Probability

- Random Experiments
- Random variables -> Probability distributions

Random experiments:

- Experiments where the outcome cannot be determined in advance
  - -> Tossing a coin
  - -> # left-handed in a group
  - -> # service disruptions in a month.
  - -> The monthly rainfall

Sample space : Ω : all possible outcomes of an experiment

-> Tossing a dice : $\Omega = \{H, T\}$

-> Cast dice twice : $\Omega = \{(1,1), (1,2), (2,1) \dots (6,6)\}$

-> # service disruptions $\Omega = \mathbb{Z}_0^+$

-> Monthly rainfall in mm : $\Omega = \mathbb{R}_0^+$

-> Elementary events: distinct outcomes that can happen.

Event

A, B, C, D : pre-defined set of outcomes. If outcome is in this set, the event occurs.

-> Sum of 2 dice is 10 : $A = \{(6,4), (5,5), (4,6)\}$

-> Monthly rainfall is at least 15mm : $A = [15, \infty)$

-> After observing the outcome, we have to be able to decide!

-> It rains a lot. ↯

-> The sum will be a small number ↯

<u>Events</u> : sets of elementary events. Operators:

- $\to A \cup B$: Union. Occurs if A, B or both occur
- $\to A \cap B$: Intersection : Occurs if both A & B occur
- $\to A^c$: Complement : Occurs if A does not occur.

If $A \cap B = \emptyset \Rightarrow A$ & B are disjoint events.

A : Odd number

B : Even number

Probability

Number between 0 & 1 telling us how likely it is that a certain event occurs.

- $\rightarrow P(A) \geq 0$
- $\rightarrow P(\Omega) = 1$
- $\rightarrow$ For disjoint events $P(\bigcup_i A_i) = \sum_i P(A_i)$

Example

Toss coin 3 times, recording the result. P(getting tails twice)?

$$\Omega = \{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(T,T,H),(T,H,T),(H,T,T),(T,T,T)\}$$

→ Elementary events   Probability of each is $\underline{\underline{\frac{1}{8}}}$
→ Disjoint
→ $P(\Omega) = 1$
$P(\bigcup_i A_i)$

→ $B = \{(T,T,H),(T,H,T),(H,T,T)\}$

$$P(B) = P(T,T,H) + P(T,H,T) + P(H,T,T) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \underline{\frac{3}{8}}$$

1) Cast a dice twice. Probability of getting at least once 6.

2) Toss a coin 4 times. Probability of getting 4 heads / 4 tails?

