P \longrightarrow Q$ is a PL map
if each point $p \in P$
has a closed star $N = p * L$
so that $f(\lambda p + (1 - \lambda)x) = \lambda f(p) + (1 - \lambda)f(x)$
for $x \in L$ and $\lambda \in [0, 1]$.

It locally maps conical rays to conical rays.

Convexity

Definition

A subset $C \in \mathbb{R}^n$ is **convex**
if for any $p, q \in C$,
the segment $\{p\} * \{q\}$
is contained in C.

Cells

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is a k-dimensional **cell**
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Write $D < C$ if D is a face of C.

Cell complex

A **cell complex** is
a finite collection K of cells such that

- If $C \in K$, and $D < C$, then $D \in K$.
- If $C, D \in K$, then $C \cap D$ is a face of C and D.

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The **underlying polyhedron**, $|K|$ is
the union of all cells in K.

A **cellular map** $f : K \to L$ is
a PL map $|f| : |K| \to |L|$
which is linear on cells of K,
and sends cells to cells.

Subdivision

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Write $L \triangleleft K$ if L is a subdivision of K.

Simplicial complexes, not abstract

A cell complex is a **simplicial complex**
if each $C \in K$ is a **simplex**
(i.e., an n-cell which is the
join of $n+1$ independent points).

A **triangulation** of a polyhedron P
is a simplicial complex K
with a PL homeomorphism $f : |K| \to P$.

From now on,

complex

means

simplicial complex,

(not abstract)

Simplicial Collapse

Principal Simplexes

Definition

Let K be a complex, and
$\sigma \in K$ a simplex.
Call σ a **principal simplex**
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Problem

Does every complex have a principal simplex?

Free faces

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Let K be a complex,
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and $\tau < \sigma$ a face.

Call τ a **free face** of σ
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Does every complex have a simplex with a free face?

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Problem

Does every complex have a simplex with a free face?
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Elementary simplicial collapse

Definition

Let L and $K = L \cup \mathrm{cl}\{\sigma, \tau\}$ be complexes

If σ is a principal simplex of K, and
τ is a free face of σ, then
L is an **elementary simplicial collapse** of K.

Simplicial collapse

Definition

Let $K_1, K_2, \ldots, K_n$ be complexes, with K_{i+1} an elementary simplicial collapse of K_i.

Call K_n a **simplicial collapse** of K_1, and write $K_1 \searrow K_n$.

Call K_1 a **simplicial expansion** of K_n, and write $K_n \nearrow K_1$.

Simple homotopy equivalence

Definition

K is **simple homotopy equivalent** to L
(sometimes abbreviated s.h.e.)
if you can reach transform K into L
via a sequence of

- PL homeomorphisms,
- simplicial collapses,
- simplicial expansions.

In this case, we write $K \searrow\!\nearrow L$.

Full subcomplex

$L \subset K$ are complexes.

Define $f_L : K \longrightarrow [0,1]$ on vertices by

$$f_L(v) = \begin{cases} 0 & \text{if } v \in L \\ 1 & \text{if } v \notin L \end{cases}$$

and extending linearly to simplexes.

If $L = {f_L}^{-1}(0)$
we say L is a **full subcomplex** of K,
and write $L \Subset K$.

Derived subdivision

$L \subset K$ are complexes.

Choose $a_i \in \operatorname{int} A_i$ for each $A_i \in K$ with $A_i \notin L$.

$K' \triangleleft K$ is "K derived away from L"
is defined inductively over skeleta by

$$A_i{}' = \begin{cases} \{a_i\} * \partial A_i{}' & \text{if } A_i \notin L, \\ A_i & \text{if } A_i \in L. \end{cases}$$

$L \subset K$ are complexes.

Define the simplicial neighborhood of L in K as

$$N(L, K) = \{A \in K : A < B, B \cap |L| \neq \varnothing\}.$$

The **simplicial complement** of L in K is

$$C(L, K) = \{A \in K : A \cap |L| = \varnothing\}.$$

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$\partial N(L, K) := N(L, K) \cap C(L, K)$

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i.e., deriving simplexes meeting $|L|$ but not in L.

$L \Subset K'$ if K' is K derived near L.

Why?

If K' is K derived near L,
$N(L, K')$ is a
derived neighborhood of L in K.

Theorem

*If K_1 and K_2 are deriveds of K near L,
then $s : K_1 \to K_2$
throws $N(L, K_1)$ onto $N(L, K_2)$
and is the identity on $L \cup C(L, K)$.*

Regular neighborhoods

Suppose $X \subset Y$ are polyhedra.
$|K|$ a neighborhood of X in Y,
$|L| = X$
$L \Subset K$
K' derived of K near L.

$|N(L, K')|$ is called a **regular neighborhood**

Theorem

If N_1 and N_2 are regular neighborhoods of X in Y,
then there is a homeomorphism $h : Y \to Y$
which throws N_1 onto N_2, and
which is the identity on X
and the identity outside a compact subset of Y.

Regular neighborhoods in manifolds

Theorem

A regular neighborhood N of a polyhedron X in a manifold M is a manifold with boundary.

Theorem (Simplicial neighborhood theorem)

X a compact polyhedron
M a manifold
$X \subset \operatorname{int} M$
N a neighborhood of X in $\operatorname{int} M$.
Then N is a regular neighborhood iff

- *N is a compact manifold with boundary*
- *there are triangulations (K, L, J) of $(N, X, \partial N)$ with $L \Subset K$, $K = N(L, K)$ and $J = \partial N(L, K)$.*

Theorem (SNT version 2)

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M a manifold
$X \subset \operatorname{int} M$
N a polyhedral neighborhood of X in $\operatorname{int} M$.
Then N is a regular neighborhood iff

- *N is a compact manifold with boundary*
- *$N \searrow X$.*

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e.g., if $X \searrow$ point, N is a ball.

e.g., a collapsible manifold is a ball.

Corollary

If $B^n \subset S^n$, *then*
$\mathrm{cl}(S^n - B^n)$ *is a ball.*

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Proof.

$S^n = \partial \Delta^{n+1}$.
A point $p \in B^n$ is contained in some $\Delta^n \subset \Delta^{n+1}$,
and both Δ^n and B are regular neighborhoods of P.
Regular neighborhoods are unique. $\square$

next time
we will introduce the work of

E. C. Zeeman