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Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space

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ABSTRACT. – A Wiener process, defined as the coordinate process X on Wiener space, remains a semimartingale if the canonical filtration is enlarged by the information about the endpoint X_1. Elimination of the resulting drift by means of a Girsanov transformation leads us to a new measure under which X is again a Wiener process. But this measure does not coincide with Wiener measure P; in fact it is singular to P.

We study this apparent paradox in the general case where X_1 is replaced by a random variable G such that X remains a semimartingale in the filtration enlarged by G.

Key words : Wiener space; enlargement of filtration; Girsanov's theorem; *h*-transforms.

RÉSUMÉ. – Un processus de Wiener, défini comme processus des coordonnées X sur l'espace de Wiener, reste une semimartingale dans la filtration canonique augmentée par l'information sur le point final X_1. L'élimination du drift moyennant une transformation de Girsanov produit une nouvelle mesure sous laquelle X est toujours un processus de Wiener.

Classification A.M.S. : primary 60 J 65, 60 G 44; secondary 60 H 05, 60 J 45.

Mais cette mesure n'est pas identique à la mesure de Wiener P; en fait, elle est singulière par rapport à P.

On étudie ce paradoxe dans le cas général où X_1 est remplacé par une variable aléatoire G telle que X reste une semimartingale dans la filtration grossie par G.

1. INTRODUCTION

Anticipation of some random variable G on Wiener space means that we pass from the canonical filtration $(\mathscr{F}_t)_{0 \leqq t \leqq 1}$ to the larger filtration $(\mathscr{G}_t)_{0 \leqq t \leqq 1}$ with

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(G).$$

With respect to this new filtration, the coordinate process X on $\Omega = C([0, 1])$ is no longer a martingale under Wiener measure P. However, under some regularity conditions on G, X will be a semimartingale of the form

$$X_t = W_t + \int_0^t \alpha_s \, ds, \tag{1}$$

where W is a Wiener process with respect to $(\mathscr{G}_t)_{0 \leqq t \leqq 1}$ and P. If, for example, we anticipate the endpoint $G = X_1$ of the Brownian path, then the decomposition (1) is given by

$$\alpha_s = \frac{X_1 - X_s}{1 - s}. \tag{2}$$

What happens if we try to eliminate the drift appearing in (1) by means of a Girsanov transformation? In the case $G = X_1$, the densities

$$M_t = \exp\left[-\int_0^t \alpha_s \, dX_s - \frac{1}{2}\int_0^t \alpha_s^2 \, ds\right], \qquad 0 \leqq t < 1, \tag{3}$$

define consistently a new probability measure Q on each σ-field $\mathscr{G}_t$, $0 \leqq t < 1$, which turns the coordinate process into a Wiener process up to each time $t < 1$. But any such measure Q on $C([0, 1])$ should coincide with Wiener measure P, in constrast to the fact that M_t is not equal to 1.

The purpose of the present paper is to investigate this apparent paradox. In section 2 we consider the case $G = X_1$. In this case, the martingale in (3) does determine a probability measure. But this measure will not live on $\Omega = C([0, 1])$. Instead, it can be constructed as the product measure

$$\bar{Q} = P \otimes N(0, 1)$$

on the product space

$$\bar{\Omega} = \Omega \times \mathbf{R}.$$

This corresponds to a decoupling of the Brownian path and the endpoint X_1. The projection of $\bar{Q}$ on the first coordinate coincides with Wiener measure P. On the other hand, if Wiener measure is identified with the joint distribution $\bar{P}$ of (X, X_1) on $\bar{\Omega}$, then $\bar{Q}$ is singular to $\bar{P}$. These are the two sides of the paradox.

In section 3 we consider a general random variable G such that X is a semimartingale with respect to $(\mathscr{G}_t)_{0 \leqq t \leqq 1}$ and P. In general, the process M defined by (3) will only be a local martingale. The associated measure $\bar{Q}$ will again be the product of Wiener measure and the distribution of G, but it will be determined only up to some life time $\zeta \leqq 1$. This fits into the general construction of the measure associated with a nonnegative supermartingale in [3], [4]. But in our particular situation, we can give an explicit description of the projective limit structure which is involved in the general case. This is illustrated by an explicit example.

The question discussed in the present paper came up in the study of anticipating Girsanov transformations in [2], and we thank R. Buckdahn for various discussions. Of course the question could be posed in a general framework, where the random variable G on Wiener space is replaced by an anticipating process (G_t) on some filtered probability space. As pointed out by a referee, such an extension would involve the techniques developped by Song [12].

2. THE "PARADOX"

Our basic probability space is the canonical Wiener space $(\Omega, \mathscr{F}, P)$, with $\Omega = C([0, 1])$ and Wiener measure P. The coordinate process is denoted by $(X_t)_{0 \leqq t \leqq 1}$, the canonical right-continuous filtration by $(\mathscr{F}_t)_{0 \leqq t \leqq 1}$. Let

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(X_1), \qquad 0 \leqq t \leqq 1.$$

It is well known that X is a semimartingale in the filtration $(\mathscr{G}_t)_{0 \leqq t \leqq 1}$, and that its canonical decomposition

$$X = W + A \tag{4}$$

is given by the process of bounded variation

$$A_t = \int_0^t \alpha_s(X)\, ds, \tag{5}$$

where

$$\alpha_s(X)=\frac{X_1-X_s}{1-s}, \qquad 0\leqq s<1, \tag{6}$$

and W is a Wiener process with respect to the measure P and the filtration $(\mathcal{G}_t)_{0\leqq t\leqq 1}$ (*see* Jeulin [7]).

