Topology of Piecewise-Linear Manifolds

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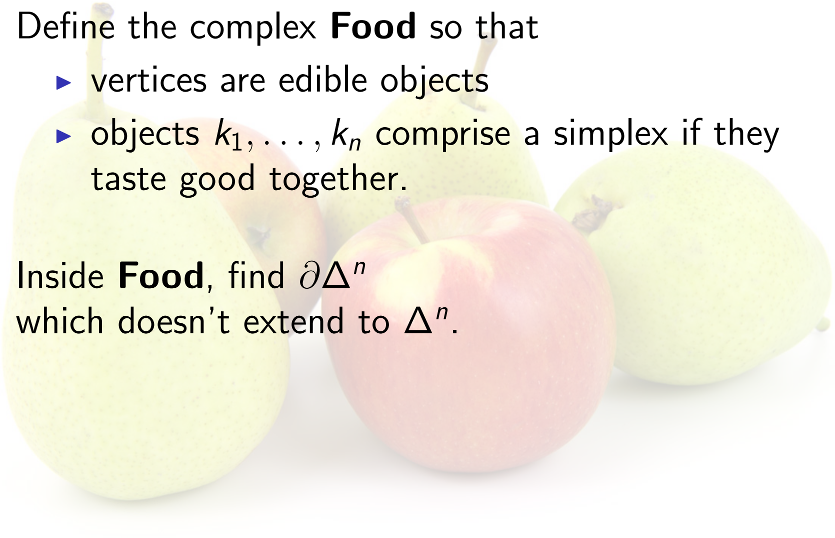
Lecture 3
Summer 2010

A simplicial complex joke

Define the complex **Food** so that

- vertices are edible objects
- objects $k_1, \ldots, k_n$ comprise a simplex if they taste good together.

A simplicial complex joke



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Inside **Food**, find $\partial\Delta^n$
which doesn't extend to Δ^n.

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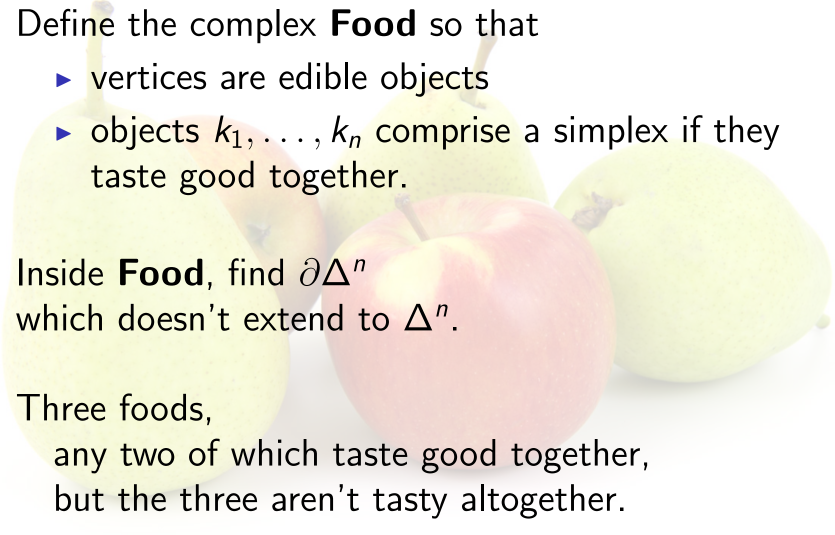

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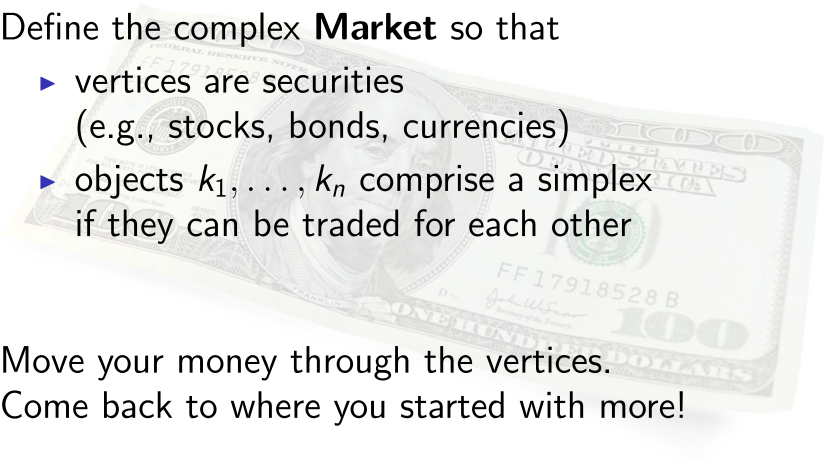
Three foods,
any two of which taste good together,
but the three aren't tasty altogether.

A simplicial complex which isn't funny

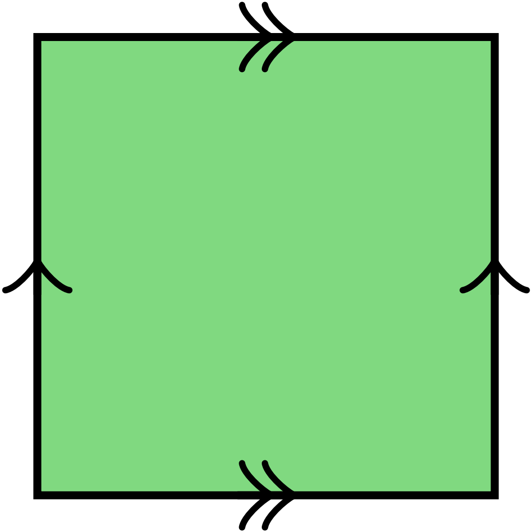


Define the complex **Market** so that

- vertices are securities
  (e.g., stocks, bonds, currencies)
- objects $k_1, \ldots, k_n$ comprise a simplex
  if they can be traded for each other

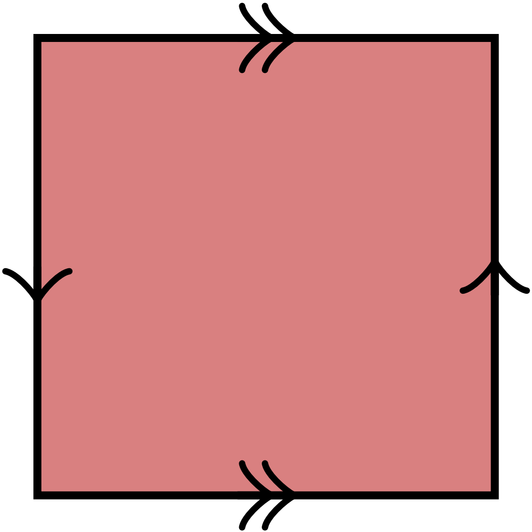
Move your money through the vertices.
Come back to where you started with more!

