6.3 The matrix of a linear transformation

Theorem 6.9 Let $L: V \to W$ be a linear transformation of an n-dimensional vector space V into an m-dimensional v.s. W.

And let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a ordered basis for W and let $T = \{\vec{w}_1, \vec{w}_2, \cdots \vec{w}_m\}$ be an ordered basis for W. Then the $m \times n$ matrix A whose j^{th} column is the coordinate $[\mathcal{L}(\vec{v}_j)]_T$ of $\mathcal{L}(\vec{v}_j)$ with respect to T has the property $[\mathcal{L}(\vec{x})]_T = A[\vec{x}]_S$ for all $\vec{x} \in V$

A is the only matrix with this property.

Example 5:

Let $\mathcal{L}: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$\mathcal{L}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix}$$

Standard basis.

consider the ordered basis $S = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

and $T = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ $\mathcal{L}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\mathcal{L}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\mathcal{L}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

To find the coordinate of S basis in the T basis.

$$\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\vec{b} \quad \text{where } \vec{b}=\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}$$

Solve:

$$\left[\begin{array}{cc|c|c|c}1&1&2&2&1\\2&3&3&5&3\end{array}\right]\to\left[\begin{array}{cc|c|c|c}1&1&2&2&1\\0&1&-1&1&1\end{array}\right]$$

$$\left[\begin{array}{cc|c|c|c}1&0&3&1&0\\0&1&-1&1&1\end{array}\right]$$

$$\boxed{\begin{bmatrix}3&1&0\\-1&1&1\end{bmatrix}}$$

7.1 Eigenvalues and Eigenvectors

Given a linear transformation $\mathcal{L}$ ($\mathcal{L}: V \to V$) we say that $\vec{x}$ ($\in V$) is an eigenvector for $\mathcal{L}$ with eigenvalue $\lambda \in \mathbb{R}$, if

$$\mathcal{L}(\vec{x}) = \lambda \cdot \vec{x} \qquad \vec{x} \neq \vec{0}$$

Given a square matrix $A_{n\times n}$, we say that $\vec{x}$ is an eigenvector for A with eigenvalue λ provided

$$A\vec{x} = \lambda\vec{x} \qquad \vec{x} \neq \vec{0}$$

Q: How to find these?

Method: First, find the eigenvalues

Theory: $A\vec{x} = \lambda\vec{x}$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

characteristic equation

need $\det(A - \lambda I_n) = 0$

solve this to find the eigenvalues λ, it would be a polynomial, called the characteristic polynomial.

$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $\quad A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 2$$

$$= 3 - 3\lambda - \lambda + \lambda^2 - 2$$

$$= \lambda^2 - 4\lambda + 1 = 0$$

$$\frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

eigenvalues are $2 \pm \sqrt{3}$.

$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ $\quad (A - \lambda I_n) = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$

$$\det(A-\lambda I_n) = (1-\lambda)(4-\lambda)+2$$

$$= 4-4\lambda-\lambda+\lambda^2+2$$

$$= \lambda^2-5\lambda+6=0$$

$$= (\lambda-3)(\lambda-2)=0$$

$$\lambda_1=3, \ \lambda_2=2.$$

Eigenvalues: $\lambda=3$, $\lambda=2$.

To find eigenvectors, solve $A\vec{x}=\lambda\vec{x}$

for $\lambda=2$,

$$A\vec{x}=2\vec{x}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1+x_2 \\ -2x_1+4x_2 \end{bmatrix}=\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\begin{bmatrix} -x_1+x_2 \\ -2x_1+2x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1+x_2=0$$

$$x_1=x_2$$

Eigenvector for Eigenvalue 2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any nonzero scaler multiple.

for $\lambda = 3$.

$$A\vec{x} = 3\vec{x}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\begin{bmatrix} -2x_1 + x_2 \\ -2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0 \qquad \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_2 = 2x_1$$

Eigenvector for Eigenvalue 3 is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any nonzero scaler multiple.

Example: Find the eigenvalue and eigenvectors for A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -\lambda \\ 4 & -4 \end{vmatrix}$$

$$= (1-\lambda)(-5\lambda+\lambda^2+4) - 2(5-\lambda-4) - (-4+4\lambda)$$

$$= (1-\lambda)(-5\lambda+\lambda^2+4) - 10+2\lambda+8+4-4\lambda$$

$$= -5\lambda+\lambda^2+4+5\lambda^2-\lambda^3-4\lambda-10+2\lambda+8+4-4\lambda$$

$$= 5\lambda+6\lambda^2-\lambda^3+6$$

$$= -\lambda^3+6\lambda^2-11\lambda+6$$

let $\det(A-\lambda I)=0$, $\quad -\lambda^3+6\lambda^2-\lambda 11+6=0$

candidates for roots are $\pm 1, \pm 2, \pm 3, \pm 6$.

Plug in $\lambda=1$

$$-1+6-11+6=0$$

Polynomial has a factor of $\lambda-1$

$$(\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$\lambda=1,2,3$ are the eigenvalues.

when $\lambda=1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1+2x_2-x_3 \\ x_1+0+x_3 \\ 4x_1-4x_2+5x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 0+2x_2-x_3 \\ x_1-x_2+x_3 \\ 4x_1-4x_2+4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solutions are of the form $\begin{bmatrix} -a \\ a \\ 2a \end{bmatrix}$ are all eigenvector for $\lambda=1$

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$\det(A-\lambda I) = (2-\lambda)(2-\lambda)(3-\lambda)$ already factor

eigenvalues are 2 and 3.

double root.

$\lambda=2$ $\quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2x_1 + x_2 + x_3 \\ 0 + 2x_2 + x_3 \\ 0 + 0 + 3x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 0x_1 + x_2 + x_3 \\ 0 + 0 + x_3 \\ 0 + 0 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0,\ x_2 = 0,\ x_1 = \alpha.$

Eigenvectors are $\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$