Let us now try to eliminate the drift appearing in (4) by means of a Girsanov transformation. As a natural candidate for the density on $\mathcal{G}_t$ of a new measure Q, which turns X into a martingale, we define

$$M_t=\exp\left[-\int_0^t \alpha_s(X)\,dW_s-\frac{1}{2}\int_0^t \alpha_s^2(X)\,ds\right]. \tag{7}$$

Clearly $(M_t)_{0\leqq t<1}$ is a non-negative supermartingale with respect to $(\mathcal{G}_t)_{0\leqq t<1}$ and P, with continuous paths and initial value $M_0=1$. In the sequel, the transition densities of the Wiener process are denoted by

$$p_s(x, y)=\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{1}{2s}(x-y)^2\right), \qquad s\geqq 0, \quad x, y\in\mathbf{R}.$$

LEMMA 1. — *For* $0\leqq t<1$ *we have*

$$\begin{aligned} M_t&=\frac{p_1(0, X_1)}{p_{1-t}(X_t, X_1)} \\ &=\exp\left[-\frac{X_1^2}{2}+\frac{(X_1-X_t)^2}{2(1-t)}+\frac{1}{2}\ln(1-t)\right]. \end{aligned} \tag{8}$$

In particular,

$$E(M_t)=1, \qquad 0\leqq t<1, \tag{9}$$

and

$$\lim_{t\uparrow 1} M_t=0 \qquad P-a.s., \tag{10}$$

and so $(M_t)_{0\leqq t\leqq 1}$ *is a martingale which is not uniformly integrable.*

Proof. — Itô's formula, applied to the function

$$f(x, y, s)=\frac{(x-y)^2}{2(1-s)}, \qquad x, y\in\mathbf{R}, \quad 0\leqq s<1,$$

yields

$$
\begin{aligned}
& f(X_1, X_t, t) - f(X_1, 0, 0) \\
& = \int_0^t \frac{\partial f}{\partial y}(X_1, X_s, s)\, dW_s + \int_0^t \frac{\partial f}{\partial s}(X_1, X_s, s)\, ds \\
& \quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(X_1, X_s, s)\, ds \\
& = \int_0^t \alpha_s(X)\, dX_s - \frac{1}{2}\int_0^t \alpha_s^2(X)\, ds + \frac{1}{2}\int_0^t \frac{1}{1-s}\, ds.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& -\int_0^t \alpha_s(X)\, dW_s - \frac{1}{2}\int_0^t \alpha_s^2(X)\, ds \\
& = -f(X_1, 0, 0) + f(X_1, X_t, t) + \frac{1}{2}\ln(1-t) \\
& = -\frac{X_1^2}{2} + \frac{(X_1 - X_t)^2}{2(1-t)} + \frac{1}{2}\ln(1-t),
\end{aligned}
$$

and so we have (8). To prove $E(M_t) = 1$, just note that

$$
\begin{aligned}
& E(p_1(0, X_1)\, p_{1-t}(X_t, X_1)^{-1}) \\
& = \int_{\mathbf{R}^2} p_1(0, y)\, p_{1-t}(x, y)^{-1}\, p_{1-t}(x, y)\, p_t(0, x)\, dx\, dy \\
& = 1.
\end{aligned}
$$

This implies that $(M_t)_{0 \leq t < 1}$ is a martingale. By the martingale convergence theorem, M_t converges a.s. to an integrable random variable M_1 as $t \to 1$. To identify the limit, we note that, for a.a. ω, there is a sequence $(t_n)_{n \in \mathbf{N}}$ converging to 1 such that $X_1(\omega) = X_{t_n}(\omega)$. Then

$$
\lim_n M_{t_n}(\omega) = \exp\left(-\frac{X_1^2}{2}\right)\sqrt{1 - t_n} = 0,
$$

and so we have $M_1 = 0$, P-a.s. In particular, this implies that $(M_t)_{0 \leq t < 1}$ is not uniformly integrable.

