Quantum Mechanics

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Abstract

Quantum Mechanics Lecture Notes.

1 Introduction

1.1 Schrödinger formalism

$$\hat{f}\Phi = E\Phi \tag{1.1}$$

$$\Phi \to dP = |\Phi|^2 dq \tag{1.2}$$

$$x \leftrightarrow \hat{x} \tag{1.3}$$

$$p_x \leftrightarrow -i\hbar\frac{\partial}{\partial x} \tag{1.4}$$

$$f \leftrightarrow \hat{f} \tag{1.5}$$

$$\bar{f} = \int \hat{f} dp = \int \Phi^* \hat{f} \Phi dq \tag{1.6}$$

1.2 Heisenberg formalism

> Schrödinger was good at math, which is why his quantum mechanics formalism is full of complex mathematical constructs. Heisenberg, on the other hand, had a lot of difficulty with math, which is why his matrix quantum mechanics formalism is limited almost exclusively to linear algebra constructs
>
> Roman ...

Name	Schrödinger	Heisenberg
State Basis	Wave function of basis states $\{\Phi_n\}$	Column vector of basis states $\begin{pmatrix}\phi_1 \\ ... \\ \phi_n\end{pmatrix}$
Observables	Operator $\bar{f} = \int \Phi_n^* \hat{f} \Phi_m$	Operator matrix $\begin{pmatrix}\phi_{11} & ... & \phi_{n1} \\ & ... & \\ \phi_{1n} & ... & \phi_{nn}\end{pmatrix}$
Shrödinger Equation	$\hat{f}\Phi = E\Phi$	$\begin{pmatrix}\phi_{11} & ... & \phi_{n1} \\ & ... & \\ \phi_{1n} & ... & \phi_{nn}\end{pmatrix} \begin{pmatrix}\psi_1 \\ ... \\ \psi_n\end{pmatrix} = \lambda \begin{pmatrix}\psi_1 \\ ... \\ \psi_n\end{pmatrix}$

Building an operator's matrix

For a system with a discrete state basis, $\{\Psi_n\}$, any state of the system can be described as a linear combination of the basis' wave functions:

$$\Psi = \sum_n a_n \Psi_n \tag{1.7}$$

Observable $\bar{f}$ for such a wave function can be decomposed into a sum over the basis state wave functions:

$$\begin{aligned}
\bar{f} &= \int \Psi^* \hat{f} \Psi dq = \int \sum_n a_n^* \Psi_n^* \hat{f} \sum_m a_m \Psi_m dq \\
&= \sum_n \sum_m a_n^* a_m \int \Psi_n^* \hat{f} \Psi_m dq = \sum_n \sum_m a_n^* a_m f_{nm}(t) \\
&= \sum_n \sum_m a_n^* f_{nm}(t) a_m
\end{aligned} \tag{1.8}$$

Where f_{nm} is the operator matrix.

$f_n m(t)$ time dependence

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Solutions to the time-independent Shrödinger equation:

$$\hat{H}\Psi_n = E_n \Psi_n \tag{1.9}$$

$$\Psi_n(t) = e^{-\frac{i}{\hbar}E_n t}\Phi_n \tag{1.10}$$

Which, in operator matrix terms translates into

$$\begin{aligned}
f_{nm}(t) &= \int \Psi_n^* \hat{f} \Psi_m dq = \int \Phi_n^* (e^{-\frac{i}{\hbar}E_n t})^* \hat{f} \Phi_m e^{-\frac{i}{\hbar}E_m t} dq \\
&= e^{+\frac{i}{\hbar}E_n t} e^{-\frac{i}{\hbar}E_m t} \int \Phi_n^* \hat{f} \Phi_m dq = e^{i\frac{E_n - E_m}{\hbar}t} \int \Phi_n^* \hat{f} \Phi_m dq \\
&= f_{nm} e^{i\omega_{nm}t}
\end{aligned} \tag{1.11}$$

