Blatt 1

1) $A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 4 \\ 1 & 0 \\ 1 & 3 \end{pmatrix}$ $C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\cancel{B}$ $D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

a) $A \cdot B = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 - 2 \cdot 1 + 0 \cdot 1 & 1 \cdot 4 - 2 \cdot 0 + 0 \cdot 3 \\ -1 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 & -1 \cdot 4 + 0 \cdot 0 + 2 \cdot 3 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}}$

b) $B \cdot A = \begin{pmatrix} 2 & 4 \\ 1 & 0 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} =$

$= \begin{pmatrix} 2 \cdot 1 + 4 \cdot (-1) & 2 \cdot (-2) + 4 \cdot 0 & 2 \cdot 0 + 4 \cdot 2 \\ 1 \cdot 1 + 0 \cdot (-1) & 1 \cdot (-2) + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 2 \\ 1 \cdot 1 + 3 \cdot (-1) & 1 \cdot (-2) + 3 \cdot 0 & 1 \cdot 0 + 3 \cdot 2 \end{pmatrix} = \boxed{\begin{pmatrix} -2 & -4 & 8 \\ 1 & -2 & 0 \\ -2 & -2 & 6 \end{pmatrix}}$

c) $A \cdot C = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \boxed{\text{unmöglich}}$

d) $A^T \cdot (C + D)$

$A^T = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 0 & 2 \end{pmatrix}$

$C + D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$A^T \cdot (C + D) = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 - 1 \cdot 0 & 1 \cdot 0 - 1 \cdot 2 \\ -2 \cdot 2 + 0 \cdot 0 & -2 \cdot 0 + 0 \cdot 2 \\ 0 \cdot 2 + 2 \cdot 0 & 0 \cdot 0 + 2 \cdot 2 \end{pmatrix} =$

$= \boxed{\begin{pmatrix} 2 & -2 \\ -4 & 0 \\ 0 & 4 \end{pmatrix}}$

e) $(C + D) \cdot A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 0 \cdot (-1) & 2 \cdot (-2) + 0 \cdot 0 & 2 \cdot 0 + 0 \cdot 2 \\ 0 \cdot 1 + 2 \cdot (-1) & 0 \cdot (-2) + 2 \cdot 0 & 0 \cdot 0 + 2 \cdot 2 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 2 & -4 & 0 \\ -2 & 0 & 4 \end{pmatrix}}$

2) $A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a) $\vec{y} A \vec{x} =$

$\vec{y} A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \boxed{\text{unbestimmt}} \Rightarrow \vec{y} A \vec{x}$ - unbestimmt

b) $\vec{y}^T A \vec{x} =$

$\vec{y}^T = (1 \ 1)$

$\vec{y}^T A = (1 \ 1) \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = (1 \cdot 1 + 1 \cdot (-1) \quad 1 \cdot (-2) + 1 \cdot 0 \quad 1 \cdot 0 + 1 \cdot 2) =$

$= \boxed{(0 \ -2 \ 2)}$

$\vec{y}^T A \vec{x} = (0 \ -2 \ 2) \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = (0 \cdot 1 - 2 \cdot (-1) + 2 \cdot 1) = \boxed{(4)}$

c) $\vec{x}^T A \vec{y} =$

$\vec{x}^T = (1 \ -1 \ 1)$

$\vec{x}^T A = (1 \ -1 \ 1) \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ - unbestimmt $\Rightarrow \vec{x}^T A \vec{y}$ - $\boxed{\text{unbestimmt}}$

d) $\vec{x}^T (\vec{y}^T A)^T$

$\vec{y}^T A = (0 \ -2 \ 2)$

$(\vec{y}^T A)^T = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$

$\vec{x}^T (\vec{y}^T A)^T = (1 \ -1 \ 1) \cdot \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = (1 \cdot 0 - 1 \cdot (-2) + 1 \cdot 2) = \boxed{(4)}$

3) $A=\begin{pmatrix}4 & -1 & 0\\ 8 & -1 & -3\\ -1 & 7 & -4\end{pmatrix}$ $B=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -4 & 3 & 1\end{pmatrix}$ $C=\begin{pmatrix}4 & -1 & 0\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}$

