domain : X

image : Y

function : set of ordered pairs (x, y) where $x \in X$ and $y \in Y$

$\in$: element of a set

$f : X \to Y$ $\qquad$ $y = f(x)$

f(fruit) = color

f(lemon) = yellow

f(plum) = purple

f(animal) = speed

f(cheetah) = 103 km/h

f(sea anemone) = 0.004 km/h

square: $f(x) = x^2$

absolute value $f(x) = |x|$

$f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}_0^+$

Domain & Image are number sets.

$\mathbb{N}$: natural numbers : $1, 2, 3, 4, 5 \ldots$

$\mathbb{Z}$: integer : $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

$\mathbb{R}$: real numbers : every number with a decimal representation : 1.23, -5, $-0.25 \ldots$ π

$\mathbb{R}^+$: positive real numbers

$\mathbb{R}_0^+$: $\mathbb{R}^+$ + zero included

$\mathbb{R}_0^+$

sets : $\{ \}$

$\mathbb{R} \setminus \{-2\}$

$\mathbb{R} \setminus \{\mathbb{Z}\}$

$\setminus$: exclusion from a set

1) $f(x) = 100\sqrt{x} + 500$

$$f(16) = 100\sqrt{16} + 500 = 100 \cdot 4 + 500 = 900$$

$$f(100) = 100\sqrt{100} + 500 = 100 \cdot 10 + 500 = 1500$$

2) $f(x) = \dfrac{1}{x+3}$ $\quad x+3 \neq 0$ $\quad x: \mathbb{R} \setminus \{-3\}$

$$x \neq -3$$

3) $g(x) = \sqrt{2x+4}$ $\quad 2x+4 \geq 0$

$$2x \geq -4$$

$$\boxed{x \geq -2}$$

4) $f(x) = \dfrac{3x+6}{x-2}$

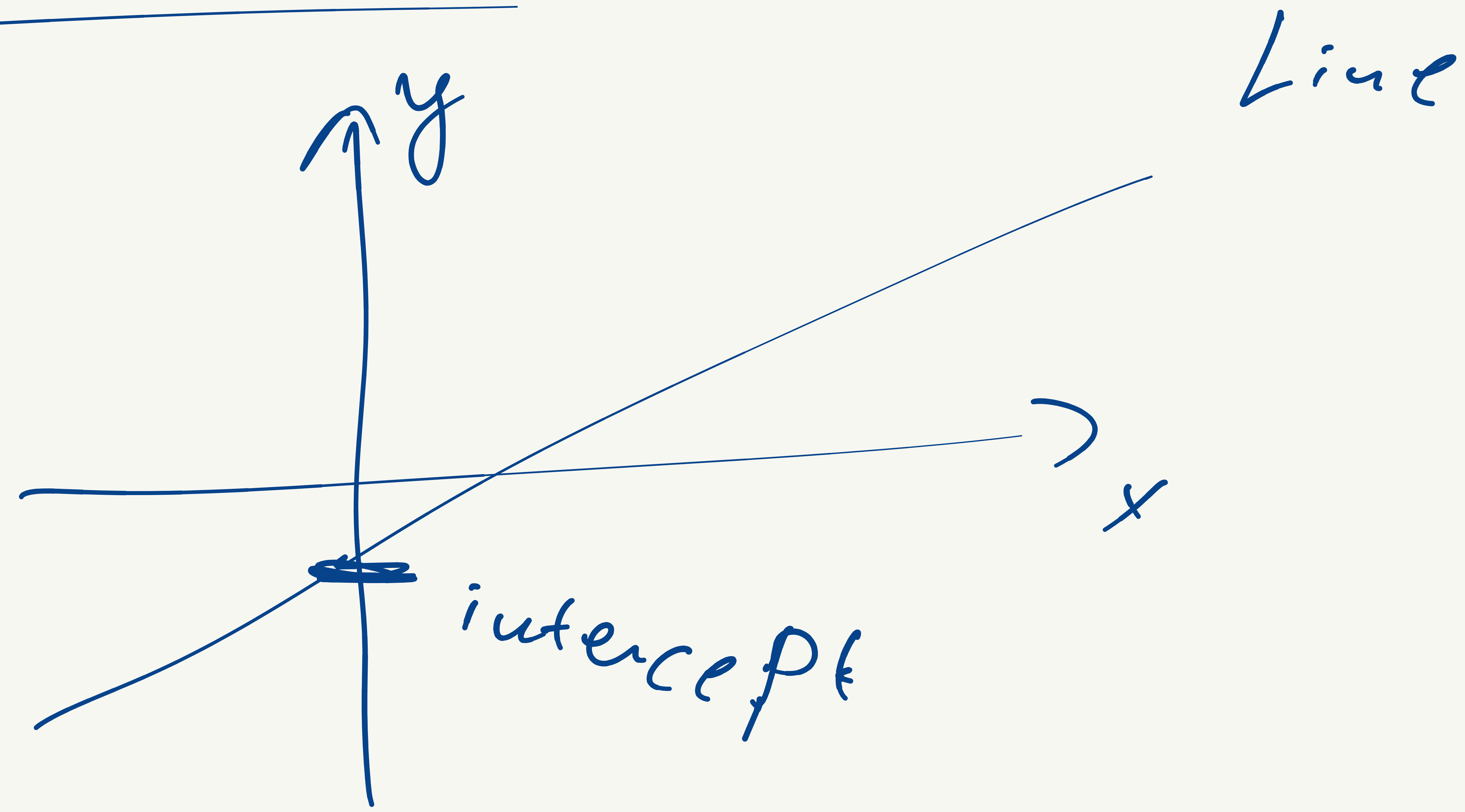
$$5 = \frac{3x+6}{x-2} \qquad 3 = \frac{3x+6}{x-2}$$

