## Exercise 4.2.5

a) Define $W_i$ : time for the $i$-th call to arrive after the $(i-1)$-th call ($i = 1, 2, \dots$, and $W_0 := 0$).

So $W_i \sim Exp(1)$.

And $P(W_4 \le 2) = 1 - e^{-1.2} \approx 0.86$

b) Define $N(t)$: # of calls from beginning to the $t$-th second.

and $T_k = \sum_{i=1}^{k} W_i$ is the arrival time of the $k$-th call from beginning.

Since $N(t) \sim Poisson(t)$, $T_4 \sim Gamma(4, 1)$.

$$
\begin{aligned}
P(T_4 \le 5) &= P(N(5) \ge 4) \\
&= 1 - P(N(5) \le 3) \\
&= 1 - e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!}\right) \\
&\approx 0.73
\end{aligned}
$$

c)
$$
\begin{aligned}
E(T_4) &= E\left(\sum_{i=1}^{4} W_i\right) \\
&= \sum_{i=1}^{4} E(W_i) \\
&= \sum_{i=1}^{4} 1 \qquad (W_i \sim Exp(1), \text{ and } EW_i = \tfrac{1}{1} = 1) \\
&= 4
\end{aligned}
$$

$\square$

**Exercise 4.2.15**

a) As the lifetime of the original and spare components follow the Same Poisson process, then we know

$$T_i \sim \text{Exp}(0.05) \text{ for } i = 1,2,3,4.$$

and $T_{total} = \sum_{i=1}^{4} T_i \sim \text{Gamma}(4, 0.05)$

So $E(T_{total}) = E(\sum_{i=1}^{4} T_i)$

$= \sum_{i=1}^{4} \frac{1}{0.05}$ $\quad$ ( $T_i \sim \text{Exp}(0.05)$, So $ET_i = \frac{1}{0.05}$ )

$= 20 \cdot 4$

$= 80$ (days)

b). Since $T_i \sim \text{Exp}(0.05)$, $\text{Var}(T_i) = \frac{1}{0.05^2} = 400$.

And $T_1, T_2, T_3, T_4$ are independent,

$SD(T_{total}) = \sqrt{\text{Var}(T_{total})}$

$= \sqrt{\sum_{i=1}^{4} \text{Var}(T_i)}$

$= \sqrt{4 \cdot 400}$

$= 40$ (days)

c) Define $N(t)$ : # of components failed in $t$ days.

We know $N(t) \sim \text{Poisson}(0.05t)$.

and thus $N(60) \sim \text{Poisson}(3)$

So $P(T_{total} \geq 60 \text{ days}) = P(N(60) \leq 3)$

$$= e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)$$

$$\approx 0.65$$

$\square$

**Exercise 4.4.2.**

Let's first find the distribution of $cT$, for any $c>0$.

Since $T \sim \text{Gamma}(r, \lambda)$, its pdf is

$$f_T(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t} \quad \text{for } t>0.$$

Let $Y = cT = g(T)$, and $Y>0$.

The inverse mapping is $T = \frac{1}{c} Y = g^{-1}(Y)$.

And $(g^{-1}(y))' = \frac{1}{c}$.

Therefore, by applying the change of variable formula, we can get the pdf for $Y = cT$ as

$$f_{cT}(y) = f_T(g^{-1}(y)) \left| (g^{-1}(y))' \right|$$

$$= \frac{\lambda^r}{\Gamma(r)} \left(\frac{y}{c}\right)^{r-1} e^{-\lambda \frac{y}{c}} \cdot \frac{1}{c}$$

$$= \frac{\left(\frac{\lambda}{c}\right)^r}{\Gamma(r)} y^{r-1} e^{-\frac{\lambda}{c} y} \quad \text{for any } y>0.$$

Therefore, by comparing the pdf of the Gamma distribution, we know $Y = cT \sim \text{Gamma}(r, \frac{\lambda}{c})$. $(*)$

To show $T \sim \text{Gamma}(r, \lambda)$ if and only if $T_1 = \lambda T \sim \text{Gamma}(r, 1)$, we need to show both sides.