1) $\Omega = \{(1,1), (1,2) \ldots, (6,6)\}$

	1	2	3	4	5	6
1						x
2						x
3						x
4						x
5						x
6	x	x	x	x	x	x

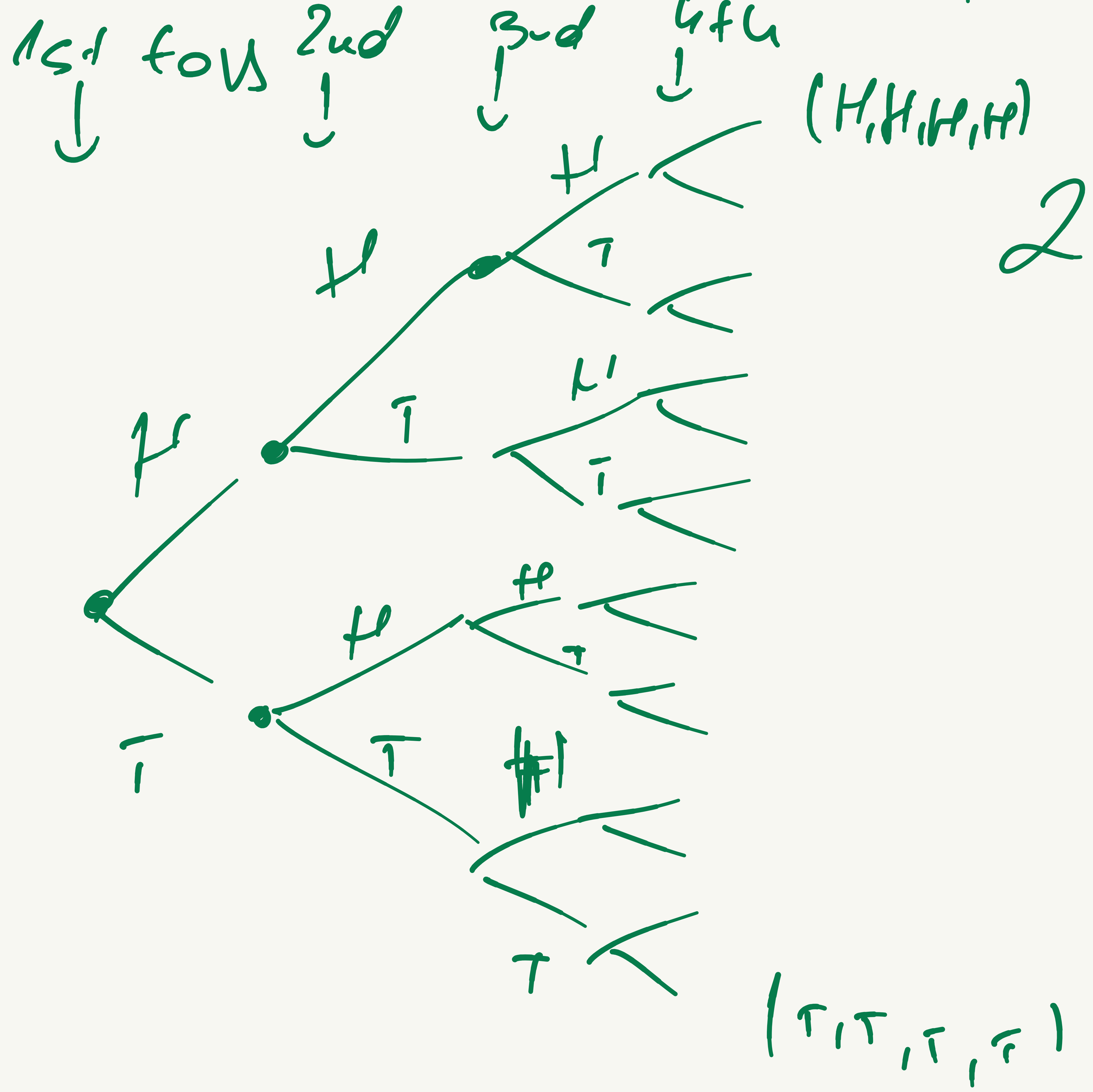
6 rows, 6 columns

$6 \times 6 = 36$ elementary events.

A occurs at 11 elementary events.

$P = \frac{11}{36}$

2) $\Omega = \{(H,H,H,H), (H,H,H,T) \ldots\}$



$2^4 = 16$ elementary events.

$A = \{(H,H,H,H), (T,T,T,T)\}$

$P(A) = \frac{2}{16} = \frac{1}{8}$

Random variables

If the outcome of an experiment is a number, then it is called a random variable.

- Discrete: sample space is finite / countably infinite:
  - → Throw a dice: $\{1,2,3,4,5,6\}$
  - → Number of days that humanity has left.
- Continuous: sample space is infinite:
  - → All experiments where $\Omega = \mathbb{R}$
  - → Tomorrow's humidity / exact temperature
  - → Randomly selected person's height

