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INSIDER TRADING IN A CONTINUOUS TIME MARKET MODEL

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This paper uses the enlargement of Brownian filtrations and a probability change for modelling the observation of a financial market by an insider trader. A characterization of admissible strategies and a criterion for optimization are given. Then a statistical test is proposed to test whether or not the trader is an insider.

1. Introduction

This paper deals with the problem of the insider trading: some financial agent knows something about the future. Thus, a market model is built on a filtered probability space $(\Omega, (\mathcal{F}_t, t \in [0,T]), \mathbb{P})$, the prices of assets being given by the equation:

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i (\sigma_s^i, dW_s), \quad 0 \le t \le T, \quad S_0 \in \mathbb{R}^d, i = 1, \ldots, d\,,$$

where W is d-dimensional Brownian motion and $(.,.)$ denotes the scalar product in $\mathbb{R}^d$.

From the beginning, $t = 0$, the investor knows a random variable $L \in L^1(\Omega, \mathcal{F}_T; \mathbb{R}^\kappa), \kappa \in \mathbb{N}^*$, (for instance, he knows that some trading will be done and when it will be done); for two assets of prices S^1 et S^2, the random variable could be their ratio at time T: $L = \ln S_T^1 - \ln S_T^2$. The "natural" filtration known by the insider trader is $\mathcal{F}_t \vee \sigma(L)$. To apply the standard results, we use the associated right continuous filtration, denoted by $\mathcal{Y} : \mathcal{Y}_t = \cap_{s>t}(\mathcal{F}_s \vee \sigma(L)), t \in [0,T]$.

But on the filtered probability space $(\Omega, (\mathcal{Y}_t, t \in [0,T]), \mathbb{P})$, the process W is no longer a semi-martingale. Following Föllmer and Imkeller [5], an equivalent probability measure Q is built such that under Q, for all $t < T$, the σ-algebra $\mathcal{F}_t$ is independent of $\sigma(L)$. Thus W is a $(\mathcal{Y}, Q)$-Brownian motion. Another useful method is the initial enlargement of filtrations, it allows to find some conditions on L so that there exist a $\mathcal{Y}$-Brownian motion B and an increasing process A satisfying $W_t = B_t + A_t$. This was studied when L is a Gaussian random variable by Yor [21], Chaleyat–Maurel and Jeulin [3].

More generally, Jacod [10] did the same when the family of conditional laws $Q_t(\omega, .)$ of L given $\mathcal{F}_t$ is dominated almost surely by a non-random measure; see also Song [20]. The Bouleau–Hirsch [2] results give some simple conditions on L so that these conditional laws are dominated by the Lebesgue measure. With some extra hypotheses, Imkeller [7], specifies the decomposition of the semi-martingale W.

Karatzas and Pikovsky [11, 12] studied similar problems on some examples of real or vectorial random variables: $L = W_1$, $L = (\lambda_i W_1^i + (1-\lambda_i)\varepsilon_i)_{i=1,d}$ with a family of independent Gaussian variables ε, $L = S^1$ the price at time 1, and $L = \mathbb{1}_{\{S^1 < p\}}$. All these cases satisfy our hypothesis HJ, so our machinery runs. An interesting point of their paper is the consideration of an optimization problem with constraints on the portfolios. But, our contribution, moreover our relatively general condition HJ, is the statistical test on the hypothesis if the trader is an insider trader or not. Let us also quote Back [1] and Kyle [13].

The market model notations and hypotheses are given in Sec. 2. In Sec. 3, the insider trader wealth equation is justified on the new probability space $(\Omega, \mathcal{Y}, Q)$. The admissible strategies are characterized in Sec. 4 and an insider trader optimal strategy is produced on $[0, A]$ when $A < T$, but some uncertainty remains on the price assets between times A and T, (cf. [11]).

When $A = T$, a risk-neutral probability measure is not so easy to find. Perhaps no such measure exists since an arbitrage strategy can be given with trading at times 0 and T on the two assets of prices S^1 and S^2 ($L = \ln \frac{S_T^1}{S_T^2}$): suppose that $S_0^1 = S_0^2 = 1$, then the strategy $\theta^1 = \text{sgn}(L)$; $\theta^2 = -\text{sgn}(L)$ is $\mathcal{Y}_0$-mesurable, and the initial wealth $X_0 = \pi^1 + \pi^2$ is 0. At time T the wealth: $X_T = \theta^1 S_T^1 + \theta^2 S_T^2 = |S_T^1 - S_T^2|$ is strictly positive. So θ is an arbitrage strategy, but it is not necessarily admissible (cf. Dana and Jeanblanc-Piqué [4]) because $X_t = \theta^1 S_t^1 + \theta^2 S_t^2$ could be non positive on a non-negligible set of $[0,T] \times \Omega$.

In Sec. 5 we show other general examples, and finally, in Sec. 6 a statistical test is proposed: the hypothesis H_0 is "the financial agent is not an insider trader"; the alternative hypothesis H_1 is "the financial agent is an insider trader".

2. Market Model

Let W be a d-dimensional Brownian motion, let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0;T]), \mathbb{P})$ a filtered probability space, with $\Omega = C([0,T]; \mathbb{R}^d)$. Consider a financial market with

d assets, where prices are described by

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i (\sigma_s^i, dW_s), \quad 0 \le t \le T\,, \tag{1}$$

and the bond evolves according to the equation: $S_t^0 = 1 + \int_0^t S_s^0 r_s ds$.

The parameters b, σ, r are supposed to be bounded on $[0, T]$, to be adapted, and to take their values in $\mathbb{R}^d, \mathbb{R}^{d\times d}, \mathbb{R}$; the matrix σ_t is invertible $dt \otimes dP$ almost surely and let $\eta_t = \sigma_t^{-1}(b_t - r_t \mathbb{1})$, for $t \in [0; T]$.

We denote, for a probability measure P, **H1(P)** the following hypothesis:

H1(P): η_t verifies

$$\exists A \in]0; T[, \quad \exists C \in \mathbb{R}, \quad \exists k > 0, \quad \forall s \in [0, A], \quad E_P[\exp k\|\eta_s\|^2] \le C\,.$$

This hypothesis is a Novikov's type criterion (cf. [19, p. 323]) to obtain an equivalent probability using the Girsanov's transform.

Remark 2.1. Along this paper, we shall define hypotheses which suppose the existence of a "final" time $A \in]0; T[$. We can suppose, without loss of generality that the final time A is the same in each hypothesis, if not we could take the minimum of the finite number of times.

