Population III Stars Dark Matter Capture

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This document details the derivations of dark matter capture rate for simulation-wise theoretical checking from "Premature Black Hole Death of Population III Stars by Dark Matter".

1 Non-relativistic Head-on Collision of Dark Matter Particle and mass

Consider a single collision between a right-moving DM particle of mass m_χ with initial velocity $\vec{v}$ and a left-moving target of mass m_i moving with velocity $\vec{v}_i$. In the DM rest frame, assu,img elastic collisions with zero impack parameter, the final velocity of Dark Matter Particle denoted as $\vec{v}'$ is aimed to be calculated.

To transform to DM rest frame before collision, boost the frame to the moving frame is needed. Thus, superscript b denotes the quantity under boosted frame.

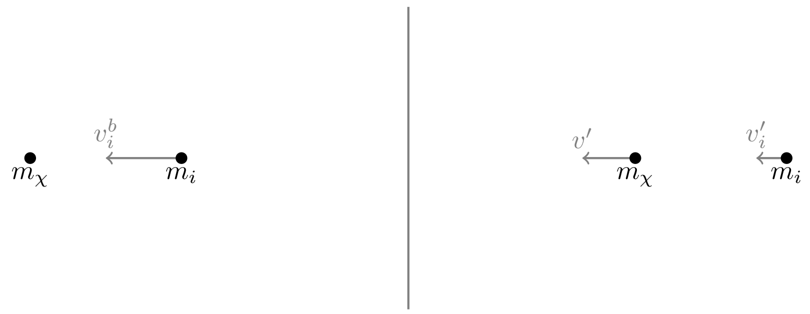

Figure 1: Single collision in the rest frame of DM particle before collision

The conditions of being elastic collision and conserving linear momentum, the following equations can be written down.

$$\text{elastic collision} \quad \tfrac{1}{2}m_i|\vec{v}_i^b|^2 = \frac{1}{2}m_\chi|\vec{v}'|^2 + \frac{1}{2}m_i|\vec{v}_i{}'|^2 \tag{1}$$

$$\text{momentum conservation} \quad m_i\vec{v}_i^b = m_\chi\vec{v}' + m_i\vec{v}_i{}' \tag{2}$$

Without loss of generality, by rearranging the terms in Equation (1), the equation becomes

$$m_i\left(|\vec{v}_i^b|^2 - |\vec{v}_i'|^2\right) = m_\chi|\vec{v}'|^2 \tag{3}$$

Using dot products, Equation (3) can be rewritten as follows.

$$m_i\left(\vec{v}_i^b + \vec{v}_i'\right)\cdot\left(\vec{v}_i^b - \vec{v}_i'\right) = m_\chi\left(\vec{v}'\cdot\vec{v}'\right) \tag{4}$$

Rearranging terms in Equation (2),

$$m_i\left(\vec{v}_i^b - \vec{v}_i'\right) = m_\chi\vec{v}' \tag{5}$$

the difference in velocity for the initial target particle can be expressed in terms of the final velocity of the DM particle in the boosted frame of initial velocity travelling frame.

$$\vec{v}_i^b - \vec{v}_i' = \frac{m_\chi}{m_i}\vec{v}' \tag{6}$$

Yielding the difference in velocity to be as follows.

$$|\vec{v}_i^b|^2 - |\vec{v}_i'|^2 = \left(\vec{v}_i^b + \vec{v}_i'\right) \cdot \left(\frac{m_\chi}{m_i}\vec{v}'\right) = \frac{m_\chi}{m_i}\left(\vec{v}_i^b + \vec{v}_i'\right) \cdot \vec{v}' \tag{7}$$

Putting this expression in Equation (4),

$$m_\chi \left(\vec{v}_i^b + \vec{v}_i'\right) \cdot \vec{v}' = m_\chi \left(\vec{v}' \cdot \vec{v}'\right) \tag{8}$$

Resulting in the following form.

$$\vec{v}' = \vec{v}_i^b + \vec{v}_i' \tag{9}$$

Making use of momentum conservation Equation (5), from relations from Equation (9), eliminates the variable for the final velocity of DM particle in boosted frame.

