## Section 15.9 Change of Variables in Multiple Integrals

We've already seen changing variables when we transformed integrals from rectangular to polar and from rectangular to spherical coordinates.

Consider $\iint_R f(x,y)\,dA$, where $R$ is a region in the $xy$-plane. Suppose we make the substitution $x = g(u,v)$ and $y = h(u,v)$, where $x$ and $y$ are functions of $u$ and $v$ that have continuous first-order partial derivatives. These equations describe a relationship between a region in the $xy$-plane and a region in the new $uv$-plane.

However, just like when we changed from rectangular to polar or from rectangular to spherical, there is an extra "factor" that we have to use in the integrand. That comes from something called the Jacobian:

The **Jacobian** of the transformation $T$ given by $x = x(u,v)$ and $y = y(u,v)$ is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \qquad \text{OR} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

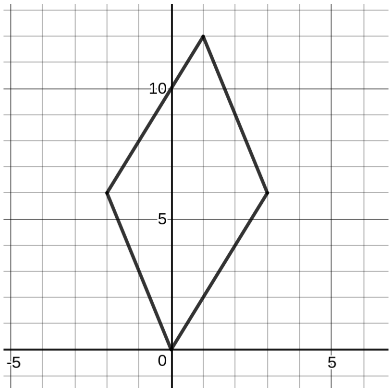
Find the Jacobian of the transformation $x = 2u + v^2$, $y = vu^2$.

Find the Jacobian of the transformation $x = r\cos\theta$, $y = r\sin\theta$.

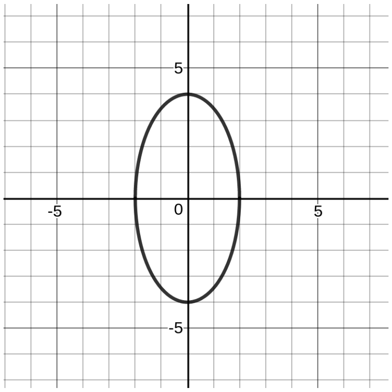
Suppose $T$ is a transformation whose Jacobian is nonzero and that $T$ maps a region $R$ in the $xy$-plane onto a region $S$ in the $uv$-plane. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

Use the change of variables $x = u - v$, $y = 2u + 3v$ to evaluate $\iint_R (2x + y)\, dA$ where $R$ is the region bounded by $y = 2x$, $y = 2x + 10$, $y = -3x$, and $y = -3x + 15$.



Use the change of variables $x = 2v$, $y = 4u$ to evaluate $\iint_R \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{16}}\, dA$ where $R$ is the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$.



Use the change of variables $u = x+y$, $v = x-y$ to evaluate $\displaystyle\iint_R (x+y)\sin(x^2-y^2)\,dA$ where $R$ is the region bounded by $x-y=0$, $x-y=4$, $x+y=0$ and $x+y=1$.

Use the change of variables $u = y - x$, $v = y + x$ to evaluate $\displaystyle\iint_R e^{\frac{y-x}{y+x}}\, dA$ where $R$ is the trapezoidal region with vertices $(2, 0)$, $(4, 0)$, $(0, 4)$ and $(0, 2)$.

For a triple integral, we would need to use a $3 \times 3$ determinant to find the Jacobian. Given the transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ then the Jacobian is

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Find the Jacobian of the transformation $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$.

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = \ldots = \rho^2 \sin\phi$$