A financial agent has a positive amount X_0 at time $t = 0$ and he wants to optimize his consumption investment strategy; $(\mathcal{Y}_t = \cap_{s>t} \mathcal{F}_s \vee \sigma(L))_{t \in [0;T]}$ is the filtration of his knowledge. His consumption rate is c, a $\mathcal{Y}$-adapted non-negative process such that $\int_0^T c_s ds < \infty, \mathbb{P}$ *a.s.* He has also θ^i units of the ith asset. His wealth at time t is then $X_t = \sum_{i=0}^d \theta_t^i S_t^i$.

Consider the standard hypothesis:

$$\textbf{H2} \qquad \text{"self-financing"}: \qquad dX_t = \sum_0^d \theta_t^i dS_t^i - c_t dt\,, \tag{2}$$

i.e. the consumption is only financed by profits from portfolio, and not by external endowments (as wages, for instance). Then, the wealth of the insider should satisfy the equation:

$$dX_t = \theta_t^0 S_t^0 r_t dt + \sum_{i=1}^d \theta_t^i S_t^i b_t^i dt + \sum_{i=1}^d \theta_t^i S_t^i (\sigma_t^i, dW_t) - c_t dt\,,$$

with initial wealth X_0 which is a positive and $\mathcal{Y}_0$-measurable variable. So the wealth equation should be an anticipative one, and the next calculus is only formal.

Denote $\pi_t^i = \theta_t^i S_t^i$ the amount invested on the ith asset, for $i = 1, \dots, d$, and remark that $\theta_t^0 S_t^0 = X_t - \sum_1^d \pi_t^i$. Let $\pi = (\pi_t^i, i = 1, \dots, d)$ denote the portfolio. Thus, the wealth is the solution of the following stochastic differential equation:

$$dX_t = (X_t r_t - c_t)dt + (\pi_t, b_t - r_t \mathbb{1})dt + (\pi_t, \sigma_t dW_t), \; X_0 \in L^0(\mathcal{Y}_0)\,.$$

Let $R_t = (S_t^0)^{-1}$ be the discounting factor; then, the discounted wealth satisfies the equation:

$$X_t R_t + \int_0^t R_s c_s ds = X_0 + \int_0^t R_s(\pi_s, b_s - r_s \mathbb{I})ds + \int_0^t R_s(\pi_s, \sigma_s dW_s)\,. \quad (3)$$

By an **admissible** consumption investment strategy (π, c) we mean a process such that π is $\mathcal{Y}$-predictable and c is $\mathcal{Y}$-adapted, $c \geq 0, \int_0^T c_s ds < \infty$ and $\sigma^* \pi$ is in $L^2[0;T]$ $\mathbb{P}$-almost surely; and so that the wealth $X^{\pi,c}$ associated to this strategy is non-negative $dt \otimes d\mathbb{P}$ almost surely.

If the process (π, c) is $\mathcal{Y}$-adapted, the stochastic integral in the last equation is anticipating. To give a meaning to Eq. (3), our main hypothesis will be:

H3: There exist $A \in]0;T[$ and a probability measure Q equivalent to $\mathbb{P}$ on the σ-algebra $\mathcal{F}_A \vee \sigma(L)$ such that under Q, for any $t \leq A$, the σ-algebras $\mathcal{F}_t$ and $\sigma(L)$ are independent.

3. Cancellation of the Anticipation

In this section, we give a set of hypotheses which allows us to get a classical framework for the insider trader wealth equation.

Lemma 3.1 (T. Jeulin). *Assume*:
HJ: *There exist $A \in]0;T[$ and a $\mathcal{F}_A \otimes \mathcal{B}(\mathbb{R}^\kappa)$-measurable function $q(A,.)$ such that for any bounded borelian function f on $\mathbb{R}^\kappa$:*
$E_{\mathbb{P}}[f(L)/\mathcal{F}_A] = \int_{\mathbb{R}^\kappa} f(x) q(A, \omega, x) \mathbb{P}_L(dx)$, and $q(A, \omega, x) > 0$ $\mathbb{P} \otimes \mathbb{P}_L$ almost surely, where $\mathbb{P}_L$ is the probability law of L.
Then, if we denote $Q = \frac{1}{q(A,L)} \mathbb{P}$, for any $t \leq A$, $Q_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}$ and the probability measure Q satisfies the hypothesis **H3**.

Proof. Let f a bounded borelian function on $\mathbb{R}^\kappa$ and $B \in \mathcal{F}_A$:

$$\begin{aligned} E_Q[f(L)\mathbb{I}_B] &= E_{\mathbb{P}}\left[\frac{f(L)}{q(A,L)}\mathbb{I}_B\right] \\ &= E_{\mathbb{P}}\left[E_{\mathbb{P}}\left[\frac{f(L)}{q(A,L)}\mathbb{I}_B \Big/ \mathcal{F}_A\right]\right] \\ &= E_{\mathbb{P}}\left[\int_{\mathbb{R}^\kappa} \frac{f(x)}{q(A,x)} q(A,x)\mathbb{P}_L(dx)\mathbb{I}_B\right], \end{aligned}$$

using that $B \in \mathcal{F}_A$ and the definition of Q we obtain

$$E_Q[f(L)\mathbb{I}_B] = E_{\mathbb{P}}[f(L)]\mathbb{P}(B)\,. \quad (4)$$

If $f = 1$, we get $Q(B) = \mathbb{P}(B)$; if $B = \Omega$, we get $E_Q[f(L)] = E_{\mathbb{P}}[f(L)]$. So the lemma is proved. □

Proposition 3.2. *Under the hypothesis* **HJ**, *the process* $(W_t, t \leq A)$ *is a* $(\mathcal{Y}, Q)$*-Brownian motion.*

The first step of the proof is the following lemma.