Überprüfen wir, dass $A=B\cdot C$:

$$B\cdot C=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -4 & 3 & 1\end{pmatrix}\cdot\begin{pmatrix}4 & -1 & 0\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}=$$

$$=\begin{pmatrix}1\cdot 4+0\cdot 0+0\cdot 0 & 1\cdot(-1)+0\cdot 1+0\cdot 0 & 1\cdot 0+0\cdot(-3)+0\cdot 5\\ 2\cdot 4+1\cdot 0+0\cdot 0 & 2\cdot(-1)\ 1\cdot 1+0\cdot 0 & 2\cdot 0+1\cdot(-3)+0\cdot 5\\ -4\cdot 4+3\cdot 0+1\cdot 0 & -4\cdot(-1)+3\cdot 1+1\cdot 0 & -4\cdot 0+3\cdot(-3)+1\cdot 5\end{pmatrix}=$$

$$=\begin{pmatrix}4 & -1 & 0\\ 8 & -1 & -3\\ -1 & 7 & -4\end{pmatrix}$$

a) $$\det A=4\cdot\begin{vmatrix}-1 & -3\\ 7 & -4\end{vmatrix}-(-1)\cdot\begin{vmatrix}8 & -3\\ -1 & 4\end{vmatrix}+\underbrace{0\cdot\begin{vmatrix}8 & -1\\ -1 & 7\end{vmatrix}}_{0}=$$

$$=4\cdot[(-1)(-4)-7\cdot(-3)]-(-1)\cdot(8\cdot 4-(-1)(-3))=$$

$$=4\cdot 25+1\cdot(-35)=100-35=\boxed{65}$$

$$\det B=1\cdot\begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}-\underbrace{0\cdot\begin{vmatrix}2 & 0\\ -4 & 1\end{vmatrix}}_{0}+\underbrace{0\cdot\begin{vmatrix}2 & 1\\ -4 & 3\end{vmatrix}}_{0}=$$

$$=1\cdot(1\cdot 1+3\cdot 0)=1\cdot 1=\boxed{1}$$

$$\det C=4\cdot\begin{vmatrix}1 & -3\\ 0 & 5\end{vmatrix}-(-1)\cdot\begin{vmatrix}0 & -3\\ 0 & 5\end{vmatrix}+\underbrace{0\cdot\begin{vmatrix}0 & -3\\ 0 & 5\end{vmatrix}}_{0}=$$

$$=4\cdot(1\cdot 5-(-3)\cdot 0)+1(0\cdot 5-0\cdot(-3))=$$

$$=4\cdot 5+0=\boxed{20}$$

b) $\det(A^2)$ – ?

$$A^2=\begin{pmatrix}4 & -1 & 0\\ 8 & -1 & -3\\ -1 & 7 & -4\end{pmatrix}\cdot\begin{pmatrix}4 & -1 & 0\\ 8 & -1 & -3\\ -1 & 7 & -4\end{pmatrix}=$$

$$=\begin{pmatrix}4\cdot 4+(-1)\cdot 8+0\cdot(-1) & 4\cdot(-1)+(-1)(-1)\overset{+0\cdot 7}{+(-3)\cdot 7} & 4\cdot 0+(-1)(-3)+0\cdot(-4)\\ 8\cdot 4+(-1)\cdot 8+(-3)(-1) & 8\cdot(-1)+(-1)(-1)-3(0) & 8\cdot 0+(-1)(-3)+(-3)(-4)\\ -1\cdot 4+7\cdot 8+(-4)(-1) & -1(-1)+7(-1)+(-4)\cdot 7 & -1(0)+7\cdot(-3)+(-4)(-4)\end{pmatrix}=$$

$$=\begin{pmatrix}8 & -3 & 3\\ 27 & -28 & 15\\ 56 & -34 & -5\end{pmatrix};\ \det(A^2)=8\cdot\begin{vmatrix}-28 & 15\\ -34 & -5\end{vmatrix}-(-3)\begin{vmatrix}27 & 15\\ 56 & -5\end{vmatrix}+3\begin{vmatrix}27 & 15\\ 56 & -5\end{vmatrix}=\boxed{4225}$$

c) $\det(A^{-1})$ - ?