$$m_i \left(\vec{v}_i^b - \vec{v}_i'\right) = m_\chi \left(\vec{v}_i^b + \vec{v}_i'\right) \tag{10}$$

The final velocity of the initial mass can be found by rearranging the orders from Equation (10).

$$\boxed{\vec{v}_i' = \frac{m_i - m_\chi}{m_i + m_\chi}\vec{v}_i^b} \tag{11}$$

Plugging in Equation (11) into Equation (9) results in final velocity of DM particle after collision.

$$\boxed{\vec{v}' = \frac{2m_i}{m_i + m_\chi}\vec{v}_i^b = \frac{2\mu_{i,\chi}}{m_\chi}\vec{v}_i^b} \tag{12}$$

1.1 Relativistic picture

In the upper derivations, it has taken into effect the boosted velocity, without regarding to lorentz transformations.

Recalling that the lorentz boost to the frame with velocity of DM particle velocity before collision, where each measurement made is relative to this frame s.t. the lorentz factor is

$$\gamma = \left(1 - \frac{{v_{rel}}^2}{c^2}\right)^{-1/2}. \tag{13}$$

Therefore, the set of equations to solve becomes the following.

$$\begin{aligned} \text{elastic collision} \quad & (\gamma_i - 1)\, m_i c^2 && = (\gamma' - 1)\, m_\chi c^2 + (\gamma_i' - 1)\, m_i c^2 \\ \text{momentum conservation} \quad & \gamma_i m_i \vec{v}_i^b && = \gamma_\chi m_\chi \vec{v}' + \gamma_i' m_i \vec{v}_i{}' \end{aligned}$$

The lorentz factors are

$$\gamma_i = \left(1 - \frac{|v_i - v|^2}{c^2}\right)^{-1/2} \quad \gamma_\chi = \left(1 - \tfrac{|v_\chi - v|^2}{c^2}\right)^{-1/2} \quad \gamma_i' = \left(1 - \frac{|v_i' - v|^2}{c^2}\right)^{-1/2}$$

In the process of simplifying the equations, it is noticed that it is inevitable for the variables of v_i, v_i' and v_i^b to become inseparable.

Instead of brute-force algebra to solving the equations, it is more reasonable to say DM particle moves with comparable velocity to target mass.

1.2 Collision with deflection angles

Consider the scattering problem as a general case with impact parameter. If we consider the initial position of the target particle, the scatter picture becomes the following.

