6.1 Linear Transformation and matrices

Definition:

let V and W be two vector spaces. There is a function

$L: V \longrightarrow W$ is called a linear transformation if

a. $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v})$ For every $\vec{u}$ and $\vec{v} \in V$

b. $L(c \cdot \vec{u}) = c \cdot L(\vec{u})$ For every $c \in \mathbb{R}$ and $\vec{u} \in V$

$V = W \rightarrow$ L is a linear operator.

Examples

Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$

Define $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ x_2 - x_3 \end{bmatrix}$ $\qquad L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

is L a linear transformation. $\qquad L\left(\begin{bmatrix} 16 \\ 9 \\ 25 \end{bmatrix}\right) = \begin{bmatrix} 41 \\ -16 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $L(\vec{u}+\vec{v}) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1+u_3+v_3 \\ u_2+v_2-u_3-v_3 \end{bmatrix}$

$L(\vec{u}) + L(\vec{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+u_3 \\ u_2-u_3 \end{bmatrix} + \begin{bmatrix} v_1+v_3 \\ v_2-v_3 \end{bmatrix} = \begin{bmatrix} u_1+u_3+v_1+v_3 \\ u_2-u_3+v_2-v_3 \end{bmatrix}$

$= L(\vec{u}+\vec{v})$

$$L(c\vec{u}) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 + cu_3 \\ cu_2 - cu_3 \end{bmatrix} = c\begin{bmatrix} u_1 + u_3 \\ u_2 - u_3 \end{bmatrix} = c \cdot L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = c \cdot L(\vec{u})$$

Therefore, L is a linear transformation.

Example: $V = \mathbb{R}^2$, $W = \mathbb{R}^2$

Define $L: V \to W$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 1 \\ 0 \end{bmatrix}$

Is L a linear transformation?

let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$L(\vec{u}+\vec{v}) = L\left(\begin{bmatrix} 4+6+1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$

$L(\vec{u}) + L(\vec{v}) = L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + L\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \neq L(\vec{u}+\vec{v})$

L fails rule (a), so it is not a linear transformation

Example:

Let $T: \mathbb{R}^2 \to \mathbb{R}$ be defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2 + 3x_2$

is T a linear transformation? No.

$T(2\begin{bmatrix}1\\1\end{bmatrix}) = T(\begin{bmatrix}2\\2\end{bmatrix}) = 4+6 = 10$

$2T(\begin{bmatrix}1\\1\end{bmatrix}) = 8 \neq T(2\begin{bmatrix}1\\1\end{bmatrix})$.

T fails (b), not a linear transformation.

$T: \mathbb{R}^n \to \mathbb{R}^m$

Let A be an mxn matrix

$T(\vec{u}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{u}}$

$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$ ✓ (a)

$T(c\vec{u}) = Ac\vec{u} = cA\vec{u} = cT(\vec{u})$ ✓ (b)

T is a linear transformation

Theorem

Let $L: V \to W$ be a linear transformation. Then

a. $L[\vec{0}_V] = \vec{0}_W$

b. $L(\vec{u}-\vec{v}) = L(\vec{u}) - L(\vec{v})$

Proof

(a) let $\vec{v} \in V$

$L(\vec{0} \cdot \vec{v}) = 0 \cdot L(\vec{v})$

$\| \quad\quad\quad \|$

$L(\vec{0}_V) = \vec{0}_W$

(b) $L(\vec{u} - \vec{v}) = L(\vec{u} + (-1)\vec{v})$

$= L(\vec{u}) + L(-1\vec{v})$

$= L(\vec{u}) + (-1)L(\vec{v})$

$= L(\vec{u}) - L(\vec{v})$

Other examples

① $V = P_n$ (polymomials of degreen $\leq n$) $T: P_n \to P_n$

multiplicatio by other polynomials

② V = space of differenttable functions $\frac{d}{dx}: V \to W$

derivaties

(Things in W may not be differentiable)

③ $V = \mathcal{C}[a,b]$ (integrable functions on $[a,b]$)

$T: \mathcal{C}[a,b] \to \mathbb{R}$ defined by

$$T(f) = \int_a^b f(x)\, dx$$

Theorem

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and consider the basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$

Let A be the $m \times n$ matrix whose columns are $L(\vec{e}_1), L(\vec{e}_2) \cdots L(\vec{e}_n)$

The matrix A has the property that

$$L(\vec{x}) = A\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$