Problem Set 7 — Piecewise-Linear Topology

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Regular neighborhoods. Here we summarize the main results on simplicial neighborhoods from the textbook *Introduction to Piecewise-Linear Topology*. We will focus on applications of this theory.

Definition (Some neighborhoods)**.** The **simplicial neighborhood** of L in K is

$$N(L, K) = \{A \in K : A < B, B \cap |L| \neq \varnothing\}.$$

The **simplicial complement** of L in K is

$$C(L, K) = \{A \in K : A \cap |L| = \varnothing\}.$$

Definition (Some subdivisions)**.** Suppose $L \subset K$ are complexes, and $a_i \in \operatorname{int} A_i$ for each $A_i \in K$ with $A_i \notin L$. We define a subdivision $K' \triangleleft K$ inductively by

$$A_i{}' = \begin{cases} \{a_i\} * \partial A_i & \text{if } A_i \notin L, \\ A_i & \text{if } A_i \in L. \end{cases}$$

This is called K **derived away** from L.

On the other hand, the derived of K **near** L is obtained by deriving K away from $L \cup C(L, K)$, i.e., subdividing those simplexes meeting $|L|$ but not in L.

If K' is K derived near L, then $N(L, K')$ is a **derived neighborhood** of L in K.

Definition (Regular neighborhood)**.** Suppose $X \subset Y$ are polyhedra.

- $|K|$ a neighborhood of X in Y.
- $|L| = X$.
- $L \Subset K$.
- K' derived of K near L.

Then $|N(L, K')|$ is called a **regular neighborhood** of X in Y.

Theorem. If N_1 and N_2 are regular neighborhoods of X in Y, then there is a homeomorphism $h : Y \to Y$, which throws N_1 onto N_2, and which is the identity on X, and the identity outside a compact subset of Y.

Theorem. A regular neighborhood N of a polyhedron X in a manifold M is a manifold with boundary.

Theorem (Simplicial neighborhood theorem)**.** Suppose X is a compact polyhedron, M is a manifold, and $X \subset \operatorname{int} M$. Then a polyhedral neighborhood N of X in $\operatorname{int} M$ is a regular neighborhood if and only if

- N is a compact manifold with boundary
- there are triangulations (K, L, J) of $(N, X, \partial N)$ with $L \Subset K$, $K = N(L, K)$ and $J = \partial N(L, K)$.

Theorem. As before, suppose X is a compact polyhedron, M is a manifold, and $X \subset \operatorname{int} M$. Then a polyhedral neighborhood N of X in $\operatorname{int} M$ is a regular neighborhood if and only if

- N is a compact manifold with boundary
- $N \searrow X$.

Corollary. A collapsible manifold is a ball.

Regular neighborhoods and Simplicial collapse

Problem 1 (From *Introduction to Piecewise-Linear Topology*). Show that $\mathbb{R}^n$ is PL homeomorphic to $S^n -$ point.

Problem 2. Suppose M^2 is a 2-manifold, with $\partial M = S^1 \cup S^1$, and $M \searrow S^1 \subset \partial M$. Identify M.

• **Problem 3.** What are the possible regular neighborhoods of S^1 inside a 2-manifold? Use this to describe those surfaces which collapse to a circle.

Problem 4. Can you find two different 3-manifolds which both collapse to S^1? Which both collapse to S^2?

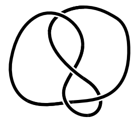
• **Problem 5.** Is a subcomplex of a collapsible complex necessarily collapsible?

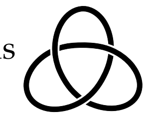
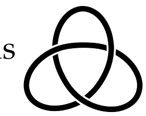
Problem 6. Let X and Y both be $S^1 \vee S^1 \vee S^1$, namely, three circles connected together at a single point; find embeddings of X and Y into T^3, so that $T^3 -$ regular neighborhood of X collapses onto Y.

• **Problem 7.** Show that $T^2 -$ a disk and $S^3 - 3$ disks collapse onto $S^1 \vee S^1$, the union of two circles along one point.

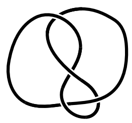
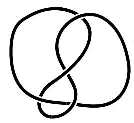
Knot theory

If you want to try your hand at Reidemeister moves, here are some exercises to help you to do so.

Problem 8. Show that the figure eight knot  is not the unknot.

Problem 9. Is the trefoil knot identical to its mirror image: is  the same as  ?

Problem 10. Is the figure eight knot the same as its mirror image, i.e., is

 the same as  ?

Problem 11. Suppose $S^1 \subset S^3$ is a knot; show that the suspension $SS^1 \subset SS^3$ gives a knotted S^2 in S^4.