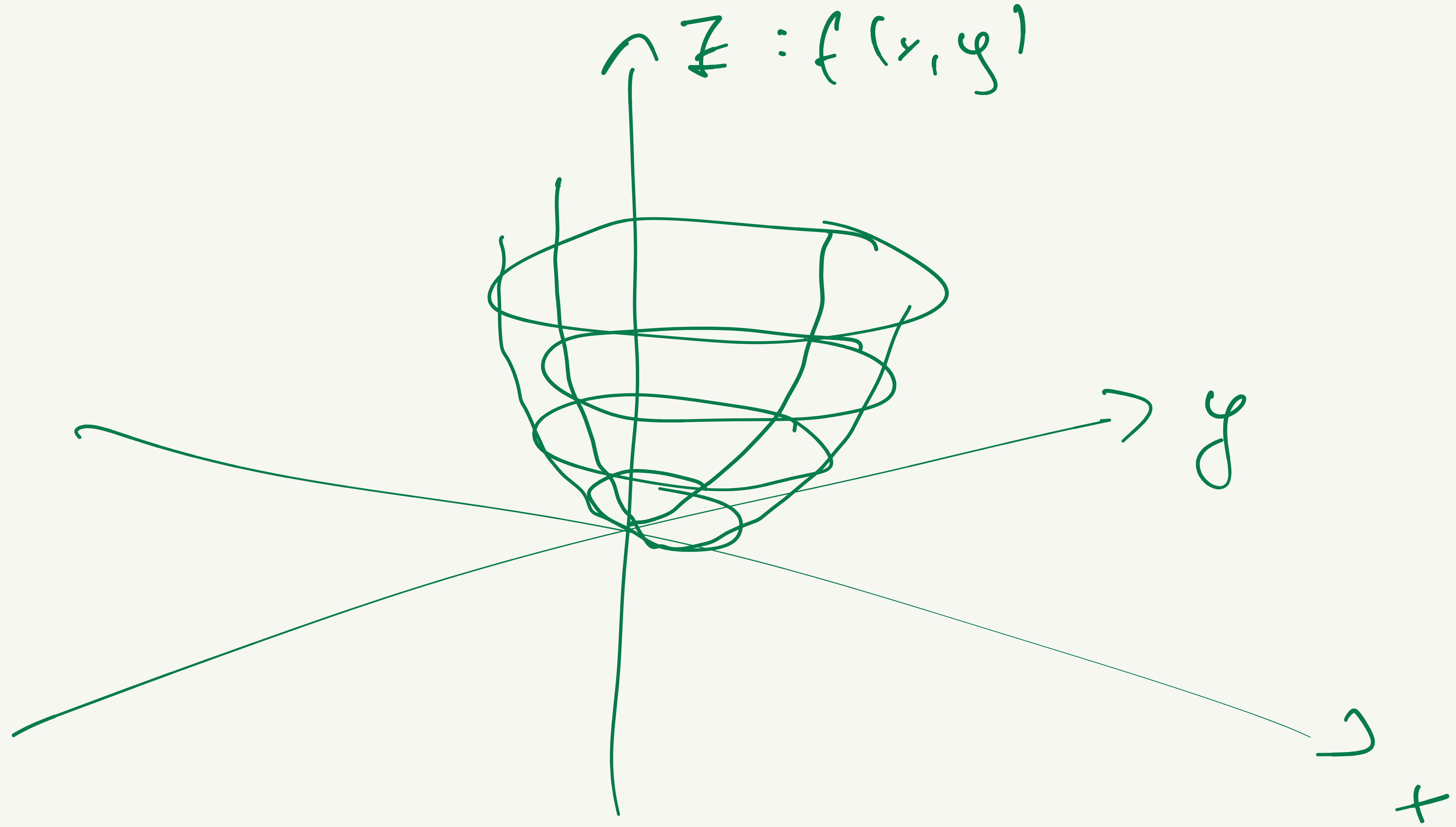
$f(x,y) = x^2 + 3y$

$f(1,2) = 1^2 + 3 \cdot 2 = \underline{\underline{7}}$



$f(x,y) = x^2 + y^2$

minima/maxima of multivariate functions?