Lemma 3.3. *Let* $N \in L^\infty(\Omega, \mathcal{F}_A, Q)$*. Then, for any* $t \leq A$*:*

$$E_Q[N/\mathcal{F}_t \vee \sigma(L)] = E_{\mathbb{P}}[N/\mathcal{F}_t]\,.$$

Proof. Let B a $\mathcal{F}_t$-measurable bounded random variable and $f \in L^1(\mathbb{R}^\kappa; \mathbb{P}_L)$, the independence of the σ-algebras $\mathcal{F}_t$ and $\sigma(L)$ and the fact that $Q_{|\mathcal{F}_t} = P_{|\mathcal{F}_t}$ (cf. Lemma 3.1) imply

$$\begin{aligned} E_Q[NBf(L)] &= E_Q[NB]E_Q[f(L)] \\ &= E_Q[E_{\mathbb{P}}[NB/\mathcal{F}_t]]E_Q[f(L)] \\ &= E_Q[E_{\mathbb{P}}[N/\mathcal{F}_t]Bf(L)]\,. \end{aligned}$$

□

Proof of the Proposition. Compute the characteristic function of an increment of W given the past, $\forall u \in \mathbb{R}^d,\ \forall s > 0,\ \forall s' \in]s, t+s[$:

$$E_Q[e^{iu.(W_{t+s}-W_s)}/\mathcal{Y}_s] = E_Q[E_Q[e^{iu.(W_{s+t}-W_{s'}+W_{s'}-W_s)}/\mathcal{F}_{s'} \vee \sigma(L)]/\mathcal{Y}_s]$$

since $\mathcal{Y}_s \subset \mathcal{F}_{s'} \vee \sigma(L)$ for any $s' > s$. Using the lemma and the $\mathcal{F}_{s'}$-measurability of $W_{s'} - W_s$, we get

$$\begin{aligned} E_Q[e^{iu.(W_{t+s}-W_s)}/\mathcal{Y}_s] &= E_Q[e^{iu.(W_{s'}-W_s)}E_{\mathbb{P}}[e^{iu.(W_{s+t}-W_{s'})}/\mathcal{F}_{s'}]/\mathcal{Y}_s] \\ &= e^{-\frac{\|u\|^2}{2(t+s-s')}}E_Q[e^{iu.(W_{s'}-W_s)}/\mathcal{Y}_s]\,. \end{aligned}$$

When s' decreases to s, the result follows from the dominated convergence Lebesgue theorem. □

Then, the discounted wealth is well defined, with a $\mathcal{Y}$-predictable portfolio, under the probability measure Q:

$$X_tR_t + \int_0^t R_sc_sds = X_0 + \int_0^t R_s(\pi_s, b_s - r_s\mathbb{1})ds + \int_0^t R_s(\pi_s, \sigma_sdW_s)\,.$$

Lemma 3.4. *The hypothesis* **H3**: *"There exists* $A \in]0;T[$ *and a probability measure* P' *equivalent to* $\mathbb{P}$ *on* $\mathcal{F}_A \vee \sigma(L)$ *such that for any* $t \leq A$ *and under* P' *the* σ*-algebras* $\mathcal{F}_t$ *and* $\sigma(L)$ *are independent" implies* **HJ**.

Obviously, this lemma and Lemma 3.1 prove the equivalence between **H3** and **HJ**.

Proof. Let $M = \frac{d\mathbb{P}}{dP'}_{|\mathcal{F}_A \vee \sigma(L)}$. So there exists a $\mathcal{F}_A \otimes \mathcal{B}$-integrable strictly positive function g such that $M(\omega) = g(\omega, L(\omega))$. Moreover, under P', the σ-algebras $\mathcal{F}_t$ and $\sigma(L)$ are independent; thus, the probability measure P' is the product $P'_{|\mathcal{F}_A} \otimes P'_{|\sigma(L)}$. Let f be a bounded borelian function, then

$$E_{\mathbb{P}}[f(L)/\mathcal{F}_A] = \frac{E_{P'}[g(.,L)f(L)/\mathcal{F}_A]}{E_{P'}[g(.,L)/\mathcal{F}_A]} = \frac{\int g(.,x)f(x)dP'_L(x)}{\int g(.,x)dP'_L(x)}\,. \tag{5}$$

The strict positivity of $\frac{g(.,x)}{\int g(.,x)dP'_L(x)}$ shows that the conditional law of L given $\mathcal{F}_A$ under $\mathbb{P}$ is equivalent to the law of L under P'. But $\mathbb{P}$ and P' are equivalent, so the law of L under $\mathbb{P}$ is equivalent to the conditional law of L given $\mathcal{F}_A$ under $\mathbb{P}$. □

The hypotheses **HJ** or **H3** are not so easy to verify. But, with the enlargement of filtration techniques, more tractable sufficient conditions can be obtained with the following propositions.

Proposition 3.5 (Jacod [10]). *Assume* **H'**: $\exists A \in]0;T[$ *such that for* $t \leq A$, *the conditional law* Q_t *of* L *given* $\mathcal{F}_t$ *is absolutely continuous with respect to a* σ*-finite measure* ν *on* $\mathbb{R}^\kappa$. *Then*

— for $t \leq A$, *there exists a measurable version of the conditional density* $p(t,x) = \frac{dQ_t}{d\nu}(x)$ *such that for all* $x \in \mathbb{R}^\kappa$, $(\omega, t) \mapsto p(\omega, t, x)$ *is a martingale and can be written as*

$$p(t,x) = p(0,x) + \int_0^t (\alpha(s,x), dW_s)\,;$$

moreover $p(s,L) > 0$ $d\mathbb{P}$ *almost surely, for all* $s \leq A$,

— if M_t *is a* $\mathcal{F}$*-continuous local martingale* $M_0 + \int_0^t (\beta_s, dW_s)$, *for* $t \leq A$, *then the quadratic variation* $d\langle M, p\rangle_t$ *is equal to* $\langle \alpha, \beta\rangle_t dt$ *and the process*

$$\tilde{M}_t = M_t - \int_0^t \frac{\langle \alpha(.,x), \beta\rangle_u|_{x=L}}{p(u,L)}\,du, \quad 0 \leq t \leq A$$

is a $\mathcal{Y}$*-continuous local martingale.*

As a corollary, the vectorial process

$$B_t = W_t - \int_0^t l_u du,\ 0 \leq t \leq A,\ \text{where } l^i_s = \frac{\alpha^i(s,L)}{p(s,L)}, i = 1, \ldots, d\,, \tag{6}$$

is a Brownian motion on the filtered probability space $(\Omega, \mathcal{Y}, \mathbb{P})$.

