

# Elementary Algebra

Olivér Kiss

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## Elementary Algebra

- Olivér Kiss

$$a^n = \underbrace{a \cdot a \dots a}_{n \text{ times}}$$

$$a^0 = 1 \quad \forall a \neq 0$$

- What if  $a = 0$ ?

$$a^{-n} = \frac{1}{a^n} \quad \forall a \neq 0$$

- What if  $a = 0$ ?

# General rules of exponentiation

$$\blacktriangleright a^n \cdot a^m = a^{n+m}$$

$$\blacktriangleright \frac{a^n}{a^m} = a^{n-m}$$

$$\blacktriangleright (a^n)^m = a^{n \cdot m}$$

$$\blacktriangleright (a \cdot b)^n = a^n b^n$$

$$\blacktriangleright \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$



# Roots

- ▶  $\sqrt[n]{x}$  is the  $n$ th root of  $x$ , where  $x$  is the base and  $n$  is the degree.
- ▶  $\sqrt[n]{x}$  is a number  $z$  such that

$$z^n = x$$

- ▶ Taking the  $\frac{1}{n}$ th power of both sides yields

$$z^{n \cdot \frac{1}{n}} = z = \sqrt[n]{x} = x^{\frac{1}{n}}$$

- ▶ Thus instead of the radical notation we can use (although a mathematician would disagree)

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

# Exercise

Using the rules of exponentiation show that

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

## Exercise - Solution

Using the rules of exponentiation show that

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

- ▶ By definition if  $\sqrt[b]{x^a} = z$  then

$$z^b = x^a$$

- ▶ Raising both sides to the  $\frac{1}{b}$ th power yields

$$z = x^{\frac{a}{b}}$$

- ▶ Since  $\sqrt[b]{x^a} = z$ , we have

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$



# Properties of roots

- ▶  $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- ▶  $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \sqrt[n]{b} \quad \forall a, b \in \mathcal{R}^+$
- ▶  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \forall a, b \in \mathcal{R}^+$

Solve the following problems [SYD Appendix A]

1. Compute  $\sqrt{1600}$
2. Compute  $125^{\frac{1}{3}}$
3. Solve for  $x$ :  $x^{0.25} = 2$
4. True or false:  $(a + b)^{-0.5} = \frac{1}{\sqrt{a+b}}$

# Rules of algebra

You are probably familiar with these.

- ▶ Commutativity of addition:  $a + b = b + a$
- ▶ Commutativity of multiplication:  $a \cdot b = b \cdot a$
- ▶ Multiplication is distributive over addition:  $a(b + c) = ab + ac$
- ▶ Associativity of addition:  $(a + b) + c = a + (b + c)$
- ▶ Associativity of multiplication:  $(ab)c = a(bc)$

Some useful identities:

- ▶  $(a + b)^2 = a^2 + 2ab + b^2$
- ▶  $(a - b)^2 = a^2 - 2ab + b^2$
- ▶  $(a + b)(a - b) = a^2 - b^2$

Solve the following problems [SYD Appendix A]

1. Compute  $\frac{1000^2}{252^2 - 248^2}$  without a calculator.
2. Calculate  $-3[4 - (-2)]$
3. Expand  $(\frac{1}{2}x + \frac{1}{3}y)(\frac{1}{2}x - \frac{1}{3}y)$

# Fractions

- ▶ A fraction can be written as a numerator over a denominator:

$$a \div b = \frac{a}{b}$$

- ▶ If  $a < b$ , it is a proper fraction
- ▶ If  $a \geq b$ , it is an improper fraction
- ▶ Improper fractions can be written as mixed numbers. E.g:

$$\frac{15}{7} = 2 + \frac{1}{7} = 2\frac{1}{7}$$

- ▶ To avoid ambiguity whether  $2\frac{1}{7}$  means  $2 + \frac{1}{7}$  or  $2 \cdot \frac{1}{7}$  you should not use mixed notation!

# Reducing fractions

You can reduce fractions by canceling common factors. E.g.:

$$\blacktriangleright \frac{24}{36} = \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2}{\cancel{2} \cdot 3 \cdot \cancel{2} \cdot 2} = \frac{2}{3}$$

$$\blacktriangleright \frac{xy+x^2y}{xy^2+x^3y^2} = \frac{\cancel{xy}(1+x)}{\cancel{xy}(y+x^2y)} = \frac{1+x}{y+x^2y}$$

- Notice that by canceling factors they don't disappear! We only use the fact that  $\frac{x}{x} = 1$ . Thus:

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} \neq \frac{0}{x+1}$$

It is actually:

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$$

# Multiplying and dividing fractions

- Just multiply the numerators and denominators

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$$

- To divide, just change up the numerator and the denominator in the divisor and then multiply

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

# Adding fractions

- If they have a common denominator, simply add numerators:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

- Otherwise you need common denominators first.

