Parametric Equations and the Parabola (Extension 1)

Parametric Equations

- Parametric equations are a set of equations in terms of a parameter that represent a relation.
- Each value of the parameter, when evaluated in the parametric equations, corresponds to a point along the curve of the relation.
- To convert equations from parametric form into a single relation, the parameter needs to be eliminated by solving simultaneous equations.

Locus of a Point

If the position of a point is given in terms of the parameter, to find the locus of the point is to find the equation of the path that the point takes as the parameter's value changes.

- If the point's x- or y-coordinate is independent of the parameter, then a vertical or horizontal line is the locus, although it might be restricted to just part of the line.
- If not, eliminate the parameter by solving the equations simultaneously.

Parametric Representation of a Parabola

Parametric equations

$$\begin{aligned} x &= 2ap & (1) \\ y &= ap^2 & (2) \end{aligned}$$

A variable point on the parabola is given by $(2ap, ap^2)$, for constant a and parameter p.

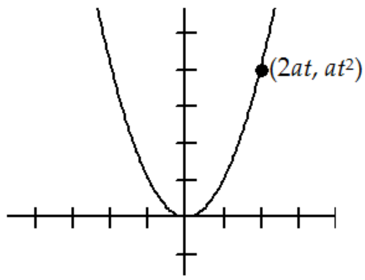
Conversion into Cartesian equation

Rearrange (1) to give:

$$p = \frac{x}{2a} \qquad (3)$$

Then substitute (3) into (2):

$$\begin{aligned} y &= a\left(\frac{x}{2a}\right)^2 \\ &= \frac{x^2}{4a} \\ x &= 4ay \end{aligned}$$



which is the equation of a parabola with vertex $(0, 0)$ and focal length a.

Gradient of Tangent

The gradient of the tangent to $x^2 = 4ay$ at $P(2ap, ap^2)$ is $\boxed{p}$.

$$y = \frac{x^2}{4a} \Rightarrow m = y' = \frac{x}{2a} = \frac{2ap}{2a} = p \qquad (4)$$

Equation of Tangent

The equation of the tangent to $x^2 = 4ay$ at $P(2ap, ap^2)$ is $\boxed{y = px - ap^2}$.

1. Find the gradient of the tangent at P; using (4):

$$m = p$$

2. Use $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y - ap^2 &= p(x - 2ap) \\ y &= px - ap^2 \end{aligned} \tag{5}$$

Equation of Normal

The equation of the normal to $x^2 = 4ay$ at $P(2ap, ap^2)$ is $\boxed{x + py = ap^3 + 2ap}$.

1. Find the gradient of the normal at P:

$$m = -\frac{1}{p}$$

2. Use $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y - ap^2 &= -\frac{1}{p}(x - 2ap) \\ py - ap^3 &= -x + 2ap \\ x + py &= ap^3 + 2ap \end{aligned} \tag{6}$$

Intersection of Tangents

The intersection of the tangents to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ is $\boxed{(a(p+q), apq)}$.

1. Obtain the equations of the tangents using (5):

$$y = px - ap^2 \tag{7}$$

$$y = qx - aq^2 \tag{8}$$

2. Substitute (7) into (8):

$$\begin{aligned} px - ap^2 &= qx - aq^2 \\ px - qx &= ap^2 - aq^2 \\ x(p - q) &= a(p - q)(p + q) \\ x &= a(p + q) \end{aligned} \tag{9}$$

3. Substitute (9) back into (7):

$$\begin{aligned} y &= p(ap + aq) - ap^2 \\ &= apq \end{aligned} \tag{10}$$

Intersection of Normals

The intersection of the normals to $x^2 = 4ay$ at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ is $\boxed{(-apq(p+q), a(p^2+pq+q^2+2))}$.

1. Obtain the equations of the normals using (6):

$$\begin{aligned} x + py &= ap^3 + 2ap \qquad (11) \\ x + qy &= aq^3 + 2aq \qquad (12) \end{aligned}$$

2. Subtract (12) from (11):

$$\begin{aligned} py - qy &= ap^3 - aq^3 + 2ap - 2aq \\ y(p-q) &= a(p-q)(p^2+pq+q^2) + 2a(p-q) \\ y &= a(p^2+pq+q^2+2) \qquad (13) \end{aligned}$$

3. Substitute (13) back into (11):

$$\begin{aligned} x + pa(p^2+pq+q^2+2) &= ap^3 + 2ap \\ x + ap^3 + ap^2q + apq^2 + 2ap &= ap^3 + 2ap \\ x &= -(ap^2q + apq^2) \\ &= -apq(p+q) \qquad (14) \end{aligned}$$

Equation of Chord

The equation of the chord from $P(2ap, ap^2)$ to $Q(2aq, aq^2)$ is $\boxed{y - \frac{1}{2}x(p+q) + apq = 0}$.
Using $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$:

$$\begin{aligned} \frac{y-ap^2}{x-2ap} &= \frac{aq^2-ap^2}{2aq-2ap} \\ (y-ap^2)2a(q-p) &= (x-2ap)a(q-p)(q+p) \\ 2(y-ap^2) &= (x-2ap)(p+q) \\ 2y - 2ap^2 &= px + qx - 2ap^2 - 2apq \\ y + apq &= \frac{1}{2}x(p+q) \\ y - \frac{1}{2}x(p+q) + apq &= 0 \qquad (15) \end{aligned}$$

Focal Chord Properties

A focal chord is a chord that goes through the focus of the parabola at $(0, a)$.

