Section 16.3 The Fundamental Theorem for Line Integrals

In general, when calculating a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ of a vector field $\mathbf{F}$ from some intial point A to some terminal point B, the value of the integral depends on the curve C, i.e. it depends on the path we take to get from A to B.

Given a force field $\mathbf{F}(x, y) = \langle y^2,\ x \rangle$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where:

The particle travels along the line segment from $(0, 0)$ to $(2, 4)$.

The particle travels along the curve $y = x^2$ from $(0, 0)$ to $(2, 4)$.

If $\mathbf{F}$ is a continuous vector field, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what** curve you travel on as long as the starting and ending points are the same.

Definition: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function f. In other words, there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function** of $\mathbf{F}$.

Consider $f(x, y) = x^3y^2 - xy$.

Recall the Fundamental Theorem of Calculus tells us that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$. Since $\nabla f = \langle f_x, f_y \rangle$, we can think of ∇f as a kind of derivative of the potential function, f. The theorem below is a version of the Fundamental Theorem of Calculus, but for line integrals.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

This means the value of the line integral of a gradient vector field depends ONLY on the starting and ending points. So, line integrals of gradient vector fields are independent of path.

Let $f(x, y) = 3x + yx^2 - y$. Evaluate $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \left\langle 2t,\ t^2 \right\rangle$, $1 \le t \le 2$.

If a vector field $\mathbf{F}$ is conservative, then we know $\mathbf{F} = \nabla f$ for some potential function f and so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

This tells us that **line integrals of conservative vector fields are independent of path**.

But there are two questions to answer. 1) How do we know in general when a vector field is conservative? and 2) If it is conservative, how do we find its potential function, so we can use the Fundamental Theorem?

In $\mathbb{R}^2$, the vector field $\mathbf{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$, where P and Q have continuous first-order partial derivatives on a domain D, is conservative if and only if $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$.

Note: This criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Is $\mathbf{F}(x,y) = \left\langle 3x^2 - 4y, 4y^2 - 2x \right\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

Is $\mathbf{F}(x,y) = \langle x + y, x - 2\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

If a vector field is conservative, it becomes MUCH easier to evaluate line integrals by using the Fundamental Theorem. So check to see if $\mathbf{F}$ is conservative before diving in to a problem.

Given $\mathbf{F}(x, y) = \langle 2xy^3, 3x^2y^2 \rangle$, where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \leq t \leq 1$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

If $\mathbf{F}$ is a **conservative** vector field and C is a **closed path** (one in which the starting point and ending point is the same), what is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

We'll learn in Section 16.5 how to determine if a 3-dimensional vector field is conservative, but for now, assume that the 3D vector field below is conservative.

Given that $\mathbf{F} = \langle y^2z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2z \rangle$ is conservative, find the potential function for $\mathbf{F}$. Then, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \langle \sqrt{t},\ t + 1,\ t^2 \rangle, 0 \leq t \leq 1$.

Section 16.4 Green's Theorem

In this section, we are only considering curves that are closed. A **closed curve** is a curve in which its terminal point coincides with its initial point. A **simple closed curve** is a closed curve that does not cross itself.

Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the xy- plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

To denote that the curve is closed and positively orientated, we sometimes write $\oint_C$ instead of just $\int_C$.

This says that **the line integral over a simple closed curve C is equal to a double integral over the area of the region D that the curve C encloses**.

We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral.

Evaluate $\oint_C yx^2\ dx + x^3\ dy$ where C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$ followed by the line segments from $(2,4)$ to $(0,2)$ and then from $(0,2)$ to $(0,0)$.

Evaluate $\oint_C y^2\ dx + y^3\ dy$ where C is the line segment from $(-2, 0)$ to $(2, 0)$ and then the top half of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(-2, 0)$.

For line integrals over vector fields, Green's Theorem also applies. Given a vector field $\mathbf{F} = \langle P,\ Q \rangle$ and C, a **simple closed curve** defined by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \left\langle 1 - y^3, x^3 + e^{y^2} \right\rangle$ and C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Find the work done by the force field $\mathbf{F} = \langle x^3 - y^2,\ xy \rangle$ in moving a particle counterclockwise along the boundary of the triangle with vertices $(0, 0)$, $(4, 2)$, $(6, 0)$.

What is the work done if we are moving a particle along a closed curve in a conservative vector field?