$$A^{-1} = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 8 & -1 & -3 & 0 & 1 & 0 \\ -16 & 7 & -4 & 0 & 0 & 1 \end{array}\right) - I = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 4 & 0 & -3 & -1 & 1 & 0 \\ -16 & 7 & -4 & 0 & 0 & 1 \end{array}\right) + 7I =$$

$$= \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 4 & 0 & -3 & -1 & 1 & 0 \\ 12 & 0 & -4 & 7 & 0 & 1 \end{array}\right) /4 = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 4 & 0 & -3 & -1 & 1 & 0 \\ 3 & 0 & -1 & \frac{7}{4} & 0 & \frac{1}{4} \end{array}\right) -3III =$$

$$= \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ -5 & 0 & 0 & -\frac{25}{4} & 1 & -\frac{3}{4} \\ 3 & 0 & -1 & \frac{7}{4} & 0 & \frac{1}{4} \end{array}\right) /5 = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 3 & 0 & -1 & \frac{7}{4} & 0 & \frac{1}{4} \end{array}\right) -4II =$$

$$= \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -4 & \frac{4}{5} & -\frac{3}{5} \\ 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 3 & 0 & -1 & \frac{7}{4} & 0 & \frac{1}{4} \end{array}\right) -3II = \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -4 & \frac{4}{5} & -\frac{3}{5} \\ 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 0 & 0 & -1 & -2 & \frac{3}{5} & -\frac{1}{5} \end{array}\right) \begin{array}{l} \cdot(-1) \\ \\ \cdot(-1) \end{array} =$$

$$= \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 0 & 0 & 1 & 2 & -\frac{3}{5} & \frac{1}{5} \end{array}\right) \updownarrow = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 0 & 1 & 0 & 4 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & 1 & 2 & -\frac{3}{5} & \frac{1}{5} \end{array}\right)$$

$$\det(A^{-1}) = \begin{vmatrix} \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 4 & -\frac{4}{5} & \frac{3}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = \frac{5}{4} \cdot \begin{vmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} + \frac{1}{5} \begin{vmatrix} 4 & \frac{3}{5} \\ 2 & \frac{1}{5} \end{vmatrix} + \frac{3}{20} \begin{vmatrix} 4 & -\frac{4}{5} \\ 2 & -\frac{3}{5} \end{vmatrix} =$$

$$= \frac{5}{4} \cdot \frac{1}{5} + \frac{1}{5}\left(-\frac{2}{5}\right) + \frac{3}{20}\left(-\frac{4}{5}\right) = \frac{1}{4} - \frac{2}{25} - \frac{3}{25} = \boxed{\frac{1}{20} = 0{,}05}$$

d) $\det(ABC) = ?$

$$AB = \begin{pmatrix} 4 & -1 & 0 \\ 8 & -1 & -3 \\ -16 & 7 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 4\cdot 1 + (-1)\cdot 2 + 0\cdot(-4) & 4\cdot 0 + (-1)\cdot 1 + 0\cdot 3 & 4\cdot 0 + (-1)\cdot 0 + 0\cdot \\ 8\cdot 1 + (-1)\cdot 2 + (-3)(-4) & 8\cdot 0 + (-1)\cdot 1 + (-3)(3) & 8\cdot 0 + (-1)\cdot 0 + (-3 \\ -16\cdot 1 + 7\cdot 2 + (-4)(-4) & -16\cdot 0 + 7\cdot 1 + (-4)\cdot 3 & -16\cdot 0 + 7\cdot 0 + (-4) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 0 \\ 18 & -10 & -3 \\ 14 & -5 & -4 \end{pmatrix}; \quad ABC = \begin{pmatrix} 2 & -1 & 0 \\ 18 & -10 & -3 \\ 14 & -5 & -4 \end{pmatrix} \cdot \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 2\cdot 4 + (-1)\cdot 0 + 0\cdot 0 & 2\cdot(-1) + (-1)\cdot 1 + 0\cdot 0 & 2\cdot 0 + (-1)(-3) + 0\cdot 5 \\ 18\cdot 4 + (-10)\cdot 0 + (-3)\cdot 0 & 18(-1) + (-10)\cdot 1 + (-3)\cdot 0 & 18\cdot 0 + (-10)(-3) + (-3)\cdot 5 \\ 14\cdot 4 + (-5)\cdot 0 + (-4)\cdot 0 & 14(-1) + (-5)\cdot 1 + (-4)\cdot 0 & 14\cdot 0 + (-5)(-3) + (-4)\cdot 5 \end{pmatrix} = \begin{pmatrix} 8 & -3 & 3 \\ 42 & -28 & 15 \\ 56 & -19 & -5 \end{pmatrix}$$

