$S=\{v_1, v_2, \ldots, v_n\}$, $T=\{w_1, w_2, \ldots, w_n\}$

if v is in V: $v = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \longrightarrow [v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Compute $[v]_S$ in terms of $[v]_T$

$[v]_S = [c_1 w_1 + c_2 w_2 + \cdots + c_n w_n]_S = c_1 [w_1]_S + \cdots + c_n [w_n]_S$

For $j=1, \ldots, n$ let $[w_j]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$, $w_j = a_{1j} v_1 + \cdots + a_{nj} v_n$

Def: The transition matrix from T basis to S basis in the $n \times n$ matrix $P_{S \leftarrow T}$

whose j^{th} column is $[w_j]_S$

Note: $[v]_S = P_{S \leftarrow T} [v]_T$

Similarly can construct $Q_{T \leftarrow S}$, the transition matrix from the S-basis to the T-basis, whose j^{th} column is $[v_j]_T$

Note $Q_{T \leftarrow S} = P_{S \leftarrow T}^{-1}$

Example: $V = \mathbb{R}^3$, basis

$S = \{v_1, v_2, v_3\}$ $v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T = \{w_1, w_2, w_3\}$ $w_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$, $w_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $w_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$

Compute $P_{S \leftarrow T}$ express w_j's in terms of v_i's solve $a_1v_1 + a_2v_2 + a_3v_3 = w_1$ for a_1, a_2, a_3

do row operations on $[v_1, v_2, v_3 \mid w_1]$ → augmented matrix

$\downarrow$ reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \end{array}\right]$$ first column of $P_{S\leftarrow T}$

do the same for w_2, w_3 to find the other columns of $P_{S\leftarrow T}$

shortcut: Consider the matrix

$$[v_1, v_2, v_3 \mid w_1 \mid w_2 \mid w_3] \xrightarrow{RREF} [I_3 \mid P_{S\leftarrow T}]$$

Rank of a matrix

$\underset{m\times n}{A}$ m rows, n columns

$\downarrow$ elements of $\mathbb{R}_n$ $\downarrow$ elements of $\mathbb{R}^n$

Row(A) is the subspace of $\mathbb{R}_n$ spanned by the rows of A.
row space of A

Col (A) —— " —— of $\mathbb{R}^m$ —— " —— columns of A.

Q? What is dim Row(A)?, dim Col(A)?

Fact: If A and B are row (column) equivalent, then Row(A) = Row(B) (Col(A) = Col(B))

Ex. $V = \mathrm{span}\{v_1, v_2, v_3, v_4\} \in \mathbb{R}_5$. Find a basis of V

$v_1 = [1\ {-2}\ 0\ 3\ {-4}]$ $\quad v_2 = [3\ 2\ 8\ 1\ 4]$

$v_3 = [2\ 3\ 7\ 2\ 3]$ $\quad v_4 = [-1\ 2\ 0\ 4\ {-3}]$

$V = \mathrm{Row}(A)$ where $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \xrightarrow{RREF} B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$

$\mathrm{Row}(A) = \mathrm{Row}(B) \rightsquigarrow$ dim 3 with basis the non-zero rows of V.

Remark: $\mathrm{Col}(A)$ has a basis made of col 1, 2 and 4 of A.

So $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$

Definition: $\dim \mathrm{Row}(A) =$ row rank of A

$\dim \mathrm{Col}(A) =$ column rank of A

Theorem: If A is a $m \times n$ matrix, then the row rank and column rank are equal. (Their common value is called rank of A)

$A \xrightarrow{RREF} B$

row rank of A = row rank of B = # non-zero rows of B

= # leading 1's (pivot)

= # pivot columns of B

= column rank of A.

might on quiz.

Ex. V-subspace of P_4, $V = \operatorname{span} S$, $S = \{v_1, v_2, v_3, v_4\}$

$v_1 = t^4 + t^2 + 2t + 1$ $\quad v_3 = 2t^4 + t^3 + t + 2$

$v_2 = t^4 + t^2 + 2t + 2$ $\quad v_4 = t^4 + t^3 - t^2 - t$

Find a basis for V

Define $L: P_4 \to \mathbb{R}_5$ is an isomorphism

$$at^4 + bt^3 + ct^2 + dt + e \to [a \ b \ c \ d \ e]$$

Restate the problem: find the basis of the rowspace of matrix A with rows given by the coefficient of vectors in S.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Basis of V corresponds to the non-zero rows of B:

$w_1 = t^4 + t^2 + 2t$

$w_2 = t^3 - 2t^2 - 3t$

$w_1 = 1$

Theorem $\quad$ A is an $m \times n$ matrix $\longrightarrow$ rank A + nullity A $= n$

when solving a homogeneous system $A\underline{x} = \underline{0}$, divide the unknowns between those corresponding to the pivot columns (those are called principle variables) and the rest, to which one assigns an arbitrary value (these are called the secondary variables)

rank A = # principles variables

nullity A = # secondary variables

Easy consequences A $n \times n$ square matrix

- rank $= A \longleftrightarrow A$ is row equivalent to I_n

  $\longleftrightarrow A$ is non-singular / invertible

  $\longleftrightarrow \det A \neq 0$

  $\longleftrightarrow A\underline{x} = \underline{0}$ has only the trivial solution.

  $\longleftrightarrow A\underline{x} = \underline{b}$ has a unique solution $\forall b$ $(x = A^{-1}b)$

  $\longleftrightarrow$ rows of A are independent

  $\longleftrightarrow$ cols of A are independent

Ex. $A\underline{x} = \underline{b}$ has a solution $\longleftrightarrow \operatorname{rank}(A) = \operatorname{rank}[A \vdots \underline{b}]$