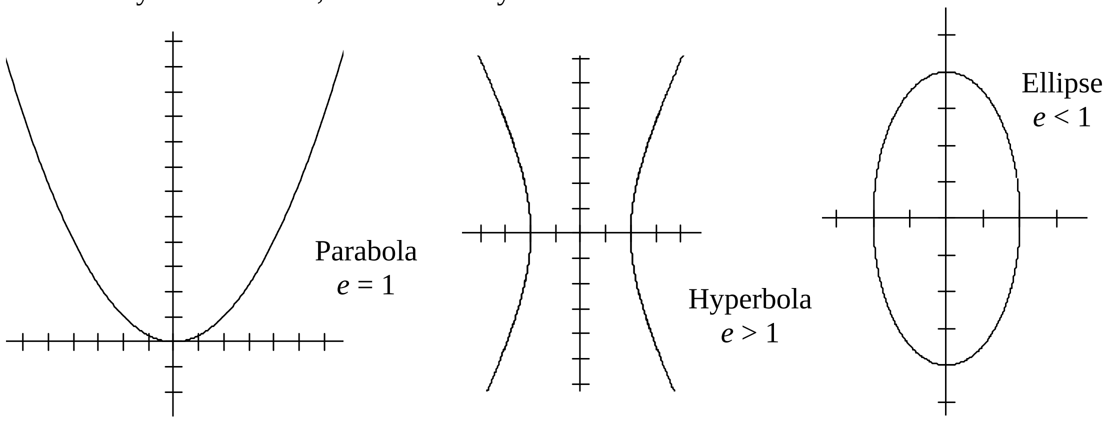
Conics (Mathematics Extension 2)

This document contains notes about Extension 2 Conics instead of providing a summary. Examples are given.

Locus Definitions

Let S = focus m = directrix

Define the locus of the point P in the plane by the condition that the ratio of the distances from P to S, and from P to m is e:1 for some constant e > 0. The shape of the curve is determined by the value of e, the eccentricity of the curve.



Notes on Ellipse and Hyperbola

	Ellipse	Hyperbola
Eccentricity formula	$b^2 = a^2(1-e^2)$	$b^2 = a^2(e^2-1)$
Cartesian equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

- If $P(x, y)$ lies on the curve, then $Q(-x, y)$ also lies on the curve
- The ellipse and hyperbola both have two foci $S(ae, 0)$ and $S'(-ae, 0)$ and two directrices $x=\frac{a}{e}$ and $x'=-\frac{a}{e}$
- The major (transverse) and the minor (conjugate) axes are $2a$ and $2b$ units respectively
- The hyperbola has two oblique asymptotes:

  $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{1}{a^2}-\frac{y^2}{b^2x^2}=\frac{1}{x^2}$ (Dividing both sides by x^2)

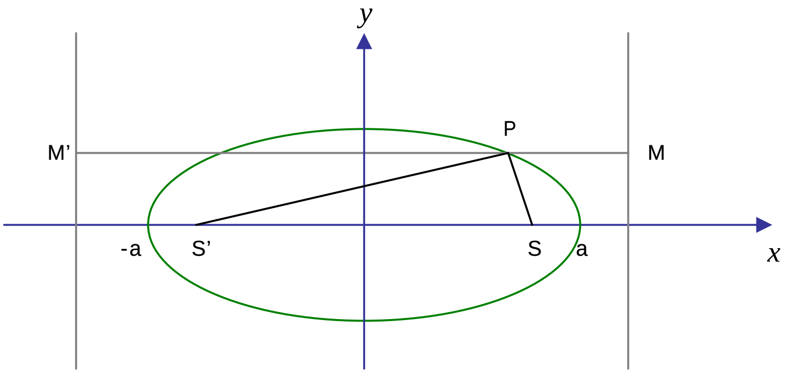
  $x \to \pm\infty \Rightarrow \frac{1}{x^2} \to 0$, so