$$\det(ABC) = 8 \cdot \begin{vmatrix} -28 & 15 \\ -19 & -5 \end{vmatrix} + 3 \begin{vmatrix} 42 & 15 \\ 56 & -5 \end{vmatrix} + 3 \begin{vmatrix} 42 & -28 \\ 56 & -19 \end{vmatrix} = \boxed{2560}$$

e) $\det(C^T C) - ?$

$$C^T = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & 5 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 4\cdot4+0\cdot0+0\cdot0 & 4\cdot(-1)+0\cdot1+0\cdot0 & 4\cdot0+0\cdot(-3)+0\cdot5 \\ -1\cdot4+1\cdot0+0\cdot0 & -1\cdot(-1)+1\cdot1+0\cdot0 & -1(0)+1\cdot(-3)+0\cdot5 \\ 0\cdot4+(-3)\cdot0+5\cdot0 & 0(-1)+(-3)\cdot1+5\cdot0 & 0\cdot0+(-3)(-3)+5\cdot5 \end{pmatrix} =$$

$$= \begin{pmatrix} 16 & -4 & 0 \\ -4 & 2 & -3 \\ 0 & -3 & 34 \end{pmatrix}; \det(C^T C) = 16\cdot\begin{vmatrix} 2 & -3 \\ -3 & 34 \end{vmatrix} + 4\begin{vmatrix} -4 & -3 \\ 0 & 34 \end{vmatrix} +$$

$$+ \underbrace{0\cdot\begin{vmatrix} -4 & 2 \\ 0 & -3 \end{vmatrix}}_{0} = 16\cdot59+4(-136) = \boxed{400}$$

4) $A = \begin{pmatrix} 4 & -1 & 0 \\ 8 & -1 & -3 \\ -16 & 7 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$

a) $A^{-1} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{5} & \frac{3}{5} \\ 4 & -\frac{4}{5} & \frac{3}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$ – aus 3 c)

b) $$B^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -4 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -2I \\ +4I \end{array} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & 4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -3II \end{array} =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 10 & -3 & 1 \end{array}\right); \boxed{B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -3 & 1 \end{pmatrix}}$$

c) $C^{-1} - ?$

$$C^{-1} = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ /5 \end{array} = \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} \\ +3III \\ \end{array} =$$

$$= \left(\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} +II \\ \\ \end{array} = \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 1 & \frac{3}{5} \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} /4 \\ \\ \end{array} =$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{20} \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right); \boxed{C^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{3}{20} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{pmatrix}}$$

d) $C^{-1}B^{-1}$ – ?

$$C^{-1}B^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{3}{20} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -3 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot(-2) + \frac{3}{20}\cdot 10 & \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{3}{20}\cdot(-3) & \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 0 + \frac{3}{20}\cdot 1 \\ 0\cdot 1 + 1\cdot(-2) + \frac{3}{5}\cdot 10 & 0\cdot 0 + 1\cdot 1 + \frac{3}{5}\cdot(-3) & 0\cdot 0 + 1\cdot 0 + \frac{3}{5}\cdot 1 \\ 0\cdot 1 + 0\cdot(-2) + \frac{1}{5}\cdot 10 & 0\cdot 0 + 0\cdot 1 + \frac{1}{5}\cdot(-3) & 0\cdot 0 + 0\cdot 0 + \frac{1}{5}\cdot 1 \end{pmatrix} =$$

$$\boxed{= \begin{pmatrix} \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 4 & -\frac{4}{5} & \frac{3}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{pmatrix}}$$

5) $A = \begin{pmatrix} 4 & -1 & 0 \\ 8 & -1 & -3 \\ -16 & 7 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$

a) $$B^{-1}C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{3}{20} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot\frac{1}{4} + 0\cdot 0 + 0\cdot 0 & 1\cdot\frac{1}{4} + 0\cdot 1 + 0\cdot 0 & 1\cdot\frac{3}{20} + 0\cdot\frac{3}{5} + 0\cdot\frac{1}{5} \\ -2\cdot\frac{1}{4} + 1\cdot 0 + 0\cdot 0 & -2\cdot\frac{1}{4} + 1\cdot 1 + 0\cdot 0 & -2\cdot\frac{3}{20} + 1\cdot\frac{3}{5} + 0\cdot\frac{1}{5} \\ 10\cdot\frac{1}{4} + (-3)\cdot 0 + 1\cdot 0 & 10\cdot\frac{1}{4} + (-3)\cdot 1 + 1\cdot 0 & 10\cdot\frac{3}{20} + (-3)\cdot\frac{3}{5} + 1\cdot\frac{1}{5} \end{pmatrix} =$$

