Dog on the beach

Math 1181H

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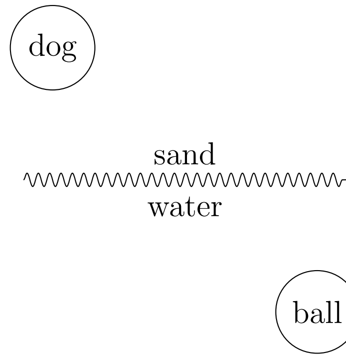
Yesterday, we talked about the following problem.

> A beach runs perfectly east-west; a perfectly spherical dog is 5 m north of the edge of the water, and a ball floats motionlessly 10 m south and 10 m east of the dog; the dog is somewhat injured, and runs at speed of 1 m/s on the sand, and swims at a speed of s m/s. Along what path should the dog travel to get to the ball as quickly as possible?



There are two ways of approaching this problem.

Via algebra and the Pythagorean theorem

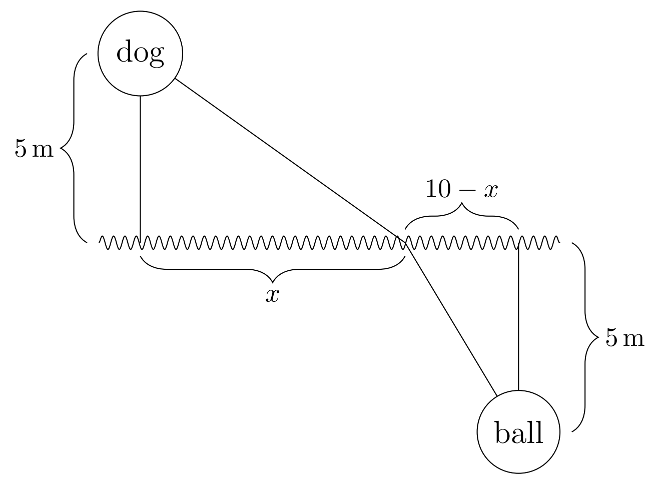
Suppose the dog enters the water x m along the coastline. Then, by the Pythagorean theorem, the total travel time in seconds is

$$f(x) = \sqrt{x^2+25} + \frac{\sqrt{(x-10)^2+25}}{s}.$$

Differentiating, we get

$$f'(x) = \frac{x}{\sqrt{x^2+25}} + \frac{x-10}{\sqrt{(x-10)^2+25}\,s}.$$

Now we look for a value of x which makes $f'(x)=0$.



To look for places where $f'(x)=0$, we want

$$\frac{x-10}{\sqrt{(x-10)^2+25}\,s} = -\frac{x}{\sqrt{x^2+25}}$$

Since $0 \leq x \leq 10$, we can rewrite both sides as a square root, co

$$\sqrt{\frac{(x-10)^2}{\left((x-10)^2+25\right)s^2}} = \sqrt{\frac{x^2}{x^2+25}},$$

and then take reciprocals to get

$$\sqrt{\frac{\left((x-10)^2+25\right)s^2}{(x-10)^2}} = \sqrt{\frac{x^2+25}{x^2}},$$

and then simplify what is under the radicals, to get

$$\sqrt{\left(\frac{25}{(x-10)^2}+1\right)s^2} = \sqrt{\frac{25}{x^2}+1}.$$

Squaring both sides,

$$\left(\frac{25}{(x-10)^2}+1\right)s^2 = \frac{25}{x^2}+1.$$

Now move everything to one side, producing

$$-s^2 - \frac{25\,s^2}{(x-10)^2} + \frac{25}{x^2} + 1 = 0.$$

This I can then put over a common denominator, specifically

$$-\frac{(s^2-1)x^4 - 20\,(s^2-1)x^3 + 125\,(s^2-1)x^2 + 500\,x - 2500}{x^4 - 20\,x^3 + 100\,x^2} = 0,$$

which vanishes provided the numerator (which we'll $p(x)$) does, that is,

$$p(x) = -\left(s^2-1\right)x^4 + 20\left(s^2-1\right)x^3 - 125\left(s^2-1\right)x^2 - 500\,x + 2500 = 0.$$

We'll call the denominator $q(x) = x^4 - 20\,x^3 + 100\,x^2$ and note that $q(x)$ factors as $(x-10)^2x^2$, so $q(x)$ is nonnegative when $0 \le x \le 10$.

