Planetary motion around the Sun

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1 Abstract

We aim to study planetary motion around the Sun of the Earth by numerical methods, using both python and excel spreadsheets as two very different methods.
We investigated the trajectory of the Earth around the Sun from a circular to elliptical orbit alongside with Kepler's Third Law. And have evaluated the remedy approach for scaling time interval by considering the problem on energy and angular momentum conservation point of view.

2 Introduction

We aim to solve the problem of solving for orbits of the Earth revolutions around the Sun by numerical methods. Aiming to search for the relationships between semi major axis a and period, as well as the dependence of initial velocity to the shape of orbit. We also aim to use to data for further studies in the topic of planetary motion of the Earth.

3 Methodology

To investigate the planetary motion, we have used both Excel spreadsheet and one programming language: Python. For the programming approach, it made use of numpy with panda, matplotlib, pandas and scipy as the tools to cope with the problems.

4 Problem

Let $\vec{r}=(x,y), \vec{v}=(v_x,v_y), \vec{a}=(a_x,a_y)$. Mass of Sun $=M$; mass of planet $=m$. Assume $M \gg m$ so that the motion of the sun can be neglected.
Consider the Earth resolute the Sun in a circular orbit of radius R and period T, s.t.

$$\begin{aligned} F = \frac{GMm}{R^2} &= \frac{mv^2}{R}, v = \frac{2\pi R}{T} \\ \frac{GM}{R} &= \frac{4\pi^2 R^2}{T^2} \\ R^3 &= \frac{GM}{4\pi^2}T^2 \end{aligned}$$

Hence, when measured length in AU and time in years, then

$$\begin{aligned} (1)^3 &= \frac{GM}{4\pi^2}(1)^2 \\ \Rightarrow GM &= 4\pi^2 \end{aligned}$$

i.e. $v_0 = \frac{2\pi R}{T} = \frac{2\pi(1)}{1} = 2\pi$ AUyear^{-1}
For both approaches to the problem, the initial conditions are all chose to be $x=1$, $y=0$, $v_x=0$, and $v_y=v_0$, with $\Delta t = 0.002$ or smaller. And made use of equations

$$x(n+1) = x(n) + v_x(n)\Delta t, \tag{1}$$
$$v_x(n+1) = v_x(n) + a_x(n)\Delta t, \tag{2}$$
$$a_x = \frac{-4\pi^2 x}{(x^2+y^2)^{\frac{3}{2}}} \tag{3}$$

and likewise to the y component.

5 Result analysis

5.1 Energy conservation and angular momentum conservation

From calculation, it is found that the total energy and angular momentum do not conserve from the two equations, that is

$$\frac{E}{m} = \frac{1}{2}(v_x^2+v_y^2) - \frac{GM}{r}, \frac{L_z}{m} = (\vec{r}\times\frac{\vec{p}}{m})_z = xv_y - yv_x$$

It is observed that when $\Delta t = 0.002$, the orbit is certainly not closed.
However, angular momentum and energy should be conserved. However, from the iterated equation, it can be shown that error would accumulate in the numerical process where
Energy-wise,

$$v(n+1)^2 = v_x(n+1)^2 + v_y(n+1)^2$$

$v_x(n+1)^2 + v_y(n+1)^2$

$$= v_x(n) + a_x(n)\Delta t)^2 + (v_y(n) + a_y(n)\Delta t)^2$$

$$= v_x(n)^2 + v_y(n)^2) + 2\Delta t(v_x(n)a_x(n) + v_y(n)a_y(n)) + \Delta t^2(a_x(n)^2 + a_y(n)^2)$$

