Inner product spaces

Def: Let V be a vector space (real scalars). An inner product on V is a function assigning a number $(u,v,)$ to any pair u, v of vectors in V, with the following properties

a) $(u,v) \geq 0$; $(u,u) = 0 \longleftrightarrow u = \vec{0}$

b) $(u,v) = (v,u)$, for any u, v in V

c) $(u+v, w) = (u,w) + (v,w)$ for all u, v in V

d) $(cu, v) = c(u,v)$ for all u, v in V and $c \in \mathbb{R}$

Ex. $\mathbb{R}^n$, $(u,v) = u \cdot v$ dot product $\rightsquigarrow$ Inner product

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \rightarrow u \cdot v = u^T \cdot v = (u_1 \cdots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \cdots + u_n v_n$$

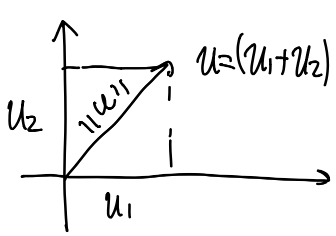
$$u \cdot u = u_1^2 + \cdots + u_n^2 \geq 0$$

$$u \cdot u = 0 \longleftrightarrow u_1 = \cdots = u_n \longleftrightarrow u = \vec{0}$$

$u \cdot u = \|u\|^2$

$\|u\|$ ~ length of $\vec{u}$

$$v \cdot u = v_1 u_1 + \cdots + v_n u_n = u \cdot v = u_1 v_1 + \cdots + u_n v_n$$



Ex. If V is a vector space of dim n and $S=\{u_1, \cdots u_n\}$ is a basis

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \longrightarrow [v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ in } \mathbb{R}^n$$

$$w = b_1u_1 + b_2u_2 + \cdots + b_nu_n \longrightarrow [w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \text{ in } \mathbb{R}^n$$

define $(v, w) = [v]_S \cdot [w]_S = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

↳ inner product on V (depends on S)

Ex.

$V = \mathscr{C}([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ continuous}\}$

for f, g in V: $(f, g) = \int_0^1 f(t)g(t)\,dt \rightsquigarrow$ inner product

$$\cdot\ (f_1, f_1) = \int_0^1 \underbrace{f^2(t)}_{\geq 0}\,dt \geq 0 \quad \& \quad \int_0^1 [f(t)]^2\,dt = 0 \longleftrightarrow f(t)^2 = 0 \Leftrightarrow f = 0$$

V-vector space with a basis $S = \{u_1 \cdots u_n\}$ and an inner product $(-,-)$.

Let v and w be in V

$$v = a_1u_1 + \cdots + a_nu_n \quad , \quad w = b_1u_1 + \cdots + b_nu_n$$

$$(v, w) = (a_1 u_1 + \cdots + a_n u_n), w \qquad (u, v_1 + v_2) = (v_1 + v_2, u)$$

$$= (a_1 u_1, w) + \cdots + (a_n u_n, w) \qquad = (v_1, u) + (v_2, u)$$

$$= (u, v_1) + (u, v_2)$$

$$= a_1 (u_1, w) + \cdots + a_n (u_n, w) \qquad (u, cv) = (cv, u) = c(v, u)$$

$$= a_1 (u_1, b_1 u_1 + \cdots + b_n u_n) + \underbrace{\quad}_{a_i (u_i, w)} a_n (u_n, b_1 u_1 + \cdots + b_n u_n)$$

$$= a_1 (u_1, b_1 u_1) + \cdots + a_1 (u_1, b_j u_j) + \cdots + a_1 (u_1, b_n u_n)$$

$$+ a_i (u_i, b_1 u_1) + \cdots + a_i (u_i, b_j u_j) + \cdots + a_i (u_i, b_n u_n)$$

$$+ a_n (u_n, b_1 u_1) + \cdots + a_n (u_n, b_j u_j) + \cdots + a_n (u_n, b_n u_n)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i (u_i, u_j) b_j$$

Let $c_{ij} = (u_i, u_j)$, $i, j = 1, \ldots, n$ let $C = [c_{ij}]_{i,j=1,\ldots,n}$

$= (u_j, u_i)$ $\hookrightarrow$ symmetric square matrix

$= c_{ji}$

$$(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j = [v]_S^T \cdot C \cdot [w]_S$$

Ex. If $(v, w) = [v]_S \cdot [w]_S$

$$C_{ij} = (u_i, u_j) = \underbrace{\begin{array}{c} \\ i^{th} \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_i} \cdot \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}\begin{array}{c} \\ j^{th} \\ \\ \end{array}}_{e_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

So. $C = [C_{ij}]$ is an identity matrix.