Integral needed

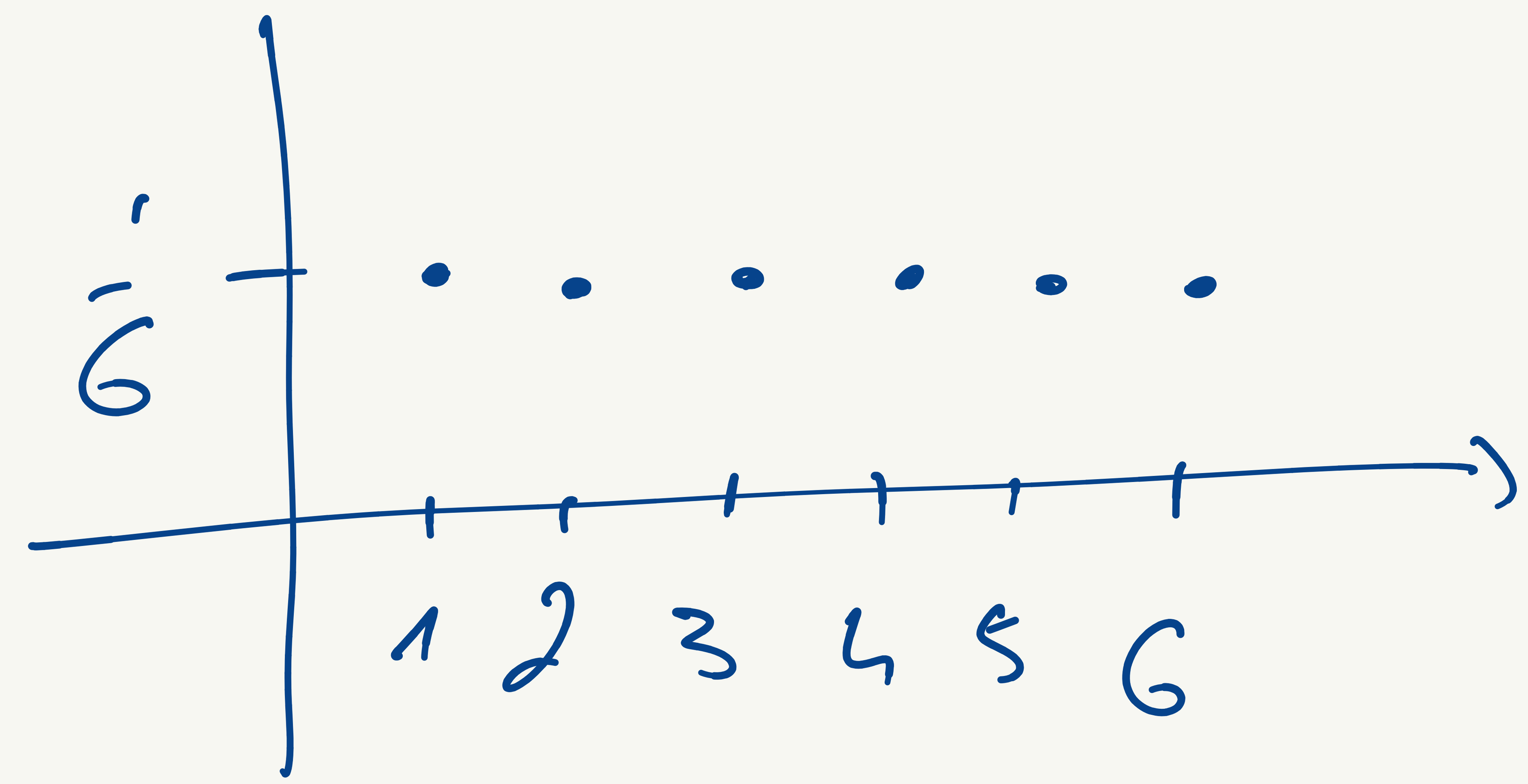
Probability mass function :

Shows the probability of each elementary event.

Result of a throw with a dice

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$P(x) = \frac{1}{6} \quad \forall\, x \in \{1, 2, 3, 4, 5, 6\}$



