Topology of Piecewise-Linear Manifolds

Jim Fowler

Lecture 7
Summer 2010

Regular neighborhoods

Suppose $X \subset M$.

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But there is an "essentially unique" regular neighborhood.

E. C. Zeeman

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Goal: *"Unknotting combinatorial balls"*

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a paper in the **Annals**

Main result

Theorem

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Corollary

Any embedding $S^1 \subset S^4$ is unknotted.

Some background

Can S^1 be knotted in S^3?

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What does *knotted* even mean?

One point compactification

$$\mathbb{R}^1 \cong S^1 - \text{point}$$

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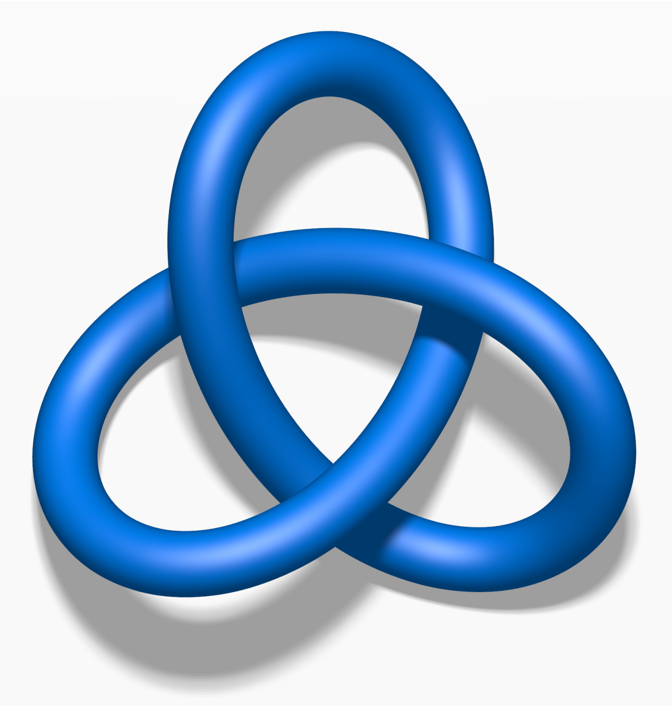
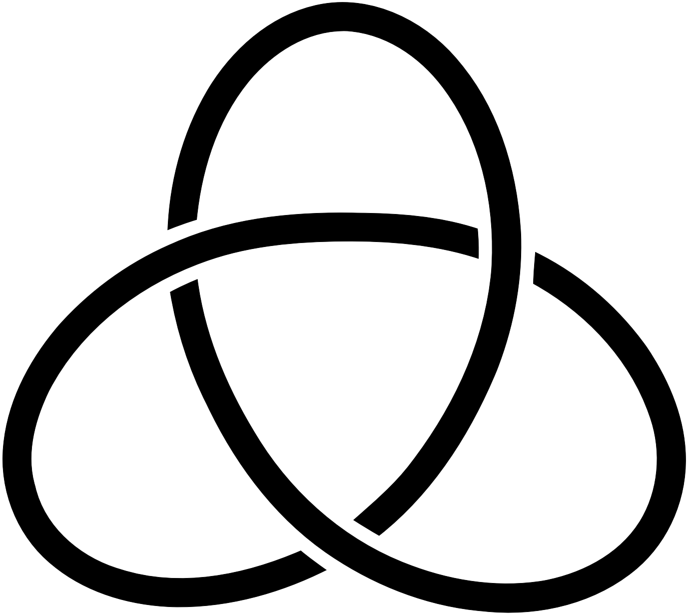
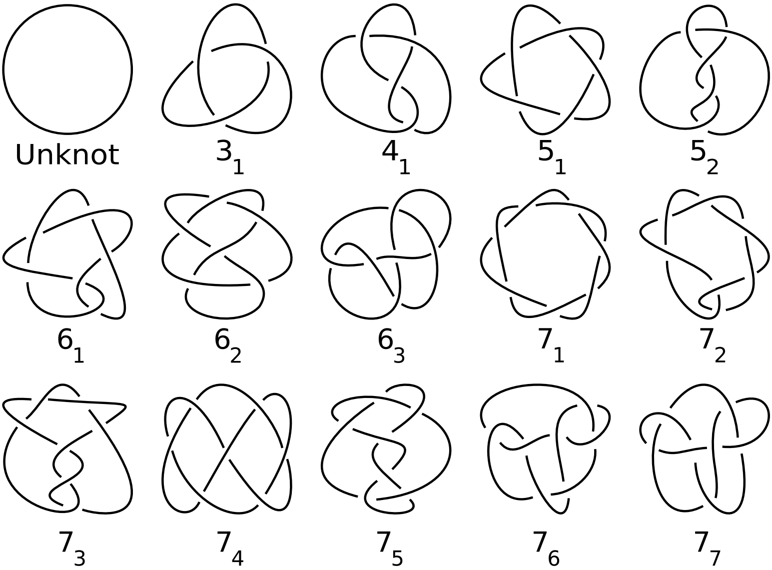
So we can draw $S^1 \subset S^3$ as if the circle were in $\mathbb{R}^3$.