The torus

$T^2 =$ 

Klein Bottle

$K =$



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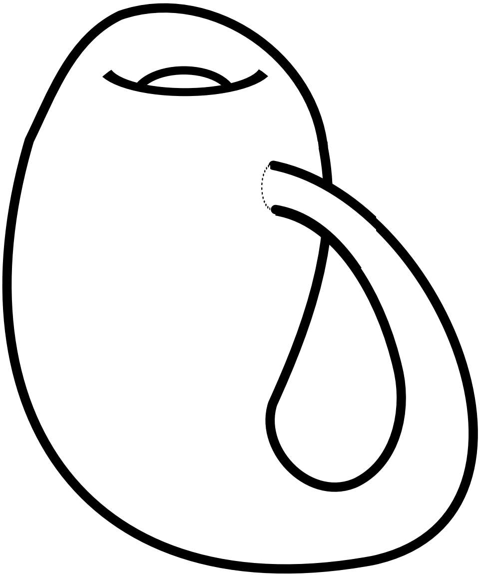
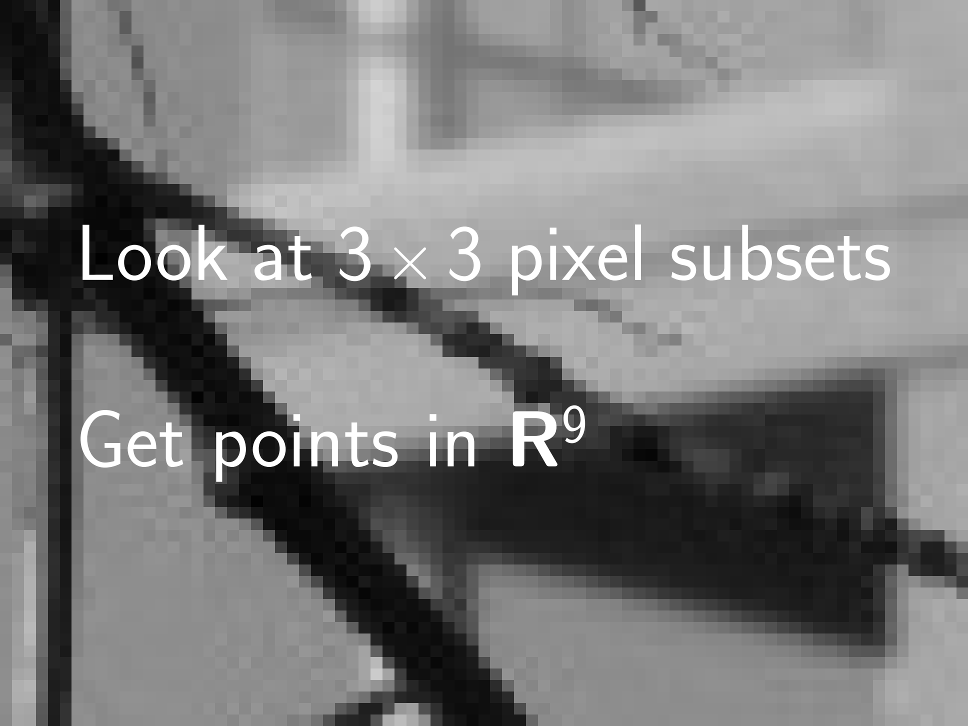
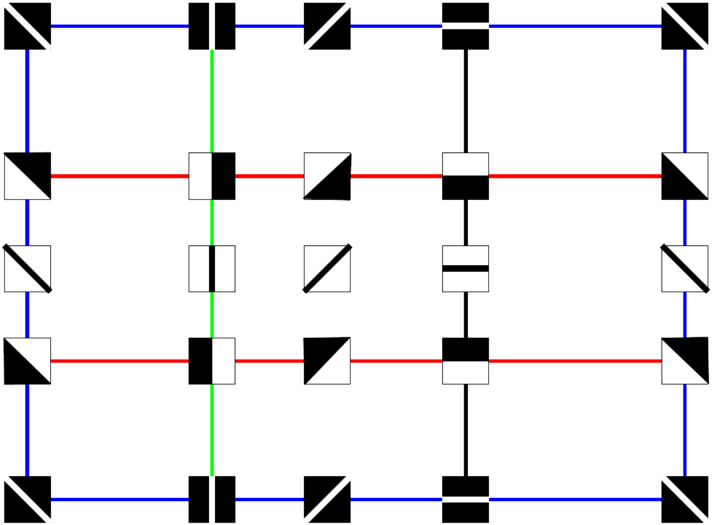



