1(a). Define $N_{loc}(t)$ = # of local calls in t minutes.

$N_{dis}(t)$ = # of long-distance calls in t minutes.

We know $N_{loc}(t) \sim \text{Poisson}(\lambda_{loc} t)$,

$N_{dis}(t) \sim \text{Poisson}(\lambda_{dis} t)$,

and $N_{loc}(t_1) \perp\!\!\!\perp N_{dis}(t_2)$ for any $t_1, t_2 \geq 0$.

Hence $P(N_{loc}(1) = 5,\ N_{dis}(1) = 3)$

$$= P(N_{loc}(1) = 5) \cdot P(N_{dis}(1) = 3)$$

$$= \left[\frac{\lambda_{loc}^5}{5!} e^{-\lambda_{loc}} \right] \cdot \left[\frac{\lambda_{dis}^3}{3!} e^{-\lambda_{dis}} \right] \quad \square$$

1(b) Since $N_{loc}(3) \sim \text{Poisson}(3\lambda_{loc})$,

$N_{dis}(3) \sim \text{Poisson}(3\lambda_{dis})$

and $N_{loc}(3) \perp\!\!\!\perp N_{dis}(3)$,

$N_{loc}(3) + N_{dis}(3) \sim \text{Poisson}(3(\lambda_{loc} + \lambda_{dis}))$

So $P(N_{loc}(3) + N_{dis}(3) = 50)$

$$= \frac{[3(\lambda_{loc} + \lambda_{dis})]^{50}}{50!} e^{-3(\lambda_{loc} + \lambda_{dis})} \quad \square$$

2. Method I: Distribution function technique:

Step 1): Find the values that Y can take.

Let $Y = \tan(\pi U - \frac{\pi}{2})$.

Since $U \in (0,1)$, $\pi U \in (0, \pi)$, $\pi U - \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$Y \in (-\infty, +\infty)$.

Step 2): Find the cdf of Y, and express it as the cdf of U.

And we know function $\tan(x)$ is monotonely increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

So the c.d.f. of Y is, for any $y \in (-\infty, +\infty)$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(\tan(\pi U - \tfrac{\pi}{2}) \leq y) \\
&= P(\pi U - \tfrac{\pi}{2} \leq \arctan(y)) \\
&= P(U \leq \tfrac{1}{\pi}\arctan(y) + \tfrac{1}{2}) \\
&= \tfrac{1}{\pi}\arctan(y) + \tfrac{1}{2}
\end{aligned}
$$

Step 3): Take derivative to get the pdf of Y.

So pdf of Y is
$$
\begin{aligned}
f_Y(y) &= F_Y'(y) \\
&= [\tfrac{1}{\pi}\arctan(y) + \tfrac{1}{2}]' \\
&= \frac{1}{\pi(1+y^2)} \quad \text{for any } y \in (-\infty, +\infty)
\end{aligned}
$$

Hence $Y \sim$ Cauchy distribution.

$\square$

Method II: Change of variable formula:

Step 1): write down the pdf of U and the range of Y:

pdf of U: $f_U(u) = \begin{cases} 1 & \text{if } 0<u<1 \\ 0 & \text{o.w.} \end{cases}$

range of Y: $Y \in (-\infty, +\infty)$. (already be shown)

Step 2): Find the inverse function of U with respect to Y. $U = g^{-1}(Y)$.

Since $Y = \tan(\pi U - \frac{\pi}{2})^{=g(U)}$, and $\pi U - \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\pi U - \frac{\pi}{2} = \arctan(Y),$$

So $$\pi U = \arctan(Y) + \frac{\pi}{2}$$

and $$U = \frac{1}{\pi}\arctan(Y) + \frac{1}{2} = g^{-1}(Y).$$

Step 3): Find the derivative of the inverse function.

$$[g^{-1}(y)]' = \frac{1}{\pi(1+y^2)}$$

Step 4): Calculate the pdf of Y using the change of variable formula:

$$f_Y(y) = f_U(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= 1 \cdot \left| \frac{1}{\pi(1+y^2)} \right|$$

$$= \frac{1}{\pi(1+y^2)} \quad \text{for any } y \in (-\infty, +\infty)$$

Don't forget to indicate the domain of $f_Y(y)$!!! $\square$