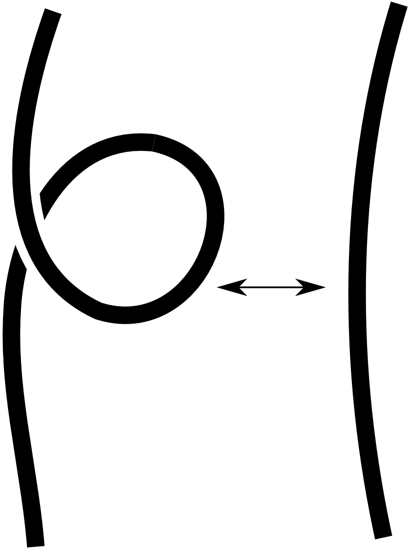




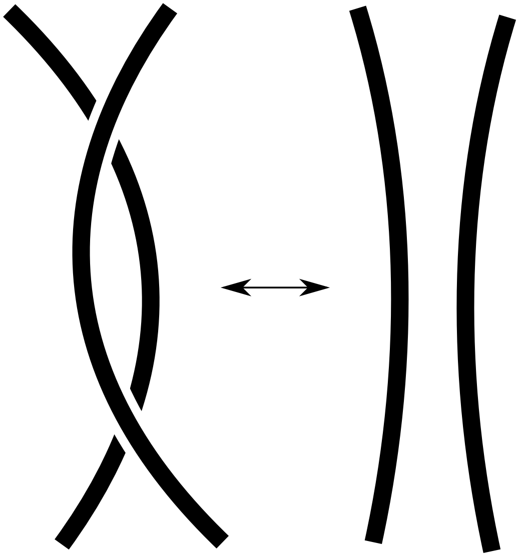


Unknot
3_1
4_1
5_1
5_2
6_1
6_2
6_3
7_1
7_2
7_3
7_4
7_5
7_6
7_7

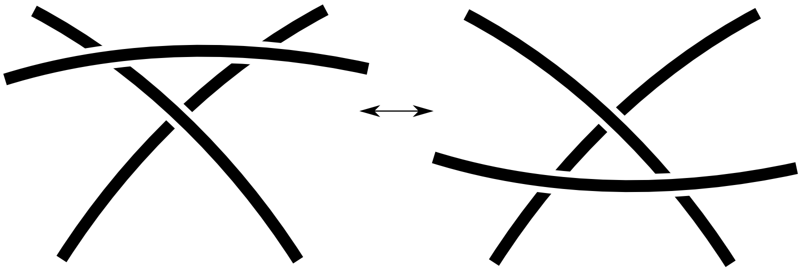
Reidemeister Move—Type I



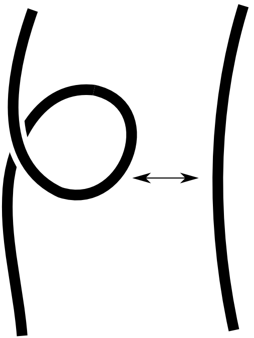
Reidemeister Move—Type II

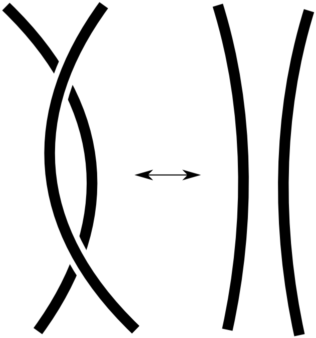


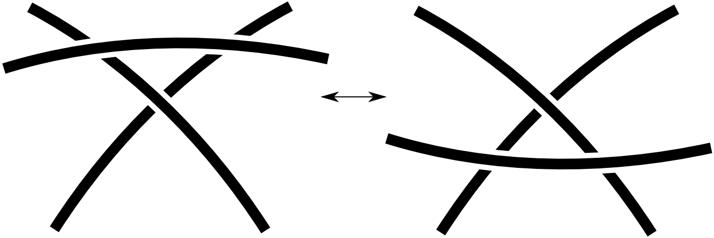
Reidemeister Move—Type III



Reidemeister Moves





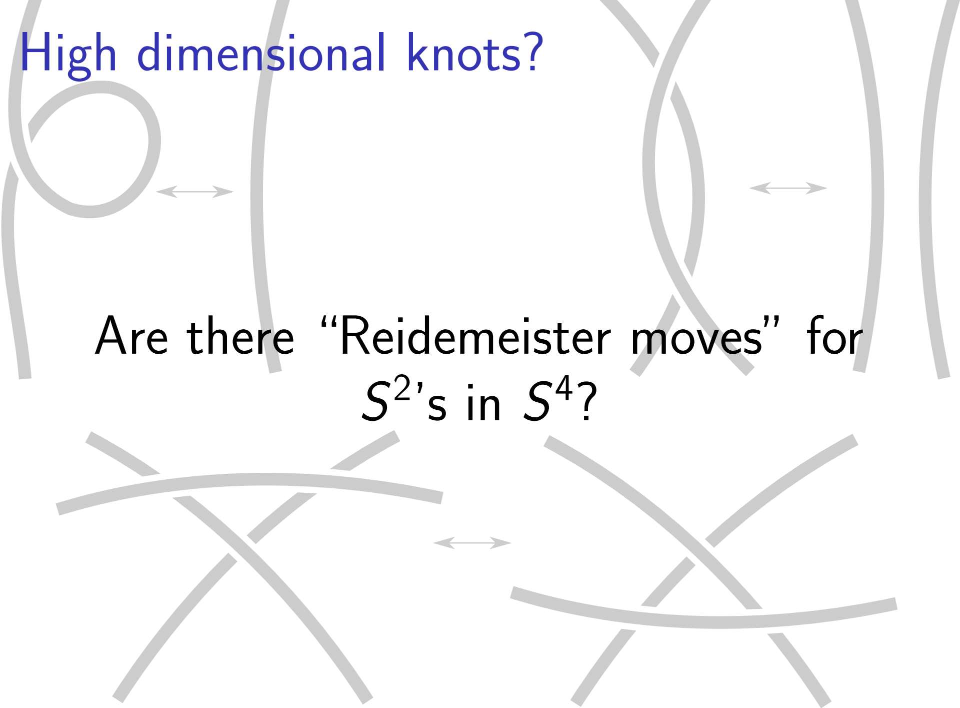


Knots

Theorem

*Two knots are the same
if a diagram for the one
can be transformed into the other