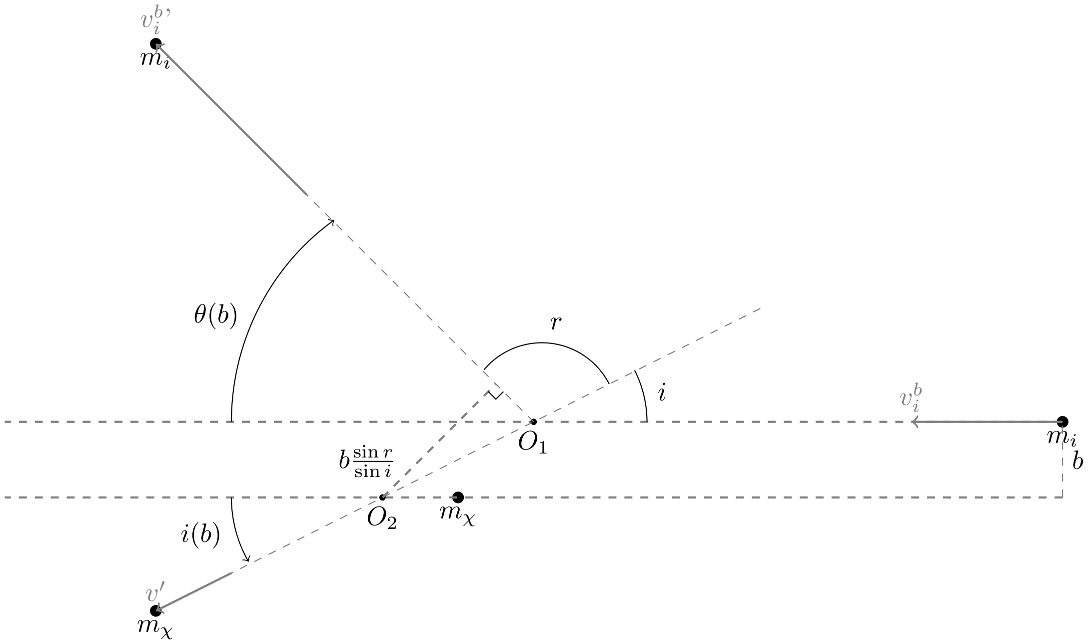

Figure 2: Scattering picture in initial DM moving frame (with consideration of angular momentum)

Here, there are three conservation relations. Energy, linear momentum and angular momentum conservation.

Angular momentum conservation

The initial momentum about the initial position of the DM mass (particle) is

$$L = m_i|\vec{v}_i^b|b. \tag{14}$$

The final angular momentum is contributed by the final momentum vectors and their relative initial position of DM particle. Since the momental center O_2 coincides with the $b = 0$ line, there is no final contribution to from $m_\chi\vec{v}'$. Around momental center O_1, the final momentum of target mass $m_i\vec{v}_i^b$ contributes to the final angular momentum only, thus, from vector diagram conservation.

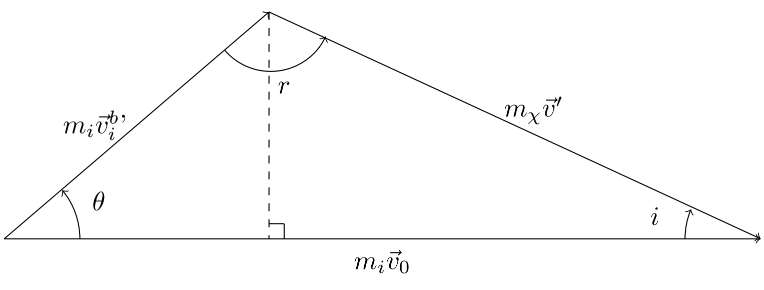

Figure 3: Linear momentum conservation of in moving frame of initial DM speed

By sine rule, the linear momentum magnitudes are related by

$$m_i|\vec{v}_i^b|\sin i = m_i|\vec{v}_i^{b}\text{'}|\sin r, \tag{15}$$

either from Figure 2, we can get the final momentum by geometric calculation or from linear momentum conservation, the conservation of angular momentum is derived as

$$L = m_i|\vec{v}_i^b|b = m_i|\vec{v}_i^{b}\text{'}|b\frac{\sin r}{\sin i} \tag{16}$$

1.3 Scattering with deflection angles: generalization

Consider the momentum conservation by decomposing the directions of vectors in Figure 3, since we are using scalars here, I will drop the vector and absolute symbol for readability.

$$
\begin{aligned}
m_i v_i^b &= m_i v_i^{b\text{'}} \cos\theta + m_\chi v' \sin r && (17)\\
0 &= m_i v_i^{b\text{'}} \sin\theta - m_\chi v' \sin i && (18)
\end{aligned}
$$

Changing subjects for the second equation above, θ can be written as a function of i, and ratios of magnitudes of momentums $m_\chi v'$ and $m_i v_i^{b\text{'}}$,

$$\sin\theta = \frac{m_\chi}{m_i}\frac{v'}{v_i^{b\text{'}}}\sin i, \tag{19}$$

but this is not very useful, since seeking for the direct relation of angles without the velocity ratios are simpler.

By rewriting and combining the terms of $\left(m_i v_i^{b\text{'}}\right)^2$, yields simplified combined expression

$$\left(m_i v_i^b - m_\chi v' \cos(i)\right)^2 + \left(m_\chi v' \sin(i)\right)^2 = \left(m_i v_i^{b\text{'}}\right)^2 \tag{20}$$

Using energy conservation

$$m_i v_i^{b^2} = m_i v_i^{b\text{'}2} + m_\chi v'^2$$

Plugging into the previous expression to eliminate final velocity of target mass $v_i^{b\text{'}}$,

$$\left(m_i v_i^b - m_\chi v' \cos(i)\right)^2 + \left(m_\chi v' \sin(i)\right)^2 = m_i^2 v_i^{b^2} - m_i m_\chi v'^2 \tag{21}$$

