Def: $\vec{q} = w + x\vec{i} + y\vec{j} + z\vec{k}$

I、multiplication (Hamilton product)

$$
\begin{aligned}
\boldsymbol{q}_1\boldsymbol{q}_2 &= (w_1+\boldsymbol{u}_1)(w_2+\boldsymbol{u}_2) \\
&= (w_1+x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})(w_2+x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}) \\
&= \quad w_1w_2+w_1x_2\mathbf{i}+w_1y_2\mathbf{j}+w_1z_2\mathbf{k} \\
&\quad +x_1w_2\mathbf{i}+x_1x_2\mathbf{i}^2+x_1y_2\mathbf{ij}+x_1z_2\mathbf{ik} \\
&\quad +y_1w_2\mathbf{j}+y_1x_2\mathbf{ji}+y_1y_2\mathbf{j}^2+y_1z_2\mathbf{jk} \\
&\quad +z_1w_2\mathbf{k}+z_1x_2\mathbf{ki}+z_1y_2\mathbf{kj}+z_1z_2\mathbf{k}^2 \\
&= \quad w_1w_2-(x_1x_2+y_1y_2+z_1z_2) \\
&\quad +w_1(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})+w_2(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}) \\
&\quad +(y_1z_2-y_2z_1)\mathbf{i}+(z_1x_2-z_2x_1)\mathbf{j}+(x_1y_2-x_2y_1)\mathbf{k} \\
&= (w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)+(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2),
\end{aligned} \tag{2.27}
$$

$q_1 \cdot q_2 \neq q_2 q_1$

e.g. $q_1 = 1 + 2i + 3j + 4k$

$q_2 = 5 + 6i + 7j + 8k$

$q_1 q_2 = -60 + 12i + 30j + 24k$

$q_2 q_1 = -60 + 20i + 14j + 32k$

Thm 1: $(q_1 q_2) q_3 = q_1 (q_2 q_3)$ s.t. $\forall q_1, q_2, q_3 \in \mathbb{H}$