$$= v(n)^2 + 2\Delta t(v_x(n)a_x(n) + v_y(n)a_y(n)) + a(n)^2\Delta t^2$$

$$\approx v(n)^2 + 2\Delta t(v_x(n)a_x(n) + v_y(n)a_y(n))$$

Since $a_x(n+1), a_y(n+1) < 0$, i.e. $v(n+1)^2 < v(n)^2$
The sign of acceleration is always negative in this motion since the radial acceleration is centripetal. Therefore the orbit is not closed when $\Delta t = 0.0002$ since potential energy increases when kinetic energy decreases due to energy loss. The shape of orbit therefore is spiralling outwards.
However, as $\Delta t \to 0$, $v(n+1)^2 \to v(n)^2$.
Angular-momentum-wise,

$$\frac{L_z}{m} = (\vec{r} \times \frac{\vec{p}}{m})_z = xv_y - yv_x$$

$x(n+1)v_y(n+1) - y(n+1)v_x(n+1)$

$$= x(n) + v_x(n)\Delta t)(v_y(n) + a_y(n)\Delta t) - (y(n) + v_y(n)\Delta t)(v_x(n) + a_x(n)\Delta t)$$

$$= x(n)v_y(n) - y(n)v_x(n) + \Delta t(x(n)a_y(n) - y(n)a_x(n)) + \Delta t^2(v_x(n)a_y(n) - v_y(n)a_x(n))$$

$$\approx x(n)v_y(n) - y(n)v_x(n) + \Delta t(x(n)a_y(n) - y(n)a_x(n))$$

i.e. $x(n+1)v_y(n+1) - y(n+1)v_x(n+1) > x(n)v_y(n) - y(n)v_x(n)$
However, as $\Delta t \to 0$, $x(n+1)v_y(n+1) - y(n+1)v_x(n+1) \to x(n)v_y(n) - y(n)v_x(n)$.
Therefore a small interval is chosen s.t. the max error < 0.1.
The orbit generated by computers is not exactly closed, however, the difference is very small with an order of magnitude of -7 and the difference drops with an reduced interval.

5.2 Elliptical orbits and Kepler's Third Law

For elliptical orbits, with $\alpha = 0.8$, 0.9, 1.05,and 1.1 respectively, as changing $v_y = \alpha v_0$. It is observed that the elliptical orbit increases in size and periods with increasing value of α in these investigates values.

Table 1: respective periods and semi major axis a values (spreadsheet)

α	a per AU	T per year
0.80	0.7384	0.6174
0.90	0.8419	0.7603
1.05	1.1153	1.1853
1.10	1.2669	1.4328

Table 2: respective periods and semi major axis a values (numpy)

α	a per AU	T per year
0.80	0.7353	0.6305
0.90	0.8403	0.7703
1.05	1.1142	1.1761
1.10	1.2658	1.4241

In order to investigate upon the Keller's third law for elliptical orbit, R is then replaced by a, the semi major axis of the orbit.

$$a^3 = \frac{GM}{4\pi^2}T^2$$
$$\Rightarrow loga = \frac{2}{3}logT + \frac{1}{3}log\frac{GM}{4\pi^2}$$

From the graph, the slope found from using python is 0.6667 ($4d.p.$); using excel spreadsheets is 0.6392 ($4d.p.$). The percentage error for python is about 6.39755×10^{-6} while for excel spreadsheets is about -4.
Both methods, however, shows a straight line slanting upwards.
To conclude, the slope of the graph agrees with the law quite well.

5.3 The Escape case

Under unchanged initial condition, the maximum initial speed s.t. $v_y = v_{esc}$, is found by

$$
\begin{aligned}
(K+U)_i &= (K+U)_f \\
\frac{1}{2}mv_{esc}^2 - \frac{GMm}{r} &= 0 \\
\frac{1}{2}mv_{esc}^2 &= \frac{GMm}{r} \\
\Rightarrow v_{esc} &= \sqrt{\frac{2GM}{r}}
\end{aligned}
$$

i.e. $v_{esc} = \sqrt{\frac{2(4\pi^2)}{1}} = 2\pi\sqrt{2}$ AU year^{-1}

6 Discussion and further study

6.1 Correlation to Kepler's First and Second law

6.1.1 Kepler's First Law

Kepler's First Law is not agreed.

