Population III Stars Dark Matter Capture

Ying Chan

June 14, 2022

This document details the derivations of dark matter capture rate for simulation-wise theoretical checking from "Premature Black Hole Death of Population III Stars byu Dark Matter".

1 Non-relativistic Head-on Collision of Dark Matter Particle and mass

Consider a single collision between a right-moving DM particle of mass m_{DM} with initial velocity $\vec{v}$ and a left-moving target of mass m_i moving with velocity $\vec{v}_i$. In the DM rest frame, assu,img elastic collisions with zero impack parameter, the final velocity of Dark Matter Particle denoted as $\vec{v}'$ is aimed to be calculated.

To transform to DM rest frame before collision, boost the frame to the moving frame is needed. Thus, superscript b denotes the quantity under boosted frame.

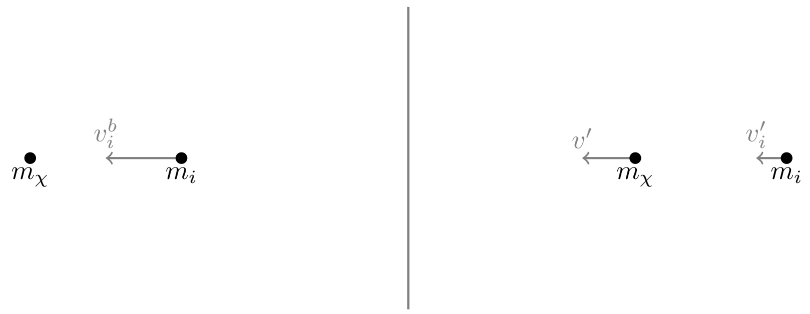

Figure 1: Single collision in the rest frame of DM particle before collision

The conditions of being elastic collision and conserving linear momentum, the following equations can be written down.

$$\begin{aligned}
\text{elastic collision} \quad \tfrac{1}{2}m_i|\vec{v}_i^b|^2 &= \frac{1}{2}m_\chi|\vec{v}'|^2 + \frac{1}{2}m_i|\vec{v}_i{}'|^2 \\
\text{momentum conservation} \quad m_i\vec{v}_i^b &= m_\chi\vec{v}' + m_i\vec{v}_i{}'
\end{aligned}$$

Since this is a 1 dimensional problem, momentum conservation equation vector signs can be dropped for convenience.

$$\text{elastic collision} \quad \tfrac{1}{2}m_i {v_i^b}^2 = \frac{1}{2}m_\chi v'^2 + \frac{1}{2}m_i v_i'^2 \tag{1}$$

$$\text{momentum conservation} \quad m_i v_i^b = m_\chi v' + m_i v_i' \tag{2}$$

Simplifying equation (1), by dropping the fraction and rearragment of terms, can allow further factorization for the velocities.

$$m_i\left(v_i^b + v_i'\right)\left(v_i^b - v_i'\right) = m_\chi v'^2 \tag{3}$$

The relationship between the final velocities of DM particle and target mass is resulted by dividing equation (3) by equation (2).

$$v_i^b + v_i' = v' \tag{4}$$

Applying this condition to equation (2) and eliminate the variable v'.

$$m_i v_i^b = m_\chi \left(v_i^b + v_i' \right) + m_i v_i' \tag{5}$$

The expression for the final velocity of the target mass in terms of v_i^b is as given

$$v_i' = \frac{m_i - m_\chi}{m_i + m_\chi} v_i^b . \tag{6}$$

Then, the final velocity of the DM particle is found represented in the following form.

$$v' = \frac{2m_i}{m_i + m_\chi} v_i^b \tag{7}$$

Considering this as a sanity check for a scattering problem, in the scattering picture, it is more natural to express it as reduced mass given as

$$\mu_{i,\chi} = \left(\frac{1}{m_\chi} + \frac{1}{m_i} \right)^{-1} = \frac{m_\chi m_i}{m_\chi + m_i} . \tag{8}$$

Therefore, the equation (3.3) in the paper is reproduced.

$$\boxed{|\vec{v}'| = \frac{2|v_i^b|}{m_\chi} \mu_{i,\chi}} \tag{9}$$

1.1 Relativistic picture

In the upper derivations, it has taken into effect the boosted velocity, without regarding to lorentz transformations.