  $$\frac{1}{a^2}-\frac{y^2}{b^2x^2} \to 0$$

  $$y^2 \to \frac{b^2x^2}{a^2}$$

  $$y \to \frac{b}{a}x$$

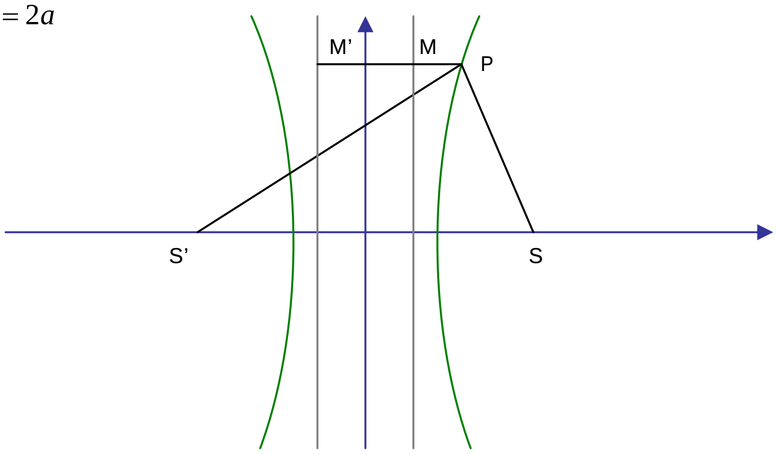
  Hence, the equations of the asymptotes are $y = \pm\frac{b}{a}x$.



For an ellipse, sum of distances from a point P on the curve to the foci is equal to the length of the major axis.

$PS + PS' = e(PM + PM') = e.MM'$

$\therefore PS + PS' = 2a$



For a hyperbola, the difference of the distances to the foci is equal to the length of the major axis.

$|PS - PS'| = e|PM - PM'| = e.MM$

$\therefore |PS - PS'| = 2a$

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a < b$,

- Foci at $(ae, 0)$ and $(-ae, 0)$
- Directrices at $x = \frac{a}{e}$ and $x' = -\frac{a}{e}$
- $b^2 = a^2(1 - e^2) \Rightarrow e = \sqrt{1 - \frac{b^2}{a}}$

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b < a$,

- Foci at $(be, 0)$ and $(-be, 0)$
- Directrices at $x = \frac{b}{e}$ and $x' = -\frac{b}{e}$
- $a^2 = b^2(1 - e^2) \Rightarrow e = \sqrt{1 - \frac{a^2}{b}}$

For $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the roles of a and b have been interchanged.

Parametric Equations for the Ellipse

$$\boxed{x = a \cos \theta \qquad y = b \sin \theta}$$

This is derived from the auxiliary circle radius a units, centred (0, 0).

Parametric Equations for the Hyperbola

$$\boxed{x = a\sec\theta \qquad y = b\tan\theta}$$

Example

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. Find Gradient PQ

$$m_{PQ} = \frac{b(\sin\theta-\sin\phi)}{a(\cos\theta-\cos\phi)} = \frac{b}{a}\cdot\frac{2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)}{-2\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)} = -\frac{b}{a}\cdot\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\theta+\phi}{2}\right)}$$

2. Find Equation of PQ

$$y-b\sin\theta = -\frac{b}{a}\cdot\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\theta+\phi}{2}\right)}(x-a\cos\theta)$$

$$\therefore y\sin\left(\frac{\theta+\phi}{2}\right)-b\sin\theta\sin\left(\frac{\theta+\phi}{2}\right) = -\frac{b}{a}x\cos\left(\frac{\theta+\phi}{2}\right)+b\cos\theta\cos\left(\frac{\theta+\phi}{2}\right)$$

$$\therefore \frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\cdot\sin\left(\frac{\theta+\phi}{2}\right) = \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)+\cos\theta\cos\left(\frac{\theta+\phi}{2}\right)$$

$$\therefore \frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\cdot\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

SPECIAL CASE: When chord passes through (0, 0), $|\theta-\phi| = \pi$.

For the hyperbola,

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\cdot\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

SPECIAL CASE: When chord passes through (0, 0), $|\theta+\phi| = \pi$.