Remark. — Note that the martingale $(M_t)_{0 \leq t < 1}$ is of the form

$$
M_t = h^{X_1}(X_t, t)^{-1}, \tag{11}
$$

where the functions

$$
h^y(x, t) = \frac{p_{1-t}(x, y)}{p_1(0, y)}, \qquad y \in \mathbf{R}, \tag{12}
$$

are the extreme points in the convex set of non-negative space-time harmonic functions $h(x, t)$ with $h(0, 0)=1$. We recall that the distribution P^y of the Brownian bridge with initial point 0 and terminal point y at time 1 is equivalent to P on each σ-field $\mathscr{F}_t$ with $0 \leqq t < 1$, and that

$$(13) \qquad \left.\frac{dP^y}{dP}\right|_{\mathscr{F}_t} = h^y(X_t, t), \qquad 0 \leqq t < 1.$$

Now suppose that the Girsanov transformation works in the usual manner, *i.e.*, that there exists a probability measure Q on the same space $\Omega = C([0, 1])$ such that

$$(14) \qquad \left.\frac{dQ}{dP}\right|_{\mathscr{G}_t} = M_t, \qquad 0 \leqq t < 1.$$

Then $(X_t)_{0 \leqq t < 1}$ would be a Wiener process under Q with respect to $(\mathscr{G}_t)_{0 \leqq t < 1}$, and this would extend to the terminal time $t=1$. But $\mathscr{G}_1 = \mathscr{F}_1$, and so Q may be viewed as the distribution of $(X_t)_{0 \leqq t \leqq 1}$. This would imply that Q is identical to Wiener measure P, in obvious contradiction to (14). In fact, (10) would imply that Q is singular to P.

Here is the solution of this apparent paradox. It is true that the martingale $(M_t)_{0 \leqq t < 1}$, resp. the supermartingale $(M_t)_{0 \leqq t < 1}$ with $M_1 = 0$, determines a probability measure. But, as in the general situation of [4], this measure will live on a suitable projective limit space. In our specific situation we can give an explicit description. In fact, the measurable space $(\Omega, \mathscr{G}_t)$ can be identified with the space

$$\bar{\Omega} = C([0, 1)) \times \mathbf{R},$$

endowed with its product σ-field $\bar{\mathscr{G}}_t = \mathscr{F}_t \otimes \mathscr{B}$. Thus, the projective limit space is given by $\bar{\Omega}$ endowed with the σ-field

$$\bar{\mathscr{G}} = \mathscr{F} \otimes \mathscr{B}.$$

We identify Wiener measure with the joint distribution of (X, X_1) under P, *i.e.*, with the measure $\bar{P}$ on $\bar{\Omega}$ defined by

$$\bar{P}(A \times B) = \int_B P^y(A)\, \mu(dy),$$

where μ denotes the standard normal distribution $N(0, 1)$. Thus, $\bar{P}$ couples the two coordinates by identifying the second one, namely y, as the endpoint $X_1(\omega) = \lim_{t \uparrow 1} X_t(\omega)$ of the first one. The density in (8) can be identified with

$$\bar{M}_t(\omega, y) = \frac{p_1(0, y)}{p_{1-t}(X_t(\omega), y)} = h^y(X_t(\omega), t)^{-1}.$$

Let us now define $\bar{Q}$ on $\bar{\Omega}$ as the product measure $\bar{Q}=P\otimes\mu$. Under $\bar{Q}$, the two coordinates are completely decoupled. Since

$$\left.\frac{dP}{dP^y}\right|_{\mathscr{F}_t}=\bar{M}_t(.,y), \qquad y\in\mathbf{R}$$

due to (13), we have

$$\begin{aligned}\bar{Q}(A_t\times B)&=\int_B P(A_t)\,\mu(dy)\\&=\int_B E^y(\bar{M}_t(.,y);A_t)\,\mu(dy)\\&=\bar{E}(\bar{M}_t;A_t\times B)\end{aligned}$$

for $A_t\in\mathscr{F}_t$ and $B\in\mathscr{B}$, and so

$$\left.\frac{d\bar{Q}}{d\bar{P}}\right|_{\mathscr{G}_t}=\bar{M}_t. \tag{15}$$

In other words, the probability measure induced by the martingale $(M_t)_{0\leqq t<1}$ under P, or rather by $(\bar{M}_t)_{0\leqq t<1}$ under $\bar{P}$, is given by $\bar{Q}$. For the diagonal

$$D=\{\bar{\omega}=(\omega,y):\lim_{t\uparrow 1}X_t(\omega)=y\}$$

in $\bar{\Omega}$ we have

$$\bar{P}(D)=\bar{Q}(D^c)=1,$$

and so $\bar{Q}$ is singular to P:

$$\bar{Q}\perp\bar{P}. \tag{16}$$

On the other hand, the projections Q and P of $\bar{Q}$ and $\bar{P}$ on the first coordinate both coincide with Wiener measure P:

$$Q=P. \tag{17}$$

This explains the two sides of the "paradox".

We will now investigate the structure of the measure $\bar{Q}$ in a more general setting.