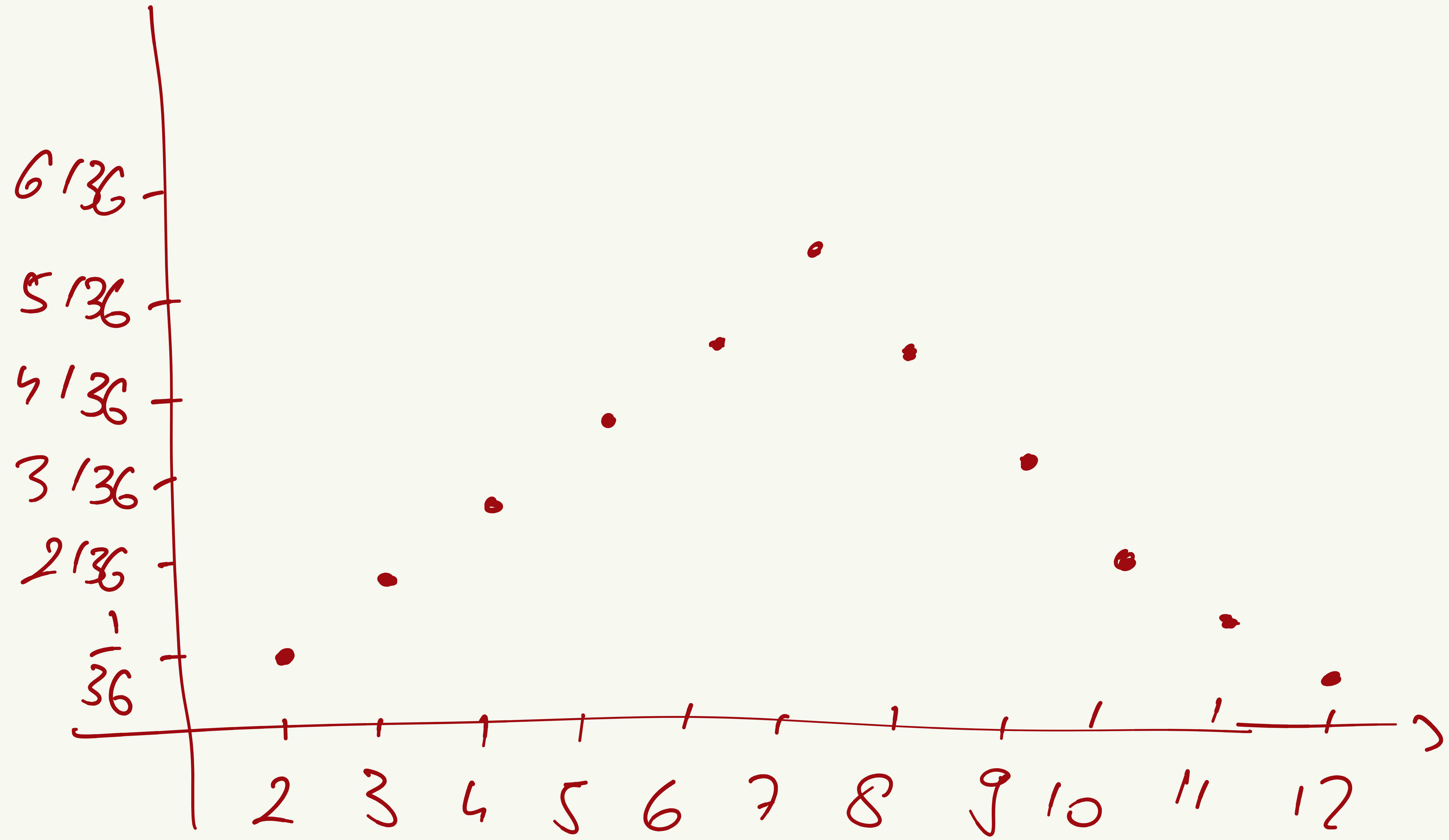
$\rightarrow$ random variable : x

$P(1) = \frac{1}{6} \quad P(2) = \frac{1}{6} \quad \ldots$

x : the sum of 2 dice that I toss.

$$x \in \{2,3,4,5,6,7,8,9,10,11,12\}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12



36 elementary events

$P(x) =$

Expected value:

$$E(x) = \sum x \, P(x)$$

-> x: outcome of throw with a dice

$x \in \{1, 2, 3, 4, 5, 6\}$

$P(x) = \frac{1}{6}$

$$E(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \underline{\underline{3.5}}$$

-> If we repeat many times and average the results it will be very close to $E(x)$.

-> Law of large numbers. Repeat experiment n times.

$$\lim_{n \to \infty} \frac{\sum x}{n} = E(x)$$

Specific types

Uniform r.v. :

- Equally likely events.
- E.g. tossing dice
- n different outcomes
- probability mass function

$$P(x) = \frac{1}{n}$$

- Expected value:

$$E(x) = \frac{\sum x}{n}$$

Bernoulli random variable

$\Omega = \{0, 1\}$

$P(1) = p$

$P(0) = 1 - p$

$$\boxed{P(x) = p^x (1-p)^{1-x}}$$

$$P(0) = p^0 (1-p)^{1-0} = 1 \cdot (1-p)^1 = 1-p$$

$$P(1) = p^1 (1-p)^{1-1} = p \cdot (1-p)^0 = p$$

$$E(x) = p$$

Binomial random variable:

Run an experiment n times, each time the probability that an event happens is p. The probability that out of n the event will happen exactly x times is a binomially distr. r.v.