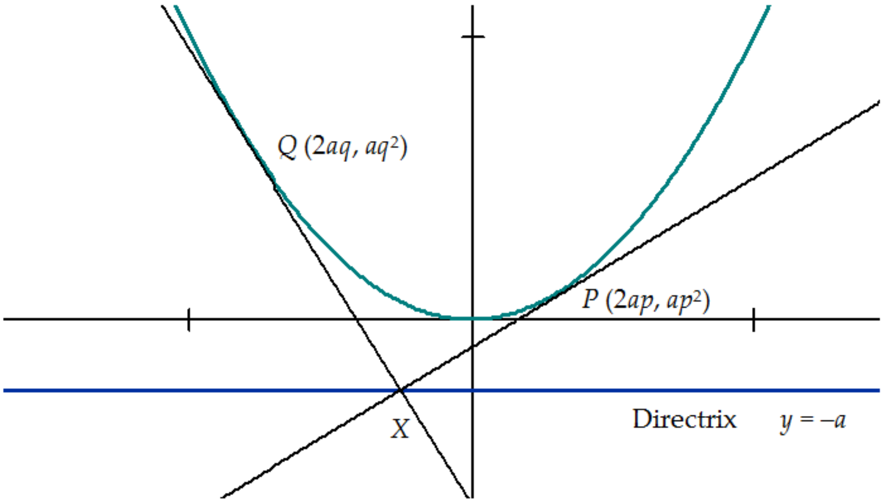
- $pq = -1$
  1. Obtain the equation of the chord using (15):

  $$y - \frac{1}{2}x(p+q) + apq = 0$$

  2. Substitute in $(0, a)$:

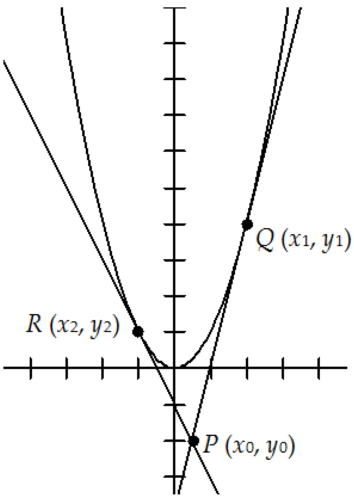
  $$\begin{aligned} a + apq &= 0 \\ apq &= -a \\ pq &= -1 \qquad (16) \end{aligned}$$

- If PQ is a focal chord, the tangents at P and Q will meet at the directrix at right angles.



Using (9) and (10), the tangents will intersect at $(a(p+q), apq) = (a(p+q), -a)$, since $pq = -1$, by (16). The directrix of the parabola $x^2 = 4ay$ is $y = -a$, and so the point lies on the directrix. Also, $m_{PX}m_{QX} = pq = -1$, and hence $PX \perp QX$, where X is the point of intersection.

Chord of Contact



For a given point $P(x_0, y_0)$, the chord of contact (the chord that joins the two points whose tangents pass through P) is $\boxed{xx_0 = 2a(y + y_0)}$.

1. Suppose $Q(x_1, y_1)$ and $R(x_2, y_2)$ are the two points with tangents passing through P. The equation of the tangent at Q can be found by applying $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y - y_1 &= \frac{x_1}{2a}(x - x_1) \\ 2a(y - y_1) &= xx_1 - x_1^2 \\ 2a(y - y_1) &= xx_1 - 4ay_1 \qquad \text{since the point lies on } x^2 = 4ay \\ xx_1 &= 2a(y + y_1) \qquad (17) \end{aligned}$$