① If $T \sim \text{Gamma}(r, \lambda)$, let $c = \lambda$ in $(*)$, we can get

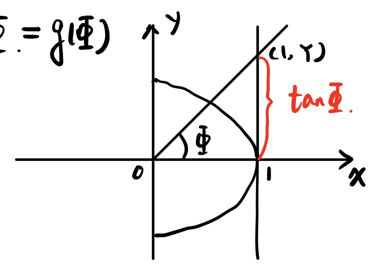
$$T_1 = \lambda T \sim \text{Gamma}(r, \tfrac{\lambda}{\lambda}) = \text{Gamma}(r, 1).$$

② If $T_1 \sim \text{Gamma}(r, 1)$, let $c = \frac{1}{\lambda}$ in $(*)$, we know

$$T = \tfrac{1}{\lambda} T_1 \sim \text{Gamma}(r, \tfrac{1}{\frac{1}{\lambda}}) = \text{Gamma}(r, \lambda). \quad \square$$

## Exercise 4.4.6

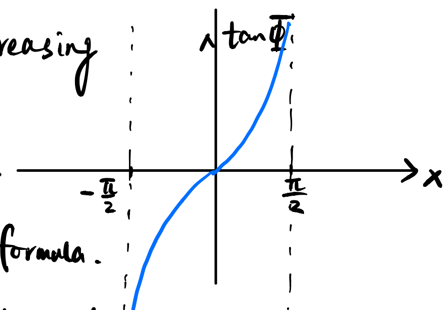
From the graph, we can see $Y = \tan \Phi = g(\Phi)$




Since $\Phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $Y \in (-\infty, +\infty)$

See the graph of $\tan\Phi$ at $(-\frac{\pi}{2}, \frac{\pi}{2})$

We can see $\tan\Phi$ is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$ as well.




To find the pdf of $Y$, we want to apply the change of variable formula.

Since $\Phi \sim \text{Uniform}(-\frac{\pi}{2}, \frac{\pi}{2})$, its pdf is

$$f_\Phi(\phi) = \begin{cases} \frac{1}{\pi} & \text{if } \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ 0 & \text{o.w.} \end{cases}$$

The inverse mapping is $\Phi = \arctan(Y) = g^{-1}(Y)$, and

$$(g^{-1}(y))' = \frac{1}{1+y^2}$$

So the pdf of $Y$ is,

$$
\begin{aligned}
f_Y(y) &= f_{\Phi}(g^{-1}(y)) \left| (g^{-1}(y))' \right| \\
&= \frac{1}{\pi} \cdot \left| \frac{1}{1+y^2} \right| . \\
&= \frac{1}{\pi(1+y^2)} \quad \text{for any } y \in (-\infty, +\infty).
\end{aligned}
$$

Since $f_Y(-y) = \frac{1}{\pi(1+(-y)^2)} = \frac{1}{\pi(1+y^2)} = f_Y(y)$, we can see this Cauchy distribution is symmetric about $0$.

To check whether its expectation is well defined, we just need to check whether $E(|Y|)$ is finite.

Since

$$
\begin{aligned}
E(|Y|) &= \int_{-\infty}^{+\infty} |y| \frac{1}{\pi(1+y^2)} \, dy \\
&= 2 \int_0^{+\infty} \frac{y}{\pi(1+y^2)} \, dy \\
&= \frac{1}{\pi} \int_0^{+\infty} \frac{1}{1+y^2} \, d(1+y^2) \\
&= \frac{1}{\pi} \ln(1+y^2) \Big|_0^{+\infty} = \infty
\end{aligned}
$$

So the required integral does not converge absolutely, and thus $EY$ is undefined. □

**Exercise 4.4.10.**

Assume the pdf and cdf for $N(0,1)$ are $\phi(x)$ and $\Phi(x)$, respectively.

a). Let $Y_1 = |Z| \geq 0$ then for any $y_1 \geq 0$, its c.d.f. is

$$F_{Y_1}(y_1) = P(Y_1 \leq y_1)$$
$$= P(|Z| \leq y_1)$$
$$= P(-y_1 \leq Z \leq y_1)$$
$$= \Phi(y_1) - \Phi(-y_1)$$
$$= \Phi(y_1) - (1 - \Phi(y_1)) \quad (\text{Symmetry of } N(0,1))$$
$$= 2\Phi(y_1) - 1$$

And $F_{Y_1}(y_1) = 0$ for any $y_1 < 0$.