$$\boxed{= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{3}{20} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{10} \end{pmatrix}}$$

b) und c) $A$ ist invertierbar (3c ist eine Inverse) $\Rightarrow (A^T)^{-1} = (A^{-1})^T$

$$A^{-1} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{5} & \frac{3}{20} \\ 4 & -\frac{4}{5} & \frac{3}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{pmatrix}; \quad (A^{-1})^T = (A^T)^{-1} = \boxed{\begin{pmatrix} \frac{5}{4} & 4 & 2 \\ -\frac{1}{5} & -\frac{4}{5} & -\frac{3}{5} \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}}$$

6) $A=\begin{pmatrix}1&-1&0\\1&1&0\\0&0&1\end{pmatrix}$ $B=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&\frac{1}{2}\end{pmatrix}$ $C=\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{\sqrt{2}}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{\sqrt{2}}\end{pmatrix}$

a) $A\cdot A^T=I$ - ortogonal

$$A\cdot A^T=\begin{pmatrix}1&-1&0\\1&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\-1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}\text{ - nicht orthogonal}$$

$$B\cdot B^T=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\1&-1&0\\0&0&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&\frac{1}{4}\end{pmatrix}\text{ - nicht orthogonal}$$

$$C\cdot C^T=\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{\sqrt{2}}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1&-\frac{1}{4}&\frac{\sqrt{2}}{2}+\frac{1}{4}\\-\frac{1}{4}&1&0\\\frac{\sqrt{2}}{2}+\frac{1}{4}&0&1\end{pmatrix}\text{ - nicht ortogonal}$$

b) $B^T=B$ - symmetrisch

c) $a\cdot\begin{pmatrix}3&4\\4&b\end{pmatrix}=M$

$M=\begin{pmatrix}3a&4a\\4a&4ab\end{pmatrix}$ ; $M^T=\begin{pmatrix}3a&4a\\4a&4ab\end{pmatrix}$

$$MM^T=\begin{pmatrix}3a&4a\\4a&4ab\end{pmatrix}\cdot\begin{pmatrix}3a&4a\\4a&4ab\end{pmatrix}=\begin{pmatrix}25a^2&12a^2+16a^2b\\12a^2+16a^2b&16a^2+16a^2b^2\end{pmatrix}$$

$25a^2=1$ $\quad$ $\cancel{4a=\pm\frac{1}{5}}$

$a^2=\frac{1}{25}$ ; $\quad 16\cdot\frac{1}{25}+16\cdot\frac{1}{25}b^2=1$ ; $b=\pm\frac{3}{4}$

$\cancel{a=\pm\frac{1}{5}}$

$a=\pm\frac{1}{5}$

1) $12a^2+16a^2b=0$ ; $b=\frac{3}{4}$
$12\cdot\frac{1}{25}+16\cdot\frac{1}{25}\cdot\frac{3}{4}=\frac{24}{25}\neq 0$

2) $12a^2+16a^2b=0$ ; $b=-\frac{3}{4}$
$12\cdot\frac{1}{25}+16\cdot\frac{1}{25}\cdot\left(-\frac{3}{4}\right)=0$

3) Also: $a=\pm\frac{1}{5}$ ; $b=-\frac{3}{4}$

7) a) $A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix}$; $M_1 = -4$; $M_2 = 4$; $M_3 = -20$;

A - negativ definit

b) $B = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 0 \end{pmatrix}$; $M_1 = 0$; $M_2 = \begin{vmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 2 \end{vmatrix} = -2$

$M_3 = 0 \cdot 2 \cdot 0 + \sqrt{2} \cdot 2\sqrt{2} \cdot 0 + 0 \cdot \sqrt{2} + 2\sqrt{2} -$
$- 0 \cdot 2 \cdot 0 - 0 \cdot 2\sqrt{2} \cdot 2\sqrt{2} - \sqrt{2}\sqrt{2} \cdot 0 = 0$

$$\left(\begin{matrix} 0 & \sqrt{2} & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 2 & 2\sqrt{2} & \sqrt{2} & 2 \\ 0 & 2\sqrt{2} & 0 & 0 & \sqrt{2} \end{matrix}\right)$$

B - ist indefint

c) $C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$; $M_1 = 2$; $M_2 = 2 \cdot 1 - 0 = 2$;

$M_3 = 2 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 2 \cdot 4 + 1(-1) = 8 - 1 = 7$

C - positiv definit

d) $D = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$; $M_1 = 4$; $M_2 = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 4 \cdot 2 = 8$

$M_3 = 4 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} =$
$= 4 \cdot 4 - 0 + 1 \cdot 0 = 16$

D - positiv definit

8) $A = \begin{pmatrix} 2 & -2 & \alpha \\ -2 & 6 & -1 \\ \alpha & -1 & 2 \end{pmatrix}$ ?

a) positiv definit?

