EE1390 Project - 1

Solving a Coordinate Geometry Problem with Matrices

Raagini Vishnubotla EE17BTECH11050
Sai Manasa Pappu EE17BTECH11036

IIT Hyderabad, Feb 2019

Index

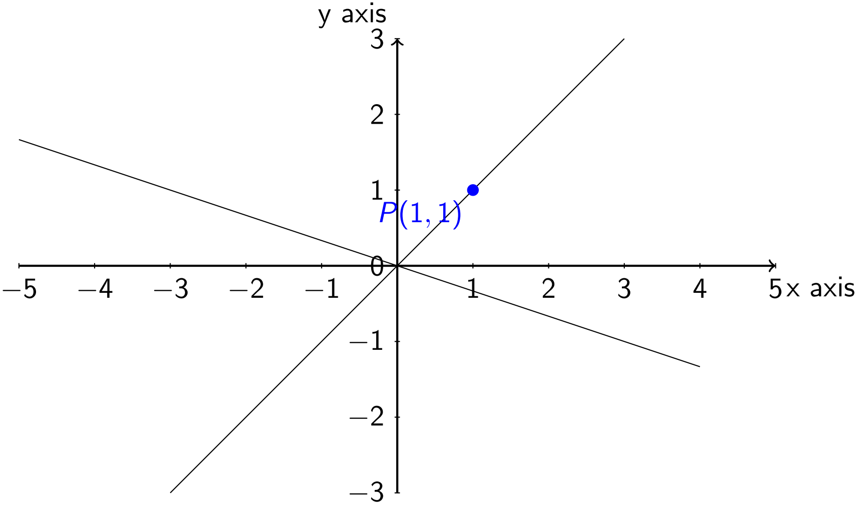
Geometry Question

Question 52 in JEE Main 2015 Code D

The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at $(1, 1)$

(1) does not meet the curve again.
(2) meets the curve again in the second quadrant.
(3) meets the curve again in the third quadrant.
(4) meets the curve again in the fourth quadrant.

Diagram



Matrix Transformation of Question

Rewriting in the form of matrices...

Find the direction vectors, and normal vectors (as column vectors), to the pair of lines represented by the equation $x^2 + 2xy - 3y^2 = 0$.
If a normal is drawn at (1,1), find out in which quadrant it intersects the curve again, i.e.,
If M stores the normal vector of the normal at (1,1) as M_1, and the normal vector of the other line as M_2 as$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$, and
$K = \begin{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T M_1 & \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T M_2 \end{pmatrix}$. Find X such that
$K = X^T M$
$X^T = KM^{-1}$

Approach/Strategy

Approach

Given equation: $x^2 + 2xy - 3y^2 = 0$
The equation is in the form of a general second degree equation
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
with g, f, c = 0.
In such a case, the condition to be satisfied for the equation to represent a pair of straight lines is $h^2 \geq ab$, which holds for the given equation.
Also, the above lines pass through the origin as (0,0) satisfies the equation.

Strategy

- Split the above combined equation into two individual lines.
- Find the line on which the given point (1,1) lies, say *l1*.
- Using the direction vector of this line, find the normal vector.
- The line passing through (1,1) with direction vector as normal vector as computed above is our required new line.
  - If this line is parallel to *l2*, they do not intersect.
  - Else, they intersect.

Solution

Given combined line equation: $x^2 + 2xy - 3y^2 = 0$
It can be rewritten as

$$\frac{x^2}{y^2} + 2\frac{x}{y} - 3 = 0$$

which is in the form $at^2 + bt + c = 0$.

It can be split into individual lines using the quadratic roots formula,

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{x}{y} = \frac{-2 \pm \sqrt{(-2)^2 - 4 \times (-3)}}{2}$$

Therefore, we get the two following lines:

$$\frac{x}{y} = 1$$
$$\frac{x}{y} = -3$$

Let M be a matrix that stores the direction vectors of the lines as M $= \begin{pmatrix} M_1 & M_2 \end{pmatrix}$. Therefore,

$$M = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$$

Let N denote a matrix that stores the normal vectors of the lines as N $= \begin{pmatrix} N_1 & N_2 \end{pmatrix}$. So,

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

The point P (1,1) lies on that line whose normal is $\perp^r$ to P-O = (1-0 , 1-0), since two lines intersect at the origin.

$P^T N = \begin{pmatrix} 1 & 1 \end{pmatrix} N = \begin{pmatrix} 0 & -4 \end{pmatrix}$

Let the new line obtained at P by drawing a normal to the curve at that point, be *l3*.

$$l3 : (x^T - \begin{pmatrix} 1 & 1 \end{pmatrix}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \text{ or } x^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

$$l2 : x^T \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0$$

Since *l3* is not parallel to *l2*, they have a point of intersection. Let this point be x_o.

Since x_o satisfies both the above equations, we have

$$x_o^T \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

$$x_o^T = \begin{pmatrix} 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}^{-1}$$

$$x_o^T = \begin{pmatrix} 2 & 0 \end{pmatrix} \times -0.5 \times \begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix} = -0.5 \times \begin{pmatrix} -6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \end{pmatrix}$$

Therefore, the point of Intersection is

$$x_o = \begin{pmatrix} 3 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

which lies in the IV quadrant.

Hence, the correct answer is -
(4) Meets the curve again in the fourth quadrant.

Figure of Solution