Graphical introduction to calculus

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Mathematics and Informatics Pre-session for Business Analytics

2023

Topics

- Limits
- Differentiation
- Unconstrained optimization

Limits of functions

A limit is the value that a function approaches as its input variable approaches a specified value. It helps us analyze the behavior of a function near a point, even if it may not be defined at that point. Limits are crucial in understanding continuity, derivatives, and integrals in calculus.

Limits of functions

We can define the limits for functions the following way.

Definition

A function $f(x)$ has a limit L when x approaches to p IF for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x that satisfies $|x - p| < \delta$ it holds that $|f(x) - L| < \varepsilon$. The notation is

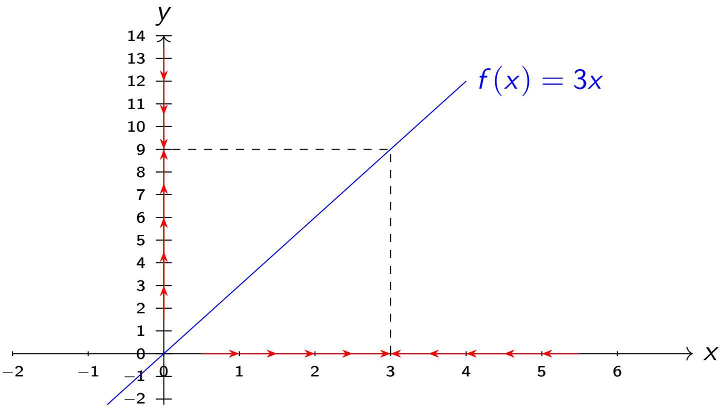
$$\lim_{x \to p} f(x) = L$$

Intuition

As we get closer and closer to a point p on the x axis, the value of the function shown on the y axis might get infinitesimally close to a value L. This is the limit of the function at point p.

Limits of functions

Example: $f(x) = 3x$. Find $\lim_{x \to 3} f(x)$.



Limits of functions

- We don't really want to use the formal definition in most cases to find the limits.
- The graphical approach often helps.
- An important property: For continuous functions the limit is the same as the value of the function.
- We can also use the following properties:

$$\lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$$
$$\lim_{x \to p} (f(x) - g(x)) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$$
$$\lim_{x \to p} (f(x) \cdot g(x)) = \lim_{x \to p} f(x) \cdot \lim_{x \to p} g(x)$$
$$\lim_{x \to p} (f(x) / g(x)) = \lim_{x \to p} f(x) / \lim_{x \to p} g(x)$$

Examples

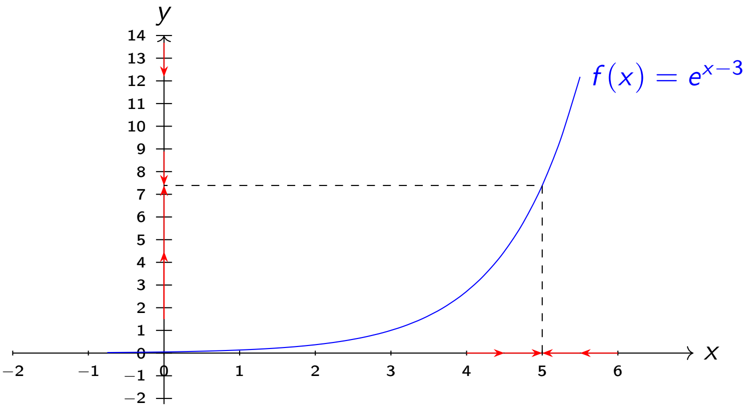
Find $\lim_{x \to 5} e^{x-3}$. Notice that this is a standard exponential function, which is continuous. Thus

$$\lim_{x \to 5} e^{x-3} = e^{5-3} = e^2$$

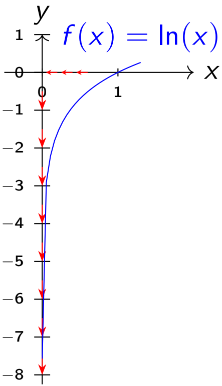
Find $\lim_{x \to 0} \ln(x)$. Now notice, that $In(0)$ is not defined. However the $\ln(x)$ function is monotonically increasing, thus as we get closer and closer to zero, its value gets closer and closer to minus infinity. Thus

$$\lim_{x \to 0} \ln(x) = -\infty$$

Examples



Examples



Solve the following problems

1. $\lim_{x \to 0} (3 + 2x^2)$
2. $\lim_{x \to -1} \frac{3+2x}{x-1}$
3. $\lim_{x \to 1} \frac{x^2+7x-8}{x-1}$

