Ex.

\$1 \$0

Event E = we originally picked the \$1 bag

$P(E) = P(\bar{E}) = \frac{1}{2}$

R = ball is red

$P(R \mid E) = \frac{1}{2}$ $\quad$ $P(\bar{R} \mid E) = \frac{1}{2}$

$P(R \mid \bar{E}) = 0$ $\quad$ $P(\bar{R} \mid \bar{E}) = 1$

$$P(R) \overset{\text{marginalize}}{=} P(R, E) + P(R, \bar{E})$$
$$\overset{\text{conditional}}{=} P(E)P(R \mid E) + P(\bar{E})P(R \mid \bar{E})$$
$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 0 = \frac{1}{4}$$

switch if $P(E \mid \bar{R}) < \frac{1}{2}$, hold otherwise $\boxed{\text{why}}$?

$$P(E \mid \bar{R}) = \frac{P(E, \bar{R})}{P(\bar{R})} \overset{\text{Bayes}}{=} \frac{P(\bar{R} \mid E)P(E)}{P(\bar{R})} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{1 - P(R)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$$

If we do not switch, the EXPECTED money we get

$$P(E \mid \bar{R})\cdot 1 + P(\bar{E} \mid \bar{R})\cdot 0 = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 0 = \$\frac{1}{3}$$

If we switch, the Expected money we get is $\$\frac{2}{3}$

In general, assume $P(E \mid \bar{R}) = x$, the $\overset{\text{not switch}}{\text{expected amount}} = x\cdot 1 + (1-x)\,0 = x$

the switch expected amount $= (1-x)1 + x0 = 1-x$

Therefore, decision rule, switch if $x < \frac{1}{2}$

Probability: $\Uparrow$

Statistics: $\Downarrow$

$$\begin{tikzcd} P \arrow[r, bend left, "\text{prob}"] & \text{data} \arrow[l, bend left, "\text{stat}"] \end{tikzcd}$$

coin head prob $p \in (0,1)$, flip n times, x heads observed.

outcomes $O_1, O_2, \cdots, O_n \in \{H, T\}$

$$P(O_1, O_2, \cdots, O_n) \xrightarrow{\text{independence}} P(O_1)\cdot P(O_2)\cdots P(O_n)$$

$= p^x(1-p)^{n-x}$ likelihood

likelihood function $L(p) = p^x(1-p)^{n-x}$

One way to estimate p: Maximum Likelihood Estimate (MLE)

$$\hat{p} = \arg\max_{p \in (0,1)} L(p)$$

arguments p that archieve $L(p)$ max