Proposition 3.6. *Let* **HN** *be the following hypothesis*:

$$\exists A \in]0;T[, \quad \exists k > 0, \quad \exists C, \forall s \in [0,A], E_{\mathbb{P}}[\exp k\|l_s\|^2] < C\,. \tag{7}$$

Assume **H'** *and* **HN**, *then* **H3** *holds.*

Proof. Processes l and B do exist, thanks to hypothesis **H'**. The hypothesis **HN** implies the existence of a $(\mathcal{Y}, \mathbb{P})$-uniformly integrable martingale $M_t^1,\ t \le A$, such that

$$dM_t^1 = -M_t^1(l_t, dB_t), M_0^1 = 1\,,$$

thus there is an equivalent probability measure $\mathbb{P}^1 = M_A^1 \mathbb{P}$ on $\mathcal{Y}_A \supset \mathcal{F}_A \vee \sigma(L)$. Then, $W_t = B_t + \int_0^t l_s ds$ for $t \le A$ is a $(\mathcal{Y}, \mathbb{P}^1)$-Brownian motion. Thus, under $\mathbb{P}^1$, W is independent of $\mathcal{Y}_0$ and so $\mathcal{Y}_0$ and $\mathcal{F}_t$ are independent for any $t \le A$. □

To end this section, we get a sufficient condition for **H'**. Let $\Omega = C([0,T];\mathbb{R}^d)$ and $H = \{h \in \Omega / \dot{h} \in L^2([0;T], \mathbb{R}^d)\}$. Let $w(h) = \int_0^1 (\dot{h}_s, dW_s)$, for $h \in H$, and let $\mathcal{S}$ be the set of real Wiener functionals (cf. P. Malliavin [14] or D. Nualart [16]):

$$\mathcal{S} = \{F \in L^2(\Omega) / \exists n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), \text{ such that}$$
$$F = f(w(h^1), \ldots, w(h^n)), \text{ with } h^1, \ldots, h^n \in H\}\,.$$

Let $F \in \mathcal{S}$ and $DF \in L^2(\Omega \times [0;T]; \mathbb{R}^d)$ be defined by

$$D_t F = \sum_{i=1}^{i=n} \frac{\partial f}{\partial x_i}(w(h^1), \ldots, w(h^n)) \dot{h}_t^i\,.$$

D is the usual stochastic gradient associated to the Wiener process W. D^* denotes its dual operator. Let $\mathbb{D}_{2,1}([0;T])$ be the Sobolev space closure of $\mathcal{S}$ with respect to the norm $\|F\|_{2,1}^2 = \|F\|_2^2 + E[\int_0^T \|D_s F\|^2 ds]$.

Let **HC** denotes the hypothesis

$$L \in \mathbb{D}_{2,1}([0;T]) \text{ such that } \int_t^T \|D_s L\|^2 ds > 0, \mathbb{P} \text{ almost surely for any } t \in [0,T[\,. \tag{8}$$

Proposition 3.7. *Under* **HC** *we have that* $\forall t < T$, *the conditional law* Q_t *of* L *given* $\mathcal{F}_t$ *is absolutely continuous with respect to the Lebesgue measure.*

Proof. (i) $\mathbb{D}_{2,1}$ is actually the standard Dirichlet space on Wiener space Ω (cf. [2]). Let $\mathcal{X}$ be the set $\{W_s^i,\ 0 \le s \le t;\ i = 1, \ldots, d\}$ and $\mathbb{D}(\mathcal{X})$ the Dirichlet subspace of $\mathbb{D}$ generated by this family (cf. [2]). Let $\mathcal{H}$ be the orthogonal subspace in $L^2(\Omega; L^2([0;T]); \mathbb{P})$ of the set

$$\{ZDU; \quad U \in \mathbb{D}(\mathcal{X}); \quad Z \in L^\infty(\Omega)\}\,,$$

and let $D^{\mathcal{X}}$ be the operator $P^{\mathcal{H}}D$ defined on $\mathbb{D}_{2,1}$, where $P^{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$.

Let us remark that for any $f \in \mathbb{D}_{2,1}$, $D^{\mathcal{X}} f = Df \mathbb{1}_{[t;T]}$. Then the condition (b) in the Proposition 5.2.5 of [2] is easily verified (i.e. $gP^{\mathcal{H}}h \in \text{Dom } D^*$ when g (resp. h) belongs to a dense subset of $L^2(\Omega)$ (resp. $L^2([0;T]))$). Thus the operator $(\mathbb{D}_{2,1}, D^{\mathcal{X}})$ is closable as an operator from $L^2(\Omega)$ to $L^2(\Omega; L^2([0;T]); \mathbb{P})$.

Then, the hypothesis **HC**, and Theorem 5.2.7 (a) [2] give the result. □

Example. We can see that $L = \ln S_T^1 - \ln S_T^2$ verifies the set of hypotheses **H1**, **H2**, **HJ** in a market with continuous and deterministic coefficients. Actually, L is then a Gaussian variable and we can use the results of [3]; if the vector $\sigma^1 - \sigma^2 \neq 0$ in $[t, T]$ for all t:

$$L = \int_0^T \beta_s ds + \int_0^T (\gamma_s, dW_s),$$

where $\beta_s = (b_s^1 - b_s^2) - \frac{1}{2}(\|\sigma_s^1\|^2 - \|\sigma_s^2\|^2)$ and $\gamma_s = \sigma_s^1 - \sigma_s^2$.

In this way we get the result given by Chaleyat–Maurel and Jeulin [3]:

$$B_t = W_t - \int_0^t \frac{\gamma_r(\int_r^T \gamma_s dW_s)}{\int_r^T \|\gamma_u\|^2 du} dr$$

is a $(\mathcal{Y}, \mathbb{P})$-Brownian motion.

Moreover the distribution of l_s is $N\left(0, \frac{\|\gamma_s\|^2}{\int_s^T \|\gamma_u\|^2 du}\right)$. See also [11].

4. Admissible and Optimal Strategies

We want now to make a risk neutral change of probability measure on which we could compute a "fair" price for an option in an insider trader setting.

Proposition 4.1. *Assume* **H1(Q)** *and let* $M_t = \exp[-\int_0^t (\eta_s, dW_s) - \frac{1}{2}\int_0^t \|\eta_s\|^2 ds]$, $t \in [0, A]$. *Then,* M *is a* $(\mathcal{Y}, Q)$*-uniformly integrable martingale and, under* $Q^1 = M_A Q$, *the process* $\tilde{B}_t = W_t + \int_0^t \eta_s ds$, *for* $t \leq A$, *is a* $(\mathcal{Y}, Q^1)$*-Brownian motion and the discounted prices are* $(\mathcal{Y}, Q^1)$*-local martingales.*

The proof is quite standard (Girsanov theorem).