Expanding and eliminating the terms for the initial velocity of target mass v_i^b,

$$m_i^2 v_i^{b^2} - 2 m_i m_\chi v_i^b v' \cos(i) + m_\chi^2 v'^2 = m_i^2 v_i^{b^2} - m_i m_\chi v'^2 \tag{22}$$

Thus, simplifying give

$$\boxed{m_\chi v' = 2\frac{m_i m_\chi}{m_i + m_\chi} v_i^b \cos(i) = 2\mu v_i^b \cos(i)} \tag{23}$$

There we have the magnitude for the final velocity of DM particle in terms of scattering angles.

To cross check the vector form (more general) for the final velocity of DM particle, using dot product

$$v_i^b \cos(i) = \frac{\vec{v}' \cdot \vec{v}_i^{b\text{'}}}{|\vec{v}'|} \tag{24}$$

Combing it to the boxed equation of final velocity expression

$$m_\chi v' = 2\mu \frac{\vec{v}' \cdot \vec{v}_i^{b\text{'}}}{|\vec{v}'|} \tag{25}$$

Again, recalling the dot product identity $\vec{a} \cdot \vec{a} = a^2$,

$$m_\chi |\vec{v}'| \cdot \vec{v}' = 2\mu \left(\frac{|\vec{v}'|^2}{|\vec{v}'|}\right) \vec{v}_i^b \tag{26}$$

Thus the vector form of the boxed equation for the final momentum of the DM particle will be

$$m_x \vec{v}' = 2\mu \vec{v}_i^b \tag{27}$$

which matches the one we derived from the previous section.

1.4 More on scattering angles

Besides verifying the vector form of the final velocity of the dark matter, we also seek for the relationship between scattering angles θ and i.

From Equation (19), we can plug in the expression to Equation (17).

$$\begin{aligned} m_i v_0 &= m_i \left(\frac{m_\chi}{m_i} \frac{v'}{v_i^{b},} \sin i \right) \cos\theta + m_\chi v_2 \cos i \\ &= m_\chi v_2 \left(\frac{\sin i}{\tan\theta} + \cos i \right) \end{aligned}$$

So we can have v_0 as a function of mass ratios and as functions of scattering angles θ and i,

$$v_i^b = \frac{m_\chi}{m_i} v' \left(\frac{\sin i}{\tan\theta} + \cos i \right) \tag{28}$$

While we also have the energy conservation condition we have

$${v_i^b}^2 = v_i^{b,2} + \frac{m_\chi}{m_i} v'^2$$

Combining the two for ${v_i^b}^2$, and Equation (19) yield

$$\left(\frac{m_\chi}{m_i} \right)^2 v'^2 \left(\frac{\sin i}{\tan\theta} + \cos i \right)^2 = \left(\frac{m_\chi}{m_i} \right)^2 \frac{\sin^2 i}{\sin^2\theta} v'^2 + \left(\frac{m_\chi}{m_i} \right) v'^2$$

Dividing the whole equation with the common factor,

$$\frac{m_\chi}{m_i} \left(\frac{\sin^2 i}{\tan^2\theta} + \frac{\sin(2i)}{\tan\theta} + \cos^2 i \right) = \frac{m_\chi}{m_i} \frac{\sin^2 i}{\sin^2\theta} + 1 \tag{29}$$

Further simplifications are shown as follows

$$\begin{aligned} \frac{\sin^2 i}{\tan^2\theta} - \frac{\sin^2 i}{\sin^2\theta} + \frac{\sin(2i)}{\tan\theta} + \cos^2 i &= \frac{m_i}{m_\chi} \\ \frac{\sin^2 i}{\sin^2\theta} \left(\cos^2\theta - 1 \right) + \frac{\sin(2i)}{\tan\theta} + \cos^2 i &= \frac{m_i}{m_\chi} \\ \frac{\sin^2 i}{\sin^2\theta} \left(\cos^2 i - \sin^2 i \right) + \frac{\sin(2i)}{\tan\theta} &= \frac{m_i}{m_\chi} \end{aligned}$$

There we have the relation of θ and i,

$$\tan\theta = \sin i / \left\{ \left(\frac{m_i}{m_\chi} \right) - \cos(2i) \right\} \tag{30}$$

1.5 Energy

Since the expression for the velocity in the rest frame of DM particle before collision is obtained, it is possible to boost back the stellar rest frame to get the final velocity of the DM particle.