Def A real vector space with an inner product is called an inner product space.

If V is finite dimensional with inner product, it is called an Euclidean space.

Notation $\|u\| = \sqrt{u, u}$ length of u in V

$\|\vec{0}\| = 0$

Theorem Cauchy - Schwarz inequality

If u, v are vectors in an inner product space V,

then $|(u, v)| \leq \|u\| \cdot \|v\|$

If $u = 0$, both sides are 0. ✓

If $u \neq 0$, let $w = cu + v$, c-scalar

$$0 \leq (w, w) = (cu + v, cu + v) = (cu, cu) + (cu, v) + (v, cu) + (v, u)$$
$$= c^2 (u, u) + c(u, v) + c(v, u) + (v, v)$$

$$= c^2\|u\|^2 + 2c(u,v) + \|v\|^2$$

$$\Downarrow$$

$$4(u,v)^2 - 4\|u\|^2\|v\|^2 \le 0$$

$$\Downarrow$$

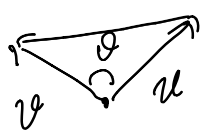
$$|(u,v)| \le \|u\|\cdot\|v\|$$

$$\Longrightarrow -\|u\|\cdot\|v\| \le (u,v) \le \|u\|\cdot\|v\|$$

$$\Longrightarrow -1 \le \frac{(u,v)}{\|u\|\cdot\|v\|} \le 1$$

For any $-1 \le t \le 1$, there is a unique angle $0 \le \theta \le \pi$

so that $\cos\theta = t$



def: The angle between u and v

is the unique $0 \le \theta \le \pi$ so that

$$\cos\theta = \frac{(u,v)}{\|u\|\cdot\|v\|}$$

Say that u and v are orthogonal (or perpendicular) if $(u,v) = 0$.

equivalently, $\theta = \frac{\pi}{2}$

Cauchy-Schwarz: $|(u,v)| \le \|u\|\cdot\|v\|$

Ex. $V = (\mathbb{R}^n, \cdot)$

$$|(u,v)| = |u_1v_1 + \cdots + u_nv_n| \le \sqrt{u_1^2 + \cdots + u_n^2}\cdot\sqrt{v_1^2 + \cdots + v_n^2}$$

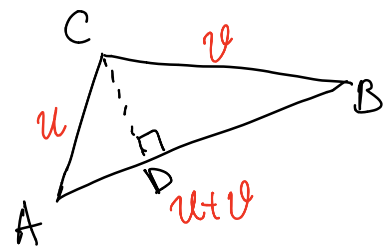
$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\|u\| = \sqrt{(u,u)} = \sqrt{u \cdot u} = \sqrt{u_1^2 + \cdots + u_n^2}$

$$(u_1v_1 + \cdots + u_nv_n)^2 \le (u_1^2 + \cdots + u_n^2)(v_1^2 + \cdots + v_n^2)$$

Ex. $V = \mathcal{C}[0,1]$, $(f,g) = \int_0^1 f(t)g(t)\,dt$

C-S → $\left(\int_0^1 f(t)g(t)\,dt\right)^2 \le \int_0^1 [f(t)]^2\,dt \cdot \int_0^1 [g(t)]^2\,dt$



AC+BC > AB

AC>AD, BC>BD, AD+DB=AB

Corollary: Triangular inequality:

If u, v are vectors in an inner product space V, then

$$\|u+v\| \le \|u\| + \|v\|$$

$\|u+v\|^2 = (u+v, u+v) = (u,u) + (u,v) + (v,u) + (v,v)$

$= \|u\|^2 + 2(u,v) + \|v\|^2$

$\le \|u\|^2 + 2|u,v| + \|v\|^2$

$$\overset{C-W}{\leq} \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$