$$5x-10 = 3x+6 \qquad 3x-6 = 3x+6$$

$$2x = 16 \qquad -6 = 6 \quad ⚡$$

$$\underline{\underline{x = 8}}$$

Linear function



$$f(x) = a + bx$$

intercept   slope

F   C

$$F = C$$

$$F = 32 + 1.8 \cdot C$$

$$C = 32 + 1.8 \cdot C$$

$$-32 = 0.8 \cdot C$$

$$-40 = C$$

$$F = a + b \cdot C$$

$$32 = a + b \cdot 0$$

$$32 = a$$

$$212 = a + b \cdot 100$$

$$212 = 32 + b \cdot 100$$

$$180 = b \cdot 100$$

$$b = 1.8$$

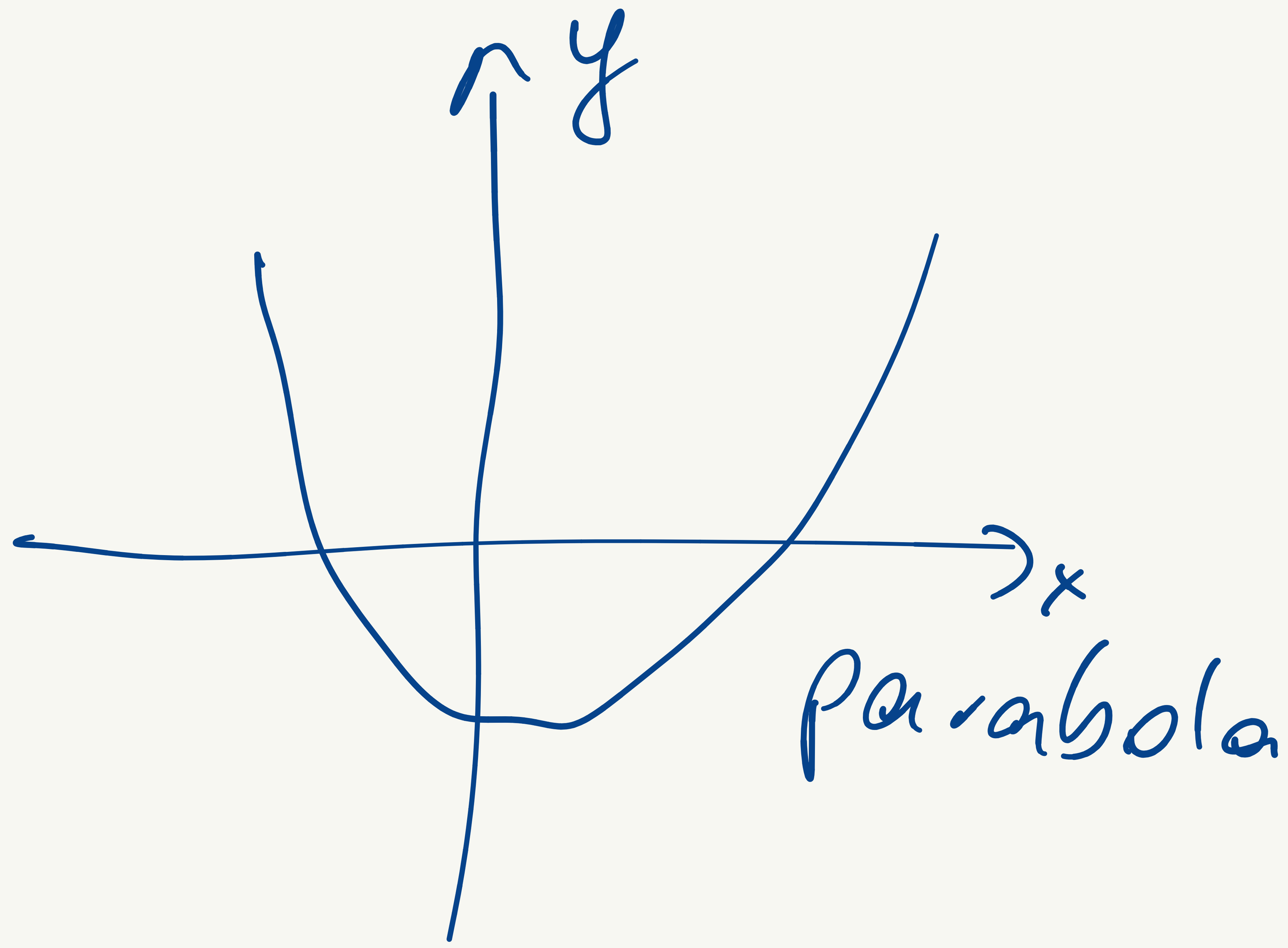
$$\boxed{F = 32 + 1.8C}$$

$$F - 32 = 1.8 \cdot C$$