The notation can be a bit confusing, since v' from the previous derivation is in fact the velocity after collision **in the boosted frame $\tilde{\mathbf{v}}'$**.

$$\tilde{\mathbf{v}}' = \vec{v}' - \vec{v} = \frac{2\mu_{i,\chi}}{m_\chi}\left(\vec{v}_i - \vec{v}\right) \tag{31}$$

Rearranging the terms, the final velocity is as follows.

$$\vec{v}' = \frac{2m_i}{m_i + m_\chi}\vec{v}_i + \left(1 - \frac{2m_i}{m_i + m_\chi}\right)\vec{v} = \frac{2m_i}{m_i + m_\chi}\vec{v}_i + \frac{m_\chi - m_i}{m_i + m_\chi}\vec{v} \tag{32}$$

Thus, the exact change of kinetic energy can be expressed in the following way.

$$\begin{aligned}
|\vec{v}'|^2 - |\vec{v}|^2 &= \left(\vec{v}' + \vec{v}\right) \cdot \left(\vec{v}' - \vec{v}\right) \\
&= \left(\frac{2m_i}{m_i + m_\chi}\vec{v}_i + \frac{2m_\chi}{m_i + m_\chi}\vec{v}\right) \cdot \left(\frac{2m_i}{m_i + m_\chi}\vec{v}_i - \frac{2m_i}{m_i + m_\chi}\vec{v}\right) \\
&= \left(\frac{2m_i}{m_i + m_\chi}\right)^2 |\vec{v}_i|^2 + \left(\frac{4m_i m_\chi}{\left(m_i + m_\chi\right)^2} - \frac{4m_i^2}{\left(m_i + m_\chi\right)^2}\right)\vec{v}\cdot\vec{v}_i - \frac{4m_i m_\chi}{\left(m_i + m_\chi\right)^2}|\vec{v}|^2
\end{aligned}$$

Multiplying the previous expression by a factor of $\frac{1}{2}m_\chi$ gives the exact kinetic energy loss.

$$\Delta T = \frac{2m_i^2 m_\chi}{\left(m_i + m_\chi\right)^2}|\vec{v}_i|^2 + \left(\frac{2m_i m_\chi^2}{\left(m_i + m_\chi\right)^2} - \frac{2m_i^2 m_\chi}{\left(m_i + m_\chi\right)^2}\right)\vec{v}\cdot\vec{v}_i - \frac{2m_i m_\chi^2}{\left(m_i + m_\chi\right)^2}|\vec{v}|^2 \tag{33}$$

Introducing a new variable for simpler estimation steps,

$$\beta_\pm \equiv \frac{4m_i m_\chi}{\left(m_i \pm m_\chi\right)^2} \tag{34}$$

Expressing the exact energy loss in terms of new variable β_+ is as follows.

$$\Delta T = \frac{\beta_+}{2}m_i|\vec{v}_i|^2 + \left(\frac{\beta_+}{2}m_\chi - \frac{\beta_+}{2}m_i\right)\vec{v}\cdot\vec{v}_i - \frac{\beta_+}{2}m_\chi|\vec{v}|^2 \tag{35}$$

If we find the exact ratio of fractional energy change, we have the following.

$$\frac{\Delta E}{E_0} = \beta_+\left[\frac{m_i}{m_\chi}\frac{|\vec{v}_i|^2}{|\vec{v}|^2} + \frac{m_\chi - m_i}{m_\chi}\frac{\vec{v}\cdot\vec{v}_i}{|\vec{v}|^2} - 1\right] \tag{36}$$

Rewriting give

$$\frac{\Delta E}{E_0} = \beta_+\left[\frac{m_i}{m_\chi}\frac{|\vec{v}_i|^2}{|\vec{v}|^2} + \frac{m_\chi - m_i}{m_\chi}\frac{|\vec{v}_i|}{|\vec{v}|}\cos\theta - 1\right] \tag{37}$$

, where θ is the angle between the initial DM particle incoming velocity and that from initial target mass.