Recalling that the lorentz boost to the frame with velocity of DM particle velocity before collision, where each measurement made is relative to this frame s.t. the lorentz factor is

$$\gamma = \left(1 - \frac{{v_{rel}}^2}{c^2} \right)^{-1/2} . \tag{10}$$

Therefore, the set of equations to solve becomes the following.

$$\begin{array}{rll} \text{elastic collision} & (\gamma_i - 1)\, m_i c^2 & = (\gamma' - 1)\, m_\chi c^2 + (\gamma_i' - 1)\, m_i c^2 \\ \text{momentum conservation} & \gamma_i m_i \vec{v}_i^b & = \gamma_\chi m_\chi \vec{v}' + \gamma_i' m_i \vec{v_i}' \end{array}$$

The lorentz factors are

$$\gamma_i = \left(1 - \frac{|v_i - v|^2}{c^2} \right)^{-1/2} \quad \gamma_\chi = \left(1 - \tfrac{|v_\chi - v|^2}{c^2} \right)^{-1/2} \quad \gamma_i' = \left(1 - \frac{|v_i' - v|^2}{c^2} \right)^{-1/2}$$

In the process of simplifying the equations, it is noticed that it is inevitable for the variables of v_i, v_i' and v_i^b to become inseparable.

Instead of brute-force algebra to solving the equations, it is more reasonable to say DM particle moves with comparable velocity to target mass. Huh ?[1]

[1] shou shou pei sin?

1.2 Energy

Since the expression for the velocity in the rest frame of DM particle before collision is obtained, it is possible to boost back the stellar rest frame to get the final velocity of the DM particle.

The notation can be a bit confusing, since v' from the previous derivation is in fact the velocity after collision **in the boosted frame $\tilde{\mathbf{v}}'$**.

$$\tilde{\mathbf{v}}' = \vec{v}' - \vec{v} = \frac{2\mu_{i,\chi}}{m_\chi}\left(\vec{v}_i - \vec{v}\right) \tag{11}$$

Multiplying both sides with a minus sign yields the following

$$\vec{v} - \vec{v}' = \frac{2\mu_{i,\chi}}{m_\chi}\left(\vec{v} - \vec{v}_i\right)$$

Thus, giving us the final velocity of DM particle in stellar rest frame to be

$$\vec{v}' = \vec{v} - \frac{2\mu_{i,\chi}}{m_\chi}\left(\vec{v} - \vec{v}_i\right) \tag{12}$$

It is therefore obtainable that the velocity modulus squared in the following way.

$$|\vec{v}'|^2 = \vec{v}' \cdot \vec{v}' = |\vec{v}|^2 - \frac{4\mu_{i,\chi}}{m_\chi}\left(|\vec{v}|\left(|\vec{v}| - |\vec{v}_i|\cos\theta\right) - \frac{\mu_{i,\chi}}{m_\chi}|\vec{v} - \vec{v}_i|^2\right) \tag{13}$$

where θ is the angle between the initial DM particle and the target mass initial velocity in stellar rest frame. The average change in kinetic energy is thus

$$\Delta T = \frac{1}{2}m_\chi\left(|\vec{v}'|^2 - |\vec{v}|^2\right) = -2\mu_{i,\chi}\left(|\vec{v}|\left(|\vec{v}| - |\vec{v}_i|\cos\theta\right) - \frac{\mu_{i,\chi}}{m_\chi}|\vec{v} - \vec{v}_i|^2\right) \tag{14}$$

Again, making use of the equality for dot product giving the modulus squared,

$$|\vec{v} - \vec{v}_i|^2 = (\vec{v} - \vec{v}_i) \cdot (\vec{v} - \vec{v}_i) = |\vec{v}|^2 - 2|\vec{v}||\vec{v}_i|\cos\theta + |\vec{v}_i|^2$$

