

# Indifference Problem Notes

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## 7/15 Post-Meeting Notes:

Overview: Looking at how indifferent voters not showing up to vote affects the expected social welfare of voting rules' selected winning candidate

$u$  = distribution of voters,  $f(u)$  = density function of distribution  $u$ ,  $F(u)$  = CDF,  $F(\frac{1}{2})$  = number of votes for candidate  $a$ ,  $u'$  = voters who actually show up to vote

People in  $u$  have utilities  $u_i = (p_a, p_b)$ , show up to vote with prob  $u_{\text{choice \#1}} - u_{\text{choice \#2}}$  aka strength of preference between the two candidates (eventually maybe entropy( $u$ ))

$$\text{Probability a person will vote: } u_1 - u_2 = \begin{cases} u_1 - (1 - u_1) & \text{if } u \geq \frac{1}{2} \\ (1 - u_1) - u_1 & \text{if } u < \frac{1}{2} \end{cases} = \begin{cases} 2u_1 - 1 & \text{if } u \geq \frac{1}{2} \\ 1 - 2u_1 & \text{if } u < \frac{1}{2} \end{cases}$$

Problem: Investigating "distortion" =  $\frac{\mathbb{E}[sw(f(u), u)]}{\mathbb{E}[sw(f(u'), u)]}$  for different voting rules

$sw(f(u'), u)$  = total social welfare (from all of  $u$ ) in an election where only  $u'$  vote

Distortion  $\geq 1 \rightarrow$  people not showing up harmed soc. welfare, distortion  $< 1 \rightarrow$  people not showing up improved outcome's sw, find examples?

Eventually try to find bounds on distortion for different voting rules

Starting subproblem: imagine  $u$  as drawn from a Beta distribution  $u_i \sim \text{Beta}(\alpha, \beta)$  (because can extend to more dimensions with Dirichlet( $\vec{\delta}$ ), doesn't necessarily have to be)

Find  $\alpha, \beta$  such that  $\begin{cases} F(\frac{1}{2}) \geq \frac{1}{2} & \text{(aka more people in } u \text{ prefer } a \text{ over } b) \\ \mathbb{E}_{u'}[f(u')] = B & \text{(expected winner if only } u' \text{ vote is } B) \end{cases}$

$\mathbb{E}_{u'}[f(u')] = b$  equiv to  $\int_0^{\frac{1}{2}} (1 - 2u)f(u)du \leq \int_{\frac{1}{2}}^1 (2u - 1)f(u)du$  (ex. votes for A  $\leq$  ex. votes for B)

$(1 - 2u)$  = probability you vote if located at  $u$ ,  $f(u)$  = prob a person's prefs are located at  $u$ ,

$\int_0^{\frac{1}{2}} (1 - 2u)f(u)du$  = expected people who will vote (i.e. are in  $u'$ ) and who prefer A

Constraints:

1.  $\frac{1}{B(\alpha, \beta)} \frac{\int_0^{\frac{1}{2}} t^{\alpha-1}(1-t)^{\beta-1} dt}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt} \leq \frac{1}{2}$
2.  $\int_0^{\frac{1}{2}} (1 - 2u)(u^{\alpha-1}(1-u)^{\beta-1})du \leq \int_{\frac{1}{2}}^1 (2u - 1)(u^{\alpha-1}(1-u)^{\beta-1})du$

## To-Do List (starting 7/16):

1. Find subproblem's example of  $\alpha, \beta$  for various voting rules
  - (a) Investigate whether to do it analytically or with numerical integration
  - (b) Try for plurality voting, then IRV, Borda, etc.
  - (c) Write up examples
2. Analyze distortion for examples
3. Investigate bounds for examples
4. Find examples where distortion  $< 1$

**No distribution  $u \sim \text{Beta}(\alpha, \beta)$  exists such that A strictly wins with full turnout and B strictly wins with turnout weighted according to  $|2u - 1|$ :**

*Proof.* Let  $f(u) = \frac{1}{B(\alpha, \beta)} u^{\alpha-1} (1-u)^{\beta-1}$ ,  $u \in [0, 1]$ . Let  $F(x) = \int_0^x f(u) du$ . A wins under full turnout and B wins with weighted turnout  $\iff$