Partial derivative : slope of a multivariate function along a certain axis.

min/max : the slope along each axis has to be equal to zero.

$f(x,y)$ partial derivative w.r.t. x : $f'_x(x,y)$ | $\frac{\partial f(x,y)}{\partial x}$

$f(x,y)$ — " — y : $f'_y(x,y)$ | $\frac{\partial f(x,y)}{\partial y}$

All rules still hold !!!

If I calculate the partial derivative w.r.t. x, I treat all other variables as if they were constant numbers.

$$f(x,y) = x^2 + 2xy + y^2$$

$$f'_x(x,y) = 2x + 2y \cdot 1 + 0 = \underline{\underline{2x+2y}}$$

$$f'_y(x,y) = 0 + 2x \cdot 1 + 2y = \underline{\underline{2x+2y}}$$

$$f(x,y) = \underline{x \cdot \ln(y)} + \frac{e^{x}}{y^{4}} = x \cdot \ln(y) + y^{-4} \cdot e^{x}$$

$$\frac{\partial f(x,y)}{\partial x} = \ln(y) \cdot 1 + y^{-4} \cdot e^{x}$$

$$\frac{\partial f(x,y)}{\partial y} = x \cdot \frac{1}{y} + e^{x} \cdot (-4) \cdot y^{-5}$$

$$f(x,y,z) = xyz + e^y \ln(x) z^5 + e^x$$

$$f'_x(x,y,z) = yz \cdot 1 + e^y z^5 \cdot \frac{1}{x} + e^x$$

$$f'_y(x,y,z) = xz \cdot 1 + \ln(x) z^5 e^y + 0$$

$$f'_z(x,y,z) = xy \cdot 1 + e^y \ln(x) 5 \cdot z^4 + 0$$

$$g(x,y) = 42x + 47y$$

$$g'_x(x,y) = 42$$

$$g'_y(x,y) = 42$$

$$f(x,y) = x^2 \ln(y) + \frac{e^y}{\ln(x)} = x^2 \ln(y) + e^y (\ln(x))^{-1}$$

$$f'_x(x,y) = \ln(y) 2 \cdot x + e^y \left(-1 \cdot (\ln(x))^{-2}\right) \cdot \frac{1}{x}$$

$$f(x) = x^{-1} \qquad f'(x) = -1 \cdot x^{-2}$$

$$g(x) = \ln(x) \qquad g'(x) = \frac{1}{x}$$

$$f'_y(x,y) = x^2 \frac{1}{y} + (\ln(x))^{-1} \cdot e^y$$

$$h(x,y,z) = \frac{z^5 \cdot e^y}{y^2 \ln(z)\, x} = \frac{z^5 e^y}{y^2 \ln(z)} \cdot x^{-1} = \frac{z^5}{\ln(z)\, x} \cdot \frac{e^y}{y^2} = \frac{e^y}{y^2 x} \cdot \frac{z^5}{\ln(z)}$$

$$\frac{\partial h(x,y,z)}{\partial x} = \frac{z^5 e^y}{y^2 \ln(z)} (-1) x^{-2}$$

$$\frac{\partial h(x,y,z)}{\partial y} = \frac{z^5}{\ln(z)\, x} \cdot \left(\frac{e^y \cdot y^2 - e^y (2y)}{(y^2)^2} \right)$$

$$\frac{\partial h(x,y,z)}{\partial z} = \frac{e^y}{y^2 x} \cdot \left(\frac{5z^4 \ln(z) - z^5 \cdot \frac{1}{z}}{(\ln(z))^2} \right)$$

Minima / maxima:

1. Get each partial derivative
2. These will form a system of equations, as they should equal 0
3. Solve system of equations => potential points of minima / maxima

$f(x,y)$

vector

$$\begin{bmatrix} f'_x(x,y) \\ f'_y(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$: GRADIENT

$h(x,y,z)$

$$\begin{bmatrix} h'_x(x,y,z) \\ h'_y(x,y,z) \\ h'_z(x,y,z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