$$\text{So } F_{Y_1}(y_1) = \begin{cases} 2\Phi(y_1) - 1 & \text{if } y_1 \geq 0 \\ 0 & \text{if } y_1 < 0 \end{cases}$$

Take derivative to get pdf of $Y_1$ as

$$f_{Y_1}(y_1) = F'_{Y_1}(y_1) = \begin{cases} 2\phi(y_1) & \text{if } y_1 \geq 0 \\ 0 & \text{if } y_1 < 0 \end{cases}$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-\frac{y_1^2}{2}} & \text{if } y_1 \geq 0 \\ 0 & \text{if } y_1 < 0 \end{cases}$$

b) Let $Y_2 = Z^2 \geq 0$. For any $y_2 \geq 0$, its cdf is

$$F_{Y_2}(y_2) = P(Y_2 \leq y_2)$$
$$= P(Z^2 \leq y_2)$$
$$= P(-\sqrt{y_2} \leq Z \leq \sqrt{y_2})$$

$$= \Phi(\sqrt{y_2}) - \Phi(-\sqrt{y_2})$$

$$= \Phi(\sqrt{y_2}) - (1 - \Phi(\sqrt{y_2}))$$

(Symmetry of $N(0,1)$)

$$= 2\Phi(\sqrt{y_2}) - 1$$

If $y_2 < 0$, $F_{Y_2}(y_2) = 0$.

$$\text{So } F_{Y_2}(y_2) = \begin{cases} 2\Phi(\sqrt{y_2}) - 1 & \text{if } y_2 \geq 0 \\ 0 & \text{if } y_2 < 0. \end{cases}$$

Take derivative to get its pdf

$$f_{Y_2}(y_2) = F'_{Y_2}(y_2) = \begin{cases} 2\phi(\sqrt{y_2}) \cdot \frac{1}{2\sqrt{y_2}} & \text{if } y_2 \geq 0 \\ 0 & \text{if } y_2 < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} y_2^{-\frac{1}{2}} e^{-\frac{y_2}{2}} & \text{if } y_2 \geq 0 \\ 0 & \text{if } y_2 < 0. \end{cases}$$

c) Let $Y_3 = \frac{1}{Z} \in (-\infty, 0) \cup (0, +\infty)$.

For any $y_3 < 0$, its cdf is

$$F_{Y_3}(y_3) = P(Y_3 \leq y_3)$$

$$= P\left(\frac{1}{Z} \leq y_3\right)$$

$$= P\left(\frac{1}{y_3} \leq Z < 0\right)$$

$$= \Phi(0) - \Phi\left(\frac{1}{y_3}\right)$$

For any $y_3 > 0$, its cdf is

$$F_{Y_3}(y_3) = P(Y_3 \le y_3)$$
$$= P\left(\frac{1}{Z} \le y_3\right)$$
$$= P\left(Z \geq \frac{1}{y_3}\right)$$
$$= 1 - \Phi\left(\frac{1}{y_3}\right).$$

And $F_{Y_3}(0) = P(Y_3 \le 0) = P\left(\frac{1}{Z} \le 0\right) = P(Z < 0) = \Phi(0)$.

$$\text{So } F_{Y_3}(y_3) = \begin{cases} \Phi(0) - \Phi\left(\frac{1}{y_3}\right) & \text{if } y_3 < 0 \\ \Phi(0) & \text{if } y_3 = 0 \\ 1 - \Phi\left(\frac{1}{y_3}\right) & \text{if } y_3 > 0 \end{cases}$$

Since its pdf is not well-defined at $y_3 = 0$, we only compute $F'_{Y_3}(y_3)$ when $y_3 \neq 0$. So the pdf of $Y_3 = \frac{1}{Z}$ is

$$f_{Y_3}(y_3) = F'_{Y_3}(y_3) = \frac{1}{y_3^2}\phi\left(\frac{1}{y_3}\right)$$
$$= \frac{1}{\sqrt{2\pi}} y_3^{-2} e^{-\frac{1}{2y_3^2}} \quad \text{for } y_3 \neq 0.$$

d) Let $Y_4 = \frac{1}{Z^2} > 0$. For any $y_4 > 0$, its cdf is

$$F_{Y_4}(y_4) = P(Y_4 \le y_4)$$
$$= P\left(\frac{1}{Z^2} \le y_4\right)$$
$$= P\left(Z^2 \geq \frac{1}{y_4}\right)$$

$$= P\left(Z \le -\frac{1}{\sqrt{y_4}}\right) + P\left(Z \ge \frac{1}{\sqrt{y_4}}\right)$$

$$= \Phi\left(-\frac{1}{\sqrt{y_4}}\right) + 1 - \Phi\left(\frac{1}{\sqrt{y_4}}\right)$$