   Similarly, the equation of the tangent at R is:

$$xx_2 = 2a(y + y_2) \qquad (18)$$

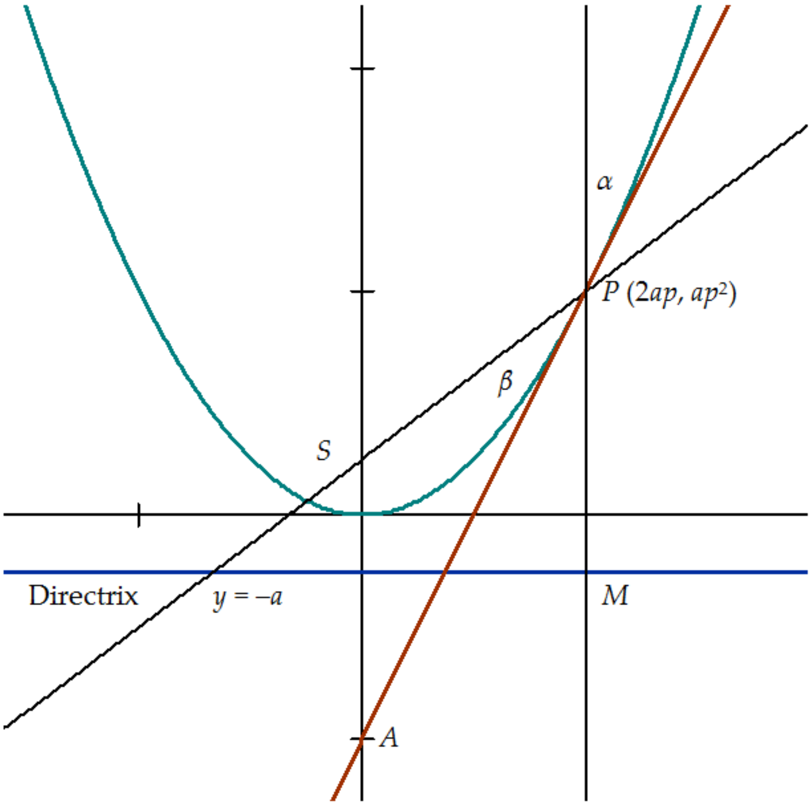
2. However, since $P(x_0, y_0)$ lies on both tangents, we substitute P into (17) and (18):

$$\begin{aligned} x_0x_1 &= 2a(y_0 + y_1) \qquad (19) \\ x_0x_2 &= 2a(y_0 + y_2) \qquad (20) \end{aligned}$$

3. From (19) and (20), it is clear that Q and R lie on the line:

$$xx_0 = 2a(y + y_0) \qquad (21)$$

Angle of Incidence Equals Angle of Reflection



1. Find the position of A by substituting $x = 0$ into the equation of the tangent at P from (5): $y = -ap^2 \Rightarrow A$ is at $(0, -ap^2)$.
2. Find distances: $AS = a + ap^2$, $SP = PM = a + ap^2$.
3. Therefore, $\triangle ASP$ is isosceles, and so $\angle SAP = \beta$.
4. However, $\angle SAP = \alpha$ (corresponding angles on parallel lines), and hence $\alpha = \beta$.

Parametric Representation of a Circle

Parametric equations

$$x = r\cos\theta \tag{22}$$
$$y = r\sin\theta \tag{23}$$

A variable point on the circle is given by $(r\cos\theta, r\sin\theta)$, for constant r and parameter θ.

Conversion into Cartesian equation

Squaring (22) and (23) gives $x^2 = r^2\cos^2\theta$ and $y^2 = r^2\sin^2\theta$, and thus:

$$\begin{aligned} x^2 + y^2 &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 \end{aligned}$$

which is the equation of a circle with centre $(0, 0)$ and radius r.

Parametric Representation of an Ellipse

Parametric equations

$$
\begin{aligned}
x &= a\cos\theta && (24)\\
y &= b\sin\theta && (25)
\end{aligned}
$$

A variable point on the ellipse is given by $(a\cos\theta, b\sin\theta)$, for constants a and b, and parameter θ.

Conversion into Cartesian equation

Squaring (24) and (25):

$$
\begin{aligned}
x^2 &= a^2\cos^2\theta \\
&= a^2\left(1-\sin^2\theta\right) && (26)\\
y^2 &= b^2\sin^2\theta \\
\sin^2\theta &= \frac{y^2}{b^2} && (27)
\end{aligned}
$$

Substitute (27) into (26):

$$
\begin{aligned}
x^2 &= a^2\left(1-\frac{y^2}{b^2}\right)\\
\frac{x^2}{a^2} &= 1-\frac{y^2}{b^2}\\
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1
\end{aligned}
$$

which is the equation of an ellipse with centre $(0,0)$, length of major axis (which is along the x-axis) $2a$ and length of minor axis $2b$.

Ellipse as Generalisation of Circle

A circle is an ellipse with equal lengths for the semi-major and semi-minor axes, and thus (24) and (25) reduce to (22) and (23) respectively, when $a = b = r$.

Use of parameterisation

It is convenient to express curves in parameterised form, because it allows you to differentiate and integrate termwise. For example, if we have the x- and y-coordinates of a particle expressed as a function of the parameter time, t:

$$r(t) = (x(t), y(t)) \tag{28}$$

we can obtain the x- and y-components of the velocity and acceleration. These are, respectively:

$$
\begin{aligned}
v(t) &= (x'(t), y'(t)) && (29)\\
a(t) &= (x''(t), y''(t)) && (30)
\end{aligned}
$$