Let us remark that $Q^1 = \frac{M_A}{q(A,L)}\mathbb{P}$ and under Q^1, the discounted wealth can be written, with a $\mathcal{Y}$-predictable portfolio:

$$X_t R_t + \int_0^t R_s c_s ds = X_0 + \int_0^t R_s(\pi_s, \sigma_s d\tilde{B}_s),\ t \in [0, A]. \tag{9}$$

Under **H1(Q)** and **HJ** we have defined the risk-neutral probability Q^1 on the probability space $(\Omega, \mathcal{Y}, Q)$ and under **H'**, **H1(**$\mathbb{P}$**)** and **HN** we can define the probability $\tilde{Q}$ on the probability space $(\Omega, \mathcal{Y}, \mathbb{P})$ by

$$\frac{d\tilde{Q}}{d\mathbb{P}} = \tilde{M}_A = \exp\left[-\int_0^A (l_s + \eta_s, dB_s) - \frac{1}{2}\int_0^A \|l_s + \eta_s\|^2 ds\right].$$

We have supposed, without loss of generality that the final time A is equal in each hypothesis. It is easy to prove that $\tilde{Q}$ is also a risk-neutral probability.

Define $q(t,x)$ as the the conditional density of L given $\mathcal{F}_t$ with respect to the law of L; hypothesis **HJ** insures that for all $t \leq A$, $q(t,x) > 0$ $\mathbb{P} \otimes \mathbb{P}_L$ almost surely, where $\mathbb{P}_L$ is the probability law of L.

Let $\tilde{M}_t = \exp[-\int_0^t (l_s + \eta_s, dB_s) - \frac{1}{2}\int_0^t \|l_s + \eta_s\|^2 ds]$, for $t \leq A$, we get:

Proposition 4.2. *Assume* **H'**, **H1($\mathbb{P}$)**, **H1(Q)** *and* **HN**, *then* $Q^1 = \tilde{Q}$ *on* $\mathcal{Y}_A$ *and the two* $(\mathcal{Y}, Q)$*-uniformly integrable martingales* M *and* $q(A,L)\tilde{M}$ *coincide on* $[0, A]$.

Proof. First, on the probability space $(\Omega, \mathcal{Y}, Q)$ we have by the previous proposition:

$$dM_s = -M_s(\eta_s, dW_s), \quad M_0 = 1\,.$$

Second, using the definition of $\tilde{M}$ just above, and writing it on $(\Omega, \mathcal{Y}, Q)$ with respect to W instead of $(\Omega, \mathcal{Y}, \mathbb{P})$ with respect to B, yields

$$d\tilde{M}_s = -\tilde{M}_s[(l_s + \eta_s, dW_s) - (l_s + \eta_s, l_s)ds], \quad \tilde{M}_0 = 1\,.$$

Moreover, the hypothesis **H'** allows us to define the conditional density $p(t,x)$ of L given $\mathcal{F}_t$ with respect to the measure $\nu(dx)$ and p satisfies (cf. Proposition 3.5):

$$p(t,x) = p(0,x) + \int_0^t (\alpha(s,x), dW_s)\,.$$

On the probability space $(\Omega, \mathcal{Y}, Q)$, L and W are independent ($\sigma(L) \subset \mathcal{Y}_0$), so we can replace x by L in the equation above, then $p(t,L) = p(0,L) + \int_0^t (\alpha(s,L), dW_s)$.

The conditional density $q(t,x)$, of L given $\mathcal{F}_t$ with respect to the law of L, satisfies $p(t,x) = q(t,x)p(0,x)$ for $x \in \mathbb{R}^\kappa$. Since we have $p(t,L) > 0$ almost surely [10], Corollary (1.11) and recalling that $l_t = \frac{\alpha(t,L)}{p(t,L)}$, we get for $t \leq A$:

$$q(t,L) = 1 + \int_0^t q(s,L)(l_s, dW_s)\,.$$

So, $(q(t,L))_{(t \leq A)}$ is the martingale $(E_Q[q(A,L)/\mathcal{Y}_t])_{(t \leq A)}$. Then, using Itô formula, for $t \leq A$, we get

$$d(\tilde{M}_t q(t,L)) = \tilde{M}_t dq(t,L) + q(t,L)d\tilde{M}_t + d[\tilde{M}, q(.,L)]_t = dM_t\,.$$

So $M_t = q(t,L)\tilde{M}_t$ and $Q^1 = \tilde{Q}$ on $\mathcal{Y}_A$. $\square$

To characterize the admissible strategies, we need a martingale representation theorem which is not a classical one because the filtration is not the natural filtration of the Brownian motion $\tilde{B}$.

Theorem 4.3. *Suppose that* **H1(Q)** *and* **HJ** *hold. Let* $Z \in L^1(\Omega, \mathcal{Y}_A, Q^1)$; *then there exists a unique* $\mathcal{Y}$*-predictable* ψ *such that*

$$Z = E_{Q^1}[Z/\mathcal{Y}_0] + \int_0^A (\psi_s, d\tilde{B}_s)\,.$$

Proof. W is a $(\mathcal{Y}, Q)$-Brownian motion. The process W is also $\mathcal{F}$-adapted, so it is also a $(\mathcal{F}, Q)$-Brownian motion, and any local $(\mathcal{F}, Q)$-martingale has the representation property with respect to W. Then, Theorem 4.33, p. 176 of [8] shows that any local $(\mathcal{Y}, Q)$-martingale has the representation property with respect to W. Finally, the equivalence of the probability measures Q^1 and Q implies that any local $(\mathcal{Y}, Q^1)$-martingale has the representation property with respect to $\tilde{B}$: indeed, we have

$$E_{Q^1}[Z/\mathcal{Y}_t] = \frac{E_Q[M_A Z/\mathcal{Y}_t]}{E_Q[M_A/\mathcal{Y}_t]}\,,$$

and $M_t = E_Q[M_A/\mathcal{Y}_t]$ satisfies the equation $M_t = 1 - \int_0^t M_s(\eta_s, dW_s)$.