Look at 3×3 pixel subsets



Look at 3×3 pixel subsets

Get points in $\mathbf{R}^9$

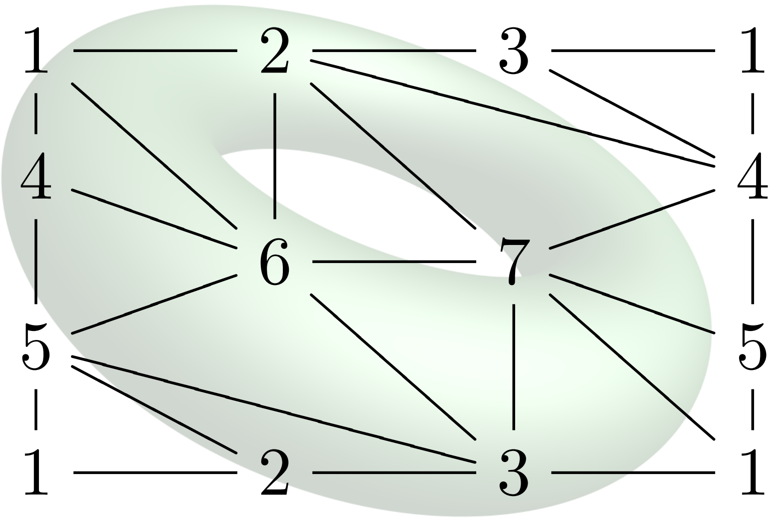


From a paper of Gunnar Carlsson and Tigran Ishkhanov

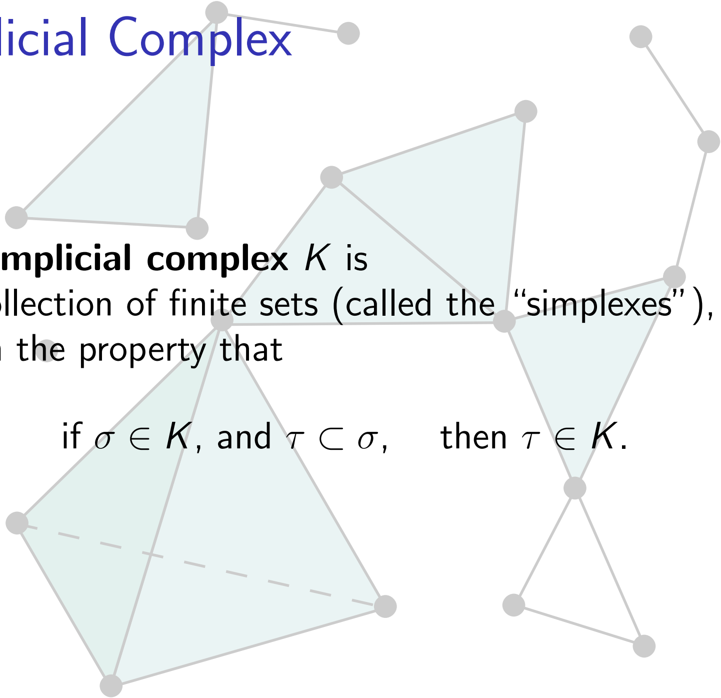
I'm a cheerleader for geometry!

Dynamics	and mixing taffy
Biology	and yeast
Chemistry	and isomers
Physics	and symmetry
Neurology	and Klein bottles
Economics	and "least action"
Engineering	and robots
Astronomy	and the shape of space
Linguistics	and document clustering

Triangulate a torus



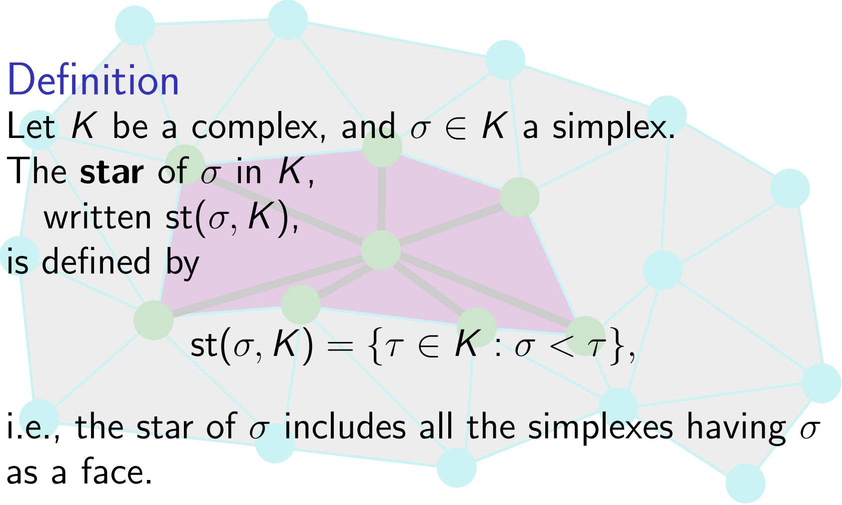
Simplicial Complex



A **simplicial complex** K is
a collection of finite sets (called the "simplexes"),
with the property that

$$\text{if } \sigma \in K, \text{ and } \tau \subset \sigma, \quad \text{then } \tau \in K.$$

Star



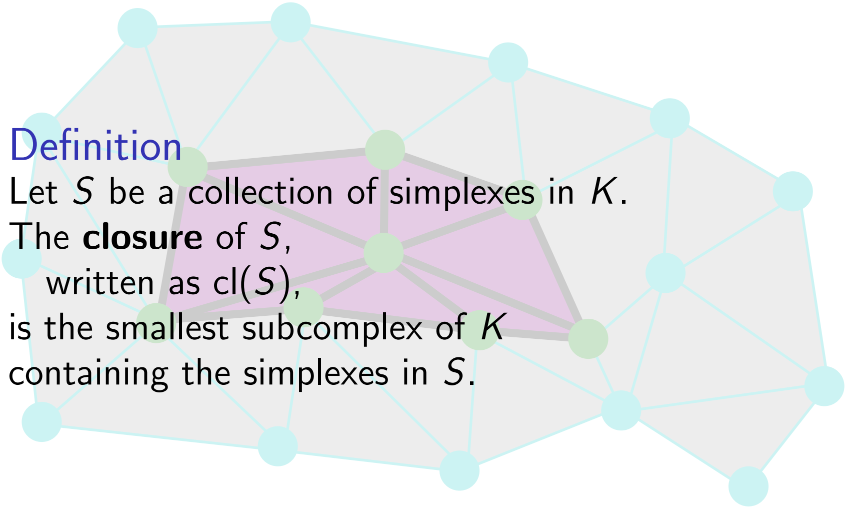
Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **star** of σ in K,
written $\mathrm{st}(\sigma, K)$,
is defined by

$$\mathrm{st}(\sigma, K) = \{\tau \in K : \sigma < \tau\},$$

i.e., the star of σ includes all the simplexes having σ as a face.