via Reidemeister moves.*

High dimensional knots?



Are there "Reidemeister moves" for S^2's in S^4?

Pairs

Definition

“(p, q)-sphere pair” means $S^q \subset S^p$.
We sometimes write (S^p, S^q).

“(p, q)-ball pair” means $B^q \subset B^p$,
with B^q properly embedded in B^p,
meaning $\partial B^q \subset \partial B^p$ and $\operatorname{int} B^q \subset \operatorname{int} B^p$.
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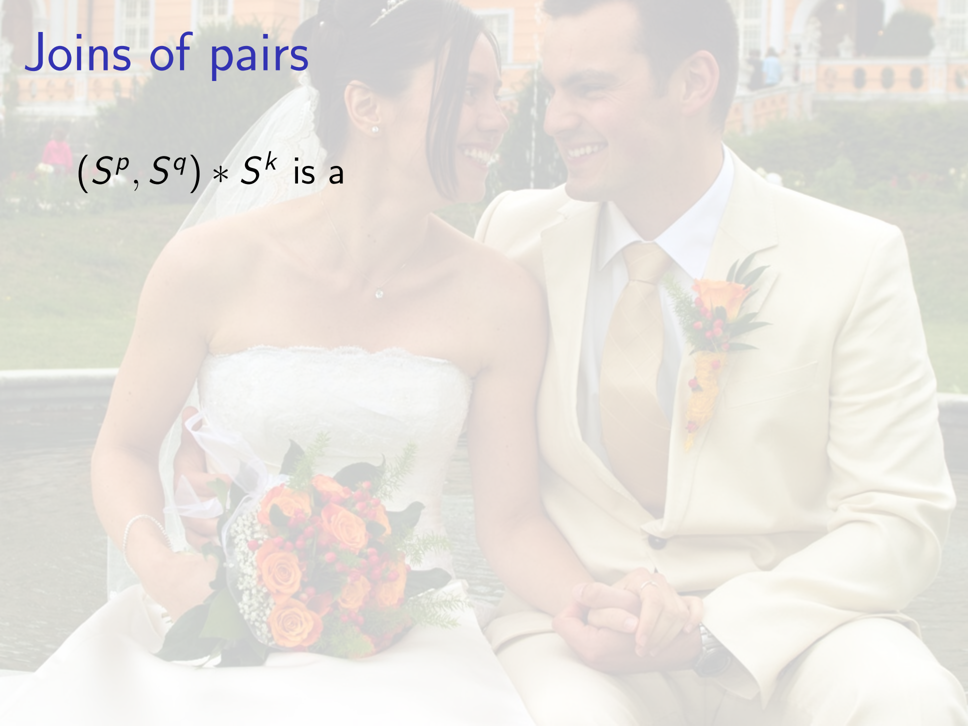
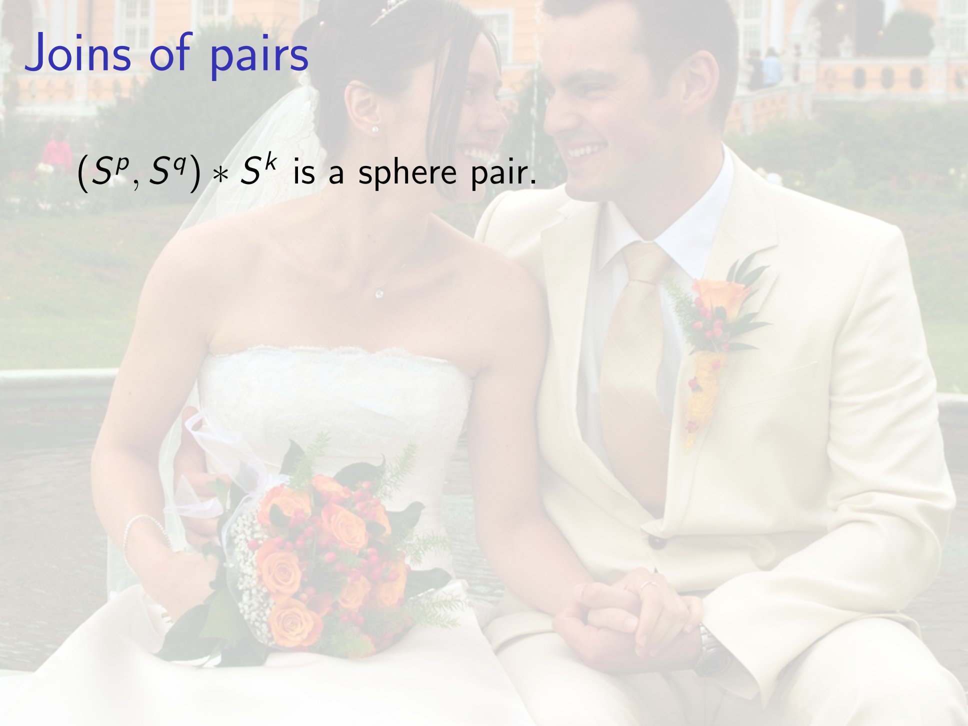
$\partial(B^p, B^q) = (S^{p-1}, S^{q-1})$.

