Lecture 13: Convergence tests

Math 153 Section 57

Monday October 27, 2008

Following chapter 12.3 and starting 12.4.

Bare hands proof that harmonic series diverges

Estimate $\sum_{n=1}^{K} 1/n$. $1/1$. $1/2$. $1/3 + 1/4$ underestimated by $1/2$. $1/5 + 1/6 + 1/7 + 1/8$ underestimates by $1/2$. And so on. Therefore, $s_{2^k} \geq 1 + k/2$. This argument is about 700 years old.

Use this to estimate how many terms we have to add up. How big does K have to be in order for $\sum_{n=1}^{K} 1/n > 10$. If I want $1 + k/2 \geq 10$, I need $k \geq 18$. But $2^{18} = 262144$.

This will do, but I actually only need $k = 12367$. Better approximation from Euler-Mascheroni constant:

$$\gamma = \lim_{K\to\infty} \left(\left(\sum_{n=1}^{K} 1/n \right) - \log K \right) \approx 0.577.$$

Since $e^{10-0.577} = 12369.6$, guess that $k = 12367$ suffices.

Be careful about series with negative terms

For example, $\sum 1/n$ diverges, but as we will see soon, $\sum (-1)^n/n$ converges—even though $1/n$ and $(-1)^n/n$ have the same magnitude. More about this on Friday.

Limit comparison test

Limit comparison test: let $a_n \geq 0$ and $b_n \geq 0$. If $\lim_{n\to\infty} a_n/b_n = L > 0$, then $\sum a_n$ converges iff $\sum b_n$ converges.

Proof: Choose ϵ so that $0 < \epsilon < L$.

Then there exist K so that for $n \geq K$, $|a_n/b_n - L| < \epsilon$.

That means,

$$(L - \epsilon)b_n < a_n < (L + \epsilon)b_n$$

So if $\sum a_n$ converges, then $\sum (L - \epsilon)b_n$ converges.

The slogan: converge is all about how quickly those terms are vanishing.

Example

$$\sum_n \frac{3n^2+n+17}{n^4+n^2+8}$$

On the one hand, we could try to bound this and apply comparison tes... but even easier is to use the limit comparison test! Use $\sum 1/n^2$.

Buyer beware!

The limit comparison test does not tell you what the series equals—only whether it converges.

Like all tools, use it properly

Limit comparison says nothing if $L = 0$, or $L = \infty$.

Review

Does a series converge?

First, is it a series we recognize? Geometric series? Harmonic series or p-series?

Apply integral test (which involves coming up with a function), comparison test (which involve coming up with another series), or limit comparison test (which involves coming up with another series and taking a limit).

More tests

Let $a_n \geq 0$, and $\lim_{n\to\infty} {a_n}^{1/n} = L$. Then if $L < 1$, then $\sum a_n$ converges, and if $L > 1$, then $\sum a_n$ diverges. If $L = 1$, inconclusive.

Proof: suppose $L < 1$. Choose ϵ small so that $L + \epsilon < 1$. Then for n large,

$${a_n}^{1/n} < L + \epsilon$$

so for n large, $a_n < (L+\epsilon)^n$. By comparison with $\sum (L+\epsilon)^n$, the series converges.

On the other hand, if $L > 1$, then choose ϵ small so that $L - \epsilon > 1$. Again, for n large, ${a_n}^{1/n} > L - \epsilon$, so $a_n > (L-\epsilon)^n$. Since $\sum (L-\epsilon)^n$ diverges, comparison proves $\sum a_n$ diverges.

Example

$$\sum \frac{1}{n^n}.$$

Or again,

$$\sum \frac{1}{(\cos n + \sin \log n + n)^n}.$$

counterexample: $\sum 1/n$ is not detected, since $\lim 1/n^{1/n} = 1$. The same is true of $\sum 1/n^2$.

This is nothing more than an application of the comparison test (or, if you like, a modification of the limit comparison test) with a geometric series.