6.1.2 Kepler's Second Law

Also, from the analysis, it is able to see that angular momentum is not conserved implying Kepler's Second Law is neither agreed.

6.1.3 Relationship of extreme velocity and perihelion and aphelion

Another method to see if Kepler's Second law is obeyed reasonable is to investigate the relationship of extreme velocity and perihelion and aphelion. Comparing the r values to Table 3, it is found that the respective r values with extreme velocities are with very little or no difference when compared to perihelion and aphelion values. It is observed that maximum velocity occurs at perihelion while minimum velocity occurs at aphelion. This correspond to agreeing with Kepler's Second law quite reasonably. Although it is found that the angular momentum do not entirely conserve at all orbits plotted, however, it agrees to the part where area swept at same time interval is the same in an ideal situation.

Table 3: List of extreme velocity and respective x-coordinated to each of the investigated α values using spreadsheets

α	v_{max} in AU year^{-1}	v_{min} in AU year^{-1}	$r_{v_{max}}$ in AU	$r_{v_{min}}$ in AU	t in year	t_{diff} in year
0.8	10.6761	5.0245	0.4712	1.0028	0.3021	-0.0066
0.9	8.3041	5.6498	0.6813	1.0021	0.3752	-0.0050
1.05	6.5972	5.3683	1.0000	1.2293	0.5888	-0.0038
1.10	6.9113	4.5105	1.0000	1.5327	0.7128	-0.0036

Another point worth noting is that the time difference, taking periods of respective α values as the accurate value, shows very little difference.
It is found that t, the time taken of the Earth travelling between position having maximum speed and minimum speed once, has very little difference with respective $\frac{T}{2}$. Implying the two points must be intersecting the center of the elliptical orbit, such that it bisects the orbit.
These two points can conclude that at perihelion, the planet obtains highest velocity, while at aphelion obtains lowest velocity such that the time difference is half a period, in conclusion.

6.2 Shape of elliptical orbits

From the obtained orbits with α = 0.8, 0.9, 1.05,and 1.1 respectively, by fixing the focus of the sun at $(0, 0)$ and the initial position of the Earth, the semi major axis a increases with α such that the perihelion increases with α as well.

Table 4: List of perihelion and aphelion to respective α values (spreadsheets)

α	**perihelion**	**aphelion**
0.80	0.4712	1.0028
0.90	0.6813	1.0021
1.05	1.0000	-1.2293
1.10	1.0000	-1.5327

It is known that all the plotted orbits are elliptical, however, the shapes are all a little bit different in shape, owing to varying eccentricity.

Table 5: List of perihelion and aphelion to respective α values (numpy)

α	**perihelion**	**aphelion**
0.80	0.4706	1.0000
0.90	0.6807	1.0000
1.05	1.0000	-1.2284
1.10	1.0000	-1.5316

Table 6: List of e values to respective α values using both spreadsheet and numpy

α	e_s	e_n
0.80	0.3611	0.3600
0.90	0.1908	0.1900
1.05	0.1028	0.1025
1.10	0.2103	0.2100

The different shapes are accounted for the different eccentricity values, since it differs by range from 0.866 times to 2.524 times corrected to 3 decimal places.

6.3 A more realistic case of escape

Considering the more realistic case for the maximum escape initial velocity of the Earth, that is with rotational Energy involved. Assuming each shell is of uniform mass distribution and is a perfect sphere itself.

$$
\begin{aligned}
E_{tot} &= \frac{1}{2}mv^2 - \frac{GMm}{r} + \frac{1}{2}I\omega^2 \\
\Rightarrow \frac{E_{tot}}{m} &= \frac{1}{2}(v^2 + \frac{2}{5}(\frac{r_E}{1AU})^2(\frac{2\pi^2}{T_{year}})^2) - \frac{GM}{r}
\end{aligned}
$$

The energy per mass difference is of 2×3.81566 measured in units of AU and year (5 sig.fig.).

