Linear algebra

- Vectors
- Matrices
- Systems of linear equations

Vectors

- a vector of order n is an ordered set of n numbers.
- It can be
  - row vector $x = [x_1\ x_2\ x_3\ \dots\ x_n]$
  - column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

"vector" ⇒ column vector

Transpose:

Transposed x is x^T

$$x = [x_1\ x_2\ x_3\ \dots\ x_n] \qquad x^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \qquad y^T = [y_1\ y_2\ y_3\ \dots\ y_n]$$

$$(x^T)^T = x$$

Add/subtract

2 column vectors of the same order n OR
" row " " "

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \quad x \pm y = \begin{bmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ x_3 \pm y_3 \\ \vdots \\ x_n \pm y_n \end{bmatrix}$$

- different orders ↯
- row ± column ↯

Multiply by scalar:

multiply each element

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad c \cdot x = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ \vdots \\ cx_n \end{bmatrix}$$

Division

can't divide by 0

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \frac{x}{c} = \begin{bmatrix} x_1/c \\ x_2/c \\ \vdots \\ x_n/c \end{bmatrix} \quad c \neq 0$$

Multiply 2 vectors: inner product of 2 vectors

- sumproduct of their components

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

inner product: $(x,y) = x^T y = \underline{\sum_{i=1}^{n} x_i y_i} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad (x,y) = 1 \cdot 4 + 2 \cdot 5 = \underline{\underline{14}}$$

x & y has to be of the same order.

Norm

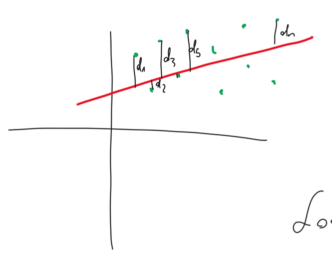
inner product of the vector with itself

$$\|x\| = (x,x) = \sum_{i=1}^{n} x_i x_i = \sum_{i=1}^{n} x_i^2$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \|x\| = 1^2 + 2^2 + 3^2 = \underline{\underline{14}}$$

L2 norm

Loss functions: norms of distance vectors



$$d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}$$

$$Loss = d_1^2 + d_2^2 + \dots d_n^2 = \|d\| \to \min$$

Equality

$x = y$ if x & y are both column vectors / row vectors of order n

and $x_i = y_i \ \forall i \in \{1, 2 \dots n\}$

$$\|x\| = 1^2 + 3^2 + 7^2 = 1 + 9 + 49 = \underline{\underline{59}}$$

$$y^T - 3x^T = [-1\ -5\ -15]$$

$$x = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad y^T = [2\ 4\ 6] \quad 3.5x = \begin{bmatrix} 3.5 \\ 10.5 \\ 24.5 \end{bmatrix}$$

$$(x,y) = 1 \cdot 2 + 3 \cdot 4 + 7 \cdot 6 = 2 + 12 + 42 = \underline{\underline{56}}$$

$$(2x,y) = 112 = 2 \cdot 2 + 6 \cdot 4 + 14 \cdot 6$$

$$x^T = [1\ 3\ 7] \qquad (x^T)^T = x = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \qquad 2y = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$$(x,y) \to (cx, dy) = c \cdot d \cdot (x,y) \qquad y - x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad x + y = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} \qquad x + 2y = \begin{bmatrix} 5 \\ 11 \\ 19 \end{bmatrix}$$

Matrices

- Rectangular array of numbers
- A matrix A with k rows and n columns is a $k \times n$ matrix
- in row i and column j is (i,j)th element

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Equality

A & B equal if both A & B are $k \times n$

and $a_{i,j} = b_{i,j} \ \forall i,j$

Transpose

$$A \quad k \times n \to A^T \quad n \times k \qquad a^T_{i,j} = a_{j,i}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \to A^T = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \qquad (A^T)^T = A$$

