Elementary Algebra

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Topics

- Powers
- Roots
- Rules of algebra
- Fractions
- Simple equations
- Quadratic equations
- Basic systems of equations

Powers

- a^n is the nth power of a, where a is the base and n is the exponent.
- a^n is defined as:

$$a^n = \underbrace{a \cdot a \ldots a}_{n \text{ times}}$$

- By definition

$$a^0 = 1 \quad \forall a \neq 0$$

- What if $a = 0$?
- We define negative exponents as

$$a^{-n} = \frac{1}{a^n} \forall a \neq 0$$

- What if $a = 0$?

General rules of exponentiation

- $a^n \cdot a^m = a^{n+m}$
- $\frac{a^n}{a^m} = a^{n-m}$
- $(a^n)^m = a^{n \cdot m}$
- $(a \cdot b)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Solve the following problems [SYD Appendix A]

1. Compute $(-1)^5$
2. Express as powers: $(a-b)(a-b)(a-b)$
3. Simplify $\frac{z^2 \div z^5}{z^3 \cdot z^{-4}}$
4. Compute $\frac{4^2 \cdot 6^2}{3^3 \cdot 2^3}$
5. Solve for x: $10^x \div 10^5 = 10^{-2}$
6. If $\left(\frac{xy}{z}\right)^{-2} = 3$ then $\left(\frac{z}{xy}\right)^6 = ?$

Roots

- $\sqrt[n]{x}$ is the nth root of x, where x is the base and n is the degree.
- $\sqrt[n]{x}$ is a number z such that
$$z^n = x$$
- Taking the $\frac{1}{n}$th power of both sides yields
$$z^{n \cdot \frac{1}{n}} = z = \sqrt[n]{x} = x^{\frac{1}{n}}$$
- Thus instead of the radical notation we can use (although a mathematician would disagree)
$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

Exercise

Using the rules of exponentiation show that

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

Exercise - Solution

Using the rules of exponentiation show that

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

- By definition if $\sqrt[b]{x^a} = z$ then
$$z^b = x^a$$
- Raising both sides to the $\frac{1}{b}$th power yields
$$z = x^{\frac{a}{b}}$$
- Since $\sqrt[b]{x^a} = z$, we have
$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

Properties of roots

- $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- $\sqrt[n]{a \cdot b} = \sqrt[n]{a}\sqrt[n]{b} \quad \forall a, b \in \mathcal{R}^+$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \forall a, b \in \mathcal{R}^+$

Solve the following problems [SYD Appendix A]

1. Compute $\sqrt{1600}$
2. Compute $125^{\frac{1}{3}}$
3. Solve for x: $x^{0.25} = 2$
4. True or false: $(a + b)^{-0.5} = \frac{1}{\sqrt{a+b}}$

Rules of algebra

You are probably familiar with these.

- Commutativity of addition: $a + b = b + a$
- Commutativity of multiplication: $a \cdot b = b \cdot a$
- Multiplication is distributive over addition: $a(b + c) = ab + ac$
- Associativity of addition: $(a + b) + c = a + (b + c)$
- Associativity of multiplication: $(ab)c = a(bc)$

Some useful identities:

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$

Solve the following problems [SYD Appendix A]

1. Compute $\frac{1000^2}{252^2 - 248^2}$ without a calculator.
2. Calculate $-3[4 - (-2)]$
3. Expand $\left(\frac{1}{2}x + \frac{1}{3}y\right)\left(\frac{1}{2}x - \frac{1}{3}y\right)$

Fractions

- A fraction can be written as a numerator over a denominator:

$$a \div b = \frac{a}{b}$$

- If $a < b$, it is a proper fraction
- If $a \geq b$, it is an improper fraction
- Improper fractions can be written as mixed numbers. E.g:

$$\frac{15}{7} = 2 + \frac{1}{7} = 2\frac{1}{7}$$

- To avoid ambiguity whether $2\frac{1}{7}$ means $2 + \frac{1}{7}$ or $2 \cdot \frac{1}{7}$ you should not use mixed notation!

Reducing fractions

You can reduce fractions by canceling common factors. E.g.:

- $\frac{24}{36} = \frac{\cancel{3}\cdot\cancel{2}\cdot\cancel{2}\cdot 2}{\cancel{3}\cdot 3\cdot\cancel{2}\cdot\cancel{2}} = \frac{2}{3}$
- $\frac{xy+x^2y}{xy^2+x^3y^2} = \frac{\cancel{xy}(1+x)}{\cancel{xy}(y+x^2y)} = \frac{1+x}{y+x^2y}$
- Notice that by canceling factors they don't disappear! We only use the fact that $\frac{x}{x} = 1$. Thus:

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} \neq \frac{0}{x+1}$$

It is actually:

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$$

Multiplying and dividing fractions

- Just multiply the numerators and denominators

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$$

- To divide, just change up the numerator and the denominator in the divisor and then multiply

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Adding fractions

- If they have a common denominator, simply add numerators:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$

- Otherwise you need common denominators first.