Now let the ratios be a variable and see the behaviour of these,

$$\epsilon \equiv \frac{m_i}{m_\chi}; \quad \delta \equiv \frac{|\vec{v}_i|}{|\vec{v}|} \tag{38}$$

The previous expression now becomes

$$\frac{\Delta E}{E_0} = \beta_+\left[\epsilon\delta^2 + \left(1 - \epsilon\right)\delta\cos\theta - 1\right] \tag{39}$$

2 Scattering in different limits

The kinetic theory illustrats collisions between DM particles in a medium in a simple language. Using the mean free path or average distance to describe collisions in between medium molecules of different regimes is a fair approach.

The mean free path denoted as ℓ_χ can be characterised by the cross-sectional area[1,2] for collision in medium.

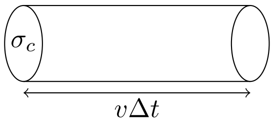

Figure 4: Volume of medium crossed by cross-sectional area

Say a medium as above only contain one type of identical target mass m_i, that it contains total mass M s.t.

$$M = Nm_i$$

where N is the total number of target mass.

In order to estimate the mean free path, it is essential to find out the number density of target mass.

$$n_i = \frac{N}{V} = \frac{M}{m_i}\frac{1}{V} = \frac{\rho_i}{m_i}$$

Recalling the mean free path refers to the distance of particle travelled without change of direction.

$$\ell_\chi \approx \frac{\text{length of path}}{\text{number of collisions}}$$

The number of collisions is approximately the same to the number of target mass inside the medium in the volume shown in the previous diagram.[3]

$$\ell_\chi \approx \frac{v\Delta t}{(\sigma_c \times v\Delta t)n_i} = \frac{1}{n_i\sigma_c} \tag{40}$$

Classification of regimes

The quantity Knudsen number Kn (alternatively the optical depth $\tau_\star$) is defined as follows

$$\text{Kn} \equiv \frac{\ell_\chi}{2R_\star} \equiv \frac{1}{\tau_\star} \tag{41}$$

This is visually equivalently to having a cross section of the star and estimating the number of collisions, to estimating the number density of the content (target mass) in the star.

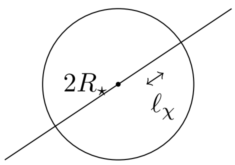

Figure 5: Cross-sectional area of spherical star with mean free path (Not to scale)

Considering the optical depth to DM particle, it refers to how far the DM particle can see without bumping into or scattering from the molecules through the star. The smaller number of mean free path that can fit into the diameter, implies the farther the DM particle can "see". And therefore, is more dilute, also requiring larger number of Kn.

According to the article, Knudsen numbers below $\text{Kn} \lesssim 10^{-2}$ in a 75% hydrogen Population III star (assuming ther target nuclei to be protons), corresponds to cross-sections $\sigma_c \gtrsim 10^{-34}\text{cm}^2$, regarded as the fluid regime. On the other hand, the Knudsen numbers $\text{Kn} \gtrsim 1$ corresponds to cross-sections $\sigma_c \lesssim 10^{-36}\text{cm}^{-2}$, regarded as particle regime.

[1]Often in kinetic theory we regard this as effective collision area. Since we are considering DM particle collistion with different molecules like Hydrogen and Helium, it is still a variable considering which case it lies in.

[2]More sepcifically speaking, in scattering language, $\sigma_c \equiv \frac{\text{Number of particles scattered per atom per sec}}{\text{Number of beam particles per } cm^3 \text{per sec}}$

[3]v is the exact velocity before collision.

