The law of total probability

$$P(A)=\sum_{i=1}^{n} P(B_i)P(A|B_i)$$

if B_i are disjoint events
and $\bigcup B_i = \Omega$

3 factories: 1st 20%
2nd 45% of production
3rd 35%

Faulty products: 1st 10%
2nd 15%
3rd 5%

randomly select a product. P(product is faulty)?

A: Product is faulty

$$\left.\begin{array}{ll} B_1: & \text{product is from factory } 1 \\ B_2: & \text{— " — } 2 \\ B_3: & \text{— " — } 3 \end{array}\right\} \text{disjoint cover } \Omega$$

$P(B_1)=0.2 \quad P(B_2)=0.45 \quad P(B_3)=0.35$

$P(A|B_1)=0.1 \quad P(A|B_2)=0.15 \quad P(A|B_3)=0.05$

$$P(A)=\sum_{i=1}^{3} P(B_i)P(A|B_i)=$$

$$=0.2\cdot 0.1+0.45\cdot 0.15+0.35\cdot 0.05=0.105$$

Bayes' rule

- Partition Ω to disjoint $B_1, B_2 \ldots B_n$
- I know $P(A|B_1), P(A|B_2) \ldots P(A|B_n)$
- I know $P(B_1), P(B_2) \ldots P(B_3)$

$$P(B_j|A)=\frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$$

3 factories
$$\left.\begin{array}{ll} \text{1st} & 20\% \\ \text{2nd} & 45\% \\ \text{3rd} & 35\% \end{array}\right\} \text{of production} \qquad \text{faulty in} \begin{array}{ll} \text{1st} & 10\% \\ \text{2nd} & 15\% \\ \text{3rd} & 5\% \end{array}$$

What is the probability that a randomly selected faulty product came from the first factory?

$$P(B_1|A)=\frac{P(A|B_1)P(B_1)}{\underbrace{\boxed{\sum_{n=1}^{3} P(A|B_i)P(B_i)}}_{\text{law of total probability}}}=\frac{0.1\cdot 0.2}{0.105}=\frac{0.02}{0.105}=\frac{0.1\cdot 0.2}{0.1\cdot 0.2+0.45\cdot 0.15+0.35\cdot 0.05}$$

$$P(B_2|A)=\frac{0.15\cdot 0.45}{0.105} \qquad P(B_1|A)+P(B_2|A)+P(B_3|A)=1$$

$$P(B_3|A)=\frac{0.35\cdot 0.05}{0.105}$$

$$P(B_j|A)=\frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$$

$$\sum_{k=1}^{n} P(B_k|A)=\sum_{k=1}^{n}\frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}=\frac{1}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}\cdot\sum_{k=1}^{n} P(B_k)P(A|B_k)=1$$