THE GRADIENT HAS TO BE ~~A~~ NULL-VECTOR

$$f(x,y) = x^2y^2 - 5x - 5y$$

$$f'_x(x,y) = y^2\cdot 2x - 5 = 0 \quad \Rightarrow \quad y^2\cdot 2x = 5$$

$$f'_y(x,y) = x^2\,2y - 5 = 0 \quad \Rightarrow \quad x^2\cdot 2y = 5$$

$$y^2\cdot 2x = x^2\cdot 2y$$

$$y = x$$

MINIMUM

$$y^2\cdot 2x = 5$$

$$2x^3 = 5$$

$$x^3 = 2.5$$

$$x = \sqrt[3]{2.5} = y$$

$$f(x,y) = -xy\cdot e^{-x^2-y^2} = -y\left[x\cdot e^{-x^2-y^2}\right]$$

$$f'_x(x,y) = -y\left((x)'_x\cdot e^{-x^2-y^2} + x\cdot\left(e^{-x^2-y^2}\right)'_x\right) =$$

$$= -y\left(e^{-x^2-y^2} + x\cdot e^{-x^2-y^2}\cdot(-2x)\right) = -y\,e^{-x^2-y^2}\left(1+(-2x)x\right) =$$

$$f(x) = e^x \qquad f'(x) = e^x \qquad = -y\,e^{-x^2-y^2}\left(1-2x^2\right)$$

$$g(x,y) = -x^2-y^2 \qquad g'_x(x,y) = -2x$$

$$f'_y(x,y) = -v\,e^{-x^2-y^2}\left(1-2y^2\right)$$

$$f'_x(x,y) = -y \boxed{e^{-x^2-y^2}} (1-2x^2) = 0$$

$\to y = 0$

$\to 1-2x^2 = 0 \to 1 = 2x^2 \to x = \pm\sqrt{0.5}$

$$f'_y(x,y) = -x \boxed{e^{-x^2-y^2}} (1-2y^2) = 0$$

$\to x = 0$

$\to 1-2y^2 = 0 \to 1 = 2y^2 \to y = \pm\sqrt{0.5}$

$e^{-x^2-y^2} > 0$

1. $y = 0$	1. $x = \sqrt{0.5}$	1. $x = \sqrt{0.5}$	1. $x = -\sqrt{0.5}$	1. $x = -\sqrt{0.5}$
2. $x = 0$	2. $y = \sqrt{0.5}$	2. $y = -\sqrt{0.5}$	2. $y = \sqrt{0.5}$	2. $y = -\sqrt{0.5}$
1	2	3	4	5

minima / maxima ?

$h(x,y,z) \Rightarrow 9$ combinations

$f''_{xx}(x,y)$ $\quad f''_{xy}(x,y)$

$f''_{yy}(x,y)$ $\quad f''_{yx}(x,y)$

Hesse matrix:

$$H = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} : \text{matrix}$$

$\Rightarrow$ positive definite?
negative definite?