3. ENLARGEMENT BY A RANDOM VARIABLE

Let G be a random variable on $(\Omega,\mathscr{F})$ with distribution P_G. G will take the role that X_1 was playing in section 2. Correspondingly, we take

$$\mathscr{G}_t=\mathscr{F}_t\vee\sigma(G), \qquad 0\leqq t<1.$$

In general, X need not be a semimartingale under P in this new filtration (*see* for example [7]). Thus, we need additional regularity assumptions on G. Let

$$(y, \mathrm{A}) \mapsto \mathrm{P}^y(\mathrm{A}), \qquad y \in \mathbf{R}, \quad \mathrm{A} \in \mathscr{F}$$

be a regular conditional probability distribution of X given G. We assume that

$$(18) \qquad \mathrm{P}^y \ll \mathrm{P} \text{ on } \mathscr{F}_t \text{ for } \mathrm{P_G}\text{-a.e. } y \in \mathbf{R}, \qquad 0 \leqq t < 1.$$

Then we can choose a measurable function $\alpha_t^y(\omega)$ on $\mathbf{R} \times [0, 1] \times \Omega$ such that, for any $y \in \mathbf{R}$,

(i) α^y is an adapted process with

$$(19) \qquad \int_0^t (\alpha_s^y)^2\, ds < \infty \ \mathrm{P}^y\text{-a.s.}, \qquad 0 \leqq t < 1,$$

(ii) the martingale

$$(20) \qquad \mathrm{Z}_t^y = \frac{d\mathrm{P}^y}{d\mathrm{P}}\bigg|_{\mathscr{F}_t}, \qquad 0 \leqq t < 1,$$

is given by

$$(21) \qquad \mathrm{Z}_t^y = \exp\left[\int_0^t a_s^y\, d\mathrm{X}_s - \frac{1}{2}\int_0^t (\alpha_s^y)^2\, ds\right] \quad \text{on } \{\mathrm{Z}_t^y > 0\}$$

(*see* appendix).

Consider the process α^{G} defined by

$$\alpha_t^{\mathrm{G}}(\omega) = \alpha_t^{\mathrm{G}(\omega)}(\omega), \qquad 0 \leqq t < 1, \quad \omega \in \Omega.$$

Note that α^{G} is adapted with respect to $(\mathscr{G}_t)_{0 \leqq t < 1}$ and satisfies

$$(22) \qquad \int_0^t (\alpha_s^{\mathrm{G}})^2\, ds < \infty$$

P-a.s., for $0 \leqq t < 1$. Now X decomposes in the following manner.

THEOREM 1. — *Under condition* (18), X *is a semimartingale with respect to* $(\mathscr{G}_t)_{0 \leqq t < 1}$ *and* P. *Its decomposition is given by*

$$(23) \qquad \mathrm{X} = \mathrm{W} + \mathrm{A},$$

where

$$(24) \qquad \mathrm{A}_t = \int_0^t \alpha_s^{\mathrm{G}}\, ds, \qquad 0 \leqq t < 1,$$

and W *is a Wiener process with respect to* $(\mathscr{G}_t)_{0 \leqq t < 1}$ *and* P.

Proof. — Under each measure P^y, the process

$$W = X - \int_0^{\cdot} \alpha_s^G \, ds$$

is of the form

$$W = X - \int_0^{\cdot} \alpha_s^y \, ds,$$

and so W is a Wiener process with respect to $(\mathcal{F}_t)_{0 \leqq t < 1}$ and P^y. Choose $0 \leqq s < t < 1$. For $A \in \mathcal{F}_s$ and $B \in \mathcal{B}$,

$$\begin{aligned} & E(W_t - W_s;\, A_s \cap \{G \in B\} \\ & = \int_B E^y(W_t - W_s;\, A_s) \, P_G(dy) \\ & = 0, \end{aligned}$$

and so W is martingale with respect to $(\mathcal{G}_t)_{0 \leqq t < 1}$ and P. Since the pathwise quadratic variation of W is given by

$$[W]_t = [X]_t = t, \qquad t \in [0, 1], \quad P-\text{a. s.},$$

Lévy's theorem implies that W is a Wiener process with respect to $(\mathcal{G}_t)_{0 \leqq t < 1}$ and P.

Remark. — Theorem 1 is a variant of well-known results on the enlargement of filtration (*see* Jacod [6], and also Jeulin [7], Jeulin, Yor [8], Jacod [5]). Jacod [6] formulates conditions in terms of a regular conditional probability distribution

$$(\omega, B) \mapsto P_t^{\omega}(B)$$

of G given X on $\mathcal{F}_t$, $0 \leqq t < 1$. He shows that if

$$(25) \qquad P_t^{\omega} \ll P_G \quad \text{for } P-\text{a. e. } \omega, \qquad 0 \leqq t < 1,$$

then W is a semimartingale with respect to $(\mathcal{G}_t)_{0 \leqq t < 1}$. It is easy to see that both (18) and (25) are equivalent to the condition that the joint distribution $P_{(X, G)}$ of X and G is absolutely continuous with respect to the product measure $P \otimes P_G$ on $\mathcal{F}_t \otimes \mathcal{B}$:

$$(26) \qquad P_{(X, G)} \ll P \otimes P_G \quad \text{on } \mathcal{F}_t \otimes \mathcal{B}, \quad 0 \leqq t < 1.$$

For our purposes, it is convenient to use version (18).

