Indifference Problem Notes

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7/15 Post-Meeting Notes:

Overview: Looking at how indifferent voters not showing up to vote affects the expected social welfare of voting rules' selected winning candidate

u = distribution of voters, $f(u)$ = density function of distribution u, $F(u)$ = CDF, $F(\frac{1}{2})$ = number of votes for candidate a, u' = voters who actually show up to vote
People in u have utilities $u_i = (p_a, p_b)$, show up to vote with prob $u_{\text{choice } \#1} - u_{\text{choice } \#2}$ aka strength of preference between the two candidates (eventually maybe entropy(u))

Probability a person will vote: $u_1 - u_2 = \begin{cases} u_1 - (1-u_1) & \text{if } u \geq \frac{1}{2} \\ (1-u_1) - u_1 & \text{if } u \geq \frac{1}{2} \end{cases} = \begin{cases} 2u_1 - 1 & \text{if } u \geq \frac{1}{2} \\ 1 - 2u_1 & \text{if } u \geq \frac{1}{2} \end{cases}$

Problem: Investigating "distortion" $= \frac{\mathbb{E}[sw(f(u),u)]}{\mathbb{E}[sw(f(u'),u)]}$ for different voting rules
$sw(f(u'),u)$ = total social welfare (from all of u) in an election where only u' vote
Distortion $\geq 1 \rightarrow$ people not showing up harmed soc. welfare, distortion $< 1 \rightarrow$ people not showing up improved outcome's sw, find examples?
Eventually try to find bounds on distortion for different voting rules

Starting subproblem: imagine u as drawn from a Beta distribution $u_i \sim \text{Beta}(\alpha, \beta)$ (because can extend to more dimensions with Dirichlet$(\vec{\delta})$, doesn't necessarily have to be)

Find α, β such that $\begin{cases} F(\frac{1}{2}) \geq \frac{1}{2} & \text{(aka more people in } u \text{ prefer } a \text{ over } b) \\ \mathbb{E}_{u'}[f(u')] = \text{ B} & \text{(expected winner if only } u' \text{ vote is B)} \end{cases}$

$\mathbb{E}_{u'}[f(u')] = b$ equiv to $\int_0^{\frac{1}{2}} (1-2u)f(u)du \leq \int_{\frac{1}{2}}^1 (2u-1)f(u)du$ (ex. votes for A $\leq$ ex. votes for B)
$(1-2u)$ = probability you vote if located at u, $f(u)$ = prob a person's prefs are located at u, $\int_0^{\frac{1}{2}} (1-2u)f(u)du$ = expected people who will vote (i.e. are in u') and who prefer A

Constraints:

1. $\frac{1}{B(\alpha,\beta)} \frac{\int_0^{\frac{1}{2}} t^{\alpha-1}(1-t)^{\beta-1}dt}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt} \leq \frac{1}{2}$
2. $\int_0^{\frac{1}{2}} (1-2u)(u^{\alpha-1}(1-u)^{\beta-1})du \leq \int_{\frac{1}{2}}^1 (2u-1)(u^{\alpha-1}(1-u)^{\beta-1})du$

To-Do List (starting 7/16):

1. Find subproblem's example of α, β for various voting rules
   (a) Investigate whether to do it analytically or with numerical integration
   (b) Try for plurality voting, then IRV, Borda, etc.
   (c) Write up examples
2. Analyze distortion for examples
3. Investigate bounds for examples
4. Find examples where distortion < 1

No distribution $u \sim \text{Beta}(\alpha, \beta)$ exists such that A strictly wins with full turnout and B strictly wins with turnout weighted according to $|2u-1|$:

Proof. Let $f(u) = \frac{1}{B(\alpha,\beta)} u^{\alpha-1}(1-u)^{\beta-1}, u \in [0,1]$. Let $F(x) = \int_0^x f(u)du$. A wins under full turnout and B wins with weighted turnout $\iff$
Constraint 1: $F(\frac{1}{2}) < \frac{1}{2}$ and
Constraint 2: $\int_0^{\frac{1}{2}} (1-2u)f(u)du < \int_{\frac{1}{2}}^1 (2u-1)f(u)du$

Fulfilling Constraint 1:
$F(\frac{1}{2}) < \frac{1}{2}$
$\implies \int_0^{\frac{1}{2}} f(u)du > \frac{1}{2}$
$\implies \alpha < \beta$.

