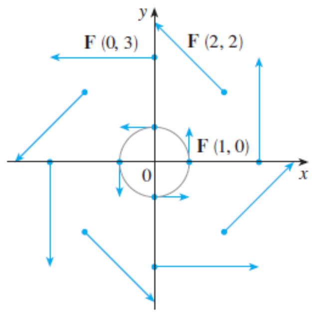
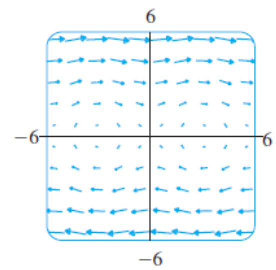
Section 16.1 Vector Fields

Consider a region D in the xy-plane. A **vector field** on D, is a function $\mathbf{F}$ that assigns to *every* point (x, y) in D a two dimensional vector $\mathbf{F}(x, y)$.

$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

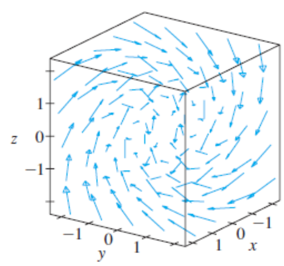


$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

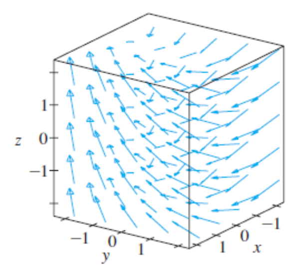


In three dimensions, if E is a region in 3D space, a **vector field** is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimenstional vector $\mathbf{F}(x, y, z)$.

$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$



$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$



Vector fields arise in many applications. They can represent force fields (such as gravity), electrical fields, velocity fields, etc.

Recall that the gradient of a function $f(x, y)$ is $\nabla f = \langle f_x, f_y \rangle$. This actually defines a vector field, which we call the gradient vector field.

Find the gradient vector field of $f(x, y) = \sqrt{x^2 + y^2}$.

Find the gradient vector field of $f(x, y, z) = x \ln(y^2 + z)$.

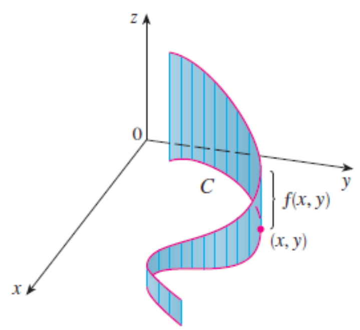
16.2 Line Integrals

A line integral is an integral where instead of integrating over a region in the xy-plane, we integrate along a curve C.

Recall that the arc length of a curve C from $t = a$ to $t = b$ is given by $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. This actually came from the fact that if we partition the curve C into segments, then the arc length of one of these segments is approximated by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

One interpretation of a line integral is that it gives us the area of a "curtain" where the height of the curtain at any point along C is given by $f(x, y)$. Or, if we think of the curve C as a wire with density $f(x, y)$ at any point, then the line integral gives the mass of the wire.



If $f(x, y)$ is defined on a smooth curve C defined by $x = x(t)$, $y = y(t)$, then the line integral of $f(x, y)$ along C is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This integral is also called the line integral with respect to arc length because of the ds involved.

Notice that a line integral is a single integral of ONE variable. This is because ANY curve in space can be represented as a function of a single variable or parameter.

If C is made up of two or more curve pieces, then the line integral along C is just the sum of the line integrals along each piece.

$\int_C \frac{x}{y}\, ds$, where C is defined by $x = t^3$, $y = t^4$, $1 \leq t \leq 2$.

$\int_C (x + y)\; ds$ where C is the line segment from the point $(2, 1)$ to $(6, 4)$.

$\int_C x^2 y\; ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Set up $\displaystyle\int_C xy\ ds$, where C is the part of the curve $y = x^2$ from $(1, 1)$ to $(3, 9)$ followed by the line $y = 9$ from $(3, 9)$ to $(5, 9)$.