Joins of pairs

$(S^p, S^q) * S^k$ is a





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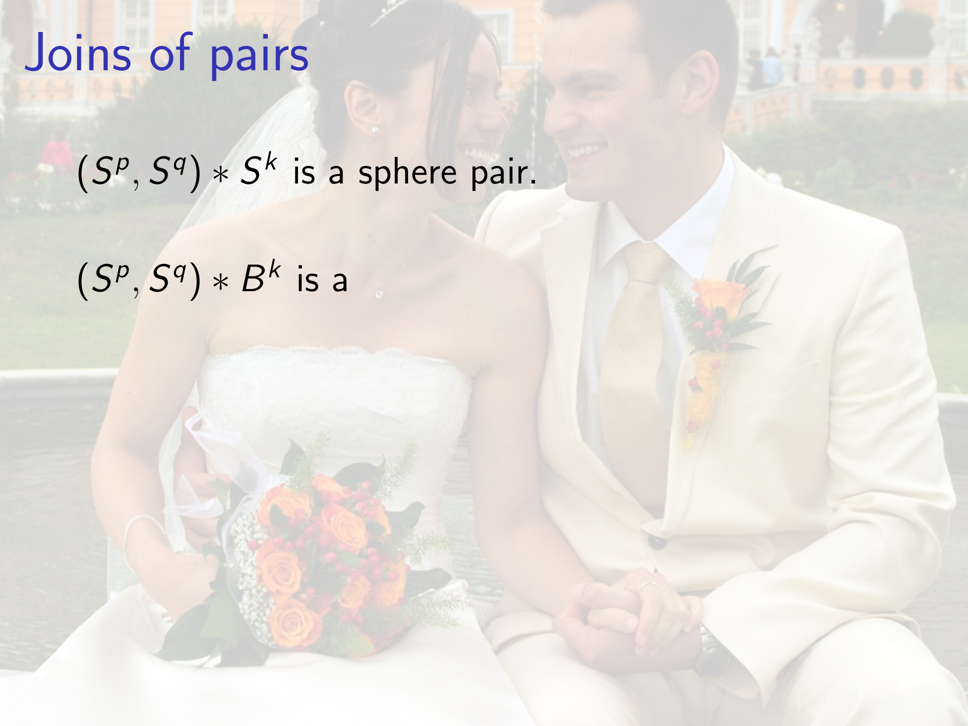

