Section 16.5 Curl and Divergence

There are two more operations that can be performed on vector fields, called divergence and curl, which are measures of how fluid/flow (or whatever the vector field represents) is behaving at a point (x, y, z) in the vector field.

The **del operator**, denoted by ∇, is defined as $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$.
Note that this is **not** the same as the gradient.

If $\mathbf{F} = \langle P,\ Q,\ R \rangle$ is a a vector field and the partial derivatives of P, Q, and R all exist, then the **divergence of** F is defined as $\nabla \cdot \mathbf{F}$.

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P,\ Q,\ R \rangle \ = \ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a vector field $\mathbf{F}$ is a measurement of how much fluid/flow enters the neighborhood around a point P compared to how much fluid/flow exits the neighborhood around P. In other words, it is a measure of the *net change* in fluid/flow entering and exiting.

If more fluid/flow leaves the neighborhood around P than enters it, the divergence will be positive.

If the same amount of fluid/flow leaves the neighborhood as enters it, the divergence will be zero.

If less fluid/flow leaves the neighborhood around P than enters it, the divergence will be negative.

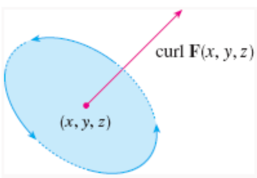
Find the divergence of $\mathbf{F} = \langle x^2y^2,\ xz,\ yz^3 \rangle$.

Find the divergence of $\mathbf{F} = \langle e^z \cos y,\ 0,\ \sin(xy) \rangle$.

If $\mathbf{F} = \langle P,\ Q,\ R\rangle$ is a a vector field and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is defined by $\nabla \times \mathbf{F}$.

$$\text{curl } \mathbf{F} \;=\; \nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The curl measures the tendency of the fluid/flow to rotate in a neighborhood around a point.



Find the curl of $\mathbf{F} = \left\langle e^z \cos y,\ x^2yz^2,\ \sin(xy)\right\rangle$.

Theorem: If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives, then $\mathbf{F}$ is conservative if and only if and curl $\mathbf{F} = \mathbf{0}$.

Determine if the vector field $\mathbf{F} = \left\langle 2xy,\ x^2 + z^3,\ 3yz^2\right\rangle$ is conservative.

Section 16.6 Parametric Surfaces and their Areas

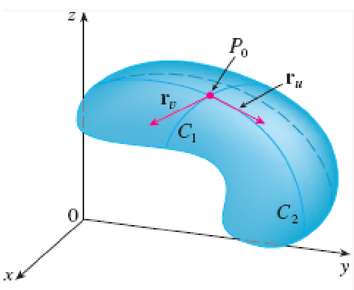
Recall that a *curve* can be parameterized with ONE parameter, which we usually called t. However, to parameterize a *surface*, we need TWO parameters, which we usually call u and v (but could be anything). So, x, y, and z can all be expressed in terms of two parameters/variables.

Parameterize the paraboloid $z = 9 - x^2 - y^2$.

Parameterize the cone $x = \sqrt{y^2 + z^2}$.

Parameterize the cylinder $x^2 + y^2 = 4$.

Given a parametrically-defined surface $\mathbf{r}(u, v)$, the tangent plane to the surface at any point has normal vector given by $\mathbf{r}_u \times \mathbf{r}_v$.



Find the tangent plane to the surface given by $\mathbf{r}(u, v) = \langle u^2 + 1,\ v^3 + 1,\ u + v \rangle$ at the point where $u = 2$ and $v = 1$.

In the case where z can be written in terms of x and y, then $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x,\ -f_y,\ 1\rangle$.

Find the tangent plane to the paraboloid $z = 9 - x^2 - y^2$ at the point $(1, 1, 7)$.

<u>Surface Area</u>

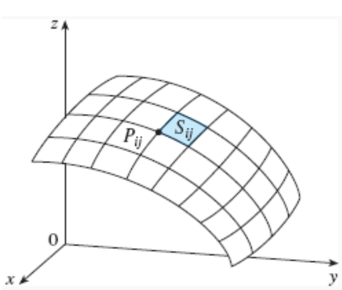
We learned in Chapter 15 that given a surface, $z = f(x, y)$, its surface area is given by

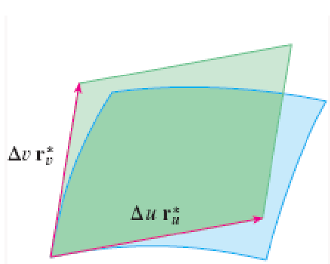
$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\ dA$$

For parametrically-defined surfaces in which z cannot be written explicitly as a function of x and y, we can use the following.

If a smooth parametric surface S is given by $\mathbf{r}(u, v)$, and S is covered just once as (u, v) ranges throughout the parametric domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA$$





(b)

Set up an integral to find the surface area of the surface $\mathbf{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$.

Find the surface area of the part of the cylinder $x^2 + y^2 = 9$ that lies above the xy-plane and below the plane $z = 10 - y$.

