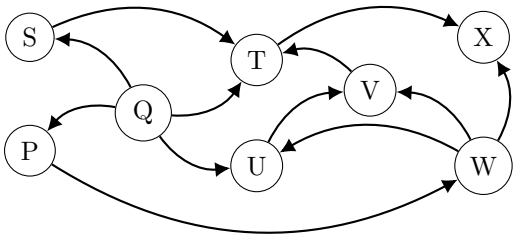
Week 8. Problem set

1. Consider the following directed graph $\mathcal{G}_1$ and answer questions:



   (a) Write down **all** possible topological sortings for the vertices of $\mathcal{G}_1$.
   (b) Write down **all** edges that can be safely removed from $\mathcal{G}_1$ without affecting the set of possible topological sortings.
   (c) What is the maximum possible number of distinct topological sortings for the vertices of a graph with 8 vertices? Briefly justify your answer (1–2 sentences).
   (d) What is the maximum possible number of distinct topological sortings for the vertices of a graph with 8 vertices and 12 edges? Briefly justify your answer (1–2 sentences).

2. Consider a large undirected graph $\mathcal{G}_2 = (V, E)$ where vertex labels are integers. We represent this graph with a variation of the adjacency-list graph representation [GTG, §14.2.2], where instead of lists we use maps, represented by B-trees [CLRS, §18] with minimum degree t and vertex labels serving as keys in the search tree. What are the worst-case time complexities of the following Graph ADT operations over this modified representation, in terms of $|V|$, $|E|$, and t?

   (a) `areAdjacent(v, u)`
   (b) `insertVertex(v)`
   (c) `insertEdge(from, to, e)`
   (d) `removeEdge(from, to)`
   (e) `removeVertex(v)`

   Assume that `v`, `from`, and `to` arguments above are *references* to vertex objects, not just labels.

   Briefly justify your answer (1–2 sentences).

3. Provide a directed graph $\mathcal{G}_3 = (V, E)$, vertex $s \in V$, and a subset of edges $T \subseteq E$ such that

   - both (V, E) and (V, T) are connected graphs,
   - $|V| \leq 5$,
   - T forms a tree with root s,
   - the set of edges T cannot be produced by running BFS [CLRS, §20.2] on $\mathcal{G}_3$, no matter how the vertices are ordered in the adjacency lists,
   - the set of edges T cannot be produced by running DFS [CLRS, §20.3] on $\mathcal{G}_3$, no matter how the vertices are ordered in the adjacency lists.

   Briefly justify your answer (1–2 sentences).

References

[CLRS] Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., 2022. *Introduction to algorithms, Fourth Edition*. MIT press.

[GTG] M. T. Goodrich, R. Tamassia, and M. H. Goldwasser. *Data Structures and Algorithms in Java*. WILEY 2014.