Gerolamo Cardano was born in Pavia on September 24, 1501. His father was a well-educated lawyer who made sure his son was equally well educated. Gerolamo graduated from the University of Padua in 1526 and decided to become a physician. However, because of doubts about the legitimacy of his birth, he was denied admission to the College of Physicians in Milan and practiced in the provinces until 1539 when the college changed the rules so that he could be admitted. He wrote on a variety of subjects, including mathematics and astrology, and gambled at chess for forty years and at dice for twenty-five. The results were at times unpleasant, but eventually, in 1526, his experience allowed him to write *Liber de ludo aleae* (*The Book on Games of Chance*), which contained the beginnings of probability theory, including a preliminary statement of the law of large numbers. Toward the end of 1538, Cardano was completing a book that he hoped would supplant Pacioli's and desperately wanted Tartaglia's secret. On March 25, 1539, enticed by Cardano's promise of an introduction to the Spanish governor of Lombardy, Tartaglia met with Cardano.

According to Tartaglia's notes (which may of course be biased), the following exchange took place:

Tartaglia: "I plan to publish the work for practical application together with a new algebra. . . . If I give it some theorist (such as your Excellency), then he could easily find other chapters with the help of this explanation . . . and publish the fruit of my discovery under another name."

Cardano: "I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them."

Tartaglia: "If I did not believe an oath such as yours, then of course I myself would deserve to be considered an atheist."[2]

And after speaking these words, Tartaglia imparted his secret to Cardano.

Tartaglia left the meeting apprehensive, but on May 12, he received a copy of the *Practica arithmeticae generalis* without the secret method and an accompanying note from Cardano which said "I have verified the formula and believe that it has broad significance."[3] Over the next few years, Cardano worked hard to extend the method given him by Tartaglia. He managed to generalize it to equations of the form $x^3 = ax + b$ and $x^3 + b = ax$. (Since negative numbers were not in use at the time, these equations, along with the earlier one, were all considered different!) But he was bound by his oath not to publish any of these methods.

In 1536, Cardano took on **Lodovico Ferrari** (1522–1565) as a servant and student. Ferrari had a temper to match his phenomenal mathematical ability; when he was seventeen, he returned from a brawl with no fingers left on his right hand! Ferrari was devoted to Cardano and in 1543, the two of them traveled to Bologna, where del Ferro's son-in-law della Nave allowed them to peruse del Ferro's papers. They convinced themselves that del Ferro had had the solution before Tartaglia and that this absolved Cardano of the oath he had given Tartaglia. The result was *Ars magna* (*The Great Art*), which appeared in 1545 and included the solution of the general cubic equation, as well as Ferrari's contribution, the solution of the quartic equation. It presented the algorithms by example. For instance, the general solution of the equation $x^3 + ax = b$ was given by solving $x^3 + 6x = 20$. Cardano did not neglect his predecessors. He mentions del Ferro, Tartaglia, and Fiore and continues: "Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things which could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil."[4]

Not surprisingly, Cardano's publication of *Ars magna* incensed Tartaglia. In 1546, he published his correspondence with Cardano and heaped abuse on him. Cardano did not react, but Ferrari challenged Tartaglia to a public debate. Hoping to draw out Cardano, Tartaglia demurred. But finally, in 1548, possibly to strengthen his position at Brescia, he agreed to a mathematical duel with Ferrari. The details of the duel are unknown, but it is certain that Tartaglia suffered a humiliating defeat. He returned to Venice a year later and died there in 1557. Ferrari became very famous, gave public lectures in Rome, headed the taxation department in Milan, and helped to bring up the emperor's son. However, he did no further scientific research. He died in 1565, possibly poisoned by his sister.

Cardano lived until 1576, but it was a difficult life. One of his sons poisoned his wife and was executed; another turned criminal and robbed his own father. In 1570, Cardano himself was sent to prison and his property confiscated, possibly on the initiative of the Inquisition. He finished his life in Rome, with a modest pension from the Pope, and spent his last years writing his autobiography. In its last pages, he finally replied to the now deceased Tartaglia when he wrote, "I confess that in mathematics I received a few suggestions, but very few, from brother Niccolò."[5]

References

1. Gindikin, Semyon Grigorevich. *Tales of Physicists and Mathematicians.* 2nd ed. Boston: Birkhäuser, 1985, p. 3.
2. *Ibid.*, pp. 9–10.
3. *Ibid.*, p. 10.
4. *Ibid.*, p. 12.
5. *Ibid.*, p. 18.