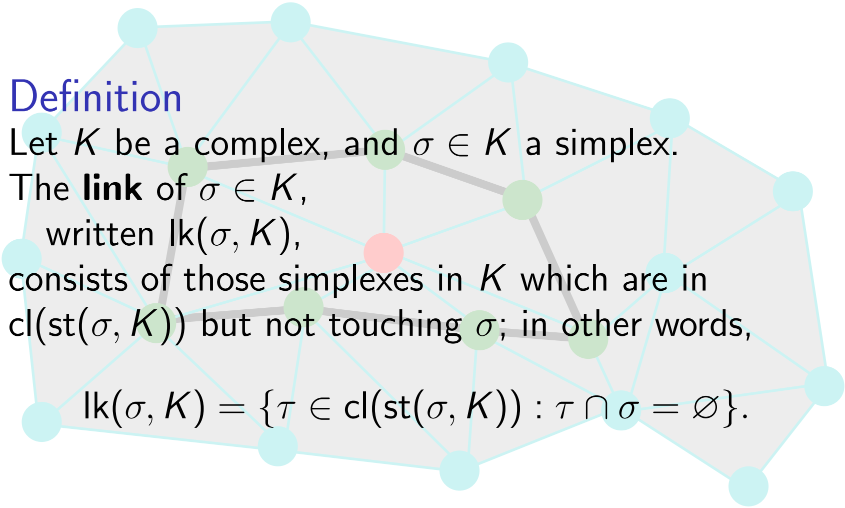
Closure



Definition

Let S be a collection of simplexes in K.
The **closure** of S,
written as $\mathrm{cl}(S)$,
is the smallest subcomplex of K
containing the simplexes in S.

Link



Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **link** of $\sigma \in K$,
written $\mathrm{lk}(\sigma, K)$,
consists of those simplexes in K which are in $\mathrm{cl}(\mathrm{st}(\sigma, K))$ but not touching σ; in other words,

$$\mathrm{lk}(\sigma, K) = \{\tau \in \mathrm{cl}(\mathrm{st}(\sigma, K)) : \tau \cap \sigma = \varnothing\}.$$

Stellar Subdivision

Definition

Let K be a complex, and $\sigma \in K$ a simplex.
The **stellar subdivision** of K at σ is a new complex K_σ with:

- the vertices of K with a new vertex v.
- the simplexes of K not in $\mathrm{st}(\sigma, K)$,
  along with the simplexes in $v * (\partial\sigma) * \mathrm{lk}(\sigma, K)$.

We might say:

$$K_\sigma := (K - \mathrm{st}(\sigma, K)) \cup (v * (\partial\sigma) * \mathrm{lk}(\sigma, K))$$

subdivision = repeated stellar subdivision

Definition

Let K, L be complexes.
If K can be produced through a (possibly empty) sequence of stellar subdivisions of L,
we say that K is a **subdivision** of L,
and write $K \triangleleft L$.

Piecewise linear maps

Definition

Let K, K', L, L' be complexes,
with $K' \triangleleft K$ and $L' \triangleleft L$.

If $f : K' \longrightarrow L'$ is a simplicial map,
we call $f : K \longrightarrow L$ a **piecewise linear map**
(or a **PL map** for short).

We call $f : K' \longrightarrow L'$ an **underlying simplicial map**.

Piecewise linear maps

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As we will see, the *real* definition of subdivision is more general than this.

from the last

homework

Goal. Problem Set 2 introduces *abstract simplicial complexes,* the main object of study. You should be warned that what follows is not the only way to formalize our intuition. As usual, **problems marked with a • should be written up and handed in.**

Definition. Geometrically, an n-dimensional **simplex** (written Δ^n, and usually called an n-simplex for short) is the n-dimensional analog of a triangle; just as a triangle is the smallest convex set containing 3 points which do not lie on a line, the n-simplex Δ^n is the smallest convex set containing $n+1$ points in "general position."

But for us, the important feature of a simplex is the relationship between the faces. Indeed, topologically, we do not care about the size of the simplexes, or where they are sitting in space—so we will abstract away the geometry, leaving only the combinatorics behind.

A simplex A is a **face** of a simplex B if the vertices determining A are a subset of the vertices determining B. We write $A < B$ if A is a face of B. Specifically, some v for which $\{v\} \in B$ is called a **vertex** of B.

A **simplicial complex** K is a collection of finite sets (called the simplexes) with the property that if $\sigma \in K$, and $\tau < \sigma$ (i.e., if $\tau \subset \sigma$ when thought of as finite sets), then $\tau \in K$.

In words, a simplicial complex is a collection of simplexes, where any face of a simplex is also in the complex. We can think of a simplicial complex as a geometric object by gluing together actual simplexes along their faces (the geometric realization) by following the pattern of the combinatorial data encoded by the finite sets.

Anytime we define a mathematical object, we must also describe the maps between such objects. Let K and L be complexes. A **simplicial map** $f : K \to L$ is a function $f : \text{vert}(K) \to \text{vert}(L)$ with the property that if $\sigma \in K$, then $f(\sigma) \in L$.

• **Problem 1.** Find an injective simplicial map $f : K_7 \to T^2$; here, T^2 is the torus, and K_7 denotes 7 points connected in pairs by all $\binom{7}{2} = 21$ edges. Use this to triangulate T^2 as a simplicial complex with as few triangles as possible.

Problem 2. Does there exist an injective map $f : K_{3,3} \to S^2$? What about $f : K_{3,3} \to T^2$? Here, $K_{3,3}$ denotes six points, connected by nine lines, as shown on the right.

Subcomplexes

Definition. If K and L are complexes, and every simplex in K is a simplex of L, then we write $K \subset L$ and say that K is a **subcomplex** of L.

