Problem 1

1(a) For any $i = 1, 2, 3, \ldots, n$,

$$E(X_i) = 1 \cdot p + 0 \cdot (1-p) = p$$

$$\begin{aligned} Var(X_i) &= E(X_i^2) - (EX_i)^2 \\ &= [1^2 \cdot p + 0^2 \cdot (1-p)] - p^2 \\ &= p(1-p) \end{aligned}$$

Then

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} E(X_i) \quad \text{(by the linearity of expectation)} \\ &= E(X_1) \quad \text{(identical distribution for } \{X_1, \ldots, X_n\}\text{)} \\ &= p \end{aligned}$$

$$\begin{aligned} Var(\bar{X}) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})\right) \\ &= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} (X_i - \bar{X})\right) \end{aligned}$$

(by the property of variance)

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i - \bar{X})$$

($\{X_1, \ldots, X_n\}$ are independent)

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

(a shift of a constant will not affect the variance)

$$= \frac{Var(X_1)}{n} = \frac{p(1-p)}{n} \qquad \square$$

(identical distribution)

1(b).

$$\begin{aligned} E(S^2) &= E\left[\frac{1}{n-1}\sum_{i=1}^{n} (X_i - \bar{X})^2\right] \\ &= \frac{1}{n-1} E\left[\sum_{i=1}^{n} (X_i - \bar{X})^2\right] \end{aligned}$$

(by the linearity of expectation)

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} X_i^2 - 2\sum_{i=1}^{n} X_i \bar{X} + n(\bar{X})^2 \right]$$
$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} X_i^2 - 2n(\bar{X})^2 + n(\bar{X})^2 \right]$$

(by the definition of sample mean)

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} X_i^2 - n(\bar{X})^2 \right]$$
$$= \frac{1}{n-1} \left[\sum_{i=1}^{n} E(X_i^2) - n\, E(\bar{X})^2 \right]$$
$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} p - n\left[Var(\bar{X}) + (E\bar{X})^2 \right] \right\}$$

(by the calculating formula for $Var(\bar{X})$)

$$= \frac{1}{n-1} \left\{ np - n\left[\frac{p(1-p)}{n} + p^2 \right] \right\}$$
$$= \frac{1}{n-1} \left\{ np - p(1-p) - np^2 \right\}$$
$$= p(1-p) = Var(X_i) \;!!! \qquad \square$$

Problem 2

We want $E(N)$, where N is the number of floors at which the elevator makes a stop to let out one or more of the people.

Define event A_i = at least one person chooses floor i, $i = 1, 2, \dots, 10$.

Then $N = \sum_{i=1}^{10} I_{A_i}$ is a counting variable.

So by linearity of expectation,

$$E(N) = \sum_{i=1}^{10} P(A_i)$$

And for any $i = 1, 2, \dots, 10$,

$$P(A_i) = 1 - P(A_i^c) = 1 - P(\text{"nobody chooses floor } i\text{"})$$
$$= 1 - \left(\frac{9}{10}\right)^{12}$$

by the independence of the people's choices.

Hence $E(N) = 10 \times [1-(\frac{9}{10})^{12}] \approx 7.18$ $\square$