Operator matrix properties

1. The operator matrix is hermitian [1] Transposed operator:

$$(\int \Phi \hat{f} \Psi dq \quad)^T = \int \Psi (\hat{f})^T \Phi dq \tag{1.12}$$

Complex conjugate:

$$(\hat{f})^* = \hat{f}^* \tag{1.13}$$

Hermitian conjugate:

$$\bar{f^*} = \int \Psi^* \hat{f}^\dagger \Psi dq \tag{1.14}$$

[1] $H^\dagger = H = (H^*)^T$

In operator matrix terms:

$$(f^*_{nm}) = \int \varphi^*_n \hat{f}^\dagger \varphi_m dq = \int \varphi^*_n (\hat{f}^*)^T \varphi_m dq$$
$$= \int \varphi_m (\hat{f}^* \varphi^*_n) dq = (\int \varphi^*_m \hat{f}^\dagger \varphi_n dq)^* = (f_{mn})^* \tag{1.15}$$

Which means, if f_{nm} is real, meaning $f^*_{nm} = f_{nm}$ that

$$f_{nm} = f^*_{mn} = f^\dagger_{nm} \tag{1.16}$$

2. The matrix' diagonal elements are time-independent and real

$$f_{nn} = \int \Psi_n \hat{f} \Psi_n dq \equiv \bar{f}_n \tag{1.17}$$

Where $\bar{f}_n$ is the value of observable f in basis state n.

3. The matrix of the product of two operators is the product of their matrices

For operators $\hat{f}$ and $\hat{g}$, what is the operator matrix for operator $\hat{f} \times \hat{g}$ — $(\hat{f} \times \hat{g})_{nm}$?

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$$\hat{f}\varphi_n = \sum_m f_{mn}\varphi_m \tag{1.18}$$
$$\int \varphi^*_k dq \times \hat{f}\varphi_n = \int \varphi^*_k dq \times \sum_m f_{mn}\varphi_m$$
$$\int \varphi^*_k \hat{f}\varphi_n dq = \sum_m f_{mn} \int \varphi^*_k \varphi_m dq f_{kn} = \sum_m f_{mn}\delta_{km} = f_{kn} \tag{1.19}$$

Because for state basis φ_n, φ_n and φ_m are orthogonal for all $m \neq n$.

Using 1.18, we can write:

$$\hat{f}\hat{g}\varphi_n = \hat{f}(\hat{g}\varphi_n) = \hat{f}\sum_k g_k n \varphi_k = \sum_k g_{kn}\hat{f}\varphi_k$$
$$= \sum_k g_{kn} \sum_m f_{mk}\varphi_m = \sum_{k,m} g_{kn} f_{mk}\varphi_m$$
$$= \sum_{k,m} f_{mk} g_{kn}\varphi_m \tag{1.20}$$

And knowing that:

$$(\hat{f}\hat{g})\varphi_n = \sum_m (\hat{f}\hat{g})_{nm}\varphi_m \tag{1.21}$$

We end up with:

$$(\hat{f}\hat{g})_{mn} = \sum_k f_{mk} g_{kn} \tag{1.22}$$

4. The operator's matrix is equivalent to the operator

1.3 Switching to a different state basis

1.4 Pauli uncertainty principle

Single-slit electron diffraction

Black holes

Quantum Pencil

[1]

1.5 Problems

2 Analytical Solutions

2.1 Rectangular quantum well

2.2 Harmonic oscillator

2.3 Spherically symmetric potential

2.4 Problems

3 Quasi-classical approximation

3.1 Problems

4 Spin

4.1 Problems

5 Perturbation theory

5.1 Time-independent

5.2 Time-dependent

5.3 Problems

6 Problem Solutions

6.1 Introduction

6.2 Analytical solutions

6.3 Quasiclassical approximation

6.4 Spin

6.5 Petrubation theory

References

[1] D. Easton, "The quantum mechanical tipping pencil – a caution for physics teachers," European Journal of Physics, vol. 28, no. 6, p. 1097, 2007.