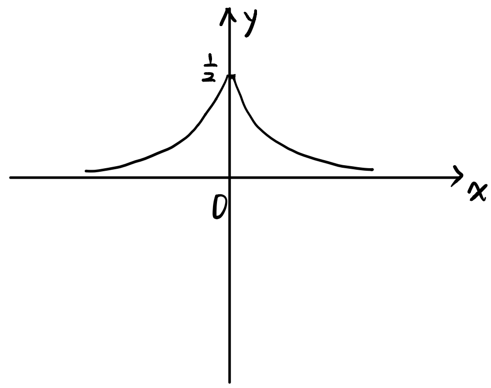
1.(a)



(b). $P(-1 < X < 2) = \int_{-1}^{2} \frac{1}{2(1+|x|)^2}\,dx$

$$= \int_{-1}^{0} \frac{1}{2(1-x)^2}\,dx + \int_{0}^{2} \frac{1}{2(1+x)^2}\,dx$$

$$= -\int_{-1}^{0} \frac{1}{2(1-x)^2}\,d(1-x) - \frac{1}{2}(1+x)^{-1}\Big|_0^2$$

$$= \frac{1}{2}(1-x)^{-1}\Big|_{-1}^{0} - \frac{1}{2}(1+x)^{-1}\Big|_0^2$$

$$= \frac{1}{2}\left[(1-0)^{-1} - (1-(-1))^{-1}\right] - \frac{1}{2}\left[(1+2)^{-1} - (1+0)^{-1}\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{3} - 1\right)$$

$$= \frac{7}{12}$$

(c). $P(|X| > 1) = 2P(X > 1)$

$$= 2\int_{1}^{+\infty} \frac{1}{2(1+x)^2}\,dx$$

$$= \int_{1}^{+\infty} \frac{1}{(1+x)^2}\,dx$$

$$= -(1+x)^{-1}\Big|_1^{+\infty}$$

$$= -\left(0 - \frac{1}{2}\right) = \frac{1}{2}$$

(d). No, because

$$E(|x|) = \int_{-\infty}^{+\infty} \frac{|x|}{2(1+|x|)^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{-x}{2(1-x)^2}\,dx + \int_{0}^{+\infty} \frac{x}{2(1+x)^2}\,dx$$

$$= -\int_{-\infty}^{0} \frac{-x}{2(1-x)^2}\,d(-x) + \int_{0}^{+\infty} \frac{x}{2(1+x)^2}\,dx$$

$\downarrow$ Change of variable: $u=-x$

$$= \int_{0}^{+\infty} \frac{u}{2(1+u)^2}\,du + \int_{0}^{+\infty} \frac{x}{2(1+x)^2}\,dx$$

$$= 2\int_{0}^{+\infty} \frac{x}{2(1+x)^2}\,dx$$

$$= \int_{0}^{+\infty} \frac{x}{(1+x)^2}\,dx$$

$$= \int_{0}^{+\infty} \frac{1+x-1}{(1+x)^2}\,dx$$

$$= \int_{0}^{+\infty} \frac{1}{1+x}\,dx - \int_{0}^{+\infty} \frac{1}{(1+x)^2}\,dx$$

$$= \underbrace{\ln(1+x)\Big|_{0}^{+\infty}}_{=\infty} + \underbrace{\frac{1}{(1+x)}\Big|_{0}^{+\infty}}_{0-1=-1}$$

$$= \infty.$$

$\square$

2 As we know, if $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$.

$$\begin{aligned}\text{Since } \frac{1}{3} &= P(X \le 0) \\ &= P\left(\frac{X-\mu}{\sigma} \le -\frac{\mu}{\sigma}\right) \\ &= \Phi\left(-\frac{\mu}{\sigma}\right),\end{aligned}$$

$$\Phi\left(\frac{\mu}{\sigma}\right) = 1 - \Phi\left(-\frac{\mu}{\sigma}\right) = 1 - \frac{1}{3} = \frac{2}{3}.$$

By the Normal table, $\frac{\mu}{\sigma} = 0.43$ ①

$$\begin{aligned}\text{Similarly, } \frac{2}{3} &= P(X \le 1) \\ &= P\left(\frac{X-\mu}{\sigma} \le \frac{1-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{1-\mu}{\sigma}\right)\end{aligned}$$

By the Normal table, $\frac{1-\mu}{\sigma} = 0.43$ ②

By equations ① and ②, we can solve μ and σ as

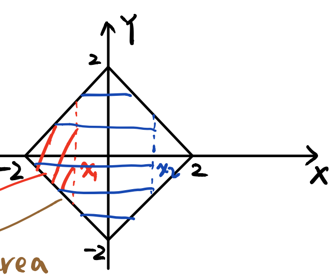
$$\begin{cases} \mu = 0.5 \\ \sigma = 1.162 \end{cases}$$

3. First, let's find its C.D.F.

① If $x < -2$, $F(x) = P(X \le x) = 0$

② If $-2 \le x < 0$,

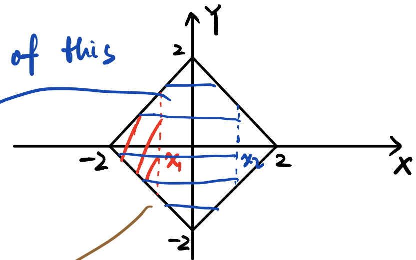
$$F(x) = P(X \le x) = \frac{2 \cdot \frac{1}{2}[x-(-2)]^2}{(2\sqrt{2})^2}$$



$$= \frac{1}{8}(x+2)^2$$

③ If $0 \le x \le 2$,

the area of this blue area



$$F(x) = P(X \le x)$$

$$= \frac{(2\sqrt{2})^2 - 2 \cdot \frac{1}{2}(2-x)^2}{(2\sqrt{2})^2}$$

the area of the whole square.

$$= 1 - \frac{1}{8}(2-x)^2$$

④ If $x > 2$, $F(x) = P(X \le x) = 1$

Hence the pdf

$$f(x) = F'(x) = \begin{cases} 0 & \text{if } x < -2 \text{ or if } x > 2 \\ \frac{2+x}{4} & \text{if } -2 \le x < 0 \\ \frac{2-x}{4} & \text{if } 0 \le x \le 2 \end{cases}$$

$$= \begin{cases} \frac{2-|x|}{4} & \text{if } -2 \le x \le 2 \\ 0 & \text{o.w.} \end{cases}$$

$\square$