associative
$$
\begin{aligned}
(\boldsymbol{q}_1\boldsymbol{q}_2)\boldsymbol{q}_3 &= ((w_1+\boldsymbol{u}_1)(w_2+\boldsymbol{u}_2))(w_3+\boldsymbol{u}_3)\\
&= ((w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)+(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2))(w_3+\boldsymbol{u}_3)\\
&= \quad ((w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)w_3-(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2)\cdot\boldsymbol{u}_3)\\
&\quad +((w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\boldsymbol{u}_3+w_3(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2)\\
&\quad +(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2)\times\boldsymbol{u}_3)\\
&= \quad w_1w_2w_3-w_3\boldsymbol{u}_1\cdot\boldsymbol{u}_2-w_1\boldsymbol{u}_2\cdot\boldsymbol{u}_3-w_2\boldsymbol{u}_1\cdot\boldsymbol{u}_3-(\boldsymbol{u}_1\times\boldsymbol{u}_2)\cdot\boldsymbol{u}_3\\
&\quad +w_1w_2\boldsymbol{u}_3-(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\boldsymbol{u}_3+w_3w_1\boldsymbol{u}_2+w_2w_3\boldsymbol{u}_1+w_3\boldsymbol{u}_1\times\boldsymbol{u}_2\\
&\quad +w_1\boldsymbol{u}_2\times\boldsymbol{u}_3+w_2\boldsymbol{u}_1\times\boldsymbol{u}_3+(\boldsymbol{u}_1\times\boldsymbol{u}_2)\times\boldsymbol{u}_3\\
&= \quad w_1w_2w_3-(\boldsymbol{u}_1\times\boldsymbol{u}_2)\cdot\boldsymbol{u}_3+(\boldsymbol{u}_1\times\boldsymbol{u}_2)\times\boldsymbol{u}_3-(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)\boldsymbol{u}_3\\
&\quad +w_2w_3\boldsymbol{u}_1+w_1\boldsymbol{u}_2\times\boldsymbol{u}_3-w_1\boldsymbol{u}_2\cdot\boldsymbol{u}_3\\
&\quad +w_3w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1\times\boldsymbol{u}_3-w_2\boldsymbol{u}_1\cdot\boldsymbol{u}_3\\
&\quad +w_1w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_1\times\boldsymbol{u}_2-w_3\boldsymbol{u}_1\cdot\boldsymbol{u}_2\,. \qquad (2.33)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{q}_1(\boldsymbol{q}_2\boldsymbol{q}_3) &= (w_1+\boldsymbol{u}_1)((w_2+\boldsymbol{u}_2)(w_3+\boldsymbol{u}_3))\\
&= (w_1+\boldsymbol{u}_1)((w_2w_3-\boldsymbol{u}_2\cdot\boldsymbol{u}_3)+(w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_2+\boldsymbol{u}_2\times\boldsymbol{u}_3))\\
&= \quad (w_1(w_2w_3-\boldsymbol{u}_2\cdot\boldsymbol{u}_3)-\boldsymbol{u}_1\cdot(w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_2+\boldsymbol{u}_2\times\boldsymbol{u}_3))\\
&\quad +(w_1(w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_2+\boldsymbol{u}_2\times\boldsymbol{u}_3)+(w_2w_3-\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\boldsymbol{u}_1\\
&\quad +\boldsymbol{u}_1\times(w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_2+\boldsymbol{u}_2\times\boldsymbol{u}_3))\\
&= \quad w_1w_2w_3-w_1\boldsymbol{u}_2\cdot\boldsymbol{u}_3-w_2\boldsymbol{u}_1\cdot\boldsymbol{u}_3-w_3\boldsymbol{u}_1\cdot\boldsymbol{u}_2-\boldsymbol{u}_1\cdot(\boldsymbol{u}_2\times\boldsymbol{u}_3)\\
&\quad +w_1w_2\boldsymbol{u}_3+w_1w_3\boldsymbol{u}_2+w_1\boldsymbol{u}_2\times\boldsymbol{u}_3+w_2w_3\boldsymbol{u}_1-(\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\boldsymbol{u}_1\\
&\quad +w_2\boldsymbol{u}_1\times\boldsymbol{u}_3+w_3\boldsymbol{u}_1\times\boldsymbol{u}_2+\boldsymbol{u}_1\times(\boldsymbol{u}_2\times\boldsymbol{u}_3)\\
&= \quad w_1w_2w_3-\boldsymbol{u}_1\cdot(\boldsymbol{u}_2\times\boldsymbol{u}_3)+\boldsymbol{u}_1\times(\boldsymbol{u}_2\times\boldsymbol{u}_3)-(\boldsymbol{u}_2\cdot\boldsymbol{u}_3)\boldsymbol{u}_1\\
&\quad +w_2w_3\boldsymbol{u}_1+w_1\boldsymbol{u}_2\times\boldsymbol{u}_3-w_1\boldsymbol{u}_2\cdot\boldsymbol{u}_3\\
&\quad +w_3w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1\times\boldsymbol{u}_3-w_2\boldsymbol{u}_1\cdot\boldsymbol{u}_3\\
&\quad +w_1w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_1\times\boldsymbol{u}_2-w_3\boldsymbol{u}_1\cdot\boldsymbol{u}_2\,. \qquad (2.34)
\end{aligned}
$$

