Probability

- Counting: Ω, A, equally likely outcomes $\Big\}$ $P(A) = \frac{\#A}{\#\Omega}$

- Drawing balls from an urn.
  - order matters or not
  - replace the balls or not

  $\Big\}$ n balls, k draws

- <u>No replacement, order matters</u>

  Permutation of size k from a set of size n

  20 balls, draw 3 $\quad \#\Omega = 20 \cdot 19 \cdot 18$

  1 2 3
  2 1 3

  (1 2 3) $\quad 3 \cdot 2 = 6$

  1 2
  1 3
  2 1
  2 3
  3 1
  3 2

  (1 2 3 4) $\quad 4 \cdot 3 \cdot 2 = 24$

  1 2 3
  1 2 4
  1 3 2
  1 3 4
  1 4 2
  1 4 3

$$\#P_k^n = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = \frac{\cancel{1 \cdot 2 \cdot 3} \cdots \cancel{(n-k)}(n-k+1)(n-k+2)\cdots n}{\cancel{1 \cdot 2 \cdot 3} \cdots \cancel{(n-k)}} = (n-k+1)(n-k+2)\cdots n$$

17 18 19 20

↳ Permutation

$$m \to m! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots m = \prod_{i=1}^{m} i$$

Urn $\quad n = 2000$, $k = 100$ $\quad 2000 \cdot 1999 \cdot 1998 \cdots 1991 \cdots 1901$

$$\frac{2000!}{1900!} = \frac{\cancel{1 \cdot 2} \cdots \cancel{1900} \cdot 1901 \cdot 1902 \cdots 2000}{\cancel{1 \cdot 2 \cdot 3} \cdots \cancel{1900}}$$

$n = 20$, $k = 3$

$$\frac{20!}{17!} = \frac{\cancel{1 \cdot 2 \cdot 3} \cdots \cancel{16} \cdot \cancel{17} \cdot 18 \cdot 19 \cdot 20}{\cancel{1 \cdot 2 \cdot 3} \cdots \cancel{16} \cdot \cancel{17}} = 18 \cdot 19 \cdot 20$$

<u>Races</u>

- Cars 30
- Different 1st, 2nd, 3rd place outcomes $\quad n = 30$, $k = 3$

$$\#P_3^{30} = 30 \cdot 29 \cdot 28 = \frac{30!}{27!}$$

<u>Lottery</u>

$n = 10$ numbers
$k = 5$ draws $\Big\}$ this will make a 5-digit number

not replacing the balls.

you have to guess a 5-digit number.

$$\boxed{10} \cdot \boxed{9} \cdot \boxed{8} \cdot \boxed{7} \cdot \boxed{6} \qquad \#\Omega = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 30240$$

$$P(\text{win with single ticket}) = \frac{1}{30240}$$

<u>No replacement, unordered</u>:

$n = 90$, $k = 5$

20, 1, 23, 42, 49
1, 20, 49, 23, 42 $\Big\}$ Same outcomes, order does not matter.

$n = 4$, $k = 3$ $\quad$ (1) (2) (3) (4)

123	213	312	412
124	214	314	413
132	231	321	421
134	234	324	423
142	241	341	431
143	243	342	432

$\Big\}$ 4 possible outcomes.

I counted everything 6 times.

if the order matters $\quad \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$

How many times did I count every outcome?

(1 2 3) $\to k$

123 213 312
132 231 321 $\Big\}$ all possible ordering of k numbers.

$\underbrace{\qquad}_{k!}$ $\quad 3 \cdot 2 \cdot 1 = 6$ $\quad \Big\}$ I counted everything $k!$ times.

$$\Rightarrow \#\Omega = \frac{n!}{(n-k)!} \cdot \frac{1}{k!}$$

↳ correct for double counting

↳ If order matters. Includes double counting.