Example. The n-dimensional **skeleton** (usually called the n-skeleton) of a complex K consists of all those simplexes which contain $n+1$ or fewer points (i.e., are at most n-dimensional). Write $K^{(n)}$ for the n-skeleton. We say that a complex K is an n-complex if $K = K^{(n)}$.

• **Problem 3.** Prove that the n-skeleton of a simplicial complex K is still a simplicial complex.

Joins

Definition

Let K and L be complexes
with disjoint sets of vertices
(we call such complexes **joinable**).
Define a new complex, the **join** of K and L, by

$$K * L := \{\sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

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Problem

What is $S^0 * S^0$?

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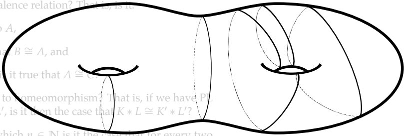
Problem

What is $S^0 * S^0$?

Problem

What is $S^n * S^m$?

on today's homework



This is the end of the beginning. Problem Set 3 introduces *manifolds*, and with that, the last of the introductory topics; Please **submit answers to problems marked with a •**.

• **Problem 1.** Suppose K is a simplicial complex, with subdivisions $K_1 \triangleleft K$ and $K_2 \triangleleft K$. Is there a subdivision K' so that $K' \triangleleft K_1$ and $K' \triangleleft K_2$?

Problem 2. Let K, L, M be complexes. If $f : K \to L$ and $g : L \to M$ are PL maps, how should we define the PL map $g \circ f : K \to M$?

Problem 3. For $n \geq 3$, let P_n be the boundary of an n-gon. Prove that $P_n \cong P_m$ for $n, m \geq 3$. Thus, any P_n is topologically an S^1.

Definition. A complex K is **path-connected** if for any two vertices $v, w \in K$, there exists a map $f : \Delta^1 \to K$ sending the boundary of Δ^1 to v and w.

Problem 4. Prove that being path-connected is a topological property, meaning it is preserved under PL homeomorphisms.

Homeomorphisms

Problem 5. Is PL homeomorphism an equivalence relation? That is, is it:

reflexive, meaning, is A homeomorphic to A,

symmetric, meaning, if $A \cong B$, is it true that $B \cong A$, and

transitive, meaning, if $A \cong B$ and $B \cong C$, is it true that $A \cong C$?

• **Problem 6.** Is join well-defined with respect to homeomorphism? That is, if we have PL homeomorphic complexes $K \cong K'$ and $L \cong L'$, is it then the case that $K * L \cong K' * L'$?

Problem 7. Let X_n consist of n points. For which $n \in \mathbb{N}$ is it the case that for every two injective maps $f, g : X_n \to S^1$, there is a homeomorphism $h : S^1 \to S^1$ so that $h \circ f = g$?

Problem 8. Let X_n be the disjoint union of n circles (e.g., $X_3 = S^1 \cup S^1 \cup S^1$). For this problem, call two maps $f, g : X_n \to S^2$ "equivalent" if there exists a homeomorphism $h : S^2 \to S^2$ so that $h \circ f = g$. Count the equivalence classes of maps from X_5 to S^2?

Problem 9. To the right, three curves are pictured on $T^2 \# T^2$, which is our notation for a two-holed surface. For which pairs of curves α and β does there exist a homeomorphism

$$f : T^2 \# T^2 \to T^2 \# T^2$$

so that $f(\alpha) = \beta$?

PL Homeomorphism

Definition

A **piecewise linear homeomorphism**
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is a PL map $f : K \longrightarrow L$
with a PL inverse.

If there exists a homeomorphism between A and B,
then A and B are **homeomorphic**.
Write $A \cong B$ if A and B are homeomorphic.

PL Manifold

A complex M is an n-dimensional **PL manifold** (for short, an n-manifold) if for every vertex v of M, $\mathrm{lk}(v, M)$ is PL homeomorphic to S^{n-1}.

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PL Manifold

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Problem

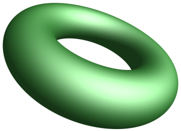
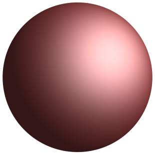
Is S^2 a manifold?

Problem

Is T^2 a manifold?

Going back, rethinking everything. . .

Is there a PL homeomorphism between S^2 and T^2?

 $\cong$  ?

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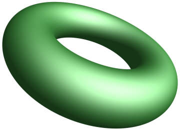
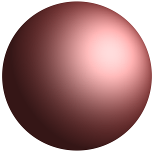
 $\cong$  ?

Check χ is unchanged after stellar subdivision
(we must also check more general subdivisions!)

Euler characteristic

$$K_\sigma := (K - \mathrm{st}(\sigma, K)) \cup (v * (\partial\sigma) * \mathrm{lk}(\sigma, K))$$

Need to check $\chi(K) = \chi(K_\sigma)$.

Euler characteristic and join

$$p_K(x) = 1/x + k_0 + k_1 x + \cdots + k_n x^n$$

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$$p_K(x) = 1/x + k_0 + k_1 x + \cdots + k_n x^n$$
$$p_L(x) = 1/x + \ell_0 + \ell_1 x + \cdots + \ell_m x^m$$

Euler characteristic and join

$$
\begin{aligned}
p_K(x) &= 1/x + k_0 + k_1 x + \cdots + k_n x^n \\
p_L(x) &= 1/x + \ell_0 + \ell_1 x + \cdots + \ell_m x^m \\
p_{K*L}(x) &= p_K(x) \cdot p_L(x) \cdot x
\end{aligned}
$$

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Since $\chi(K) = p_K(-1) + 1$,

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$$\chi(K * L) = p_{K*L}(-1) + 1$$

Euler characteristic and join

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Since $\chi(K) = p_K(-1) + 1$,

$$\begin{aligned} \chi(K * L) &= p_{K*L}(-1) + 1 \\ &= (\chi(K) - 1)\,(\chi(L) - 1) \cdot (-1) + 1 \end{aligned}$$