Following the lines of section 2, let us now try to eliminate the drift appearing in theorem 1 by means of a Girsanov transformation. We first consider the case where

$$(27) \qquad P^y \sim P \quad \text{on } \mathcal{F}_t, \text{ for } P_G-\text{a. e. } y, \qquad 0 \leqq t < 1.$$

In that case, the process

$$M^y = (Z^y)^{-1}\, 1_{\{Z^y > 0\}} \tag{28}$$

is a martingale with respect to $(\mathscr{F}_t)_{0 \leq t < 1}$ and P^y for P_G − a. e. y. As in the proof of theorem 1, this implies that the process M^G defined by

$$M^G(\omega) = M^{G(\omega)}(\omega)$$

is a martingale with respect to $(\mathscr{G}_t)_{0 \leq t < 1}$ and P. Under the measure

$$M_t^G . P,$$

the σ-fields $\mathscr{F}_t$ and $\sigma(G)$ become independent. To see this, fix $0 \leq t < 1$, and choose a set A in $\mathscr{G}_t$, say of the form

$$A = A_t \cap \{G \in B\} \quad \text{with } A_t \in \mathscr{F}_t, \qquad B \in \mathscr{B}.$$

Then we have

$$\begin{aligned}\int_A (M_t^G)\, dP &= \int_{A_t} (M_t^G)\, 1_B(G)\, dP \\ &= \int_B \left(\int_{A_t} M_t^y\, dP^y \right) dP_G \\ &= \int_B \left(\int_{A_t} M_t^y (M_t^y)^{-1}\, dP \right) dP_G \\ &= P(A_t) . P_G(B).\end{aligned} \tag{29}$$

Thus, the martingale M^G induces a decoupling between G and the behaviour of X up to each time $t < 1$.

Let us now describe the new probability measure determined by M^G. As in section 2, we imbed our initial space $(\Omega, \mathscr{F})$ into the product space

$$\bar{\Omega} = C([0, 1)) \times \mathbf{R}, \qquad \bar{\mathscr{G}} = \mathscr{F} \otimes \mathscr{B}.$$

We identify Wiener measure P with the joint distribution $\bar{P} = P_{(X, G)}$ of X and G under P:

$$\bar{P}(A \times B) = \int_B P^y(A)\, P_G(dy), \qquad A \in \mathscr{F}, \quad B \in \mathscr{B}.$$

If $\xi : (y, \omega) \mapsto \xi^y(\omega)$ is a random variable on $\bar{\Omega}$, the distribution of ξ under $\bar{P}$ is the same as the distribution of ξ^G under P, due to the fact that $\bar{P}$ is concentrated on

$$D_G = \{(\omega, y) : G(\omega) = y\}.$$

In particular, we define the processes

$$\begin{aligned}\bar{M}((\omega, y)) &= M^y(\omega), \\ \bar{\alpha}((\omega, y)) &= \alpha^y(\omega), (\omega, y) \in \bar{\Omega},\end{aligned}$$

and the filtrations

$$\bar{\mathscr{F}}_t=\mathscr{F}_t\otimes\{\phi, \mathbf{R}\}, \qquad \bar{\mathscr{G}}_t=\mathscr{F}_t\otimes\mathscr{B}, \quad 0\leqq t\leqq 1.$$

For the extended processes, this means for example

$$\bar{\mathrm{E}}(\bar{\mathrm{M}}_t\,|\,\bar{\mathscr{G}}_s)=\mathrm{E}(\mathrm{M}_t^{\mathrm{G}}\,|\,\mathscr{G}_s), \qquad 0\leqq s\leqq t\leqq 1.$$

In particular, $\bar{\mathrm{M}}$ is a martingale with respect to $(\bar{\mathscr{G}}_t)_{0\leqq t<1}$ and $\bar{\mathrm{P}}$. The process X will be considered as a process on $\bar{\Omega}$ depending only on the first coordinate. Thus,

$$\bar{\mathscr{F}}=\bar{\mathscr{F}}_1=\sigma(\mathrm{X})$$

is the σ-algebra generated by X.

THEOREM 2. – *Under condition* (27), *the measure determined by the martingale* M^{G} *resp.* $\bar{\mathrm{M}}$ *is given by the product measure*

$$\bar{\mathrm{Q}}=\mathrm{P}\otimes\mathrm{P}_{\mathrm{G}} \tag{30}$$

on $(\bar{\Omega}, \bar{\mathscr{G}})$, i.e.,

$$\left.\frac{d\bar{\mathrm{Q}}}{d\bar{\mathrm{P}}}\right|_{\bar{\mathscr{G}}_t}=\bar{\mathrm{M}}_t, \qquad 0\leqq t<1. \tag{31}$$

In particular, we have

$$\bar{\mathrm{Q}}=\bar{\mathrm{P}} \quad \text{on } \bar{\mathscr{F}}, \tag{32}$$

i. e.,

$$\bar{\mathrm{E}}(\bar{\mathrm{M}}_t\,|\,\bar{\mathscr{F}}_t)=1, \qquad 0\leqq t<1. \tag{33}$$

Proof. – Both statements follow from the decoupling in equation (29).

In the case under discussion, the relationship between $\bar{\mathrm{P}}$ and $\bar{\mathrm{Q}}$ is analogous to the case $\mathrm{G}=\mathrm{W}_1$ at the end of section 2. If G has no atoms, $\bar{\mathrm{P}}$ and $\bar{\mathrm{Q}}$ are orthogonal. If G is not a. s. constant, $\bar{\mathrm{Q}}$ is not absolutely continuous with respect to $\bar{\mathrm{P}}$. This will be proved under the more general condition (18) after theorem 3.