Recall from spherical coordinates, a sphere with radius ρ can be parameterized as

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$$

So, $\mathbf{r}(\theta, \phi) = \langle \rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi \rangle$. Note that this is a function of two parameters, θ and ϕ. The value of ρ IS the radius of the sphere because we are on the *surface* of the sphere, not within it.

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \left\langle \rho^2 \sin^2\phi \cos\theta, \rho^2 \sin^2\phi \sin\theta, \rho^2 \sin\phi \cos\phi \right\rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin\phi$$

Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

Section 16.7 Surface Integrals

Suppose we want to integrate a function $f(x, y, z)$ over a surface S given by $\mathbf{r}(u, v)$. Then the **surface integral of f over S is**

$$\iint\limits_S f(x, y, z)\, dS = \iint\limits_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

If the surface S is explicitly given as $z = g(x, y)$, then this can be simplified to

$$\iint\limits_S f(x, y, g(x, y))\, dS = \iint\limits_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2}\ dA$$

Evaluate $\displaystyle\iint\limits_S (2y+z)\,dS$ where S is defined by $\mathbf{r}(u,v) = \langle 3u+v, u-2v, 3-u+v\rangle$, $0 \le u \le 1$ and $0 \le v \le 2$.

Evaluate $\displaystyle\iint\limits_S \left(x^2+y^2\right)\,dS$ where S is the part of the paraboloid $z = 4-x^2-y^2$ that lies above the xy-plane.

Evaluate $\displaystyle\iint_S yz^2 \, dS$ where S is the part of the cone $y = \sqrt{x^2+z^2}$, with $0 \le y \le 2$.

Reminder: For a sphere of radius ρ:

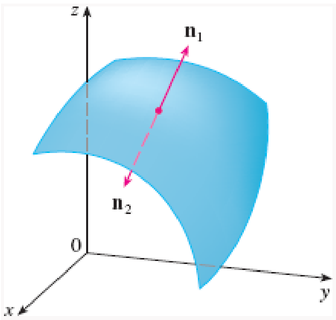
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \left\langle \rho^2 \sin^2\phi \cos\theta, \rho^2 \sin^2\phi \sin\theta, \rho^2 \sin\phi \cos\phi \right\rangle$$

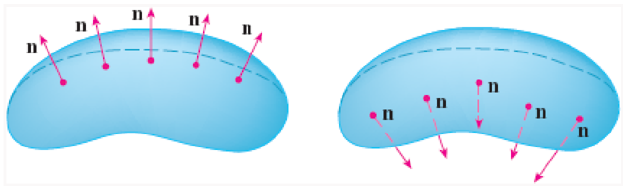
$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin\phi$$

Evaluate $\displaystyle\iint_S z \, dS$ where S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies above the cone $z = \sqrt{x^2+y^2}$.

Flux: Surface Integrals through Vector Fields

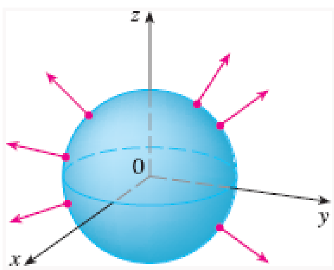
When dealing with surfaces in vector fields, we must have an orientation for the surface because we are looking at two-sided surfaces. The normal vectors to the surface provide this orientation, but there are always two normal vectors!

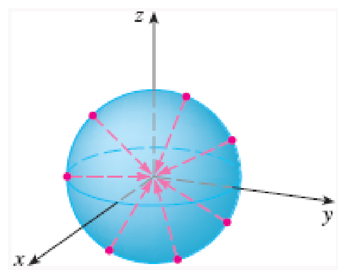




The usual orientation we choose is the upward orientation where the $\mathbf{k}$ components of the normal vector is positive. If the problem doesn't state the orientation, assume upward. However, if the problem says to use downward orientation, we want to have a negative $\mathbf{k}$ component.

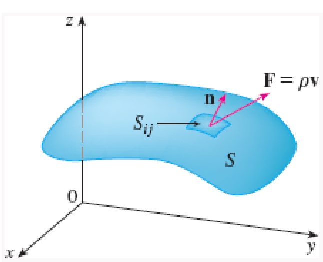
For closed surfaces, we use the normal vectors that point *outward* from the surface. This is called the positive orientation of a closed surface.





Suppose $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field that contains the surface S where S is given by $\mathbf{r}(u, v)$. The amount of fluid/flow that passes through the surface S is called the **flux** of $\mathbf{F}$ across the surface S and is given by

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$



If the surface can be written as $z = f(x, y)$, remember that $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x,\ -f_y,\ 1 \rangle$, which has upward orientation.

Find the flux of $\mathbf{F} = \langle y,\ 2z,\ x \rangle$ across S where S is defined by $\mathbf{r}(u,v) = \langle 3u+v, u-2v, 3-u+v \rangle$, $0 \le u \le 1$ and $0 \le v \le 2$, and S has upward orientation.

Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ across S where S is the part of the surface $z = 4x^2 + 4y^2$ that lies between the planes $z = 8$ and $z = 13$, and has upward orientation.

Find the flux of the vector field $\mathbf{F} = \langle x, y, -2x \rangle$ across the surface S where S is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies below the sphere $x^2 + y^2 + z^2 = 8$ and has downward orientation.