Fulfilling Constraint 2:
$\int_0^{\frac{1}{2}} (1-2u)f(u)du < \int_{\frac{1}{2}}^1 (2u-1)f(u)du$
$\implies -\int_0^{\frac{1}{2}} (2u-1)f(u)du < \int_{\frac{1}{2}}^1 (2u-1)f(u)du$
$\implies 0 < \int_{\frac{1}{2}}^1 (2u-1)f(u)du + \int_0^{\frac{1}{2}} (2u-1)f(u)du$
$\implies \int_0^1 (2u-1)f(u)du > 0$
$\implies 2\int_0^1 uf(u)du - \int_0^1 f(u)du > 0$
Because $f(u)$ is a probability density, we know $\int_0^1 f(u)du = 1$ and $\int_0^1 uf(u)du = \mathbb{E}[u]$.
Thus, $2\int_0^1 uf(u)du - \int_0^1 f(u)du > 0 \implies 2\mathbb{E}[u] - 1 > 0 \implies \mathbb{E}[u] > \frac{1}{2}$
For $u \sim \text{Beta}(\alpha, \beta)$, $\mathbb{E}[u] = \frac{\alpha}{\alpha+\beta}$.
Thus, $\mathbb{E}[u] > \frac{1}{2} \implies \frac{\alpha}{\alpha+\beta} > \frac{1}{2}$
$\implies \alpha > \beta$.
Because Constraint 1 $\iff \alpha < \beta$ and Constraint 2 $\iff \alpha > \beta$, there is no distribution $u \sim \text{Beta}(\alpha, \beta)$ that fulfills both constraints.
It follows that there is no distribution $u \sim \text{Beta}(\alpha, \beta)$ that fulfills both constraints for any distribution of votes proportional to $|2u-1|$. $\square$

No distribution $u \sim \text{Beta}(\alpha, \beta)$ exists such that A strictly wins with full turnout and B strictly wins with turnout weighted according to any function w such that $w(u)$ is symmetric around $u = \frac{1}{2}$:

Proof. Let $w(u)$ be symmetric around $u = \frac{1}{2}$.
Then, $\int_0^1 w(u)f(u)du = \int_0^1 w(1-u)f(u)du$
For $u \sim \text{Beta}(\alpha, \beta)$ we have:

- $f(u) > f(1-u)$ on $u \in [0, \frac{1}{2}) \iff \alpha < \beta$
- $f(u) = f(1-u) \iff \alpha = \beta$
- $f(u) < f(1-u)$ on $u \in [0, \frac{1}{2}) \iff \alpha > \beta$

So if $f(u)$ has more mass on the left (i.e. a wins the unweighted vote with full turnout, i.e. $F(\frac{1}{2}) > \frac{1}{2}$, i.e. $\alpha < \beta$), and $w(u)$ is symmetric, then:

We know $w(u) \geq 0$, and $f(u) > f(1-u)$ on $u \in [0, \frac{1}{2}$, so $w(u)f(u) > w(u)f(1-u)$
$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_0^{\frac{1}{2}} w(u)f(1-u)du$
Let $t = 1-u$ (while $u \in [0, \frac{1}{2}]$). $\int_0^{\frac{1}{2}} w(u)f(1-u)du = \int_1^{\frac{1}{2}} w(1-t)f(t)(-dt) = \int_{\frac{1}{2}}^1 w(t)f(t)dt$
$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_{\frac{1}{2}}^1 w(t)f(t)dt$
$\implies \int_0^{\frac{1}{2}} w(u)f(u)du > \int_{\frac{1}{2}}^1 w(u)f(u)du$
$\implies$ weighted votes for a $>$ weighted votes for b.
Thus, for votes distributed according to $u \sim \text{Beta}(\alpha, \beta)$, a wins the unweighted election $\implies$ a wins any election weighted according to a function w such that $w(u)$ is symmetric around $u = \frac{1}{2}$. $\square$