Moreover the martingale representation property with respect to W gives

$$E_Q[M_A Z/\mathcal{Y}_t] = E_Q[M_A Z/\mathcal{Y}_0] + \int_0^t (\phi_s, dW_s)\,,$$

and Ito formula yields

$$E_{Q^1}[Z/\mathcal{Y}_t] = E_{Q^1}[Z/\mathcal{Y}_0] + \int_0^t \left(\frac{\phi_s}{M_s} + E_{Q^1}[Z/\mathcal{Y}_s]\eta_s, d\tilde{B}_s\right)\,.$$

We have then $\psi_s = \frac{\phi_s}{M_s} + E_{Q^1}[Z/\mathcal{Y}_s]\eta_s,\ \forall s \leq A$. □

We have as a corollary a usual characterization of admissible strategies:

Proposition 4.4. *We suppose that* **H1(Q)**, **H2** *and* **HJ** *hold. Let* X_0 *be a positive* $\mathcal{Y}_0$*-measurable variable. Then for an "admissible" strategy* (π, c) *and the associated final wealth* $X_A^{\pi,c}$, *we have*

$$E_{Q^1}\left[X_A^{\pi,c} R_A + \int_0^A R_t c_t dt/\mathcal{Y}_0\right] \leq X_0\,.$$

Conversely, given an initial wealth $X_0 \in L^1(\mathcal{Y}_0)$, *a consumption process* c, $\mathcal{Y}$*-adapted positive and such that* $\int_0^A c_s ds < \infty$ Q^1 *almost surely, and a random variable* $Z \in L^1(\mathcal{Y}_A, Q^1)$ *such that*

$$E_{Q^1}\left[Z R_A + \int_0^A R_t c_t dt/\mathcal{Y}_0\right] = X_0\,,$$

there exists a $\mathcal{Y}$-predictable portfolio $\pi = (\pi_t, t \in [0; A])$ such that (π, c) is admissible and $X_A^{\pi,c} = Z$.

Proof. The first part is standard, under the probability Q^1, Eq. (9) is

$$dX_t R_t + R_t c_t dt = R_t(\pi_t, \sigma_t d\tilde{B}_t), \quad X_0 \in L^0(\mathcal{Y}_0), \quad 0 \le t \le A\,.$$

It is a positive $(\mathcal{Y}, Q^1)$-local martingale, therefore it is a supermartingale with initial value X_0, and we have the result.

For the converse, we use the precedent theorem, let

$$N_t = E_{Q^1}\left[ZR_A + \int_0^A R_t c_t dt/\mathcal{Y}_t\right], \quad 0 \le t \le A\,,$$

we have $N_t = E_{Q^1}[N_A/\mathcal{Y}_0] + \int_0^t (\psi_s, d\tilde{B}_s)$ for a $\mathcal{Y}$-predictable process ψ. For $0 \le s \le A$, let $\pi_s = R_s^{-1}(\sigma_s')^{-1}\psi_s$ where σ_s' denotes the transposed matrice of σ_s, the process π is then $\mathcal{Y}$-predictable. With the strategy (π, c) we get under the probability measure Q^1 the discounted wealth equation:

$$dX_t^{\pi,c} R_t + R_t c_t dt = R_t(\pi_t, \sigma_t d\tilde{B}_t), \qquad X_0 \in L^1(\mathcal{Y}_0), \qquad 0 \le t \le A\,.$$

Then $X_t^{\pi,c} R_t + \int_0^t R_s c_s ds$ is a uniformly integrable $(\mathcal{Y}, Q^1)$-martingale which is equal to the conditional expectation of its terminal value, and, thus

$$X_t^{\pi,c} R_t = E_{Q^1}\left[ZR_A + \int_t^A R_s c_s ds/\mathcal{Y}_t\right]$$

which is positive and shows the admissibility of the strategy. □

Optimization of the insider trader's strategy. We use a couple of utility functions (U_1, U_2) such that U_1 and U_2 are increasing, concave, non-negative C^1 functions verifying $\lim_{x\to\infty} U_i'(x) = 0$. (Usually $x \mapsto \log x$ is taken, or $x \mapsto x^\alpha$, $\alpha \in]0, 1[$).

Now, we have to maximize

$$(\pi, c) \mapsto J(X_0, \pi, c) = E_{\mathbb{P}}\left[\int_0^A U_1(c_t)dt + U_2(X_A^{\pi,c})/\mathcal{Y}_0\right]$$

in the set of admissible strategies under the constraint

$$E_{Q^1}\left[X_A^{\pi,c} R_A + \int_0^A R_t c_t dt/\mathcal{Y}_0\right] \le X_0\,.$$

Recall that $Q^1 = \frac{M_A}{q(A,L)}\mathbb{P}$. So the constraint becomes

$$E_{\mathbb{P}}\left[\left(X_A^{\pi,c}\frac{M_A}{q(A,L)}R_A + \int_0^A R_t E_{\mathbb{P}}\left[\frac{M_A}{q(A,L)}\Big/\mathcal{Y}_t\right]c_t dt\right)\Big/\mathcal{Y}_0\right]$$
$$\leq X_0 E_{\mathbb{P}}\left[\frac{M_A}{q(A,L)}\Big/\mathcal{Y}_0\right].$$

But using Proposition 4.2, $\frac{M_A}{q(A,L)}$ is the terminal value of a $(\mathcal{Y},\mathbb{P})$-martingale $(\tilde{M}_t)_{(t\leq A)}$ starting at 1, thus we have $E_{\mathbb{P}}[\frac{M_A}{q(A,L)}/\mathcal{Y}_0] = 1$.

This maximization is obtained by means of the Lagrange multipliers; the Lagrangian of this constrained problem is

$$E_{\mathbb{P}}\left[\int_0^A U_1(c_t)dt + U_2(X_A^{\pi,c}) + \lambda\left(\int_0^A R_t\tilde{M}_t c_t dt + X_A^{\pi,c}R_A\tilde{M}_A - X_0\right)\Big/\mathcal{Y}_0\right],$$

where λ is a $\mathcal{Y}_0$-measurable random variable. Let

$$I_i = (U_i')^{-1} \tag{10}$$

and

$$\mathcal{X}(y)(\omega) = E_{\mathbb{P}}\left[\int_0^A R_t\tilde{M}_t I_1(yR_t\tilde{M}_t)dt + R_A\tilde{M}_A I_2(yR_A\tilde{M}_A)/\mathcal{Y}_0\right](\omega). \tag{11}$$

