11) $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & -1 & 3 \end{pmatrix}$; $C = \begin{pmatrix} 4 & 4 & 0 \\ 3 & 4 & 6 \\ 0 & -2 & 4 \end{pmatrix}$

a)

1. A: $\begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} = A - \lambda I$, $\det(A-\lambda I) = 4-\lambda \begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = (4-\lambda)^3$

2. B: $\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 5-\lambda & 1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 4-\lambda \begin{vmatrix} 5-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (4-\lambda)(15-5\lambda-3\lambda+\lambda^2+1) = (4-\lambda)^3$

3. $C = \begin{vmatrix} 4-\lambda & 4 & 0 \\ 3 & 4-\lambda & 6 \\ 0 & -2 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 4-\lambda & 6 \\ -2 & 4-\lambda \end{vmatrix} - 4\begin{vmatrix} 3 & 6 \\ 0 & 4-\lambda \end{vmatrix} + 0 =$

$= (4-\lambda)(28-8\lambda+\lambda^2) - 4(12-3\lambda) = 112-32\lambda+4\lambda^2-28\lambda+8\lambda^2-\lambda^3$ $+1$

$= -\lambda^3 + 12\lambda^2 - 48 + 64 = (4-\lambda)^3$

b) $(4-\lambda)^3 = 0$

$\lambda = 4$

d) $A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$B - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}$; $\begin{cases} y + z = 0 \\ -y - z = 0 \end{cases}$; $y = -z$;

$\downarrow$

$z = 0$

$y = 0$

$v = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$C - \lambda I = \begin{pmatrix} 0 & 4 & 0 \\ 3 & 0 & 6 \\ 0 & -2 & 0 \end{pmatrix}$ $\begin{cases} 4y = 0; \\ 3x + 6z = 0; \\ -2y = 0; \end{cases}$ $y = 0; x = -2z$

$v = z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

c) Algebraische Veilfachkeite: 3, weil A, B, C sind 3x3 und haben 1 Eige

A – Geometrische Vielfachkeit: 3, da 3 Eigenvektoren

B – 1, da 1 Eigenvektor

C – 1, da 1 Eigenvektor.

12) $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{pmatrix}$

$\det(A-\lambda I) = 2-\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix} =$

$= (2-\lambda)((1-\lambda)(4-\lambda)) - (1-\lambda) = (1-\lambda)((2-\lambda)(4-\lambda)-1)$

b) $(1-\lambda)((2-\lambda)(4-\lambda)-1) = 0$

$\lambda = 1 \qquad (2-\lambda)(4-\lambda)-1 = 0$

$\lambda^2 - 6\lambda + 7 = 0 \qquad \lambda = \frac{6 \pm \sqrt{36-28}}{2} = 3 \pm \sqrt{2}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 3-\sqrt{2} \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ✓

$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1+\sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2}-1 \\ 0 \\ \sqrt{2}+3 \end{pmatrix} = (3+\sqrt{2}) \begin{pmatrix} -1+\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1-\sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2}-1 \\ 0 \\ -\sqrt{2}+3 \end{pmatrix} = (3-\sqrt{2}) \begin{pmatrix} -1-\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$

13) $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

b) $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{vmatrix} =$

$= \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 2 \cdot 4 - 1 = 7$

a) Auf die gleiche Weise wie in der vorherigen Aufgabe

14) $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix}; \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

a) $\|A\|_1 = \max\{2+1+3,\ 1+0+1,\ 1+1+4\} = \max\{6, 2, 6\} = 6$

$\|A\|_\infty = \max\{2+1+1,\ 1+0+1,\ 3+1+4\} = \max\{\cancel{6}\,4, 2, 8\}$

b) $\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -2\,II \\ \\ -3\,II \end{array} \left(\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -3 & 1 \end{array}\right) \cdot(-1) =$

$= \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -3 & 1 \end{array}\right) +I = \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right) /2 =$

$= \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{l} -III \\ -III \\ \\ \end{array} = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \updownarrow =$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$

$\|A^{-1}\|_1 = \max\left(\frac{3}{2}, \frac{9}{2}, \frac{3}{2}\right) = \frac{9}{2}$

$\|A^{-1}\|_\infty = \max\left(\frac{5}{2}, \frac{7}{2}, \frac{3}{2}\right) = \frac{7}{2}$

c) $\|A\|_1 \|A^{-1}\|_1 = 6 \cdot \frac{9}{2} = 27$

$\|A\|_\infty \|A^{-1}\|_\infty = 8 \cdot \frac{7}{2} = 28$

d) $\|x\|_1 = 1+1+1 = 3$

$\|x\|_\infty = \max\{1, 1, 1\} = 1$

e) $A\vec{x} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix}$; $\|A\vec{x}\|_1 = 4+2+8 = 14$