Add/subtract:

both $k \times n$

$$A \pm B = \begin{bmatrix} a_{1,1} \pm b_{1,1} & \dots & a_{1,n} \pm b_{1,n} \\ a_{2,1} \pm b_{2,1} & & a_{2,n} \pm b_{2,n} \\ \vdots & \ddots & \vdots \\ a_{k,1} \pm b_{k,1} & \dots & a_{k,n} \pm b_{k,n} \end{bmatrix}$$

$$\cancel{\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 4 \end{bmatrix} = \begin{matrix} 6 & 8 & 6 \\ 11 & 13 & 4 \\ 7 & 6 & 0 \\ & 2 & \end{matrix} \right)}$$

multiply by scalar

$$cA = \begin{bmatrix} ca_{1,1} & \dots & ca_{1,n} \\ ca_{2,1} & \ddots & \vdots \\ ca_{k,1} & \dots & ca_{k,n} \end{bmatrix}$$

division

same but $\frac{A}{0}$ ↯

determinant

Multiplication

$$A: k \times \underline{\underline{n}} \qquad B: \underline{\underline{n}} \times m$$

\# cols in A has to match # rows in B

$$AB_{i,j} = \sum_{h=1}^{n} A_{i,h} B_{h,j}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad A \cdot B$$

$$\begin{array}{c|ccc} & 1 & 0 & -1 \\ & 1 & -1 & 0 \\ \hline 2\ 3 & 5 & -3 & -2 \\ 4\ 5 & 9 & -5 & -4 \\ 6\ 7 & 13 & -7 & -6 \end{array}$$

$k \times m \quad 3 \times 3$

$2 \cdot 1 + 3 \cdot 1 = 5$
$2 \cdot 0 + 3(-1) = -3$
$2 \cdot (-1) + 3 \cdot 0 = -2$
$4 \cdot 1 + 5 \cdot 1 = 9$
$4 \cdot 0 + 5 \cdot (-1) = -5$
$4 \cdot (-1) + 5 \cdot 0 = -4$
$6 \cdot 1 + 7 \cdot 1 = 13$
$6 \cdot 0 + 7 \cdot (-1) = -7$
$6 \cdot (-1) + 7 \cdot 0 = -6$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 & -2 \\ 9 & -5 & -4 \\ 13 & -7 & -6 \end{bmatrix}$$

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3 products: sales price, cost

10 customers

A: what they buy: 10×3

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 4 \\ \vdots & & \\ 1 & 1 & 1 \end{bmatrix}$$

B: 3×2

$$\begin{bmatrix} p_1 & c_1 \\ p_2 & c_2 \\ p_3 & c_3 \end{bmatrix}$$

A·B 10×2

$$\begin{bmatrix} 2p_1 + 1p_2 + 0p_3 & 2c_1 + c_2 + 0c_3 \\ 3p_1 + 5p_2 + 4p_3 & 3c_1 + 5c_2 + 4c_3 \\ \vdots & \end{bmatrix}$$

Total sales price for customer 1

→ Total cost of products sold to 1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

profit on each $= A \cdot B \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

total profit $= \left(A \cdot B \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (x,y) = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 6$$

$$\begin{array}{c|c} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} & \begin{matrix} 9 \\ 18 \\ 27 \end{matrix} \end{array} \qquad \begin{array}{c|c} & \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \\ \hline [1\ 1\ 1] & 15\ \ 18\ \ 21 \end{array}$$

income	age	years of educ.
$300	20	8
$1000	24	12
$1400	27	14
$2000	30	20

income $= \underline{\underline{a}} + \underline{\underline{b}} \cdot age + \underline{\underline{c}}\, edu$

$$y = \begin{bmatrix} 300 \\ 1000 \\ 1400 \\ 2000 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 20 & 8 \\ 1 & 24 & 12 \\ 1 & 27 & 14 \\ 1 & 30 & 20 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (XX^T)^{-1} y$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \arg\min_{abc} \boxed{Loss}$$