This concludes the discussion under the special condition (27). Let us now consider the general situation (18). In this case, the process M^y defined in (28) is only a local martingale with respect to $(\mathscr{F}_t)_{0\leqq t<1}$ and P^y, and we have

$$\mathrm{E}^y(\mathrm{M}_t^y;\,\mathrm{A})=\mathrm{P}(\mathrm{A}\cap\{\mathrm{Z}_t^y>0\}) \tag{34}$$

for $\mathrm{A}\in\mathscr{F}_t$ and $0\leqq t<1$. A sequence of localizing stopping times is defined by

$$\mathrm{T}_n^y=\inf\{t\in[0,\,1):\mathrm{M}_t^y>n\}\wedge 1.$$

We set

$$\bar{\mathrm{T}}_n(\omega,\,y)=\mathrm{T}_n^y(\omega)$$

and introduce the life time

$$\zeta = \lim_{n \to \infty} \bar{T}_n$$

and the σ-field

$$\bar{\mathscr{G}}_{\zeta-} = \bigvee_{n \in \mathbb{N}} \bar{\mathscr{G}}_{\bar{T}_n}.$$

THEOREM 3. — *The measure associated with the local martingale* $\bar{M}$ *is given by the restriction of the product measure*

$$\bar{Q} = P \otimes P_G$$

to the σ-field $\bar{\mathscr{G}}_{\zeta-}$. *More precisely, we have*

$$\bar{Q}(\bar{A} \cap \{\zeta > t\}) = \bar{E}(\bar{M}_t; \bar{A}) \tag{35}$$

for $\bar{A} \in \bar{\mathscr{G}}_t$ *and* $0 \leqq t < 1$.

Proof. — Fix $0 \leqq t < 1$. have

$$\begin{aligned} &\bar{E}(\bar{M}_t; \bar{A}) \\ &= \int E^y(M_t^y; \bar{A}(., y)) P_G(dy) \\ &= \int P(\bar{A}(., y) \cap \{Z_t^y > 0\}) P_G(dy). \end{aligned}$$

But

$$\{Z_t^y > 0\} = \{\lim_{n \to \infty} T_n^y > t\},$$

since the non-negative martingale Z^y will not become positive after reaching 0. Thus,

$$\begin{aligned} &\bar{E}[\bar{M}_t; \bar{A}) \\ &= \int P(\bar{A}(., y) \cap \{\zeta(., y) > t\}) P_G(dy) \\ &= \bar{Q}[\bar{A} \cap \{\zeta > t\}). \end{aligned}$$

This completes the proof.

Remark. — For a non-negative supermartingale parametrized by [0, 1) on a space $\bar{\Omega}$ which is closed under certain projective limits, there is a unique associated measure on the predictable σ-algebra on $\bar{\Omega} \times (0, 1]$. For a local martingale, this measure is supported by the graph of a suitable life time ζ, and so it may be viewed as a measure on the underlying space $\bar{\Omega}$, equipped with the σ-field of events observable before time ζ; *see* [3] and [4]. In our special case, we have provided a direct construction on $\bar{\Omega}$, without first passing to a product space $\bar{\Omega} \times (0, 1]$. As an alternative, we could have used the general results in [1].

COROLLARY 1. — *Suppose that* (18) *is satisfied. If* G *is not* P$-$*a.s. constant, then* $\bar{Q}$ *is not absolutely continuous with respect to* $\bar{P}$*:*

$$\bar{Q} \not\ll \bar{P}. \tag{36}$$

If P_G *has no atoms, then* $\bar{Q}$ *is singular with respect to* $\bar{P}$*:*

$$\bar{Q} \perp \bar{P}. \tag{37}$$

Proof. — Using the localizing sequence of stopping times introduced before theorem 3, we define the $(\bar{\mathscr{G}}_{\bar{T}_n})_{n \in N}$-martingale

$$\bar{M}_n(\omega, y) = \bar{M}_{\bar{T}_n(\omega, y)}(\omega, y), \qquad n \in N, \quad (\omega, y) \in \Omega \times \mathbf{R}.$$

Since for $y \in \mathbf{R}$, absolute continuity of P with respect to P^y is equivalent with saying that $P(A(., y)) = 1$ for

$$A = \{ \lim_{n \to \infty} \bar{M}_n < \infty \},$$

we know that

$$\text{if } P \not\ll P^y, \quad \text{then } P(A^c(., y)) \neq 0, \tag{38}$$
$$\text{if } P \perp P^y, \quad \text{then } P((A^c(., y)) = 1. \tag{39}$$

Assume first that we have

$$P(G \neq \text{Const.}) \neq 1.$$

Then $P \not\ll P^y$ for any $y \in \mathbf{R}$. Hence (38) tells us that

$$\bar{Q}(A^c) = \int P(A^c(., y))\, P_G(dy) \neq 0,$$

whereas

$$\bar{P}(A) = \int P^y(A(., y))\, P_G(dy) = 1.$$

Hence in this case

$$\bar{Q} \not\ll \bar{P},$$

as asserted. Next assume that P_G has no atoms. Then even $P \perp P^y$ for any $y \in \mathbf{R}$. Hence by (39)

$$\bar{Q}(A^c) = 1$$

and hence

$$\bar{Q} \perp \bar{P}.$$

Let us finally compare conditions (18) and (27) by briefly discussing an example.