$\|A\vec{x}\|_\infty = \max\{4, 2, 8\} = 8$

f) $\|A\|_1 \|\vec{x}\|_1 = 6 \cdot 3 = 18$

$\|A\|_\infty \|\vec{x}\|_\infty = 8 \cdot 1 = 8$

15) $a = 0{,}23371258 \cdot 10^{-4}$

$b = 0{,}33678429 \cdot 10^{2}$

$c = -0{,}33677811 \cdot 10^{2}$

a) $a + b = 0{,}23371258 \cdot 10^{-4} + 0{,}33678429 \cdot 10^{2} =$

$= 0{,}00000023371258 \cdot 10^{2} + 0{,}33678429 \cdot 10^{2} =$

$= 0{,}33678452 \cdot 10^{2};$

$(a+b)+c = 0{,}33678452 \cdot 10^{2} - 0{,}33677811 \cdot 10^{2} =$

$= 0{,}0000641 \cdot 10^{2} = 0{,}641 \cdot 10^{-3}$

$b + c = 0{,}33678429 \cdot 10^{2} - 0{,}33677811 \cdot 10^{2} = 0{,}0000618 \cdot 10^{2} =$

$= 0{,}618 \cdot 10^{-3}$

$a + (b+c) = 0{,}23371258 \cdot 10^{-4} + 0{,}618 \cdot 10^{-3} =$

$= 0{,}023371258 \cdot 10^{-3} + 0{,}618 \cdot 10^{-3} = 0{,}64137126 \cdot 10^{-3}$

genauer, weil in a) Rundung passi[illegible]

16) $a = 0{,}0345$

$b = 29$

$c = 2$

a) $a = 0{,}345 \cdot 10^{-1}$

$b = 0{,}29 \cdot 10^{2}$

$c = 0{,}2 \cdot 10$

b) $a \cdot b = 0{,}345 \cdot 10^{-1} \cdot 0{,}29 \cdot 10^{2} =$

$= 0{,}1001 \cdot 10^{1}$ - gerundet

$0.1001 \cdot 10^{1} \cdot 0.2 \cdot 10^{1} = 0{,}2002 \cdot 10^{1}$

$b \cdot c = 0{,}58 \cdot 10^{2}$

$0.345 \cdot 10^{-1} \cdot 0.58 \cdot 10^{2} = 0.2001 \cdot 10^{1}$

genauer

17) a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{pmatrix} \begin{matrix} \\ II - 2I \\ -3I \end{matrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -2I \\ -3I \end{matrix}$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 3 \end{pmatrix} \quad I = M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 3 \end{pmatrix} \begin{matrix} \\ \\ -2II \end{matrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ -2II \end{matrix}$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad I = M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

b) U haben wir schon

$U = M_2 M_1 A; \quad M_2^{-1} U = M_1 A; \quad M_1^{-1} M_2^{-1} U = A; \quad M_1^{-1} M_2^{-1} = L$

$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}; \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}; \quad A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{pmatrix}$

c) $M_k^{-1} = I + v_k e_k^T; \quad k = 1; \ j = 2$

$M_1^{-1} = I + v_1 e_1^T;$

$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} (1 \ 0 \ 0)$

$M_1^{-1} M_2^{-1} = I + v_1 e_1^T + v_2 e_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} (1 \ 0 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} (0 \ 1 \ 0)$

18.

a) $A = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 0 & -2 \\ -2 & 4 & 6 \end{pmatrix} \begin{matrix} \\ -\mathrm{I} \\ +2\mathrm{I} \end{matrix} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & -3 & -6 \\ 0 & 10 & 14 \end{pmatrix} \begin{matrix} \\ \\ +\frac{10}{3}\mathrm{II} \end{matrix} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & -3 & -6 \\ 0 & \cancel{10}\,0 & \cancel{14}\,-6 \end{pmatrix} = U$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -\mathrm{I} \\ +2\mathrm{I} \end{matrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = M_1$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ +\frac{10}{3}\mathrm{II} \end{matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{10}{3} & 0 \end{pmatrix}$

$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}; \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{10}{3} & 1 \end{pmatrix} \Rightarrow M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -\frac{10}{3} & 1 \end{pmatrix}$

$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -\frac{10}{3} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 4 \\ 0 & -3 & -6 \\ 0 & \cancel{10}\,0 & \cancel{14}\,-6 \end{pmatrix}$

b) $\det(A) = \det(LU) = \det(L) \cdot \det(U) = 18$

$\det(L) = 1 \cdot \begin{vmatrix} 1 & 0 \\ -\frac{10}{3} & 1 \end{vmatrix} = 1$

$\det(U) = 1 \cdot \begin{vmatrix} -3 & -6 \\ 0 & -6 \end{vmatrix} = 18$

c) $Ax = b; \quad b = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & 4 \\ 1 & 0 & -2 \\ -2 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$; Sei $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, dann:

$\begin{cases} a + 3b + 4c = 1 \\ a - 2c = 0 \\ -2a + 4b + 6c = 5 \end{cases} \Rightarrow \begin{cases} a + 3b + 4c = 1 \\ a - 2c = 0 \end{cases} - \Rightarrow 3b + 6c = 1 \quad \begin{cases} 3b + 6c = 1 \\ 10b + 14c = 7 \end{cases}$

$\begin{cases} a + 3b + 4c = 1 \\ -2a + 4b + 6c = 5 \end{cases} +2 \Rightarrow 10b + 14c = 7$

$\begin{cases} 30b + 60c = 10 \\ 10b + 14c = 7 \end{cases} \quad \begin{cases} 30b + 60c = 10 \\ -30b - 42c = -21 \end{cases} \quad 18c = -11; \; c = -\frac{11}{18}; \; b = \frac{14}{9}$

$a = -\frac{11}{9}$

$x = \begin{pmatrix} -\frac{11}{9} \\ \frac{14}{9} \\ -\frac{11}{18} \end{pmatrix}$