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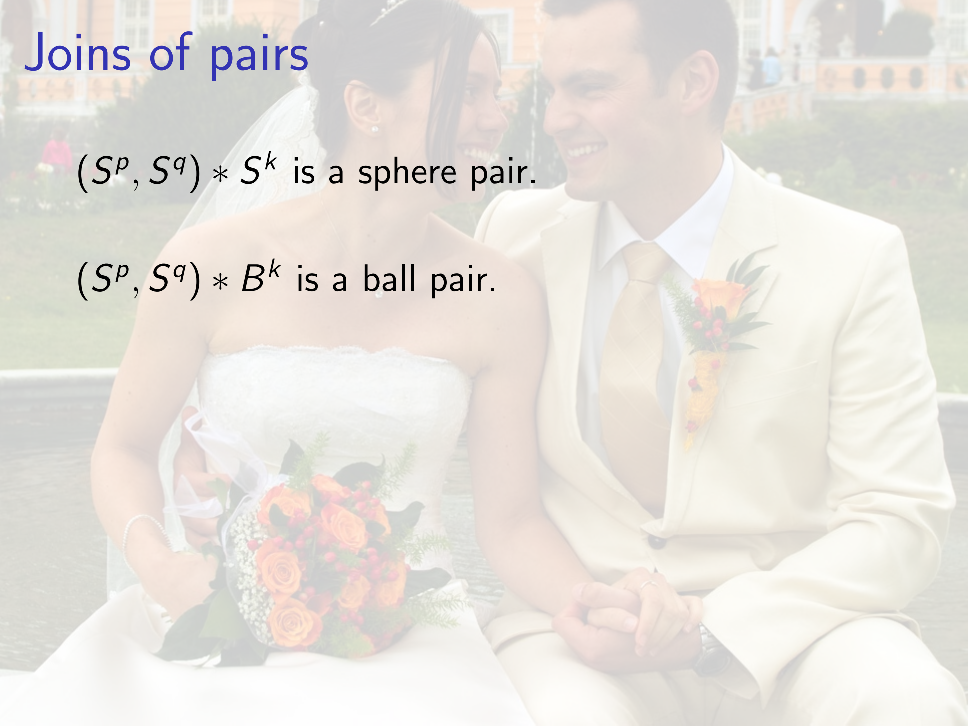

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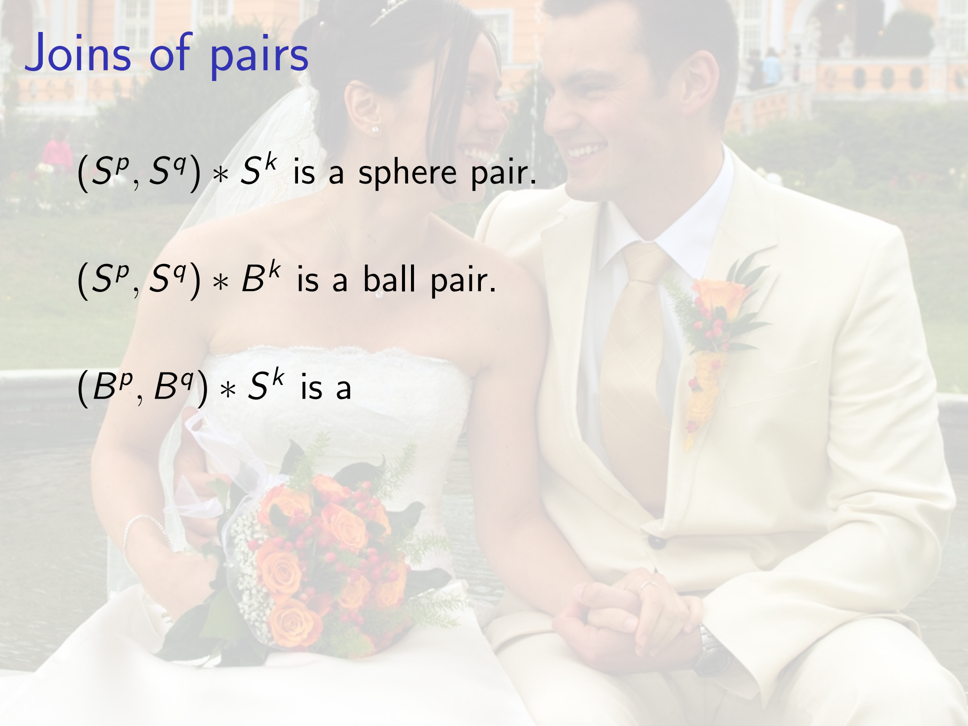

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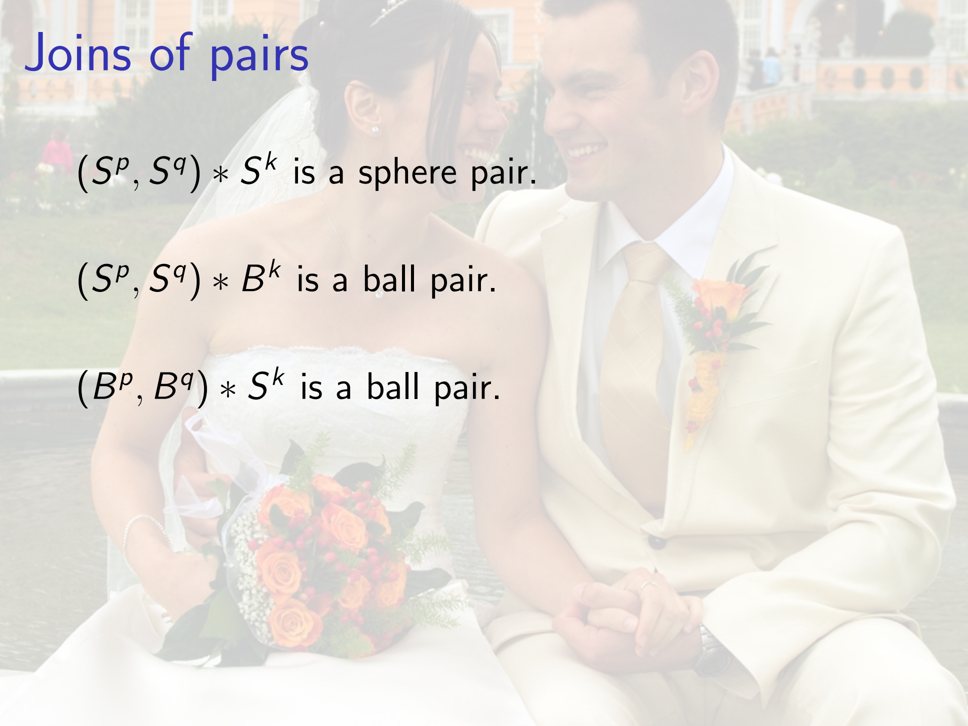