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\frac{d}{d}+\frac{c}{d}\frac{b}{b}=\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd}$$

Solve the following problems [SYD Appendix A]

1. Simplify $\frac{5}{7} + \frac{3}{7} - \frac{2}{7}$
2. Simplify $\frac{1}{8ab} + \frac{1}{8b(a-2)} + \frac{1}{b(a^2-4)}$
3. Simplify
$$\frac{\frac{1}{x-1} + \frac{1}{x^2-1}}{x - \frac{2}{x+1}}$$
4. Calculate $\left(\frac{1}{4} - \frac{1}{5}\right)^{-2}$

Simple equations

When you solve equations you can perform the following operations on both sides of the equation:

- add the same number
- subtract the same number
- multiply by the same non-zero number
- divide by the same non-zero number

Example

Solve $3x + 10 = x + 4$

$$\begin{aligned} 3x + 10 &= x + 4 && \text{(subtract 10)} \\ 3x &= x - 6 && \text{(subtract x)} \\ 2x &= -6 && \text{(divide by 2)} \\ x &= -3 \end{aligned}$$

Solve the following problems [SYD Appendix A]

1. Mr. Barne receives double pay for every hour he works over and above 38 hours a week. Last week, he worked 48 hours and earned a total of $812. What is Mr. Barne's regular hourly wage?
2. $6p - \frac{1}{2}(2p-3) = 3(1-p) - \frac{7}{6}(p+2)$
3. $\frac{x+2}{x-2} - \frac{8}{x(x-2)} = \frac{2}{x}$
4. When Ann passed away, their estate was divided in the following manner: 2/3 of the estate was left to their wife, 1/4 to their children, and the remainder, $10 000 was donated to a charitable organization. How big was Ann's estate?

Quadratic equations

A general quadratic equation takes the following form:

$$ax^2 + bx + c = 0$$

We can easily solve it using a simple formula that we will now derive together:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Deriving the solution to a quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{(Factor out a)}$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \qquad \text{(Divide by a)}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(Subtract c/a)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{(Add the same to both sides)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(Continued...)}$$

Deriving the solution to a quadratic equation

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(Complete the square LHS)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(Simplify RHS)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{b^2 - 4ac}{4a^2}$$

Thus either $x + \frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a}$ or $x + \frac{b}{2a} = -\frac{\sqrt{b^2-4ac}}{2a}$ which yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following problems [SYD Appendix A]

1. $15x - x^2 = 0$
2. $x^2 - 9 = 0$
3. $x^2 - 4x + 4 = 0$
4. $x^2 - 5x + 6 = 0$
5. In a right-angled triangle, the hypotenuse is 34 cm. One of the short sides is 14 cm longer than the other. Find the lengths of the two short sides.

Basic systems of equations

We will review two methods to solve simple systems of equations (2 linear equations, 2 unknowns).

- Substitution: Solve one equation to one variable and substitute it in the other equation
- Elimination: Add/subtract a multiple of one equation from the other to eliminate one variable

Substitution

Solve

$$2x + 3y = 18$$
$$3x - 4y = -7$$

Express x from the first equation:

$$x = 9 - 1.5y$$

Substitute x in the second equation:

$$3(9 - 1.5y) - 4y = -7$$

Solve for y:

$$y = 4$$

Solve for x:

$$x = 3$$

Elimination

Solve

$$2x + 3y = 18$$
$$3x - 4y = -7$$

Multiply the first equation by 4/3:

$$\frac{8}{3}x + 4y = \frac{72}{3}$$

Add it to the second equation:

$$\frac{17}{3}x = \frac{51}{3}$$

Solve for x:

$$x = 3$$

Solve for y:

$$y = 4$$

Solve the following problems [SYD Appendix A]

1. $x - y = 5$ and $x + 3y = 11$
2. $3x + 4y = 2.1$ and $5x - 6y = 7.3$
3. A person has two accounts with a total saving of \$10 000. The interest rates are 5% and 7.2% respectively. If the person earns \$676 interest in a year, what was the balance of these accounts?