The Urn

→ n balls
→ draw k

1. Do you replace the balls?
2. Do you care about the order?

1. No replacement, order matters.

- Race — n balls : racers
- 1st, 2nd, 3rd — k draws : first k positions

$\# \Omega = 10 \cdot 9 \cdot 8 = 720$

3 positions $\{2, 9, 8\}$
10 racers $\{2, 8, 9\}$ $P(A) = \frac{72}{720} = \underline{0.1}$

A = Person 4 wins

$\# A = 9 \cdot 8 = 72$

$\{4, 1, 2\}\{4, 1, 3\}\{4, 1, 5\}\ldots$

$\left.\begin{matrix} n \text{ balls} \\ k \text{ draws} \end{matrix}\right\}$ Permutation

$\# P_k^n = n(n-1)(n-2)\ldots(n-k+1)$

$\# P_3^{10} = 10 \cdot 9 \cdot 8 = 720$

$m!$ factorial

$m! = \prod_{i=1}^{m} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$

$\# P_k^n = n(n-1)(n-2)\ldots(n-k+1) = \boxed{\frac{n!}{(n-k)!}}$

$\frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \ldots \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \ldots \cancel{(n-k)}} = (n-k+1)(n-k+2)\ldots n$

2. No replacement, order doesn't matter.

- Lottery
- Combination of size k from a set

$n: 90$
$k: 5$

Let's assume that order matters.

$\# P_5^{90} = \frac{90!}{85!}$ $\quad \#_k^n = \frac{n!}{(n-k)!}$

$\left.\begin{matrix} \{1, 2, 3, 4, 5\} \\ \{1, 2, 3, 5, 4\} \\ \{1, 2, 4, 3, 5\} \end{matrix}\right\}$ All the same!

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 = 5! = k!$

$\# \Omega = \frac{\frac{90!}{85!}}{5!} = \frac{90!}{85!5!}$

$\# C_k^n = \frac{n!}{(n-k)!k!} = \binom{n}{k}$ n choose k

$\# C_5^{90} = \frac{90!}{85!5!} = 43\ 949\ 268$

A : I win!

$\# A = 1$ $\quad P(A) = \frac{\# A}{\# \Omega} = \frac{1}{43\ 949\ 268}$

3. Draw w/ replacement, order matters

A, B, C, D n — I choose randomly

30 questions k

$\Omega = \{\underbrace{A, A, A, A, A \ldots A}_{30}\}, \{A, A, A \ldots B\}$

$\# \Omega = 4 \cdot 4 \cdot 4 \ldots 4 = 4^{30} = n^k$ $\quad P(A) = \frac{3^{15}}{4^{30}}$

A : First 15 right, other 15 wrong

$\# A = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \ldots \cdot 1 \cdot 3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3 = 3^{15}$

4. Drawing w/ replacement, order doesn't matter

- Toss a dice k times
- # of 1s, 2s, 3s, 4s, 5s, 6s

10 tosses

$\{0, 2, 4, 1, 1, 2\}$
1s 2s 3s 4s 5s 6s

o o | o o o | o | o o | o | o
2 1s 3 2s 1 3 2 4s 1 5 1 6

o | | o o | o o | | o o o o o
1 1 0 2s 2 3s 2 4s 0 5 5 6s

o | o o | o o | o | o | o o o

n−1 separators
k circles

$n+k-1$ circles or separators

How many such patterns are there?

$\boxed{n+k-1}$ positions
$\boxed{k \text{ separators}}$

$\{1, 3, 5, 7, 10\}$: positions of separators
$\{1, 3, 5, 10, 7\}$

$\boxed{\binom{n+k-1}{k}}$ $\quad \# \Omega = \binom{n+k-1}{k}$