# 6.5 Orthogonal Complements

Def: Let $(V, ( , ))$ be an inner product space and let $W$ be a subspace.

A vector $u$ in $V$ is orthogonal to $W$ is it is orthogonal to every vector in $W$:

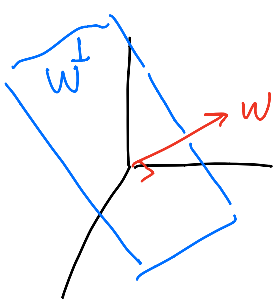
$$u \perp W \longleftrightarrow (u, w) = 0 \quad \text{for all } w \in W.$$

The set $W^\perp$ of all vectors in $V$ orthogonal to $W$ is called the orthogonal complement of $W$.

Ex. $W$ = subspace of $\mathbb{R}^3$ spanned by $w = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

$$W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| 2x - 3y + 4z = 0 \right\} \text{ - plan in } \mathbb{R}^3$$




Facts $W^\perp$ is a subspace of $V$

- if $u, v$ in $W^\perp$, show $u+v$, $cu$ in $W^\perp$

for any $w$ in $W$:

$$(u+v, w) = \underbrace{(u, w)}_{0} + \underbrace{(v, w)}_{0} = 0$$

$$(cu, w) = c(u, w) = 0$$

- $\vec{0}$ is the only vectors in both $W$ and $W^{\perp}$

if $u$ in in $W$ and $W^{\perp}$:

$(u, u) = 0 \rightarrow u$ is $\vec{0}$

Theorem: If $W$ is a subspace of a inner product space $V$,

then any $v$ in $V$ can be written uniquely as

$$v = w + w^{\perp}$$

with $w$ in $W$ and $w^{\perp}$ in $W^{\perp}$

say $\dim W = m$, let $S = \{w_1, \dots, w_m\}$ be an orthonormal basis for $W$.

project $v$ on $W$: get $w = (v, w_1)w_1 + \cdots + (v, w_m)w_m$

let $w^{\perp} = v - w$ $\qquad w = \sum_{j=1}^{m} (v, w_j) w_j.$

to check $w^{\perp} \in W^{\perp}$, suffices to show $(w^{\perp}, w_i) = 0$ for $i \dots m$.

$(w^{\perp}, w_i) = (v - w, w_i) = (v, w_i) - \boxed{(w, w_i)} \rightarrow$ coeff of $w$ in the orthogonal basis.

$= (v, w_i) - (v, w_i)$

$= 0$

$$(w, w_i) = \left( \sum_{j=1}^{n} (v, w_j) w_j \, , \, w_i \right)$$

$$= \sum_{j=1}^{n} \left( (v, w_j) w_j \, , \, w_i \right)$$

$$= \sum_{j=1}^{n} (v, w_j) \underbrace{(w_j, w_i)}_{= \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}}$$

$$= (v, w_i)$$

Relation to matrices

solve $A\underline{x} = \underline{0}$, $\mathrm{Null}(A)$ = solutions

$A_{m \times n}$ $\qquad = \{ \underline{x} \text{ in } \mathbb{R}^n \mid A\underline{x} = \underline{0} \}$

$$\begin{pmatrix} \underline{R_1} \\ \underline{R_2} \\ \vdots \\ \underline{R_m} \end{pmatrix} \Bigg| \underline{x} = \begin{pmatrix} R_1 \cdot \underline{x} \\ R_2 \cdot \underline{x} \\ \vdots \\ R_m \cdot \underline{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad = (\mathrm{Row}(A))^{\perp}$$

$\mathrm{Null}(A) = (\mathrm{Row}(A))^{\perp}$ $\qquad$ $\mathrm{Col}(A) = \mathrm{Row}(A^T) = (\mathrm{Null}(A^T))^{\perp}$

$\mathrm{Null}(A)^{\perp} = \mathrm{Row}(A)$ $\qquad$ $\mathrm{Col}(A)^{\perp} = (\mathrm{Null}(A^T))$

Ex Find the basis for the orthogonal complement of the subspace $W$ of $\mathbb{R}_5$ spanned by

$w_1 = [2\ \ -1\ \ 0\ \ 1\ \ 2]$

$w_2 = [1\ \ 3\ \ 1\ \ -2\ \ -4]$

$w_3 = [3\ \ 2\ \ 1\ \ -1\ \ -2]$

$w_4 = [7\ \ 7\ \ 3\ \ -4\ \ -8]$

$w_5 = [1\ \ -4\ \ -1\ \ -1\ \ -2]$

easy way Let $A$ be the matrix with rows $R_1 = w_1, \dots R_5 = w_5$

Then $W = \text{Row}(A)$

So $W^{\perp} = \text{Row}(A)^{\perp} = \text{Null}(A)$

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 2 \\ 1 & 3 & 1 & -2 & -4 \\ 3 & 2 & 1 & -1 & -2 \\ 7 & 7 & 3 & -4 & -8 \\ 1 & -4 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \quad \text{sol} = x = r \underbrace{\begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u_1} + s \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{u_2}$$

$\text{Null}(A) = \text{span}\{u_1, u_2\}$

long way

Let $w = [a \ b \ c \ d \ e]$ in $W^{\perp} \longrightarrow (w, w_1) = 0 \cdots (w, w_5) = 0$

$$
\begin{aligned}
(w, w_1) &= w \cdot w_1 = \\
(w, w_2) &= w \cdot w_2 =
\end{aligned}
\begin{bmatrix}
2a - b + d + 2e = 0 \\
a + 3b + c - 2d - 4e = 0 \\
\vdots
\end{bmatrix}
$$

$\sim$ matrix $A$,
solve $Ax = 0$
find Null($A$)