2.1 The Fluid Regime

From the previous sections, the average change in kinetic energy per distance travelled between scattering events ℓ_χ is given as

$$\frac{\Delta E}{\ell_\chi} = n_i \sigma_c \Delta E. \tag{42}$$

In the fluid limit, it is relatively opaque to the DM particle, where $\ell_\chi \ll R_\star$ and $\mathrm{Kn} \to 0$, under the scale of $R_\star$, consider energy along the scattering path, that is demonstrated in the figure as follows.

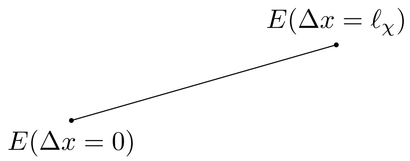

Figure 6: Energy as a function of position in a head-on collision under classical approach (Not to scale)

Making use of Equation (42), for the third equality, the energy change per average scattering event distance is as follows

$$\frac{\Delta E}{\ell_\chi} = \frac{E(x_0 + \ell_\chi) - E(x_0)}{\ell_\chi} \approx \frac{dE}{dx} \sim \frac{\rho_i}{m_i} \sigma_c \Delta E. \tag{43}$$

By requiring, $\Delta E = \Delta T$ which implies the microscopic potential energy (intermolecular potential energy) remains the same.

2.1.1 Large DM initial energy

Recall the form of the change of kinetic energy in Equation () that contains two terms with averaged collision angles.

Consider when the initial DM particle dominates, s.t.

$$\Delta T = -\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 + \frac{2m_\chi m_i^2}{(m_\chi + m_i)^2} |\vec{v}_i|^2 \tag{44}$$

$$\Delta E \approx -\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 \tag{45}$$

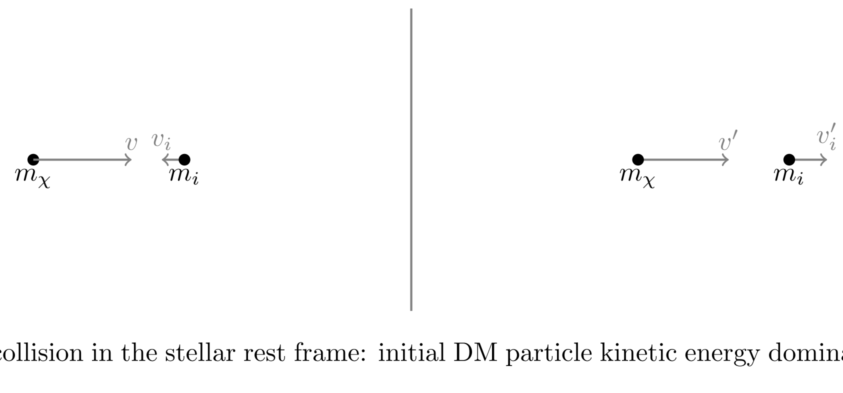

Figure 7: Single collision in the stellar rest frame: initial DM particle kinetic energy dominates (Not to scale)

Plugging the approximated change of energy into Equation (43),

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i} \sigma_c \left(-\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 \right) \tag{46}$$

Since we are to estimate maximal velocity the DM particle can have far away from the star. The general expression of the DM particle can be written as follows.

$$E(x) = \frac{1}{2} m_\chi \left(v_{\mathrm{esc},\star}^2 + v(x)^2 \right) \tag{47}$$

Using equation (46) with the above expression for energy in general from equation (47).

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i}\sigma_c \left(-\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} \left(\frac{2E}{m_\chi} - v_{\text{esc},\star}^2 \right) \right) \tag{48}$$

Simplifying the expression, the differential equation becomes,

$$\frac{dE}{dx} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) \left(E - \frac{1}{2}m_\chi v_{\text{esc},\star}^2 \right) \tag{49}$$

where we can perform change of variables and further simplify and solve for the solution.