Thm2:
distributive

$$q_1(q_2+q_3) = q_1q_2 + q_1q_3$$
$$(q_1+q_2)q_3 = q_1q_3 + q_2q_3$$

$$\begin{aligned}
\boldsymbol{q}_1(\boldsymbol{q}_2+\boldsymbol{q}_3) &= (w_1+\boldsymbol{u}_1)((w_2+\boldsymbol{u}_2)+(w_3+\boldsymbol{u}_3)) \\
&= (w_1+\boldsymbol{u}_1)((w_2+w_3)+(\boldsymbol{u}_2+\boldsymbol{u}_3)) \\
&= \quad (w_1(w_2+w_3)-\boldsymbol{u}_1\cdot(\boldsymbol{u}_2+\boldsymbol{u}_3)) \\
&\quad +(w_1(\boldsymbol{u}_2+\boldsymbol{u}_3)+(w_2+w_3)\boldsymbol{u}_1+\boldsymbol{u}_1\times(\boldsymbol{u}_2+\boldsymbol{u}_3)) \\
&= \quad (w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)+(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2) \\
&\quad +(w_1w_3-\boldsymbol{u}_1\cdot\boldsymbol{u}_3)+(w_1\boldsymbol{u}_3+w_3\boldsymbol{u}_1\boldsymbol{u}_1\times\boldsymbol{u}_3) \\
&= \boldsymbol{q}_1\boldsymbol{q}_2+\boldsymbol{q}_1\boldsymbol{q}_3\,. \qquad (2.39)
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{q}_1+\boldsymbol{q}_2)\boldsymbol{q}_3 &= ((w_1+\boldsymbol{u}_1)+(w_2+\boldsymbol{u}_2))(w_3+\boldsymbol{u}_3) \\
&= ((w_1+w_2)+(\boldsymbol{u}_1+\boldsymbol{u}_2))(w_3+\boldsymbol{u}_3) \\
&= \quad ((w_1+w_2)w_3-(\boldsymbol{u}_1+\boldsymbol{u}_2)\cdot\boldsymbol{u}_3) \\
&\quad +((w_1+w_2)\boldsymbol{u}_3+w_3(\boldsymbol{u}_1+\boldsymbol{u}_2)+(\boldsymbol{u}_1+\boldsymbol{u}_2)\times\boldsymbol{u}_3) \\
&= \quad (w_1w_3-\boldsymbol{u}_1\cdot\boldsymbol{u}_3)+(w_1\boldsymbol{u}_3+w_3\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_3) \\
&\quad +(w_2w_3-\boldsymbol{u}_2\cdot\boldsymbol{u}_3)+(w_2\boldsymbol{u}_3+w_3\boldsymbol{u}_2+\boldsymbol{u}_2\times\boldsymbol{u}_3) \\
&= \boldsymbol{q}_1\boldsymbol{q}_3+\boldsymbol{q}_2\boldsymbol{q}_3\,. \qquad (2.40)
\end{aligned}$$

Thm3:
conjugate

$$\overline{q_1q_2} = \overline{q_2}\cdot\overline{q_1}$$

real   imaginary

$$\begin{aligned}
\overline{\boldsymbol{q}_1\boldsymbol{q}_2} &= \underbrace{(w_1w_2-\boldsymbol{u}_1\cdot\boldsymbol{u}_2)}_{}-\underbrace{(w_1\boldsymbol{u}_2+w_2\boldsymbol{u}_1+\boldsymbol{u}_1\times\boldsymbol{u}_2)}_{} \\
&= (w_2w_1-(-\boldsymbol{u}_2)\cdot(-\boldsymbol{u}_1))+(w_2(-\boldsymbol{u}_1)+w_1(-\boldsymbol{u}_2)+(-\boldsymbol{u}_2)\times(-\boldsymbol{u}_1)) \\
&= (w_2-\boldsymbol{u}_2)(w_1-\boldsymbol{u}_1) \\
&= \bar{\boldsymbol{q}}_2\bar{\boldsymbol{q}}_1\,. \qquad (2.42)
\end{aligned}$$

(recall that $q_1q_2 = (w_1+u_1)(w_2+u_2)$
$= (w_1w_2 - u_1\cdot u_2)\ (w_1u_1 + w_2u_2$
$+ u_1\times u_2)$

$(w+u)\cdot(w-u)$

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{w^2+x^2+y^2+z^2}$$

$$= \sqrt{(w^2+u\cdot u)+(wu - wu + (-1)\,u\times u)} = \sqrt{w^2+(u\cdot u)}$$

$\dfrac{q}{\|q\|}$ versor

Thm 4.

$$\|\bar{q}\| = \|q\|$$

Thm5:

$$\|\lambda q\| = |\lambda|\|q\| \quad \forall \lambda \in \mathbb{R}, \forall q \in \mathbb{H}$$