Differentiation

- We are often interested in the slope of the tangent line of a curve at a given point.
- To get this, we use differentiation.
- It is especially useful in case of optimization problems.
- Why? Consider for example the case when you are looking for the maximum of $f(x) = 3 - x^2$.
- What is the slope of the tangent line at the maximum point?

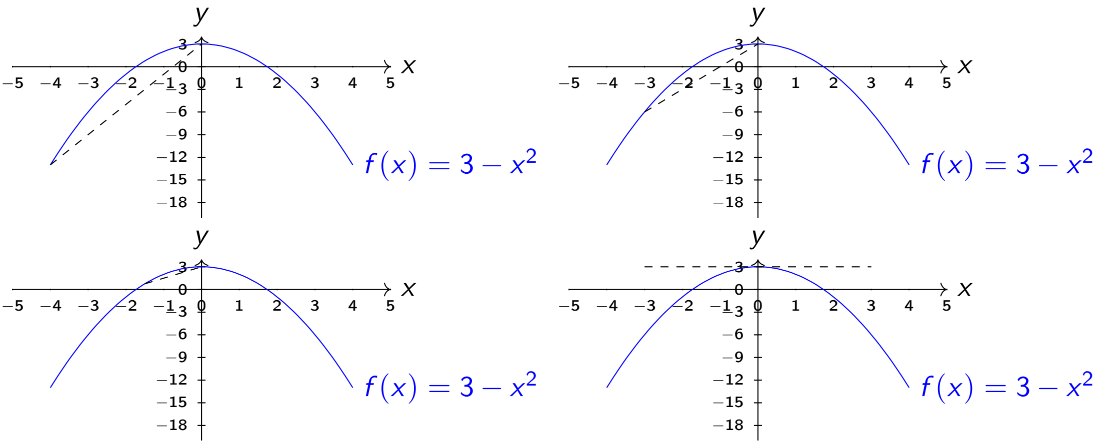
Differentiation

- The first differential $f'(x_0)$ of a function $f(x)$ at a given point x_0 is given by the limit:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Notice that $\frac{f(a)-f(b)}{a-b}$ is the slope of the section connecting the function at a and b.
- What we do here, is we get these two points closer and closer.
- Once they are infinitesimally close, it gives the slope of the tangent line.

Visually


y
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−5 −4 −3 −2 −1 0 1 2 3 4 5
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$f(x) = 3 - x^2$
y
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y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
3 0 −3 −6 −9 −12 −15 −18
$f(x) = 3 - x^2$

The derivative function

- The first derivative functions give the derivative of a function at any point.
- The usual notation is either $f'(x)$ or

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x}$$

- You can find these functions using tools like WolframAlpha.

Unconstrained optimization

We often want to solve so-called unconstrained optimization problems. Examples:

- What is the optimal quantity to produce in order to maximize your profit?
- What is the optimal length of sleep if you want to be as productive as possible?

If we can characterize these problems with functions, we can optimize them.

- We want to find their minima/maxima
- At these points, the tangent line should be horizontal
- Thus the derivative should be equal to zero

A quick example

Assume that you want to find the minimum of $f(x) = x^2 - 2x - 3$.

- We can either notice that it is equivalent to $f(x) = (x+1)(x-3)$ and infer that its minimum is at $x = 1$
- Or take its first derivative and find its root

$$\begin{aligned} \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} &= 0 \\ 2x - 2 &= 0 \\ x &= 1 \end{aligned}$$

Minimum or maximum? Maybe neither?

