1(a). $$P(X=Y) = \sum_{k=1}^{n} P(X=k, Y=k)$$
$$= \sum_{k=1}^{n} P(X=k)P(Y=k) \quad \text{(independence)}$$
$$= \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{1}{n}$$
$$= \boxed{\frac{1}{n}}$$

1(b) Method 1:
$$P(X<Y) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(X=i, Y=j)$$
$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(X=i)P(X=j) \quad \text{(independence)}$$
$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{n^2}$$
$$= \sum_{i=1}^{n-1} \frac{n-i}{n^2}$$
$$= \frac{1}{n}\cdot(n-1) - \frac{1}{n^2}\cdot\frac{(n-1)\cdot[(n-1)+1]}{2}$$
$$= \frac{n-1}{n} - \frac{n-1}{2n}$$
$$= \boxed{\frac{n-1}{2n}}$$

Method 2: By symmetry (X and Y follows the same distribution),
$$P(X>Y) = P(X<Y).$$
So $$1 = P(X>Y) + P(X<Y) + P(X=Y)$$
$$= 2P(X<Y) + P(X=Y)$$
And $$P(X<Y) = \frac{1}{2}[1-P(X=Y)]$$
$$= \frac{1}{2}\left(1-\frac{1}{n}\right) \quad \text{(from 1(a))}$$
$$= \boxed{\frac{n-1}{2n}}$$

1(c) $P(\max(X,Y)=k)$

$$= P(X=k, Y<k) + P(X<k, Y=k) + P(X=k, Y=k)$$

$$= P(X=k)P(Y<k) + P(X<k)P(Y=k) + P(X=k)P(Y=k)$$

$$= \frac{1}{n}\cdot\frac{k-1}{n} + \frac{k-1}{n}\cdot\frac{1}{n} + \frac{1}{n^2}$$

$$= \frac{2(k-1)+1}{n^2}$$

$$= \frac{2k-1}{n^2}$$

$\square$

2(a) For $z>0$,

$$P(Z>z) = P(\min(X,Y)>z)$$

$$= P(X>z, Y>z)$$

$$= P(X>z)P(Y>z)$$

$$= e^{-\lambda z}\cdot e^{-\mu z}$$

$$= e^{-(\lambda+\mu)z}$$

Hence the cdf of Z is, for $z>0$,

$$F_Z(z) = 1 - P(Z>z) = 1 - e^{-(\lambda+\mu)z}.$$

and the pdf is

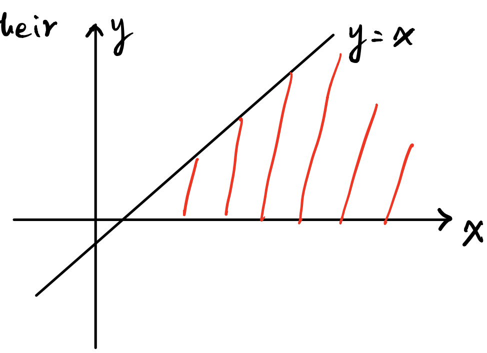
$$f_Z(z) = F_Z'(z) = (\lambda+\mu)e^{-(\lambda+\mu)z}, \text{ for } z>0.$$

So $Z \sim \text{Exp}(\lambda+\mu)$.

2(b). Since X, Y are independent, their joint pdf is

$$f(x,y) = \lambda\mu e^{-\lambda x - \mu y}$$

The shaded area in red is the region of $\{X \geq Y\}$.



And hence

$$P(X \geq Y) = \int_0^{+\infty}\int_0^{x} \lambda\mu e^{-\lambda x - \mu y}\, dy\, dx$$

$$= -\int_0^{+\infty} \lambda e^{-\lambda x}\left[\int_0^{x} e^{-\mu y}\, d(-\mu y)\right] dx$$

$$= -\int_0^{+\infty} \lambda e^{-\lambda x}\left(e^{-\mu y}\Big|_{y=0}^{y=x}\right) dx$$

$$= \int_0^{+\infty} \lambda e^{-\lambda x}\left(1 - e^{-\mu x}\right) dx$$

$$= \lambda\int_0^{+\infty} e^{-\lambda x} dx - \lambda\int_0^{+\infty} e^{-(\lambda+\mu)x}\, dx$$

$$= -e^{-\lambda x}\Big|_{x=0}^{x=+\infty} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)x}\Big|_{x=0}^{x=+\infty}$$

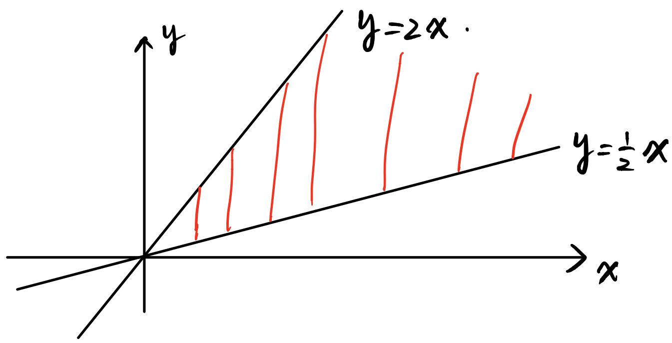
$$= 1 - \frac{\lambda}{\lambda+\mu}$$

$$= \frac{\mu}{\lambda+\mu}.$$

2(c). When $\lambda = \mu$, the joint pdf of (X, Y) becomes

$$f(x,y) = \lambda^2 e^{-\lambda(x+y)}.$$

Then the region of $\{\frac{1}{2} < \frac{X}{Y} < 2\} = \{\frac{1}{2}X < Y < 2X\}$ is



Therefore, $P(\frac{1}{2} < \frac{X}{Y} < 2)$

$$= \int_0^{+\infty} \int_{\frac{1}{2}x}^{2x} \lambda^2 e^{-\lambda(x+y)} \, dy \, dx$$

$$= -\int_0^{+\infty} \lambda e^{-\lambda x} \int_{\frac{1}{2}x}^{2x} e^{-\lambda y} \, d(-\lambda y) \, dx$$

$$= -\int_0^{+\infty} \lambda e^{-\lambda x} \left(e^{-\lambda y} \Big|_{y=\frac{1}{2}x}^{y=2x} \right) dx$$

$$= -\int_0^{+\infty} \lambda e^{-\lambda x} \left(e^{-2\lambda x} - e^{-\frac{1}{2}\lambda x} \right) dx$$

$$= \int_0^{+\infty} \lambda e^{-\frac{3}{2}\lambda x} dx - \int_0^{+\infty} \lambda e^{-3\lambda x} \, dx$$

$$= -\frac{2}{3} e^{-\frac{3}{2}\lambda x} \Big|_{x=0}^{x=+\infty} + \frac{1}{3} e^{-3\lambda x} \Big|_{x=0}^{x=+\infty}$$

$$= -\frac{2}{3}(0-1) + \frac{1}{3}(0-1)$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$

□