Example. — Let $\tau = \inf\{s \in [0, 1] : W_s = -1\} \wedge 1$, $A = \{\tau = 1\}$. Take $G = 1_A$, and let us consider

$$P^1 = P(.\,|\,G=1) = P(.\,|\,A), \qquad P^0 = P(.\,|\,G=0) = P(.\,|\,A^c).$$

If we set $\alpha = P(G=0)$, $\beta = P(G=1)$, we clearly have

$$P = \alpha P^0 + \beta P^1$$

and therefore

$$1 = \alpha M^0 + \beta M^1 \tag{40}$$

for the respective martingales. To compute M^1, we use the Markov property of X to derive for $0 \leqq t < 1$

$$M_t^1 = \left.\frac{dP^1}{dP}\right|_{\mathscr{F}_t} = \left.\frac{1}{\beta} 1_A\right|_{\mathscr{F}_t} = 1_{\{\tau \geqq t\}} \frac{P_{W_t}(\tau \geqq 1-t)}{\beta}, \tag{41}$$

where P_x denotes Wiener measure starting in x. From this equation, it is immediate that

$$\zeta = \inf\{s \in [0, 1] : M_s^1 = 0\} = \tau.$$

Moreover, (40) and (41) yield

$$\zeta(., 0) = 1.$$

This implies

$$\begin{aligned} \zeta(\omega, 0) = \tau(\omega) &\quad \text{on } A \times \{0\}, \\ \zeta(\omega, 1) = \tau(\omega) &\quad \text{on } A^c \times \{1\}, \\ \zeta = 1 &\quad \text{on } A \times \{1\} \cup A^c \times \{0\}. \end{aligned}$$

Thus $\bar{\mathscr{G}}_{\zeta-}$ is given by $\mathscr{F}$ in the first coordinate on the set $A \times \{1\} \cup A^c \times \{0\}$, and by $\mathscr{F}_{\tau-}$ on the complement. The measure $\bar{Q}$ is given by the restriction of $(\alpha\delta_{\{0\}} + \beta\delta_{\{1\}}) \otimes P$ to $\bar{\mathscr{G}}_{\zeta-}$. Hence τ acts as explosion time on the set $A \times \{0\} \cup A^c \times \{1\}$, on which $\bar{Q} \perp \bar{P}$. There is no explosion on the complement, since there we have $\bar{Q} \sim \bar{P}$.

4. APPENDIX

Under condition (18) we will now justify the properties of the exponential martingale Z given by formula (21) which were claimed in section 3. The main problem we have to face lies in the fact that completion of the natural filtration of X by the nullsets of $\mathscr{F}$ may disturb the local absolute continuity condition (18); for example, Brownian bridges live on nullsets of P. For this reason, we choose the following variant of completion. For $0 \leqq t < 1$ let $\mathscr{N}_t$ be the class of null sets of the right regularization of $\sigma(X_s : s \leqq t)$, and suppose that each $\mathscr{F}_t$ is completed by the system of countable unions of sets in $\bigcup_{0 \leqq t < 1} \mathcal{N}_t$. Thus, the σ-algebra property carries over to the completions, and, more importantly, (18) is inherited from the natural filtration of X, thus not affecting generality. This choice has one advantage: it restores the "usual conditions" on any subinterval $[0, t]$ with $t<1$, if we take $\sigma(X_s : s \leqq t)$ as the global σ-algebra. We therefore are able to just quote the results needed from the general theory of processes. With a little more effort, we could work quite as well in the right regularization of the natural filtration.

PROPOSITION 1. — *Let* (18) *be satisfied. Then there is a measurable function*

$$\mathbb{R} \times [0, 1[\times \Omega \to \mathbb{R}, \qquad (y, t, \omega) \mapsto Z_t^y(\omega),$$

which is non-negative and continuous in t such that the following is true

(i) *for any* $y \in \mathbb{R}$, Z^y *is a martingale with respect to* $((\mathcal{F}_t)_{0 \leqq t < 1}, P)$,

(ii) *for any* $0 \leqq t < 1$, $Z_t^y = \left.\dfrac{dP^y}{dP}\right|_{\mathcal{F}_t}$ for $P_G - a.e.\ y \in \mathbb{R}$,

Proof. — Fix $0 \leqq t_0 < 1$. Due to our choice of filtration it is obviously enough to prove corresponding statements on the space $\mathbf{R} \times [0, t_0] \times \Omega$, with respect to the filtration $(\mathcal{H}_t)_{0 \leqq t \leqq t_0}$, which is defined to be the natural filtration of X on $[0, t_0]$, modified so as to fulfill the "usual conditions" with respect to the nullsets of $\sigma(X_s : s \leqq t_0)$. First of all, these conditions guarantee that for any $y \in \mathbf{R}$ we may choose a non-negative continuous martingale N^y such that

$$N_t^y = \left.\frac{dP^y}{dP}\right|_{\mathcal{H}_t}, \qquad 0 \leqq t \leqq t_0.$$