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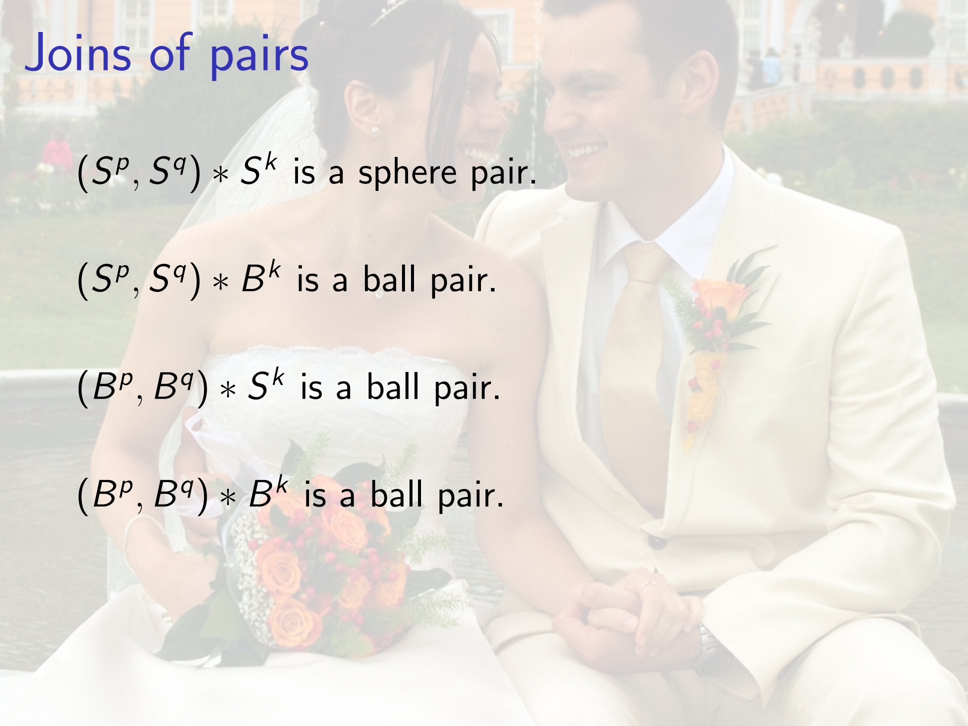

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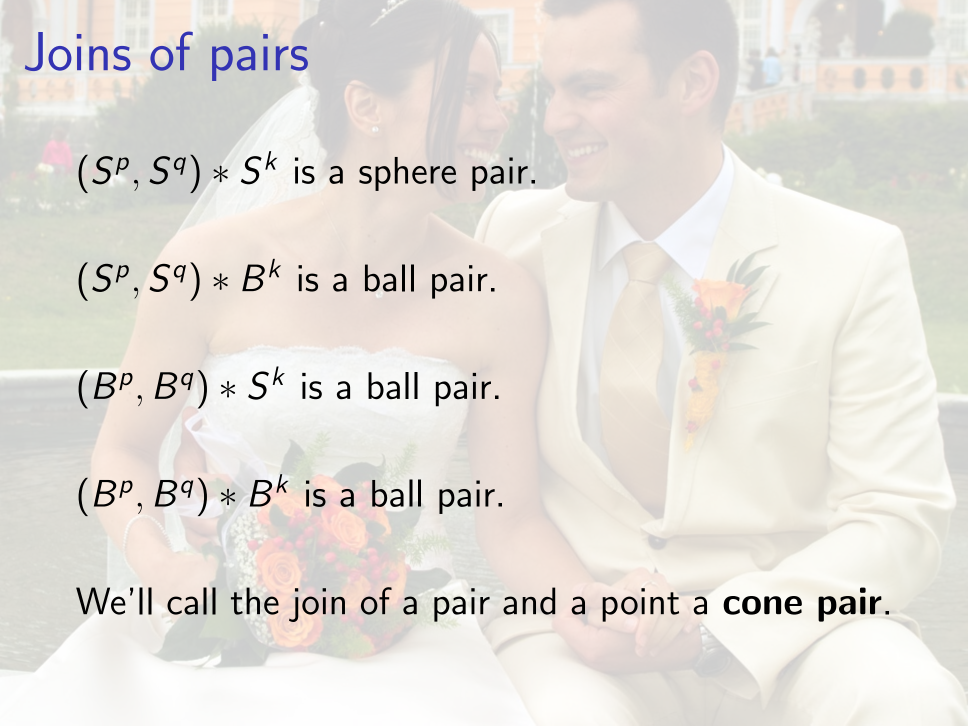

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We'll call the join of a pair and a point a **cone pair**.