3 Complex Numbers

The solutions to the equations in Examples 2.1, 2.3, 2.4, and 2.5 all involve square roots of negative numbers. In general, the methods for solving polynomial equations given in Chapter 1 yield solutions of this kind, i.e., solutions that may not live in the real numbers. As described in the historical note at the end of this chapter, this observation spurred the development of the numbers that can encompass all such solutions, i.e., the complex numbers.

Formally adjoining the nonreal number $\sqrt{-1}$ to the real numbers struck some mathematicians as both mysterious and suspicious. But these doubts, along with the specific adjunction of $\sqrt{-1}$, disappeared with the discovery that the set of formal sums $a + b\sqrt{-1}$, for real numbers a and b, can be viewed, with no loss of algebraic content, as the set of ordered pairs (a,b). From this point of view, $\sqrt{-1}$ is just the ordered pair $(0,1)$ and thus presents neither mystery nor difficulty. In this chapter, we will adopt the standard notation $a + bi$, where i represents $\sqrt{-1}$. But it is important to note that this is merely convention; we are really looking at the set of ordered pairs (a,b) with particular algebraic operations of addition and multiplication. In fact, we will not abandon the notation (a,b) entirely. On the contrary, we will see that valuable geometric information can be gained by associating the complex number $a + bi$ with the point (a,b) on the Euclidean plane.

Notation. We let

$\mathbb{Z}$ denote the set of integers,
$\mathbb{Q}$ denote the set of rational numbers,
$\mathbb{R}$ denote the set of real numbers.

DEFINITION

The **complex numbers** are the elements of the set

$$\mathbb{C} = \{a + bi \mid a,b \in \mathbb{R}\}.$$

If $a + bi, c + di \in \mathbb{C}$, we add and multiply $a + bi$ and $c + di$ as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

For notational convenience, we denote

$$a + 0i \text{ by } a, \qquad 0 + bi \text{ by } bi, \qquad a + (-b)i \text{ by } a - bi.$$

This makes algebraic as well as notational sense because, as we show next, addition and multiplication in $\mathbb{C}$ obey many of the same algebraic laws that govern them in $\mathbb{Q}$ and $\mathbb{R}$.

Proposition 3.1 *The following laws hold in* $\mathbb{C}$.

A1. ADDITIVE CLOSURE: *For all* $a + bi, c + di \in \mathbb{C}$, $(a + bi) + (c + di) \in \mathbb{C}$.

A2. ADDITIVE ASSOCIATIVITY: *For all* $a + bi, c + di, e + fi \in \mathbb{C}$,
$(a + bi) + [(c + di) + (e + fi)] = [(a + bi) + (c + di)] + (e + fi)$.

A3. EXISTENCE OF AN ADDITIVE IDENTITY: *There is a complex number* 0, *such that for all* $a + bi \in \mathbb{C}$, $(a + bi) + 0 = a + bi = 0 + (a + bi)$.

A4. EXISTENCE OF ADDITIVE INVERSES: *For all* $a + bi \in \mathbb{C}$, *there is a complex number,* $-a - bi$, *such that* $(a + bi) + (-a - bi) = 0 = (-a - bi) + (a + bi)$.

A5. ADDITIVE COMMUTATIVITY: *For all* $a + bi, c + di \in \mathbb{C}$, $(a + bi) + (c + di) = (c + di) + (a + bi)$.

M1. MULTIPLICATIVE CLOSURE: *For all* $a + bi, c + di \in \mathbb{C}$, $(a + bi)(c + di) \in \mathbb{C}$.

M2. MULTIPLICATIVE ASSOCIATIVITY: *For all* $a + bi, c + di, e + fi \in \mathbb{C}$,
$(a + bi)[(c + di)(e + fi)] = [(a + bi)(c + di)](e + fi)$.