Recall our goal is to find x so that $f'(x) = 0$, and now it suffices that $p(x) = 0$, but $p(x)$ is a *quartic*, so, yes, it is solvable, but no, it won't be easy to solve. The quartic formula is terrible! If you think it will be easy, just try setting $s = 2\,\mathrm{m/s}$. In that case, the polynomial $p(x)$ is

$$-3\,x^4 + 60\,x^3 - 375\,x^2 - 500\,x + 2500$$

which does not factor into quadratics with rational coefficients. Our dog, even armed with a straightedge and compass, would have trouble figuring out the best place to cross into the water.

There are some things we can say. It's "physically" clear that this polynomial $p(x)$ can only have one root in the interval $[0, 10]$, so that

$$\begin{aligned} p(0) &= 2500 > 0, \\ p(5) &= -1250\,s^2 + 1250, \text{ and} \\ p(10) &= -2500\,s^2 < 0 \end{aligned}$$

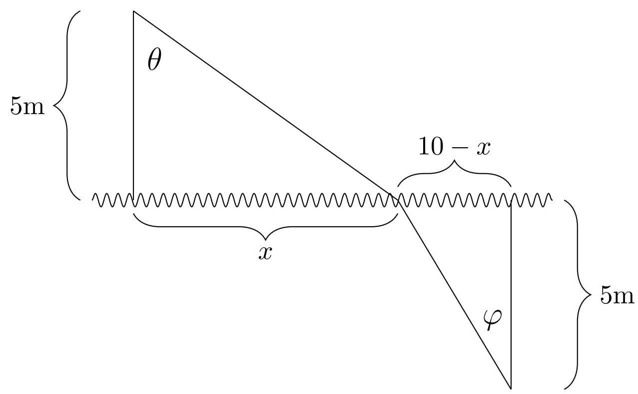
hold is qualitatively significant, in light of the intermediate value theorem; note that $p(5) < 0$ when $s > 1$ and $p(5) > 0$ when $s < 1$. This reflects the fact that when $s > 1$, the dog can swim faster than he can "run," so the root to $p(x)$ must lie between 0 and 5.

Via trigonometry

Perhaps you did not find the previous algebraic solution very enlightening. I don't blame you. Remember we differentiated to get

$$f'(x) = \frac{x}{\sqrt{x^2+25}} + \frac{x-10}{\sqrt{(x-10)^2+25s}}$$

and now we will reinterpret this in terms of trigonometry. Let's label a couple angles as θ and φ as shown below.



Then we can interpret the terms in $f'(x)$ via trigonometry. The first term,

$$\frac{x}{\sqrt{x^2+25}},$$

is $\sin\theta$. The second term,

$$\frac{x-10}{\sqrt{(x-10)^2+25s}},$$

is $\frac{-1}{s}\cdot\sin\varphi$. So we can write

$$f'(x) = \sin\theta - \frac{1}{s}\sin\varphi.$$

Now we are looking for an x which makes $f'(x) = 0$, meaning we want

$$s\cdot\sin\theta = \sin\varphi.$$

At this point, we've stumbled upon *Snell's law* describing refraction, but we still don't have instructions to give to the dog. There's a constraint we can express in terms of θ and φ: since $x = 5\tan\theta$ and $10 - x = 5\tan\varphi$, we must have

$$5\tan\theta + 5\tan\varphi = 10,$$

So for the dog to know which way to run, it suffices to solve the system of equations

$$\begin{aligned} s\cdot\sin\theta &= \sin\varphi \\ \tan\theta &= 2 - \tan\varphi \end{aligned}$$

which, superficially, looks a lot nicer than the terrible polynomial $p(x)$ we encountered earlier. But is it any better?