Simplifying and regrouping terms from equation (14), the average change in kinetic energy over collision angle gives

$$\Delta T = -\frac{2m_\chi^2 m_i}{\left(m_\chi + m_i\right)^2}|\vec{v}|^2 + \frac{2m_\chi m_i^2}{\left(m_\chi + m_i\right)^2}|\vec{v}_i|^2\left(+\frac{6m_i^2 m_\chi + 2m_i m_\chi^2}{\left(m_i + m_\chi\right)^2}|\vec{v}||\vec{v}_i|\frac{\int_0^\pi \cos\theta}{\pi}\right)$$

Therefore, the average change in kinetic energy yields

$$\boxed{\Delta T = -\frac{2m_\chi^2 m_i}{\left(m_\chi + m_i\right)^2}|\vec{v}|^2 + \frac{2m_\chi m_i^2}{\left(m_\chi + m_i\right)^2}|\vec{v}_i|^2} \tag{15}$$

2 Scattering in different limits

The kinetic theory illustrats collisions between DM particles in a medium in a simple language. Using the mean free path or average distance to describe collisions in between medium molecules of different regimes is a fair approach.

The mean free path denoted as ℓ_χ can be characterised by the cross-sectional area[2,3] for collision in medium.

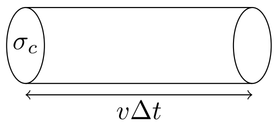

Figure 2: Volume of medium crossed by cross-sectional area

Say a medium as above only contain one type of identical target mass m_i, that it contains total mass M s.t.

$$M = Nm_i$$

where N is the total number of target mass.

In order to estimate the mean free path, it is essential to find out the number density of target mass.

$$n_i = \frac{N}{V} = \frac{M}{m_i}\frac{1}{V} = \frac{\rho_i}{m_i}$$

Recalling the mean free path refers to the distance of particle travelled without change of direction.

$$\ell_\chi \approx \frac{\text{length of path}}{\text{number of collisions}}$$

The number of collisions is approximately the same to the number of target mass inside the medium in the volume shown in the previous diagram.[4]

$$\ell_\chi \approx \frac{v\Delta t}{(\sigma_c \times v\Delta t)n_i} = \frac{1}{n_i\sigma_c} \tag{16}$$

Classification of regimes

The quantity Knudsen number Kn (alternatively the optical depth $\tau_\star$) is defined as follows

$$\mathrm{Kn} \equiv \frac{\ell_\chi}{2R_\star} \equiv \frac{1}{\tau_\star} \tag{17}$$

This is visually equivalently to having a cross section of the star and estimating the number of collisions, to estimating the number density of the content (target mass) in the star.

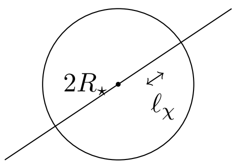

Figure 3: Cross-sectional area of spherical star with mean free path (Not to scale)

Considering the optical depth to DM particle, it refers to how far the DM particle can see without bumping into or scattering from the molecules through the star. The smaller number of mean free path that can fit into the diameter, implies the farther the DM particle can "see". And therefore, is more dilute, also requiring larger number of Kn.

According to the article, Knudsen numbers below $\mathrm{Kn} \lesssim 10^{-2}$ in a 75% hydrogen Population III star (assuming ther target nuclei to be protons), corresponds to cross-sections $\sigma_c \gtrsim 10^{-34}\mathrm{cm}^2$, regarded as the fluid regime. On the other hand, the Knudsen numbers $\mathrm{Kn} \gtrsim 1$ corresponds to cross-sections $\sigma_c \lesssim 10^{-36}\mathrm{cm}^{-2}$, regarded as particle regime.

[2] Often in kinetic theory we regard this as effective collision area. Since we are considering DM particle collistion with different molecules like Hydrogen and Helium, it is still a variable considering which case it lies in.

[3] More sepcifically speaking, in scattering language, $\sigma_c \equiv \frac{\text{Number of particles scattered per atom per sec}}{\text{Number of beam particles per } cm^3 \text{per sec}}$

[4] v is the exact velocity before collision.

2.1 The Fluid Regime

From the previous sections, the average change in kinetic energy per distance travelled between scattering events ℓ_χ is given as

$$\frac{\Delta E}{\ell_\chi} = n_i \sigma_c \Delta E. \tag{18}$$

In the fluid limit, it is relatively opaque to the DM particle, where $\ell_\chi \ll R_\star$ and $\mathrm{Kn} \to 0$, under the scale of $R_\star$, consider energy along the scattering path, that is demonstrated in the figure as follows.