Constraint 1:  $F(\frac{1}{2}) < \frac{1}{2}$  and

Constraint 2:  $\int_0^{\frac{1}{2}} (1-2u)f(u) du < \int_{\frac{1}{2}}^1 (2u-1)f(u) du$

**Fulfilling Constraint 1:**

$$F(\frac{1}{2}) < \frac{1}{2}$$

$$\implies \int_0^{\frac{1}{2}} f(u) du > \frac{1}{2}$$

$$\implies \alpha < \beta.$$

**Fulfilling Constraint 2:**

$$\int_0^{\frac{1}{2}} (1-2u)f(u) du < \int_{\frac{1}{2}}^1 (2u-1)f(u) du$$

$$\implies -\int_0^{\frac{1}{2}} (2u-1)f(u) du < \int_{\frac{1}{2}}^1 (2u-1)f(u) du$$

$$\implies 0 < \int_{\frac{1}{2}}^1 (2u-1)f(u) du + \int_0^{\frac{1}{2}} (2u-1)f(u) du$$

$$\implies \int_0^1 (2u-1)f(u) du > 0$$

$$\implies 2 \int_0^1 u f(u) du - \int_0^1 f(u) du > 0$$

Because  $f(u)$  is a probability density, we know  $\int_0^1 f(u) du = 1$  and  $\int_0^1 u f(u) du = \mathbb{E}[u]$ .

Thus,  $2 \int_0^1 u f(u) du - \int_0^1 f(u) du > 0 \implies 2\mathbb{E}[u] - 1 > 0 \implies \mathbb{E}[u] > \frac{1}{2}$

For  $u \sim \text{Beta}(\alpha, \beta)$ ,  $\mathbb{E}[u] = \frac{\alpha}{\alpha+\beta}$ .

Thus,  $\mathbb{E}[u] > \frac{1}{2} \implies \frac{\alpha}{\alpha+\beta} > \frac{1}{2}$

$$\implies \alpha > \beta.$$

Because Constraint 1  $\iff \alpha < \beta$  and Constraint 2  $\iff \alpha > \beta$ , there is no distribution  $u \sim \text{Beta}(\alpha, \beta)$  that fulfills both constraints.

It follows that there is no distribution  $u \sim \text{Beta}(\alpha, \beta)$  that fulfills both constraints for any distribution of votes proportional to  $|2u - 1|$ .  $\square$

**No distribution  $u \sim \text{Beta}(\alpha, \beta)$  exists such that A strictly wins with full turnout and B strictly wins with turnout weighted according to any function  $w$  such that  $w(u)$  is symmetric around  $u = \frac{1}{2}$ :**

*Proof.* Let  $w(u)$  be symmetric around  $u = \frac{1}{2}$ .

Then,  $\int_0^1 w(u)f(u) du = \int_0^1 w(1-u)f(u) du$

For  $u \sim \text{Beta}(\alpha, \beta)$  we have:

- $f(u) > f(1-u)$  on  $u \in [0, \frac{1}{2}) \iff \alpha < \beta$
- $f(u) = f(1-u) \iff \alpha = \beta$
- $f(u) < f(1-u)$  on  $u \in [0, \frac{1}{2}) \iff \alpha > \beta$

So if  $f(u)$  has more mass on the left (i.e.  $a$  wins the unweighted vote with full turnout, i.e.  $F(\frac{1}{2}) > \frac{1}{2}$ , i.e.  $\alpha < \beta$ ), and  $w(u)$  is symmetric, then:

We know  $w(u) \geq 0$ , and  $f(u) > f(1-u)$  on  $u \in [0, \frac{1}{2}]$ , so  $w(u)f(u) > w(u)f(1-u)$

$$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_0^{\frac{1}{2}} w(u)f(1-u)du$$

Let  $t = 1 - u$  (while  $u \in [0, \frac{1}{2}]$ ).  $\int_0^{\frac{1}{2}} w(u)f(1-u)du = \int_1^{\frac{1}{2}} w(1-t)f(t)(-dt) = \int_{\frac{1}{2}}^1 w(t)f(t)dt$

$$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_{\frac{1}{2}}^1 w(t)f(t)dt$$

$$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_{\frac{1}{2}}^1 w(u)f(u)du$$

$\implies$  weighted votes for a  $>$  weighted votes for b.

