Definition 6.4

If $\mathcal{L}: V \to W$ is a linear transformation, then the range of $\mathcal{L}$ or image of V under $\mathcal{L}$, denoted by range $\mathcal{L}$ is the set of all vectors $\vec{w} \in W$ such that $\mathcal{L}(\vec{v}) = \vec{w}$ for some $\vec{v} \in V$.

$\mathcal{L}$ is called onto whenever range $\mathcal{L} = W$

Example: Define $\mathcal{L}$ from $\mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathcal{L}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_3 \\ x_2 \end{bmatrix}$$

What is ker. $\mathcal{L}$

$$\ker.\mathcal{L} = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix},\ x_1 \in \mathbb{R} \right\}$$

What is range $\mathcal{L}$?

$$\text{Range}\,\mathcal{L} = \mathbb{R}^2$$

Why?

Given $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

$$\mathcal{L}\left(\begin{bmatrix} 0 \\ v_2 \\ -v_1 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

So, any vector in $\mathbb{R}^2$ can be achieved by $\mathcal{L}$,

alternately $\underbrace{\mathcal{L}\left(\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_2 \text{ and } \mathcal{L}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2}_{\text{standard basis.}}$

In this case, $\mathcal{L}$ is onto.

Example: $\mathcal{L}: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\mathcal{L}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(a) Is $\mathcal{L}$ onto?

Given any $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$

If $\mathcal{L}$ is onto, there exists $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$

such that $L(\vec{u}) = \vec{w}$

$$\mathcal{L}(\vec{u}) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

look at linear system,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 1 & 1 & c-2a \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 & a \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{bmatrix}$$

not all a, b, c behaves like $c-b-a=0$.

so, $\mathcal{L}$ is not onto.

(b) Find a basis for Range $\mathcal{L}$

Range of $\mathcal{L}$ is the set of vectors $\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\}$

such that $c-b-a=0$

$$c=a+b$$

so, range of $\mathcal{L}$ is all vectors of the form

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

therefore, the basis for Range $\mathcal{L}$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

Dim (Range $\mathcal{L}$) $= 2$

range $\mathcal{L}$ is the column space of A.

(c) Find ker $\mathcal{L}$

need solutions to

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right]$ i.e. Find the null space of the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\ker \mathcal{L} = \begin{bmatrix} u_1 \\ u_1 \\ -u_1 \end{bmatrix}$ $\quad$ $\mathrm{Dim}(\ker \mathcal{L}) = 1$, Basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

(d) Is $\mathcal{L}$ one-to-one?

No, because $\mathrm{Dim}(\ker \mathcal{L}) > 0$.

Theorem 6.6 (Rank-Nullity)

If $\mathcal{L}: V \to W$ is a linear transformation, and $\mathrm{Dim}\, V = n$.

Then $\mathrm{Dim}(\ker. \mathcal{L}) + \mathrm{Dim}(\mathrm{Range}\, \mathcal{L}) = \mathrm{Dim}\, V$

Proof: let $k = \dim(\ker. \mathcal{L})$, $k \le n$,

suppose $k = n$, in this case, $\ker \mathcal{L} = V$

so $\mathrm{range}\, \mathcal{L} = \{\vec{0}_W\}$ $\mathrm{Dim}(\mathrm{Range}\, \mathcal{L}) = 0$

$\text{Dim}(\text{Ker } \mathcal{L}) + \text{Dim}(\text{Range } \mathcal{L}) = n + 0 = n = \dim V$

Now, suppose $1 < k < n$, let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a basis for kernel

Extend this to a full basis for V,

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \dots, \vec{v}_n\}$$

Show

$$\{\mathcal{L}(\vec{v}_{k+1}), \mathcal{L}(\vec{v}_{k+2}) \dots \mathcal{L}(\vec{v}_n)\} \text{ is a basis for range } \mathcal{L}$$

suppose $\vec{w} \in \text{Range } \mathcal{L}$

$\mathcal{L}(\vec{v}) = \vec{w}$ for some $\vec{w} \in W$

$\mathcal{L}(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k + a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n) = \vec{w}$

$a_1 \underset{0}{\underset{\parallel}{\mathcal{L}(\vec{v}_1)}} + a_2 \underset{0}{\underset{\parallel}{\mathcal{L}(\vec{v}_2)}} + \dots + a_k \underset{0}{\underset{\parallel}{\mathcal{L}(\vec{v}_k)}} + a_{k+1}\mathcal{L}(\vec{v}_{k+1}) + \dots + a_n\mathcal{L}(\vec{v}_n) = \vec{w}$

So. $\{\mathcal{L}(\vec{v}_1), \mathcal{L}(\vec{v}_2) \dots, \mathcal{L}(\vec{v}_n)\}$ spans Range of $\mathcal{L}$

but $a_{k+1}\mathcal{L}(\vec{v}_{k+1}) + \dots + a_n\mathcal{L}(\vec{v}_n) = \vec{w}$

then $\{\mathcal{L}(\vec{v}_{k+1}), \dots, \mathcal{L}(\vec{v}_n)\}$ spans Range of $\mathcal{L}$

need $\{L(\vec{v}_{k+1}), \ldots, L(\vec{v}_n)\}$ linearly independent.

suppose there exists $c_{k+1} \cdots c_n \in \mathbb{R}$

$$c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = \vec{0}$$

$$L(c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n) = \vec{0}$$

$$c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n \in \ker L$$

there is no way for this to be in the $\ker L$ with

basis $\{\vec{v}_1 \cdots \vec{v}_k\}$ unless $c_{k+1} \cdots c_n = 0$

hence $\{L(\vec{v}_{k+1}), \ldots L(\vec{v}_n)\}$ is linear independent,

hence it is the basis for the range L

$\mathrm{Dim}(\mathrm{Range}\, L) = n-k$

When $k=0$, similar argument as before, shows

$\mathrm{Dim}(\mathrm{Range}\, L) = n$

Corollary: 6.2.

If $L: V \to W$ is a linear transformation and $\dim V = \dim W$

then $\mathcal{L}$ is ① one-to-one and ② onto.

Proof ①: Dim ker $L = 0$ so Dim Range $\mathcal{L} = n =$ Dim W

② $\mathcal{L}$ is onetoone $\Rightarrow$ onto.

Range $\mathcal{L} = W$

Dim Range $\mathcal{L} =$ Dim $W =$ Dim V,

Dim ker $\mathcal{L} = 0$.

Theorem 6.7 A linear transformation $\mathcal{L}: V \to W$ is invertible if and only if $\mathcal{L}$ is one-to-one and onto.