Euler characteristic and join

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Since $\chi(K) = p_K(-1) + 1$,

$$\begin{aligned} \chi(K * L) &= p_{K*L}(-1) + 1 \\ &= (\chi(K) - 1)(\chi(L) - 1) \cdot (-1) + 1 \\ &= \chi(K) + \chi(L) - \chi(K)\chi(L). \end{aligned}$$

More joins!

$$\chi(K * L * M)$$

More joins!

$$\chi(K * L * M) = \chi(K * L) + \chi(M) - \chi(K * L)\chi(M)$$

More joins!

$$\begin{aligned}\chi(K * L * M) &= \chi(K * L) + \chi(M) - \chi(K * L)\chi(M) \\ &= \chi(K) + \chi(L) - \chi(K)\chi(L) + \chi(M) - \\ &\quad \left(\chi(K) + \chi(L) - \chi(K)\chi(L)\right)\chi(M)\end{aligned}$$

More joins!

$$
\begin{aligned}
\chi(K * L * M) &= \chi(K * L) + \chi(M) - \chi(K * L)\chi(M) \\
&= \chi(K) + \chi(L) - \chi(K)\chi(L) + \chi(M) - \\
&\quad \left(\chi(K) + \chi(L) - \chi(K)\chi(L)\right)\chi(M) \\
&= \chi(K) + \chi(L) + \chi(M) \\
&\quad - \chi(K)\chi(L) - \chi(K)\chi(M) - \chi(L)\chi(M) \\
&\quad + \chi(K)\chi(L)\chi(M)
\end{aligned}
$$

Euler characteristic and union

The usual inclusion-exclusion business gives

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$$

Euler characteristic

$\chi(K_\sigma)$

$= \chi(K).$

Euler characteristic

$$\chi(K_\sigma)$$

$$= \chi\left((K - \mathrm{st}(\sigma, K)) \cup \left(v * (\partial\sigma) * \mathrm{lk}(\sigma, K)\right)\right)$$

$$= \chi(K).$$

Euler characteristic

$$\chi(K_\sigma)$$

$$\begin{aligned}
&= \chi\left((K - \mathrm{st}(\sigma, K)) \cup \left(v * (\partial\sigma) * \mathrm{lk}(\sigma, K)\right)\right) \\
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&\quad + \chi\left(v * (\partial\sigma) * \mathrm{lk}(\sigma, K)\right) \\
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\end{aligned}$$

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&\quad + \chi\left(v * (\partial\sigma) * \mathrm{lk}(\sigma, K)\right) \\
&\quad - \chi\left(\mathrm{lk}(\sigma, K)\right) \\
&= \chi\left(K - \mathrm{st}(\sigma, K)\right) + \chi\left(v * (\partial\sigma)\right) \\
&\quad - \chi\left(v * (\partial\sigma)\right) \cdot \chi\left(\mathrm{lk}(\sigma, K)\right)
\end{aligned}$$

$$= \chi(K).$$

Euler characteristic

$$
\begin{aligned}
&\chi(K_\sigma) \\
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&\\
&= \chi(K).
\end{aligned}
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&= \chi\left(K - \mathrm{st}(\sigma, K)\right) + \chi\left(v * (\partial\sigma)\right) \\
&\quad - \chi\left(v * (\partial\sigma)\right) \cdot \chi\left(\mathrm{lk}(\sigma, K)\right) \\
&= \chi\left(K - \mathrm{st}(\sigma, K)\right) + 1 - \chi\left(\mathrm{lk}(\sigma, K)\right) \\
&= \chi\left(K - \mathrm{st}(\sigma, K)\right) + \chi(\mathrm{cl}(\mathrm{st}(\sigma, K))) - \chi\left(\mathrm{lk}(\sigma, K)\right) \\
&= \chi(K).
\end{aligned}
$$

Upshot

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$\chi(S^2) = 2$ but $\chi(T^2) = 0$,
and χ is a PL homeo invariant,
so $S^2 \ncong T^2$.

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Haiku

Upshot

$\chi(S^2) = 2$ but $\chi(T^2) = 0$,
and χ is a PL homeo invariant,
so $S^2 \not\cong T^2$.

Haiku

The sphere and torus,
what with their differing χ,
are not the same space.

The torus versus the Klein bottle

Since $\chi(T^2) = \chi(K) = 0$,

The torus versus the Klein bottle

Since $\chi(T^2) = \chi(K) = 0$,
and ... um ...

The torus versus the Klein bottle

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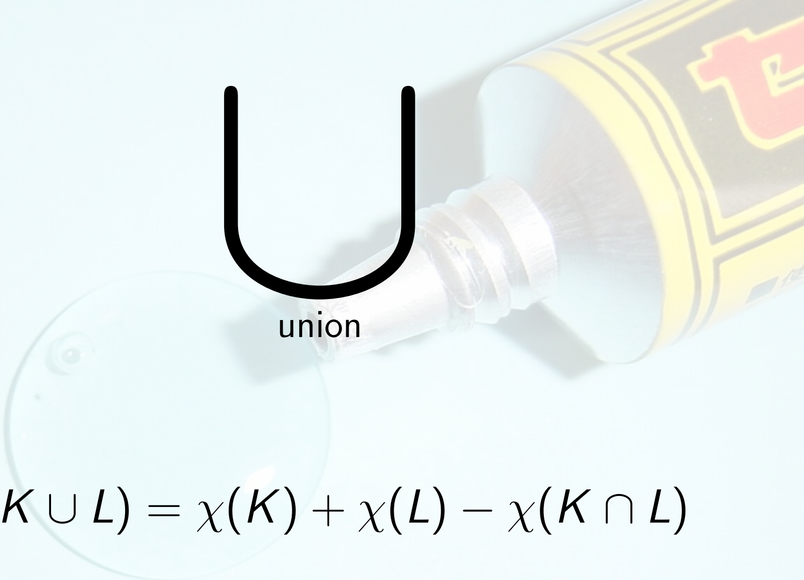
χ is not a complete invariant.

$$\cancel{\chi}$$

Let's think about Euler characteristic



$$\bigcup$$

union

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$$

A new invariant

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Similarly, $F(\Delta^n) = F(I)$.

