Theorem 6.2

Let $L: V \to W$ be a linear transformation where $\dim(V) = n$.

Let $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ be a basis for V.

If $\vec{v} \in V$, then $L(\vec{v})$ is completely determined by

$\{L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)\}$

Proof:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \qquad c_i \in \mathbb{R} \quad i = 1 \cdots n$$

$$L(\vec{v}) = L(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)$$

$$= L(c_1\vec{v}_1) + L(c_2\vec{v}_2) + \dots + L(c_n\vec{v}_n)$$

$$= c_1L(\vec{v}_1) + c_2L(\vec{v}_2) + \dots + c_nL(\vec{v}_n)$$

Example: $L: \mathbb{R}^4 \to \mathbb{R}^2$ and $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$

with $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

suppose that

$L(\vec{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $L(\vec{v}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $L(\vec{v}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $L(\vec{v}_4) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Let $\vec{v} = \begin{bmatrix} 2 \\ -5 \\ -5 \\ 0 \end{bmatrix}$ $\vec{v} = 2\vec{v}_1 + \vec{v}_2 - 3\vec{v}_3 + \vec{v}_4$

$L(\vec{v}) = 2L(\vec{v}_1) + L(\vec{v}_2) - 3L(\vec{v}_3) + L(\vec{v}_4)$ By proof of Theorem 6.2.

$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

Theorem 6.3

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and consider the standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$. Let A be the $m \times n$ matrix whose j^{th} column is $L(\vec{e}_j)$

If $\vec{x} \in \mathbb{R}^n$, $L(\vec{x}) = A\vec{x}$

Moreover, A is the only such matrix.

Proof:

write $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

$L(\vec{x}) = x_1L(\vec{e}_1) + x_2L(\vec{e}_2) + \cdots + x_nL(\vec{e}_n)$ by Theorem 6.2.

$L(\vec{x}) = A\vec{x}$

Uniqueness:

Suppose there is a matrix B such that $L(\vec{x}) = B\vec{x}$

since $L(\vec{x}) = A\vec{x}$ combine these, show $B = A$.

$L: \mathbb{R}^3 \to \mathbb{R}^2$ be $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ x_2 - x_3 \end{bmatrix}$

Find matrix A such that $L(\vec{x}) = A\vec{x}$ $\quad A_{2\times3}$

$$\underbrace{L(\vec{e}_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L(\vec{e}_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad L(\vec{e}_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{columns of } A}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

6.2 Kernel and Range of a Linear Transformation

Definition 6.2:

A linear transformation $L: V \to W$ is called one-to-one

if $v_1, v_2 \in V$ such that $\vec{v}_1 \neq \vec{v}_2$ then $L(\vec{v}_1) \neq L(\vec{v}_2)$

equivalently, L is one-to-one $\longleftrightarrow$ $L(\vec{v}_1) = L(\vec{v}_2) \to \vec{v}_1 = \vec{v}_2$

$L: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 3x_1 \\ 2x_2 \\ -x_1 \end{bmatrix}$ $\ker(L) = \{\vec{0}\}$

is L one-to-one ?

Suppose $L(\vec{v}) = L(\vec{u})$ with $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$L(\vec{v}) = L(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})$ $L(\vec{u}) = L(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix})$

$= \begin{bmatrix} 3v_1 \\ 2v_2 \\ -v_1 \end{bmatrix}$ $= \begin{bmatrix} 3u_1 \\ 2u_2 \\ -u_1 \end{bmatrix}$

if $L(\vec{v}) = L(\vec{u})$, then $\begin{bmatrix} 3v_1 \\ 2v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 3u_1 \\ 2u_2 \\ -u_1 \end{bmatrix}$

$3v_1 = 3u_1$, $2v_2 = 2u_2$, $-v_1 = -u_1$, So. $v_1 = u_1$, $v_2 = u_2$.

Therefore. L is one-to-one

$P: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\ker(L) = \{\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}, x_2 \in \mathbb{R}\}$

$P(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ P is not one-to-one.

because

$P(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}) = P(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, same output different input.

Definition 6.3.

Let $\mathcal{L}: V \to W$ be a linear transformation.

The kernel of $\mathcal{L}$ is the subset of all elements $\vec{v} \in V$ such that $\mathcal{L}(\vec{v}) = \vec{0}_W$

The kernel is denoted $\ker(\mathcal{L})$

Theorem 6.4

Let $\mathcal{L}: V \to W$ be a linear transformation Then:

① $\ker \mathcal{L}$ is a subspace of V.

② $\mathcal{L}$ is one-to-one $\longleftrightarrow$ $\ker \mathcal{L} = \{\vec{0}_V\}$

Proof ①: Gi $\vec{u}$ and $\vec{v} \in \ker \mathcal{L}$.

Then $\mathcal{L}(\vec{u}) = \mathcal{L}(\vec{v}) = \vec{0}_W$

$\mathcal{L}(\vec{u}+\vec{v}) = \mathcal{L}(\vec{u}) + \mathcal{L}(\vec{v}) = \vec{0}_W + \vec{0}_W = \vec{0}_W$

so $\vec{u}+\vec{w} \in \ker \mathcal{L}$.

Similarly, Given any scalar c,

$\mathcal{L}(c\vec{u}) = c\mathcal{L}(\vec{u}) = c\,\vec{0}_W = \vec{0}_W$, $c\vec{u} \in \ker \mathcal{L}$

Since ker. $\mathcal{L}$ is closed under addition and multiplicat,

it is a subspace of V.

② If $\mathcal{L}$ is one-to-one, then

suppose $\vec{v} \in \ker \mathcal{L}$

since $\vec{0}_V \in \ker \mathcal{L}$

$L(\vec{0}_V) = L(\vec{v})$ since $\mathcal{L}$ is one-to-one,

$\vec{v} = \vec{0}_V$.

Conversely. $\ker \mathcal{L} = \{\vec{0}_V\}$

suppose $\exists \vec{u}, \vec{v} \in V$ such that $\mathcal{L}(\vec{u}) = \mathcal{L}(\vec{v})$

$\mathcal{L}(\vec{u}) - \mathcal{L}(\vec{v}) = \vec{0}_W$

$\mathcal{L}(\vec{u} - \vec{v}) = \vec{0}_W$

therefore, $\vec{u} - \vec{v} \in \ker \mathcal{L}$.

$\vec{u} - \vec{v} = \vec{0}_0$, hence $\vec{u} = \vec{v}$

therefore, $\mathcal{L}$ is one-to-one.

Corollary. 6.1

$\mathcal{L}(\vec{x}) = \vec{b}$ and $\mathcal{L}(\vec{y}) = \vec{b} \longrightarrow \vec{x} - \vec{y} \in \ker \mathcal{L}$