$\Rightarrow$ Long calculation

$f(x,y) = x^2 + 9x + y^2 - 6y + 15$

$f'_x(x,y) = 2x + 9 = 0 \rightarrow \underline{\underline{x = -4.5}}$

$f'_y(x,y) = 2y - 6 = 0 \rightarrow \underline{\underline{y = 3}}$

$\Big\}$ MINIMUM

$h(x,y) = e^{-x^2-y^2}$

$g(x,y) = -x^2 - y^2$

$h'_x(x,y) = e^{-x^2-y^2}(-2x) = 0 \rightarrow x = 0$

$h'_y(x,y) = e^{-x^2-y^2}(-2y) = 0 \rightarrow y = 0$

$g'_x(x,y) = -2x = 0$ $\quad x = 0$

$g'_y(x,y) = -2y = 0$ $\quad y = 0$

$f(x) = e^x$ $\quad f'(x) = e^x$

$g(x,y) = -x^2 - y^2$

$\rightarrow -2x$

$\rightarrow -2y$

Constrained optimization

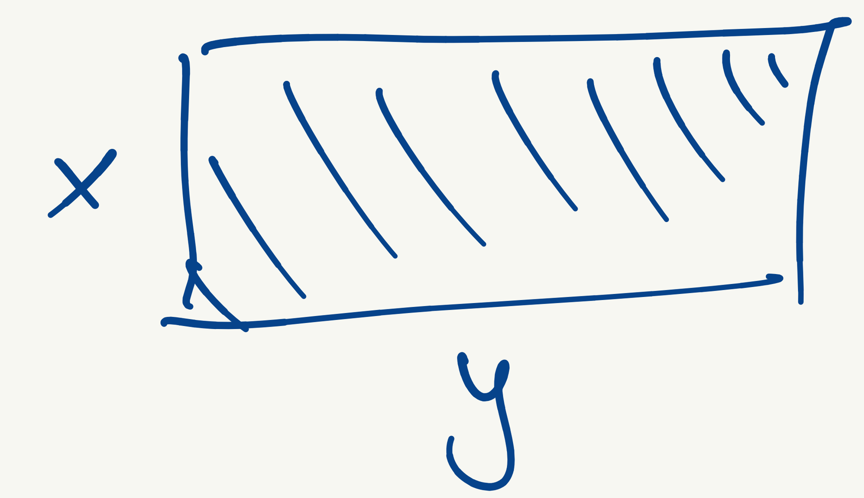
- Max / Min a function
- Under a constraint

$$f(x,y) = 2x + 2y$$

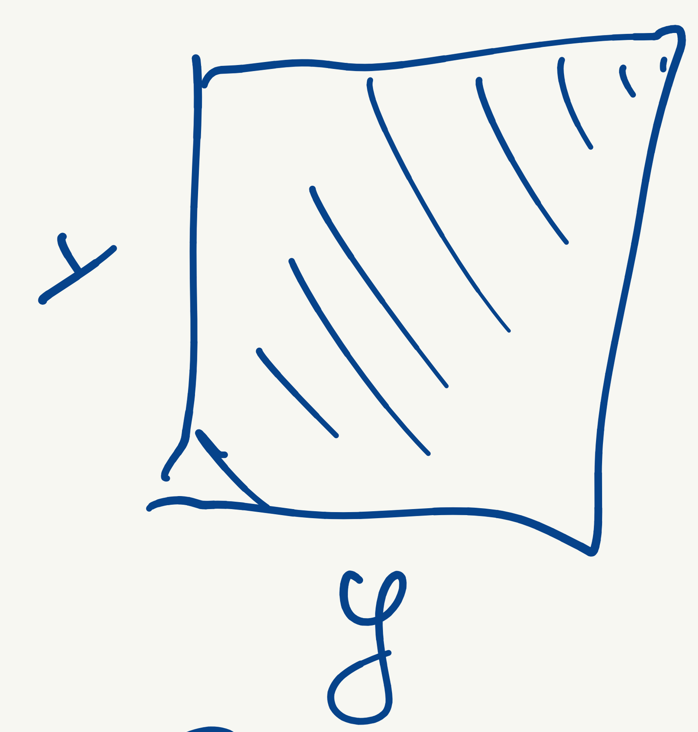
s.t. $x + y = 20$

$$g(x,y) = x \cdot y$$

s.t. $x + y = 20$



$x = 10$
$y = 10$



- **Lagrangian**
  - function $f(x,y)$
  - constraint $g(x,y) = 0$

$\mathcal{L} = f(x,y) - \lambda g(x,y) \Rightarrow$ partial derivatives $= 0$

Linear Algebra

- Vectors
- Matrices
- System of linear equations

Vector

A vector of order n is a set of n numbers:

- row vector: $\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}$
- column vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & 2 & 54 & \pi \end{bmatrix}$$

$$y = \begin{bmatrix} e \\ -5 \\ 10.55 \\ 4 \end{bmatrix}$$

Transposition :

$$x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \qquad x^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad y^T = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$$

$$(x^T)^T = x$$

Equality

$$x = y$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$x = y$ if and only if $x_i = y_i \ \forall i$

- We cannot compare row vectors w/ col vectors

  except: $n = 1$

Addition

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad x \pm y = \begin{bmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{bmatrix}$$

Multiplication by scalar

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad c \qquad\qquad c \cdot x = \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ \vdots \\ c\,x_n \end{bmatrix}$$

Division: same.