Since the initial conditions remains unchanged. At $t = 0$,

$$\frac{1}{2}mv_{esc}'^2 + E_{rot} - \frac{GMm}{r} = 0$$
$$\Rightarrow v_{esc}' = \sqrt{2 \times [\frac{GM}{1} - E_{rot}]}$$

i.e. $v_{esc}' \approx 8.885 AU year^{-1}$ (3 d.p.)
Percentage error= $\frac{v_{esc}' - 2\sqrt{2}\pi}{2\sqrt{2}\pi} \times 100\% = -9.66 \times 10^{-3}\%$ (3 d.p.)
However, this solution for maximum initial velocity adds to the dependency of mass distribution and shape of the Earth (the planet) while using neglecting the rotational energy depends only on the Sun (the planet is orbiting around). This shows that neglecting the rotational energy is actually reasonable, for the first cause, it has low percentage difference implying high accuracy, second, one fewer factors to be concerned about.

6.4 Algebraic relation of planetary motion to numerical results

By Newton's Second Law,

$$\vec{F} = \frac{-GMm}{\|\vec{r}\|^2}\frac{\vec{r}}{\|\vec{r}\|}$$
$$mv'(t) = \frac{-GMm}{\|\vec{r}\|^3}\vec{r}$$
$$\frac{dv}{dt} = \frac{-GM}{\|\vec{r}\|^3}\vec{r}$$
$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

Since $\vec{v} \times \vec{v} = 0$ and $\frac{d\vec{v}}{dt} \parallel \vec{r}$, thus $\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$. This implies that $(\vec{r} \times \vec{r}) = \vec{C}$ such that $\vec{C}$ is a constant vector function and $\vec{r}, \vec{v} \perp \vec{C}\ \forall t$.

$$\Rightarrow \frac{d\vec{v}}{dt} \times \vec{C} = \frac{-GMm}{\|\vec{r}\|^3}\vec{r} \times (\vec{r} \times \vec{v})$$
$$= \frac{-GMm}{\|\vec{r}\|^3}[(\vec{r} \cdot \vec{v})\vec{r} - (\vec{r} \cdot \vec{r})\vec{v}]$$
$$= \frac{-GMm}{\|\vec{r}\|^3}[(\vec{r} \cdot \vec{v})\vec{r} - \|\vec{r}\|^2\vec{v}]$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d\vec{r}}{dt}\vec{r} + \vec{r}\frac{d\vec{r}}{dt} = 2\vec{r}\cdot\vec{v} \tag{4}$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d}{dt}\|\vec{r}\|^2 = 2\|\vec{r}\|\frac{d}{dt}\|\vec{r}\| \tag{5}$$

From (4), (5),

$$2\vec{r}\cdot\vec{v} = 2\|\vec{r}\|\frac{d}{dt}\|\vec{r}\|$$

$$\Rightarrow \vec{r}\cdot\vec{v} = \|\vec{r}\|\frac{d}{dt}\|\vec{r}\| \tag{6}$$

$$\frac{d\vec{v}}{dt}\times\vec{C} = \frac{-GM}{\|\vec{r}\|^3}[\vec{r}\|\vec{r}\|\frac{d}{dt}(\|\vec{r}\|) - \|\vec{r}\|^2\vec{v}]$$

$$\Rightarrow \frac{d\vec{v}}{dt}\times\vec{C} = GM[\frac{\vec{v}}{\|\vec{r}\|} - \frac{\vec{r}}{\|\vec{r}\|^2}\frac{d}{dt}(\|\vec{r}\|)] \tag{7}$$