A concrete example

Let's set $s = 2$. In that case, the polynomial is

$$\begin{aligned} p(x) &= -\left(s^2-1\right)x^4 + 20\left(s^2-1\right)x^3 - 125\left(s^2-1\right)x^2 - 500\,x + 2500 \\ &= -3\,x^4 + 60\,x^3 - 375\,x^2 - 500\,x + 2500. \end{aligned}$$

We can try plugging in some values, such as

$$\begin{aligned} p(2) &= -3\,\cdot 2^4 + 60\,\cdot 2^3 - 375\,\cdot 2^2 - 500\,\cdot 2 + 2500 \\ &= 432 > 0 \\ p(3) &= -3\,\cdot 3^4 + 60\,\cdot 3^3 - 375\,\cdot 3^2 - 500\,\cdot 3 + 2500 < 0 \\ &= -998 \end{aligned}$$

and so the root lies between 2 and 3. More work would reveal that the root is between 2.3086 and 2.3087, that is, about 30/13. The actual value is

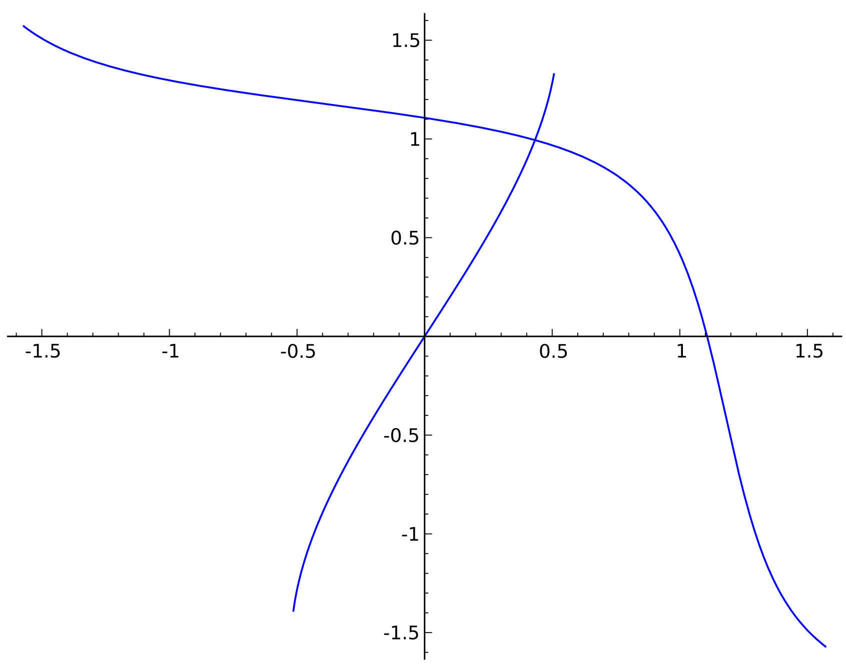
$$\frac{1}{2}\sqrt{-r^{\left(\frac{1}{3}\right)} + \frac{2500\,r^{\left(\frac{1}{6}\right)}}{\sqrt{9\,r^{\left(\frac{2}{3}\right)} + 150\,r^{\left(\frac{1}{3}\right)} + 15625}} - \frac{15625}{9\,r^{\left(\frac{1}{3}\right)}} + \frac{100}{3}} - \frac{\sqrt{9\,r^{\left(\frac{2}{3}\right)} + 150\,r^{\left(\frac{1}{3}\right)} + 15625}}{6\,r^{\left(\frac{1}{6}\right)}} + 5$$