M3. EXISTENCE OF A MULTIPLICATIVE IDENTITY OR UNIT ELEMENT: *For all* $a + bi \in \mathbb{C}$, *there is a complex number,* 1, *such that* $(a + bi)(1) = a + bi = (1)(a + bi)$.

M4. EXISTENCE OF MULTIPLICATIVE INVERSES: *For all* $0 \neq a + bi \in C$, *there is a complex number,* $x + yi$, *such that* $(x + yi)(a + bi) = 1 = (a + bi)(x + yi)$.

M5. MULTIPLICATIVE COMMUTATIVITY: *For all* $a + bi, c + di \in \mathbb{C}$,
$(a + bi)(c + di) = (c + di)(a + bi)$.

D1. LEFT DISTRIBUTIVITY: *For all* $a + bi, c + di, e + fi \in \mathbb{C}$,
$(a + bi)[(c + di) + (e + fi)] = (a + bi)(c + di) + (a + bi)(e + fi)$.

D2. RIGHT DISTRIBUTIVITY: *For all* $a + bi, c + di, e + fi \in \mathbb{C}$,
$[(a + bi) + (c + di)](e + fi) = (a + bi)(e + fi) + (c + di)(e + fi)$.

Proof By definition, $\mathbb{C}$ is additively and multiplicatively closed. With the exception of the existence of a multiplicative inverse, the remaining properties all involve similar calculations. As examples of such calculations, we prove additive commutativity and left distributivity. We conclude with a proof of the existence of multiplicative inverses.

A5. Addition of complex numbers is commutative because addition of real numbers is commutative:

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ &= (c + a) + (d + b)i = (c + di) + (a + bi).\end{aligned}$$

D1. Multiplication of complex numbers distributes over addition from the left because multiplication of real numbers distributes over addition from the left and because addition of real numbers is commutative:

$$\begin{aligned}(a + bi)((c + di) + (e + fi)) &= (a + bi)((c + e) + (d + f)i) \\ &= (a(c + e) - b(d + f)) + (b(c + e) + a(d + f))i \\ &= ((ac - bd) + (ae - bf)) + ((bc + ad) + (be + af))i \\ &= ((ac - bd) + (bc + ad)i) + ((ae - bf) + (be + af)i) \\ &= (a + bi)(c + di) + (a + bi)(e + fi).\end{aligned}$$

M4. Each nonzero complex number has a multiplicative inverse. Note that since $a + bi \neq 0$, either $a \neq 0$ or $b \neq 0$, and hence $a^2 + b^2 \neq 0$. Then

$$\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$

is a well-defined complex number, and

$$\begin{aligned}(a + bi)\left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i\right) &= \left(\frac{a^2 + b^2}{a^2 + b^2}\right) + \left(\frac{-ab + ba}{a^2 + b^2}\right)i = 1, \\ \left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i\right)(a + bi) &= \left(\frac{a^2 + b^2}{a^2 + b^2}\right) + \left(\frac{-ab + ba}{a^2 + b^2}\right)i = 1. \quad \blacksquare\end{aligned}$$

Note that associating the real number a and the complex number $a + 0i$ embeds the real numbers in the complex numbers in such a way that addition and multiplication of real numbers is the same as addition and multiplication of the corresponding complex numbers. This justifies our use of the notation a for $a + 0i$, bi for $0 + bi$, and $a - bi$ for $a + (-b)i$. And for this reason, in the sequel, we will always treat $\mathbb{R}$ as a subset of $\mathbb{C}$.

Note as well that, as we indicated before,

$$i^2 = (0 + i)(0 + i) = (0 - 1) + (0 + 0)i = -1.$$

In practice, it is usually easiest to multiply complex numbers by combining this observation with the distributive laws, rather than by using the definition. For example,

$$\begin{aligned}(2 + 3i)(1 + 5i) &= 2(1 + 5i) + 3i(1 + 5i) = 2 \cdot 1 + 2 \cdot 5i + 3 \cdot 1i + 3 \cdot 5i^2 \\ &= (2 \cdot 1 - 3 \cdot 5) + (2 \cdot 5 + 3 \cdot 1)i = -13 + 13i.\end{aligned}$$

We may divide complex numbers in several ways. The preceding proof supplies a formula for the multiplicative inverse of the nonzero complex number $a + bi$:

$$(a + bi)^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i,$$

and we may use this formula to divide $c + di$ by $a + bi$:

$$\frac{c + di}{a + bi} = (c + di)(a + bi)^{-1} = \frac{(c + di)(a - bi)}{a^2 + b^2}.$$

An alternative method is to use an associated complex number called the complex conjugate.

