

Probability

- Random experiments
- Probability distributions

Random experiments:

- Experiment \rightarrow outcome cannot be determined in advance

- Examples:
- Tossing a coin
 - Rolling dice
 - Roulette
 - Pulling stuff out of a bag
 - Monthly rainfall in Vienna
 - How long do I have to wait for the train?

- Sample space: set of possible outcomes.

Ω

- Toss a coin: $\Omega = \{H, T\}$
- Roll a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Monthly rainfall $\Omega = \mathbb{R}_0^+$
- # of forest fires in Canada this year: \mathbb{Z}_0^+
- Event: A pre-defined set of outcomes.
If the outcome of the experiment is in this set
 \Rightarrow Event occurs

- Roll a dice
Event: outcome is odd. $A = \{1, 3, 5\}$

- Rainfall
Event: less than 15 mm $A = [0, 15)$
 \hookrightarrow included \hookrightarrow excluded

- Given the outcome, we have to be able to decide if event occurred or not.

- It rained a lot
- I got a high number

- Operations:

If I have 2 events A and B

- $A \cap B$: intersection. Occurs if both A & B occur.
- $A \cup B$: union. Occurs if A or B or both occur.
- A^c : complement. Occurs if A does not occur.

If $A \cap B = \emptyset$ then A & B are disjoint events.
cannot occur at the same time.

- Toss a coin: $A: \{H\}$ $B: \{T\}$
- Roll a dice: $A: \{1, 3, 5\}$ $B: \{2, 4, 6\}$
- $A: \{2\}$ $B: \{4\}$

Probability

- Number between 0 and 1 corresponding to how likely it is that an event occurs. P

$$P(A) \geq 0$$

$$P(\Omega) = 1$$

$$\text{For disjoint events } P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

probability of the union sum of individual probabilities

- Experiment: Roll a dice

$$\Omega = \{1, 2, 3, 4, 5, 6\} \quad P(\Omega) = 1$$

$$P(\{1\}) = \frac{1}{6}$$

$$6 \text{ disjoint events: } \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$$

$$P(\{1, 2, 3, 4, 5, 6\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) + P(\{5\}) + P(\{6\})$$

$$1 = \sim$$

$$\Rightarrow P(\{1\}) = \frac{1}{6} \quad P(\{2\}) = \frac{1}{6} \quad P(\{3\}) = \frac{1}{6} \dots$$

$$P(\{1, 3, 5\}) = P(\{1\}) + P(\{3\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Toss a coin 3 times:

$$\Omega = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$$

- All are disjoint
- equally likely
- I have 8 disjoint events
- each has a probability of $\frac{1}{8}$

$$A: \text{At least 2 T}$$

$$A: \{(H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$$

$$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

- Experiment: Roll a dice twice. Record the outcomes.

Event: I have 6 at least once. A

$$P(A) = ?$$

$$\Omega = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \dots, (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$$\# \Omega = 36$$

\hookrightarrow cardinality of a set
 $\#$ elements

$$\hookrightarrow \text{All disjoint}$$

$$\hookrightarrow \text{equally likely}$$

$$\hookrightarrow \text{probability } \frac{1}{36}$$

$$A = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$$

$$\# A = 11$$

$$P(A) = \frac{1}{36} + \frac{1}{36} + \dots + \frac{1}{36} = 11 \cdot \frac{1}{36} = \frac{11}{36}$$

B : First one is 6

$$P(B) = \frac{1}{6}$$

C : Second one is 6

$$P(C) = \frac{1}{6}$$

$$B = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$$C = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}$$

B & C are not disjoint (6, 6)

Example:

Multiple choice exam: 4 options
10 questions

$$\Omega = \{(A, A, A, A, A, A, A, A, A, A), (A, A, A, A, A, A, A, A, B, A), \dots\}$$

Fill this in randomly.

What is the probability that you get a perfect score?

A : perfect score

$$\# \Omega = 4 \cdot 4 \cdot 4 \cdot 4 \cdot \dots \cdot 4 = 4^{10}$$

$$\# A = 1$$

$$\# \Omega$$

$$P(A) = \frac{1}{4^{10}} = \frac{\# A}{\# \Omega}$$

Urn, draw balls

- n balls

- draw 2

$\# \Omega$ depends on 2 things

- Does the order matter?
- Replace balls between draws or not?

Exam

$$\text{Urn: } (A), (B), (C), (D)$$

Draw 2 = 10

replace them

order matters

race b/w 20 horses: (H1), (H2), ..., (H20)

first 3 horses: 2 = 3 draws
no replacement order matters

CEU library 200 000 books borrow random 5

2 = 5 draws order doesn't matter
no replacement