- In the previous case we knew that we had a minimum, as it was a simple convex parabola
- But the derivative is 0 at minima and maxima as well
- Also, there is something called an inflection point that we will see by checking $f(x) = x^3$
- If we try to find its minimum/maximum we get

$$\begin{aligned} \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} &= 0 \\ 3x^2 &= 0 \\ x &= 0 \end{aligned}$$

- Thus we should have a minimum/maximum at $x = 0$
- But we don't have one! The derivative can be zero, where the function changes convexity (inflection point)

How to decide?

- Notice that if it is a minimum point, the function has to be convex around the point
- For a maximum point, the function has to be concave around the point
- In case of an inflection point, the function is convex on one side but concave on the other side
- We should look at convexity

How to decide convexity?

- Notice that for convex functions the slope of the tangent line is continuously increasing (or at least not decreasing).
- For concave functions, this is the opposite. The slope of the tangent line is continuously decreasing (or at least not increasing).
- We already know a method to show whether a function is increasing or decreasing: taking its derivative
- Thus if the derivative shows the slope of the function (how the function values change), the derivative of the derivative shows how the slope of the function changes (convexity).
- Therefore to decide we need to check the sign of the second derivative denoted by $f''(x)$ or $\frac{\mathrm{d}^2 f(x)}{\mathrm{d}\, x^2}$

An example

Look at $f(x) = \frac{1}{3}x^3 - 1.5x^2 - 4x + 10$ on WolframAlpha.
The function is concave at $x = -1$, and that point should be a local maximum. It is convex at $x = 4$, and it should be a local minimum.

Local versus global

- If you look at the previous example, you can see that the function actually takes higher values than the maximum we found
- It also takes lower values than the minimum we found
- By looking at the derivatives, we find so-called local minima/maxima
- These are the highest/lowest values of the function in its surrounding
- It is not necessarily the same as the global maximum/minimum
- We should also check the limits of the function at the endpoints of the domain

Partial derivatives

A partial derivative is the derivative of a multivariate function with respect to one of its variables, while we consider all other variables to be constant. An example for the notation if we have a function of two variables $f(x, y)$:

- The partial derivative w.r.t. x is $f'_x(x, y)$ or $\frac{\partial f(x,y)}{\partial x}$
- The partial derivative w.r.t. y is $f'_y(x, y)$ or $\frac{\partial f(x,y)}{\partial y}$

Multivariate unconstrained optimization

Just like in the one variable case, the derivatives at the minima/maxima have to be equal to zero. The difference is that now we need all partial derivatives to be equal to zero. If we put all these partial derivatives in a vector, it is called the **gradient**. You don't have to use it right now, but it is good to know, as it will come up later (E.g.: Gradient descent in ML courses).

Examples

Let's find the maxima/minima of the following function using WolframAlpha: $f(x, y) = x^2y^2 - 5x - 5y$

We find that the solution is $x = y = \sqrt[3]{5/2}$

Let's find the maxima/minima of the following function: $f(x, y) = -xye^{-x^2-y^2}$

We find 5 solutions: $(x, y) = (0, 0)$, $(x, y) = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $(x, y) = \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$, $(x, y) = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $(x, y) = \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

Are these minima or maxima?

- That, again, depends on the convexity
- But it is significantly more difficult to check the convexity here
- We would need to check whether the so-called Hessian matrix (see below) is positive or negative definite, which we can not do without a decent knowledge on linear algebra

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Constrained optimization

- Sometimes we want to maximize/minimize a function, but we face a constraint
- For example: You have 20m of fence-material and you want to fence the largest possible rectangle area
- In this case the sides of the rectangle are a and b, thus we want to maximize $f(a, b) = ab$. But $2a + 2b = 20$ also has to hold.
- To solve such problems we can use a function called the Lagrangian

The Lagrangian (good to know, worth looking up)

If we want to find the minima/maxima of a function $f(x, y)$ with a constraint $g(x, y) = 0$, then we can formulate the following function:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

We call this function the Lagrangian, and λ is called a Lagrange multiplier. We can find the solution(s) to out constrained optimization problem by taking the partial derivatives of the Lagrangian w.r.t. x, y and λ. These all should be equal to zero at the minimum/maximum.