Subpairs

$X = (X^p, X^q)$ and $Y = (Y^r, Y^s)$ are pairs
we say Y is a subpair of X
(written $Y \subset X$ or $X \supset Y$)
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If $P = (S^p, S^q) \supset Q = (B^p, B^q)$, then

$$P - \operatorname{int} Q = (S^p - \operatorname{int} B^p, S^q - \operatorname{int} B^q)$$

is a ball pair (via regular neighborhood machinery).

Faces of pairs

If $Q' = (B^{p-1}, B^{q-1})$ is
contained in the boundary of $Q = (B^p, B^q)$,
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Standard pairs

$\Gamma^{p,q} = (S^{p-q}\Delta^q, \Delta^q)$ is the standard (p,q)-ball pair.
$\partial\Gamma^{p+1,q+1}$ is the standard (p,q)-sphere pair.

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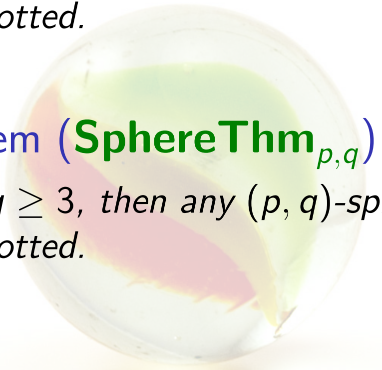
A pair is **unknotted**
if it is homeomorphic to a standard pair.

Theorem ($\mathbf{BallThm}_{p,q}$)

If $p - q \geq 3$, then any (p, q)-ball pair is unknotted.

Theorem ($\mathbf{SphereThm}_{p,q}$)

If $p - q \geq 3$, then any (p, q)-sphere pair is unknotted.



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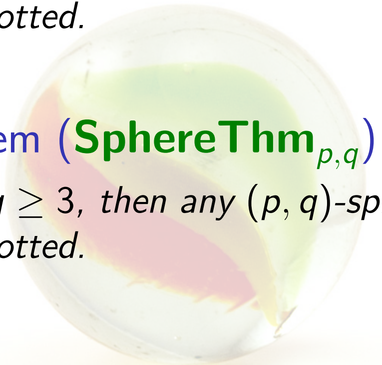

Theorem ($\mathbf{SphereThm}_{p,q}$)

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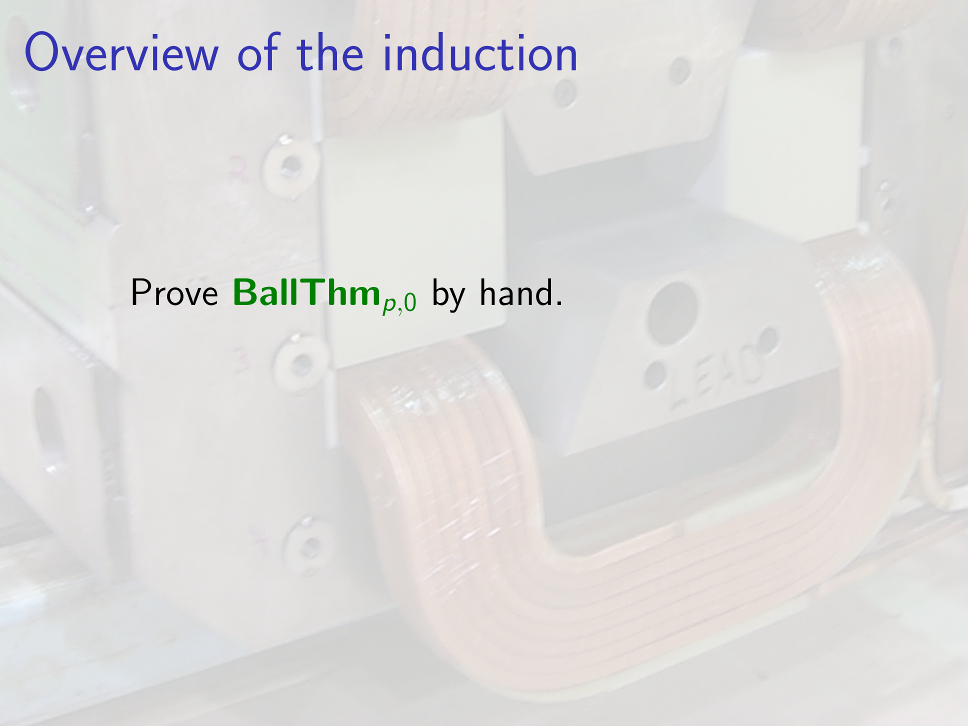
Proof.