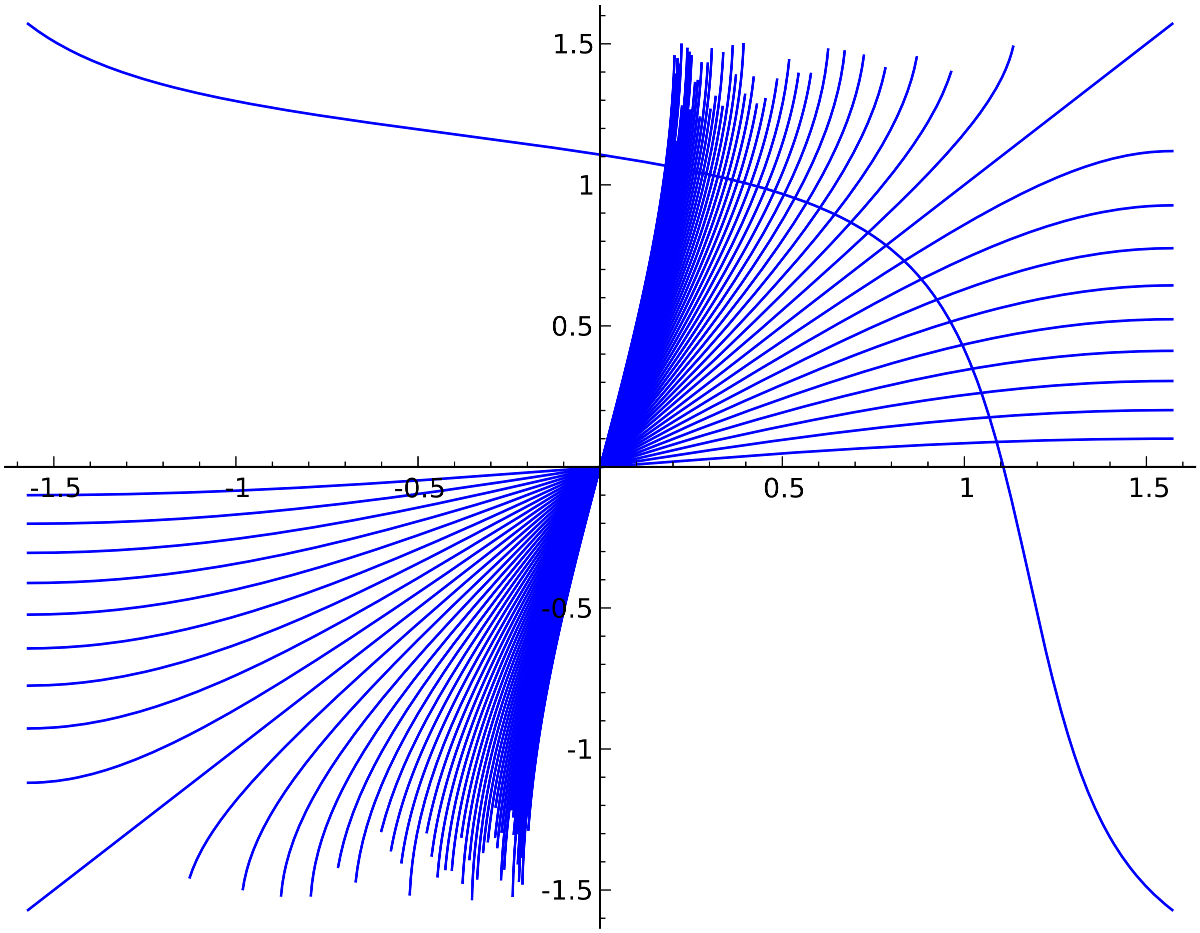
where $r = (125000\,\sqrt{3}\sqrt{173} + 3453125)/27$.

Perhaps it is easier to solve the system of trigonometric equations? We want to find a θ which satisfies

$$\begin{aligned} 2 \cdot \sin\theta &= \sin\varphi \text{ and} \\ \tan\theta &= 2 - \tan\varphi \end{aligned}$$

for some φ. One way to get a sense of this is to graph the points in the (θ,φ)-plane which satisfy these equations.

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