Thm 6:

$$\|q_1 q_2\| = \|q_1\|\|q_2\| \quad \forall q_1, q_2 \in \mathbb{H}$$

Pf:

$$\begin{aligned}
\|\boldsymbol{q}_1\boldsymbol{q}_2\|^2 =& \|(w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2) \\
& + (w_1x_2 + w_2x_1 + y_1z_2 - y_2z_1)\mathbf{i} \\
& + (w_1y_2 + w_2y_1 + z_1x_2 - z_2x_1)\mathbf{j} \\
& + (w_1z_2 + w_2z_1 + x_1y_2 - x_2y_1)\mathbf{k}\|^2 \\
=& (w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2)^2 \\
& + (w_1x_2 + w_2x_1 + y_1z_2 - y_2z_1)^2 \\
& + (w_1y_2 + w_2y_1 + z_1x_2 - z_2x_1)^2 \\
& + (w_1z_2 + w_2z_1 + x_1y_2 - x_2y_1)^2 \\
=& (w_1^2 + x_1^2 + y_1^2 + z_1^2)(w_2^2 + x_2^2 + y_2^2 + z_2^2) \\
=& \|\boldsymbol{q}_1\|^2\|\boldsymbol{q}_2\|^2,
\end{aligned}$$

Thm7:

$$q \cdot \bar{q} = \bar{q} \cdot q = \|q\|^2$$

$$\begin{aligned}
\boldsymbol{q}\bar{\boldsymbol{q}} &= (w + \boldsymbol{u})(w - \boldsymbol{u}) \\
&= (w^2 - \boldsymbol{u} \cdot (-\boldsymbol{u})) + (w(-\boldsymbol{u}) + w\boldsymbol{u} + \boldsymbol{u} \times (-\boldsymbol{u})) \\
&= w^2 + \boldsymbol{u} \cdot \boldsymbol{u} \\
&= \|\boldsymbol{q}\|^2.
\end{aligned}$$

Def 1: $q^{-1} = \frac{1}{\|q\|^2} \cdot q$

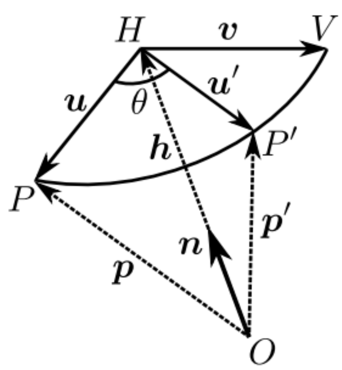
Corellary 1: $qq^{-1} = q^{-1} \cdot q = 1$

corallary 2: $\|q\|=1 \Rightarrow q^{-1} = \bar{q}$

Thm 8: $\bar{q} p q = p'$

$p \xrightarrow{\theta} p'$

$q = \cos\frac{\theta}{2} + \vec{n} \sin\frac{\theta}{2}$



$$\boldsymbol{h} = (\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}, \tag{2.56}$$

$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{h}, \tag{2.57}$$

$$\boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{n}, \tag{2.58}$$

$$\boldsymbol{u}' = \boldsymbol{u}\cos\theta + \boldsymbol{v}\sin\theta, \tag{2.59}$$

$$\boldsymbol{p}' = \boldsymbol{h} + \boldsymbol{u}'. \tag{2.60}$$

$$\begin{aligned} \boldsymbol{p}' &= (\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} + (\boldsymbol{p} - (\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n})\cos\theta + (\boldsymbol{p} - (\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}) \times \boldsymbol{n}\sin\theta \\ &= \boldsymbol{p}\cos\theta + (1-\cos\theta)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} + \boldsymbol{p}\times\boldsymbol{n}\sin\theta. \end{aligned} \tag{2.61}$$

let $q = c + s = \cos\frac{\theta}{2} + n \sin\frac{\theta}{2}$

Corollary 3 : $\forall w \in \mathbb{R}$

$\bar{q}(w+p)q = w + p'$

$$\begin{aligned}\bar{q}(w+p)q &= w\bar{q}q + \bar{q}pq \\ &= w\|q\|^2 + \bar{q}pq \\ &= w + \bar{q}pq \\ &= w + p'. \end{aligned} \tag{2.64}$$

