Linear algebra

- Special matrices
  - square matrix: $m = n$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

  - Diagonal matrix: Every element not on the diagonal is 0

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3.14 \end{bmatrix}$$

  - upper-triangular: everything below the diagonal is 0

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - lower triangular: everything above the diagonal is 0

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

  - symmetric matrix: $A = A^T$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 8 & 14 \end{bmatrix} \qquad 2A + B^T = \begin{bmatrix} 4 & 7 \\ 5 & 7 \\ 12 & 16 \end{bmatrix}$$

- A^T, B^T
- $A + B^T$, $A^T - B$
- $2A$, $3B$
- $2A + B^T$
- $A \cdot B$, $B \cdot A$

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} \qquad 3B = \begin{bmatrix} 6 & 3 & 12 \\ 9 & 3 & 6 \end{bmatrix}$$

$$A + B^T = \begin{bmatrix} 3 & 5 \\ 3 & 4 \\ 8 & 9 \end{bmatrix} \qquad A^T - B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

$A \cdot B$

$$\begin{array}{cc|ccc} & & 2 & 1 & 4 \\ & & 3 & 1 & 2 \\ \hline 1 & 2 & 8 & 3 & 8 \\ 2 & 3 & 13 & 5 & 14 \\ 4 & 7 & 29 & 11 & 30 \end{array}$$

$B A$

$$\begin{array}{ccc|cc} & & & 1 & 2 \\ & & & 2 & 3 \\ & & & 4 & 7 \\ \hline 2 & 1 & 4 & 20 & 35 \\ 3 & 1 & 2 & 13 & 23 \end{array}$$

$2 \cdot 1 + 1 \cdot 2 + 4 \cdot 4$
$2 \cdot 2 + 1 \cdot 3 + 4 \cdot 7$
$3 \cdot 1 + 1 \cdot 2 + 2 \cdot 4$
$3 \cdot 2 + 1 \cdot 3 + 2 \cdot 7$

The inverse:

$A \rightarrow$ the inverse of A: A^{-1} is an other matrix such that

$$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

- Gauss elimination
- Basis transformation

Systems of linear equations

$(1) - (2) \Rightarrow x = 4$
$y = -0.5$

$$\begin{aligned} 3x + 4y &= 10 \\ 2x + 4y &= 6 \end{aligned}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} = \text{some matrix} = ($$

$$\begin{array}{cc|c} & & x \\ & & y \\ \hline 3 & 4 & 3x+4y \\ 2 & 4 & 2x+4y \end{array}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3x+4y \\ 2x+4y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

$$\begin{array}{cc|c} & & x \\ & & y \\ \hline 1 & 0 & 1 \cdot x + 0 \cdot y \\ 0 & 1 & 0 \cdot x + 1 \cdot y \end{array}$$

$$\begin{aligned} 5x_1 + 3x_2 - x_3 + 2x_4 &= 10 \\ x_1 - 2x_2 + 14x_3 + x_4 &= 22 \\ 7x_1 + x_2 - 21x_3 + 5x_4 &= 41 \\ 10x_1 - 5x_2 + 4x_3 + 7x_4 &= 5 \end{aligned}$$

$$\begin{bmatrix} 5 & 3 & -1 & 2 \\ 1 & -2 & 14 & 1 \\ 7 & 1 & -21 & 5 \\ 10 & -5 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \\ 41 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 & 3 & -1 & 2 \\ 1 & -2 & 14 & 1 \\ 7 & 1 & -21 & 5 \\ 10 & -5 & 4 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 22 \\ 41 \\ 5 \end{bmatrix}$$

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in general:

$$A \cdot x = y \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

A: $n \times n$ matrix

$$x = A^{-1} y$$

if A^{-1} exists.

$$\begin{aligned} (1)\ 2x + 4y &= 6 \\ (2)\ 3x + 6y &= 9 \end{aligned}$$

$(1) \cdot 1.5 = (2)$

$(1)\ 2x + 4y = 6 \quad /:2$
$x + 2y = 3 \quad /-2y$
$\underline{x = 3 - 2y}$

$3(3 - 2y) + 6y = 9$
$9 - 6y + 6y = 9$
$9 = 9$ ✓

$x = 1$, $y = 1$ $\quad$ $x = 3$, $y = 0$
$x = 10$, $y = -3.5$ $\quad$ $x = 0$, $y = 1.5$

(1) & (2) are linearly dependent.
↳ there is a constant c such that $(1) \cdot c = (2)$
$c = 1.5$
↳ in these cases A^{-1} doesn't exist
↳ A is singular

$$\left.\begin{aligned} (1)\ 2x + 4y + z &= 6 \\ (2)\ 3x + 6y - 3z &= 9 \\ (3)\ 5x + 10y - 2z &= 15 \end{aligned}\right\} \qquad \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & -3 \\ 5 & 10 & -2 \end{bmatrix}^{-1} \rightarrow \text{does not exist}$$

$(1) + (2) = (3)$ $\qquad$ $\underbrace{(3) = c_1 \cdot (1) + c_2 (2)}_{\text{linearly dependent}}$

now $c_1 = 1$ $c_2 = 1$

The determinant

- a single number.
- if it is 0, the matrix is singular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = a \cdot d - b \cdot c$$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad \det(A) = 2 \cdot 6 - 3 \cdot 4 = \underline{\underline{0}}$$

$\Rightarrow$ singular

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \qquad \det(A) = 2 \cdot 4 - 7 \cdot 1 = 8 - 7 = 1$$

$$B = \begin{bmatrix} 12 & e \\ \pi & 5 \end{bmatrix} \qquad \det(B) = 12 \cdot 5 - \pi \cdot e = 60 - \pi \cdot e$$

$$C = \begin{bmatrix} 5 & 0.1 \\ 10 & 0.2 \end{bmatrix} \qquad \det(C) = 5 \cdot 0.2 - 10 \cdot 0.1 = 1 - 1 = \underline{\underline{0}}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix} \qquad + \quad -$$

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix} \qquad \det(A) = 2 \cdot 4 \cdot 1 + 7 \cdot 3 \cdot 3 + 1 \cdot 1 \cdot 2 - 1 \cdot 4 \cdot 3 - 2 \cdot 3 \cdot 2 - 7 \cdot 1 \cdot 1$$

$= 8 + 63 + 2 - 12 - 12 - 7$
$= \underline{\underline{42}}$

$$\begin{matrix} 2 & 7 & 1 & 2 & 7 \\ 1 & 4 & 3 & 1 & 4 \\ 3 & 2 & 1 & 3 & 2 \end{matrix}$$

from row 1	from row 2	from row 3
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1