- Combination of size k from a set of n

$$\#C_k^n = \frac{\#P_k^n}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k} \leftarrow n \text{ choose } k$$

$n = 4$, $k = 3$

$$\#C_3^4 = \frac{4!}{1!\,3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 1 \cdot 2 \cdot 3} = 4$$

Lottery

$n = 90$, $k = 5$

$$\#C_5^{90} = \binom{90}{5} = \frac{90!}{85!\,5!} = 43\,949\,268$$

$$P(\text{win}) = \frac{1}{43949268}$$

Draw with replacement, order matters

- multiple choice test (A) (B) (C) (D)
- 10 questions

$\Big\}$ $n = 4$, $k = 10$

$$\#\Omega = n^k$$

$$P(\text{totally random answers} \Rightarrow \text{perfect score}) = \frac{1}{4^{10}}$$

AAAAAAAAAA
— 11 — B
— 11 — C $\Big\}$ 4
— 11 — D
— 11 — BA
— 11 — BB
— 11 — BC
— 11 — BD
CA
CB
CC
CD
DA
DB
DC
DD

$\Big\}$ $4 \cdot 4$ $\quad \Big\}$ whatever the first 7 are $4 \cdot 4 \cdot 4$ $\quad \Big\}$ whatever first 6 are $4 \cdot 4 \cdot 4 \cdot 4$ $\quad \Big\}$ $\underline{\underline{4^{10}}}$

<u>Draw with replacement, unordered</u>

$$\#\Omega = \binom{k+n-1}{k}$$

- Roll a dice 10 times.
  - Count the number of 1s, 2s, 3s, 4s, 5s, 6s

(4, 3, 0, 0, 2, 1) $\to$ sum of this numbers is 10

↳ 4 1s
↳ 3 2s
↳ 0 3s
↳ 0 4s
↳ 2 5s
↳ 1 6

1265112251
1526112251 $\Big\}$ order does not matter

Roll 10 times outcome token in a tray $\quad$ token = O

OOOO	OOO			OO	O
1	2	3	4	5	6

O O OOOOOOOOOO $\quad$ 10 tokens

OOOO | OOO | | | OO | O $\quad$ 6 trays $\quad \underbrace{\qquad}$ 5 separators |

4 1s $\quad$ 3 2s $\quad$ 0 3s $\quad$ 0 4s $\quad$ 2 5s $\quad$ 1 6

(2, 2, 1, 1, 3, 1)

$\underbrace{\text{OO|OO|O|O|OOO|O}}_{15 \quad k+n-1 \text{ symbols}}$ $\to$ sequence O |

altogether k tokens
$n-1$ separators

1 2 3 4 5 6 . . . .

$k+n-1$ positions

each a O or |

$n-1$ positions are separators

2 3 6 7 10
6 7 2 3 10

k will be O
$n-1$ will be |

how many patterns can I come up with?

I place separators to 5 places out of 15 positions

order does not matter, no 2 separators in same position

↳ no replacement $\quad n^* = n + k - 1$

↳ order doesn't matter $\quad k^* = n - 1$

$$\#C = \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!\,k!} = \binom{n+k-1}{k}$$

$$\binom{20}{2} = \frac{20!}{2! \cdot 18!}$$

$$\parallel$$

$$\binom{20}{18} = \frac{20!}{18! \cdot 2!}$$

- CEU library 200 000 books.
  - borrow 5.
  - How many outcomes can I have?

- 1. Replacement: No $\quad n = 200\,000$
  2. Order matter: No $\quad k = 5$

$$\#\Omega = \frac{200000!}{199995!\,5!} = \text{LAAARGE number.}$$

- Roll a dice 3 times. Record how many 1s 2s 3s.... we have
  number of outcomes we can have?

  w replacement $\quad n = 6$
  no ordering $\quad k = 3$

$$\#\Omega = \binom{6+3-1}{3} = \frac{(6+3-1)!}{(6-1)!\,3!} = \frac{8!}{5!\,3!} = \frac{6 \cdot 7 \cdot 8}{3 \cdot 2} = 2 \cdot 7 \cdot 4 = \underline{\underline{56}}$$

- Lottery 35 numbers 7 has to be right

  ↳ no replacement $\quad n = 35$
  ↳ no ordering $\quad k = 7$

$$\#\Omega = \frac{35!}{(35-7)!\,7!} = \binom{35}{7} = 6\,724\,520$$