Corollary 4 : n clockwise θ $\equiv$ $-n$ Counter clockwise θ

$$\begin{aligned}&(\cos(-\frac{\theta}{2}) - (-n)\sin(-\frac{\theta}{2}))p(\cos(-\frac{\theta}{2}) + (-n)\sin(-\frac{\theta}{2})) \\ =&(\cos\frac{\theta}{2} - n\sin\frac{\theta}{2})p(\cos\frac{\theta}{2} + n\sin\frac{\theta}{2}).\end{aligned}$$

Thm

$$R_n(\theta) = I\cos\theta + (1-\cos\theta)nn^{\mathrm{T}} + A_n \sin\theta.$$

$$A_n = \begin{bmatrix} & n_z & -n_y \\ -n_z & & n_x \\ n_y & -n_x & \end{bmatrix}.$$

recall that: $$= \boldsymbol{p}\cos\theta + (1-\cos\theta)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n} + \boldsymbol{p}\times\boldsymbol{n}\sin\theta$$
$$= \boldsymbol{p}'.$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cos\theta + (1-\cos\theta)(n_x)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} + \begin{vmatrix} i & j & k\\ 1 & 0 & 0\\ n_x & n_y & n_z\end{vmatrix}\sin\theta$$

$$\begin{array}{ccccc}
\cos\theta & + & (1-\cos\theta)\,n_x^2 & + & 0\\
0 & + & (1-\cos\theta)\,n_x n_y & + & -n_z\sin\theta\\
0 & + & (1-\cos\theta)\,n_x n_z & + & n_y\sin\theta
\end{array}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix}\cos\theta + (1-\cos\theta)(n_y)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} + \begin{vmatrix} i & j & k\\ 0 & 1 & 0\\ n_x & n_y & z\end{vmatrix}\sin\theta$$

$$\begin{array}{ccccc}
0 & + & (1-\cos\theta)\,n_x n_y & + & n_z\sin\theta\\
\cos\theta & + & (1-\cos\theta)\,n_y^2 & + & 0\\
0 & + & (1-\cos\theta)\cdot n_y n_z & + & -n_x\sin\theta
\end{array}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\cos\theta + (1-\cos\theta)(n_z)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} + \begin{vmatrix} i & j & k\\ 0 & 0 & 1\\ n_x & n_y & n_z\end{vmatrix}\sin\theta$$

$$\begin{array}{ccccc}
0 & + & (1-\cos\theta) & + & (-n_y)\sin\theta\\
0 & + & (1-\cos\theta) & + & n_x\sin\theta\\
\cos\theta & + & (1-\cos\theta)\cdot & + & 0
\end{array}$$

$$
\left[\begin{array}{ccc}
\begin{array}{l}
\cos\theta + (1-\cos\theta)\, n_x^2 + 0 \\
0 + (1-\cos\theta)\, n_x n_y + -n_z \sin\theta \\
0 + (1-\cos\theta)\, n_x n_z + n_y \sin\theta
\end{array}
&
\begin{array}{l}
0 + (1-\cos\theta)\, n_x n_y + n_z \sin\theta \\
\cos\theta + (1-\cos\theta)\, n_y^2 + 0 \\
0 + (1-\cos\theta)\, n_y n_z + -n_x \sin\theta
\end{array}
&
\begin{array}{l}
0 + (1-\cos\theta) + (-n_y) \sin\theta \\
0 + (1-\cos\theta) + n_x \sin\theta \\
\cos\theta + (1-\cos\theta) . + 0
\end{array}
\end{array}\right]
$$

$$
\begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix}
+ (1-\cos\theta)
\begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_z n_y \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix}
+
\begin{bmatrix} 0 & n_z & -n_y \\ -n_z & 0 & -n_x \\ n_y & -n_x & 0 \end{bmatrix} \sin\theta
$$