Lecture 15: Kinds of convergence

Math 153 Section 57

Friday October 31, 2008

Following chapter 12.5.

0.1 Ratio test

sometimes ratio test fails because limit does not exist.

0.2 Recall from last time

If $\sum |a_n|$ conmverges, then $\sum a_n$ converges.

0.3 Definitions

We say $\sum a_n$ is "absolutely convergent" if $\sum |a_n|$ converges.

Absolute convergence implies convergence.

0.4 Converse is false

There are series which converge, but not absolutely.

0.5 Alternating series

Definition (signs alternate).

General form: $\sum(-1)^n a_n$ for $a_n > 0$.

Very easy to determine convergence: $\sum(-1)^n a_n$ converges provided $\lim a_n = 0$ and a_n are decreasing.

Proof: one direction is obviousl. The other direction: first look at partial sums s_{2k}. Then

$$s_{2k+2} = s_{2k} - (a_{2k+1} - a_{2k+2})$$

so $s_{2k+2} < s_{2k}$. So decreasing, and bounded below by 0, so converges to some number, say, L.

Also, $s_{2k+1} = s_{2k} + a_{2k+1}$, so taking limits, the odd terms also converge to L. But if both even and odd subsequences converge to L, then the sequence converges to L.

0.6 Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2$$

Proof?

$$e^{\sum_{n=0}^{k} \frac{(-1)^n}{n+1}} = \frac{e^{1/1} e^{1/3} \cdots}{e^{1/2} e^{1/4} \cdots}$$

$$\sum \frac{(-1)^k}{\sqrt{k}}$$

0.7 Estimates

If $\sum (-1)^n a_n = L$, then $|s_k - L| < a_{k+1}$.

$\log 2 \approx 0.6931471805599453094172321214581765680755001343$

$$\sum_{n=0}^{5} \frac{(-1)^n}{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$

and $1/1 + 1/3 + 1/5 = 23/15$ and $1/2 + 1/4 + 1/6 = 11/12$, so the answer is $23/15 - 11/12 = (92 - 55)/60 = 37/60$, which is $0.61\overline{6}$.

The true answer is off by no more than $1/7 \approx 0.14$, so we know $\log 2$ is between 0.47 and 0.76.

0.8 Another estimate

$\sum (-1)^n / n! = 1/e$.

$$1/1 - 1/1 + 1/2 - 1/6 + 1/24 = 3/8 = 0.375$$

The next term is $1/120$, so we know that e is between $3/8 - 1/120 = 22/60 = 0.3\overline{6}$, and $3/8 + 1/120 = 23/60$.

And indeed, $e \approx 22.0727/60$.

0.9 Yet another estimates

We have

$$\sum (-1)^k / (2k)! = \cos 1 \approx 0.54030230586813971$$

But

$$1/1 - 1/2 + 1/24 - 1/720 = 389/720$$

So $\cos 1$ is between $389/720 - 1/40320 \approx 0.54025$ and $389/720 + 1/40320 \approx 0.54031$.

$$1/1 - 1/2 + 1/24 = 13/24$$

So $\cos 1$ is between $13/24 - 1/720 = 389/720 \approx 0.54027$ and $13/24 + 1/720 = 391/720 \approx 0.543057$.

In fact, $\cos 1 \approx 389.018/720$.

0.10 Rearranging conditionally convergent series

The order matters.

Example: rearrange the terms of $(-1)^n/n$.