$$\frac{F - 32}{1.8} = \boxed{C = \frac{1}{1.8} F - 17.777}$$

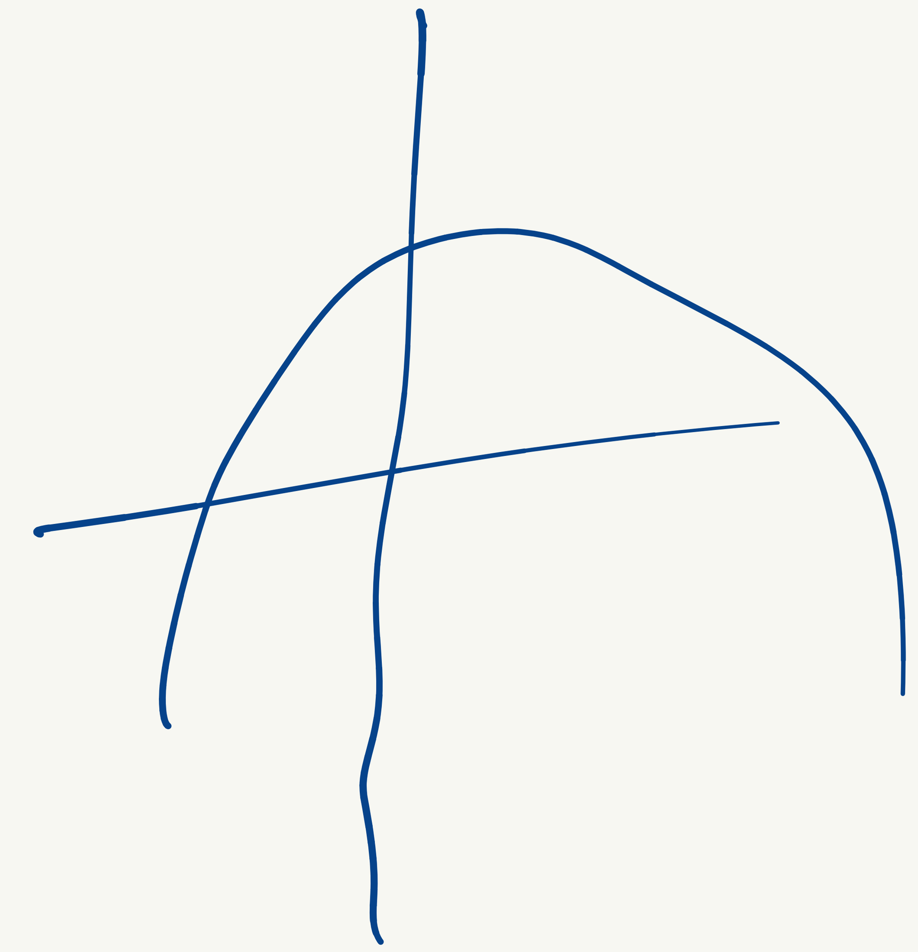
Quadratic functions

$f(x) = ax^2 + bx + c$



if $a > 0$ happy

if $a < 0$ sad



if $a > 0$: convex

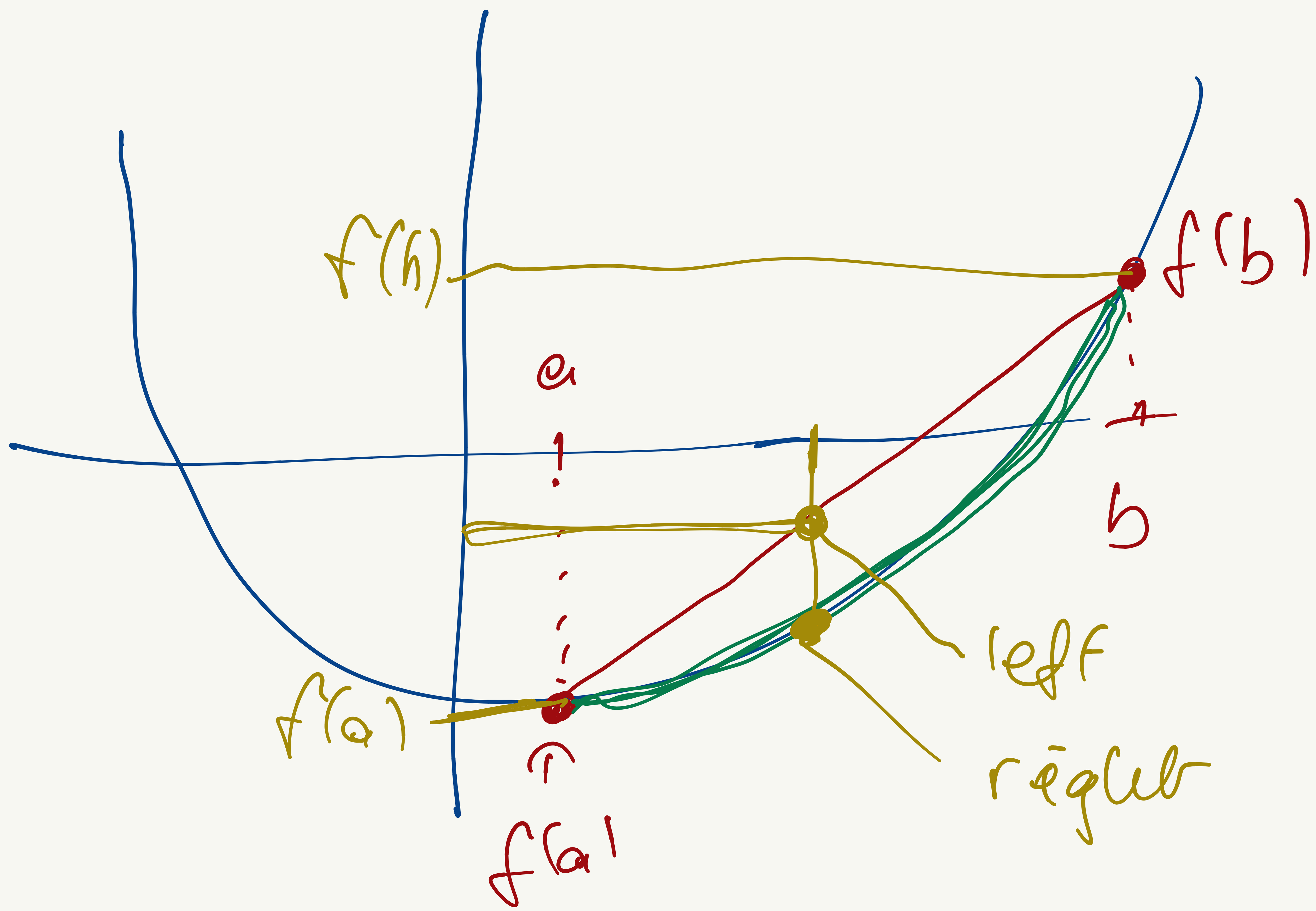
if $a < 0$: concave

min/max point: $\frac{-b}{2a}$

Convex :  $\lambda f(a) + (1-\lambda) f(b) \geq f(\lambda a + (1-\lambda) b)$

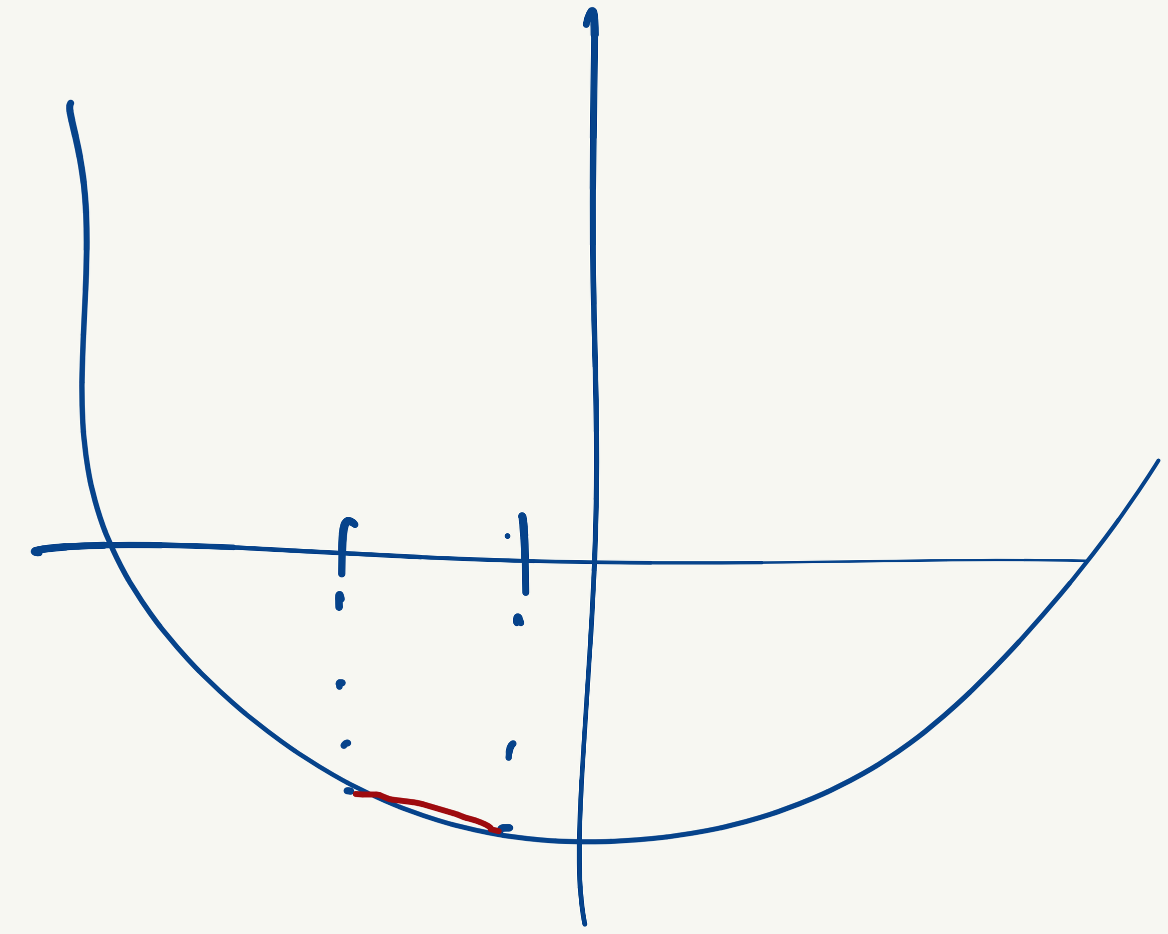
Concave :  $\lambda f(a) + (1-\lambda) f(b) \leq f(\lambda a + (1-\lambda) b)$