1) $M_1 > 0$ -?; $2 > 0$ V; 2) $M_2 > 0$?; $\begin{vmatrix} 2 & -2 \\ -2 & 6 \end{vmatrix} = 2 \cdot 6 - 2 \cdot 2 = 8$ V

3) $M_3 > 0$ -?

$2 \cdot \begin{vmatrix} 6 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \cdot \begin{vmatrix} -2 & -1 \\ \alpha & 2 \end{vmatrix} + \alpha \begin{vmatrix} -2 & 6 \\ \alpha & -1 \end{vmatrix} = 2 \cdot 11 + 2 \cdot (-4 + \alpha) + \alpha(2 - 6\alpha)$

$= 22 - 8 + 2\alpha + 2\alpha - 6\alpha^2 = -6\alpha^2 + 4\alpha + 14$

$-6\alpha^2 + 4\alpha + 14 > 0$ $\quad \frac{1 \pm \sqrt{22}}{3}$

$$\boxed{\alpha \in \left( \frac{1 - \sqrt{22}}{3} ; \frac{1 + \sqrt{22}}{3} \right)}$$

6) $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 6 & -1 \\ 0 & -1 & 2 \end{pmatrix} = M$

1) $M - \lambda I = \begin{pmatrix} 2-\lambda & -2 & 0 \\ -2 & 6-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix}$

2) $\det(M-\lambda I) = (2-\lambda)\begin{vmatrix} 6-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & -1 \\ 0 & 2-\lambda \end{vmatrix} =$

$= (2-\lambda)((6-\lambda)(2-\lambda)-1) + 2(-2)(2-\lambda) =$

$= (2-\lambda)(11-8\lambda+\lambda^2) - 4(2-\lambda) = (2-\lambda)(\lambda^2-8\lambda+7)$

$\lambda^2 - 8\lambda + 7 = 0$; $2-\lambda = 0$

$\lambda = 7$; $\lambda = 1$ $\quad \lambda = 2$

$\boxed{c) \det A = -6\lambda^2 + 4\lambda + 14}$

Eigenwerte: $\lambda_1 = 1$; $\lambda_2 = 2$; $\lambda_3 = 7$

Eigenvektore:

1) $\lambda_1 = 1$

$M - \lambda I = \begin{pmatrix} 2-1 & -2 & 0 \\ -2 & 6-1 & -1 \\ 0 & -1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

Lass $v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, dann

1. $\begin{cases} x - 2y + 0z = 0 \Rightarrow x = 2y \\ -2x + 5y - 1z = 0 \\ 0x - 1y + 1z = 0 \Rightarrow z = y \end{cases} \Rightarrow -2(2y) + 5y - y = 0;\ -4y + 4y =$

wäh

Also: für $\lambda_1 = 1$; $v_1 = y\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

2) $\lambda_2 = 2$

$M - \lambda I = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$; $\begin{cases} -2y = 0 \\ -2x + 4y - z = 0 \\ -1 = 0 \end{cases}$ $y = 0$; $z = -2x$

für $\lambda_2 = 2$; $v_2 = x\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

3) $\lambda_3 = 7$; $M - \lambda I = \begin{pmatrix} -5 & -2 & 0 \\ -2 & -1 & -1 \\ 0 & -1 & -5 \end{pmatrix}$; $\begin{cases} -5x - 2y = 0 \Rightarrow -5x + 10z = 0 \Rightarrow x \\ -2x - y - z = 0 \Rightarrow -4z + 5z - z = 0 - w \\ -y - 5z = 0 \Rightarrow y = -5z \end{cases}$

für $\lambda_3 = 7$; $v_3 = z\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$

Antwort: Eigenwerte: $\lambda = 1$, $\lambda = 2$, $\lambda = 7$; Eigenvektore: $\lambda = 1$: $v_1 = y($; $\lambda = 2$: $v_2 = x$; $\lambda = 3$: $v_3 = z($