Lecture 29: More differential equations

Math 153 Section 57

Wednesday December 3, 2008

Following chapter 9.2.
It is the end.
Gottfried Leibniz and Isaac Newton's epic battle—commemorated in cookies.
Review sheet for differential equations.
Solution curves for first order equations
Symmetries in solving differential equations:

Factoring the operator

What about something like

$$f''(x) - 2f'(x) + f(x) = 0$$

This is $(\frac{d}{dx} - 1)\, f(x) = 0$, so e^x is a solution. But is there another?

Yes! $x\, e^x$.

In general, an n-th order linear homogeneous equation has n "distinct" solutions.

Factor the operator—do not fear this. This is just taking a complicated procedure, and breaking it up into pieces, each of which can be done in order (and, in fact, in any order!).

Factors of the form $(\frac{d}{dx} - r)$ contribute a e^{rx}.

Factors of the form $(\frac{d}{dx} - r)^n$ contribute a e^{rx}, and xe^{rx}, x^2e^{rx}, through $x^{n-1}e^{rx}$.

Knowing that this is everything requires some fancier stuff ("determinants, Wronksians, linear algebra").

Encoding Fibonacci numbers

Shift operator

$$(S^2 - S - 1)f_n = 0$$

But $x^2 - x - 1$ has solutions $x = (1 \pm \sqrt{5})/2$. Call these $A \approx -0.62$ and $B \approx 1.62$.

So we can write this as

$$(S - A)(S - B)f_n = 0$$

What sorts of sequences have $(S - A)f_n = 0$? Have $(S - B)f_n = 0$?

So $CA^n + DB^n = f_n$.

If $f_0 = 1$, then we want $C + D = 1$.

If $f_1 = 1$, then we want $CA + DB = 1$. But $A = 1 - B$ and $C = 1 - D$, so

$$(1 - D)(1 - B) + DB = 1$$

so

$$D = \frac{1}{\sqrt{5}}\rho$$

and so

$$f_n = (1 - \sqrt{5}\rho)(1 - \rho)^n - \frac{1}{\sqrt{5}}\rho\rho^n$$

Qualitative questions about solutions

Suppose $f''(x) = -f(x)$.

Can you tell that f is periodic? That f is bounded?

Find a conserved quantity!

$f'(x)^2 + f(x)^2$ is constant. How do we know? Differentiate!

$$\frac{d}{dx}\left(f'(x)^2 + f(x)^2\right) = 2f''(x)f'(x) + 2f'(x)f(x) = 0$$

More than studying solutions: we know that f must be bounded. We have discovered conservation of energy—as a consequence of the equation.

More importantly, we have discovered that sine and cosine are related by the Pythagorean identity! At last, the "circle" is closed.

More complicated example

What next?

Hand out.

Course evaluations

Farewell