$\lambda \in [0,1]$



Right hand side

Left hand side



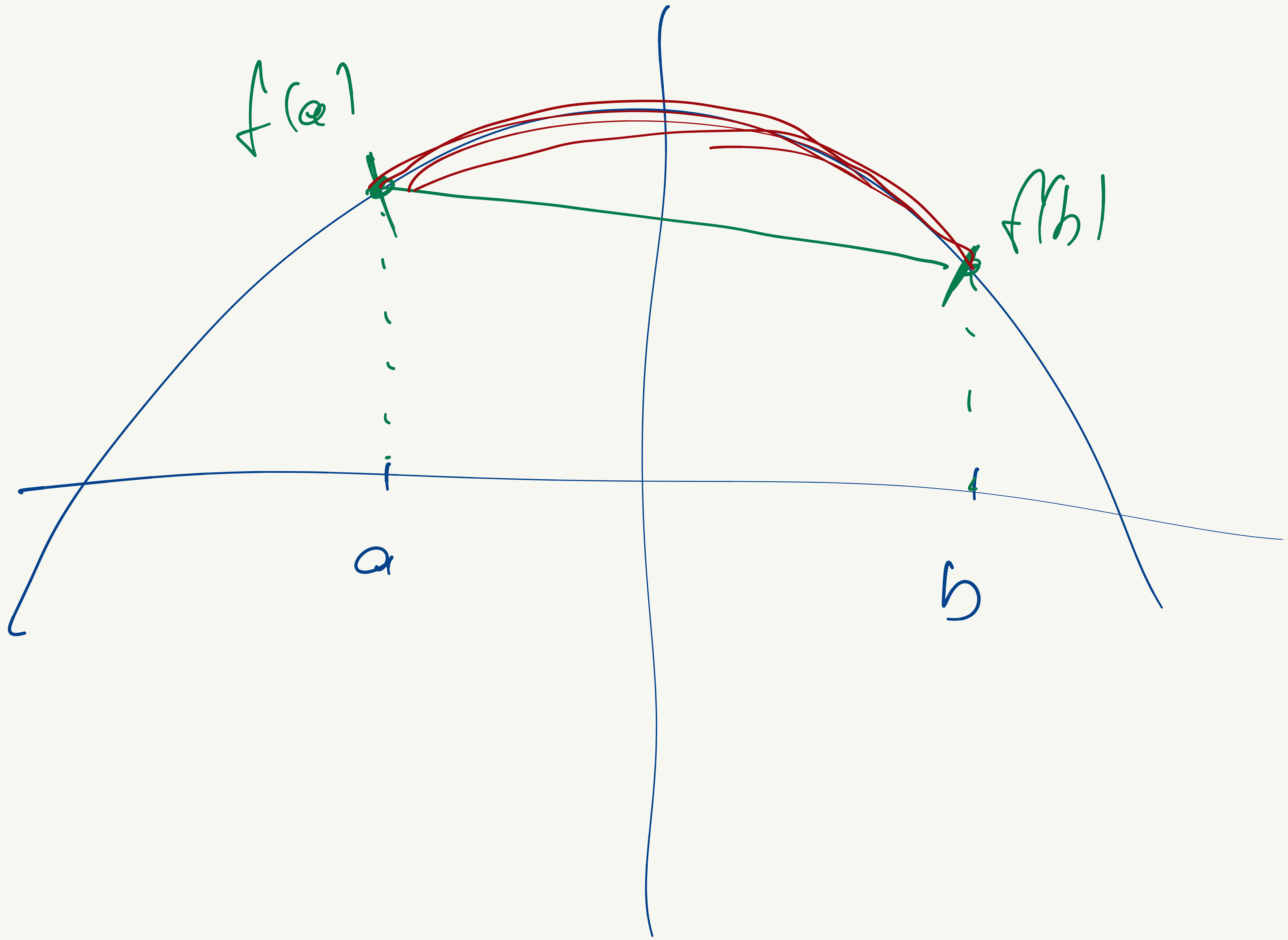
$\lambda = 0.5$

$0.5 f(a) + 0.5 f(b)$

$\lambda = 0.75$

$0.75 f(a) + 0.25 f(b)$

Red is above green
0.5



2 numbers x & y

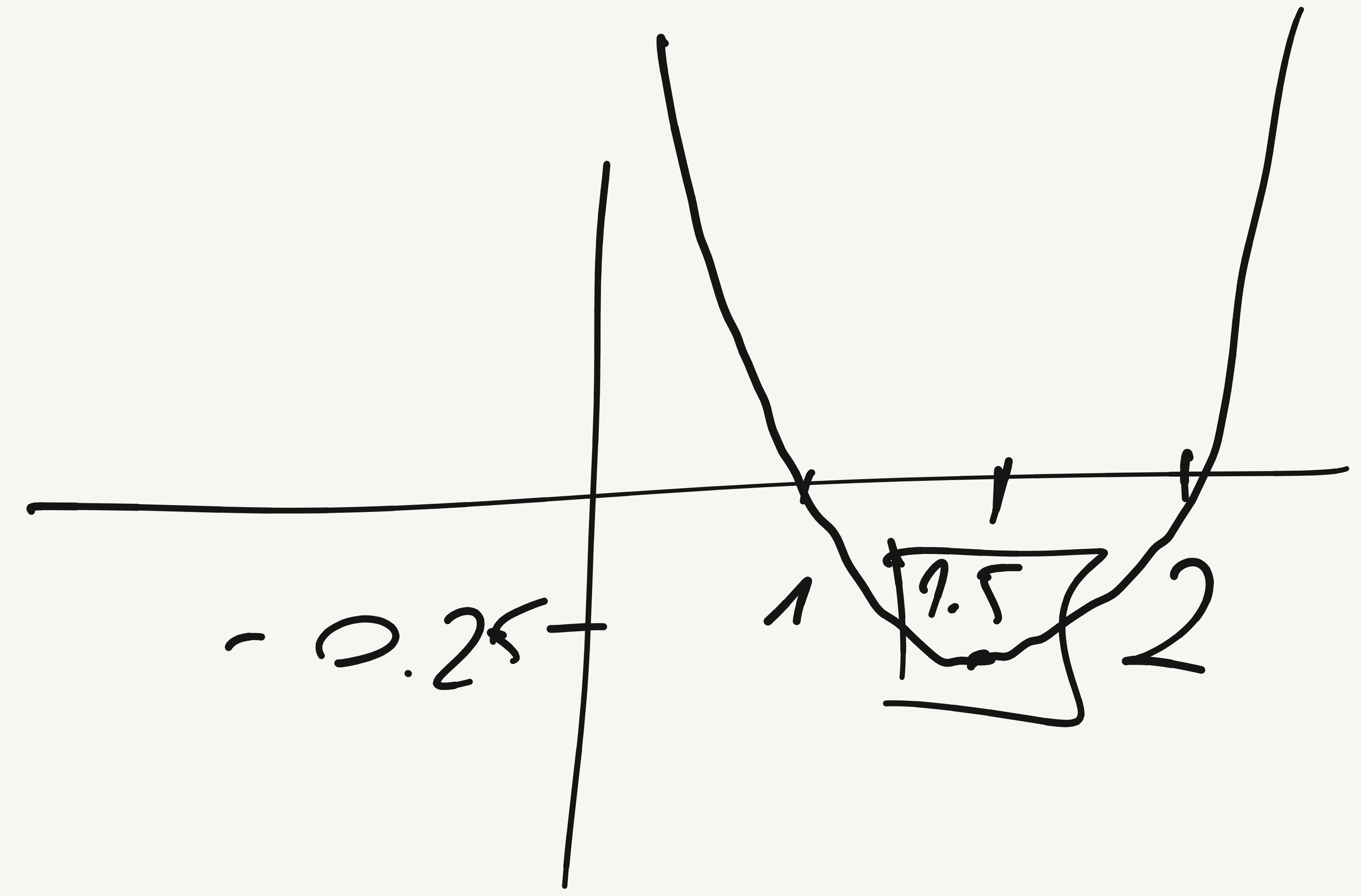
$\lambda x + (1-\lambda) y$ if $\lambda \in [0,1]$: convex combination of x & y

1)

$$f(x) = x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$(1.5-2)(1.5-1) = -0.25$$



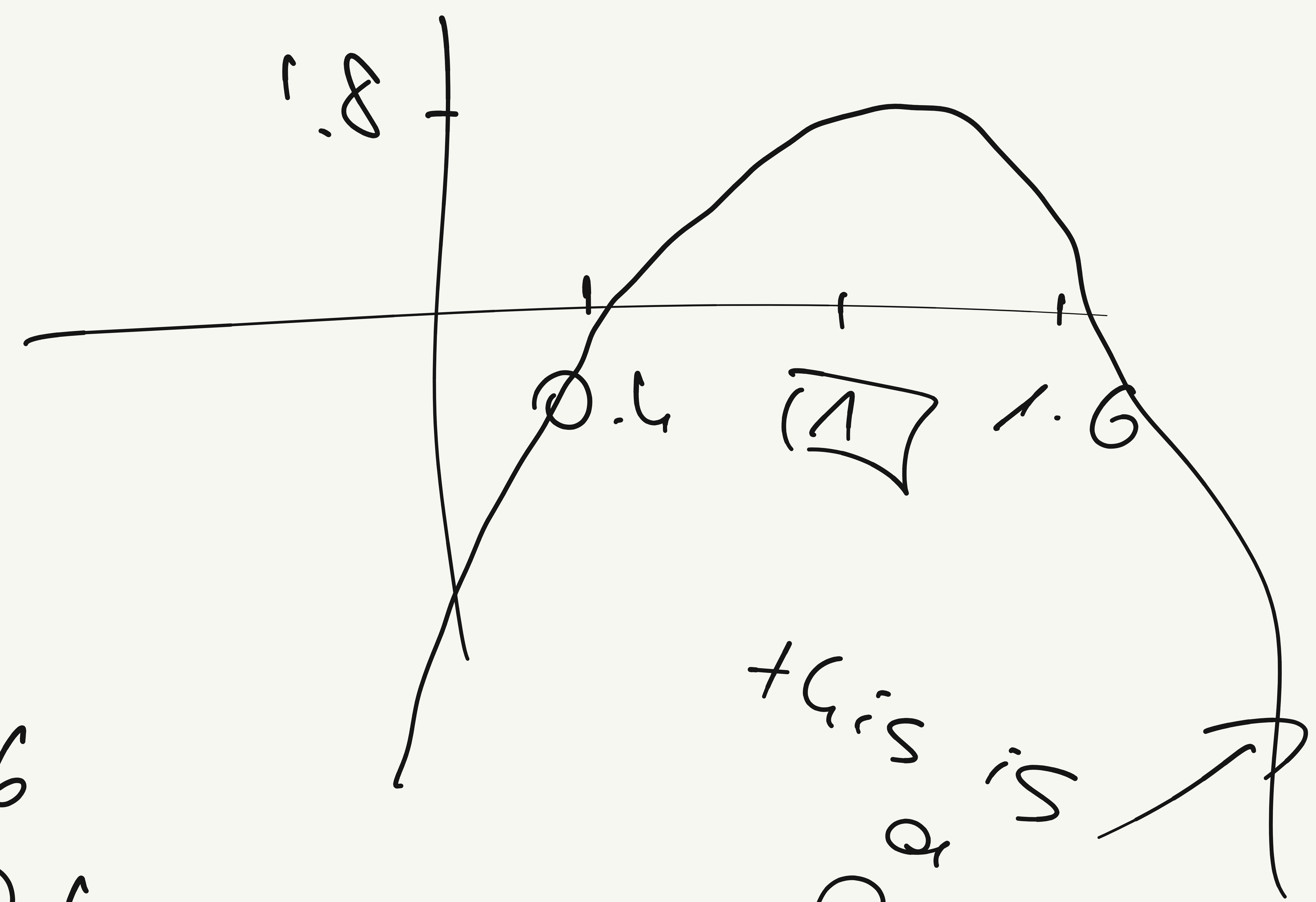
2)