Next fix a rational $q \in [0, t_0]$. Due to Stricker, Yor [13] there exists a $\mathcal{H}_q \otimes \mathcal{B}$-measurable function

$$(\omega, y) \mapsto Q_q^y(\omega)$$

such that

$$N_q = Q_q \quad P_G \otimes P - a.s.\,.$$

Note that we use (18) here. More precisely, there exists a set $B_q \in \mathcal{H}_q \otimes \mathcal{B}$ such that

$$P_G \otimes P(B_q) = 0 \qquad \text{and} \qquad B_q \supset \{(\omega, y) : N_q^y(\omega) \neq Q_q^y(\omega)\}.$$

Let

$$A_q = \bigcup_{\mathbf{Q} \ni p \leqq q} B_p.$$

Then also

$$P_G \otimes P(A_q) = 0,$$

and $A_q \in \mathscr{H}_q \otimes \mathscr{B}$ is increasing in q. Let now

$$S^y(\omega) = \inf\{0 \leqq q \leqq t_0 : q \in \mathbf{Q}, (\omega, y) \in A_q\} \wedge t_0.$$

Then

$$(\omega, y) \mapsto S^y(\omega)$$

is measurable and for any $y \in \mathbf{R}$ S^y is a stopping time with respect to $(\mathscr{H}_t)_{0 \leqq t \leqq t_0}$. With these prerequisites we may now define M, first on the rationals, then on the reals. For rational $q \in [0, t_0]$ let

$$M_q^y(\omega) = \begin{cases} Q_q^y(\omega) & \text{if } q \leqq S^y(\omega), \\ \lim\limits_{\mathbf{Q} \ni p \uparrow S^y(\omega)} Q_q^y(\omega), & \text{if } q > S^y(\omega). \end{cases}$$

Note that for any $(\omega, y) \in \Omega \times \mathbf{R}$, the functions $M^y(\omega)$ and $N^y(\omega)$ agree on the interval $[0, S^y(\omega)]$. We may therefore continue with our definition. For $0 \leqq t \leqq t_0$ let

$$M_t^y(\omega) = \lim_{\mathbf{Q} \ni q \downarrow t} M_q^y(\omega), \qquad (\omega, y) \in \Omega \times \mathbf{R}.$$

We thus obviously obtain a function which is measurable in its three variables, which is non-negative and continuous in t and which fulfills (i). To prove (ii), it is enough to remark that $S^y = 1$ P-a.s. for $P_G - a.e.\, y \in \mathbf{R}$, as follows from Fubinis theorem.

In the following proprosition, Z is described as an exponential martingale.

PROPOSITION 2. — *Let* (18) *be satisfied, and let* Z *be given by proposition* 1. *Then there exists a measurable function*

$$\mathbb{R} \times [0, 1] \times \Omega \to \mathbb{R}, \qquad (y, t, \omega) \mapsto \alpha_t^y(\omega)$$

which is adapted in (t, ω) *such that for* $y \in \mathbf{R}$ *the following is true*

(i) $\displaystyle\int_0^t (\alpha_s^y)^2\, ds < \infty$ P^y-a. s., $0 \leqq t < 1$,

(ii) $\displaystyle M_t^y = \exp\left(\int_0^t \alpha^y\, dX - \frac{1}{2}\int_0^t (\alpha^y)^2\, d\lambda\right)$ on $\{Z_t^y > 0\}, 0 \leqq t < 1$.

Proof. — We may fix $0 \leqq t_0 < 1$ again and work on the interval $[0, t_0]$ as in the preceding proof. In the same way as before we may construct a measurable function

$$(y, t, \omega) \mapsto A_t^y(\omega)$$

which is adapted in (t, ω) and fulfills

(iii) $A_t^y = [Z^y, X]_t$, $0 \leqq t \leqq t_0$, $y \in \mathbf{R}$,

(iv) the measure induced by $A^y(\omega)$ on the Borel sets of $[0, t_0]$ is absolutely continuous with respect to Lebesgue measure for all $(\omega, y) \in \Omega \times \mathbf{R}$. Here $[X, Y]$ stands for the mutual variation of the processes X and Y. We may therefore choose a measurable function

$$(y, t, \omega) \mapsto \gamma_t^y(\omega)$$

which is adapted in (t, ω) such that for any $y \in \mathbf{R}$

$$\gamma^y = \frac{dA_\cdot^y}{d\lambda} = \frac{d[Z^y, X]}{d\lambda}. \tag{42}$$

Now set

$$\alpha^y(\omega) = (Z^y)^{-1}\gamma^y(\omega)\,1_{\{Z^y > 0\}}, \qquad (\omega, y) \in \Omega \times \mathbf{R}. \tag{43}$$

Observe that for $0 \leqq t \leqq t_0$ we have $Z_t^y > 0$ $P^y - a.s.$ to conclude that (i) holds. Finally, to obtain (ii) from (42) and (43), it is enough to argue in the usual way (*see* Rogers, Williams [11], p. 81).

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