A “new” invariant

Congratulations!

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New invariants cannot be precisely additive.

Connected components

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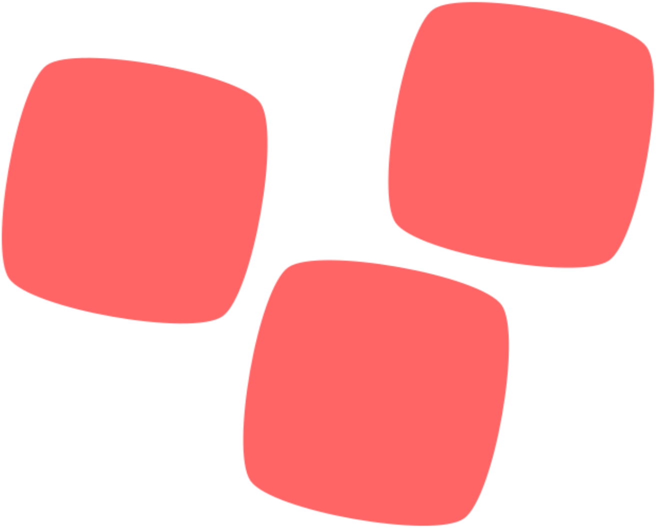
Problem

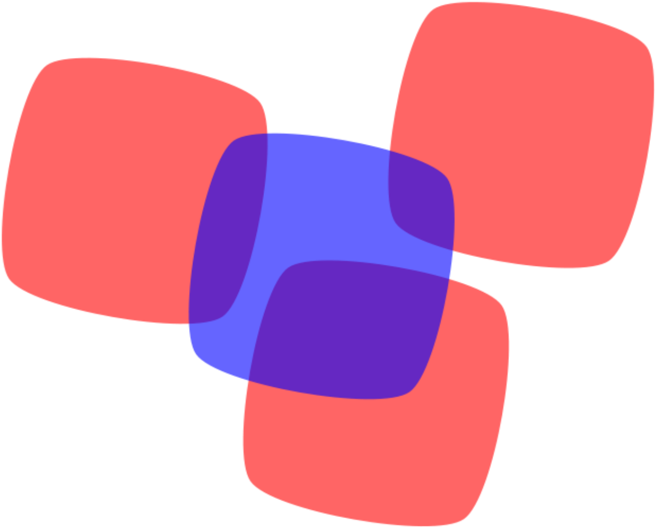
Is "belong to the same component"
an equivalence relation
on the vertices of K?

Connected components



three

Connected components

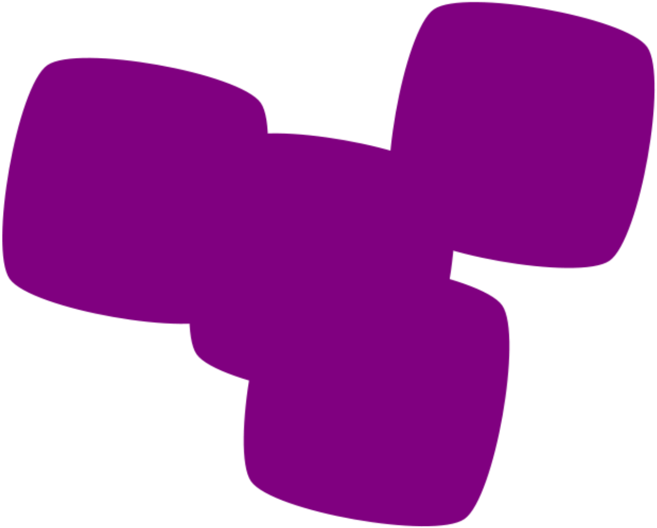


three + one

Connected components

three + one − three

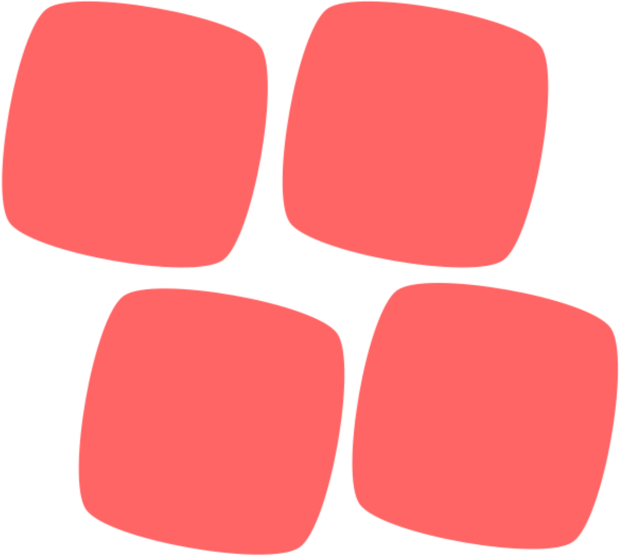
Connected components



three + one − three = one

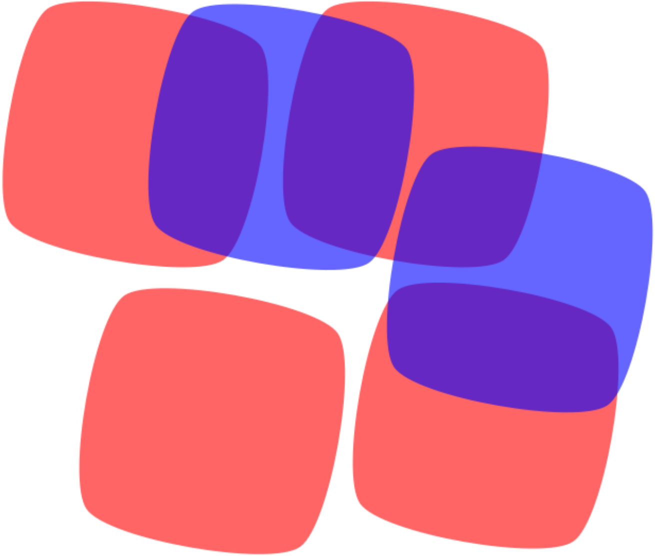
$$b_0(K \cup L) = b_0(K) + b_0(L) - b_0(K \cap L)?$$

