Def If V is an inner product space, a set S of vectors in V is called orthogonal if any two distinct vectors in S are orthogonal (that is, $(u,v)=0$ for any $u \neq v$ in S) If, in addition, any vectors in S has length 1 ($\|u\|=1$, for all u in S). Then S is called orthonormal

Ex. $V=(\mathbb{R}^n, \cdot)$, $S=\{e_1, \dots, e_n\}$ standard basis

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ place} \qquad e_i \cdot e_j = \begin{cases} 1, & \text{if } i=j \Rightarrow \|e_i\|=1 \\ 0, & \text{if } i \neq j \end{cases}$$

So S is an orthonormal basis.

Note if x is a nonzero vector, then $u=\frac{1}{\|x\|}x$ has length 1

$$\|u\| = \sqrt{(u,u)} = \sqrt{\left(\tfrac{1}{\|x\|}x, \tfrac{1}{\|x\|}x\right)} = \sqrt{\tfrac{1}{\|x\|}\cdot\tfrac{1}{\|x\|}(x,x)} = \sqrt{\tfrac{1}{\|x\|^2}\cdot\|x\|^2} = \sqrt{1} = 1$$

Ex. $x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $(\mathbb{R}^3, \cdot)$

$x_1 \cdot x_2 = 1(-2)+0\cdot 0+2(1)$ $= 0$ $\quad x_1 \cdot x_3 = 0 \quad x_2 \cdot x_3 = 0$ $\Big\} \Rightarrow \{x_1, x_3, x_3\}$ orthogonal set

$$\|x_1\| = \sqrt{x_1 \cdot x_1} = \sqrt{5} \qquad u_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\|x_2\| = \sqrt{x_2 \cdot x_2} = \sqrt{5} \qquad u_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\|x_3\| = \sqrt{x_3 \cdot x_3} = 1 \qquad u_3 = x_3$$

$\{u_1, u_2, u_3\}$ - orthonormal set.

Theorem : Let $S = \{u_1, \cdots, u_n\}$ be a finite orthogonal set of nonzero vectors in an inner product space V, Then S is linearly independent.

Say $a_1u_1 + \cdots + a_nu_n = \vec{0}$, show $a_1 = \cdots = a_n = 0$

$$0 = (a_1u_1 + \overset{a_iu_i}{\cdots} + a_nu_n \ , u_i)$$

$$= (a_1u_1, u_i) + \cdots + (a_iu_i, u_i) + \cdots + (a_nu_n, a_iu_i)$$

$$= a_1\underbrace{(u_1, u_i)}_{=0} + \cdots + a_i\underbrace{(u_i, u_i)}_{\neq 0} + \cdots + a_n\underbrace{(u_n, u_i)}_{=0}$$

$$= a_i\underbrace{(u_i, u_i)}_{\neq 0} \longrightarrow a_i \text{ has to be zero}$$

Ex. $V = \mathcal{C}([-\pi,\pi]) = \{f: [-\pi,\pi] \to \mathbb{R},\ \text{continuous}\}$

$$(f,g) = \int_{-\pi}^{\pi} f(t)g(t)\,dt$$

Claim $1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt$

are linearly independent.

Suffices to check they are pairwise orthogonal:

if n, m are non-negative, $n \neq m$, then $\int_{-\pi}^{\pi} \cos nt \cos mt\,dt = 0$,

$\int_{-\pi}^{\pi} \cos nt \sin mt\,dt = 0$

(check!)

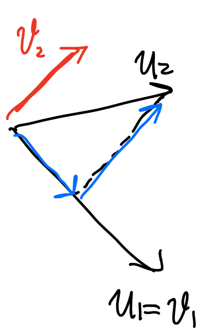
Section 5.4 Gram-Schmidt Process

Theorem Let V be an inner product space and $W \neq \{\vec{0}\}$ an m-dimensional subspace of V. Then W has an orthonormal basis $T = \{w_1, w_2, \ldots, w_m\}$

sketch Let $S = \{u_1, u_2, \ldots, u_m\}$ be any basis of W.

First build an orthogonal basis $T^* = \{v_1, v_2, \ldots, v_m\}$ of W

then take $w_i = \frac{v_i}{\|v_i\|}$, $i = 1 \ldots m$ to get T



Let $v_1 = u_1 \neq 0$

v_2 is in $\text{span}\{u_1, u_2\} = \text{span}\{v_1, u_2\}$

and $(v_2, v_1) = 0$

$v_2 = a_1 v_1 + a_2 u_2$, a_1, a_2 - reals

$0 = (v_2, v_1) = (a_1 v_1 + a_2 u_2, v_1) = a_1 (v_1, v_1) + a_2 (u_2, v_1)$

$\Rightarrow a_1 = -a_2 \frac{(u_2, v_1)}{(v_1, v_1)}$ Let $a_2 = 1$, then $a_1 = -\frac{(u_2, v_1)}{(v_1, v_2)}$

$$\Rightarrow \boxed{v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_2)} v_1}$$

note, $\text{span}\{u_1 u_2\} = \text{span}\{v_1, u_2\}$

Similarly, $v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$ and so on

Corollary: Let V be an Euclidian space, and let $S = \{u_1, \dots, u_n\}$ be an orthonormal basis. If $v = a_1 u_1 + \cdots + a_n u_n$

$w = b_1 u_1 + \cdots + b_n u_n$

$\rightarrow (v, w) = [v]_S \cdot [w]_S = [v]_S^T [w]_S$

recall if S is any basis of V, then $(v, w) = [v]_S \cdot C \cdot [w]_S$,

important

with $C = [C_{ij}]$, $C_{ij} = (u_i, u_j)$

If S-orthonormal, then $C = I_n$

Ex W subspace of $(\mathbb{R}^4, \cdot)$ with basis $S = \{u_1, u_2, u_3\}$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

First build $T^* = \{v_1, v_2, v_3\}$ – orthogonal basis. Exam II

$$v_1 = u_1, \quad v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} \cdot v_1$$

$$v_2 = u_2 - \frac{-2}{3} v_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \left(-\frac{1}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(-\frac{2}{\cancel{3}} \cdot \frac{\overset{3}{\cancel{9}}}{15}\right)\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ \frac{1}{5} \\ -\frac{3}{5} \end{bmatrix}$$

Divide each of v_1, v_2, v_3 by their length.

$\|v_1\| = \|u_1\| = \sqrt{3}$

$$w_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -\frac{1}{\sqrt{15}} \\ 2\frac{1}{\sqrt{15}} \\ -1\frac{1}{\sqrt{15}} \\ 3\frac{1}{\sqrt{15}} \end{bmatrix} \quad w_3 = \begin{bmatrix} -\frac{4}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \\ -\frac{3}{\sqrt{35}} \end{bmatrix}$$

$T = \{w_1, w_2, w_3\} \rightarrow$ orthonormal basis. unique.