Thus, for votes distributed according to  $u \sim \text{Beta}(\alpha, \beta)$ , a wins the unweighted election  $\implies$  a wins any election weighted according to a function  $w$  such that  $w(u)$  is symmetric around  $u = \frac{1}{2}$ .  $\square$

## 7/23 Post-Meeting Notes

Distortion =  $\frac{sw(a^*, u)}{\mathbb{E}[sw(f(P'), u)]}$ ,  $a^*$  = optimal welfare alternative,  $P$  = fixed preference profile,  $P' \subseteq P$  = after dropout according to  $w(u)$

Next goals:

1. Axiomatize/parametrize  $w$ 
  - $w$  should be symmetric around  $u = \frac{1}{2}$ , need  $w(x) = w(\sigma(x))$  (permutation of  $x$ )
  - Look at goals of voting, give characteristics of  $w$  maybe
  - Conj: if  $w$  is symmetric and single-peaked at  $\frac{1}{2}$  then  $\mathbb{E}[sw(f(P'), u)] \geq \mathbb{E}[sw(f(P), u)]$
2. Two "types of voters"/model with two lines of different proportions
  - $v_a : u_a$  and  $v_b : u_b$  (two utility locations),  $Pr[v_a] = p_a$  and  $Pr[v_b] = p_b$  (two proportions of each group)
  - Find restrictions on  $u_a, u_b, p_a$  (lin eq) where  $F(\frac{1}{2}) \geq \frac{1}{2}$  and  $\int_{\frac{1}{2}}^1 w(x)f(x)dx \geq \int_0^{\frac{1}{2}} w(x)f(x)dx$
  - Fix  $w(x) = (2x - 1)^2 \rightarrow$  parametrize as  $(cx - \frac{c}{2})^2$  (different steepnesses of curve)
3. Characterize continuous (Lipschitz) distributions  $f$  where for symmetric  $w$  weighted turnout can flip outcome
  - $\{f : F(\frac{1}{2}) \geq \frac{1}{2}, \int_{\frac{1}{2}}^1 w(x)f(x)dx \geq \int_0^{\frac{1}{2}} w(x)f(x)dx\}$
  - $\iff \int_{\frac{1}{2}}^1 w(x)[f(x) - f(1-x)]dx \geq 0$ , parametrize somehow?

## Two groups model:

Majority prefers a and b wins election with turnout  $\iff$

$$\begin{cases} p_a > 1 - p_a \\ w(u_b)(1 - p_a) > w(u_a)p_a \end{cases}$$

$$w(u_b)(1 - p_a) > w(u_a)p_a$$

$$\implies w(u_b) - p_a(w(u_b) + w(u_a)) > 0$$

$$\implies \frac{1}{2} < p_a < \frac{w(u_b)}{w(u_b) + w(u_a)}$$

For  $w(u) = |2u - 1|$ :

$$\begin{aligned}
p_a &< \frac{w(u_b)}{w(u_b) + w(u_a)} \\
\implies p_a &< \frac{2u_b - 1}{2u_b - 1 + 1 - 2u_a} \\
\implies p_a &< \frac{2u_b - 1}{2u_b - 2u_a} \\
\implies \frac{1}{2} &< \frac{2u_b - 1}{2u_b - 2u_a} \\
\implies u_b - u_a &< 2u_b - 1 \\
\implies u_a + u_b &> 1
\end{aligned}$$

### Investigating possible $f$ :

Goal: characterize continuous  $f$  where weighting turnout according to a symmetric  $w$  can flip the outcome of an election,  $\{f : F(\frac{1}{2}) \geq \frac{1}{2}, \int_{\frac{1}{2}}^1 w(x)f(x)dx \geq \int_0^{\frac{1}{2}} w(x)f(x)dx\}$

Approaching as a convex optimization problem:

$$D(f) = \int_{\frac{1}{2}}^1 w(u)[f(u) - f(1 - u)]du$$

Feasible space:  $S = \{f \in C^0([0, 1]) : f \geq 0, \int_0^1 f = 1, \int_0^{\frac{1}{2}} f > \frac{1}{2}\}$

Maximize  $D(f)$  over  $S$  (aka asking question:  $\exists f \in S$  s.t.  $D(f) > 0$ ?)