By letting a new variable $\tilde{E}$,

$$\tilde{E} = E - \frac{1}{2}m_\chi v_{\text{esc},\star}^2 \tag{50}$$

The first order ODE becomes,

$$\frac{d\tilde{E}}{dx} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) \tilde{E} \tag{51}$$

that allows us to solve by separation of variables.

$$\frac{1}{\tilde{E}} d\tilde{E} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) dx$$

Integrating both sides gives use, with $r \to \infty$,

$$\ln \left(\frac{E\,(r)}{E\,(R_\star)} \right) \sim \frac{2\rho_i\sigma_c R_\star}{m_\chi} \left(\frac{2m_\chi^2}{(m_\chi + m_i)^2} \right) \tag{52}$$

the equation for the velocity at far away from star can be written by using Equation (50).

$$\boxed{v \sim v_{\text{esc},\star} \left(\exp\left(\frac{2\rho_i\sigma_c R_\star}{m_\chi} \left(\frac{2m_\chi^2}{(m_\chi + m_i)^2} \right) \right) - 1 \right)^{1/2}} \tag{53}$$

It is however different from the expression from our article.

$$v \sim v_{\text{esc},\star} \left(\exp\left(\frac{2\bar{\rho}_\star\sigma_c R_\star}{m_\chi} \right) - 1 \right)^{1/2} \tag{54}$$

The understanding to the velocity under the limit of dominating initial DM particle energy having ρ_i as $\bar{\rho}_s tar$ appears to be quite straight forward. As mentioned from the second footnote before, the composition of star is not of only one type of molecule, which corresponds to the mean value from the notation. Therefore, it does not make sense to me of having the two expression as the same expressions.

It agrees to the qualitative understanding of the behavior of velocity for the incoming dark matter particle where it loses energy exponentially fast as it penetrates the star.

2.1.2 Large traget mass initial energy

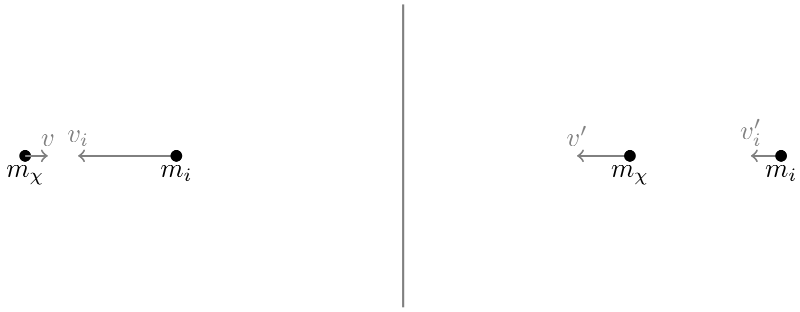

Figure 8: Single collision in the stellar rest frame: initial target mass energy dominates (Not to scale)

Again, making use of Equation (43), we consider the initial energy of the colliding system having dominating energy the target mass which yields the following differential equation.

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i}\sigma_c \left(\frac{2m_\chi m_i^2}{\left(m_\chi + m_i\right)^2} |\vec{v}_i|^2 \right) \tag{55}$$

This equation can be solved directly using separation of variables,

$$E\left(r\right) - E\left(R_\star\right) \sim \left(2\rho_i\sigma_c \frac{m_\chi m_i}{\left(m_\chi + m_i\right)^2} |\vec{v}_i|^2 \right) \left(r - R_\star\right) \tag{56}$$

Taking the energy of DM particle at $x = R_\star$ to become

$$E\left(R_\star\right) = \frac{1}{2} m_\chi v_{\text{esc},\star}^2. \tag{57}$$

The velocity equation w.r.t. distance from the star is as follows.

$$\boxed{v \sim v_i \frac{\left(2m_i\rho_i\sigma_c\right)^{1/2}}{m_\chi + m_i} \left(r - R_\star\right)^{1/2}} \tag{58}$$

It appears to be very strange since it diverges, which is weird when we are seeking for a maximal velocity of the DM particle to be captured. In this limit, it only agrees with the article by its linear relationship for velocity with the distance.

Meanwhile the expression of the velocity of the DM particle in the article is

$$v \sim v_i \frac{\left(2m_i\rho_i R_\star \sigma_c\right)^{1/2}}{m_\chi + m_i}? \tag{59}$$

2.2 Capture Probability

Then it all appears natural that the capture probability P_{cap} is

$$P_{\text{cap}} = \Theta\left(v_{\max} - v\right) \tag{60}$$

with $\Theta\left(x\right)$ as the Heaviside step function.