$$f(x) = -5x^2 + 10x - 3.2 = 0$$

$$x_{1,2} = \frac{-10 \pm \sqrt{100 - 4(-5)(-3.2)}}{-10} =$$

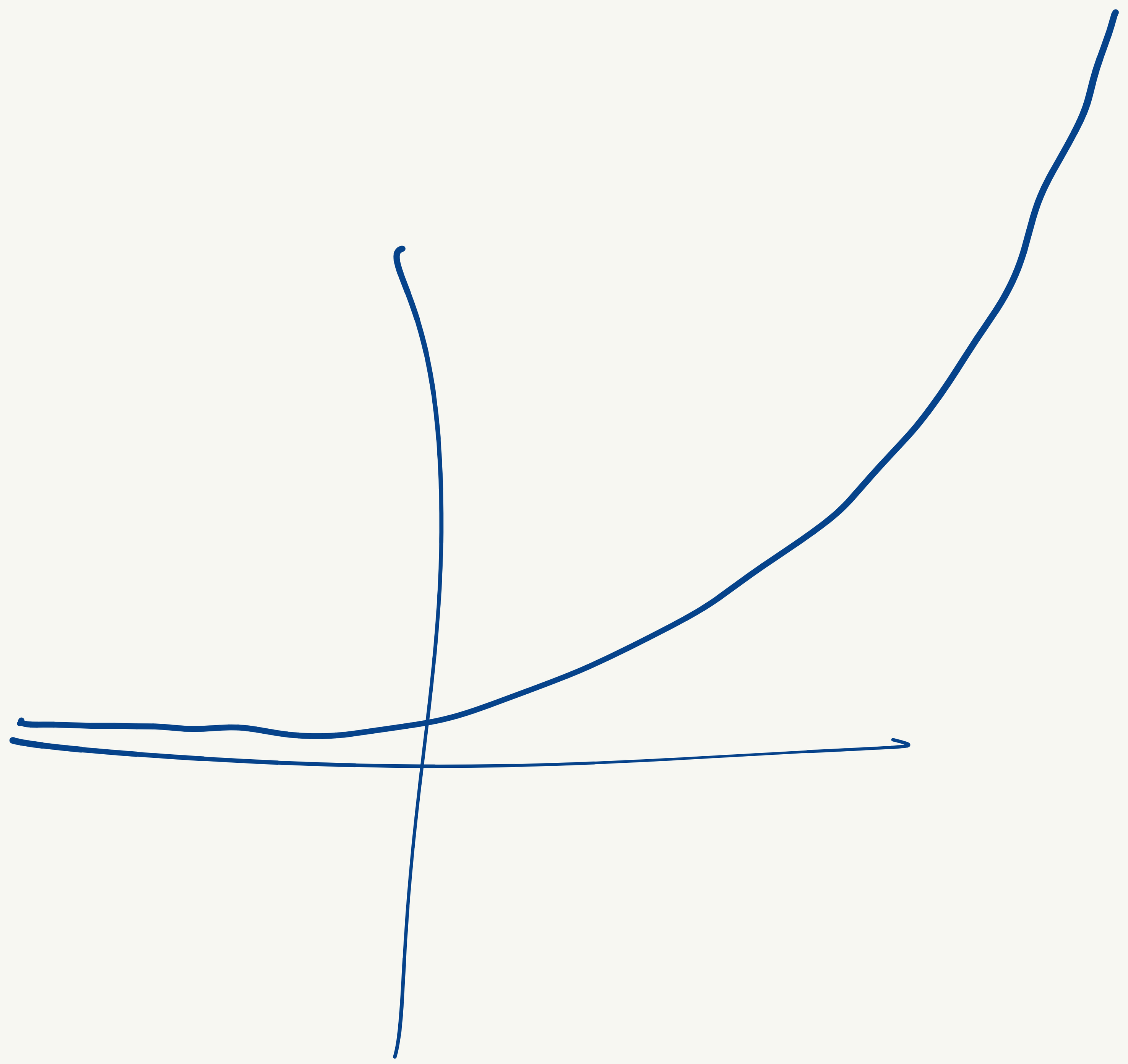
$$= \frac{-10 \pm \sqrt{100-64}}{-10} = \frac{-10 \pm 6}{-10} = \begin{cases} 1.6 \\ 0.4 \end{cases}$$

$$-5 \cdot 1^2 + 10 \cdot 1 - 3.2 = \underline{\underline{1.8}}$$



this is a parabola

1.) $f(x) = 2^x$



$a : 1 \qquad f(a) = f(1) = 2^1 = 2$

$b : 3 \qquad f(b) = f(3) = 2^3 = 8$

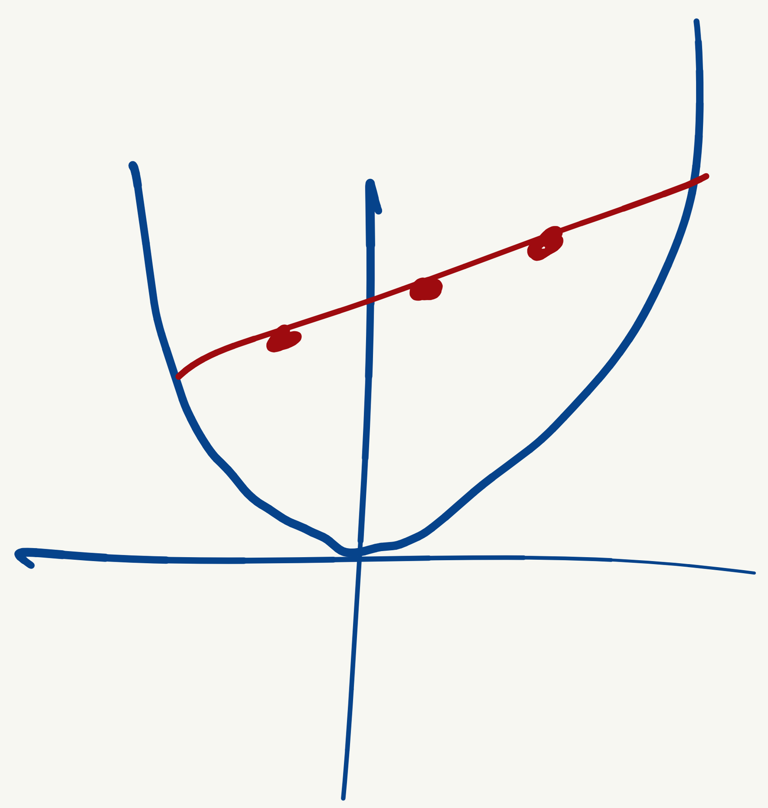
$\lambda = 0.5$

$0.5 \cdot f(1) + 0.5 f(3) \qquad f(0.5 \cdot 1 + 0.5 \cdot 3)$

$0.5 \cdot 2 + 0.5 \cdot 8 \qquad f(2)$

$5 > 2^2 = 4$ convex

2.) $f(x) = x^2$



$a : -5 \qquad f(a) = 25$

$b : 5 \qquad f(b) = 25$

$\lambda = 0.5$

$0.5 \cdot 25 + 0.5 \cdot 25 \qquad f(0.5 \cdot (-5) + 0.5 \cdot 5)$

$f(0)$

$25 > 0$ convex

$\lambda = 0.75$

$0.75 \cdot 25 + 0.25 \cdot 25 \qquad f(0.75 \cdot (-5) + 0.25(5))$

$f(-2.5)$

$25 > 6.25$

$$f(x) = ax^3 + bx^2 + cx + d$$

<u>Polynomials</u> : a polynomial of degree n is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_n, a_{n-1} \dots a_1, a_0 \in \mathbb{R}$

$a_n \neq 0$

degree 4 : $g(x) = -3x^4 + 2x^2 - 5$

degree n -> n roots : points where $f(x) = 0$

there might be multiplicities

some roots might be complex

Power functions

$$f(x) = a x^b$$

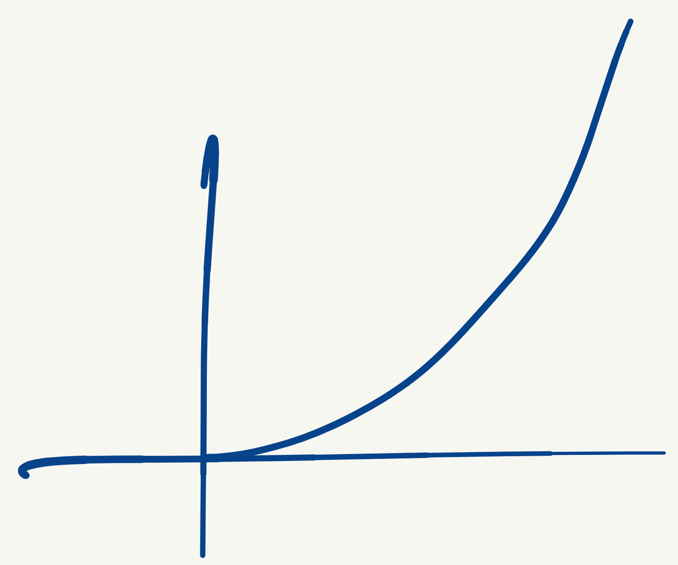
$$a, b \in \mathbb{R}$$

- $S = 4.88V^{2/3}$

if $b > 1$ convex

if $b < 1$ concave

if $v > 0$

$2v^2$ 

$x^{0.5} = \sqrt{x}$ 

Exponential functions

$$f(x) = b \cdot a^x$$

$$a, b \in \mathbb{R}$$

- Invest \$100, 5% interest. After t years

$$f(t) = 100 \cdot 1.05^t$$

- Half-life of x, I_0 initial units:

$$I(t) = I_0 \cdot 0.5^{t/x}$$

The natural exponential function:

base: e

$$f(x) = e^x$$

e: Euler's number: irrational number like π

$$e \approx 2.71828$$

$1 interest rate 100%

100%

!