For any $y_4 \le 0$, $F_{Y_4}(y_4) = 0$

So $$F_{Y_4}(y_4) = \begin{cases} \Phi\left(-\frac{1}{\sqrt{y_4}}\right) + 1 - \Phi\left(\frac{1}{\sqrt{y_4}}\right) & \text{if } y_4 > 0 \\ 0 & \text{if } y_4 \le 0 \end{cases}$$

Take derivative to get its pdf: for any $y_4 \neq 0$

$$f_{Y_4}(y_4) = F'_{Y_4}(y_4) = \begin{cases} \frac{1}{2}\phi\left(-\frac{1}{\sqrt{y_4}}\right) y_4^{-\frac{3}{2}} + \frac{1}{2}\phi\left(\frac{1}{\sqrt{y_4}}\right) y_4^{-\frac{3}{2}} & \text{if } y_4 > 0 \\ 0 & \text{if } y_4 \le 0 \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} y_4^{-\frac{3}{2}} e^{-\frac{1}{2y_4}} & \text{if } y_4 > 0 \\ 0 & \text{if } y_4 \le 0 \end{cases}$$

All pdfs in a) – d) can be calculated using the change of variable formula, and you can get the same expressions. $\square$

**Exercise 3.1.6.**

There are 8 equally likely outcomes for 3 fair coin tosses:

| outcome | probability | X | Y | X+Y |
|---|---|---|---|---|
| HHH | $\frac{1}{8}$ | 2 | 2 | 4 |
| HHT | $\frac{1}{8}$ | 2 | 1 | 3 |
| HTH | $\frac{1}{8}$ | 1 | 1 | 2 |
| HTT | $\frac{1}{8}$ | 1 | 0 | 1 |
| THH | $\frac{1}{8}$ | 1 | 2 | 3 |
| THT | $\frac{1}{8}$ | 1 | 1 | 2 |
| TTH | $\frac{1}{8}$ | 0 | 1 | 1 |
| TTT | $\frac{1}{8}$ | 0 | 0 | 0 |

a) Joint distribution table for $(X, Y)$

| Y \ X | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\frac{1}{8}$ | $\frac{1}{8}$ | 0 |
| 1 | $\frac{1}{8}$ | $\frac{2}{8}$ | $\frac{1}{8}$ |
| 2 | 0 | $\frac{1}{8}$ | $\frac{1}{8}$ |

b). X and Y are not independent.

Since $P(X=2, Y=0) = 0$,

$$P(X=2) = 0 + \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

$$P(Y=0) = \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4}.$$

$$P(X=2, Y=0) = 0 \neq \frac{1}{16} = P(X=2)P(Y=0).$$

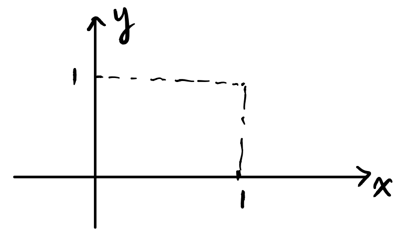
c). The distribution table for $X+Y$ is

| $z$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X+Y=z)$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{8}$. |

$\square$

**Exercise 5.2.3**

a) Since




$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy \\
&= \int_0^1 \int_0^1 c(x^2+4xy)\,dx\,dy \\
&= \int_0^1 \left( \frac{c}{3}x^3 \Big|_{x=0}^{x=1} + 2cy\cdot x^2 \Big|_{x=0}^{x=1} \right) dy \\
&= \int_0^1 \left(\frac{c}{3}+2cy\right) dy \\
&= \frac{c}{3}y \Big|_{y=0}^{y=1} + cy^2 \Big|_{y=0}^{y=1} \\
&= \frac{c}{3}+c = \frac{4}{3}c,
\end{aligned}$$