Scalar product / Inner product / Dot product :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$(x, y) = x^T y = \sum_{i=1}^{n} x_i y_i$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad (x, y) = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = \underline{\underline{32}}$$

x : Quantities
y : prices

(x, y) : value of all goods

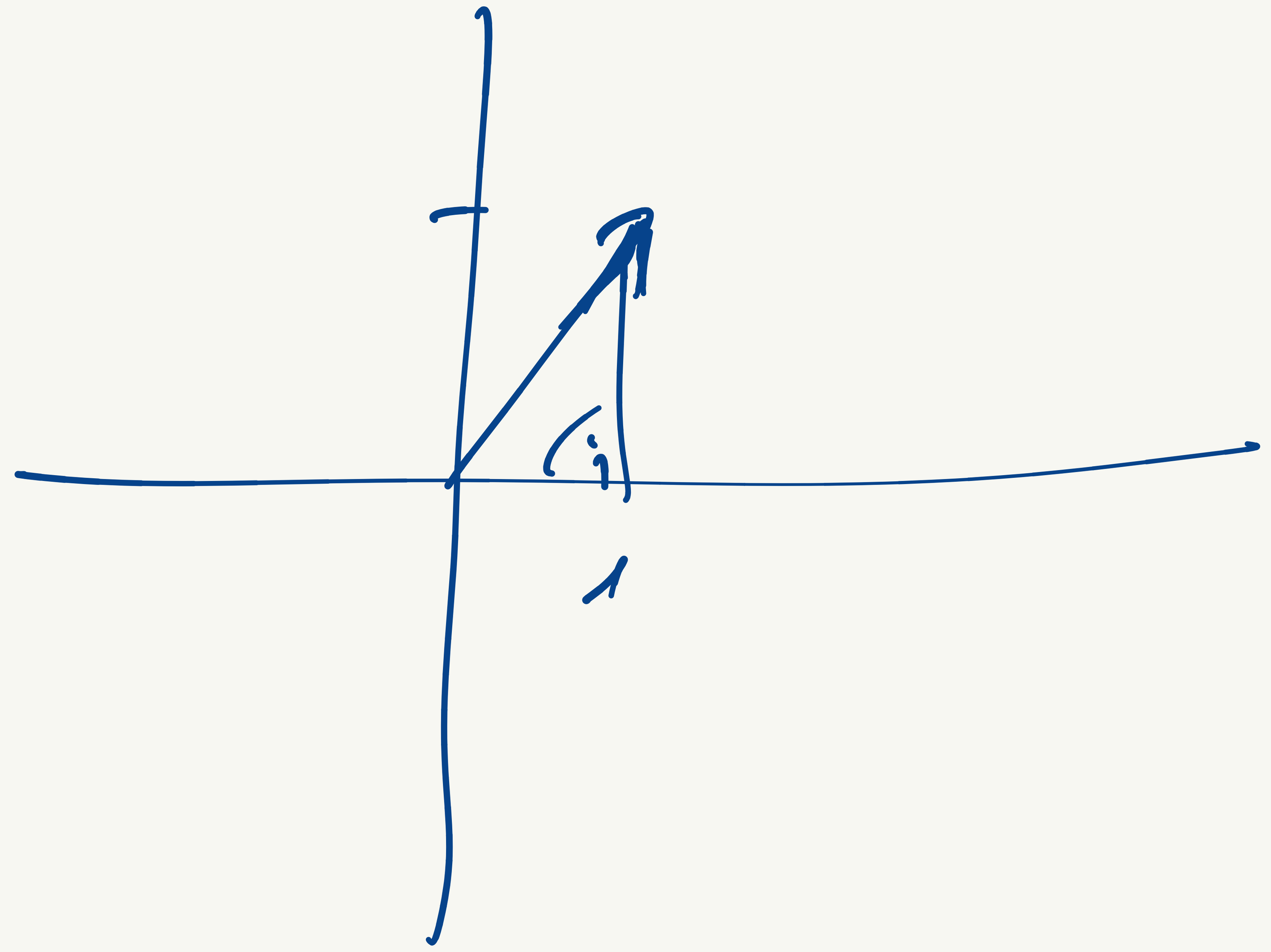
Norm :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \|x\| = (x . x)$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \|x\| = 1^2 + 2^2 + 3^2 = \underline{\underline{14}}$$