DEFINITION

Let $z = a + bi \in \mathbb{C}$. The **complex conjugate** of z is the complex number

$$\bar{z} = \overline{a + bi} = a - bi.$$

We can use the complex conjugate to divide two complex numbers in the following way: multiply the quotient of two numbers written in standard form by 1 written in the form of the complex conjugate of the denominator over itself.

Example 3.2 We have

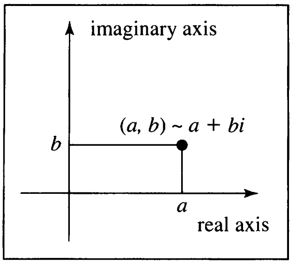
$$\frac{2 - 3i}{4 + i} = \frac{2 - 3i}{4 + i} \cdot \frac{4 - i}{4 - i} = \frac{(8 - 3) + (-2 - 12)i}{16 + 1} = \frac{5}{17} - \frac{14}{17}i. \quad ❖$$

We observe in passing that the operation of conjugation behaves very well with respect to multiplication and addition.

Proposition 3.3 *If $z, w \in \mathbb{C}$, then $\bar{z}\,\bar{w} = \overline{zw}$, and $\bar{z} + \bar{w} = \overline{z + w}$.*

Proof The proof is left to the reader (see Exercise 46). ■

FIGURE 3.1

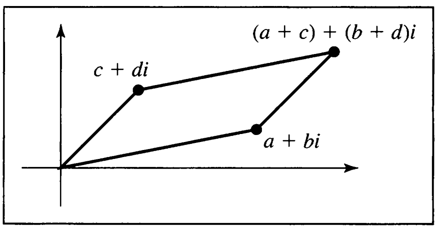


As previously noted, we can construct a picture of $\mathbb{C}$ by associating each $a + bi \in \mathbb{C}$ with the point (a,b) in the Euclidean plane $\mathbb{R}^2$ (as shown in Figure 3.1). When we think of complex numbers in this way, as points in the plane, we refer to $\mathbb{C}$ as the complex plane This identification allows us to view complex numbers geometrically.

If we further associate each point in the complex plane with its corresponding vector (as shown in Figure 3.2), then we can see that addition of complex numbers corresponds to addition of vectors:

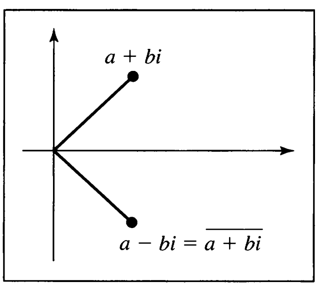
$$(a + bi) + (c + di) = (a + c) + (b + d)i;$$
$$(a,b) + (c,d) = (a + c, b + d).$$

FIGURE 3.2



The complex conjugate of $a + bi$ is very easy to find on the complex plane; it is merely the reflection of $a + bi$ about the x-axis (as shown in Figure 3.3).

FIGURE 3.3



We can interpret multiplication, as well as addition, of complex numbers geometrically. To do this, recall that a nonzero vector (a, b) may be expressed in terms of its polar coordinates r (length) and θ (angle). The analogous coordinates for complex numbers are the modulus and the argument. (See Figure 3.4.)

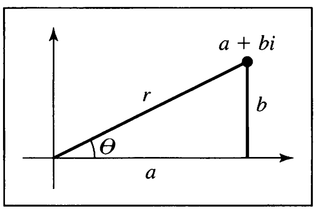
DEFINITION

Let $0 \neq a + bi \in \mathbb{C}$.

(*a*) The **modulus** of $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$.

(*b*) If $a + bi \neq 0$, then the **argument** of $a + bi$, denoted $\arg(a + bi)$, is the angle in the appropriate quadrant whose tangent is $\frac{b}{a}$.

