Linear Algebra

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Topics

- Vectors
- Matrices
- Systems of linear equations

Vectors

A vector of order n is a set of n numbers. It can be

- a row vector: $x = [x_1 x_2 \ldots x_n]$
- a column vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

If we only say "vector", we usually mean a column vector.

Vector operations

- Transposition: The transpose of a row vector is a column vector with the same elements. The transpose of a column vector is a row vector with the same elements. Notation: $\mathbf{x}^{\mathsf{T}}$.

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \qquad \mathbf{x}^{\mathsf{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{y}^{\mathsf{T}} = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$$

Thus $(\mathbf{x}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{x}$

Vector operations

- Equality: Two column vectors x and y of equal order n are said to be equal if and only if all their components are equal, $x_i = y_i \quad \forall i \in 1, 2, \ldots, n$
- The same holds for row vectors
- We cannot compare vectors of different order
- We cannot compare row vectors with column vectors (except $n = 1$)

Vector operations

- Addition/subtraction: Two column vectors x and y of equal order n can be added/subtracted by adding/subtracting their corresponding components:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \mathbf{x} \pm \mathbf{y} = \begin{bmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{bmatrix}$$

- The same holds for row vectors
- We cannot add/subtract vectors of different order
- We cannot add/subtract row vectors with column vectors (except $n = 1$)
- Notice that commutativity and associativity still hold

Vector operations

- Multiplication/division by scalar: Multiply/divide each component of the column vector by the scalar

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{cx} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

- The same holds for row vectors
- We cannot divide by zero

Vector operations

- Inner product: The inner product of two column vectors x and y is the sumproduct of their components. The inner product is usually denoted by (x, y) or $x^T y$

$$(x, y) = x^T y = \sum_{i=1}^{n} x_i y_i$$

- Vectors of different order have no inner product

Vector operations

- Norm: The norm of a vector x is the inner product of the vector with itself. It is usually denoted by $||x||$

$$||x|| = (x, x) = x^T x = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2$$

- It is zero if and only if all the components of the vector are zeros (null vector)

Solve the following problems

If $x = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, find:

- x^T, y^T, $(x^T)^T$
- $x + y$, $y - x$
- $3.5x$, $2y$
- $x + 2y$
- $y^T - 3x^T$
- (x, y), $(2x, y)$
- $||x||$, $||y - x||$

Matrices

- A matrix is a rectangular array of numbers
- A matrix **A** with k rows and n columns is called a k by n or $k \times n$ matrix
- The number in row i and column j is called the (i, j)th element

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \end{bmatrix}$$

Matrix operations

- Equality: Two matricas X and Y of equal size $k \times n$ are said to be equal if and only if all their components are equal
  $x_{i,j} = y_{i,j} \quad \forall i \in 1, 2, \ldots, k$ and $j \in 1, 2, \ldots, n$
- We cannot compare matrices of different order

Matrix operations

- Transposition: The transpose of a $k \times n$ matrix A is an $n \times k$ matrix A^T where $A^T_{j,i} = A_{i,j} \quad \forall i \in 1, 2, \ldots, k$ and $j \in 1, 2, \ldots, n$
- Notice that $(X^T)^T = X$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \dots & a_{1,k} \\ a_{1,2} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,k} \end{bmatrix}$$

Matrix operations

- Addition/subtraction: Two matrices A and B of equal order $k \times n$ can be added/subtracted by adding/subtracting their corresponding components:

$$A \pm B = \begin{bmatrix} a_{1,1} \pm b_{1,1} & a_{1,2} \pm b_{1,2} & \dots & a_{1,n} \pm b_{1,n} \\ a_{2,1} \pm b_{2,1} & a_{2,2} \pm b_{2,2} & \dots & a_{2,n} \pm b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} \pm b_{k,1} & a_{k,2} \pm b_{k,2} & \dots & a_{k,n} \pm b_{k,n} \end{bmatrix}$$

- We cannot add/subtract matrices of different size
- Notice that commutativity and associativity still hold

Matrix operations

- Multiplication/division by scalar: Multiply/divide each component of the matrix by the scalar

$$cA = \begin{bmatrix} ca_{1,1} & ca_{1,2} & \dots & ca_{1,n} \\ ca_{2,1} & ca_{2,2} & \dots & ca_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{k,1} & ca_{k,2} & \dots & ca_{k,n} \end{bmatrix}$$

- We cannot divide by zero

Matrix operations

- Multiplication of two matrices: we can multiply two matrices A and B if A is $k \times n$ and B is $n \times m$, that is, the number of columns in matrix A has to be the same as the number of rows in matrix B. Then AB will be a $k \times m$ matrix where

$$AB_{i,j} = \sum_{h=1}^{n} A_{i,h} B_{h,j}$$

- Example:

$$A = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad AB = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \\ Ea + Fc & Eb + Fd \end{bmatrix}$$

