Answer key for Dis 4.

Problem 1 ① p.m.f. of X ?

1) First find the range: X could take value in $\{0, 1, 2, 3\}$.

2) Then its p.m.f. can be got: $P(X=x) = \underbrace{\binom{3}{x}}_{\text{Count \# of situations with } x \text{ heads}} \overbrace{\left(\frac{1}{2}\right)^x}^{\text{prob. for } x \text{ heads}} \underbrace{\left(\frac{1}{2}\right)^{3-x}}_{\text{prob. for rest of } 3-x \text{ tails}}$

$$= \binom{3}{x}\left(\frac{1}{2}\right)^3$$

for any $x = 0, 1, 2, 3$, and is 0 otherwise.

We can also list its distribution by a table.

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

② c.d.f. of $|X-1|$?

Define $Y = |X-1|$

Step 1): First find the range $Y = \begin{cases} 0 & \text{if } X=1 \\ 1 & \text{if } X = 0 \text{ or } 2 \\ 2 & \text{if } X=3 \end{cases}$

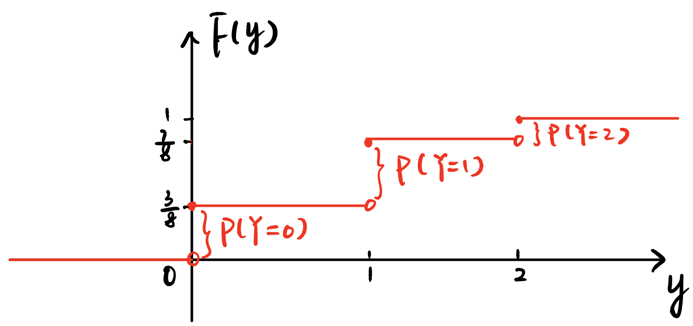
Step 2): Find its p.m.f.

$$P(Y=y) = \begin{cases} P(X=1) & \text{if } y=0 \\ P(X=0)+P(X=2) & \text{if } y=1 \\ P(X=3) & \text{if } y=2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3}{8} & \text{if } y=0 \\ \frac{1}{2} & \text{if } y=1 \\ \frac{1}{8} & \text{if } y=2 \\ 0 & \text{o.w.} \end{cases}$$

Step 3: Find its c.d.f.

$$F(y) = P(Y \le y) = \begin{cases} 0 & \text{if } y<0 \\ \frac{3}{8} & \text{if } 0 \le y < 1 \\ \frac{7}{8} & \text{if } 1 \le y < 2 \\ 1 & \text{if } y \geq 2 \end{cases}$$



Problem 2

The key idea is to simplify the solution by independence.

① c.d.f. of Y_1.

For $y<1$, $F_{Y_1}(y) = P(\max(X_1, X_2) \le y) = 0$

For $y \ge 6$, $F_{Y_1}(y) = 1$

For any $y \in [1,6)$,

$$\begin{aligned} F_{Y_1}(y) &= P(\max(X_1,X_2) \le y) \\ &= P(X_1 \le y, X_2 \le y) \\ \text{(indep.)} &= P(X_1 \le y) P(X_2 \le y) \\ &= \left(\frac{\lfloor y \rfloor}{6}\right)^2 \end{aligned}$$

Therefore, the c.d.f. of Y_1 is $F_{Y_1}(y) = \begin{cases} 0 & \text{if } y<1 \\ \left(\frac{\lfloor y \rfloor}{6}\right)^2 & \text{if } 1 \le y < 6 \\ 1 & \text{if } y \ge 6 \end{cases}$

② c.d.f. of Y_2.

For $y<1$, $F_{Y_2}(y) = P(\min(X_1,X_2) \le y) = 0$

For $y \ge 6$, $F_{Y_2}(y) = P(\min(X_1,X_2) \le y) = 1$

For $1 \le y < 6$,

$$\begin{aligned} F_{Y_2}(y) &= P(\min(X_1,X_2) \le y) \\ &= 1 - P(\min(X_1,X_2) > y) \end{aligned}$$

$$= 1 - P(X_1 > y,\ X_2 > y)$$

(indep.) $$= 1 - P(X_1 > y)P(X_2 > y)$$

$$= 1 - [1 - P(X_1 \le y)][1 - P(X_2 \le y)]$$

$$= 1 - \left(1 - \frac{\lfloor y \rfloor}{6}\right)^2$$

Therefore the c.d.f. of Y_2 is

$$F_{Y_2}(y) \begin{cases} 0 & \text{if } y < 1 \\ 1 - \left(1 - \frac{\lfloor y \rfloor}{6}\right)^2 & \text{if } 1 \le y < 6 \\ 1 & \text{if } y \geqslant 6 \end{cases}$$

Problem 3

(a). Since $p(x)$ is a p.d.f.,

by the property $\sum_{x=2}^{\infty} p(x) = 1$, we can solve c :

$$
\begin{aligned}
1 &= \sum_{x=2}^{\infty} p(x) \\
&= \sum_{x=2}^{\infty} \frac{c}{x^2-1} \\
&= \frac{c}{2} \sum_{x=2}^{\infty} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) \\
&= \frac{c}{2} \left(1 - \cancel{\tfrac{1}{3}} + \tfrac{1}{2} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}} + \ldots\ldots \right) \\
&= \frac{3}{4} c .
\end{aligned}
$$

So $c = \frac{4}{3}$

(b). According to the definition of the expectation,

$$
\begin{aligned}
E(X) &= \sum_{x=2}^{\infty} x\, p(x) \\
&= \frac{4}{3} \sum_{x=2}^{\infty} \frac{x}{x^2-1} \\
&\geq \frac{4}{3} \sum_{x=2}^{\infty} \frac{x}{x^2} \\
&= \frac{4}{3} \sum_{x=2}^{\infty} \frac{1}{x} \\
&= \infty .
\end{aligned}
$$

Thus $E(X)$ doesn't exist.