There are two other line integrals that can arise when ds is replaced by dx or dy. These are called the line integrals with respect to x and y.

Let C be a smooth curve defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$.

The line integral of f along C with respect to x is

$$\int_C f(x, y)\ dx = \int_a^b f(x(t), y(t))\ x'(t)\ dt$$

The line integral of f along C with respect to y is

$$\int_C f(x, y)\ dy = \int_a^b f(x(t), y(t))\ y'(t)\ dt$$

Let C be given by $x = t^3$, $y = t^2$, $0 \le t \le 2$.

Evaluate $\displaystyle\int_C x^2 y\ dx$.

Evaluate $\displaystyle\int_C (x + y)\ dy$.

Many times these two integrals will be evaluated together and we usually combine them in this case:

$$\int_C P(x,y)\ dx + \int_C Q(x,y)\ dy = \int_C P(x,y)\ dx + Q(x,y)\ dy$$

Evaluate $\displaystyle\int_C y^2\ dx + x\ dy$, where C is given by $x = e^t$, $y = 2e^{2t}$, $0 \le t \le 2$.

Evaluate $\displaystyle\int_C y^2\ dx + x\ dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

Note: In general for line integrals with respect to x and y, the orientation of the curve, i.e., which direction the curve is traversed is important. If C and $-C$ represent traversing the same curve but in different directions, then $\displaystyle\int_{-C} f(x,y)\ dx = -\int_C f(x,y)\ dx$.

Evaluate $\displaystyle\int_C y^2\ dx + x\ dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(3, 1)$ to $(-5, -3)$.

In three dimensions, all of the above ideas still hold.

Let $f(x,y,z)$ be a function defined on a smooth curve C defined by $x=x(t), y=y(t), z=z(t)$ for $a \leq t \leq b$.

The line integral of f along C with respect to arc length is

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

The line integral of f along C with respect to x is $\displaystyle\int_C f(x,y,z)\ dx = \int_a^b f(x(t),y(t),z(t))\ x'(t)\ dt.$

The line integral of f along C with respect to y is $\displaystyle\int_C f(x,y,z)\ dy = \int_a^b f(x(t),y(t),z(t))\ y'(t)\ dt.$

The line integral of f along C with respect to z is $\displaystyle\int_C f(x,y,z)\ dz = \int_a^b f(x(t),y(t),z(t))\ z'(t)\ dt.$

Evaluate $\displaystyle\int_C xz\ ds$ where C consists of the line segment from $(1,1,0)$ to $(2,3,1)$ followed by the line segment from $(2,3,1)$ to $(4,2,3)$.

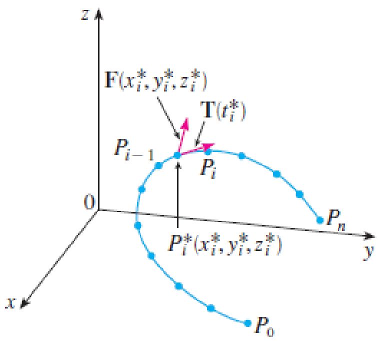
Evaluate $\displaystyle\int_C y\ dx + z^2\ dy + x^2\ dz$ where C is given by $x=t^2$, $y=t^3$, $z=t^4$, $0 \leq t \leq 1$.

Line Integrals over Vector Fields

Suppose a particle is moving along a curve C through a vector field or force field, $\mathbf{F}$.

We define the **line integral of F along C** to be $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt$.

This line integral of $\mathbf{F}$ along C represents the **work** done in moving the object along a curve through the force field.



Find $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by $\mathbf{r}(\mathbf{t}) = \langle t^3, t^2, t\rangle$, $0 \le t \le 1$ and $\mathbf{F}(x, y, z) = \langle 2y, z^2, xy\rangle$.

Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 4$.