pays once : $1 \cdot (1+1) = 2$

pays twice : $1 \cdot \left(1+\frac{1}{2}\right)^2 = 2.25$

pays 3 times : $1 \cdot \left(1+\frac{1}{3}\right)^3 = \ldots$

pays n times : $1 \cdot \left(1+\frac{1}{n}\right)^n$

pays in every moment? ∞ $= \$e$

Logarithmic functions

$$b^x = a$$

Let's start with $b = e$

$$e^x = a$$

$$x = \ln a$$

$\ln a$ is a number s.t.

$$e^{\ln a} = a$$

Domain is $\mathbb{R}^+$

$$\ln 1 = 0$$

$$e^{\ln 1} = 1 = e^0$$

$$\ln e = 1$$

$$e^{\ln e} = e = e^1$$

$$\ln e^5 = 5$$

$$e^{\ln e^5} = e^5 = e^5$$

$$\ln \frac{1}{e} = -1$$

$$e^{\ln \frac{1}{e}} = \frac{1}{e} = e^{-1}$$

$\ln(-6) = ↯$ doesn't exist

$$e^{\ln(-6)} = -6$$

Properties:

$$\ln(xy) = \ln(x) + \ln(y)$$

$$e^{\ln(xy)} = xy = \underbrace{e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x)+\ln(y)}}_{\ln(xy) = \ln(x) + \ln(y)}$$

$$e^{\ln(x)} = x$$

$$e^{\ln(y)} = y$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$e^{\ln\left(\frac{x}{y}\right)} = \frac{x}{y} = \frac{e^{\ln(x)}}{e^{\ln(y)}} = e^{\ln(x) - \ln(y)}$$

$$\ln(x^y) = y \ln x$$

$$e^{\ln(x^y)} = x^y = \left(e^{\ln(x)}\right)^y = e^{y \ln(x)}$$

1)

$$5 \cdot e^{-3x} = 16 \quad | :5$$

$$e^{-3x} = \frac{16}{5} \quad | \ln$$

$$\ln(e^{-3x}) = \ln(3.2)$$

$$-3x = \ln(3.2)$$

$$\underline{\underline{x = -\frac{\ln(3.2)}{3}}}$$

2.)

$$1.08^{x} = 10 \quad | \ln$$

$$\ln(1.08^{x}) = \ln(10)$$

$$x \ln(1.08) = \ln(10)$$

$$\underline{\underline{x = \frac{\ln(10)}{\ln(1.08)}}}$$

3) $3^x \, 4^{x+2} = 8 \qquad | \ln$

$$\ln(3^x \, 4^{x+2}) = \ln(8)$$

$$\ln(3^x) + \ln(4^{x+2}) = \ln(8)$$

$$x \ln(3) + (x+2)\ln(4) = \ln(8)$$

$$x \ln(3) + x \ln(4) + 2\ln(4) = \ln(8)$$

$$x(\ln(3) + \ln(4)) = \ln(8) - 2\ln(4)$$

$$x = \frac{\ln(8) - 2\ln(4)}{\ln(3) + \ln(4)}$$

$$\log_a b$$

$$a^{\log_a b} = b$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$a^{\underline{\log_a(xy)}} = xy = a^{\log_a x} \cdot a^{\log_a y} = a^{\underline{\log_a x + \log_a y}}$$

$$a^{\log_a x} \quad a^{\log_a y}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^y) = y \log_a x$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

ln : base e   log : base e or base 10

$$\log_5 25 = 2$$

$$5^{\log_5 25} = 25 = 5^2$$

$$\log_{125} 5 = \frac{1}{3}$$

$$125^{\log_{125} 5} = 5 = 125^{\frac{1}{3}} = \sqrt[3]{125}$$

$$\log_{11}(x+5) = -1$$

$$11^{\log_{11}(x+5)} = 11^{-1}$$

$$x+5 = \frac{1}{11}$$

$$x = \frac{1}{11} - 5 = -\frac{54}{11}$$

$$\log_7(-8x) = 1$$

$$7^{\log_7(-8x)} = 7^1$$

$$-8x = 7$$

$$x = -\frac{7}{8}$$

$e \qquad e^2 \qquad e^3 \qquad e^4 \qquad e^5$

$\ln \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$

$100 \longleftrightarrow 10\,000\,000$

$\downarrow \qquad\qquad \downarrow$

$\log_{10} \qquad 2 \longleftrightarrow 7$

Calculus

- Sequences
- Limits
- Differentiation
- Optimization

(- Plotting functions)

Sequences

collection of numbers

a_n : nth element of sequence a

Prime numbers:

$$a_1 = 2$$
$$a_2 = 3$$
$$a_3 = 5$$
$$a_4 = 7$$
$$\vdots$$

$$a_n = \frac{n}{2} \rightarrow a_1 = \frac{1}{2} \qquad a_{153} = \frac{153}{2}$$
$$a_2 = \frac{2}{2}$$

Finite / Infinite :

- Finite # elements : 1-digit positive integers

  $a_1 = 1 \quad \ldots\ldots \quad a_9 = 9$

- Infinite : Prime numbers, $a_n = \frac{n}{2}$

Increasing / Decreasing :

- Increasing if $a_{n+1} \geq a_n \quad \forall n$ $\qquad \forall$: for all
- Decreasing if $a_{n+1} \leq a_n \quad \forall n$ $\qquad \exists$: exists

Boundedness :

- Bounded from above if $\exists N$ such that $a_n < N \quad \forall n$
- — " — below if $\exists M$ such that $a_n > M \quad \forall n$

Bounded from above: - positive integers < 10

- $a_n = \frac{1}{n}$

$a_1 = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{3}$

< 2

Bounded from below: - prime #

- $a_n = n$
- $a_n = n^2$

Limit of a sequence: A number to which the sequence elements get closer and closer.

$$a_n \to A$$

$$\lim_{n \to \infty} a_n = A$$

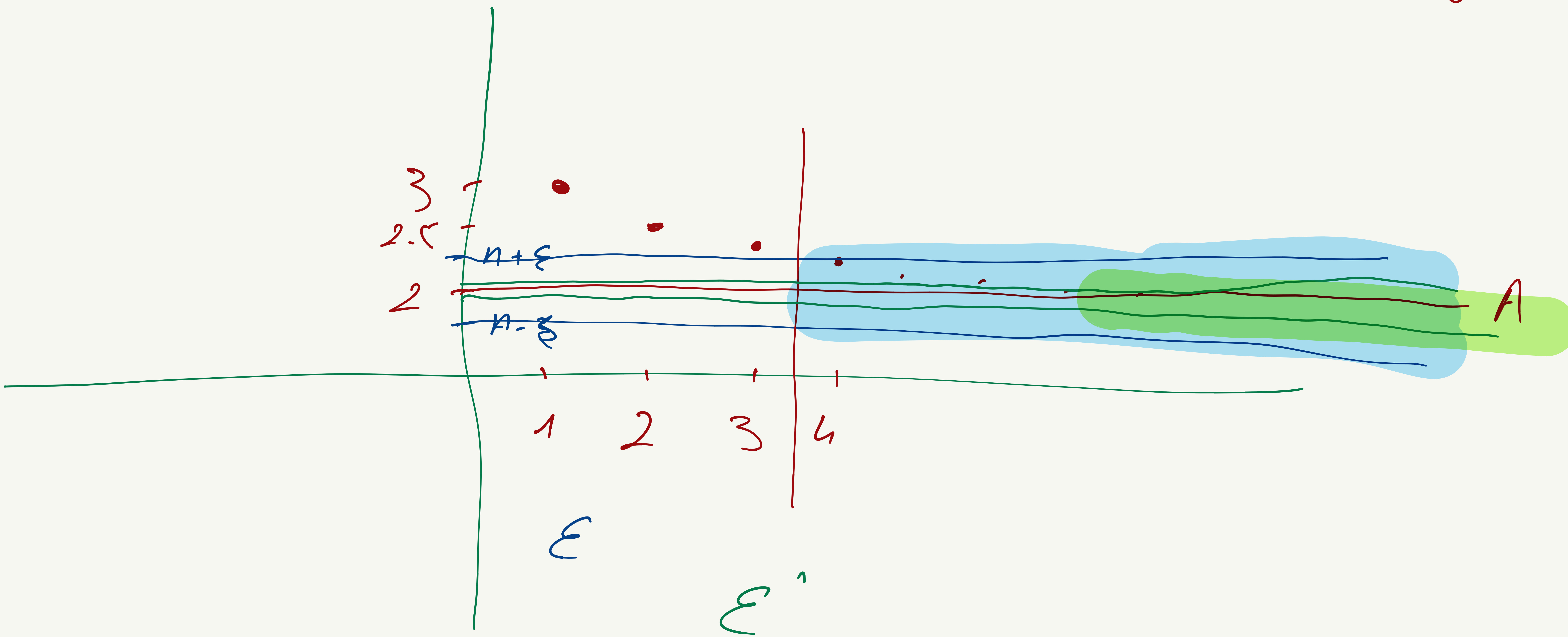
$a_n = 5$ $\quad a_1 = 5 \quad a_2 = 5 \quad a_3 = 5 \quad \ldots \quad \lim_{n \to \infty} a_n = 5$

$a_n = \frac{1}{n}$ $\quad a_1 = \frac{1}{1} \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{3} \quad \ldots \quad \lim_{n \to \infty} a_n = 0$

$$0 = \frac{1}{n}$$

The limit of a sequence a_n is A if $\forall \varepsilon > 0$ $\exists N$ such that if $n > N$ $|a_n - A| < \varepsilon$.

