Artin Algebra

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February 2023

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Theorem

this is

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Definition

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2.1 matrix multiplication

Definition

$M(T)$:matrix of a linear map;
notation:$M(T,(v_1,\cdots,v_n),(w_1,\cdots,w_m))$

$$Tv_k = A_{1,k}w_1 + \cdots A_{m,k}w_m$$

$$A_{m,n} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix}$$

for $A = M(T,(u_1,\cdots,u_p),(v_1,\cdots,v_n)), C = M(S,(v_1,\cdots,v_n),(w_1,\cdots,w_m))$,thus A is a n by p matrix, and C is a m by n matrix. We can derive the form of Matrix multiplication via mapping as followed:

$$\begin{align} (CA)u_k &= C(Au_k) \tag{1}\\ &= C(A_{1,k}v_1 + \cdots + A_{n,k}v_n) \tag{2}\\ &= A_{1,k}Cv_1 + \cdots + A_{n,k}Cv_n \tag{3}\\ &= \sum_{i=1}^{n} A_{i,k}Cv_i \tag{4}\\ &= \sum_{i=1}^{n} A_{i,k}\sum_{j=1}^{m} C_{j,i}w_j \tag{5}\\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} A_{i,k}C_{j,i}w_j \tag{6} \end{align}$$

then we get $(CA)_{j,k} = \sum_{i=1}^{n} C_{j,i}A_{i,k}$

2.2 upper triangle matrix

3 Groups

3.1 Groups and subgroups

3.1.1 group

Group

A group is a set together with a law of composition that has the following properties:

- $\forall a, b, c \in G,\ (ab)c = a(bc)$
- $\exists 1 \in G,\ \forall a \in G,$ s.t. $a1 = 1a = a$
- $\forall a \in G,\ \exists b \in G,$ s.t, $ab = ba = 1$

order of group it's the number of elements it contains.

$$|G| = numberofelements$$

Here are some intuition for associative law:

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & T \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & T \arrow[r, "h"] & T \end{tikzcd}$$

Law of Cancellation

a, b, c $\in G$, if $ab = ac$ or $ba = ca$, then $b = c$
if $ab = b$ or $ba = b$, then $a = 1$

It needs to be mentioned that a must be invertible.

3.1.2 permutation

permutation

A permutation of a set S is a bijective map p from a set S to itself:

$$p : S \to S$$

for example, p is a permutation over set $\{1, 2, 3, 4\}$:

i	1	2	3	4	5
$p(i)$	3	5	4	1	2

Since $p(3) = 4, P(4) = 1, p(1) = 3$, we have$(3 \quad 4 \quad 1)$. Similarly, we have $(2 \quad 5)$. Thus we can write P as $(2 \quad 5)$,$(3 \quad 4 \quad 1)$.

The product of 2 perturbations can be written as: $qp = [(1 \quad 4 \quad 5 \quad 2)] \circ [(2 \quad 5)(3 \quad 4 \quad 1)] = (3 \quad 5 \quad 1)$. (hint: $qp(3) = q(3) \circ p(3)$, for p $3 \to 4$, for q $4 \to 5$, thus for pq, $3 \to 5$)

3.1.3 symmetric group

symmetric group

The group of permutations of the set of indices $\{1, 2, \cdots n\}$, denoted by S_n

Attention, a symmetric group is a group of mapping, working on a set of mapping.

In S_2, one of elements(2!) must be 1, then we call it $\{1, g\}$, then we have a table gg, $1g$ $g1$ 11 only gg is not defined. Assume $gg = g$, then according to law of cancellation, we have $g = 1$, it's forbidden, thus we only have $gg = 1$.

In S_3, let $x = (1 \quad 2 \quad 3)$ and $y = (1 \quad 2)$, then we have:

$$x^3 = 1 \quad y^2 = 1 \quad yx = x^2y \quad xy = yx^2 \tag{7}$$

thus we can eliminate some perturbations and get all of them in a distinctive way:

$$S_3 = \{1, x, x, x^2, y, xy, x^2y\} \tag{8}$$

This is an important example, since $xy \neq yx$, it's the smallest group whose law of composition is not commutative.

3.1.4 Subgroup

subgroup

A subset H of group G is a subgroup if it has the following properties:

- Closure: If a and b are in H, then ab are in H
- Identity: 1 is in H
- Inverse: if a is in H then a^-1 is in H