FIGURE 3.4



Whenever possible, the argument should be expressed as an angle between 0 and 2π radians. Note that the square of the modulus of a complex number is just the product of the number and its conjugate (see Exercise 33): $|z|^2 = z\bar{z}$. Note also that care must be taken to identify the proper quadrant for $\arg(a + bi)$. For example, if $a < 0$ and $b > 0$, then $\frac{\pi}{2} < \arg(a + bi) < \pi$, and if $a > 0$ and $b < 0$, then $\frac{3\pi}{2} < \arg(a + bi) < 2\pi$.

Example 3.4 We have

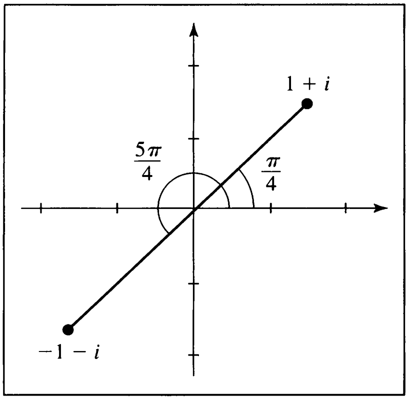
$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \frac{1}{1} = 1,$$

and $\arg(1 + i) = \frac{\pi}{4}$, while

$$|-1 - i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}, \quad \frac{-1}{-1} = 1,$$

but $\arg(-1 - i) = \frac{5\pi}{4}$ (see Figure 3.5). ❖

FIGURE 3.5



As indicated, the modulus and argument may be used to interpret multiplication of complex numbers geometrically. To describe this interpretation, let $z = a + bi$, $r = |z|$, and $\Theta = \arg(z)$. Observe that, as with polar coordinates, $a = r\cos\Theta$ and $b = r\sin\Theta$ (see Figure 3.4), and hence we may write

$$z = a + bi = r\cos\Theta + r\sin\Theta\, i = r(\cos\Theta + i\sin\Theta).$$

So if $z = r(\cos\Theta + i\sin\Theta)$ and $w = s(\cos\Psi + i\sin\Psi)$, then we may use the familiar formulas for the cosine and sine of the sum of two angles to see that

$$\begin{aligned} zw &= r(\cos\Theta + i\sin\Theta)\, s(\cos\Psi + i\sin\Psi) \\ &= rs((\cos\Theta\cos\Psi - \sin\Theta\sin\Psi) + i(\cos\Theta\sin\Psi + \sin\Theta\cos\Psi)) \\ &= rs(\cos(\Theta + \Psi) + i\sin(\Theta + \Psi)) \end{aligned}$$

and therefore that $|zw| = |z|\,|w|$ and $\arg(zw) = \arg(z) + \arg(w)$. We may summarize these observations as follows.

Proposition 3.5 *The modulus of a product of complex numbers is the product of its moduli; and the argument of a product is the sum of its arguments.*

Example 3.6 Consider the complex numbers $1 + \sqrt{3}i$ and $-2\sqrt{3} + 2i$. Then

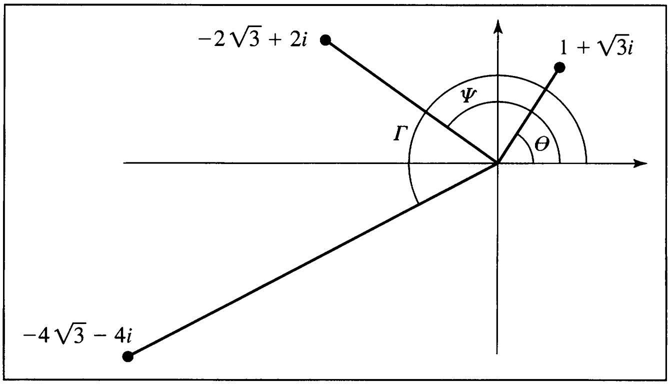
$$(1 + \sqrt{3}i)(-2\sqrt{3} + 2i) = (-2\sqrt{3} - 2\sqrt{3}) + (2 - 6)i = -4\sqrt{3} - 4i,$$

and we have:

$$\begin{aligned} &|1 + \sqrt{3}i| = 2, && \Theta = \arg(1 + \sqrt{3}i) = \tfrac{\pi}{3} \\ &|2\sqrt{3} + 2i| = 4, && \Psi = \arg(-2\sqrt{3} + 2i) = \tfrac{5\pi}{6} \\ &|-4\sqrt{3} - 4i| = 8 = 2 \cdot 4, && \Upsilon = \arg(-4\sqrt{3} - 4i) = \tfrac{7\pi}{6} = \tfrac{\pi}{3} + \tfrac{5\pi}{6}. \end{aligned}$$