It is easy to verify that I and $y \mapsto \mathcal{X}(y)(\omega)$ (for ω fixed) are surjective monotone functions on $\mathbb{R}^+$ and there exists a $\mathcal{Y}_0$-measurable solution of the implicit equation $\mathcal{X}(y) = X_0$; indeed, $y \mapsto \mathcal{X}(y)(\omega)$ (for ω fixed) is strictly decreasing on $\mathbb{R}^+$ from $+\infty$ to 0, thus there exists a unique $\lambda^*(\omega)$ such that $\mathcal{X}(\lambda^*(\omega))(\omega) = X_0(\omega)$. More precisely,

$$\lambda^*(\omega) = \sup\{y \in \mathbb{R}^+ :\ \mathcal{X}(y)(\omega) \geq X_0(\omega)\},$$

and $\lambda^*(\omega)$ is $\mathcal{Y}_0$-measurable. We obtain:

Under hypothesis **H1(Q)**, **H2** *and* **HJ**, *there exists an optimal strategy* (π^*, c^*) *such that*

$$J(X_0, \pi^*, c^*) = \sup\{J(X_0,\pi,c), (\pi,c)\ \textit{admissible}\}.$$

It is of the form: $c_t^* = I_1(\lambda^*\tilde{M}_t R_t)$; $X_A^{\pi^*,c^*} = I_2(\lambda^*\tilde{M}_A R_A)$, *where* λ^* *is a* $\mathcal{Y}_0$-*measurable random variable solution of the implicit equation*: $\mathcal{X}(\lambda^*) = X_0$. *We then have the optimal value of the maximization problem*:

$$E_{\mathbb{P}}\left[U_2 \circ I_2(\lambda^*\tilde{M}_A R_A) + \int_0^A U_1 \circ I_1(\lambda^*\tilde{M}_t R_t)dt/\mathcal{Y}_0\right].$$

5. Examples

5.1. Let $U_2 = \log$ and $U_1 = 0$. Under **H'**, **H1($\mathbb{P}$)** and **HN**, we obtain $R_A X_A^* = X_0 \tilde{M}_A^{-1}$, therefore

$$U_2(X_A^*) = \log(R_A^{-1} X_0) + \int_0^A (l_s + \eta_s, dB_s) + \frac{1}{2} \int_0^A \|l_s + \eta_s\|^2 ds\,.$$

The optimal value of this problem is

$$J(X_0, \pi^*, c^*) = \log(R_A^{-1} X_0) + \frac{1}{2} E_{\mathbb{P}} \left(\int_0^A \|l_s + \eta_s\|^2 ds / \mathcal{Y}_0 \right) \text{ for all } A < T, \quad (12)$$

since B is a $(\mathcal{Y}, \mathbb{P})$ Brownian motion, so it is independent of $\mathcal{Y}_0$.

If L is a Gaussian random variable, the optimal value tends to infinity when $A \to T$, and thus is greater than the optimal value of the portfolio of a non-insider trader with the same optimization function: $\log x + \frac{1}{2} E[\int_0^T \|\eta_s\|^2 ds]$: see [11] or the example after Proposition 3.7, where the process l is given by

$$l_r^i = \frac{\gamma_r^i \int_r^T (\gamma_s, dW_s)}{\int_r^T \|\gamma_u\|^2 du}\,.$$

Using the first formalism under **HJ**, the optimal wealth can be written as $X_A^* = X_0 \frac{q(A,L)}{M_A}$, and the optimal value is

$$J(X_0, \pi^*, c^*) = \log(R_A^{-1} X_0) + E_{\mathbb{P}} \left(\log \left(\frac{q(A, L)}{M_A} \right) \Big/ \mathcal{Y}_0 \right)\,.$$

This is equal to the non-insider optimal value plus the term $E_{\mathbb{P}}[\log q(A, L)/\mathcal{Y}_0]$.

5.2. Under **H'**, **H1($\mathbb{P}$)**, **H1(Q)** and **HN**, for $U_i(x) = \log(x),\ i = 1, 2$, we have $\lambda^* = \frac{A+1}{X_0}$ and the optimal strategy:

$$R_A X_A^* = \frac{X_0}{A+1} \tilde{M}_A^{-1};\ R_t c_t^* = \frac{X_0}{A+1} \tilde{M}_t^{-1}\,,$$

where the anticipating feature is seen in $\tilde{M}_t = \frac{M_t}{q(t,L)}$, for $t \leq A$.

It is interesting to note that we can then obtain the expression of the optimal portfolio, from the proof of Proposition 4.4,

$$X_t^* R_t + \int_0^t R_s c_s^* ds = E_{Q^1} \left[X_A^* R_A + \int_0^A R_s c_s^* ds / \mathcal{Y}_t \right]\,. \quad (13)$$

Let $\xi_s = -(l_s + \eta_s)$ and $N_t = \tilde{M}_t^{-1}$, then N_t verifies the Itô equation:

$$dN_t = -N_t(\xi_t, d\tilde{B}_t), \quad N_0 = 1\,. \quad (14)$$

Then

$$\begin{aligned}
&X_t^* R_t + \int_0^t R_s c_s ds \\
&\quad = \frac{X_0}{A+1} E_{Q^1}\left[1 - \int_0^A N_s(\xi_s, d\tilde{B}_s) + A - \int_0^A \left[\int_0^s N_u(\xi_u, d\tilde{B}_u)\right] ds / \mathcal{Y}_t\right].
\end{aligned}$$

Commuting the stochastic integral and the time integral and conditioning the stochastic integral with respect to $\mathcal{Y}_t$, we obtain

$$X_t^* R_t + \int_0^t R_s c_s ds = X_0 - \frac{X_0}{A+1}\left[\int_0^t (1+A-s) N_s(\xi_s, d\tilde{B}_s)\right],$$

and identifying with the expression of the $\tilde{B}$-martingale we get

$$R_s \sigma_s' \pi_s^* = \frac{X_0(1+A-s)}{A+1} N_s(-\xi_s). \tag{15}$$

Now, making the same computation from (13) and substracting the consumption from 0 to t, we have

$$\begin{aligned}
X_t^* R_t &= E_{Q^1}\left[X_A^* R_A + \int_t^A R_s c_s^* ds \Big/ \mathcal{Y}_t\right] \\
&= \frac{X_0}{A+1} E_{Q^1}\left[1 - \int_0^A N_s(\xi_s, d\tilde{B}_s) + A - t - \int_t^A \left[\int_0^s N_u(\xi_u, d\tilde{B}_u)\right] ds \Big/ \mathcal{Y}_t\right] \\
&= \frac{X_0}{A+1} E_{Q^1}\left[1 - \int_0^A N_s(\xi_s, d\tilde{B}_s) + (A-t)\left[1 - \int_0^t N_u(\xi_u, d\tilde{B}_u)\right] \Big/ \mathcal{Y}_t\right]. \qquad (16)
\end{aligned}$$

Using now the differential expression of N_t we get

$$X_t^* R_t = X_0 \frac{A+1-t}{A+1} N_t, \qquad 0 \le t \le A,$$

and this expression used in (15) gives an explicit expression of the optimal portfolio function of wealth:

$$\pi_s^* = X_s^* (\sigma_s')^{-1}(l_s + \eta_s),$$

where σ' is the transpose of σ. This gives a proof of a result conjectured in [11].