Special matrices

- Square matrix: any matrix where $k = n$, that is, it has the same number of rows and columns
- Diagonal matrix: A matrix where only the diagonal elements $a_{i,i}$ have non-zero values
- Upper-triangular matrix: $a_{i,j} = 0$ if $i > j$.
- Lower-triangular matrix: $a_{i,j} = 0$ if $i < j$.
- Symmetric matrix: $A = A^T$

Solve the following problems

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$, find:

- A^T, B^T
- $A + B^T$, $A^T - B$
- $2A$, $3B$
- $2A + B^T$
- AB, BA

Inverse of a matrix

- The inverse of a matrix A is an other matrix A^{-1} such that

$$A^{-1}A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- We will not calculate this, but it is useful in solving equations

Systems of linear equations

- Consider the following problem:

$$\begin{aligned} 3x + 4y &= 10 \\ 2x + 4y &= 6 \end{aligned}$$

- Notice that it is equivalent to

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

- Any system of linear equations can be written in this form
- It is useful, because it is easy to solve by something called the inverse of the coefficient matrix

Systems of linear equations

- Solution by elimination:

$$\begin{aligned} 3x + 4y &= 10 \\ 2x + 4y &= 6 \end{aligned}$$

- Subtracting the second equation from the first yields $x = 4$, then $y = -0.5$.
- In WolframAlpha try calculating:

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

- It should give you the same solution

Systems of linear equations

- If you can write any problem in the form of

$$Ax = y$$

where A is a coefficient matrix, x is a vector if variables and y is a vector of scalars, then

$$x = A^{-1}y$$

if A^{-1} exists.

Systems of linear equations

- Example:

$$
\begin{aligned}
5x_1 + 3x_2 - x_3 + 2x_4 &= 10 \\
x_1 - 2x_2 + 14x_3 + x_4 &= 22 \\
7x_1 + x_2 - 21x_3 + 5x_4 &= 41 \\
10x_1 - 5x_2 + 4x_3 + 7x_4 &= 5
\end{aligned}
$$

$$
\begin{bmatrix} 5 & 3 & -1 & 2 \\ 1 & -2 & 14 & 1 \\ 7 & 1 & -21 & 5 \\ 10 & -5 & 4 & 7 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 10 \\ 22 \\ 41 \\ 5 \end{bmatrix}
\implies
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 5 & 3 & -1 & 2 \\ 1 & -2 & 14 & 1 \\ 7 & 1 & -21 & 5 \\ 10 & -5 & 4 & 7 \end{bmatrix}^{-1}
\begin{bmatrix} 10 \\ 22 \\ 41 \\ 5 \end{bmatrix}
$$

Solve the following problems

Write the following systems of equations in a matrix form:

- $$\begin{aligned} 2y_1 + 3y_2 &= 10 \\ 4y_1 - 5y_2 &= 12 \end{aligned}$$

- $$\begin{aligned} 3x_1 + 10x_2 - 4x_3 + 5x_4 &= 2 \\ 7x_1 - 9x_2 + 7x_3 + 1x_4 &= 1 \\ 10x_1 + 2x_2 - 10x_3 + 3x_4 &= 21 \\ 4x_1 + 6x_2 + 9x_3 + 2x_4 &= 10 \end{aligned}$$

Linear independence

- Remember: $x = A^{-1}y$ **if** A^{-1} **exists**.
- What does it mean that A^{-1} exists?
- Consider:

$$2x + 4y = 6$$
$$3x + 6y = 9$$

- Try to solve it!

Linear independence

- It is also a problem in a bit more complex cases:

$$\begin{aligned} 2x + 4y + z &= 6 \\ 3x + 6y - 3z &= 9 \\ 5x + 10y - 2z &= 15 \end{aligned}$$

- If there is any combination of equations yielding an other one, then the coefficient vector is called linearly dependent or singular.
- In this case the system has either no solutions or infinitely many solutions
- Also, the inverse of the coefficient matrix doesn't exist

The determinant

- It is impossible to check all possible combinations of the equations
- Luckily, it is equivalent to checking the so-called determinant of the coefficient matrix
- The determinant is easy to compute for 2×2 and 3×3 matrices
- It is a lot more difficult in larger matrices
- If it is zero, the matrix is singular.

The determinant

- In case of 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$det(A) = ad - bc$$

- In case of 3×3 matrices $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

- For larger matrices you can use WolframAlpha

Solve the following problems

Find the determinants of the following matrices:

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 12 & e \\ \pi & 5 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 0.1 \\ 10 & \frac{1}{5} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$