$$\begin{aligned}\frac{d}{dt}\frac{\vec{r}}{\|\vec{r}\|} &= \frac{\|\vec{r}\|\frac{d\vec{r}}{dt} - \vec{r}\frac{d\|\vec{r}\|}{dt}}{\|\vec{r}\|^2} \\ &= \frac{1}{\|\vec{r}\|}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{\|\vec{r}\|^2}\frac{d\|\vec{r}\|}{dt} \\ &= \frac{\vec{v}}{\|\vec{r}\|} - \frac{\vec{r}}{\|\vec{r}\|^2}\frac{d\|\vec{r}\|}{dt}\end{aligned}$$

$$\Rightarrow \frac{d\vec{v}}{dt}\times\vec{C} = GM(\frac{d}{dt}\frac{\vec{r}}{\|\vec{r}\|}) \tag{8}$$

$$\vec{v}\times\vec{C} = GM\frac{\vec{r}}{\|\vec{r}\|} + \vec{D} \tag{9}$$

s.t. $\frac{d\vec{D}}{dt} = 0$

$$\begin{aligned}\vec{r}\cdot(\vec{v}\times\vec{C}) &= \frac{GM\|\vec{r}\|^2}{\|\vec{r}\|} + \vec{r}\cdot\vec{D} \\ &= GM\|\vec{r}\| + \vec{r}\cdot\vec{D}\end{aligned}$$

$$\begin{aligned}&\vec{r}\cdot(\vec{v}\times\vec{C}) = (\vec{r}\times\vec{v})\cdot\vec{C} \\ &\Rightarrow \|\vec{C}\|^2 = GM\|\vec{r}\| + \|\vec{r}\|\|\vec{D}\|cos\theta\end{aligned}$$

$$\|\vec{r}\| = \frac{\|\vec{C}\|^2}{GM + \|\vec{D}\|cos\theta} \qquad = \frac{\|\vec{C}\|^2}{GM(1 + ecos\theta)}, e = \frac{\|\vec{D}\|}{GM}$$

Observed from data from table 3 and 4, all values of $0 < e < 1$, which satisfy the algebraic derivation of the polar form of general equation for ellipse. Since $\vec{v} = \frac{d\vec{r}}{dt}$, it can be seen that it has much relation with e in the equation.

$$\|\vec{r}\| = \frac{\|\vec{r_0}\|}{(1 + ecos\theta)}$$

As an analogy, $r_0 = 1 \Rightarrow \|\vec{C}\| = 2\pi$ as an analogy.
This derivation [1]accounts for the orbit being elliptical.

6.5 Methodology comparison

In this report, we have adopted both numpy and excel spreadsheet as two very different methods to obtain results.

6.5.1 Accuracy Evaluation

Data listed in this report are all corrected to 4 decimal places. From all the listed ones, it is possible to observe that, in measurement units of AU and year, the differences of datas of using the very different methods occurs from the third decimal place and beyond. This introduced the differences in terms of advantages and disadvantages regarding the use of both in this report.
Talking about advantages, in numpy, the time interval is set to be 0.0001 while is 0.0002 in excel spreadsheet. It comes more handy for display in terms of numpy, since it is displays less massive amount data, easier to trace at the same time.
On the other hand, excel spreadsheet allows us to see the trends of data, giving the more easy way to generate conclusions by observation and without prior computational knowledge.
To conclude, both are suitable regarding handling fundamental questions in this report as results agreed to most relations in Kepler's Third Law reasonably well.

7 Conclusion

It is concluded that using numerical methods can find out the trajectory of orbits, elliptical and circular successfully, with the help of computational

[1]derivation adopted from Libretexts. "13.4: Motion in Space: Velocity and Acceleration." Mathematics LibreTexts. Libretexts, April 26, 2019.

tools.
However, the solution is not perfect and error depends on how small the time interval has been taken.
In this report, we have also tested for the validity of Kepler's First and Second Law by using numerical methods for testing and proving Kepler's Third Law that the result is partly true for obeying Kepler's Second Law only. We have also tried to investigate upon algebraic relations with the results and found out the relationship between the tested alpha-values with the shape of orbit.
As a whole, the result is not perfect but still have room for improvement in terms of accuracy.