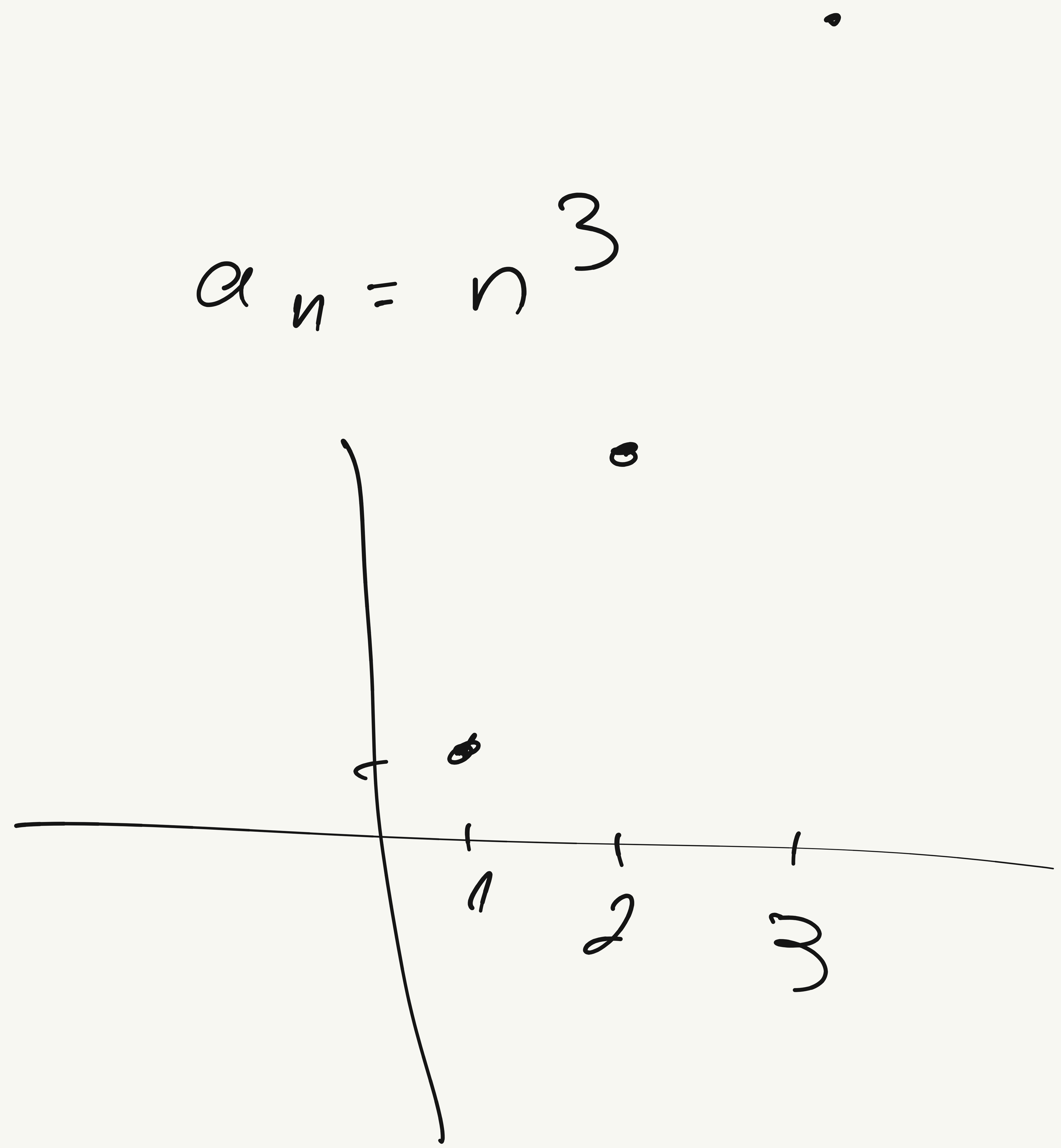
$a_n = 2 + \frac{1}{n}$



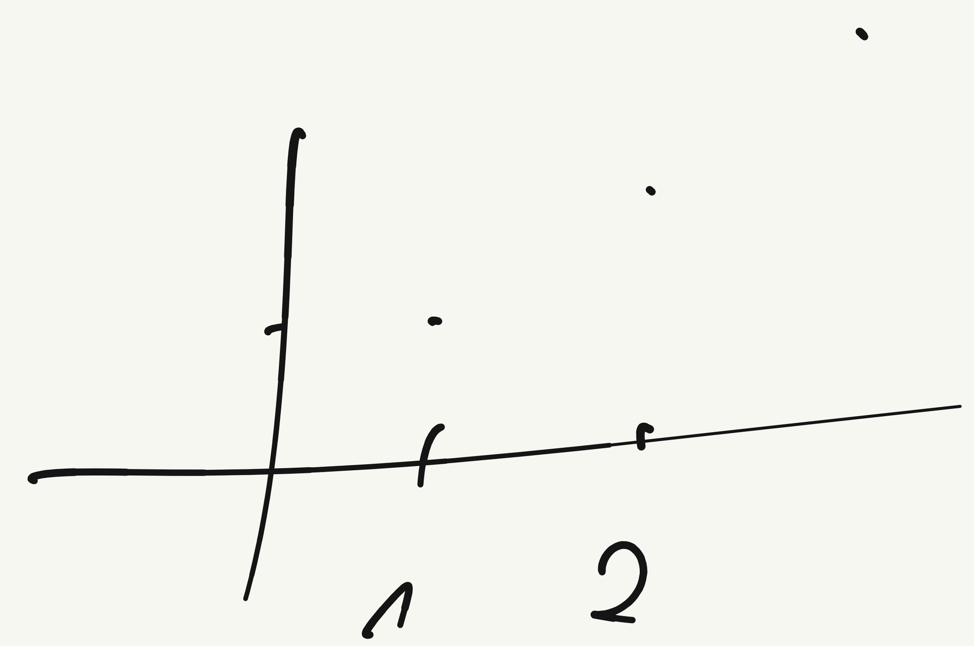
If a sequence has a limit then it is called convergent $\leftrightarrow$ divergent

Divergent sequence:

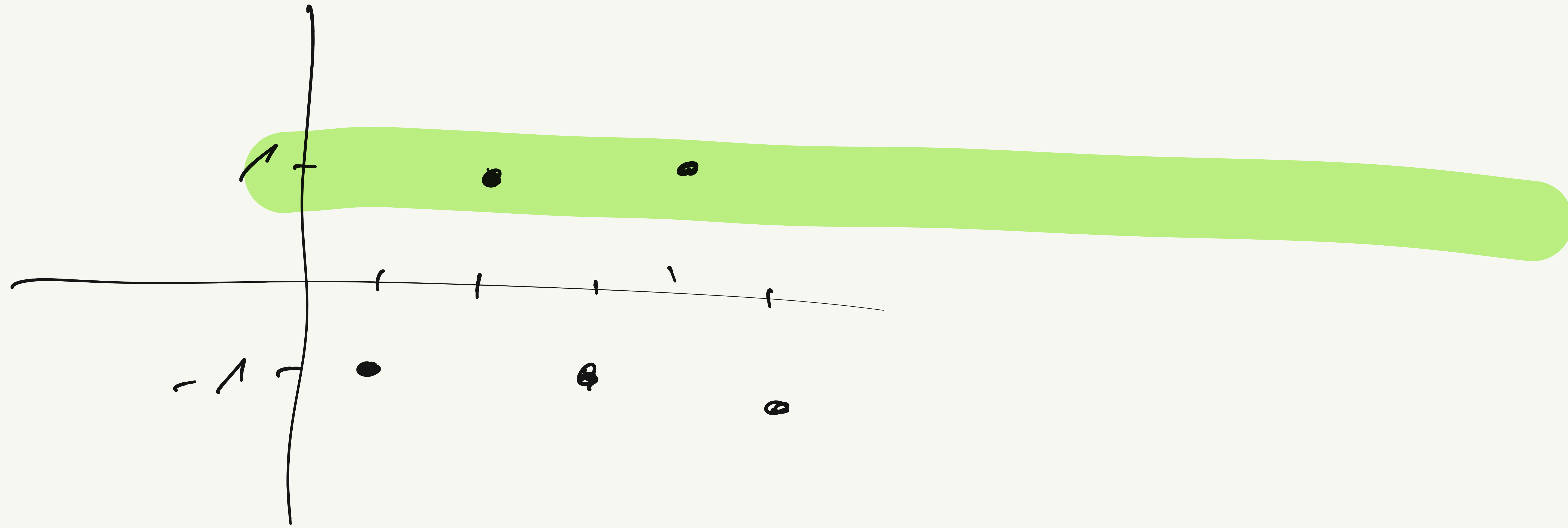
$$a_n = n^3$$



$$a_n = n$$



$$a_n = (-1)^n$$



$$a_n = (-1)^n \cdot \frac{1}{n}$$

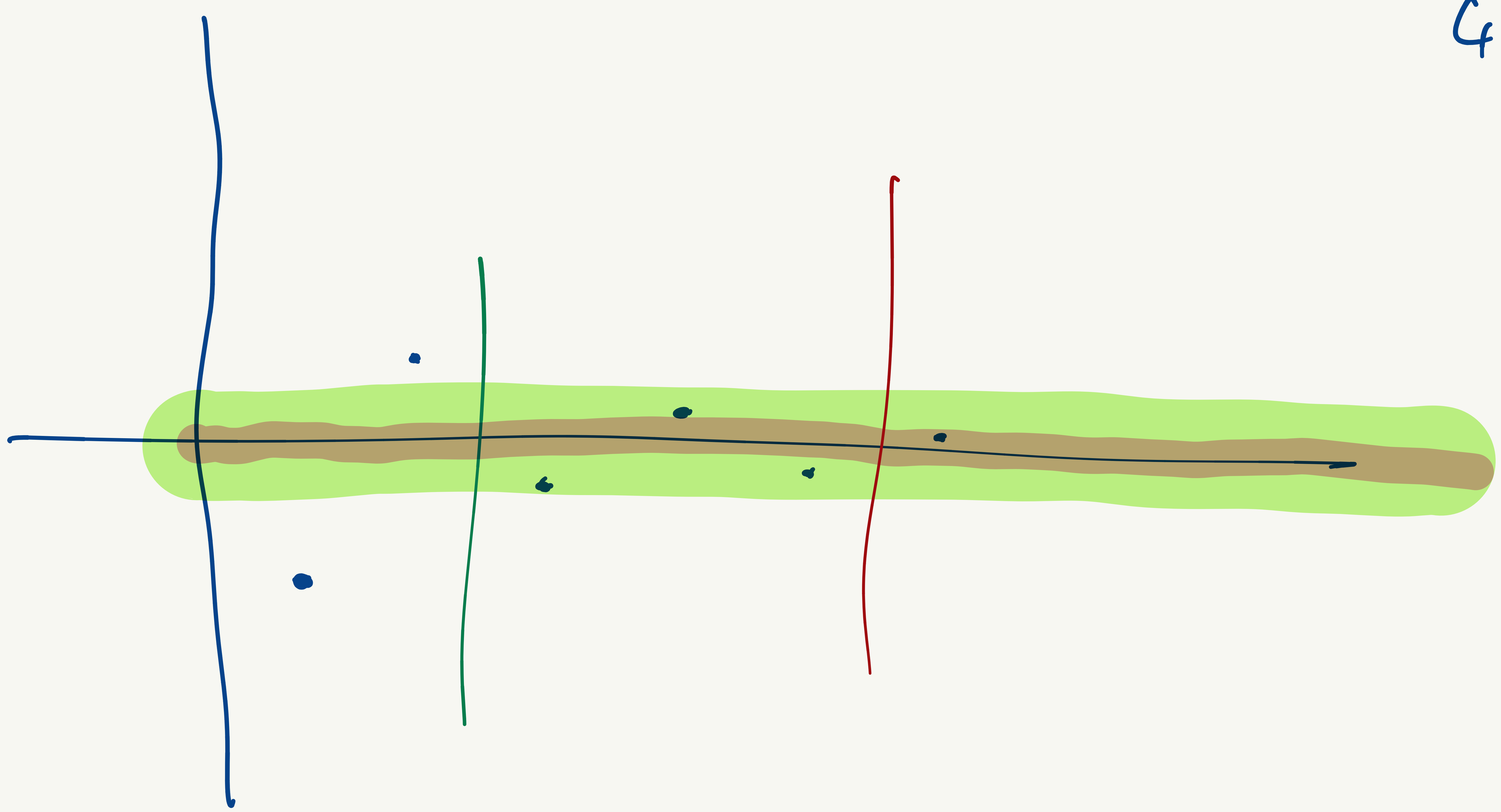
$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

$$a_4 = \frac{1}{4}$$

$$\lim_{n \to \infty} a_n = 0$$



$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n{}^p = \left(\lim_{n\to\infty} a_n\right)^p$$

$$\lim_{n\to\infty} \frac{n^2-3}{n^3-2} = \lim_{n\to\infty} \frac{\frac{1}{n^3}(n^2-3)}{\frac{1}{n^3}(n^3-2)} = \lim_{n\to\infty} \frac{\frac{1}{n}-\frac{3}{n^3}}{1-\frac{2}{n^3}} =$$

$$= \frac{\lim\limits_{n\to\infty} \frac{1}{n}-\frac{3}{n^3}}{\lim\limits_{n\to\infty} 1-\frac{2}{n^3}} = \frac{\overbrace{\lim\limits_{n\to\infty} \frac{1}{n}}^{0} - \lim\limits_{n\to\infty} \frac{3}{n^3}}{\underbrace{\lim\limits_{n\to\infty} 1}_{1} - \lim\limits_{n\to\infty} \frac{2}{n^3}} = \frac{-\lim\limits_{n\to\infty} \frac{3}{n^3}}{1-\lim\limits_{n\to\infty} \frac{2}{n^3}} =$$

$$= \frac{-3\lim\limits_{n\to\infty} \frac{1}{n^3}}{1-2\lim\limits_{n\to\infty} \frac{1}{n^3}} = \frac{-3\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^3}{1-2\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^3} = \frac{-3\Big[\overbrace{\lim\limits_{n\to\infty} \frac{1}{n}}^{0}\Big]^3}{1-2\Big[\underbrace{\lim\limits_{n\to\infty} \frac{1}{n}}_{0}\Big]^3} = \frac{-3\cdot 0^3}{1-2\cdot 0^3} = \frac{0}{1} = \boxed{0}$$

$$\lim_{n\to\infty} \frac{n^4+5n^3+3n^2-2}{3n^4-6} = \lim_{n\to\infty} \frac{\frac{1}{n^4}\left(n^4+5n^3+3n^2-2\right)}{\frac{1}{n^4}\left(3n^4-6\right)} = \frac{\lim\limits_{n\to\infty} 1+\underset{\to 0}{\frac{5}{n}}+\underset{\to 0}{\frac{3}{n^2}}-\underset{\to 0}{\frac{2}{n^4}}}{\lim\limits_{n\to\infty} 3-\underset{\to 0}{\frac{6}{n^4}}} =$$

$$= \underline{\underline{\frac{1}{3}}}$$

$$\lim_{n\to\infty} \frac{5}{n+1}+\frac{n}{n+1} = \lim_{n\to\infty}\frac{5}{n+1} + \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{\frac{5}{n}}{1+\frac{1}{n}} + \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} =$$

$$= \frac{\lim\limits_{n\to\infty}\underset{\to 0}{\frac{5}{n}}}{\underset{\to 1}{\lim\limits_{n\to\infty} 1}+\underset{\to 0}{\frac{1}{n}}} + \frac{\underset{\to 1}{\lim\limits_{n\to\infty} 1}}{\underset{\to 1}{\lim\limits_{n\to\infty} 1}+\underset{\to 0}{\frac{1}{n}}} = \frac{0}{1+0}+\frac{1}{1+0} = 0+1 = \underline{\underline{1}}$$

$$\lim_{n \to \infty} b^n$$

$b = 0$ $\quad \lim_{n \to \infty} 0^n = 0$

$b = 1$ $\quad \lim_{n \to \infty} 1^n = 1$

$b > 1$ $\quad \lim_{n \to \infty} b^n \Rightarrow$ divergent $\to \infty$

$b < -1$ $\quad \lim_{n \to \infty} b^n \Rightarrow$ divergent

$-1 < b < 1$ $\quad \lim_{n \to \infty} b^n = 0$