we get $\boxed{c = \frac{3}{4}}$

b) For $0<a<1$,

$$\begin{aligned}
P(X\le a) &= \frac{3}{4}\int_0^1 \int_0^a (x^2+4xy)\,dx\,dy \\
&= \frac{3}{4}\int_0^1 \left( \frac{1}{3}x^3 \Big|_{x=0}^{x=a} + 2y\cdot x^2 \Big|_{x=0}^{x=a} \right) dy \\
&= \frac{3}{4}\int_0^1 \left(\frac{1}{3}a^3 + 2a^2 y\right) dy
\end{aligned}$$

$$= \frac{3}{4}\left( \frac{1}{3}a^3 y \Big|_{y=0}^{y=1} + a^2 y^2 \Big|_{y=0}^{y=1} \right)$$

$$= \frac{3}{4}\left( \frac{1}{3}a^3 + a^2 \right)$$

$$= \frac{a^2(a+3)}{4}$$

c) For $0<b<1$,

$$P(Y \le b) = \frac{3}{4}\int_0^b \int_0^1 (x^2 + 4xy)\, dx\, dy$$

$$= \frac{3}{4}\int_0^b \left( \frac{1}{3}x^3 \Big|_{x=0}^{x=1} + 2y x^2 \Big|_{x=0}^{x=1} \right) dy$$

$$= \frac{3}{4}\int_0^b \left( \frac{1}{3} + 2y \right) dy$$

$$= \frac{3}{4}\left( \frac{1}{3}y \Big|_{y=0}^{y=b} + y^2 \Big|_{y=0}^{y=b} \right)$$

$$= \frac{3}{4}\left( \frac{b}{3} + b^2 \right)$$

$$= \frac{b(3b+1)}{4}$$

$\square$

**Exercise 5.2.4**

a) If $x>0$ and $y>0$,

$$P(X \le x, Y \le y) = \int_0^x \int_0^y 6e^{-2u-3v}\, dv\, du$$

$$= 6\int_0^x e^{-2u}\left( \int_0^y e^{-3v}\, dv \right) du$$

$$= 6\int_0^x e^{-2u}\left[-\frac{1}{3}\int_0^y e^{-3v}\, d(-3v)\right] du$$

$$= -2\int_0^x e^{-2u} \cdot \left(e^{-3v}\Big|_{v=0}^{v=y}\right) du$$

$$= -2(e^{-3y}-1)\int_0^x e^{-2u}\, du$$

$$= (e^{-3y}-1)\int_0^x e^{-2u}\, d(-2u)$$

$$= (e^{-3y}-1)\cdot\left(e^{-2u}\Big|_{u=0}^{u=x}\right)$$

$$= (e^{-3y}-1)\cdot(e^{-2x}-1)$$

$$= (1-e^{-2x})\cdot(1-e^{-3y})$$

If one of $x$ and $y < 0$, then $P(X \leq x, Y \leq y) = 0$.

Hence

$$P(X \leq x, Y \leq y) = \begin{cases} (1-e^{-2x})\cdot(1-e^{-3y}) & \text{if } x>0 \text{ and } y>0 \\ 0 & \text{o.w.} \end{cases}$$

b) For $x > 0$,

$$f_X(x) = \int_0^{+\infty} 6e^{-2x-3y}\, dy$$

$$= (-2)e^{-2x}\int_0^{+\infty} e^{-3y}\, d(-3y)$$

$$= (-2)e^{-2x}\cdot\left(e^{-3y}\Big|_{y=0}^{y=+\infty}\right)$$

$$= -2e^{-2x}\cdot(0-1) = 2e^{-2x}$$

So $f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$

c) For $y>0$,

$$f_Y(y) = \int_0^{+\infty} 6e^{-2x-3y}\, dx$$

$$= (-3)\, e^{-3y} \int_0^{+\infty} e^{-2x}\, d(-2x)$$

$$= (-3)\, e^{-3y} \cdot \left( e^{-2x} \Big|_{x=0}^{x=+\infty} \right)$$

$$= -3e^{-3y}(0-1)$$

$$= 3e^{-3y}$$

So $f_Y(y) = \begin{cases} 3e^{-3y} & \text{if } y>0 \\ 0 & \text{if } y \le 0. \end{cases}$

d) Yes, they are independent since

$$f_X(x) \cdot f_Y(y) = \begin{cases} 6e^{-2x-3y} & \text{if } x>0 \text{ and } y>0 \\ 0 & \text{o.w.} \end{cases}$$

$$= f(x,y)$$

$\square$.