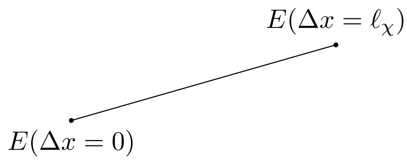

Figure 4: Energy as a function of position in a head-on collision under classical approach (Not to scale)

Making use of Equation (18), for the third equality, the energy change per average scattering event distance is as follows

$$\frac{\Delta E}{\ell_\chi} = \frac{E(x_0 + \ell_\chi) - E(x_0)}{\ell_\chi} \approx \frac{dE}{dx} \sim \frac{\rho_i}{m_i} \sigma_c \Delta E. \tag{19}$$

By requiring, $\Delta E = \Delta T$ which implies the microscopic potential energy (intermolecular potential energy) remains the same.

2.1.1 Large DM initial energy

Recall the form of the change of kinetic energy in Equation (15) that contains two terms with averaged collision angles.

Consider when the initial DM particle dominates, s.t.

$$\Delta T = -\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 + \frac{2m_\chi m_i^2}{(m_\chi + m_i)^2} |\vec{v}_i|^2 \tag{20}$$

$$\Delta E \approx -\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 \tag{21}$$

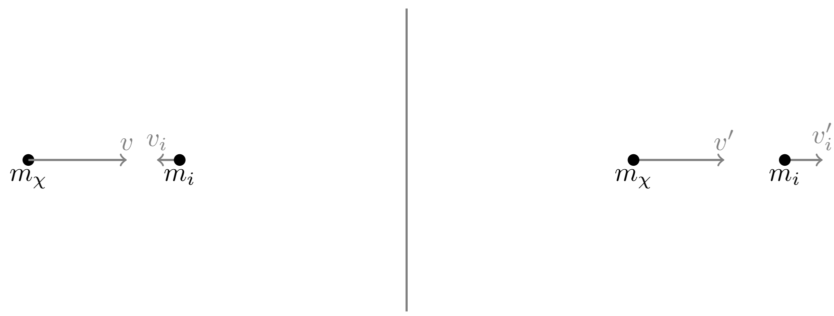

Figure 5: Single collision in the stellar rest frame: initial DM particle kinetic energy dominates (Not to scale)

Plugging the approximated change of energy into Equation (19),

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i} \sigma_c \left(-\frac{2m_\chi^2 m_i}{(m_\chi + m_i)^2} |\vec{v}|^2 \right) \tag{22}$$

Since we are to estimate maximal velocity the DM particle can have far away from the star. The general expression of the DM particle can be written as follows.

$$E(x) = \frac{1}{2} m_\chi \left(v_{\mathrm{esc},\star}^2 + v(x)^2 \right) \tag{23}$$

Using equation (22) with the above expression for energy in general from equation (23).

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i}\sigma_c \left(-\frac{2m_\chi^2 m_i}{\left(m_\chi + m_i\right)^2} \left(\frac{2E}{m_\chi} - v_{\text{esc},\star}^2 \right) \right) \tag{24}$$

Simplifying the expression, the differential equation becomes,

$$\frac{dE}{dx} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) \left(E - \frac{1}{2} m_\chi v_{\text{esc},\star}^2 \right) \tag{25}$$

where we can perform change of variables and further simplify and solve for the solution.

By letting a new variable $\tilde{E}$,

$$\tilde{E} = E - \frac{1}{2} m_\chi v_{\text{esc},\star}^2 \tag{26}$$

The first order ODE becomes,

$$\frac{d\tilde{E}}{dx} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) \tilde{E} \tag{27}$$

that allows us to solve by separation of variables.

$$\frac{1}{\tilde{E}} d\tilde{E} \sim -2\rho_i\sigma_c \left(\frac{2m_\chi}{(m_\chi + m_i)^2} \right) dx$$