5.3. In a very general context (cf. [7], for precise hypotheses), it is possible to get a formula for the conditional density $p(t,x)$ with respect to Lebesgue measure of L given $\mathcal{F}_t$:

$$p(t,x) = E\left(\mathbb{1}_{\{L>x\}} D^* \left(\frac{DL\ \mathbb{1}_{[t,T]}}{|DL\ \mathbb{1}_{[t,T]}|_H^2}\right) \Big/ \mathcal{F}_t\right),$$

and we could get also, using the Ocone–Clark formula, an expression for l_s (cf. [7]):

$$l_s(x) = \frac{D_s p(s,x)}{p(s,x)} \, .$$

So we can give a general expression of the hypotheses **HC**, **H1($\mathbb{P}$)**, **H1(Q)** and **HN**. In the Gaussian case, using these formulae, we get the same results as in [3]. But for a little harder example, like $L = \int_0^T \int_0^t (\gamma_s dW_s) dW_t$, where γ is deterministc, l_s becomes rather complicated.

5.4. To get another general example in which **HJ** is verified, let $L = Y_T$ where Y is a strong solution of a usual stochastic differential equation, for which we know that the probability law $\mathcal{L}(Y_t)$ has a strictly positive density with respect to Lebesgue measure (cf. [15]); it is not hard to see that the process $(Y_{t+s} - Y_t)_s$ is a $\mathcal{F}$-Markov process (and not only a $\sigma(Y)$-Markov process) so the conditional law of Y_T given $\mathcal{F}_t$ is the law of the solution at time $T-t$ of the same stochastic differential equation with Y_t as initial condition and then $p(t,x) > 0$ almost surely.

6. Statistical Test

We take in that section a logarithmic utility function and deterministic bounded coefficients. The statistical test is given by: the nul hypothesis is $L \in \mathcal{F}_0$.

$$H_0 : L \in \mathcal{F}_0, \text{ against } H_1 : L \notin \mathcal{F}_0 \, .$$

We construct the statistical test on the example $L = \ln S_T^1 - \ln S_T^2$. L is then a Gaussian variable and if the vector $\sigma^1 - \sigma^2 \neq 0$ in $[t,T]$ for all t:

$$L = \int_0^T \beta_s ds + \int_0^T (\gamma_s, dW_s) \, ,$$

where $\beta_s = (b_s^1 - b_s^2) - \frac{1}{2}(\|\sigma_s^1\|^2 - \|\sigma_s^2\|^2)$ and $\gamma_s = \sigma_s^1 - \sigma_s^2$.

We have $l_r = \frac{\gamma_r(\int_r^T (\gamma_s, dW_s))}{\int_r^T \|\gamma_u\|^2 du}$, that $B_t = W_t - \int_0^t l_r dr$ is a $(\mathcal{Y}, \mathbb{P})$-Brownian motion.

We can compare the optimal strategies for an insider trader and for a non-insider trader with same initial wealth $x > 0$. For the latter, the strategy is:

$$R_A X_A^* = y M_A^{-1} \; ; \; R_t c_t^* = y M_t^{-1}, \text{ where } y = \frac{x}{A+1}$$

with $M_t = \exp(-\int_0^t (\eta_s, dW_s) - \frac{1}{2}\int_0^t \|\eta_s\|^2 ds)$, where $\eta_s = \sigma_s^{-1}(b_s - r_s \mathbb{1})$.

We then have to compare the optimal consumptions under H_0:

$$\log R_t c_t^* = \log y + \int_0^t (\eta_s, dW_s) + \frac{1}{2}\int_0^t \|\eta_s\|^2 ds \, ,$$

to the optimal consumptions under H_1:

$$\log R_t c_t^* = \log y + \int_0^t (\eta_s, dW_s) + \frac{1}{2}\int_0^t \|\eta_s\|^2 ds + \log q(t,L) \, .$$

Let $0 \leq t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0;T]$, under the hypothesis H_0 we denote, for $0 \leq i \leq n-1$,

$$Y_i = \log R_{t_{i+1}} c_{t_{i+1}} - \log R_{t_i} c_{t_i} = \int_{t_i}^{t_{i+1}} \eta_s dW_s + \frac{1}{2} \int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds \,.$$

We have supposed that the coefficients b, r, σ are deterministic, thus Y is a sequence of Gaussian independent random variables with expectation $\frac{1}{2}\int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds$ and variance $\int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds$. Under hypothesis H_1, there is an additional term log $\frac{q(t_{i+1},L)}{q(t_i,L)}$. We could then construct a statistical test with critical region given by

$$RC_i = \left\{ \omega : \left| Y_i(\omega) - \frac{1}{2} \int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds \right| > C \right\} ,$$

and, for instance, a statistical test with level 0.05 is the test with critical region:

$$RC_i = \left\{ \omega : \left| Y_i(\omega) - \frac{1}{2} \int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds \right| > 1.96 \sqrt{\int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds} \right\} .$$

Note that, under H_1, for $0 \leq i \leq n-2$, the random variable

$$Y_i(\omega) - \frac{1}{2} \int_{t_i}^{t_{i+1}} \|\eta_s\|^2 ds = \int_{t_i}^{t_{i+1}} \eta_s dW_s + \log \frac{E_Q[q(t_{i+1}, L)/\mathcal{Y}_{t_{i+1}}]}{E_Q[q(t_i, L)/\mathcal{Y}_{t_i}]} \,.$$

In the case where L is Gaussian, this random variable is a Gaussian, plus a non independent combination of chi-2 law random variables plus the constant $\frac{1}{2} \log \frac{\int_{t_i}^{T} \gamma_s^2 ds}{\int_{t_{i+1}}^{T} \gamma_s^2 ds}$. Thus the power of the Neyman–Pearson test $H_0' = (L = L_0)$ (L_0 is a constant) against $H_1' = (L = \ln S_T^1 - \ln S_T^2)$ can be computed.

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