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

↳ n choose x

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

n factorial

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

$$E(x) = n \cdot p$$

Example:

- Throw dice 20 times.
- Probability of getting 6 exactly 5 times.

$$p = \frac{1}{6} \qquad n = 20 \qquad x = 5$$

$$P(5) = \binom{20}{5} \cdot \left(\frac{1}{6}\right)^{5} \cdot \left(\frac{5}{6}\right)^{15} = \underline{0.1294}$$

$$E(x) = 20 \cdot \frac{1}{6} = 3.\dot{3}$$

$$P(5) = \binom{20}{5} \cdot \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^{15} = \frac{20!}{5!\,15!} \cdot \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^{15} =$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot \ldots \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \underline{15 \cdot 14 \cdot 13 \cdot 12} \cdot \ldots} \left(\frac{1}{6}\right)^5 \cdot \left(\frac{5}{6}\right)^{15} =$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2} \cdot \frac{5^{15}}{6^{20}}$$

Geometric random variable:

Assume that you run an experiment, where the probability of an event happening is p. You repeat it, until the event occurs.

x: number of repetitions until the event occurs.

$$P(x) = p \cdot (1-p)^{x-1}$$

$$E(x) = \frac{1}{p}$$

Example: Toss a coin. Repeat until we get heads.

$$p = 1/2$$

$$E(x) = \frac{1}{1/2} = \underline{\underline{2}}$$

$$P(1) = \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right)^{1-1} = \frac{1}{2}$$

$$P(3) = \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right)^{3-1} = \frac{1}{8}$$

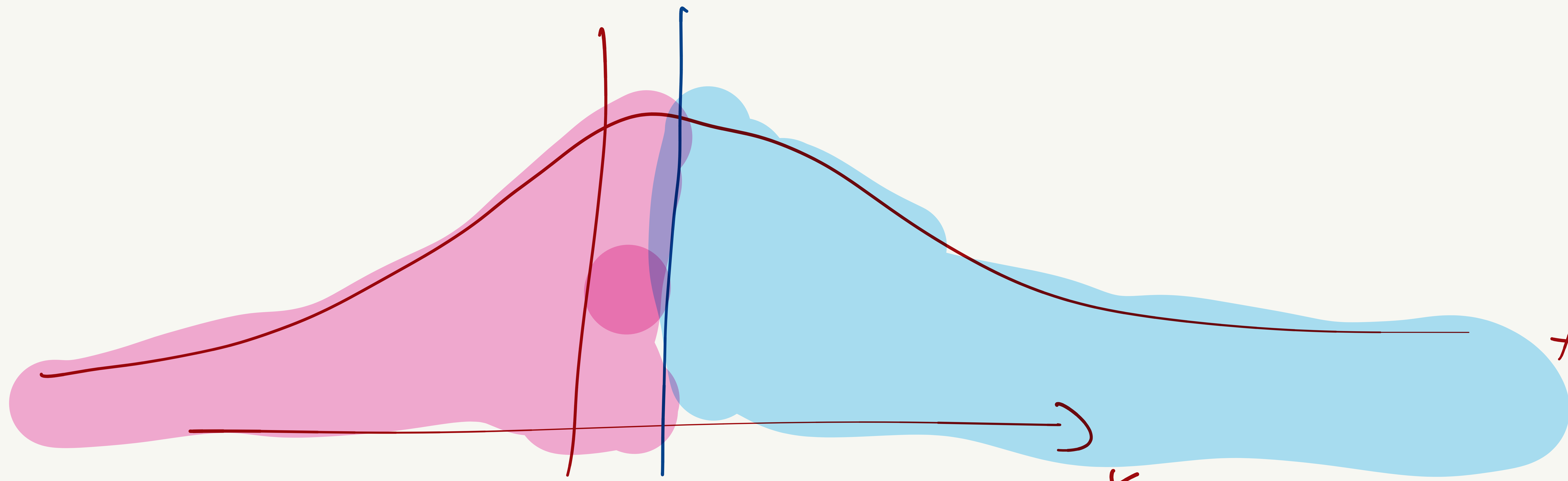
$$P(2) = \frac{1}{2} \left(1 - \frac{1}{2}\right)^{2-1} = \frac{1}{4}$$

Continuous distributions

r.v. X can take infinitely many values.

probability mass function → probability distribution function.