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \frac{d}{d} + \frac{c}{d} \frac{b}{b} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$$



Solve the following problems [SYD Appendix A]

1. Simplify  $\frac{5}{7} + \frac{3}{7} - \frac{2}{7}$

2. Simplify  $\frac{1}{8ab} + \frac{1}{8b(a-2)} + \frac{1}{b(a^2-4)}$

3. Simplify

$$\frac{\frac{1}{x-1} + \frac{1}{x^2-1}}{x - \frac{2}{x+1}}$$

4. Calculate  $\left(\frac{1}{4} - \frac{1}{5}\right)^{-2}$

# Simple equations

When you solve equations you can perform the following operations on both sides of the equation:

- ▶ add the same number
- ▶ subtract the same number
- ▶ multiply by the same non-zero number
- ▶ divide by the same non-zero number

# Example

Solve  $3x + 10 = x + 4$

$$3x + 10 = x + 4$$

(subtract 10)

$$3x = x - 6$$

(subtract x)

$$2x = -6$$

(divide by 2)

$$x = -3$$

## Solve the following problems [SYD Appendix A]

1. Mr. Barne receives double pay for every hour he works over and above 38 hours a week. Last week, he worked 48 hours and earned a total of \$812. What is Mr. Barne's regular hourly wage?
2.  $6p - \frac{1}{2}(2p - 3) = 3(1 - p) - \frac{7}{6}(p + 2)$
3.  $\frac{x+2}{x-2} - \frac{8}{x(x-2)} = \frac{2}{x}$
4. When Ann passed away, their estate was divided in the following manner:  $\frac{2}{3}$  of the estate was left to their wife,  $\frac{1}{4}$  to their children, and the remainder, \$10 000 was donated to a charitable organization. What was the total value of Ann's estate?

# Quadratic equations

A general quadratic equation takes the following form:

$$ax^2 + bx + c = 0$$

We can easily solve it using a simple formula that we will now derive together:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

# Deriving the solution to a quadratic equation

$$ax^2 + bx + c = 0$$

(Factor out a)

$$a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) = 0$$

(Divide by a)

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

(Subtract c/a)

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

(Add the same to both sides)

$$x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 = -\frac{c}{a} + \left( \frac{b}{2a} \right)^2$$

(Continued...)

# Deriving the solution to a quadratic equation

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{(Complete the square LHS)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{(Simplify RHS)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{b^2 - 4ac}{4a^2}$$

Thus either  $x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$  or  $x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$  which yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following problems [SYD Appendix A]

1.  $15x - x^2 = 0$

2.  $x^2 - 9 = 0$

3.  $x^2 - 4x + 4 = 0$

4.  $x^2 - 5x + 6 = 0$

5. In a right-angled triangle, the hypotenuse is 34 cm. One of the short sides is 14 cm longer than the other. Find the lengths of the two short sides.



# Basic systems of equations

We will review two methods to solve simple systems of equations (2 linear equations, 2 unknowns).

- ▶ Substitution: Solve one equation to one variable and substitute it in the other equation
- ▶ Elimination: Add/subtract a multiple of one equation from the other to eliminate one variable

# Substitution

Solve

$$2x + 3y = 18$$

$$3x - 4y = -7$$

Express  $x$  from the first equation:

$$x = 9 - 1.5y$$

Substitute  $x$  in the second equation:

$$3(9 - 1.5y) - 4y = -7$$

Solve for  $y$ :

$$y = 4$$

Solve for  $x$ :

$$x = 3$$

# Elimination

Solve

$$2x + 3y = 18$$

$$3x - 4y = -7$$

Multiply the first equation by  $4/3$ :

$$\frac{8}{3}x + 4y = \frac{72}{3}$$

Add it to the second equation:

$$\frac{17}{3}x = \frac{51}{3}$$

Solve for  $x$ :

$$x = 3$$

Solve for  $y$ :

$$y = 4$$

## Solve the following problems [SYD Appendix A]

1.  $x - y = 5$  and  $x + 3y = 11$
2.  $3x + 4y = 2.1$  and  $5x - 6y = 7.3$
3. A person has two accounts with a total saving of \$10 000. The interest rates are 5% and 7.2% respectively. If the person earns \$676 interest in a year, what was the balance of these accounts?