Integrating both sides gives use, with $r \to \infty$,

$$\ln \left(\frac{E\left(r\right)}{E\left(R_\star\right)} \right) \sim \frac{2\rho_i\sigma_c R_\star}{m_\chi} \left(\frac{2m_\chi^2}{(m_\chi + m_i)^2} \right) \tag{28}$$

the equation for the velocity at far away from star can be written by using Equation (26).

$$\boxed{v \sim v_{\text{esc},\star} \left(\exp\left(\frac{2\rho_i\sigma_c R_\star}{m_\chi} \left(\frac{2m_\chi^2}{(m_\chi + m_i)^2} \right) \right) - 1 \right)^{1/2}} \tag{29}$$

It is however different from the expression from our article.

$$v \sim v_{\text{esc},\star} \left(\exp\left(\frac{2\bar{\rho}_\star\sigma_c R_\star}{m_\chi} \right) - 1 \right)^{1/2} \tag{30}$$

The understanding to the velocity under the limit of dominating initial DM particle energy having ρ_i as $\bar{\rho}_s tar$ appears to be quite straight forward. As mentioned from the second footnote before, the composition of star is not of only one type of molecule, which corresponds to the mean value from the notation. Therefore, it does not make sense to me of having the two expression as the same expressions.

It agrees to the qualitative understanding of the behavior of velocity for the incoming dark matter particle where it loses energy exponentially fast as it penetrates the star.

2.1.2 Large traget mass initial energy

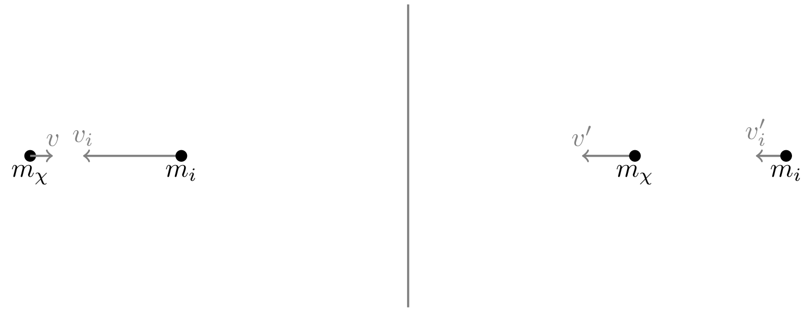

Figure 6: Single collision in the stellar rest frame: initial target mass energy dominates (Not to scale)

Again, making use of Equation (19), we consider the initial energy of the colliding system having dominating energy the target mass which yields the following differential equation.

$$\frac{dE}{dx} \sim \frac{\rho_i}{m_i}\sigma_c \left(\frac{2m_\chi m_i^2}{\left(m_\chi + m_i\right)^2} |\vec{v}_i|^2 \right) \tag{31}$$

This equation can be solved directly using separation of variables,

$$E\left(r\right) - E\left(R_\star\right) \sim \left(2\rho_i\sigma_c \frac{m_\chi m_i}{\left(m_\chi + m_i\right)^2} |\vec{v}_i|^2 \right) \left(r - R_\star\right) \tag{32}$$

Taking the energy of DM particle at $x = R_\star$ to become

$$E\left(R_\star\right) = \frac{1}{2} m_\chi v_{\mathrm{esc},\star}^2. \tag{33}$$

The velocity equation w.r.t. distance from the star is as follows.

$$\boxed{v \sim v_i \frac{\left(2m_i\rho_i\sigma_c\right)^{1/2}}{m_\chi + m_i} \left(r - R_\star\right)^{1/2}} \tag{34}$$

It appears to be very strange since it diverges, which is weird when we are seeking for a maximal velocity of the DM particle to be captured. In this limit, it only agrees with the article by its linear relationship for velocity with the distance.

Meanwhile the expression of the velocity of the DM particle in the article is

$$v \sim v_i \frac{\left(2m_i\rho_i R_\star \sigma_c\right)^{1/2}}{m_\chi + m_i} \tag{35}$$

2.2 Capture Probability

Then it all appears natural that the capture probability P_{cap} is

$$P_{\mathrm{cap}} = \Theta\left(v_{\max} - v\right) \tag{36}$$

with $\Theta\left(x\right)$ as the Heaviside step function.