By induction. $\square$

Overview of the induction



Prove **BallThm**$_{p,0}$ by hand.

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together imply **BallThm**$_{p,q}$.

Base case

Lemma

$\mathbf{BallThm}_{p,0}$ *is true.*

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Proof.

A ball B^p with a marked point B^0
is homeomorphic
to any other such
(via regular neighborhood theory). $\square$

Overview of the induction

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BallThm$_{p,q}$ $\Rightarrow$ **SphereThm**$_{p,q}$

Proof.
$P = (S^p, S^q)$. Choose vertex $x \in S^q$.
$P = Q \cup \{x\} * \partial Q$,

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Extend $Q \cong \Gamma^{p,q}$ to
a homeomorphism $P \cong \partial\Gamma^{p+1,q+1}$. $\square$

Overview of the induction

- ✓ Prove $\textbf{BallThm}_{p,0}$ by hand.
- $\textbf{BallThm}_{p,q}$ implies $\textbf{SphereThm}_{p,q}$.
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BallThm$_{p-1,q-1}$ and **SphereThm**$_{p-1,q-1}$ together imply **BallThm**$_{p,q}$.

The last step

$$\textbf{BallThm}_{p-1,q-1} \text{ and } \textbf{SphereThm}_{p-1,q-1}$$
$$\Downarrow$$
$$\textbf{BallThm}_{p,q}$$

This will require more machinery,
building on simplicial collapse
and regular neighborhoods.

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Lemma

If $p - q \geq 3$ *and* (B^p, B^q) *is any ball pair, then* $B^p \searrow B^q$.

But first. . .

Before we can proceed,
we will prove a couple of helpful lemmas.

Lemma

Q_1 and Q_2 are unknotted (p, q)-ball pairs.
Any homeomorphism $h : \partial Q_1 \xrightarrow{\cong} \partial Q_2$
extends to a homeomorphism $h' : Q_1 \xrightarrow{\cong} Q_2$.

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Proof (via *Alexander trick*).

$y =$ point in interior of Δ^q;
then since Q_i is unknotted, we have maps
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$h' : Q_1 \xrightarrow{f_1} \{y\} * \partial\Gamma^{p,q} \xrightarrow{\text{cone}\, g} \{y\} * \partial\Gamma^{p,q} \xrightarrow{{f_2}^{-1}} Q_2$ □

In short, **radial extension**.

Lemma (assume **BallThm**$_{p-1,q-1}$)

If Q_1, Q_2 are unknotted (p, q)-ball pairs,
and $Q_3 = Q_1 \cap Q_2$ is a face,
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Proof.

Choose $Q_3 \xrightarrow{\cong} \Gamma^{p-1,q-1}$, extend over $\partial Q_1 - \operatorname{int} Q_3$
to $h : \partial Q_1 \xrightarrow{\cong} \Gamma^{p-1,q-1} \cup C\partial\Gamma^{p-1,q-1}$,

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h extends to $Q_1 \rightarrow C\Gamma^{p-1,q-1}$

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Similarly produce $Q_2 \longrightarrow C\Gamma^{p-1,q-1}$

Glue together $Q_1 \cup Q_2 \cong S\Gamma^{p-1,q-1}$. $\square$

Regular neighborhoods

M, an n-manifold, $X \subset M$ a polyhedron
a **regular neighborhood** of X in M
is a subpolyhedron $N \subset M$ such that

- N is a closed neighborhood of X
- N is an n-manifold
- $N \searrow X$.

Theorem

If N_1 and N_2 are regular neighborhoods of $X \subset M$, there's a homeomorphism $N_1 \longrightarrow N_2$ keeping X fixed.

Lemma

Assuming $\mathbf{BallThm}_{p-1,q-1}$ *and* $\mathbf{SphereThm}_{p-1,q-1}$*,* (B^p, B^q) *with* $p - q \geq 3$ *is unknotted provided* $B^p \searrow B^q$*.*

Lemma

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Warning: The lemma is false if $p - q = 2$.
If $(B^4, B^2) = \text{cone}(S^3, S^1)$,
then $B^4 \searrow B^2$ because cones collapse to a subcone

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Lemma

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$Q_0 = \{x\} * L$ where $L = (\mathrm{lk}(x, B^p), \mathrm{lk}(x, B^q))$,
and L is unknotted by either $\mathbf{BallThm}_{p-1,q-1}$ or $\mathbf{SphereThm}_{p-1,q-1}$.

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$K_i - K_{i-1}$ consists of a principal simplex A with a free face C.

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If $p - q \geq 3$ and (B^p, B^q) is any ball pair, then $B^p \searrow B^q$.

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Proof Technique.

Sunny collapse. $\square$

STOP