Lecture 25: Partial fractions

Math 153 Section 57

Monday November 24, 2008

Overview

Recall that the limit comparison test allows us to determine whether

$$\sum \frac{\text{polynomial}}{\text{polynomial}}$$

converges. These quotients of polynomials are called rational functions.

Rational functions are maybe the "next simplest" class o f functions after polynomials (which we understand very well).

Derivative of a rational function is a rational function, so we can take all the derivatives we would like—no problem!

But to antidifferentiate a rational function? For this, we use **partial fractions**. This is, in fact, how the computers do these integrals.

Theory

We can integrate polynomials.

What are some rational functions we can integrate?

$$\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} + C$$

or somewhat more generally, for $n > 1$,

$$\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a} \cdot \frac{(ax+b)^{-n+1}}{-n+1} + C$$

For quadratics:

$$\int \frac{x}{x^2+b^2}\,dx = \frac{1}{2}\log\left(x^2+b^2\right) + C$$

or even

$$\int \frac{x-a}{(x-a)^2+b^2}\,dx = \frac{1}{2}\log\left((x-a)^2+b^2\right) + C$$

We can also do

$$\int \frac{1}{(x-a)^2+b^2}\,dx = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right)+C$$

We can also do these if the denominators are raised to a power (with more difficulty).

These functions suffice to do all rational functions. But we need a machine for transforming the general rational function into something of this form. What would that be?

Enter partial fractions

["I would like to see a proof of such and such." "How many alphabets are you familiar with?"]

The theorem. Let $f(x) = p(x)/q(x)$, a ratio of polynomials. We can assume that q's leading term has a 1 coefficient.

The fundamental theorem of algebra: factor denominator as

$$q(x) = \cdot \prod_{i=1}^{m}(x-a_i)^{j_i} \cdot \prod_{i=1}^{n}(x^2+b_ix+c_i)^{k_i}$$

that is, into linear and quadratic terms.

Then we can write $f(x)$ as

$$f(x) = \frac{p(x)}{q(x)} = P(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r}$$

This seems pretty awful. How can we ever hope to find all these constants?

An example

Suppose

$$\frac{1}{(x-3)^2(x-2)} = \frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{x-2}$$

The least clever (and most involved) way is to multiply through by the denominator:

$$1 = A(x-2)(x-3)+B(x-2)+C(x-3)^2$$

This gives

$$1 = (A+C)x^2+(-5A+B-6C)x+(6A-2B+9C)$$

So $A+C=0$, and $-5A+B-6C=0$, and $6A-2B+9C=1$. Solve the three equations in three unknowns (the resulting equations will always have a unique solution). Get $A=-C$. So $5C+B-6C=0$, so $B-C=0$, so $B=C$. Then, $-6C-2C+9C=1$, so $C=1$, and therefore $A=-1$ and $B=1$.

Check!

$$\frac{1}{(x-3)^2(x-2)} = \frac{-1}{x-3} + \frac{1}{(x-3)^2} + \frac{1}{x-2}$$

This is true.

This lets us do the integral

$$\int \frac{1}{(x-3)^2(x-2)}\,dx$$

because you can do

$$\int \left(\frac{-1}{x-3} + \frac{1}{(x-3)^2} + \frac{1}{x-2}\right)dx$$

So many problems

The problems involved here are many.

You are not likely to see $(x-3)^2(x-2)$. You are more likely to see

$$x^3 - 8x^2 + 21x - 18$$

How are you supposed to **solve the factorization problem**?

A tool in our toolbox, not a rule for solving all problems!

It is silly for me to give you lists of rules.

A second example

$$f(x) = \frac{x^3+16}{x^3-4x^2+8x}$$

Do long division

$$f(x) = 1 + \frac{4x^2-8x+16}{x^3-4x^2+8x} = 1 + \frac{4x^2-8x+16}{x(x^2-4x+8)}$$

$$\frac{4x^2-8x+16}{x(x^2-4x+8)} = \frac{A}{x} + \frac{Bx+C}{x^2-4x+8}$$

Multiply through by $x(x^2-4x+8)$ to get

$$4x^2-8x+16 = A(x^2-4x+8) + (Bx+C)x$$

So $A=2$. So $B=2$. And $C=0$. Therefore,

$$f(x) = 1 + 2\left(\frac{1}{x} + \frac{x}{x^2-4x+8}\right)$$

Some tricks

Multiply the partial fraction decomposition by $(x - a_i)$ and take the limit $x \to a_i$. Then only the term $A_{i,1}$ survives, so

$$A_{i,1} = \lim_{x \to a_i} (x - a_i) f(x)$$

In the case of

$$f(x) = \frac{1}{(x-3)^2(x-2)}$$

this tells us that

$$\lim_{x \to 2} (x-2) \cdot \frac{1}{(x-3)^2(x-2)} = 1$$

but of course is useless for to find the coefficients on $1/(x-3)$ and $1/(x-3)^2$.

We can find

$$\lim_{x \to 3} (x-3)^2 \cdot \frac{1}{(x-3)^2(x-2)} = 1$$

to discover that the coefficient on $1/(x-3)^2$ is 1.

Then we know that

$$\frac{1}{(x-3)^2(x-2)} = \frac{1}{x-2} + \frac{1}{(x-3)^2} + \frac{A}{x-3}$$

We could plug in a value like $x = 4$ to find:

$$\frac{1}{(4-3)^2(4-2)} = \frac{1}{4-2} + \frac{1}{(4-3)^2} + \frac{A}{4-3}$$

which becomes

$$\frac{1}{2} = -\frac{1}{2} + \frac{1}{1^2} + \frac{A}{1}$$

so $A = -1$.

Underview

Some some tricks include: taking limits, and plugging in values.