7/23 Post-Meeting Notes

Distortion $= \frac{sw(a^*,u)}{\mathbb{E}[sw(f(P'),u)]}$, a^* = optimal welfare alternative, P = fixed preference profile, $P' \subseteq P$ = after dropout according to $w(u)$
Next goals:

1. Axiomatize/parametrize w
   - w should be symmetric around $u = \frac{1}{2}$, need $w(x) = w(\sigma(x))$ (permutation of x)
   - Look at goals of voting, give characteristics of w maybe
   - Conj: if w is symmetric and single-peaked at $\frac{1}{2}$ then $\mathbb{E}[sw(f(P'),u)] \geq \mathbb{E}[sw(f(P),u)]$
2. Two "types of voters"/model with two lines of different proportions
   - $v_a : u_a$ and $v_b : u_b$ (two utility locations), $Pr[v_a] = p_a$ and $Pr[v_b] = p_b$ (two proportions of each group)
   - Find restrictions on u_a, u_b, p_a (lin eq) where $F(\frac{1}{2}) \geq \frac{1}{2}$ and $\int_{\frac{1}{2}}^1 w(x)f(x)dx \geq \int_0^{\frac{1}{2}} w(x)f(x)dx$
   - Fix $w(x) = (2x-1)^2 \to$ parametrize as $(cx - \frac{c}{2})^2$ (different steepnesses of curve)
3. Characterize continuous (Lipschitz) distributions f where for symmetric w weighted turnout can flip outcome
   - $\{f : F(\frac{1}{2}) \geq \frac{1}{2}, \int_{\frac{1}{2}}^1 w(x)f(x)dx \geq \int_0^{\frac{1}{2}} w(x)f(x)dx\}$
   - $\iff \int_{\frac{1}{2}}^1 w(x)[f(x) - f(1-x)]dx \geq 0$, parametrize somehow?

Two groups model:

Majority prefers a and b wins election with turnout $\iff$

$$\begin{cases} p_a > 1 - p_a \\ w(u_b)(1-p_a) > w(u_a)p_a \end{cases}$$

$$\begin{aligned} & w(u_b)(1-p_a) > w(u_a)p_a \\ & \implies w(u_b) - p_a(w(u_b) + w(u_a)) > 0 \\ & \implies \frac{1}{2} < p_a < \frac{w(u_b)}{w(u_b) + w(u_a)} \end{aligned}$$

For $w(u) = |2u - 1|$:

$$
\begin{aligned}
p_a &< \frac{w(u_b)}{w(u_b) + w(u_a)} \\
\implies p_a &< \frac{2u_b - 1}{2u_b - 1 + 1 - 2u_a} \\
\implies p_a &< \frac{2u_b - 1}{2u_b - 2u_a} \\
\implies \frac{1}{2} &< \frac{2u_b - 1}{2u_b - 2u_a} \\
\implies u_b - u_a &< 2u_b - 1 \\
\implies u_a + u_b &> 1
\end{aligned}
$$

Investigating possible f:

Goal: characterize continuous f where weighting turnout according to a symmetric w can flip the outcome of an election, $\{f : F(\frac{1}{2}) \geq \frac{1}{2}, \int_{\frac{1}{2}}^{1} w(x)f(x)dx \geq \int_{0}^{\frac{1}{2}} w(x)f(x)dx\}$

Approaching as a convex optimization problem:

$D(f) = \int_{\frac{1}{2}}^{1} w(u)[f(u) - f(1-u)]du$

Feasible space: $S = \{f \in C^0([0,1]) : f \geq 0, \int_0^1 f = 1, \int_0^{\frac{1}{2}} f > \frac{1}{2}\}$
Maximize $D(f)$ over S (aka asking question: $\exists f \in S$ s.t. $D(f) > 0$?)
(see Python script, not sure how to interpret)

Trying to linearize everything with nudge function:

$f(u) = f_0(u) + \epsilon g(u)$ where $f_0(u) = 1$ (uniform density for baseline), $\int_0^1 g(u)du = 0$, $\epsilon << 1$ controls deviation (still fulfills pdf requirements)
Full turnout condition:
$\int_0^{\frac{1}{2}} f(u)du = \int_0^{\frac{1}{2}} (1 + \epsilon g(u))du = \frac{1}{2} + \epsilon \int_0^{\frac{1}{2}} g(u)du$
$\implies$ A wins under full turnout $\iff \epsilon \int_0^{\frac{1}{2}} g(u)du > 0$
Weighted turnout condition:
$\int_{\frac{1}{2}}^{1} w(u)f(u)du - \int_0^{\frac{1}{2}} w(u)f(u)du = \int_0^1 w(u)f(u)\,\mathrm{sgn}(u - \frac{1}{2})du$
$\int_0^1 w(u)(1 + \epsilon g(u))\,\mathrm{sgn}(u - \frac{1}{2})du = \int_0^1 w(u)\,\mathrm{sgn}(u - \frac{1}{2})du + \epsilon \int_0^1 w(u)g(u)\,\mathrm{sgn}(u - \frac{1}{2})du$
$\int_0^1 w(u)\,\mathrm{sgn}(u - \frac{1}{2})du = 0$ because $\mathrm{sgn}(u - \frac{1}{2})$ flips from -1 to 1 and w is symmetric, so the two halves should cancel out.
$\implies$ B wins with weighted turnout $\iff \epsilon \int_0^1 w(u)g(u)\,\mathrm{sgn}(u - \frac{1}{2})du > 0$
A wins under full turnout and B wins with weighted turnout $\iff$
Constraint 1: $\epsilon \int_0^{\frac{1}{2}} g(u)du > 0$ and
Constraint 2: $\epsilon \int_0^1 w(u)g(u)\,\mathrm{sgn}(u - \frac{1}{2})du > 0$

7/29 Post-Meeting Notes

To-Do:

- Conjecture: FOSD of G_B vs. G_A is sufficient to enable swap

- $g_A : [0, \frac{1}{2}] = f(x), g_B : [0, \frac{1}{2}] = f(1-x)$ gives strength of prefs, $[0, \frac{1}{2}]$ maps strong to weak preferences
  - First order stochastic dominance: point-for-point $G_B \geq G_A$
  - Sufficient that flip over symmetric w can occur with $u \sim f$
- Explore whether FOSD is necessary
- Try to write "closest case" of f as function of w
  - Closest case minimizes votes for B - votes for A
  - Look at Python script for examples, w closest case looks quadratic when w is quadratic, etc.
- Claim: can characterize dist(V, w) for majority on two outcomes not achieving optimal
  - Comes from characterizing f from w?
  - V is voting rule, P is total population, P' is weighted voters

Conjecture: FOSD of G_B vs. G_A is sufficient to enable swap

Definition 1. *Let A, B be two random variables. A has first-order stochastic dominance over B*

$$\iff \forall \text{ outcomes } x, Pr[A \geq x] \geq Pr[B \geq x] \text{ and } \exists x \text{ such that } Pr[A \geq x] > Pr[B \geq x]$$

$$\iff \forall x \; F_A(x) \leq F_B(x) \text{ and } \exists x \text{ such that } F_A(x) < F_B(x), \text{ where } F_A, F_B \text{ are the CDFs of } A, B.$$

Let $f(x) : [0, 1]$ be some probability density from which voters' preferences are drawn, with $F(x) : [0, 1]$ as its cumulative distribution function. Let $g_A : [0, \frac{1}{2}] = f(x), g_B : [0, \frac{1}{2}] = f(1-x)$. Then, let $G_A : [0, \frac{1}{2}] = \int_0^x g_A(x)dx = \int_0^x f(x)dx = F(x)$, and $G_B : [0, \frac{1}{2}] = \int_0^x g_B(x)dx = \int_0^x f(1-x)dx = -\int_1^{1-x} f(x)dx = \int_{1-x}^1 f(x)dx = F(1) - F(1-x)$.

Proof. Suppose G_A has FOSD over G_B. □