Figure 3.6 shows these numbers plotted on the complex plane. ❖

FIGURE 3.6



We have observed that multiplying complex numbers means adding arguments. Since powers are a special kind of product, we may apply this observation in the following special situation.

Proposition 3.7 **De Moivre's Theorem**

For any positive integer n,

$$(\cos\Theta + i\sin\Theta)^n = \cos(n\Theta) + i\sin(n\Theta).$$

Proof We proceed by induction on n, where $P(n)$ is the equality $(\cos\Theta + i\sin\Theta)^n = \cos(n\Theta) + i\sin(n\Theta)$.

(*i*) For $n = 1$, clearly the equality holds.

(*ii*) Suppose that k is a positive integer and that $P(k)$ holds, i.e., that $(\cos\Theta + i\sin\Theta)^k = \cos(k\Theta) + i\sin(k\Theta)$. Then, by the induction hypothesis and Proposition 3.5,

$$\begin{aligned}(\cos\Theta + i\sin\Theta)^{k+1} &= (\cos\Theta + i\sin\Theta)^k(\cos\Theta + i\sin\Theta)\\ &= (\cos(k\Theta) + i\sin(k\Theta))(\cos\Theta + i\sin\Theta)\\ &= \cos((k+1)\Theta) + i\sin((k+1)\Theta),\end{aligned}$$

and therefore De Moivre's theorem holds by induction. ■

De Moivre's theorem is obviously useful for computation; it is also useful for finding roots.

DEFINITION

Let $\alpha \in \mathbb{C}$. An ***n*th root** of α in $\mathbb{C}$ is a complex number z such that $z^n = \alpha$.

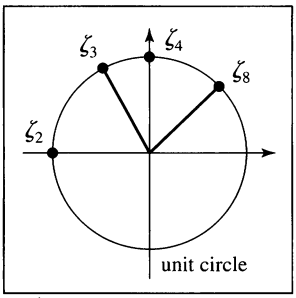
Note that z is an nth root of α if and only if z solves the equation $x^n - \alpha = 0$. Thus if we have found all nth roots of α, we have simultaneously found all solutions of the equation $x^n - \alpha = 0$. We will list all such nth roots, and hence all such solutions, in Proposition 3.8.

To phrase Propositon 3.8, we single out some special complex numbers. For any positive integer n, let

$$\boxed{\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.}$$

These complex numbers all lie on the unit circle, $\frac{1}{n}$th of the way around it (see Figure 3.7). In particular, $\zeta_2 = -1$ is half way around the circle and $\zeta_4 = i$ is a quarter of the way around the circle.

FIGURE 3.7



The numbers ζ_n will be extremely important in much of what follows. Note in particular that by De Moivre's theorem, $(\zeta_n)^n = \cos 2\pi + i\sin 2\pi$, i.e., ζ_n is an nth root of 1, or in symbols

$$\boxed{(\zeta_n)^n = 1.}$$

As well, De Moivre's theorem allows us to express all the nth roots of a complex number in terms of any particular nth root and the numbers ζ_n as follows.

Propositon 3.8 *Let n be a positive integer and let α, β be nonzero complex members such that $\beta^n = \alpha$. Then the set*

$$\{\beta\zeta_n, \beta(\zeta_n)^2, \beta(\zeta_n)^3, \dots, \beta(\zeta_n)^{n-1}, \beta\}$$

is the set of all nth roots of α in $\mathbb{C}$.

Proof As we have observed, $(\zeta_n)^n = 1$, and hence for any positive integer k,

$$(\beta(\zeta_n)^k)^n = \beta^n((\zeta_n)^n)^k = \beta^n = \alpha.$$