$\|x\| = 0$ if and only if $x_i = 0 \ \forall i$

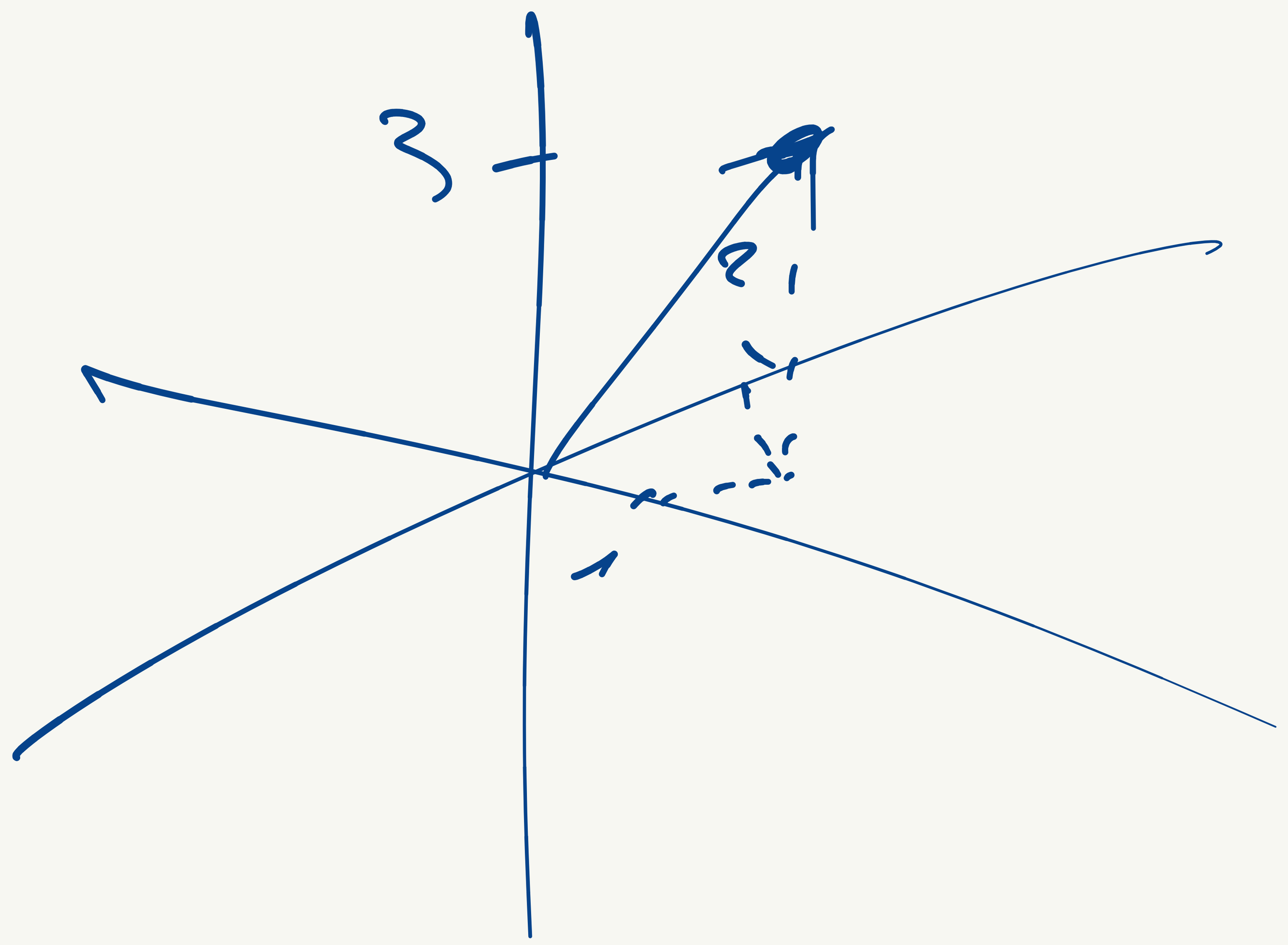
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Norm: square of the length:

$$\|x\| = 1^2 + 2^2 = 5$$

$$y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



$$z = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad (x^T)^T = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

$$x^T = [1 \; 3 \; 7]$$
$$y^T = [2 \; 4 \; 6]$$

$$x + y = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} \quad y - x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad 3.5x = \begin{bmatrix} 3.5 \\ 10.5 \\ 24.5 \end{bmatrix}$$

$$2y = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \quad x + 2y = \begin{bmatrix} 5 \\ 11 \\ 19 \end{bmatrix}$$

$$y^T - 3x^T = [-1 \; -5 \; -15]$$

$$3x^T = [3 \; 9 \; 21]$$

$$(x, y) = 1 \cdot 2 + 3 \cdot 4 + 7 \cdot 6 = \underline{\underline{56}}$$

$$(2x, y) = 112 = 2(x, y)$$

$$(ax, by) = a \cdot b\,(x, y)$$

$$\|x\| = 1^2 + 3^2 + 7^2 = 59$$

$$\|y - x\| = 1^2 + 1^2 + (-1)^2 = \underline{\underline{3}}$$

Matrices

- A rectangular array of numbers.
- A matrix A with q rows and n columns is a $q \times n$ matrix
- The number in the ith row and jth column is the (i,j)th element

$$A = \left.\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & \ddots & & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{q,1} & a_{q,2} & \cdots & a_{q,n} \end{bmatrix}}_{n \text{ columns}}\right\} q \text{ rows}$$

$$A = \begin{bmatrix} \pi & 3 & -1.2 \\ 2 & -4 & 7 \end{bmatrix}$$

2×3 matrix

Equality

Two matrices X & Y of equal size $\xi \times n$ are equal if

$$x_{i,j} = y_{i,j} \quad \forall i \in 1, 2 \ldots \xi$$
$$\forall j \in 1, 2 \ldots n$$

Transpose

The transpose of $q \times n$ matrix A is $n \times q$ matrix A^T

where $A^T_{j,i} = A_{i,j}$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$$

Addition

2 matrices of equal size A & B

$$A \pm B = \begin{bmatrix} a_{1,1} \pm b_{1,1} & \cdots & a_{1,n} \pm b_{1,n} \\ a_{2,1} \pm b_{2,1} & \ddots & \vdots \\ \vdots & & \vdots \\ a_{\ell,1} \pm b_{\ell,1} & \cdots & a_{\ell,n} \pm b_{\ell,n} \end{bmatrix}$$

Multiplication w/ scalar:

A $\quad c$

$$c \cdot A = \begin{bmatrix} c\,a_{1,1} & \cdots & c\,a_{1,n} \\ c\,a_{2,1} & \ddots & \vdots \\ \vdots & & \vdots \\ c\,a_{\ell,1} & \cdots & c\,a_{\ell,n} \end{bmatrix}$$

Multiplication of 2 matrices:

$A : \ell \times n$

$B : n \times m$

$$\boxed{AB_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}}$$

$$A = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{n=2} \qquad B = \left.\begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}\right\} n = 2$$

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{B} \\ & & 2 & 4 \\ & & 1 & 7 \\ \hline & 1\ 2 & 4 & 18 \\ A & 3\ 4 & 10 & 40 \\ & 5\ 6 & 16 & 62 \end{array}$$

$2 \cdot 1 + 1 \cdot 2 = 4$

$1 \cdot 4 + 2 \cdot 7 = 18$

$3 \cdot 2 + 4 \cdot 1 = 10$

$3 \cdot 4 + 4 \cdot 7 = 40$

$5 \cdot 2 + 6 \cdot 1 = 16$

$5 \cdot 4 + 6 \cdot 7 = 62$

- Square matrix $k = n$
- diagonal matrix: only diagonal elements can be non-zero:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