Connected components



four

Connected components

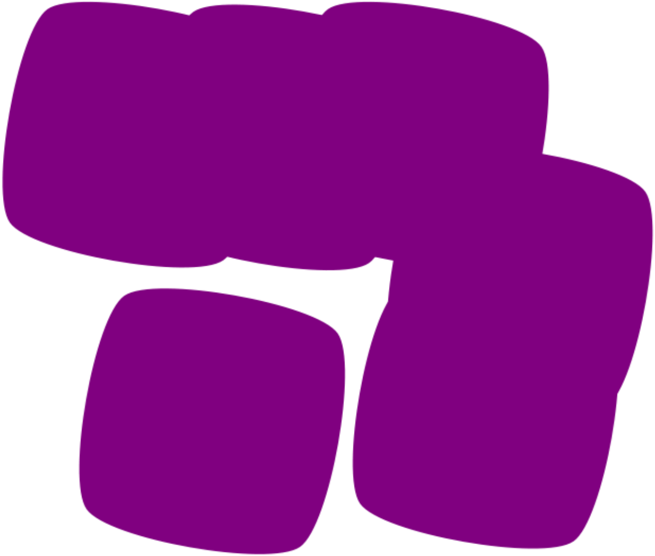


four + two

Connected components

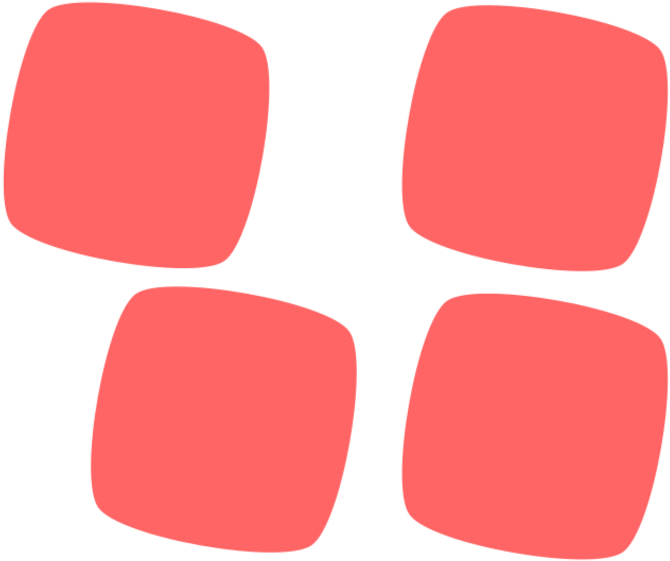
four + two − four

Connected components



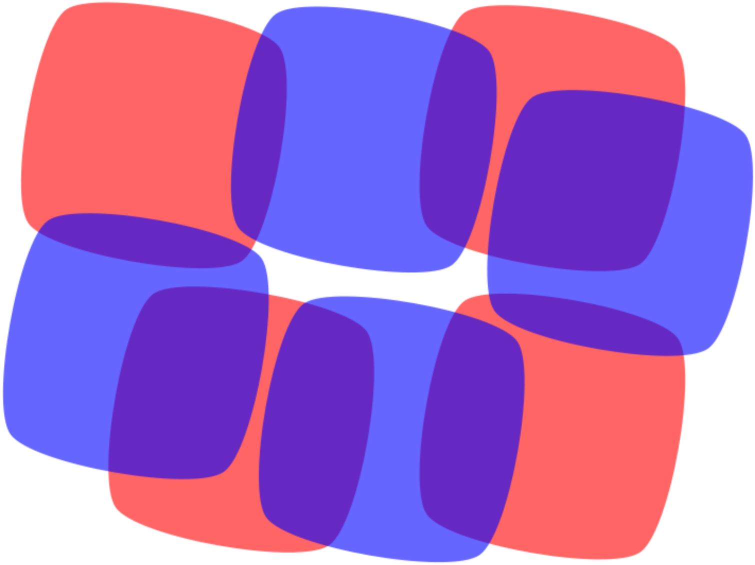
four + two − four = two

Connected components



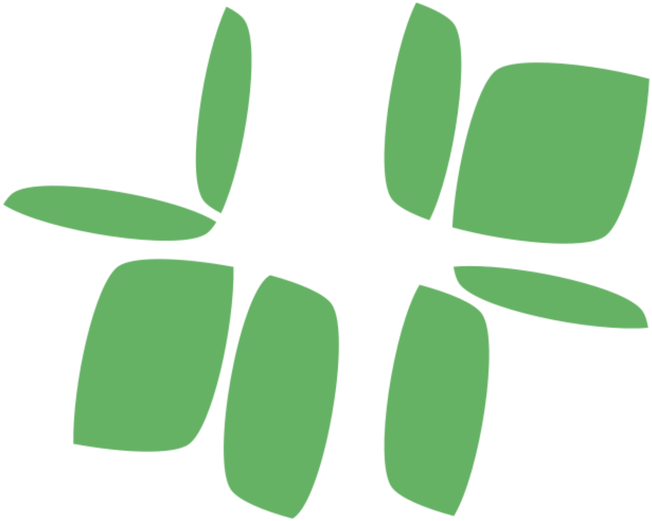
four

Connected components



four + four

Connected components



four + four − eight

Connected components

$\text{four} + \text{four} - \text{eight} = \text{zero} \neq \text{one}$

We will invent new invariants.
These invariants will not *quite* be additive.
The failure of additivity gives a new invariant,
itself not quite additive.
The failure will be captured by a deeper invariant,
itself not quite additive.
The failure will be captured by . . .

This is **homology**.

The end of the beginning

What's next?

- manifolds
- simplicial collapse
- simple homotopy equivalence
- knots
- unknotting theorems
- return to (co)homology