$P(x) = 0$ → infinitesimal, $\frac{1}{\infty}$



$$\int_{-\infty}^{\infty} f(x) = 1$$

$\int$: continuous equivalent of Σ

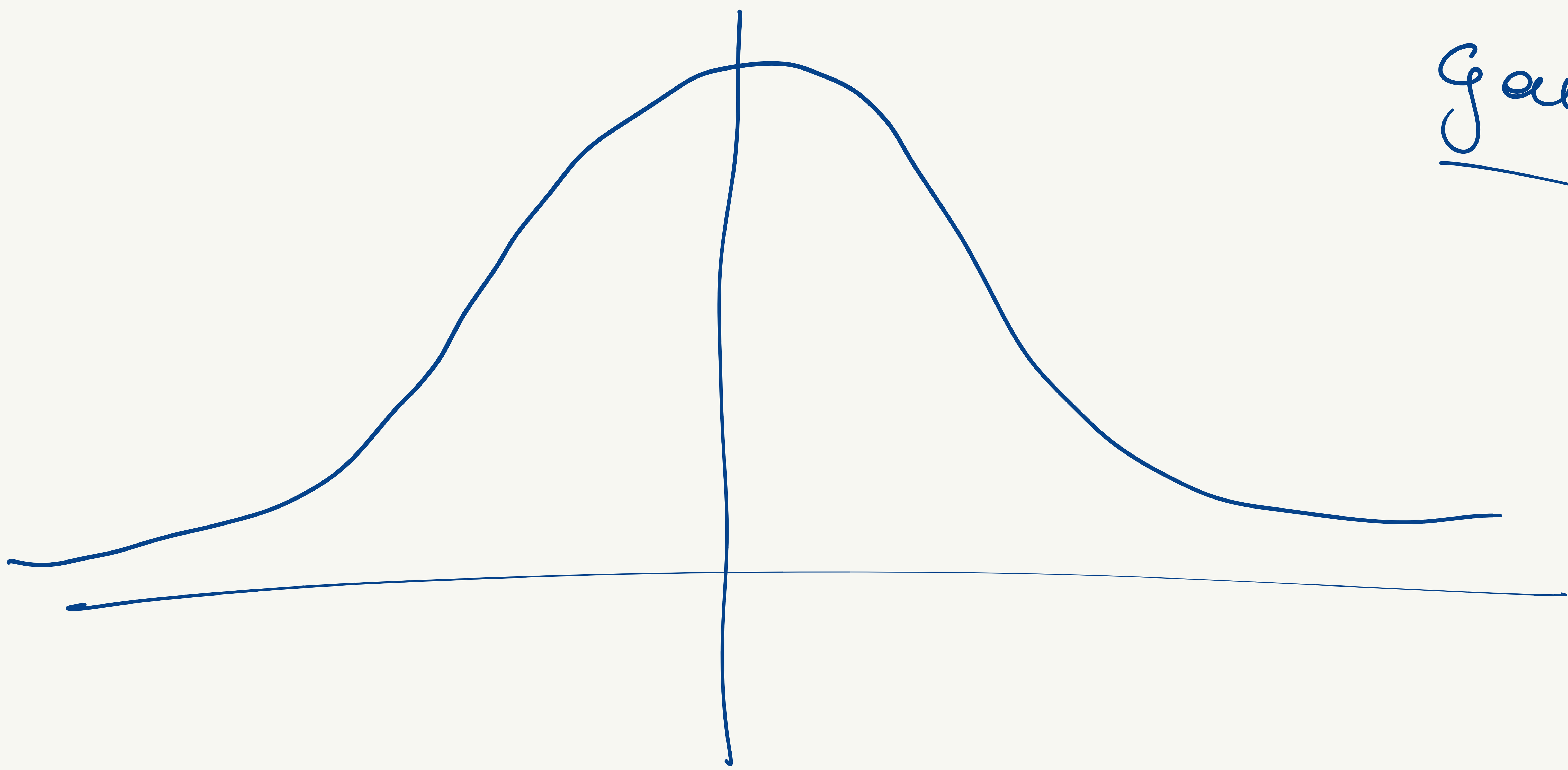
Area under $f(x)$.