$\diagdown$: Diagonal

- Upper triangular matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Lower triangular matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & \pi \end{bmatrix}$$

Symmetric matrix:

$$A = A^T$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix} \qquad 2A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 8 & 14 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} \qquad 3B = \begin{bmatrix} 6 & 3 & 12 \\ 9 & 3 & 6 \end{bmatrix}$$

$$A + B^T = \begin{bmatrix} 3 & 5 \\ 3 & 4 \\ 8 & 9 \end{bmatrix} \qquad A^T - B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & 5 \end{bmatrix} \qquad 2A + B^T = \begin{bmatrix} 4 & 7 \\ 5 & 7 \\ 12 & 16 \end{bmatrix}$$

$$\begin{array}{cc|ccc} \multicolumn{2}{c|}{A \cdot B} & 2 & 1 & 4 \\ & & 3 & 1 & 2 \\ \hline 1 & 2 & 8 & 3 & 8 \\ 2 & 3 & 13 & 5 & 14 \\ 4 & 7 & 29 & 11 & 30 \end{array} \qquad \begin{array}{ccc|cc} & & & 1 & 2 \\ \multicolumn{3}{c|}{B \cdot A} & 2 & 3 \\ & & & 4 & 7 \\ \hline 2 & 1 & 4 & 20 & 35 \\ 3 & 1 & 2 & 13 & 23 \end{array}$$

1) System of equations:

$$\boxed{\begin{aligned} 3x + 4y &= 10 \\ 2x + 4y &= 6 \end{aligned}}$$

2)

$$\boxed{\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}}$$

$$\begin{array}{cc|c} & & x \\ & & y \\ \hline 3 & 4 & 3x+4y \\ 2 & 4 & 2x+4y \end{array}$$

$$\begin{bmatrix} 3x+4y \\ 2x+4y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix} \Rightarrow \begin{aligned} 3x+4y &= 10 \\ 2x+4y &= 6 \end{aligned}$$

1) = 2)

System of linear equations

$\Rightarrow$ it will always have a matrix form

Inverse of a matrix

the inverse of A is A^{-1}

$$\boxed{A^{-1}} \cdot A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = I$$

$\downarrow$

Identity matrix

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$\underbrace{\boxed{A^{-1} A}}_{I} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{A^{-1} \cdot b}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = [\quad]$$

$$\begin{array}{cc|c} & & x \\ & & y \\ \hline 1 & 0 & x \\ 0 & 1 & y \end{array}$$

$$3x + 4y = 10 \qquad 3 \cdot 4 + 4 \cdot (-0.5) = 10$$

$$2x + 4y = 6 \qquad 2 \cdot 4 + 4(-0.5) = 6$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

→ Gauss elimination

→ Basis transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -0.5 \end{bmatrix} \checkmark$$

A^{-1} : might not exist.

$$2x + 4y = 6 \quad \rightarrow \quad 2x = 6 - 4y$$
$$3x + 6y = 9 \qquad\qquad x = 3 - 2y$$

$$3(3-2y) + 6y = 9$$
$$9 - 6y + 6y = 9$$
$$\underline{\underline{9 = 9}}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

→ Inverse does not exist

(1) $2x + 4y = 6 \quad | \cdot 1.5$

$$\underline{3x + 6y = 9}$$

redundant equations!

$\Rightarrow$ Linearly dependent!

$$\left.\begin{aligned} &(1) \quad 2x + 4y + z = 6 \\ &(2) \quad 3x + 6y - 3z = 9 \\ &(3) \quad 5x + 10y - 2z = 15 \end{aligned}\right\} \quad \begin{aligned} &3 \text{ variables} \\ &2 \text{ eqs with information} \end{aligned}$$

$$(1) + (2) = (3)$$

if

$$\underline{\underline{a(1) + b(2) = (3)}} \quad \Rightarrow \quad ↯$$

Determinant

- A number
- if it is 0 ⇒ Linearly dependent ⇒ cannot be inverted

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = a \cdot d - b \cdot c$$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$$\det(A) = 2 \cdot 6 - 4 \cdot 3 = \underline{\underline{0}}$$