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$E(x) = \mu$

standard deviation: σ



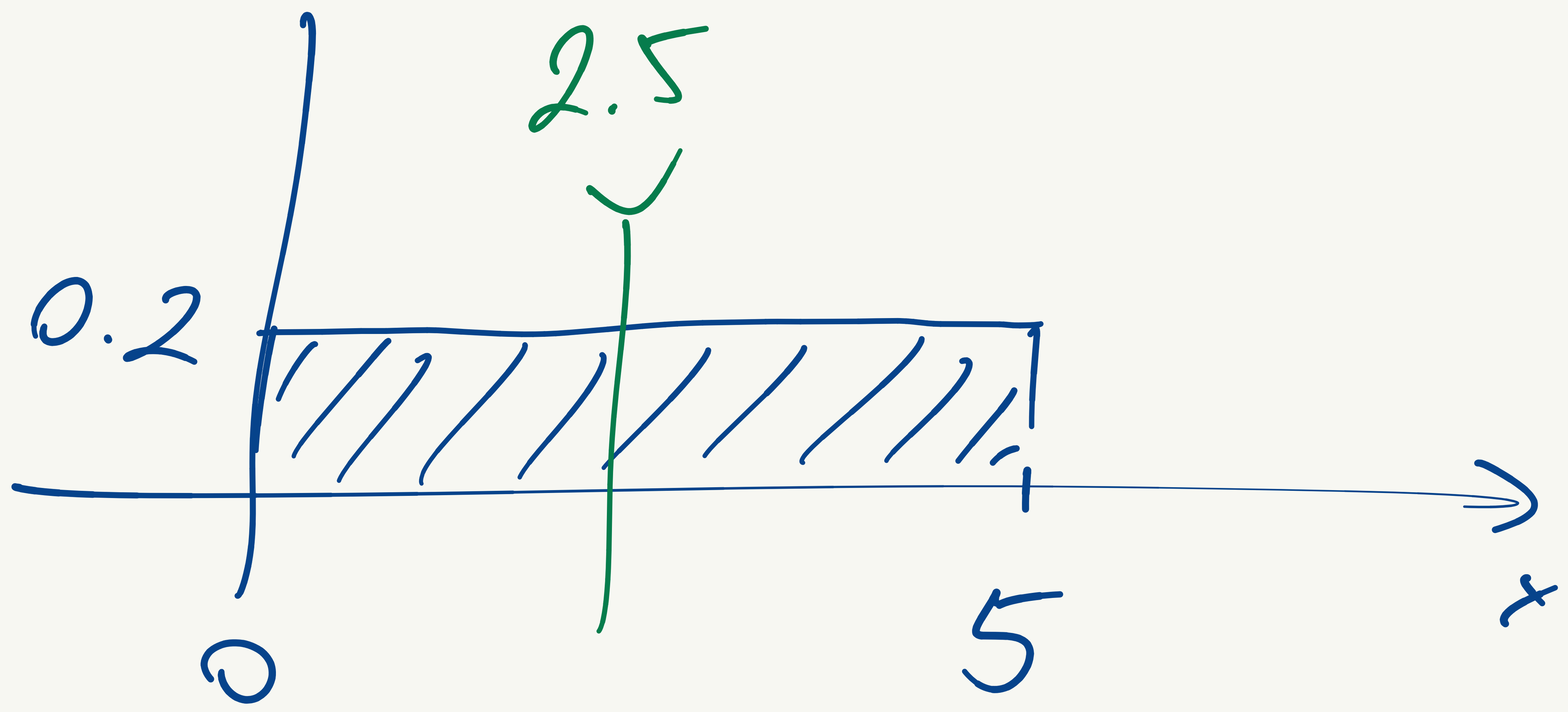
Gauss-curve

Uniform continuous distribution:

-> Tram comes at every 5 minutes.

-> I go to the stop randomly.

-> x: How much do I have to wait?



$$\int_0^5 f(x) = 0.2 \cdot 5 = \underline{\underline{1}}$$

$$E(x) = \underline{\underline{\int x f(x)}} =$$

$$0 \cdot 0.2 + 0.00001 \cdot 0.2$$

$$\int_0^5 x \cdot 0.2 = \underline{\underline{2.5}}$$